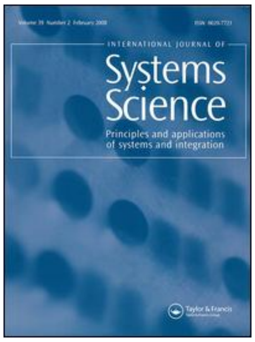

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High-order fully actuated system approaches: Part III. Robust control and high-order backstepping

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ABSTRACT

In this paper, three types of uncertain high-order nonlinear models are firstly proposed, namely, a single high-order fully actuated (HOFA) model with nonlinear uncertainties, an uncertain second-order strict-feedback system (SFS) and an uncertain high-order SFS, and the relations among these types of models are also discussed. Secondly, a direct approach for the design of robust stabilising controllers and robust tracking controllers for an uncertain single HOFA model are proposed based on the Lyapunov stability theory. Using the obtained robust control design result, the second- and high-order backstepping methods for the designs of robust stabilising controllers of the introduced second- and high-order SFSs are also proposed. The proposed approaches do not need to convert the high-order systems into first-order ones, and for a specific system design, the proposed high-order backstepping methods need fewer steps than the usual first-order backstepping method, hence are generally more direct and simpler. An illustrative example demonstrates both the effect and the application procedure of the proposed HOFA robust control approaches.

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1. Introduction

Most practical systems are subject to uncertainties. Robust control aims to preserve the stability and performance of systems under the existence of uncertainties, and has been widely recognised as one of the most important problems in the field of control systems, and eventually has attracted tremendous attention in the last several decades. Robust control of linear systems with uncertainties has been relatively well-solved (Bhattacharyya, 2017; Bhattacharyya & Keel, 1995; Dorato, 1987; Green & Limebeer, 2012; Park et al., 2006; Zhou & Doyle, 1998; Zhou et al., 1996). While for robust control of nonlinear systems by now many problems are still left open (Blongdel et al., 1995).

Nonlinear systems widely exist, and appear in various forms. On one hand, many practical applications give rise to various types of practical nonlinear systems; on the other hand, there are also numerous 'man-made' nonlinear systems of different forms (Isidori, 1995; Khalil, 2002; Lin, 2007; Qu, 1998). Among the various types of nonlinear systems, probably the strict-feedback systems (SFSs) are the largest type of nonlinear systems which have been fully investigated (see Duan (2020d) and the references therein). The specific triangular structure allows a systematic recursive design procedure of stabilising controllers, which is well-known as the method of backstepping. Robust control of SFSs using the method of backstepping has been tackled by a great number of researchers (e.g. Arcak & Kokotovic, 2001; Itou et al., 1999; Jiang & Nijmeijer, 1997; Kokotovic & Arcak, 2001; Krstic et al., 1996; Wang & Khorrami, 2000).

Like most results in control systems theory, works in robust control of nonlinear systems are mainly restricted to the first-order state-space system framework.

The first-order state-space approaches have remained in a dominant position for over a half-century (Kalman, 1959, 1960; Kalman et al., 1969). Most results in the field of control systems analysis and design are derived and presented in the first-order system framework. A state-space model focuses on the state of a system, and is more suitable and convenient for problems of state solution (response analysis) and observation (estimation). It can be used to deal with the

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control problem of dynamical systems, but is not the best choice. It was the preconceived occurrence which made it so popular, while it does have certain limitations in dealing with the control of dynamical systems (Duan, 2020a, 2020b, 2020c).

Fully-actuated systems are originally a type of physical systems, and are hence considered to be a minor portion of the whole set of control systems since under-actuated systems widely exist (Fantoni & Lozano, 2002; Spong et al., 2008). Therefore, fully actuated systems have not attracted enough attention although their control can be solved extremely simply with the help of the full-actuation feature. However, the concept of full-actuation can be generalised, as done in Duan (2020a, 2020b, 2020c), to give a type of HOFA models which can describe many practical and theoretical systems. This stimulates us to propose the so-called HOFA approaches. As a matter of fact, the key step of HOFA approaches is to derive a HOFA model for the considered nonlinear system, which is usually composed of a single HOFA model or a set of HOFA models. The control of the system can then be well solved by using the full-actuation feature of the model (Duan, 2020a, 2020b, 2020c).

This paper is a continuation along the line of HOFA approaches, and proposes an approach for robust control of nonlinear systems. The contributions of the paper include the following three aspects.

Firstly, similar to the case of HOFA models without uncertainties (Duan, 2020b, 2020c), an uncertain HOFA model is also composed of either one single HOFA model or a set of HOFA models. In the latter case, two types of more commonly and more practically used uncertain system models are proposed, namely, the second-order and high-order SFSs, which are both generalisations of the well-known uncertain first-order SFSs.

Secondly, for a single uncertain nonlinear HOFA model, it is shown that a very effective HOFA approach exists for designing a stabilising or signal tracking controller, which enables that the state and their relevant orders of derivatives all converge globally into an ellipsoid with an arbitrarily small radius. Moreover, in the case that the uncertainties do not exist, the designed control law results in a constant linear closed-loop system with an arbitrarily assignable eigenstructure.

Thirdly, a hidden advantage of the well-known first-order SFSs, as well as the newly proposed second- and high-order SFSs, is that each subsystem is actually a fully actuated system, when the x_{i+1} in the additional term of the i-th subsystem is taken as the driving (control) variable. Therefore, the proposed robust control method for a single HOFA model can be readily applied to each subsystem in an SFS. Consequently, when this process is combined with the idea of backstepping, second- and high-order methods of backstepping are proposed for the corresponding second- and high-order SFSs.

Thanks to the full-actuation feature of the HOFA models, the proposed approach for robust control of uncertain nonlinear systems is generally simpler and guarantees better properties. An illustrative example demonstrates the advantages of the proposed approach.

In the sequential sections, the r-dimensional vector space and the matrix space of dimension $m \times n$ are denoted by $\mathbb{R}^r$ and $\mathbb{R}^{m\times n}$, respectively, the spectral norm of a matrix A is denoted by $\|A\|$, and the real part of a complex number s is denoted by $\mathrm{Re}(s)$; $\lambda_i(A)$ denotes the i-th eigenvalue of the matrix A, I_n denotes the identity matrix, and

$$\overset{\circ}{I}_n = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Moreover, for $x \in \mathbb{R}^m$, and $A_i \in \mathbb{R}^{m\times m}, i = 1, 2, \ldots, n$, the following symbols are also frequently used in the paper:

$$x^{(0\sim n)} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n)} \end{bmatrix},$$

$$x_{i\sim j}^{(0\sim n)} = \begin{bmatrix} x_i^{(0\sim n)} \\ x_{i+1}^{(0\sim n)} \\ \vdots \\ x_j^{(0\sim n)} \end{bmatrix}, \quad j \geq i,$$

$$A^{0\sim n-1} = [A_0 \quad A_1 \quad \cdots \quad A_{n-1}],$$

$$\Phi(A^{0\sim n-1}) = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & -A_1 & \cdots & -A_{n-1} \end{bmatrix}.$$

2. Uncertain HOFA models

It is pointed out in Duan (2020c, 2020d) that the main step in the HOFA approaches is to derive a HOFA model for the considered nonlinear system. For robust control of uncertain nonlinear systems using HOFA approaches, obtaining a HOFA model of the uncertain nonlinear system is also a key step.

2.1. Single uncertain HOFA models

In Duan (2020a, 2020b, 2020c, 2020d), it has been shown that many systems without uncertainties can be converted into HOFA models in the following form

$$x^{(n)} = f(x^{(0\sim n-1)}) + L(x^{(0\sim n-1)})u, \tag{1}$$

where $x,\ u \in \mathbb{R}^r$ are the state vector and the control input vector, respectively, $f(x^{(0\sim n-1)}) \in \mathbb{R}^r$ is a continuous vector function, and $L(x^{(0\sim n-1)}) \in \mathbb{R}^{r\times r}$ is a continuous matrix function satisfying the following fully actuated assumption:

Assumption A1: $\det L(x^{(0\sim n-1)}) \neq 0,\ \forall\, x^{(i)} \in \mathbb{R}^r,\ i = 0, 1, \ldots, n-1.$

Correspondingly, for some nonlinear systems subject to uncertainties, often their HOFA models can be also derived, but in the following form:

$$x^{(n)} = f(x^{(0\sim n-1)}) + \Delta f(x^{(0\sim n-1)}) + L(x^{(0\sim n-1)})u, \tag{2}$$

where $\Delta f(x^{(0\sim n-1)}) \in \mathbb{R}^r$ is the nonlinear uncertainty of the system. The derivation of the above uncertain HOFA model may be of two folds: one is through physical modelling, see, e.g. the Example 2 in Duan (2020d), when the disturbances $d_i,\ i = 1, 2, 3$, are taken as uncertainties; the other is through model conversion (see the example section in this paper).

2.2. Uncertain second-order SFSs

Physical laws such as Lagrangian Equations and Theorem of Linear and Angular Momentum are often used to model physical systems. In such applications, the obtained subsystem models are generally of second-order (Duan, 2020a, 2020e). Due to this fact, we introduce the following second-order strict-feedback nonlinear system with uncertainties (refer also to Duan, 2020d):

$$\begin{cases} \ddot{x}_1 = f_1(x_1, \dot{x}_1) + \Delta f_1(x_1, \dot{x}_1) + G_1(x_1, \dot{x}_1)x_2 \\ \ddot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)}) + \Delta f_2(x_{1\sim 2}^{(0\sim 1)}) + G_2(x_{1\sim 2}^{(0\sim 1)})x_3 \\ \qquad \vdots \\ \ddot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}) + \Delta f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}) \\ \qquad + G_{n-1}(x_{1\sim n-1}^{(0\sim 1)})x_n \\ \ddot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)}) + \Delta f_n(x_{1\sim n}^{(0\sim 1)}) + G_n(x_{1\sim n}^{(0\sim 1)})u, \end{cases} \tag{3}$$

where $x_i \in \mathbb{R}^r,\ i = 1, 2, \ldots, n$ are the state vectors, $u \in \mathbb{R}^r$ is the control input vector, $f_i(x_{1\sim i}^{(0\sim 1)}) \in \mathbb{R}^r,\ i = 1, 2, \ldots, n$ are a set of sufficiently smooth vector functions, while $\Delta f_i(x_{1\sim i}^{(0\sim 1)}) \in \mathbb{R}^r,\ i = 1, 2, \ldots, n$ are a set of sufficiently smooth unknown vector functions. The coefficient matrices $G_i(x_{1\sim i}^{(0\sim 1)}) \in \mathbb{R}^{r\times r},\ i = 1, 2, \ldots, n$ are a set of sufficiently smooth matrix functions satisfying the following full-actuation assumption:

Assumption A2: $G_i(x_{1\sim i}^{(0\sim 1)}) \neq 0,\ \forall\, x_i, \dot{x}_i \in \mathbb{R}^r,\ i = 1, 2, \ldots, n.$

In the case that the nonlinear uncertainties in the above second-order SFS do not appear, it has been shown in Duan (2020d) that the model can be equivalently converted into the form of (1). However, when nonlinear uncertainties do exist, in some cases, an equivalent single HOFA model in the form of (2) may not be able to be derived, or in other words, the derived HOFA model may be too conservative in the sense that too much of certain information has to be included into the uncertain terms $\Delta f_i(x_{1\sim i}^{(0\sim 1)}),\ i = 1, 2, \ldots, n$, since it is sometimes hard to separate the certain part from the uncertain terms. Therefore, in certain applications, it may be more favourable to treat directly the set of models in (3), or, only convert a few of the subsystems in (3) into several HOFA models in the form of (2). After all, all the subsystems in (3), as they stand, are a set of cascade fully actuated subsystems, when the x_{i+1} in the i-th subsystem is looked upon as the control variable.

It is worth pointing out that the above second-order SFS (3) also has the following companion form:

$$\begin{cases} \ddot{x}_1 = f_1(x_1, \dot{x}_1) + \Delta f_1(x_1, \dot{x}_1) + G_1(x_1, \dot{x}_1)\dot{x}_2 \\ \ddot{x}_2 = f_2(x_{1\sim 2}^{(0\sim 1)}) + \Delta f_2(x_{1\sim 2}^{(0\sim 1)}) + G_2(x_{1\sim 2}^{(0\sim 1)})\dot{x}_3 \\ \qquad \vdots \\ \ddot{x}_{n-1} = f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}) + \Delta f_{n-1}(x_{1\sim n-1}^{(0\sim 1)}) \\ \qquad + G_{n-1}(x_{1\sim n-1}^{(0\sim 1)})\dot{x}_n \\ \ddot{x}_n = f_n(x_{1\sim n}^{(0\sim 1)}) + \Delta f_n(x_{1\sim n}^{(0\sim 1)}) + G_n(x_{1\sim n}^{(0\sim 1)})u. \end{cases} \tag{4}$$

As a matter of fact, the last term $G_i(x_{1\sim i}^{(0\sim 1)})\dot{x}_{i+1}$ in each subsystem can be further generalised into

$$G_{0i}(x_{1\sim i}^{(0\sim 1)})x_{i+1} + G_{1i}(x_{1\sim i}^{(0\sim 1)})\dot{x}_{i+1},$$

with $[G_{0i}(x_{1\sim i}^{(0\sim 1)}) \ \ G_{1i}(x_{1\sim i}^{(0\sim 1)})]$ being of full rank.

2.3. Uncertain high-order SFSs

Parallel to the uncertain second-order SFS (3), we introduce the following high-order (mixed-order) strict-feedback nonlinear systems with unknown nonlinearities:

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}) + \Delta f_1(x_1^{(0\sim m_1-1)}) \\ \qquad +G_1(x_1^{(0\sim m_1-1)})x_2 \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) \\ \qquad +\Delta f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) \\ \qquad +G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2})x_3 \\ \qquad \vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}) \\ \qquad +\Delta f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}) \\ \qquad +G_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1})x_n \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}) \\ \qquad +\Delta f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}) \\ \qquad +G_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n})u, \end{cases} \tag{5}$$

where $m_i, \ i = 1, 2, \ldots, n$ are a set of positive integers, $f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}), \ \Delta f_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}) \in \mathbb{R}^r$, $k = 1, 2, \ldots, n$ are two sets of sufficiently smooth vector functions, $G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}) \in \mathbb{R}^{r\times r}, \ k = 1, 2, \ldots, n$ are a set of sufficiently smooth matrix functions satisfying the following full-actuation assumption:

Assumption A3: $\det G_k(x_i^{(0\sim m_i-1)}|_{i=1\sim k}) \neq 0, \ \forall x_i^{(j)} \in \mathbb{R}^r, j = 0, 1, \ldots, m_i - 1, \ i = 1, 2, \ldots, k, \ k = 1, 2, \ldots, n.$

Regarding the derivation of the above uncertain mixed-order SFS (5), we make the following statements:

- In the modelling of physical systems, we often use physical laws such as Lagrangian Equations and Theorem of Linear and Angular Momentum. In such cases the original models of the obtained subsystems are of second-order. On the other side, there are also subsystems modelled by other physical laws, such as the well-known Hamiltonian Theorem, and the obtained original models are of first-order. Such operations give a mixed-order SFS (5), with $m_i, \ i = 1, 2, \ldots, n$, taking the values of 1 or 2.
- It has been pointed out above that the mixed-order SFS may be obtained from the second-order SFS by converting equivalently some of the subsystems in (3) into several HOFA models in the form of (2). Meanwhile, some of the subsystems in (3) can also be converted into several first-order equations. Both operations gives mixed-order SFSs.

Parallely, the above mixed-order SFS (5) also has a companion form which appears as follows:

$$\begin{cases} x_1^{(m_1)} = f_1(x_1^{(0\sim m_1-1)}) + \Delta f_1(x_1^{(0\sim m_1-1)}) \\ \qquad +G_1(x_1^{(0\sim m_1-1)})\dot{x}_2 \\ x_2^{(m_2)} = f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) + \Delta f_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2}) \\ \qquad +G_2(x_i^{(0\sim m_i-1)}|_{i=1\sim 2})\dot{x}_3 \\ \qquad \vdots \\ x_{n-1}^{(m_{n-1})} = f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}) \\ \qquad +\Delta f_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1}) \\ \qquad +G_{n-1}(x_i^{(0\sim m_i-1)}|_{i=1\sim n-1})\dot{x}_n \\ x_n^{(m_n)} = f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}) + \Delta f_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}) \\ \qquad +G_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n})u. \end{cases} \tag{6}$$

It is reminded that this form can be often converted into the form of (5) by lowering the orders of the subsystems. A by-product of doing this is that the number of equations in the system will be increased.

3. Robust control

In this section, let us consider the control of systems represented by the single HOFA model (2) under the following condition:

Condition 1: There exists a non-negative continuous scalar function $\rho(x^{(0\sim n-1)})$ such that the nonlinear uncertainty $\Delta f(x^{(0\sim n-1)}) \in \mathbb{R}^r$ satisfies

$$\left\| \Delta f(x^{(0\sim n-1)}) \right\| \leqslant \rho(x^{(0\sim n-1)}).$$

3.1. Preliminaries

In order to give a control law for the above high-order system (2) with dynamic uncertainties, we need to present the following preliminary results.

Lemma 3.1: *Let $A \in \mathbb{R}^{n\times n}$ satisfy*

$$\mathrm{Re}\lambda_i(A) \leq -\frac{\gamma}{2}, \quad i = 1, 2, \ldots, n, \tag{7}$$

where $\gamma > 0$, then there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$A^{\mathrm{T}}P + PA \leq -\gamma P. \tag{8}$$

Proof: It follows from (7) that the matrix $A + \frac{\gamma}{2}I$ is stable, thus there exists a matrix P satisfying

$$\left(A + \frac{\gamma}{2}I\right)^{\mathrm{T}} P + P\left(A + \frac{\gamma}{2}I\right) < 0.$$

Expanding the above equation gives the Equation (8). ■

Lemma 3.2: *For any $\mu > 0$, there exist a set of matrices $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$ satisfying*

$$\mathrm{Re}\lambda_i\left(\Phi\left(A^{0\sim n-1}\right)\right) < -\frac{\mu}{2}, \quad i = 1, 2, \ldots, nr. \tag{9}$$

Proof: In view of

$$\det\left(sI - \Phi\left(A^{0\sim n-1}\right)\right) = \det\left(s^n I_r + \sum_{i=0}^{n-1} A_i s^i\right),$$

the eigenvalues of $\Phi(A^{0\sim n-1})$ can be arbitrarily assigned by properly choosing A_i, $i = 0, 1, \ldots, n-1$, thus the conclusion holds. ■

It is known from Lemma 3.1 that, when the condition in (9) holds for some $\mu > 0$, there exists a positive definite matrix $P(A^{0\sim n-1})$ satisfying

$$\begin{aligned}&\Phi^{\mathrm{T}}(A^{0\sim n-1})P(A^{0\sim n-1}) + P(A^{0\sim n-1})\Phi(A^{0\sim n-1})\\ &\quad < -\mu P(A^{0\sim n-1}).\end{aligned} \tag{10}$$

Partition $P(A^{0\sim n-1})$ as

$$P(A^{0\sim n-1}) = \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix}, \quad P_i \in \mathbb{R}^{nr\times r}, \tag{11}$$

we can further introduce the following notation which will be frequently used in the paper:

$$P_L(A^{0\sim n-1}) = P(A^{0\sim n-1}) \begin{bmatrix} 0 \\ I_r \end{bmatrix} = P_n. \tag{12}$$

Finally, let us state the following simple result.

Lemma 3.3: *Let a, b be two real numbers, and $b > 0$. Then the following relation holds:*

$$a - \frac{a^2}{4b} \leqslant b. \tag{13}$$

3.2. Robust stabilisation

About the robust stabilisation for the high-order system (2) with nonlinear uncertainties, we have the following theorem.

Theorem 3.4: *Suppose that system (2) satisfies Assumption A1 and Condition 1. Let μ and ε be two arbitrarily given positive numbers, and $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$ be a set of matrices satisfying (9). Then, the following control law*

$$\begin{cases} u = -L^{-1}(x^{(0\sim n-1)})\left(A^{0\sim n-1}x^{(0\sim n-1)} + u^*\right) \\ u^* = f(x^{(0\sim n-1)}) + \dfrac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)}) \\ \quad \times P_L^{\mathrm{T}}(A^{0\sim n-1})x^{(0\sim n-1)}, \end{cases} \tag{14}$$

for the uncertain system (2) guarantees that the state $x^{(0\sim n-1)}$ converges into the following ellipsoid centred at the origin:

$$\begin{aligned}&\Theta_{\mu,\varepsilon}(0)\\ &= \left\{ x^{(0\sim n-1)} \,\middle|\, \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P(A^{0\sim n-1})x^{(0\sim n-1)} \leq \frac{\varepsilon}{\mu} \right\}.\end{aligned}$$

Proof: Substituting the control law (14) into system (2), gives the following closed-loop system

$$x^{(n)} + A^{0\sim n-1}x^{(0\sim n-1)} = \phi(x^{(0\sim n-1)}), \tag{15}$$

where

$$
\begin{aligned}
&\phi(x^{(0\sim n-1)}) \\
&\quad = -\frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)})P_L^{\mathrm{T}}(A^{0\sim n-1})x^{(0\sim n-1)} \\
&\qquad + \Delta f(x^{(0\sim n-1)}).
\end{aligned} \tag{16}
$$

The closed-loop system (15) can be written in the following state-space form:

$$
\dot{x}^{(0\sim n-1)} = \Phi(A^{0\sim n-1})x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi(x^{(0\sim n-1)}) \end{bmatrix}. \tag{17}
$$

Since $A_i \in \mathbb{R}^{r\times r},\ i = 0, 1, \ldots, n-1$ satisfy the condition (9), there exists a positive definite matrix $P(A^{0\sim n-1})$ satisfying (10). Then the following Lyapunov function can be chosen for the system (17):

$$
V = \tfrac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P(A^{0\sim n-1})x^{(0\sim n-1)}.
$$

In view of (10), (12) and (17), we have

$$
\begin{aligned}
\dot{V} &= \frac{1}{2}\left(\dot{x}^{(0\sim n-1)}\right)^{\mathrm{T}} Px^{(0\sim n-1)} + \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} \\
&\quad \times P\dot{x}^{(0\sim n-1)} \\
&= \frac{1}{2}\left(\Phi x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi(x^{(0\sim n-1)}) \end{bmatrix}\right)^{\mathrm{T}} Px^{(0\sim n-1)} \\
&\quad + \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} \\
&\quad \times P\left(\Phi x^{(0\sim n-1)} + \begin{bmatrix} 0_{(n-1)r} \\ \phi(x^{(0\sim n-1)}) \end{bmatrix}\right) \\
&= \frac{1}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} \left(\Phi^{\mathrm{T}}P + P\Phi\right) x^{(0\sim n-1)} \\
&\quad + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P \begin{bmatrix} 0_{(n-1)r} \\ \phi(x^{(0\sim n-1)}) \end{bmatrix} \\
&\leq -\frac{\mu}{2}\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} Px^{(0\sim n-1)} + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} \\
&\quad \times P_L\phi(x^{(0\sim n-1)}) \\
&= -\mu V + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L\phi(x^{(0\sim n-1)}).
\end{aligned}
$$

Next, let us consider the last term of the above equation. It follows from Condition 1, Lemma 3.3 and Equation (16) that

$$
\begin{aligned}
&\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L\phi(x^{(0\sim n-1)}) \\
&\quad = -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left[\left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L P_L^{\mathrm{T}} x^{(0\sim n-1)}\right] \\
&\qquad + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Delta f(x^{(0\sim n-1)}) \\
&\quad = -\frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)} \right\|^2 \\
&\qquad + \left(x^{(0\sim n-1)}\right)^{\mathrm{T}} P_L \Delta f(x^{(0\sim n-1)}) \\
&\quad \leq \frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)} \right\|^2 \\
&\qquad + \left\| \Delta f(x^{(0\sim n-1)}) \right\| \left\| P_L^{\mathrm{T}} x^{(0\sim n-1)} \right\| \\
&\quad \leq \frac{\rho^2(x^{(0\sim n-1)})}{4\varepsilon}\left\| P_L^{\mathrm{T}} x^{(0\sim n-1)} \right\|^2 \\
&\qquad + \rho(x^{(0\sim n-1)}) \left\| P_L^{\mathrm{T}} x^{(0\sim n-1)} \right\| \\
&\quad \leq \varepsilon.
\end{aligned}
$$

Combining the above two equations, gives

$$
\dot{V} \leq -\mu V + \varepsilon. \tag{18}
$$

It thus follows from the Comparison Theorem that

$$
V \leq V(0)\,\mathrm{e}^{-\mu t} + \frac{\varepsilon}{\mu}\left(1 - \mathrm{e}^{-\mu t}\right),
$$

which gives

$$
V \leq \left(V(0) - \frac{\varepsilon}{\mu}\right)\mathrm{e}^{-\mu t} + \frac{\varepsilon}{\mu} \to \frac{\varepsilon}{\mu}, \quad t \to \infty.
$$

Thus the state $x^{(0\sim n-1)}$ eventually converges into the ellipsoid $\Theta_{\mu,\varepsilon}(0)$. The proof is then completed. ■

In the case of $n = 1$, the system (2) becomes

$$
\dot{x} = f(x) + \Delta f(x) + L(x)u, \tag{19}
$$

and the robust control law turns into

$$
u = L^{-1}(x)\left(Ax - f(x) - \frac{1}{4\varepsilon}\rho^2(x)P_L^{\mathrm{T}}(A)x\right). \tag{20}
$$

In such a case, the state x of the uncertain control system eventually converges into the following ellipsoid:

$$
\Theta_{\mu,\varepsilon}(0) = \left\{ x \,\middle|\, x^{\mathrm{T}}P(A)x \leq \frac{\varepsilon}{\mu} \right\}.
$$

3.3. Robust tracking

Let $x^*(t) \in \mathbb{R}^r$ be a reference signal to be tracked by the state $x(t)$, and define

$$z = x - x^*,$$

then we have

$$z^{(i)} = x^{(i)} - \left(x^*\right)^{(i)}, \quad i = 0, 1, \ldots, n, \tag{21}$$

and $f(x^{(0\sim n-1)})$, $\Delta f(x^{(0\sim n-1)})$ and $L(x^{(0\sim n-1)})$ can all be transformed into functions with respect to $z^{(0\sim n-1)}$ and t, that is,

$$\begin{cases} f\left(x^{(0\sim n-1)}\right) = f\left(z^{(0\sim n-1)}, t\right) \\ \Delta f\left(x^{(0\sim n-1)}\right) = \Delta f\left(z^{(0\sim n-1)}, t\right) \\ L\left(x^{(0\sim n-1)}\right) = L\left(z^{(0\sim n-1)}, t\right). \end{cases} \tag{22}$$

Thus the system (2) can be turned into

$$\begin{aligned} z^{(n)} &= f\left(z^{(0\sim n-1)}, t\right) + \Delta f\left(z^{(0\sim n-1)}, t\right) \\ &\quad + L\left(z^{(0\sim n-1)}, t\right) u - \left(x^*\right)^{(n)}. \end{aligned} \tag{23}$$

Applying Theorem 3.4 to the above system, gives the following robust stabilising control law

$$\begin{aligned} u &= -L^{-1}\left(z^{(0\sim n-1)}, t\right)\left(A^{0\sim n-1} z^{(0\sim n-1)} + u^*\right) \\ u^* &= \frac{1}{4\varepsilon}\rho^2(z^{(0\sim n-1)}, t) P_L^{\mathrm{T}}(A^{0\sim n-1}) z^{(0\sim n-1)} \\ &\quad + f\left(z^{(0\sim n-1)}, t\right) - \left(x^*\right)^{(n)}. \end{aligned}$$

Then, using the relations in (21)–(22) again reversely, we can obtain the robust tracking control law for the original system (2).

Theorem 3.5: *Suppose that system (2) satisfies Assumption A1 and Condition 1. Let μ and ε be two arbitrarily given positive numbers, and $A_i \in \mathbb{R}^{r\times r}$, $i = 0, 1, \ldots, n-1$ be a set of matrices satisfying (9). Then, the following control law*

$$\begin{aligned} u &= -L^{-1}(x^{(0\sim n-1)})\left(A^{0\sim n-1} x^{(0\sim n-1)} + u^*\right) \\ u^* &= \frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)}) P_L^{\mathrm{T}}(A^{0\sim n-1}) x^{(0\sim n-1)} \\ &\quad + f(x^{(0\sim n-1)}) - u_0(x^*), \end{aligned}$$

with

$$\begin{aligned} u_0(x^*) &= (x^*)^{(n)} - A^{0\sim n-1}(x^*)^{(0\sim n-1)} \\ &\quad - \frac{1}{4\varepsilon}\rho^2(x^{(0\sim n-1)}) P_L^{\mathrm{T}}(A^{0\sim n-1})(x^*)^{(0\sim n-1)}, \end{aligned}$$

guarantees that the state $x^{(0\sim n-1)}$ converges globally into the following ellipsoidal belt-shaped region centred at $(x^*)^{(0\sim n-1)}$:

$$\begin{aligned} &\Theta_{\mu,\varepsilon}\left((x^*)^{(0\sim n-1)}\right) \\ &\quad = \left\{ x^{(0\sim n-1)} \,\middle|\, \left((x - x^*)^{(0\sim n-1)}\right)^{\mathrm{T}} P (x - x^*)^{(0\sim n-1)} \le \frac{\varepsilon}{\mu} \right\}. \end{aligned}$$

Remark 3.1: In applications, the radii of the ellipsoids $\Theta_{\mu,\varepsilon}(0)$ and $\Theta_{\mu,\varepsilon}((x^*)^{(0\sim n-1)})$ can be comprehensively adjusted through two ways to make it as small as desired: one is to increase the stability threshold μ of the linear part of the system, the other is to reduce the value of ε, or equivalently, to increase the magnitude of the control.

Remark 3.2: The proposed method can be extended to some sub-fully actuated systems (Duan, 2020a, 2020c). One of the natural ideas of doing this is to select properly a set of initial values and a reference signal $x^*(t)$ such that the solution of the system can pass safely around the 'dangerous' area which possibly make the matrix $L(x^{(0\sim n-1)})$ off-rank by following the 'lead' of the reference signal $x^*(t)$, so as to preserve the boundedness and realizability of the controller.

Remark 3.3: When the nonlinear uncertainty is removed, we can choose $\rho(x^{(0\sim n-1)}) = 0$. In this case the closed-loop system is in the following constant linear form:

$$x^{(n)} + A^{0\sim n-1} x^{(0\sim n-1)} = 0,$$

which can be equivalently written in the following state-space form:

$$\dot{x}^{(0\sim n-1)} = \Phi(A^{0\sim n-1}) x^{(0\sim n-1)}.$$

This tells us that, different from many other nonlinear control approaches, the proposed HOFA system approach produces a constant linear closed-loop system when the uncertainties do not present (see also, Duan, 2020a, 2020b, 2020c). This huge advantage eventually allows most of the theories and techniques for linear control systems design to be applicable.

Regarding the solution of the matrix $A^{0\sim n-1}$, we have the following result, which is the Corollary 1 in Duan (2020b) (see, also Duan, 2020a, 2020d, 2020f).

Proposition 3.6: *For an arbitrarily chosen $F \in \mathbb{R}^{nr\times nr}$, all the matrix $A^{0\sim n-1}$ and the nonsingular matrix $V \in \mathbb{R}^{nr\times nr}$ satisfying*

$$\Phi(A^{0\sim n-1}) = VFV^{-1}$$

are given by

$$A^{0\sim n-1} = -ZF^n V^{-1}(Z,F),$$

$$V = V(Z,F) = \begin{bmatrix} Z \\ ZF \\ \vdots \\ ZF^{n-1} \end{bmatrix},$$

where $Z \in \mathbb{R}^{r\times nr}$ is an arbitrary parameter matrix satisfying

$$\det V(Z,F) \neq 0.$$

Remark 3.4: It is seen from the above result that in order to meet the condition (9), it suffices only to select a matrix F satisfying

$$\mathrm{Re}\lambda_i(F) < -\frac{\mu}{2}, \quad i = 1,2,\ldots,nr. \tag{24}$$

Clearly, there are quite some degrees of freedom in the selection of F, while on the other hand we also have the parameter matrix Z, which provides $r^2 n$ degrees of freedom. All these degrees of freedom can be further utilised to achieve additional performance of the system (see, e.g. Duan, 1992, 1993; Duan et al., 2002, 2000; Duan & Zhao, 2020).

4. Backstepping for second-order SFSs

Based on Theorem 3.4, in this section we give a direct backstepping design method for the second-order SFS (3) without turning the system into a conventional first-order SFS. To do this, we need to make some preparations.

Suppose $A_i^{0\sim 1} \in \mathbb{R}^{1\times 2}$, $i = 1,2,\ldots,n$ are a set of matrices satisfying

$$\mathrm{Re}\lambda_k\left(\Phi\left(A_i^{0\sim 1}\right)\right) \leq -\frac{\mu}{2}, \quad k = 1,2, \tag{25}$$

where μ is an arbitrarily given positive number, and

$$P_i\left(A_i^{0\sim 1}\right) = \left[P_{iF}\left(A_i^{0\sim 1}\right) \quad P_{iL}\left(A_i^{0\sim 1}\right)\right] \in \mathbb{R}^{2\times 2}, \tag{26}$$

is a positive definite solution to the matrix inequality

$$\begin{aligned}\Phi^{\mathrm{T}}\left(A_i^{0\sim 1}\right)P_i\left(A_i^{0\sim 1}\right) + P_i\left(A_i^{0\sim 1}\right)\Phi\left(A_i^{0\sim 1}\right)\\ \leq -\gamma P_i\left(A_i^{0\sim 1}\right),\end{aligned} \tag{27}$$

for $i = 1,2,\ldots,n$ and some positive number γ. Further, define, for $i = 1,2,\ldots,n$,

$$\tilde{P}_i\left(A_i^{0\sim 1}\right) = \mathring{I}_2 P_i^{\mathrm{T}}\left(A_i^{0\sim 1}\right),$$

and denote

$$\begin{aligned}\tilde{P}_i^{-1}\left(A_i^{0\sim 1}\right) &= \begin{bmatrix} Q_{i1}\left(A_i^{0\sim 1}\right) \\ Q_{i2}\left(A_i^{0\sim 1}\right)\end{bmatrix} \\ &= \begin{bmatrix} Q_{i1}\left(A_i^{0\sim 1}\right) \\ Q_{iF}\left(A_i^{0\sim 1}\right) \quad Q_{iL}\left(A_i^{0\sim 1}\right)\end{bmatrix},\end{aligned}$$

where $Q_{iF}(A_i^{0\sim 1})$ and $Q_{iL}(A_i^{0\sim 1})$ are two scalars, and it can be easily shown that

$$Q_{iL}\left(A_i^{0\sim 1}\right) \neq 0, \quad i = 1,2,\ldots,n. \tag{28}$$

It is worth pointing out that all the variables defined above are independent of the system (3), and can all be easily obtained beforehand.

4.1. The first step

Let

$$z_1^{(0\sim 1)} = x_1^{(0\sim 1)}, \tag{29}$$

and

$$\tilde{P}_2\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} = x_2^{(0\sim 1)} - \begin{bmatrix}\alpha_1 \\ 0\end{bmatrix}. \tag{30}$$

In view of the symmetry property of $P_2(A_2^{0\sim 1})$, the above Equation (30) can be equivalently decomposed into

$$P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} = x_2 - \alpha_1, \tag{31}$$

and

$$P_{2F}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} = \dot{x}_2. \tag{32}$$

Therefore, by using (29) and (31), the first equation in (3) is transformed into

$$\begin{aligned}\ddot{z}_1 &= \hat{f}_1(z_1,\dot{z}_1) + \Delta\hat{f}_1(z_1,\dot{z}_1) + L_1(z,\dot{z}_1)\\ &\quad\times\left(P_{2L}^{\mathrm{T}}\left(A_2^{0\sim 1}\right) z_2^{(0\sim 1)} + \alpha_1\right),\end{aligned} \tag{33}$$

where

$$\hat{f}_1(z_1,\dot{z}_1) = f_1(z_1,\dot{z}_1)$$

$$\Delta\hat{f}_1(z_1,\dot{z}_1) = \Delta f_1(z_1,\dot{z}_1)$$
$$L_1(z_1,\dot{z}_1) = G_1(z_1,\dot{z}_1). \quad (34)$$

If there exists a non-negative scalar continuous function $\rho_1(z_1,\dot{z}_1)$ such that

$$|\Delta f_1(z_1,\dot{z}_1)| \le \rho_1(z_1,\dot{z}_1),$$

then, by applying Theorem 3.4 to the above system (33), the first virtual control law can be designed as

$$\begin{aligned}\alpha_1 &= -L_1^{-1}(z_1,\dot{z}_1)\left(A_1^{0\sim1}z_1^{(0\sim1)} + \alpha_1^*\right)\\ \alpha_1^* &= \hat{f}_1(z_1,\dot{z}_1) + \frac{1}{4\varepsilon}\rho_1^2(z_1,\dot{z}_1)P_{1L}^{\mathrm{T}}(A_1^{0\sim1})z_1^{(0\sim1)}.\end{aligned} \quad (35)$$

The closed-loop subsystem resulted in by this control law is

$$\ddot{z}_1 + A_1^{0\sim1}z_1^{(0\sim1)} = \phi_1 + w_1, \quad (36)$$

with

$$\phi_1 = -\frac{1}{4\varepsilon}\rho_1^2(z_1^{(0\sim1)})P_{1L}^{\mathrm{T}}(A_1^{0\sim1})z_1^{(0\sim1)} + \Delta\hat{f}_1(z_1^{(0\sim1)}), \quad (37)$$

and

$$w_1 = L_1(z_1,\dot{z}_1)P_{2L}^{\mathrm{T}}\left(A_2^{0\sim1}\right)z_2^{(0\sim1)}. \quad (38)$$

4.2. The second step

The virtual control law α_1 given by (35) is a function with respect to z_1 and $\dot{z}_1$, that is,

$$\alpha_1 = \alpha_1\left(z_1,\dot{z}_1\right).$$

Thus we have, using the first equation in (3) again,

$$\begin{aligned}\dot{\alpha}_1 &= \frac{\partial\alpha_1}{\partial z_1}\dot{z}_1 + \frac{\partial\alpha_1}{\partial\dot{z}_1}\ddot{z}_1\\ &= h_1(z_{1\sim2}^{(0\sim1)}) + \Delta h_1(z_{1\sim2}^{(0\sim1)}),\end{aligned} \quad (39)$$

with

$$\begin{aligned}h_1 &= \frac{\partial\alpha_1}{\partial z_1}\dot{z}_1 + \frac{\partial\alpha_1}{\partial\dot{z}_1}\left(f_1 + G_1x_2\right)\\ \Delta h_1 &= \frac{\partial\alpha_1}{\partial\dot{z}_1}\Delta f_1(x_1,\dot{x}_1).\end{aligned}$$

On the other hand, we obtain, from (30),

$$z_2^{(0\sim1)} = \tilde{P}_2^{-1}\left(A_2^{0\sim1}\right)\left(x_2^{(0\sim1)} - \begin{bmatrix}\alpha_1\\0\end{bmatrix}\right), \quad (40)$$

which gives

$$\begin{aligned}\dot{z}_2 &= Q_{22}\left(A_2^{0\sim1}\right)\left(x_2^{(0\sim1)} - \begin{bmatrix}\alpha_1\\0\end{bmatrix}\right)\\ &= Q_{2F}\left(A_2^{0\sim1}\right)(x_2-\alpha_1) + Q_{2L}\left(A_2^{0\sim1}\right)\dot{x}_2.\end{aligned}$$

Further taking differentials of the above equation, and using (39) and the second equation in (3), yield

$$\begin{aligned}\ddot{z}_2 &= Q_{2F}\left(A_2^{0\sim1}\right)(\dot{x}_2-\dot{\alpha}_1) + Q_{2L}\left(A_2^{0\sim1}\right)\ddot{x}_2\\ &= Q_{2F}\left(A_2^{0\sim1}\right)(\dot{x}_2 - h_1(z_1,\dot{z}_1,z_2) - \Delta h_1(z_1,\dot{z}_1))\\ &\quad + Q_{2L}\left(A_2^{0\sim1}\right)\\ &\quad \times\left(f_2(x_{1\sim2}^{(0\sim1)}) + \Delta f_2(x_{1\sim2}^{(0\sim1)}) + G_2(x_{1\sim2}^{(0\sim1)})x_3\right)\\ &= \hat{f}_2(z_{1\sim2}^{(0\sim1)}) + \Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)}) + L_2(z_{1\sim2}^{(0\sim1)})x_3,\end{aligned} \quad (41)$$

where

$$\begin{aligned}\hat{f}_2(z_{1\sim2}^{(0\sim1)}) &= Q_{2F}\left(A_2^{0\sim1}\right)(\dot{x}_2 - h_1(z_1,\dot{z}_1,z_2))\\ &\quad + Q_{2L}\left(A_2^{0\sim1}\right)f_2(x_{1\sim2}^{(0\sim1)})\\ \Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)}) &= -Q_{2F}\left(A_2^{0\sim1}\right)\Delta h_1(z_1,\dot{z}_1)\\ &\quad + Q_{2L}\left(A_2^{0\sim1}\right)\Delta f_2(x_{1\sim2}^{(0\sim1)})\\ L_2(z_{1\sim2}^{(0\sim1)}) &= Q_{2L}\left(A_2^{0\sim1}\right)G_2(x_{1\sim2}^{(0\sim1)})\end{aligned} \quad (42)$$

Further let

$$\tilde{P}_3\left(A_3^{0\sim1}\right)z_3^{(0\sim1)} = x_3^{(0\sim1)} - \begin{bmatrix}\alpha_2\\0\end{bmatrix}, \quad (43)$$

which can be equivalently decomposed, again in view of the symmetry property of $P_3(A_3^{0\sim1})$, into

$$P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)} = x_3 - \alpha_2, \quad (44)$$

and

$$P_{3F}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)} = \dot{x}_3. \quad (45)$$

Substituting (44) into (41), gives

$$\begin{aligned}\ddot{z}_2 &= \hat{f}_2\left(z_{1\sim2}^{(0\sim1)}\right) + \Delta\hat{f}_2\left(z_{1\sim2}^{(0\sim1)}\right) + L_2(z_{1\sim2}^{(0\sim1)})\\ &\quad\times\left(P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)} + \alpha_2\right).\end{aligned} \quad (46)$$

It follows from (28) that $L_2(z_{1\sim2}^{(0\sim1)}) \neq 0,\ \forall\ z_{1\sim2}^{(0\sim1)} \in \mathbb{R}^4$. If there exists a continuous function $\rho_2(z_{1\sim2}^{(0\sim1)})$

satisfying

$$\left|\Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)})\right| \leqslant \rho_2(z_{1\sim2}^{(0\sim1)}), \tag{47}$$

then, applying Theorem 3.4 to system (46) gives the second virtual control law of the system as

$$\begin{aligned} \alpha_2 &= -L_2^{-1}(z_{1\sim2}^{(0\sim1)})\left(A_2^{0\sim1}z_2^{(0\sim1)} + \alpha_2^*\right) \\ \alpha_2^* &= \frac{1}{4\varepsilon}\rho_2^2(z_{1\sim2}^{(0\sim1)})P_{2L}^{\mathrm{T}}(A_2^{0\sim1})z_2^{(0\sim1)} \\ &\quad + \hat{f}_2(z_{1\sim2}^{(0\sim1)}) + v_1, \end{aligned} \tag{48}$$

where

$$v_1 = L_1(z_1, \dot{z}_1)P_{1L}^{\mathrm{T}}\left(A_1^{0\sim1}\right)z_1^{(0\sim1)}. \tag{49}$$

The closed-loop subsystem resulted in by this control law is

$$\ddot{z}_2 + A_2^{0\sim1}z_2^{(0\sim1)} = \phi_2 + w_2 - v_1, \tag{50}$$

where

$$\phi_2 = -\frac{1}{4\varepsilon}\rho_2^2(z_{1\sim2}^{(0\sim1)})P_{2L}^{\mathrm{T}}(A_2^{0\sim1})z_2^{(0\sim1)} + \Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)}), \tag{51}$$

and

$$w_2 = L_2(z_{1\sim2}^{(0\sim1)})P_{3L}^{\mathrm{T}}\left(A_3^{0\sim1}\right)z_3^{(0\sim1)}. \tag{52}$$

4.3. The third step

The function α_2 defined by (48) is the function of $z_{1\sim2}^{(0\sim1)}$, that is,

$$\alpha_2 = \alpha_2\left(z_1, \dot{z}_1; z_2, \dot{z}_2\right). \tag{53}$$

Similar to the second step, taking derivative of the above equation, and further separating its certain part and uncertain part, yield

$$\dot{\alpha}_2 = h_2(z_{1\sim3}^{(0\sim1)}) + \Delta h_2(z_{1\sim3}^{(0\sim1)}), \tag{54}$$

where $\Delta h_2(z_{1\sim3}^{(0\sim1)})$ is a function directly related to the unmodeled items Δf_1 and Δf_2.

On the other hand, we can obtain, from (43),

$$z_3^{(0\sim1)} = \tilde{P}_3^{-1}\left(A_3^{0\sim1}\right)\left(x_3^{(0\sim1)} - \begin{bmatrix}\alpha_2\\0\end{bmatrix}\right), \tag{55}$$

which gives

$$\dot{z}_3 = Q_{3F}\left(A_3^{0\sim1}\right)(x_3 - \alpha_2) + Q_{3L}\left(A_3^{0\sim1}\right)\dot{x}_3.$$

Taking differentials of the above equation, and using (54) and the third equation in (3), we obtain

$$\ddot{z}_3 = \hat{f}_3(z_{1\sim3}^{(0\sim1)}) + \Delta\hat{f}_3(z_{1\sim3}^{(0\sim1)}) + L_3(z_{1\sim3}^{(0\sim1)})x_4, \tag{56}$$

Further, introducing the transformation

$$\tilde{P}_4\left(A_4^{0\sim1}\right)z_4^{(0\sim1)} = x_4^{(0\sim1)} - \begin{bmatrix}\alpha_3\\0\end{bmatrix},$$

via a similar treatment, we can convert (56) into

$$\begin{aligned} \ddot{z}_3 &= \hat{f}_3(z_{1\sim3}^{(0\sim1)}) + \Delta\hat{f}_3(z_{1\sim3}^{(0\sim1)}) + L_3(z_{1\sim3}^{(0\sim1)}) \\ &\quad \times\left(P_{4L}^{\mathrm{T}}\left(A_4^{0\sim1}\right)z_4^{(0\sim1)} + \alpha_3\right). \end{aligned} \tag{57}$$

Assume that there exists a non-negative continuous function $\rho_3(z_{1\sim3}^{(0\sim1)})$ satisfying

$$\left|\Delta\hat{f}_3(z_{1\sim3}^{(0\sim1)})\right| \leqslant \rho_3(z_{1\sim3}^{(0\sim1)}), \tag{58}$$

then, by applying Theorem 3.4 to the system (57), the third virtual control law of the system can be designed as

$$\begin{aligned} \alpha_3 &= -L_3^{-1}(z_{1\sim3}^{(0\sim1)})\left(A_3^{0\sim1}z_3^{(0\sim1)} + \alpha_3^*\right) \\ \alpha_3^* &= \frac{1}{4\varepsilon}\rho_3^2(z_{1\sim3}^{(0\sim1)})P_{3L}^{\mathrm{T}}(A_3^{0\sim1})z_3^{(0\sim1)} \\ &\quad + \hat{f}_3\left(z_{1\sim3}^{(0\sim1)}\right) - w_2 + v_2, \end{aligned} \tag{59}$$

where

$$v_2 = L_2(z_{1\sim2}^{(0\sim1)})P_{2L}^{\mathrm{T}}\left(A_2^{0\sim1}\right)z_2^{(0\sim1)}. \tag{60}$$

The closed-loop subsystem resulted in by this control law is

$$\ddot{z}_3 + A_3^{0\sim1}z_3^{(0\sim1)} = \phi_3 + w_3 - v_2, \tag{61}$$

where

$$\phi_3 = -\frac{1}{4\varepsilon}\rho_3^2(z_{1\sim3}^{(0\sim1)})P_{3L}^{\mathrm{T}}(A_3^{0\sim1})z_3^{(0\sim1)} + \Delta\hat{f}_3(z_{1\sim3}^{(0\sim1)}), \tag{62}$$

and

$$w_3 = L_3(z_{1\sim3}^{(0\sim1)})P_{4L}^{\mathrm{T}}\left(A_4^{0\sim1}\right)z_4^{(0\sim1)}. \tag{63}$$

4.4. The n-th step

Continue the above process, and suppose that the expressions of α_{n-1}, $\dot{\alpha}_{n-1}$, $h_{n-1}(z_{1\sim n}^{(0\sim 1)})$ and $\Delta h_{n-1}(z_{1\sim n}^{(0\sim 1)})$ have all been obtained. Then, similar to the previous treatment, we can obtain

$$\ddot{z}_n = \hat{f}_n(z_{1\sim n}^{(0\sim 1)}) + \Delta\hat{f}_n(z_{1\sim n}^{(0\sim 1)}) + L_n(z_{1\sim n-1}^{(0\sim 1)})u. \quad (64)$$

Assume that there exist continuous functions $\rho_i(z_{1\sim i}^{(0\sim 1)}), i = 1, 2, \ldots, n$ satisfying

$$\left|\Delta\hat{f}_i(z_{1\sim i}^{(0\sim 1)})\right| \leqslant \rho_i(z_{1\sim i}^{(0\sim 1)}), \quad i = 1, 2, \ldots, n, \quad (65)$$

then, by applying Theorem 3.4 to system (64), the control law of the system (3) can be finally designed as

$$\begin{aligned} u &= -L_n^{-1}(z_{1\sim n}^{(0\sim 1)})\left(A_n^{0\sim 1}z_n^{(0\sim 1)} + u^*\right) \\ u^* &= \frac{1}{4\varepsilon}\rho_n^2(z_{1\sim n}^{(0\sim 1)})P_{nL}^{\mathrm{T}}(A_n^{0\sim 1})z_n^{(0\sim 1)} \\ &\quad + \hat{f}_n\left(z_{1\sim n}^{(0\sim 1)}\right) + v_{n-1}, \end{aligned} \quad (66)$$

where

$$v_{n-1} = L_{n-1}(z_{1\sim n-1}^{(0\sim 1)})P_{n-1,L}^{\mathrm{T}}\left(A_{n-1}^{0\sim 1}\right)z_{n-1}^{(0\sim 1)}. \quad (67)$$

The last closed-loop subsystem resulted in by this control law is

$$\ddot{z}_n + A_n^{0\sim 1}z_n^{(0\sim 1)} = \phi_n - v_{n-1}, \quad (68)$$

where

$$\phi_n = -\frac{1}{4\varepsilon}\rho_n^2(z_{1\sim n}^{(0\sim 1)})P_{nL}^{\mathrm{T}}(A_n^{0\sim 1})z_n^{(0\sim 1)} + \Delta\hat{f}_n(z_{1\sim n}^{(0\sim 1)}). \quad (69)$$

4.5. The conclusion

The above process defines a transformation between $x_{1\sim n}^{(0\sim 1)}$ and $z_{1\sim n}^{(0\sim 1)}$, as follows:

$$\begin{aligned} z_1^{(0\sim 1)} &= x_1^{(0\sim 1)}, \\ \tilde{P}_i\left(A_i^{0\sim 1}\right)z_i^{(0\sim 1)} &= x_i^{(0\sim 1)} - \begin{bmatrix}\alpha_{i-1}(x_{1\sim i-1}) \\ 0\end{bmatrix}, \\ & i = 2, 3, \ldots, n. \end{aligned} \quad (70)$$

In order to prove that the above control law (66) stabilises the system (3), we define the following Lyapunov function:

$$V = \sum_{k=1}^{n} V_k, \quad (71)$$

with

$$V_i = \left(z_i^{(0\sim 1)}\right)^{\mathrm{T}} P_i(A_i^{0\sim 1})z_i^{(0\sim 1)}, \quad i = 1, 2, \ldots, n. \quad (72)$$

Following the proof of Theorem 3.4, it is easy to obtain

$$\begin{aligned} \dot{V}_1 &\leq -\mu V_1 + \varepsilon + 2\left(z_1^{(0\sim 1)}\right)^{\mathrm{T}} P_{1L}(A_1^{0\sim 1})w_1 \\ \dot{V}_2 &\leq -\mu V_2 + \varepsilon + 2\left(z_2^{(0\sim 1)}\right)^{\mathrm{T}} P_{2L}(A_2^{0\sim 1})\left(w_2 - v_1\right) \\ &\vdots \\ \dot{V}_{n-1} &\leq -\mu V_{n-1} + \varepsilon + 2\left(z_{n-1}^{(0\sim 1)}\right)^{\mathrm{T}} \\ &\quad \times P_{n-1,L}(A_{n-1}^{0\sim 1})\left(w_{n-1} - v_{n-2}\right) \\ \dot{V}_n &\leq -\mu V_n + \varepsilon + 2\left(z_n^{(0\sim 1)}\right)^{\mathrm{T}} P_{nL}(A_{n-1}^{0\sim 1})\left(-v_{n-1}\right). \end{aligned} \quad (73)$$

In view of

$$\begin{aligned} v_i &= L_i\left(x_i^{(0\sim 1)}\right)P_{iL}^{\mathrm{T}}\left(A_i^{0\sim 1}\right)z_i^{(0\sim 1)} \\ w_i &= L_i\left(x_i^{(0\sim 1)}\right)P_{i+1,L}^{\mathrm{T}}\left(A_{i+1}^{0\sim 1}\right)z_{i+1}^{(0\sim 1)} \\ & i = 1, 2, \ldots, n-1, \end{aligned} \quad (74)$$

we have

$$\begin{aligned} &\left(z_i^{(0\sim 1)}\right)^{\mathrm{T}} P_{iL}(A_i^{(0\sim 1)})w_i \\ &= \left(z_i^{(0\sim 1)}\right)^{\mathrm{T}} P_{iL}(A_i^{(0\sim 1)})L_i\left(x_i^{(0\sim 1)}\right) \\ &\times P_{i+1,L}^{\mathrm{T}}\left(A_{i+1}^{0\sim 1}\right)z_{i+1}^{(0\sim 1)} \\ &= \left(z_{i+1}^{(0\sim 1)}\right)^{\mathrm{T}} P_{i+1,L}\left(A_{i+1}^{0\sim 1}\right)L_i\left(x_i^{(0\sim 1)}\right) \\ &\times P_{iL}^{\mathrm{T}}(A_i^{(0\sim 1)})z_i^{(0\sim 1)} \\ &= \left(z_{i+1}^{(0\sim 1)}\right)^{\mathrm{T}} P_{i+1,L}(A_{i+1}^{(0\sim 1)})v_i \\ & i = 1, 2, \ldots, n-1. \end{aligned} \quad (75)$$

Adding the equations in (73) side by side, and using the relations in (75), give

$$\dot{V} \leqslant -\mu V + n\varepsilon. \quad (76)$$

The above process clearly proves the following result.

Theorem 4.1: *Suppose that Assumption A2 holds. Let μ and ε be two arbitrarily given positive numbers, and $A_i^{0\sim 1}, P_i(A_i^{(0\sim 1)})$ and $P_{iL}(A_i^{(0\sim 1)}),\ i=1,2,\ldots,n-1$ be three sets of matrices as described above. Assume that there exist a set of non-negative continuous scalar functions $\rho_i(x_{1\sim i}^{(0\sim 1)}),\ i=1,2,\ldots,n$ satisfying (65). Then, the controller (66) for the uncertain second-order SFS (3) guarantees that the transformed state $z_{1\sim n}^{(0\sim 1)}$ converges into the following ellipsoid:*

$$\left\{ z_{1\sim n}^{(0\sim 1)} \,\middle|\, \sum_{i=1}^{n}\left[\left(z_{1\sim i}^{(0\sim 1)}\right)^{\mathrm{T}} P_i(A_i^{(0\sim 1)}) z_{1\sim i}^{(0\sim 1)}\right] \leqslant \frac{n\varepsilon}{\mu} \right\}.$$

It follows from the above theorem that the state $z_i^{(0\sim 1)},\ i=1,2,\ldots,n$ of each subsystem eventually converges into the following ellipsoid centred at the origin:

$$\Theta_i^{\mu,\varepsilon}(0) = \left\{ z_i^{(0\sim 1)} \,\middle|\, \left(z_i^{(0\sim 1)}\right)^{\mathrm{T}} P_i(A_i^{(0\sim 1)}) z_i^{(0\sim 1)} \leqslant \frac{n\varepsilon}{\mu} \right\},$$

which implies that the state $x_1^{(0\sim 1)}$ of the original system also converges into the following ellipsoid centred at the origin:

$$\Theta_1^{\mu,\varepsilon}(0) = \left\{ x_1^{(0\sim 1)} \,\middle|\, \left(x_1^{(0\sim 1)}\right)^{\mathrm{T}} P_1(A_1^{(0\sim 1)}) x_1^{(0\sim 1)} \leqslant \frac{n\varepsilon}{\mu} \right\}.$$

Remark 4.1: The above proposed second-order backstepping method can be easily modified to suit the other companion SFS (4). As a matter of fact, for SFS (4), we can simply take $\tilde{P}_i(A_i^{0\sim 1}) = P_i(A_i^{0\sim 1}),\ i=1,2,\ldots,n$, and in this case we also have theoretically $Q_{iL}(A_i^{0\sim 1}) \neq 0,\ \ i=1,2,\ldots,n$, which ensure $L_i(z_{1\sim i}^{(0\sim 1)}) \neq 0,\ i=1,2,\ldots,n$.

5. Backstepping for high-order SFSs

Similar to Section 4, in this section we also give, based on Theorem 3.4, a direct backstepping design method for the high-order SFS (5) without turning it into a conventional first-order SFS. Parallel to the second-order case, we also make the following preparations.

Suppose $A_i^{0\sim m_i-1} \in \mathbb{R}^{1\times m_i},\ i=1,2,\ldots,n$ are a set of matrices satisfying

$$\mathrm{Re}\lambda_k\left(\Phi\left(A_i^{0\sim m_i-1}\right)\right) \leq -\frac{\mu}{2}, \quad k=1,2,\ldots,m_i, \tag{77}$$

where μ is an arbitrarily given positive number, and $P_i(A_i^{0\sim m_i-1})$ is a positive definite solution to the Lyapunov inequality

$$\Phi^{\mathrm{T}}\left(A_i^{0\sim m_i-1}\right)P_i\left(A_i^{0\sim m_i-1}\right) + P_i\left(A_i^{0\sim m_i-1}\right) \times \Phi\left(A_i^{0\sim m_i-1}\right) \leq -\gamma P_i\left(A_i^{0\sim m_i-1}\right), \tag{78}$$

for $i=1,2,\ldots,n$ and some positive number γ. Let

$$P_i\left(A_i^{0\sim m_i-1}\right) = \left[P_{iF}\left(A_i^{0\sim m_i-1}\right) \quad \cdots \quad P_{iL}\left(A_i^{0\sim m_i-1}\right)\right] \in \mathbb{R}^{m_i\times m_i},$$

where $P_{iF}(A_i^{0\sim m_i-1})$ and $P_{iL}(A_i^{0\sim m_i-1})$ are respectively the first and the last column of $P_i(A_i^{0\sim m_i-1})$.

Further, define

$$\tilde{P}_i\left(A_i^{0\sim m_i-1}\right) = \overset{\circ}{I}_{m_i} P_i^{\mathrm{T}}\left(A_i^{0\sim m_i-1}\right),$$

and denote

$$\tilde{P}_i^{-1}\left(A_i^{0\sim m_i-1}\right) = \begin{bmatrix} Q_{i1}\left(A_i^{0\sim m_i-1}\right) \\ Q_{i2}\left(A_i^{0\sim m_i-1}\right) \end{bmatrix} = \begin{bmatrix} & Q_{i1}\left(A_i^{0\sim m_i-1}\right) \\ Q_{iF}\left(A_i^{0\sim m_i-1}\right) & Q_{iM}\left(A_i^{0\sim m_i-1}\right) \\ & Q_{iL}\left(A_i^{0\sim m_i-1}\right) \end{bmatrix},$$

where $Q_{iF}(A_i^{0\sim m_i-1})$ and $Q_{iL}(A_i^{0\sim m_i-1})$ are two scalars. Note that $P_i(A_i^{0\sim m_i-1})$ is not unique, proper matrix $P_i(A_i^{0\sim m_i-1})$ can be found such that

$$Q_{iL}\left(A_i^{0\sim m_i-1}\right) \neq 0, \quad i=1,2,\ldots,n. \tag{79}$$

For simplicity, only the basic idea is given in this section.

5.1. The first step

Let

$$z_1^{(0\sim m_1-1)} = x_1^{(0\sim m_1-1)}, \tag{80}$$

and

$$\tilde{P}_2 z_2^{(0\sim m_2-1)} = x_2^{(0\sim m_2-1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}, \tag{81}$$

which implies, in view of the symmetry property of $P_2(A_2^{0\sim m_2-1})$,

$$P_{2L}^{\mathrm{T}}\left(A_2^{0\sim m_2-1}\right) z_2^{(0\sim m_2-1)} = x_2 - \alpha_1. \tag{82}$$

Thus the first equation in (5) can be transformed into

$$\begin{aligned} z_1^{(m_1)} &= \hat{f}_1(z_1^{(0\sim m_1-1)}) + \Delta\hat{f}_1(z_1^{(0\sim m_1-1)}) \\ &\quad + G_1(z_1^{(0\sim m_1-1)}) \\ &\quad \times \left(P_{2L}^{\mathrm{T}}\left(A_2^{0\sim m_1-1}\right) z_2^{(0\sim m_2-1)} + \alpha_1\right), \end{aligned} \tag{83}$$

where

$$\begin{aligned} \hat{f}_1(z_1^{(0\sim m_1-1)}) &= f_1(x_1^{(0\sim m_1-1)}) \\ \Delta\hat{f}_1(z_1^{(0\sim m_1-1)}) &= \Delta f_1(x_1^{(0\sim m_1-1)}) \\ L_1(z_1^{(0\sim m_1-1)}) &= G_1(x_1^{(0\sim m_1-1)}). \end{aligned} \tag{84}$$

Assume that there exists a non-negative continuous scalar function $\rho_1(z_1^{(0\sim m_1-1)})$ satisfying

$$\left|\Delta\hat{f}_1(z_1^{(0\sim m_1-1)})\right| \le \rho_1(z_1^{(0\sim m_1-1)}),$$

applying Theorem 3.4 to system (83) gives the first virtual control law as

$$\begin{aligned} \alpha_1 &= -L_1^{-1}(z_1^{(0\sim m_1-1)})\left(A_1^{0\sim m_1-1} z_1^{(0\sim m_1-1)} + \alpha_1^*\right) \\ \alpha_1^* &= \frac{1}{4\varepsilon}\rho_1^2(z_1^{(0\sim m_1-1)}) P_{1L}^{\mathrm{T}}(A_1^{0\sim m_1-1}) z_1^{(0\sim m_1-1)} \\ &\quad + \hat{f}_1(z_1^{(0\sim m_1-1)}). \end{aligned} \tag{85}$$

The closed-loop subsystem resulted in by this control law is

$$z_1^{(m_1)} + A_1^{0\sim m_1-1} z_1^{(0\sim m_1-1)} = \phi_1 + w_1, \tag{86}$$

where

$$\begin{aligned} \phi_1 &= -\frac{1}{4\varepsilon}\rho_1^2(z_1^{(0\sim m_1-1)}) P_{1L}^{\mathrm{T}}(A_1^{0\sim m_1-1}) z_1^{(0\sim m_1-1)} \\ &\quad + \Delta\hat{f}_1(z_1^{(0\sim m_1-1)}), \end{aligned} \tag{87}$$

and

$$w_1 = L_1(z_1^{(0\sim m_1-1)}) P_{2L}^{\mathrm{T}}\left(A_2^{0\sim m_2-1}\right) z_2^{(0\sim m_2-1)}. \tag{88}$$

5.2. The second step

The virtual control law α_1 given by (85) is a function with respect to $z_1^{(0\sim m_1-1)}$, that is,

$$\alpha_1 = \alpha_1\left(z_1^{(0\sim m_1-1)}\right).$$

Taking differentials, we can obtain

$$\dot{\alpha}_1 = h_1(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) + \Delta h_1(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}), \tag{89}$$

where $h_1(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})$ and $\Delta h_1(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})$ are two proper functions, and the latter is related to the system uncertainties.

On the other hand, we can obtain, from (81),

$$z_2^{(0\sim m_2-1)} = \tilde{P}_2^{-1}\left(A_2^{0\sim m_2-1}\right)\left(x_2^{(0\sim m_2-1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}\right), \tag{90}$$

which gives,

$$\begin{aligned} z_2^{(m_2-1)} &= Q_{22}\left(A_2^{0\sim m_2-1}\right)\left(x_2^{(0\sim m_2-1)} - \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}\right) \\ &= Q_{2F}\left(A_2^{0\sim m_2-1}\right)(x_2 - \alpha_1) \\ &\quad + Q_{2M}\left(A_2^{0\sim m_2-1}\right) x_2^{(1\sim m_2-2)} \\ &\quad + Q_{2L}\left(A_2^{0\sim m_2-1}\right) x_2^{(m_2-1)}. \end{aligned}$$

Taking differentials of the above equation, and using (89) and the second equation in (5), produce

$$\begin{aligned} z_2^{(m_2)} &= Q_{2F}\left(A_2^{0\sim 1}\right)(\dot{x}_2 - \dot{\alpha}_1) \\ &\quad + Q_{2M}\left(A_2^{0\sim 1}\right)\dot{x}_2^{(1\sim m_2-2)} \\ &\quad + Q_{2L}\left(A_2^{0\sim 1}\right) x_2^{(m_2)} \\ &= \hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) + \Delta\hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) \\ &\quad + L_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) x_3. \end{aligned} \tag{91}$$

Further let

$$\tilde{P}_3\left(A_3^{0\sim m_3-1}\right) z_3^{(0\sim m_3-1)} = x_3^{(0\sim m_3-1)} - \begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}, \tag{92}$$

which implies, in view of the symmetry property of $P_3(A_3^{0\sim m_3-1})$,

$$P_{3L}^{\mathrm{T}}\left(A_3^{0\sim m_3-1}\right) z_3^{(0\sim m_3-1)} = x_3 - \alpha_2. \tag{93}$$

Substituting (93) into (91), yields

$$z_2^{(m_2)} = \hat{f}_2\left(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}\right) + \Delta\hat{f}_2\left(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}\right) + L_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) \times \left(P_{3L}^{\mathrm{T}}\left(A_3^{0\sim m_3-1}\right)z_3^{(0\sim m_3-1)} + \alpha_2\right). \quad (94)$$

Assume that there exists a continuous function $\rho_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})$ satisfying

$$\left|\Delta\hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\right| \leqslant \rho_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}), \quad (95)$$

applying Theorem 3.4 to system (94) produces the second virtual control law of the system as

$$\alpha_2 = -L_2^{-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})\left(A_2^{0\sim 1}z_2^{(0\sim m_2-1)} + \alpha_2^*\right)$$
$$\alpha_2^* = \frac{1}{4\varepsilon}\rho_2^2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})P_{2L}^{\mathrm{T}}(A_2^{0\sim 1})z_2^{(0\sim 1)} + \hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}) + v_1, \quad (96)$$

where

$$v_1 = L_1(z_1^{(0\sim m_1-1)})P_{1L}^{\mathrm{T}}\left(A_1^{0\sim m_1-1}\right)z_1^{(0\sim m_1-1)}. \quad (97)$$

The closed-loop subsystem resulted in by this control law is

$$z_2^{(m_2)} + A_2^{0\sim m_2-1}z_2^{(0\sim m_2-1)} = \phi_2 + w_2 - v_1, \quad (98)$$

where

$$\phi_2 = -\frac{1}{4\varepsilon}\rho_2^2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})P_{2L}^{\mathrm{T}}(A_2^{0\sim m_2-1})z_2^{(0\sim m_2-1)} + \Delta\hat{f}_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2}), \quad (99)$$

and

$$w_2 = L_2(z_i^{(0\sim m_i-1)}|_{i=1\sim 2})P_{3L}^{\mathrm{T}}\left(A_3^{0\sim m_3-1}\right)z_3^{(0\sim m_3-1)}. \quad (100)$$

5.3. The n-th step

Suppose that the expressions of α_{n-1}, $\dot{\alpha}_{n-1}$, $h_{n-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim n})$ and $\triangle h_{n-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim n})$ have all been obtained in the $(n-1)$th step. Similar to the previous deduction, we can obtain

$$z_n^{(m_n)} = \hat{f}_n(z_i^{(0\sim m_i-1)}|_{i=1\sim n}) + \Delta\hat{f}_n(z_i^{(0\sim m_i-1)}|_{i=1\sim n}) + L_n(z_i^{(0\sim m_i-1)}|_{i=1\sim n})u. \quad (101)$$

Assume that there exist a set of non-negative continuous scalar functions $\rho_k(z_i^{(0\sim m_i-1)}|_{i=1\sim k})$, $k = 1, 2, \ldots, n$ satisfying

$$\left|\Delta\hat{f}_k(z_i^{(0\sim m_i-1)}|_{i=1\sim k})\right| \leq \rho_k(z_i^{(0\sim m_i-1)}|_{i=1\sim k}), \times k = 1, 2, \ldots, n. \quad (102)$$

Then, by applying Theorem 3.4 to system (101), the control law of the system (5) can be finally designed as

$$u = -L_n^{-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim n}) \times \left(A_n^{0\sim m_n-1}z_n^{(0\sim m_n-1)} + u^*\right)$$
$$u^* = \frac{1}{4\varepsilon}\rho_n^2(z_i^{(0\sim m_i-1)}|_{i=1\sim n})P_{nL}^{\mathrm{T}}(A_n^{0\sim m_n-1})z_n^{(0\sim m_n-1)} + \hat{f}_n\left(z_i^{(0\sim m_i-1)}|_{i=1\sim n}\right) + v_{n-1}, \quad (103)$$

where

$$v_{n-1} = L_{n-1}(z_i^{(0\sim m_i-1)}|_{i=1\sim n-1})P_{n-1,L}^{\mathrm{T}} \times \left(A_{n-1}^{0\sim m_{n-1}-1}\right)z_{n-1}^{(0\sim m_{n-1}-1)}. \quad (104)$$

The last closed-loop subsystem resulted in by this control law is

$$z_n^{(m_n)} + A_n^{0\sim m_n-1}z_n^{(0\sim m_n-1)} = \phi_n - v_{n-1}, \quad (105)$$

where

$$\phi_n = -\frac{1}{4\varepsilon}\rho_n^2(z_i^{(0\sim m_i-1)}|_{i=1\sim n}) \times P_{nL}^{\mathrm{T}}(A_n^{0\sim m_n-1})z_n^{(0\sim m_n-1)} + \Delta\hat{f}_n(x_i^{(0\sim m_i-1)}|_{i=1\sim n}). \quad (106)$$

5.4. The conclusion

It should be noted that the above procedure also defines a state transformation between $z_i^{(0\sim m_i-1)}$ and $x_i^{(0\sim m_i-1)}$, $i = 1, 2, \ldots, n$, in the form of

$$z_1^{(0\sim m_1-1)} = x_1^{(0\sim m_1-1)},$$

$$\tilde{P}_i\left(A_i^{0\sim m_i-1}\right) z_i^{(0\sim m_i-1)} = x_i^{(0\sim m_i-1)} - \begin{bmatrix}\alpha_{i-1}\\ 0\end{bmatrix},$$
$$i = 2, 3, \ldots, n, \tag{107}$$

where $\alpha_{i-1} = \alpha_{i-1}(x_k^{(0\sim m_k-1)}|_{k=1\sim i-1})$. Similar to the proof of Theorem 4.1, we can also prove the following conclusion.

Theorem 5.1: *Suppose that Assumption A3 holds. Let μ and ε be two arbitrarily given positive numbers, and $A_i^{0\sim m_i-1} \in \mathbb{R}^{1\times m_i}$, $i = 1, 2, \ldots, n$ be a set of matrices satisfying (77). Assume that there exist a set of non-negative continuous scalar functions $\rho_k(z_i^{(0\sim m_i-1)}|_{i=1\sim k})$, $k = 1, 2, \ldots, n$ satisfying (102). Then, the controller (103) for the uncertain high-order SFS (5) guarantees that the transformed states $z_i^{(0\sim m_i-1)}$, $i = 1, 2, \ldots, n$ finally converge into the following ellipsoid:*

$$\Theta^{\mu,\varepsilon}(0) = \left\{ z_i^{(0\sim m_i-1)}|_{i=1\sim n} \,\middle|\, \sum_{i=1}^{n}\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i z_i^{(0\sim m_i-1)} \le \frac{n\varepsilon}{\mu}\right\},$$

where $P_i = P_i(A_i^{0\sim m_i-1})$.

It is known from the above theorem that the states $z_i^{(0\sim m_i-1)}$, $i = 1, 2, \ldots, n$ of each subsystem eventually converge into the following ellipsoid centred at the origin:

$$\Theta_i^{\mu,\varepsilon}(0) = \left\{ z_i^{(0\sim m_i-1)} \,\middle|\, \left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i z_i^{(0\sim m_i-1)} \le \frac{n\varepsilon}{\mu}\right\}.$$

Since $z_1 = x_1$, the state $x_1^{(0\sim m_1-1)}$ converges into the following ellipsoid centred at the origin:

$$\Theta_1^{\mu,\varepsilon}(0) = \left\{ x_1^{(0\sim m_1-1)} \,\middle|\, \left(x_1^{(0\sim m_1-1)}\right)^{\mathrm{T}} P_1 x_1^{(0\sim m_1-1)} \le \frac{n\varepsilon}{\mu}\right\}.$$

To end this section, we make a few remarks.

Remark 5.1: The above proposed mixed-order backstepping method can also be easily modified to suit the other companion SFS (6). As a matter of fact, for SFS (6), we can simply take $\tilde{P}_i(A_i^{0\sim m_i-1}) = P_i(A_i^{0\sim m_i-1})$, $i = 1, 2, \ldots, n$, and in this case we also have theoretically $Q_{iL}(A_i^{0\sim m_i-1}) \neq 0$, $i = 1, 2, \ldots, n$, since $Q_{iL}(A_i^{0\sim m_i-1})$ can be easily shown to be a diagonal element of a positive definite matrix.

Remark 5.2: The second- and high-order (mixed-order) backstepping methods discussed in the previous and the current sections can both be properly modified to cope with the case of signal tracking. Specifically, in order to let x_1 track a given sufficiently smooth signal $x_1^*(t)$, we need only to adjust the first step of these second- and high-order methods of backstepping by substituting (30) or (81) by

$$z_1^{(0\sim 1)} = x_1^{(0\sim 1)} - \left(x_1^*\right)^{(0\sim 1)},$$

or

$$z_1^{(0\sim m_1-1)} = x_1^{(0\sim m_1-1)} - \left(x_1^*\right)^{(0\sim m_1-1)},$$

and then in the sequential steps treat the terms related to the reference signal $x_1^*(t)$ properly.

Remark 5.3: The above high-order backstepping method for high-order systems appears to be more complicated in description. It should be noted that, in many specific engineering applications, m_i, $i = 1, 2, \ldots, n$ often take the values of 1 and 2. So the fact is, in such applications, the mixed-order backstepping method is even simpler than that of the second-order systems.

Remark 5.4: For a specific practical system of high-order, direct application of the high-order backstepping generally requires fewer steps than application of the normal first-order backstepping method, since there are more subsystems in the converted first-order SFS. What is more, application of high-order backstepping also saves the process of converting the high-order subsystems originally obtained from physical modelling into first-order ones. Therefore, to solve a specific practical problem, the method of high-order (or mixed-order) backstepping is generally simpler than the normal first-order method of backstepping.

6. An illustrative example

Applications of the proposed robust HOFA approach to some practical systems will be investigated elsewhere. In this section, we only demonstrate the application procedures of the proposed approaches via an illustrative example obtained from the following first-order SFS (Kokotovic & Arcak, 2001, p.648):

$$\begin{aligned}\dot{x}_1 &= x_1^2 + x_2\\ \dot{x}_2 &= u,\end{aligned} \tag{108}$$

where all variables are scalar ones. By changing the first-order derivatives into second-order ones, and also adding uncertainties, we obtain the following second-order uncertain SFS:

$$\begin{aligned}\ddot{x}_1 &= x_1^2 + \Delta f_1(x_1) + x_2\\ \ddot{x}_2 &= \Delta f_2(x_1, \dot{x}_2) + u.\end{aligned} \tag{109}$$

Obviously, we have

$$G_1 = G_2 = 1, \quad f_1 = x_1^2, \quad f_2 = 0.$$

The uncertain terms are supposed to satisfy

$$\left|\Delta f_1(x_1)\right| \leq \sigma_1 x_1^2, \tag{110}$$

$$\left|\Delta f_2(x_1, \dot{x}_2)\right| \leq \sigma_2 \left|x_1 \dot{x}_2\right|, \tag{111}$$

and

$$\left|\frac{\partial \Delta f_1(x_1)}{\partial x_1}\right| \leq \gamma_1 \left|\dot{x}_1\right|, \tag{112}$$

$$\left|\frac{\partial^2 \Delta f_1(x_1)}{\partial x_1^2}\right| \leq \gamma_2 \left|\ddot{x}_1\right|, \tag{113}$$

with σ_i and γ_i, $i = 1, 2$, being some non-negative scalars.

6.1. Solution with backstepping

6.1.1. Step 1

Corresponding to the form of system (33), we have

$$\begin{aligned}\hat{f}_1(z_1, \dot{z}_1) &= z_1^2\\ \Delta\hat{f}_1(z_1, \dot{z}_1) &= \Delta f_1(z_1)\\ L_1(z_1, \dot{z}_1) &= 1.\end{aligned} \tag{114}$$

In view of (110), we can take

$$\rho_1(z_1, \dot{z}_1) = \sigma_1 z_1^2,$$

it then follows from Theorem 3.4 that the virtual control law can be designed as

$$\alpha_1 = -A_1^{0\sim1} z_1^{(0\sim1)} - z_1^2 - \frac{1}{4\varepsilon}\sigma_1^2 z_1^4 P_{1L}^{\mathrm{T}}(A_1^{0\sim1}) z_1^{(0\sim1)}. \tag{115}$$

6.1.2. Step 2

Partition $P_{1L}(A_1^{0\sim1})$ into

$$P_{1L}(A_1^{0\sim1}) = \begin{bmatrix} P_{1L}^{\mathrm{I}}(A_1^{0\sim1}) \\ P_{1L}^{\mathrm{II}}(A_1^{0\sim1}) \end{bmatrix},$$

then, we have, from (115),

$$\dot{\alpha}_1 = h_1(z_1, \dot{z}_1, x_2) + \Delta h_1(z_1), \tag{116}$$

with

$$\begin{aligned} h_1&(z_1, \dot{z}_1, x_2)\\ &= -\left(A_1^0 \dot{z}_1 + 2z_1\dot{z}_1 + A_1^1\left(z_1^2 + x_2\right)\right)\\ &\quad - \frac{1}{\varepsilon}\sigma_1^2 z_1^3 \dot{z}_1 P_{1L}^{\mathrm{T}}(A_1^{0\sim1}) z_1^{(0\sim1)}\\ &\quad - \frac{1}{4\varepsilon}\sigma_1^2 P_{1L}^{\mathrm{I}}(A_1^{0\sim1}) z_1^4 \dot{z}_1\\ &\quad - \frac{1}{4\varepsilon}\sigma_1^2 P_{1L}^{\mathrm{II}}(A_1^{0\sim1})\left(z_1^2 + x_2\right) z_1^4, \end{aligned} \tag{117}$$

and

$$\Delta h_1(z_1) = -\left(A_1^1 + \frac{1}{4\varepsilon}\sigma_1^2 z_1^4 P_{1L}^{\mathrm{II}}(A_1^{0\sim1})\right)\Delta f_1(z_1). \tag{118}$$

By formula (42), we can obtain

$$\ddot{z}_2 = \hat{f}_2(z_{1\sim2}^{(0\sim1)}) + \Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)}) + Q_{2L}\left(A_2^{0\sim1}\right)u, \tag{119}$$

with

$$\begin{aligned}\hat{f}_2(z_{1\sim2}^{(0\sim1)}) &= Q_{2F}\left(A_2^{0\sim1}\right)\left(\dot{x}_2 - h_1(z_1, \dot{z}_1, x_2)\right)\\ \Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)}) &= Q_{2F}\left(A_2^{0\sim1}\right)\Delta h_1(z_1) + Q_{2L}\left(A_2^{0\sim1}\right)\\ &\quad \times \Delta f_2(x_1, \dot{x}_2).\end{aligned} \tag{120}$$

Noting (118), we have

$$\begin{aligned}\left|\Delta h_1(z_1)\right| &\leq \left|A_1^1 + \frac{1}{4\varepsilon}\sigma_1^2 z_1^4 P_{1L}^{\mathrm{II}}(A_1^{0\sim1})\right| \left|\Delta f_1(z_1)\right|\\ &\leq \sigma_1\left(\left|A_1^1\right| + \frac{1}{4\varepsilon}\sigma_1^2 z_1^4 \left|P_{1L}^{\mathrm{II}}(A_1^{0\sim1})\right|\right) x_1^2\\ &\triangleq \delta(z_1).\end{aligned} \tag{121}$$

Further using (120) and (111), we have

$$\begin{aligned}&\left|\Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)})\right|\\ &\quad \leq \left|Q_{2F}\left(A_2^{0\sim1}\right)\right| \left|\Delta h_1(z_1)\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|Q_{2L}\left(A_2^{0\sim1}\right)\right|\left|\Delta f_2(x_1,\dot{x}_2)\right| \\
&\le \left|Q_{2F}\left(A_2^{0\sim1}\right)\right|\delta\left(z_1\right)+\sigma_2\left|Q_{2L}\left(A_2^{0\sim1}\right)\right|\left|x_1\dot{x}_2\right|.
\end{aligned}
$$

Thus we can choose

$$
\begin{aligned}
&\rho_2\left(z_{1\sim2}^{(0\sim1)}\right) \\
&= \left|Q_{2F}\left(A_2^{0\sim1}\right)\right|\delta\left(z_1\right)+\sigma_2\left|Q_{2L}\left(A_2^{0\sim1}\right)\right|\left|x_1\dot{x}_2\right|.
\end{aligned}
$$

Applying Theorem 3.4 to system (119) gives the robust stabilising control law of the system as

$$
\begin{aligned}
u &= -Q_{2L}^{-1}\left(A_2^{0\sim1}\right) \\
&\quad\times\left(A_2^{0\sim1}z_2^{(0\sim1)}+\hat{f}_2(z_{1\sim2}^{(0\sim1)})+\alpha_2^*+v_1\right) \\
\alpha_2^* &= \frac{1}{4\varepsilon}\rho_2^2\left(z_{1\sim2}^{(0\sim1)}\right)P_{2L}^{\mathrm{T}}(A_2^{0\sim1})z_2^{(0\sim1)} \\
v_1 &= P_{1L}^{\mathrm{T}}\left(A_1^{0\sim1}\right)z_1^{(0\sim1)}.
\end{aligned}
\tag{122}
$$

where $A_i^{0\sim1}=[A_i^0\ \ A_i^1]\in\mathbb{R}^{1\times2}$, $i=1,2$, are taken to satisfy

$$
\mathrm{Re}\lambda_i\left(\Phi\left(A_k^{0\sim1}\right)\right)<-\frac{\mu}{2},\quad i,k=1,2,
\tag{123}
$$

and $P_{iL}(A_i^{0\sim1})$, $i=1,2$, are determined by (26)–(27).

6.1.3. The closed-loop system

It follows from the steps of the second-order method of backstepping described in Section 4 that the closed-loop system resulted in by this robust control law is

$$
\begin{aligned}
\ddot{z}_1+A_1^{0\sim1}z_1^{(0\sim1)} &= \phi_1+w_1 \\
\ddot{z}_2+A_2^{0\sim1}z_2^{(0\sim1)} &= \phi_2-v_1,
\end{aligned}
\tag{124}
$$

with

$$
\begin{aligned}
\phi_1 &= -\frac{1}{4\varepsilon}\sigma_1^2z_1^2P_{1L}^{\mathrm{T}}(A_1^{0\sim1})z_1^{(0\sim1)}+\Delta f_1(z_1) \\
\phi_2 &= -\frac{1}{4\varepsilon}\rho_2^2(z_{1\sim2}^{(0\sim1)})P_{2L}^{\mathrm{T}}(A_2^{0\sim1})z_2^{(0\sim1)} \\
&\quad+\Delta\hat{f}_2(z_{1\sim2}^{(0\sim1)}),
\end{aligned}
\tag{125}
$$

and

$$
w_1=P_{2L}^{\mathrm{T}}\left(A_2^{0\sim1}\right)z_2^{(0\sim1)}.
\tag{126}
$$

It follows from Theorem 4.1 that the transformed state $z_{1\sim2}^{(0\sim1)}$ converges into the following ellipsoid:

$$
\begin{aligned}
&\Theta^{\mu,\varepsilon}(0) \\
&= \left\{z_{1\sim2}^{(0\sim1)}\left|\sum_{i=1}^{2}\left[\left(z_{1\sim i}^{(0\sim1)}\right)^{\mathrm{T}}P_i(A_i^{0\sim1})z_{1\sim i}^{(0\sim1)}\right]\leqslant\frac{2\varepsilon}{\mu}\right.\right\}.
\end{aligned}
$$

Particularly, the state $x_1^{(0\sim1)}$ of the original system converges into the following ellipsoid centred at the origin:

$$
\begin{aligned}
&\Theta_1^{\mu,\varepsilon}(0) \\
&= \left\{x_1^{(0\sim1)}\left|\left(x_1^{(0\sim1)}\right)^{\mathrm{T}}P_1(A_1^{(0\sim1)})x_1^{(0\sim1)}\leqslant\frac{2\varepsilon}{\mu}\right.\right\}.
\end{aligned}
$$

Remark 6.1: If the system (109) is converted into the first-order state-space model, then the converted model is composed of 4 subsystems, and eventually four steps of backstepping are needed to solve the problem. Experience with normal first-order backstepping designs tells us that the design process for the converted first-order systems with four steps of backstepping is definitely much more complicated than the above process.

6.2. Solution based on a single HOFA model

By the Corollary 3.1 in Duan (2020d), we have $L=1$, and

$$
\begin{aligned}
h(x_1,x_2) &= [1\quad G_1]\begin{bmatrix}\left(x_1^2+\Delta f_1(x_1)\right)'' \\ \Delta f_2(x_1,\dot{x}_2)\end{bmatrix} \\
&= [1\quad 1]\begin{bmatrix}2\dot{x}_1^2+2x_1\ddot{x}_1+\Delta\ddot{f}_1(x_1) \\ \Delta f_2(x_1,\dot{x}_2)\end{bmatrix} \\
&= 2\dot{x}_1^2+2x_1\ddot{x}_1+\Delta\ddot{f}_1(x_1)+\Delta f_2(x_1,\dot{x}_2).
\end{aligned}
\tag{127}
$$

Thus an equivalent HOFA model for system (109) can be derived, with $z=x_1$, as follows:

$$
z^{(4)}=f\left(z^{(0\sim2)}\right)+\Delta f\left(z^{(0\sim3)}\right)+u,
\tag{128}
$$

where

$$
f\left(z^{(0\sim2)}\right)=2\dot{z}^2+2z\ddot{z},
\tag{129}
$$

$$
\begin{aligned}
&\Delta f\left(z^{(0\sim3)}\right) \\
&= \Delta\ddot{f}_1(z)+\Delta f_2(z,\dot{x}_2) \\
&= \Delta\ddot{f}_1(z)+\Delta f_2(z,\left(\ddot{z}-z^2+\Delta f_1(z)\right)').
\end{aligned}
\tag{130}
$$

Proposition 6.1: *The uncertain term in system* (128) *satisfies*

$$
\left|\Delta f\left(z^{(0\sim3)}\right)\right|\le\rho\left(z^{(0\sim3)}\right),
$$

with

$$\rho\left(z^{(0\sim3)}\right)=\left(\gamma_1\left|\dot{z}\right|+\gamma_2\dot{z}^2\right)\left|\ddot{z}\right| \\ +\sigma_2\left|z\right|\left(\left|\dddot{z}-2z\dot{z}\right|+\gamma_1\left|\dot{z}\right|^2\right).$$

Proof: Since

$$\Delta\ddot{f}_1(z)=\left(\frac{\partial\Delta f_1(z)}{\partial z}\dot{z}\right)' \\ =\frac{\partial^2\Delta f_1(z)}{\partial z^2}\dot{z}^2+\frac{\partial\Delta f_1(z)}{\partial z}\ddot{z},$$

we have, using (112) and (113),

$$\left|\Delta\ddot{f}_1(z)\right|\leq\left|\frac{\partial^2\Delta f_1(z)}{\partial z^2}\right|\dot{z}^2+\left|\frac{\partial\Delta f_1(z)}{\partial z}\right|\left|\ddot{z}\right| \\ \leq\left(\gamma_1\left|\dot{z}\right|+\gamma_2\dot{z}^2\right)\left|\ddot{z}\right|.$$

Further using (111), gives

$$\left|\Delta f_2(z,\left(\ddot{z}-z^2+\Delta f_1(z)\right)')\right| \\ \leq\sigma_2\left|z\right|\left|\dddot{z}-2z\dot{z}+\frac{\partial\Delta f_1(z)}{\partial z}\dot{z}\right| \\ \leq\sigma_2\left|z\right|\left(\left|\dddot{z}-2z\dot{z}\right|+\left|\frac{\partial\Delta f_1(z)}{\partial z}\right|\left|\dot{z}\right|\right) \\ \leq\sigma_2\left|z\right|\left(\left|\dddot{z}-2z\dot{z}\right|+\gamma_1\left|\dot{z}\right|^2\right).$$

Finally, using the above two relations and the definition

(130), yields

$$\left|\Delta f\left(z^{(0\sim3)}\right)\right| \\ \leq\left|\Delta\ddot{f}_1(z)\right|+\left|\Delta f_2(z,\ddot{z}-z^2+\Delta f_1(z))\right| \\ \leq\left(\gamma_1\left|\dot{z}\right|+\gamma_2\dot{z}^2\right)\left|\ddot{z}\right| \\ +\sigma_2\left|z\right|\left(\left|\dddot{z}-2z\dot{z}\right|+\gamma_1\left|\dot{z}\right|^2\right).$$

Thus the conclusion holds. ■

Using the function $\rho(z^{(0\sim3)})$ defined in the above proposition, we can design for the system (109) the following robust stabilising control law

$$u=-\left(A^{0\sim3}z^{(0\sim3)}+f(z^{(0\sim2)})+u^*\right) \\ u^*=\frac{1}{4\varepsilon}\rho^2(z^{(0\sim3)})P_L^{\mathrm{T}}(A^{0\sim3})z^{(0\sim3)}, \tag{131}$$

where $A^{0\sim3}$ is chosen to satisfy

$$\mathrm{Re}\lambda_i\left(\Phi\left(A^{0\sim3}\right)\right)<-\frac{\mu}{2},\quad i=1,2,3,4, \tag{132}$$

μ and ε are two arbitrarily given positive numbers. The controller guarantees that the vector $z^{(0\sim3)}$ converges into the following ellipsoid centred at the origin:

$$\Theta_{\mu,\varepsilon}(0)=\left\{z^{((0\sim3))}\left|\left(z^{(0\sim3)}\right)^{\mathrm{T}}Pz^{(0\sim3)}\leq\frac{\varepsilon}{\mu}\right.\right\},$$

where $P=P(A^{0\sim3})$ and $P_L^{\mathrm{T}}(A^{0\sim3})$ are determined by (10) and (12).

The above design process demonstrates that the design based on a single HOFA model is even simpler than the proposed high-order method of backstepping, but it should be noted that the two approaches need different prior knowledge of the nonlinear uncertainties.

7. Conclusion

The paper proposes a HOFA approach for robust control of uncertain nonlinear systems, and has demonstrated the following:

- Many uncertain nonlinear systems can be represented by an uncertain HOFA model, which is either composed of a single uncertain HOFA model or a set of uncertain HOFA models. As a result, the well-known first-order uncertain SFSs, as well as the newly proposed second- and high-order uncertain SFSs are actually all ‘pseudo’ uncertain HOFA models for nonlinear systems since each of their subsystem obey a full-actuation structure.
- Due to the full-actuation feature of the uncertain HOFA models, effective designs of robust controllers for nonlinear systems are proposed. For the cases of second- and high-order uncertain SFSs, the designs turn out to be the generalised second- and high-order methods of backstepping for second- and high-order SFSs.
- The proposed approach possesses several merits, particularly, the state vectors converge globally into an ellipsoid with a radius which can be made as small as desired. Furthermore, for the case of a single HOFA model, the closed-loop system becomes a constant linear one when uncertainties do not present.

It is commonly known that the method of backstepping can hardly be applicable to an SFS with many subsystems due to the well-known differential explosion' problem. Such a problem of course also applies to the newly proposed second- and high-order methods of backstepping. Nevertheless, for a specific system physically modelled by, e.g. Lagrangian Equation, a second-order method of backstepping needs only half of the steps of the first-order method of backstepping, and also saves the process of converting the second-order system into a first-order one, and is thus much simpler to apply than the normal first-order method of backstepping.

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