ARTICLE TEMPLATE

High-order fully-actuated system approaches: Part IX. Generalized PID control and model reference tracking

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ABSTRACT
Tracking control of general dynamical systems in high-order fully actuated (HO-FA) system representation is solved. For the case of tracking a constant or slow time-varying signal in the presence of constant or slow time-varying disturbances, a generalized PID control scheme is proposed, which realizes asymptotical tracking to a prescribed signal and also guarantees that the state derivatives of certain orders converge to the origin. For the case that the signal to be tracked is generated by a reference model, a model reference tracking (MRT) controller is presented, which relies on the solution to a type of generalized Sylvester matrix equations and guarantees the desired asymptotical tracking requirement. Due to the full-actuation property of the HOFA models, closed-loop systems under both control schemes are constant and linear. Furthermore, based on a general parametric solution to the type of Sylvester matrix equations and a general parametric eigenstructure assignment result, simple and complete parameterization of the two types of control designs are provided, and feasibility conditions in terms of the system initial values for sub-fully actuated systems are also derived. An illustrative example is presented to demonstrate the application of the proposed approaches and their effects.

KEYWORDS
PID control; model reference tracking; fully actuated systems; feasibility conditions; parameterization

1. Introduction

1.1. About PID control

PID controller exhibits a lasting vitality in the domain of automatic control due to its simple structure and ease of use. It dates back to 1910 with the first PID controller developed by Elmer Sperry for the US Navy (Ang, Chong, & Li, 2005; Borase, Maghade, Sondkar & Pawar, 2020), and Ziegler-Nichols rule (Ziegler & Nichols, 1942, 1943) is one of the most prominent method to choose controller parameters which is seen as the most challenging work for PID controller design.

Most of existing works of PID control focus on linear systems (e.g., Åström & Hägglund (1995); Åström, Hägglund, & Astrom (2006)). Furthermore, Ho & Lin (2003) synthesizes a PID controller for a class of SISO systems subject to uncertainty with robust performance, and Silva, Datta, & Bhattacharyya (2002) considers stabilization

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of a first-order time-delay system using a PID controller with a complete set of PID parameters determined. Although almost all the systems in fact are nonlinear with uncertainties, there exist only a few basic theories of PID control for nonlinear systems (Zhang & Guo, 2019). Among those, Chang, Hwang, & Hsieh (2002) studies a class of nonlinear PID control systems based on the Lyapunov approach with proposing a direct adaptive tuning method, and Zhao & Guo (2017) propose a theory on PID controllers for nonlinear uncertain systems, which gives a simple and analytic design method for the PID parameters based on the knowledge of some upper bound, and guarantees the global stability and asymptotic regulation of the closed-loop system. Very recently, an extended PID control is considered by Zhao & Guo (2020) for high-order affine nonlinear uncertain systems, with the semi-global stabilization of systems and exponential convergence of regulation errors guaranteed under some suitable conditions.

1.2. About model reference tracking (MRT)

Model reference control, also termed as model following control, appeared in the late 1970s (Young, 1978). It regulates the system output to follow a reference model output and has attracted much attention. Duan, Liu, & Liu (2001) and Duan & Zhang (2007) investigate the robust model-reference control for multivariable linear systems and descriptor linear systems, respectively, with structural parameter uncertainties. A robust model following control for a class of second-order dynamical systems subject to parameter uncertainties is considered in Duan & Huang (2008). Furthermore, an linear parameter varying (LPV) model reference control scheme is designed in Abdullah & Zribi (2009) for LPV systems. Based on a multi-objective optimisation problem, Gonçalves, Bachur, Palhares, & Takahashi (2011) proposes robust $\mathrm{H}_2/\mathrm{H}_\infty$ reference model dynamic output-feedback controllers for uncertain continuous- or discrete-time linear time-invariant systems with polytopic uncertainty.

In addition, some of the recent literature focuses on model reference adaptive control (MRAC). It is used for the systems of which the parameters are unknown and/or change with time, and designs controllers with controller parameter adjustment mechanism by comparing the output of systems and the reference model (Pathak & Adhyaru, 2012). However, this method has the shortcomings of effectiveness for nonlinear systems, and the theories on sensitivity, controllability, observability, stability and robustness need further study (Shekhar & Sharma, 2018).

1.3. Fully actuated system approaches

As outlined above, the problem of tracking and regulation control in the state-space framework can be only effectively solved for linear systems and certain very special nonlinear ones with strict conditions. While the results for the nonlinear cases are also often limited to certain local senses.

High-order fully actuated (HOFA) models proposed in Duan (2020a,b), and Duan (2020d, 2021c), although subject to a full-actuation condition, are general models for dynamical control systems. A huge advantage of such models is that their full-actuation feature allows one to cancel the known nonlinearities in the system and hence to convert, to some extent, a nonlinear problem into a linear one. Consequently, the HOFA approaches for control systems design have been demonstrated to be extremely convenient and effective in dealing with nonlinear control problems (Duan, 2020f,g, 2021a,b,d).

This paper further contributes to the HOFA approaches by treating the problem of tracking and regulation in general nonlinear systems. The contributions are of the following aspects.

Firstly, the well-known PID scheme is generalized to be applicable to HOFA systems. It is shown that generalized PID scheme guarantees asymptotical tracking of the system output to an arbitrarily prescribed constant or slow time-varying signal in subjection to constant or slow time-varying disturbances. Moreover, the derivatives, of orders up to the system orders, of the system states all converge to zero in spite of the existence of the constant disturbances. The last feature eventually provides for the system very smooth and steady state responses.

Secondly, the MRT approach for linear systems is also extended to the HOFA system case. The controller is composed of a feedback term and a feed-forward term, with the latter determined by a generalized Sylvester matrix equation. Asymptotical tracking of the system output to that of the reference model is achieved. When such a design is combined with the proposed generalized PID control design, the difference between the outputs of the system and the reference model can be made to track asymptotically an arbitrary given constant signal, also in the presence of constant system disturbances.

Thirdly, as a huge advantage of the HOFA approaches, the closed-loop systems under both the above mentioned generalized PID controller and the MRT controller are constant linear. This not only assures the desired exponential stability of the systems but also allows us to establish a complete parameterization of both the two designs. As a consequence, complete and analytical closed-form parametric expressions for all the controller gains are obtained, and the design degrees of freedom provided in the parametric designs can be further utilized to achieve additional system performance.

Lastly, the sub-fully actuated system case is also dealt with. Although many systems in practical applications can be represented by complete fully actuated models, there still exist some systems which can be only represented by sub-fully actuated models (Duan, 2020a, 2021c). Such systems are relatively more difficult to handle due to a problem of feasibility. For both the above mentioned generalized PID control system and the MRT system, feasibility conditions are established for the case of sub-fully actuated systems. It turns out that these feasibility conditions are in fact restrictions on the system initial values.

In the sequential sections, I_n denotes the identity matrix, $\varnothing$ denotes the null set, and $\Omega\backslash\Theta$ represents the complement of the set Θ in set Ω. For a square matrix P, P^{-1} and $\det(P)$ denote its inverse and determinant, respectively, while for a nonsingular matrix P, its condition number is denoted by $\nu(P) = \|P\| \left\|P^{-1}\right\|$. Furthermore, for $x, x_i \in \mathbb{R}^m$, and $A_i \in \mathbb{R}^{m\times m}$, $n_0, n_i \in \mathbb{N}$, $n_0 < n_i$, $i = 1, 2, ..., n$, as in the former papers in the series, the following symbols are used in the paper:

$$x_{n_1\sim n_2} = \begin{bmatrix} x_{n_1} \\ x_{n_1+1} \\ \vdots \\ x_{n_2} \end{bmatrix}, n_1 \leq n_2,$$

$$x^{(n_1 \sim n_2)} = \begin{bmatrix} x^{(n_1)} \\ x^{(n_1+1)} \\ \vdots \\ x^{(n_2)} \end{bmatrix}, n_1 \le n_2,$$

$$x_{i\sim j}^{(n_1 \sim n_2)} = \begin{bmatrix} x_i^{(n_1\sim n_2)} \\ x_{i+1}^{(n_1\sim n_2)} \\ \vdots \\ x_j^{(n_1\sim n_2)} \end{bmatrix}, i \le j, n_1 \le n_2,$$

$$x_k^{(n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_i)} \\ x_{i+1}^{(n_{i+1})} \\ \vdots \\ x_j^{(n_j)} \end{bmatrix}, i \le j,$$

$$x_k^{(n_0\sim n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_0\sim n_i)} \\ x_{i+1}^{(n_0\sim n_{i+1})} \\ \vdots \\ x_j^{(n_0\sim n_j)} \end{bmatrix}, j \ge i,$$

$$A_{0\sim n-1} = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1} \end{bmatrix},$$

$$\Phi(A_{0\sim n-1}) = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & -A_1 & \cdots & -A_{n-1} \end{bmatrix}.$$

The paper is organized into 8 sections. The next section formulates the problems of generalized PID control and MRT control, and the solutions to these two problems are presented in Sections 3 and 4. Section 5 gives parameterizations of the two types of control system designs, while Section 6 further treats the case of sub-fully actuated systems. An illustrative example is fully studied in Section 7, followed by a brief concluding remark in Section 8. The appendix gives the proof of a technical lemma.

2. Problems formulation

2.1. The HOFA model

Consider the following general HOFA system proposed in Duan (2021c):

$$\begin{aligned}\begin{bmatrix} x_1^{(\mu_1)} \\ x_2^{(\mu_2)} \\ \vdots \\ x_\eta^{(\mu_\eta)} \end{bmatrix} &= \begin{bmatrix} f_1\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right) \\ f_2\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right) \\ \vdots \\ f_\eta\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right) \end{bmatrix} \\ &\quad +B\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right)u+\Gamma d, \end{aligned} \tag{1}$$

which, by our notations, can be also written more compactly as

$$\begin{aligned} x_k^{(\mu_k)}|_{k=1\sim\eta} &= f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right) \\ &\quad +B\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right)u+\Gamma d, \end{aligned} \tag{2}$$

where $u\in\mathbb{R}^r$ is the control vector, $d\in\mathbb{R}^\varsigma$ is the disturbance vector, $\Gamma\in\mathbb{R}^{r\times\varsigma}$ is a constant matrix, $\zeta\in\mathbb{R}^\nu$ may represent a parameter vector, an external variable vector, a time-delayed state vector, an unmodeled dynamic state vector, etc.; $\mu_k, k=1,2,...,\eta$, are a set of integers, $x_k\in\mathbb{R}^{r_k}, k=1,2,...,\eta$, are a set of vectors of proper dimensions, with $r_k, k=1,2,...,\eta$, being a set of integers satisfying

$$r_1+r_2+\cdots+r_\eta=r. \tag{3}$$

Further,

$$f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right)=\begin{bmatrix} f_1\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right) \\ f_2\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right) \\ \vdots \\ f_\eta\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right) \end{bmatrix},$$

with $f_k(\cdot)\in\mathbb{R}^{r_k}, k=1,2,...,\eta$, being a set of continuous nonlinear vector functions, and $B(\cdot)\in\mathbb{R}^{r\times r}$ is a continuous matrix function satisfying the following full-actuation condition:

Assumption A1. $\det B\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right)\neq 0,\ \text{or}\ \infty,\ \text{for all}\ x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}$, ζ, and $t\geq 0$.

The system (1) satisfying the above Assumption A1 is called a (complete) fully actuated system (Duan, 2021c).

Since $x_k\in\mathbb{R}^{r_k}$, $k=1,2,...,\eta$, we have

$$x_k^{(0\sim\mu_k-1)}\in\mathbb{R}^{\mu_k r_k},\quad k=1,2,...,\eta.$$

Denote

$$\varkappa = \sum_{k=1}^{\eta} r_k \mu_k, \tag{4}$$

then it is easy to see that

$$x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \in \mathbb{R}^{\varkappa}.$$

For the above system (2) we impose the output equation

$$y = Cx_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \tag{5}$$

where $C \in \mathbb{R}^{m\times\varkappa}$ is a known constant full-row rank matrix.

2.2. Formulation of problems

Basically, in this paper we consider the tracking control in the HOFA system (2). Two circumstances are dealt with.

Firstly, when the design objective is to let the output y track a properly given constant vector $y_{\mathrm{c}} \in \mathbb{R}^m$, we intend to solve the following generalized PID control problem.

Problem 2.1. *Under Assumption A1, find for the HOFA system (2) a controller in the form of*

$$\begin{cases} u = -B^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \zeta, t\right)\left[f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \zeta, t\right) - v\right] \\ v = K_{PD}x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + K_I \int_0^t \left[y(\sigma) - y_{\mathrm{c}}\right] \mathrm{d}\sigma, \end{cases} \tag{6}$$

such that the closed-loop system is stable and

$$\lim_{t\to\infty} y(t) = y_{\mathrm{c}}. \tag{7}$$

Obviously, $K_{PD}x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}$ represents a term of proportional plus derivative feedback, while $K_I \int_0^t [y(\sigma) - y_{\mathrm{c}}]\, \mathrm{d}\sigma$ is an integral feedback. Therefore, the above controller (6) is in a generalized PID form.

Secondly, we consider the case that the design objective is to let the output y track a properly given signal vector $y_{\mathrm{m}}(t)$ generated by the following reference model:

$$\begin{cases} \dot{x}_{\mathrm{m}} = A_{\mathrm{m}}x_{\mathrm{m}} \\ y_{\mathrm{m}} = C_{\mathrm{m}}x_{\mathrm{m}}, \end{cases} \tag{8}$$

where $x_{\mathrm{m}} \in \mathbb{R}^p$, $y_{\mathrm{m}} \in \mathbb{R}^m$, A_{m} and C_{m} are known constant matrices of appropriate dimensions.

Problem 2.2. *Under Assumption A1, find for the HOFA system (2) a controller in* *the form of*

$$\begin{cases} u = -B^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right)\left[f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},\zeta,t\right)-v\right] \\ v = v\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta},x_{\mathrm{m}},y_{\mathrm{c}}\right), \end{cases} \tag{9}$$

such that the closed-loop system is stable and

$$\lim_{t\to\infty}\left[y\left(t\right)-y_{\mathrm{m}}\left(t\right)\right]=y_{\mathrm{c}}, \tag{10}$$

where $y_{\mathrm{m}}(t)$ is generated by the reference model (8), $y_{\mathrm{c}} \in \mathbb{R}^m$ is any desired constant vector.

To end this section, let us finally make some remarks about the considered HOFA model (1), or equivalently, (2) (see also, Duan (2021d)).

Remark 1. Please note that a special case of the above general HOFA model (1) is clearly the following:

$$x^{(n)}=f\left(x^{(0\sim n-1)},\zeta,t\right)+B\left(x^{(0\sim n-1)},\zeta,t\right)u, \tag{11}$$

which forms the basic part of the system models involved in the problems of robust control, adaptive control and disturbance rejection treated in Duan (2020f,g, 2021a,b).

Remark 2. The above HOFA model (2) might be easily mistaken to represent a very small portion of systems due to the full-actuation Assumption A1. While as discussed in Duan (2020a,d, 2021c), it serves as a general model for dynamical control systems. Many systems which are in state-space forms can be converted into HOFA systems (see Duan (2020a,b,d,e)), and practical systems can also be modeled as HOFA systems (see the Remark 3 in Duan (2021d)).

3. Generalized PID Control

In this section, we consider the solution to Problem 2.1.

3.1. Deriving the linear system

Denote

$$v=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\eta \end{bmatrix}, \quad v_i \in \mathbb{R}^{r_i},$$

and

$$\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_\eta \end{bmatrix}, \quad \Gamma_i \in \mathbb{R}^{r_i \times \varsigma},$$

then, under the following control input transformation

$$u = -B^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \zeta, t\right)\left[f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \zeta, t\right) - v\right], \tag{12}$$

the system (2) is turned into the following series of linear systems:

$$x_k^{(\mu_k)} = v_k + \Gamma_k d, \quad k = 1, 2, ..., \eta, \tag{13}$$

which can be equivalently written in the state-space form

$$\begin{array}{rcl} \dot{x}_k^{(0\sim\mu_k-1)} & = & \Phi_k(0_{0\sim\mu_k-1})x_k^{(0\sim\mu_k-1)} + B_{kc}v_k + B_{kc}\Gamma_k d, \\ k & = & 1, 2, ..., \eta, \end{array} \tag{14}$$

where, by our notations,

$$\Phi_k\left(0_{0\sim\mu_k-1}\right) = \begin{bmatrix} 0 & I_{r_k} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \cdots & I_{r_k} \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_{kc} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{r_k} \end{bmatrix}. \tag{15}$$

Define

$$A_E = \operatorname{blockdiag}\left(\Phi_k\left(0_{0\sim\mu_k-1}\right), k = 1, 2, ..., \eta\right),$$

$$B_E = \operatorname{blockdiag}\left(B_{kc}, k = 1, 2, ..., \eta\right),$$

then the set of systems in (14) can be more compactly written as

$$\dot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} = A_E x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + B_E v + B_E \Gamma d. \tag{16}$$

3.2. Solution to Problem 2.1

Define the vector

$$q = \int_0^t \left[y\left(\sigma\right) - y_{\mathrm{c}}\right] \mathrm{d}\sigma, \tag{17}$$

then we have

$$\begin{aligned}\dot{q} &= y(t) - y_{\mathrm{c}} \\ &= Cx_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} - y_{\mathrm{c}}.\end{aligned} \tag{18}$$

Combining the above (18) with (16), gives the following extended system

$$\begin{aligned}\left[\begin{array}{c}\dot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \\ \dot{q}\end{array}\right] &= \tilde{A}\left[\begin{array}{c}x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \\ q\end{array}\right] + \tilde{B}v \\ &\quad + \left[\begin{array}{c}B_E\Gamma \\ 0\end{array}\right]d - \left[\begin{array}{c}0 \\ y_{\mathrm{c}}\end{array}\right].\end{aligned} \tag{19}$$

where

$$\tilde{A} = \left[\begin{array}{cc}A_E & 0 \\ C & 0\end{array}\right], \quad \tilde{B} = \left[\begin{array}{c}B_E \\ 0\end{array}\right]. \tag{20}$$

The following lemma gives a necessary and sufficient condition for the controllability of the linear system (19).

Lemma 3.1. *The matrix pair* $\left[\tilde{A}, \tilde{B}\right]$ *is controllable if and only if*

$$\mathrm{rank}\left[\begin{array}{cc}A_E & B_E \\ C & 0\end{array}\right] = \varkappa + m. \tag{21}$$

Furthermore, it is not stabilizable if

$$m = \mathrm{rank}C > r. \tag{22}$$

Proof. Since $\left[\Phi_k\left(0_{0\sim\mu_k-1}\right), B_{kc}\right], k = 1, 2, ..., \eta$, are all controllable, it is easy to show that $[A_E, B_E]$ is also controllable. Therefore, the first conclusion follows immediately from the Theorem 7.3.2 in Duan (2016).

To show the second conclusion, let condition (22) be met, and partition C as

$$C = \left[\begin{array}{c}C_1 \\ C_2\end{array}\right], \quad C_1 \in \mathbb{R}^{r\times\varkappa}.$$

Then

$$\mathrm{rank}\left[\begin{array}{cc}sI - \tilde{A} & \tilde{B}\end{array}\right] = \mathrm{rank}\left[\begin{array}{cccc}sI - A_E & 0 & 0 & B_E \\ -C_1 & sI_r & 0 & 0 \\ -C_2 & 0 & sI_{m-r} & 0\end{array}\right].$$

Clearly, the existence of C_2 adds $m - r$ number of transmission zeros $\{0\}$. It follows from the first conclusion that these added transmission zeros are not the controllable modes of the matrix pair $\left[\tilde{A}, \tilde{B}\right]$. Since they are not (asymptotically) stable ones, the system is then not stabilizable. □

The first conclusion in the above Lemma 3.1 implies $m \le r$, while the second conclusion further indicates that the system is even not stabilizable when $m > r$. Therefore, the case of $m > r$ is after all of no necessity of consideration. This clearly gives a very important insight, that is, the maximum number of signals (counting by scalar dimension) that the system can track is not greater than the number of the system inputs.

Under the condition (21), we can find for the above compound system (19) a feedback controller in the form of

$$\begin{aligned} v &= K_{PD} x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + K_I q \\ &= K_{PD} x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + K_I \int_0^t \left[y(\sigma) - y_{\mathrm{c}}\right] \mathrm{d}\sigma. \end{aligned} \tag{23}$$

Further denote

$$\begin{aligned} \tilde{A}_c &= \tilde{A} + \tilde{B} \begin{bmatrix} K_{PD} & K_I \end{bmatrix} \\ &= \begin{bmatrix} A_E + B_E K_{PD} & B_E K_I \\ C & 0 \end{bmatrix}, \end{aligned} \tag{24}$$

and

$$\begin{cases} \mathring{C} = \begin{bmatrix} \mathring{C}_1 & \mathring{C}_2 & \cdots & \mathring{C}_\eta \end{bmatrix} \\ \mathring{C}_k = \begin{bmatrix} I_{r_k} & 0_{(\mu_k-1)r_k \times r_k} \end{bmatrix}. \end{cases} \tag{25}$$

Then we can prove the following lemma which gives the property of the above controller (23).

Lemma 3.2. *Consider the system (16), with the output equation (5), satisfying condition (21). If d is also constant, and K_{PD} and K_I are matrices making $\tilde{A}_c$ Hurwitz, then the controller (23) guarantees*

$$\lim_{t\to\infty} y(t) = y_{\mathrm{c}}, \tag{26}$$

$$\lim_{t\to\infty} x_k^{(1\sim\mu_k)}|_{k=1\sim\eta} = 0, \tag{27}$$

and

$$\lim_{t\to\infty} x_{1\sim\eta}(t) = -\mathring{C}\tilde{A}_c^{-1} \begin{bmatrix} \Gamma d \\ -y_{\mathrm{c}} \end{bmatrix}. \tag{28}$$

Proof. When the controller (23) is applied to the system (16), the closed-loop system is obtained as

$$\begin{aligned} \begin{bmatrix} \dot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \\ \dot{q} \end{bmatrix} &= \tilde{A}_c \begin{bmatrix} x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \\ q \end{bmatrix} \\ &\quad + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} d - \begin{bmatrix} 0 \\ y_{\mathrm{c}} \end{bmatrix}, \end{aligned} \tag{29}$$

where $\tilde{A}_c$ is given by (24), which can be made Hurwitz by selecting the gain matrices K_{PD} and K_I due to Lemma 3.1

Recall that y_{c} is constant. If the disturbance d is also constant, then, taking first-order derivative on both sides of (29), yields

$$\left[\begin{array}{c} \ddot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \\ \ddot{q} \end{array}\right] = \tilde{A}_c \left[\begin{array}{c} \dot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \\ \dot{q} \end{array}\right]. \tag{30}$$

Therefore, when $\tilde{A}_c$ is Hurwitz, we have

$$\left[\begin{array}{c} \dot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \\ \dot{q} \end{array}\right] \to 0, \text{ as } t \to \infty,$$

which immediately gives

$$\dot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} = x_k^{(1\sim\mu_k)}|_{k=1\sim\eta} \to 0, \text{ as } t \to \infty, \tag{31}$$

and

$$\dot{q}\,(t) = y\,(t) - y_{\mathrm{c}} \to 0, \text{ as } t \to \infty. \tag{32}$$

These are respectively the conditions (27) and (26).

Using the notations in (25), we have

$$x_{1\sim\eta} = \mathring{C}\dot{x}_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}.$$

Further, using again the closed-loop system (29), we have, in the frequency domain,

$$x_{1\sim\eta}\,(s) = \frac{1}{s}\mathring{C}\left(sI - \tilde{A}_c\right)^{-1} u_{dc},$$

where

$$u_{dc} = \left[\begin{array}{c} \Gamma \\ 0 \end{array}\right] d - \left[\begin{array}{c} 0 \\ y_{\mathrm{c}} \end{array}\right].$$

Therefore, by the well-known Final Value Theorem,

$$\lim_{t\to\infty} x_{1\sim\eta}\,(t) = -\mathring{C}\tilde{A}_c^{-1}u_{dc}.$$

This is the condition (28). By now the proof is complete. □

Combining the above lemma with the results in Subsection 3.1, produces the solution to the generalized PID control problem.

Theorem 3.3. *Consider the system (2) with the output equation (5). If*

(1) Assumption A1 and condition (21) are satisfied;
(2) the disturbance vector d is also constant; and
(3) the two gain matrices K_{PD} and K_I make the matrix $\tilde{A}_c$ in (24) Hurwitz,

then the controller (6) guarantees the asymptotical tracking relations (7) and (27)-(28).

It is well-known that PID is a very popular control technique, and has been widely and successfully used in various practical processes. The above Lemma 3.2 and Theorem 3.3 reveal theoretically the reason behind such a fact, that is, such controllers guarantee asymptotical tracking of constant vectors under arbitrary constant disturbances. We also remark that such controllers often work practically well enough with slow time-varying tracked vectors and disturbances.

One more advantage of the generalized PID controller (6) is that it provides very smooth and steady transient response due to the relations in (27). It should be noted that the relation (27) is really an extra and seldom by-product. For instance, when such a generalized PID controller is applied to a second-order system, besides the velocity signals, the acceleration signals also converge to zero.

3.3. Decoupled design

The above design of the intermediate control vector v based on the overall model (16) is a coupled design (see, Duan (2021c)). In certain cases, e.g., when $y(t)$ is dependent on only some, but not all, x_k's and their derivatives, a decoupled design can be proposed, which is simpler in the sense that it deals with certain decoupled linear systems.

For convenience, let us simply assume that

$$y = Cx_{\eta}^{(0\sim\mu_{\eta}-1)}.$$

In this case, instead of converting (14) into a whole system (16), we can particularly consider the following subsystem:

$$\begin{cases} \dot{x}_{\eta}^{(0\sim\mu_{\eta}-1)} = \Phi_{\eta}(0_{0\sim\mu_{\eta}-1})x_{\eta}^{(0\sim\mu_{\eta}-1)} + B_{\eta c}v_{\eta} + B_{\eta c}\Gamma_{\eta}d \\ y = Cx_{\eta}^{(0\sim\mu_{\eta}-1)}. \end{cases} \tag{33}$$

According to our approach, a PID controller for the system can be designed as

$$v_{\eta} = K_{\eta PD}x_{\eta}^{(0\sim\mu_{\eta}-1)} + K_{\eta I}\int_{0}^{t}[y(\sigma) - y_{\mathrm{c}}]\,\mathrm{d}\sigma,$$

where $K_{\eta PD}$ and $K_{\eta I}$ are two matrices making

$$\tilde{A}_{\eta c} = \begin{bmatrix} \Phi_{\eta}(0_{0\sim\mu_{\eta}-1}) & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} B_{\eta c} \\ 0 \end{bmatrix}\begin{bmatrix} K_{\eta PD} & K_{\eta I} \end{bmatrix}$$

Hurwitz. As a result, the asymptotical tracking condition (26) is met.

Regarding the design of the other $v_k, k = 1, 2, ..., \eta - 1$, we can again adopt either the decoupled design or the coupled one. When the coupled design is adopted, we can design a control vector $v_{1\sim\eta-1}$ by controlling the following system

$$\dot{x}_{k}^{(0\sim\mu_{k}-1)}|_{k=1\sim\eta-1} = A'_{E}x_{k}^{(0\sim\mu_{k}-1)}|_{k=1\sim\eta-1} + B'_{E}v_{1\sim\eta-1}, \tag{34}$$

where

$$A_E' = \mathrm{blockdiag}\left(\Phi_k\left(0_{0\sim\mu_k-1}\right), k=1,2,...,\eta-1\right),$$

$$B_E' = \mathrm{blockdiag}\left(B_{kc}, k=1,2,...,\eta-1\right).$$

4. Model reference tracking

In this section, let us consider the problem of MRT.

4.1. The error system

Let us introduce the following assumption.

Assumption A2 There exist matrices $G \in \mathbb{R}^{\varkappa\times p}$ and $H \in \mathbb{R}^{r\times p}$ satisfying the following two equations

$$\left\{\begin{array}{l} A_E G + B_E H = G A_\mathrm{m} \\ CG = C_\mathrm{m}. \end{array}\right. \tag{35}$$

Define the generalized error variables

$$\left\{\begin{array}{l} \delta x = x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} - G x_\mathrm{m} \\ \delta v = v - H x_\mathrm{m} \\ \delta y = y - y_\mathrm{m}, \end{array}\right. \tag{36}$$

then, by taking derivatives of the above variables, and using (5) and (16), we can easily obtain the following lemma.

Lemma 4.1. *Under Assumption A2, the system (16) with the output equation (5) is equivalent to the following error system*

$$\left\{\begin{array}{l} \delta\dot{x} = A_E\delta x + B_E\delta v + B_E\Gamma d \\ \delta y = C\delta x, \end{array}\right. \tag{37}$$

where $\delta x, \delta v$ and δy are defined as in (36).

4.2. Solutions

Firstly, let us investigate the case of $d = 0$.

4.2.1. Solution for the case of $d = 0$

In this case the error system (37) becomes

$$\left\{\begin{array}{l} \delta\dot{x} = A_E\delta x + B_E\delta v \\ \delta y = C\delta x. \end{array}\right. \tag{38}$$

Choosing the following state feedback controller for this system:

$$\delta v = K_{FB}\delta x, \tag{39}$$

results in the following closed-loop system

$$\begin{cases} \delta \dot{x} = A_E^c \delta x \\ \delta y = C\delta x, \end{cases} \tag{40}$$

where

$$A_E^c = A_E + B_E K_{FB}. \tag{41}$$

If K_{FB} makes the above matrix A_E^c Hurwitz, then

$$\lim_{t\to\infty} \delta x(t) = 0,$$

which implies

$$\lim_{t\to\infty} \delta y(t) = \lim_{t\to\infty} C\delta x(t) = 0.$$

Finally, note that combination of (39) and (36) yields

$$\begin{aligned} v &= \delta v + Hx_{\mathrm{m}} \\ &= K_{FB}\delta x + Hx_{\mathrm{m}} \\ &= K_{FB} x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + (H - K_{FB}G)\, x_{\mathrm{m}}, \end{aligned}$$

we now obtain the following result about the solution to Problem 2.2.

Theorem 4.2. *Consider the system (2) with the output equation (5). Let*

(1) Assumptions A1 and A2 be met, and $d = 0$;
(2) K_{FB} be a matrix making A_E^c in (41) Hurwitz, and K_{FF} be given by

$$K_{FF} = H - K_{FB}G. \tag{42}$$

Then the controller

$$\begin{cases} u = -B^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \zeta, t\right)\left[f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \zeta, t\right) - v\right] \\ v = K_{FB} x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + K_{FF} x_{\mathrm{m}}, \end{cases} \tag{43}$$

guarantees

$$\lim_{t\to\infty} \left[y(t) - y_{\mathrm{m}}(t)\right] = 0, \tag{44}$$

and

$$\lim_{t\to\infty} \left[x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} - Gx_{\mathrm{m}}\right] = 0. \tag{45}$$

In the above controller (43), clearly, $K_{FB}x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}$ is a state-plus-derivative feedback term, while $K_{FF}x_{\mathrm{m}}$ is a feed-forward term.

4.2.2. Solution for the case of d being a constant vector

In this case, let us apply Lemma 3.2 to the error system (37), and a controller is obtained as

$$\delta v = K_{PD}\delta x + K_I \int_0^t \left[\delta y\left(\sigma\right) - y_{\mathrm{c}}\right] \mathrm{d}\sigma, \tag{46}$$

where the feedback gains K_{PD} and K_I make the matrix $\tilde{A}_c$ defined in (24) Hurwitz. In this case, we have

$$\lim_{t\to\infty} \delta y\left(t\right) = y_{\mathrm{c}}, \tag{47}$$

$$\lim_{t\to\infty} \delta \dot{x}\left(t\right) = 0, \tag{48}$$

and

$$\lim_{t\to\infty} \mathring{C}\delta x\left(t\right) = -\mathring{C}\tilde{A}_c^{-1}\left[\begin{array}{c} \Gamma d \\ -y_{\mathrm{c}} \end{array}\right]. \tag{49}$$

Substituting the expressions in (36) into (46), gives

$$\begin{aligned} v &= Hx_{\mathrm{m}} + K_{PD}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} - Gx_{\mathrm{m}}\right) \\ &\quad + K_I \int_0^t \left[y\left(\sigma\right) - y_{\mathrm{m}}\left(\sigma\right) - y_{\mathrm{c}}\right] \mathrm{d}\sigma \\ &= K_{PD}x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + \left(H - K_{PD}G\right)x_{\mathrm{m}} \\ &\quad + K_I \int_0^t \left[y\left(\sigma\right) - y_{\mathrm{m}}\left(\sigma\right) - y_{\mathrm{c}}\right] \mathrm{d}\sigma. \end{aligned} \tag{50}$$

The above process proves the following solution to Problem 2.2.

Theorem 4.3. *Consider the system (2) with the output equation (5). Let*

(1) Assumptions A1 and A2 be met, and d and y_{c} be constant vectors;
(2) K_{PD} and K_I be matrices making $\tilde{A}_c$ in (24) Hurwitz, and K_{FF} be given by

$$K_{FF} = H - K_{PD}G. \tag{51}$$

Then the controller

$$\left\{\begin{array}{l} u = -B^{-1}\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \zeta, t\right)\left[f\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}, \zeta, t\right) - v\right] \\ v = K_{PD}x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} + K_{FF}x_{\mathrm{m}} \\ \qquad\qquad + K_I \int_0^t \left[y\left(\sigma\right) - y_{\mathrm{m}}\left(\sigma\right) - y_{\mathrm{c}}\right] \mathrm{d}\sigma, \end{array}\right. \tag{52}$$

guarantees

$$\lim_{t\to\infty}\left[y\left(t\right)-y_{\mathrm{m}}\left(t\right)\right]=y_{\mathrm{c}}, \tag{53}$$

$$\lim_{t\to\infty}\left(x_{k}^{(1\sim\mu_{k})}|_{k=1\sim\eta}-G\dot{x}_{m}\right)=0, \tag{54}$$

and

$$\lim_{t\to\infty}\left(x_{1\sim\eta}\left(t\right)-\mathring{C}Gx_{\mathrm{m}}\right)=-\mathring{C}\tilde{A}_{c}^{-1}\begin{bmatrix}\Gamma d\\ -y_{\mathrm{c}}\end{bmatrix}. \tag{55}$$

Obviously, the three relations (53)-(55) are respectively the interpretations of the relations (47)-(49) in terms of the original system variables.

It can be easily recognized that the above MRT result is a generalization of the generalized PID control result given in Theorem 3.3. The above result clearly reduces to that in Theorem 3.3 when x_{m} does not exist, or, equivalently, $p=0$.

5. Parameterization

In this section we intend to give parametric solutions to the problems of generalized PID control and MRT. Before presenting the parametric expressions of the corresponding controllers, we first give some preliminaries.

5.1. Preliminary lemma

Consider the linear system

$$\dot{x}=Ax+Bu+\Gamma u_{r}, \tag{56}$$

where $x\in\mathbb{R}^{n}$ and $u\in\mathbb{R}^{r}$ are the state vector and the control input vector, respectively, $u_{r}\in\mathbb{R}^{p}$ is an external vector; A, B and Γ are real matrices of appropriate dimensions. When a controller for the system is chosen as

$$u=Kx, \tag{57}$$

the closed-loop system is obtained as

$$\dot{x}=A_{\mathrm{c}}x+\Gamma u_{r}, \tag{58}$$

where

$$A_{\mathrm{c}}=A+BK. \tag{59}$$

If the matrix pair $[A, B]$ is controllable, according to Duan (2015) (Chapter 3), there exist a pair of right coprime polynomial matrices $N\left(s\right)\in\mathbb{R}^{n\times r}[s]$ and $D\left(s\right)\in\mathbb{R}^{r\times r}[s]$

satisfying the following right coprime factorization (RCF):

$$(sI - A)^{-1} B = N(s) D^{-1}(s). \tag{60}$$

If we denote $D(s) = [d_{ij}(s)]$ and

$$\omega = \max\{\deg(d_{ij}(s)),\ i, j = 1, 2, ..., r\},$$

then $N(s)$ and $D(s)$ can be written in the form of

$$\begin{cases} N(s) = \sum\limits_{i=0}^{\omega} N_i s^i,\ N_i \in \mathbb{R}^{n\times r}, \\ D(s) = \sum\limits_{i=0}^{\omega} D_i s^i,\ D_i \in \mathbb{R}^{r\times r}. \end{cases} \tag{61}$$

The following lemma performs an important role in this section (see, Duan (2015)).

Lemma 5.1. *Let $[A, B]$ be controllable, and $N(s)$ and $D(s)$ be a pair of right coprime polynomial matrices given by (61) and satisfy the generalized RCF (60). Then,*

(1) for an arbitrarily chosen $F \in \mathbb{R}^{p\times p}$, all the matrices $W \in \mathbb{R}^{r\times p}$ and $V \in \mathbb{R}^{n\times p}$ satisfying the following Sylvester matrix equation

$$AV + BW = VF,$$

are given by

$$\begin{cases} V = N_0 Z + N_1 ZF + ... + N_\omega ZF^\omega \\ W = D_0 Z + D_1 ZF + ... + D_\omega ZF^\omega, \end{cases} \tag{62}$$

where $Z \in \mathbb{R}^{r\times p}$ is an arbitrary parameter matrix; and

(2) for an arbitrarily chosen $F \in \mathbb{R}^{n\times n}$, all the matrices $V \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{r\times n}$ satisfying $\det V \neq 0$ *and*

$$A_c = A + BK = VFV^{-1}, \tag{63}$$

are given by (62), with $p = n$, and

$$K = WV^{-1} \tag{64}$$

with $Z \in \mathbb{R}^{r\times n}$ being an arbitrary parameter matrix satisfying the following constraint:

$$\det(N_0 Z + N_1 ZF + ... + N_\omega ZF^\omega) \neq 0. \tag{65}$$

The meaning of equation (63) is very clear. If F is chosen to be a stable matrix, then K turns to be a stabilizing gain matrix for the state feedback controller (57). Therefore, the second conclusion in the above Lemma 5.1 has given all the stabilizing state feedback controllers for the system (56). The parameter matrix Z in the above lemma represents the design degrees of freedom in the design, and can be properly utilized to achieve additional system performance.

5.2. Right coprime factorizations

It is seen from the above lemma that the pair of right coprime polynomial matrices $N(s)$ and $D(s)$ satisfying the generalized RCF (60) is essential. The following two lemmas summarize the several pairs of such polynomial matrices associated with the solution to the Sylvester matrix equation in (35) and the feedback gains in the controllers (6), (43) and (52).

The conclusions of the following lemma can be easily verified (see, also Duan (2021c)).

Lemma 5.2. *Consider the matrix pairs* $[\Phi_k(0_{0\sim\mu_k-1}), B_{kc}]$, $k = 1,2,...,\eta$, *and* $[A_E, B_E]$.

(1) For $k = 1,2,...,\eta$, the pair of right coprime polynomial matrices $N_k(s)$ and $D_k(s)$ satisfying the RCF

$$[sI_{\mu_k r_k} - \Phi_k(0_{0\sim\mu_k-1})]^{-1} B_{kc} = N_k(s) D_k^{-1}(s), \tag{66}$$

are given by

$$N_k(s) = \begin{bmatrix} I_{r_k} \\ I_{r_k} s \\ \vdots \\ I_{r_k} s^{\mu_k-1} \end{bmatrix}, \quad D_k(s) = I_{r_k} s^{\mu_k}. \tag{67}$$

(2) The pair of right coprime polynomial matrices $N(s)$ and $D(s)$ satisfying the RCF

$$(sI - A_E)^{-1} B_E = N(s) D^{-1}(s) \tag{68}$$

are given by

$$\begin{cases} N(s) = \operatorname{blockdiag}(N_k(s), \quad k = 1,2,...,\eta) \\ D(s) = \operatorname{blockdiag}(I_{r_k} s^{\mu_k}, \quad k = 1,2,...,\eta). \end{cases} \tag{69}$$

The following lemma further gives the RCF associated with the matrix pair $\left[\tilde{A}, \tilde{B}\right]$.

Lemma 5.3. *Let $N(s)$ and $D(s)$ be a pair of right coprime polynomial matrices given by (61) and satisfy the generalized RCF (68). Then the matrix pair $\left[\tilde{A}, \tilde{B}\right]$ is controllable if and only if*

$$\operatorname{rank}(CN_0) = m. \tag{70}$$

In this case, the pair of right coprime polynomial matrices $\tilde{N}(s)$ and $\tilde{D}(s)$ satisfying the RCF

$$\left(sI - \tilde{A}\right)^{-1} \tilde{B} = \tilde{N}(s) \tilde{D}^{-1}(s) \tag{71}$$

are given by

$$\begin{cases} \tilde{N}(s) = \left[\begin{bmatrix} N(s) \\ CN^{\#}(s) \end{bmatrix} T_2 \quad \begin{bmatrix} sN(s) \\ CN(s) \end{bmatrix} T_1 P \right] \\ \tilde{D}(s) = \left[D(s) T_2 \quad sD(s) T_1 P \right], \end{cases} \tag{72}$$

where

$$N^{\#}(s) = \frac{1}{s}\left(N(s) - N_0\right),$$

and $P \in \mathbb{R}^{m \times m}$ *and* $[T_1 \quad T_2]$, *with* $T_1 \in \mathbb{R}^{r \times m}$ *and* $T_2 \in \mathbb{R}^{r \times (r-m)}$, *are nonsingular matrices satisfying*

$$PCN_0 \left[T_1 \quad T_2\right] = \left[I_m \quad 0\right]. \tag{73}$$

For a proof of the result, refer to the appendix.

It is obvious that, in the case of $m = r$, (72) becomes

$$\begin{cases} \tilde{N}(s) = \begin{bmatrix} sN(s) \\ CN(s) \end{bmatrix} T_1 P \\ \tilde{D}(s) = sD(s) T_1 P. \end{cases} \tag{74}$$

5.3. Parametric solutions

Let us firstly consider the Sylvester matrix equation in (35).

5.3.1. Solution of G and H

Let $N(s)$ and $D(s)$ given by (69) obey the expressions in (61), then, according to the first conclusion in Lemma 5.1, the general expression of the matrices G and H satisfying the Sylvester matrix equation in (35) is given by

$$\begin{cases} G = N_0 Z + N_1 Z A_{\mathrm{m}} + ... + N_\omega Z A_{\mathrm{m}}^{\omega} \\ H = D_0 Z + D_1 Z A_{\mathrm{m}} + ... + D_\omega Z A_{\mathrm{m}}^{\omega}, \end{cases} \tag{75}$$

where $Z \in \mathbb{R}^{r \times p}$ is an arbitrary parameter matrix.

With the matrix G given by (75), the second equation in (35) becomes

$$CN_0 Z + CN_1 Z A_{\mathrm{m}} + ... + CN_\omega Z A_{\mathrm{m}}^{\omega} = C_{\mathrm{m}}, \tag{76}$$

which is a linear equation with respect to $Z \in \mathbb{R}^{r \times p}$.

With the above understanding, Assumption A2 is clearly equivalent to

Assumption A2' There exists a parameter matrix $Z \in \mathbb{R}^{r \times p}$ satisfying constraint (76).

5.3.2. Solution of feedback gains

Firstly, it can be easily observed that the decoupled designs proposed in Section 3.3 is basically a problem as follows.

Problem 5.1. *For some $1 \leq i \leq \eta$, find a matrix K_i such that the following matrix*

$$A_{ic} = \Phi_i \left(0_{0\sim\mu_i-1}\right) + B_{ic}K_i$$

is Hurwitz.

Secondly, according to Theorem 4.2, finding the feedback gain K_{FB} in the generalized PID controller (43) is mainly to solve the following problem:

Problem 5.2. *Find a matrix K_{FB} such that the following matrix*

$$A_E^c = A_E + B_E K_{FB} \tag{77}$$

is Hurwitz.

Thirdly, according to Theorems 3.3 and 4.3, finding the gain matrices K_{PD} and K_I in the generalized PID controllers (6) and (52) is to solve the following problem.

Problem 5.3. *Find a matrix $K = \begin{bmatrix} K_{PD} & K_I \end{bmatrix}$ such that the following matrix*

$$\tilde{A}_c = \tilde{A} + \tilde{B}K$$

is Hurwitz.

All the above three problems can be solved in a parametric manner by using Lemmas 5.1 and 5.2. In the following, let us briefly illustrate the basic ideas in solving Problems 5.2 and 5.3.

To solve Problem 5.2, let $N(s)$ and $D(s)$ given by (69) obey the expressions in (61), then according to the second conclusion in Lemma 5.1, the general expression of the gain matrix K_{FB} is given by

$$\begin{cases} K_{FB} = WV^{-1} \\ V = N_0 Z + N_1 ZF + ... + N_\omega ZF^\omega \\ W = D_0 Z + D_1 ZF + ... + D_\omega ZF^\omega, \end{cases} \tag{78}$$

where $F \in \mathbb{R}^{\varkappa\times\varkappa}$ is a stable matrix, and $Z \in \mathbb{R}^{r\times\varkappa}$ is an arbitrary parameter matrix satisfying the constraint (65). As a consequence, we have

$$A_E^c = A_E + B_E K_{FB} = VFV^{-1}.$$

To solve Problem 5.3, let $\tilde{N}(s)$ and $\tilde{D}(s)$ be a pair of right coprime polynomial matrices satisfying the generalized RCF (71) and possess the following expressions:

$$\begin{cases} \tilde{N}(s) = \sum\limits_{i=0}^{\tilde{\omega}} \tilde{N}_i s^i, \ \tilde{N}_i \in \mathbb{R}^{(\varkappa+m)\times r} \\ \tilde{D}(s) = \sum\limits_{i=0}^{\tilde{\omega}} \tilde{D}_i s^i, \ \tilde{D}_i \in \mathbb{R}^{r\times r}. \end{cases} \tag{79}$$

Then according to the second conclusion in Lemma 5.1, the general expression of the gain matrix $K = \left[\begin{array}{cc} K_{PD} & K_I \end{array} \right]$ is given by

$$\begin{cases} \left[\begin{array}{cc} K_{PD} & K_I \end{array} \right] = \tilde{W}\tilde{V}^{-1} \\ \tilde{V} = \tilde{N}_0\tilde{Z} + \tilde{N}_1\tilde{Z}\tilde{F} + ... + \tilde{N}_\omega\tilde{Z}\tilde{F}^{\tilde{\omega}} \\ \tilde{W} = \tilde{D}_0\tilde{Z} + \tilde{D}_1\tilde{Z}\tilde{F} + ... + \tilde{D}_\omega\tilde{Z}\tilde{F}^{\tilde{\omega}}, \end{cases} \tag{80}$$

where $\tilde{F} \in \mathbb{R}^{(\varkappa+m)\times(\varkappa+m)}$ is a stable matrix, and $\tilde{Z} \in \mathbb{R}^{r\times(\varkappa+m)}$ is an arbitrary parameter matrix satisfying the constraint:

$$\det\left(\tilde{N}_0\tilde{Z} + \tilde{N}_1\tilde{Z}\tilde{F} + ... + \tilde{N}_\omega\tilde{Z}\tilde{F}^{\tilde{\omega}}\right) \neq 0. \tag{81}$$

As a consequence, we have

$$\tilde{A}_c = \tilde{A} + \tilde{B}K = \tilde{V}\tilde{F}\tilde{V}^{-1}. \tag{82}$$

It is important to emphasize again that the design degrees of freedom represented by (F, Z) and $\left(\tilde{F}, \tilde{Z}\right)$ can be properly utilized to achieve additional closed-loop system performance (see, e.g., Duan (1992, 1993); Duan, Irwin, & Liu (2002); Duan, Liu, & Thompson (2000); Duan & Zhao (2020)).

6. The sub-fully actuated case

Firstly, let us recall the concept of singular points of sub-fully actuated systems, introduced in Duan (2021c). For simplicity, in this paper let us consider a simpler case and impose the following assumption.

Assumption A3 The matrix $B(\cdot)$ in system (1) depends on only the state, that is,

$$B(\cdot) = B\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}\right).$$

For the general case, please refer to Duan (2021c) and Duan (2021d).

6.1. Singularity and feasibility

For a sub-fully actuated system in the form of (1) satisfying the above Assumption A3, the following concept is essential.

Definition 6.1. If $x_k^{(0\sim\mu_k-1)}(t)|_{k=1\sim\eta} \in \mathbb{R}^\varkappa$ satisfies

$$\det B\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}\right) = 0 \text{ or } \infty, \tag{83}$$

then it is called a singular point of system (1).

Let $\mathbb{S}$ be the set of all singular points of system (1), that is,

$$\mathbb{S} = \left\{ x_k^{(0\sim\mu_k-1)}(t)|_{k=1\sim\eta} \middle| \text{equation (83) holds} \right\}.$$

Then the following set

$$\mathbb{F} = \mathbb{R}^{\varkappa} \backslash \mathbb{S},$$

is called the set of feasible points of system (1). More strictly, the system (1) is called a sub-fully actuated system if $\mathbb{F}$ is a set with dimension not less than 1 (Duan, 2021c).

To implement the controllers (6), (43) and (52), the following feasibility condition

$$x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \notin \mathbb{S}, \tag{84}$$

or, equivalently,

$$x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta} \in \mathbb{F}, \tag{85}$$

needs to be met. Such a problem can be generally turned into a state constrained control problem and solved via an eigenstructure-based approach.

6.2. State constrained control

It is easily observed that, with our designed PID and MRT controllers, the closed-loop systems are linear constant ones, and generally obey the form of (58).

The state constrained control problem can be simply stated as follows:

Problem 6.1. *Let $[A, B]$ be stabilizable, and $\mathbb{F} \subset \mathbb{R}^n$ be a manifold. For the linear system (56) with u_r being a constant external input, find a state feedback controller (57) such that the following requirements are met:*

(1) the matrix $A_\mathrm{c} = A + BK$ is Hurwitz; and
(2) the state constraint $x \in \mathbb{F}$ is met.

In Duan (2021d), a solution to such a problem based on Lyapunov matrix equations is presented. In this section let us present an eigenstructure-based solution to the problem.

Clearly, the equilibrium point of the system (58) is

$$x_e = -A_\mathrm{c}^{-1}\Gamma u_r. \tag{86}$$

Let

$$z = x - x_e,$$

then the system (58) is transformed into

$$\dot{z} = A_\mathrm{c} z. \tag{87}$$

Applying Lemma 5.1 to system (56), we can choose the feedback gain

$$\begin{cases} K = WV^{-1} \\ V = N_0 Z + N_1 ZF + \ldots + N_\omega ZF^\omega \\ W = D_0 Z + D_1 ZF + \ldots + D_\omega ZF^\omega, \end{cases} \tag{88}$$

and eventually we have

$$A_{\mathrm{c}} = VFV^{-1}. \tag{89}$$

For only demonstration, let us consider the case of

$$F = \mathrm{diag}\left(s_i, i = 1, 2, ..., n\right), \tag{90}$$

where $s_i, i = 1, 2, ..., n$ are a series of negative real scalars. While the result can be easily generalized into the case that F contains a diagonal real block of the following form:

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

When (90) holds, the solution to the closed-loop system (87) is given by

$$\begin{aligned} z(t) &= e^{A_c t} z(0) \\ &= V e^{Ft} V^{-1} z(0) \\ &= V \mathrm{diag}\left(e^{s_i t}, i = 1, 2, ..., n\right) V^{-1} z(0), \end{aligned}$$

which gives

$$\|z(t)\| \leq \|V\| \left\|V^{-1}\right\| \|z(0)\|. \tag{91}$$

Therefore, for some $d_0 > 0$,

$$\|V\| \left\|V^{-1}\right\| \|z(0)\| < d_0,$$

implies

$$\|z(0)\| \leq \|V\| \left\|V^{-1}\right\| \|z(0)\| < d_0,$$

and

$$\|z(t)\| < d_0, \quad t > 0.$$

The above process clearly proves the following important lemma about state constrained control.

Lemma 6.2. *Let* $[A, B]$ *be stabilizable,* u_r *be constant,* d_0 *be the maximum number such that*

$$\Omega = \{x | \|x - x_e\| < d_0\} \subset \mathbb{F}, \tag{92}$$

and K *be the state feedback gain, of the state feedback controller (57), given by (88), with* F *being a stable diagonal matrix given by (90). Further, if the initial values are chosen to satisfy*

$$\nu(V) \|x(0) - x_e\| < d_0, \tag{93}$$

then

$$x(t) \in \Omega \subset \mathbb{F}, \quad \forall t \geq 0. \tag{94}$$

Recalling (86) and (89), we have

$$x_e = -VF^{-1}V^{-1}\Gamma u_r, \tag{95}$$

which is dependent on V. Furthermore, the condition (93) is also dependent on $\nu(V)$. In applications, it is desirable to select the free parameter matrix Z to let d_0 be as big as possible, and $\nu(V)$ be as small as possible.

6.3. Feasibility conditions

In this subsection, let us apply the above Lemma 6.2 to our generalized PID control and MRT.

6.3.1. Case of PID control

The purpose here is to provide a solution to Problem 2.1 with the feasibility requirement (85) satisfied. The general idea is as follows.

Applying Lemma 5.3, we can obtain a pair of polynomial matrices $\tilde{N}(s)$ and $\tilde{D}(s)$ in the form of (79) satisfying the generalized RCF (71). Then according to the second conclusion in Lemma 5.1, the general expression of the gain matrix $K = [K_{PD} \quad K_I]$ can be obtained as in (80). Finally, with the help of Lemma 6.2, the result about generalized PID control of sub-fully actuated systems can be obtained.

The equilibrium point of the closed-loop system is easily obtained as

$$\begin{bmatrix} X_{\mathrm{e}} \\ q_{\mathrm{e}} \end{bmatrix} = -\tilde{A}_{\mathrm{c}}^{-1} \begin{bmatrix} \Gamma d \\ -y_{\mathrm{c}} \end{bmatrix} = -\tilde{V}\tilde{F}^{-1}\tilde{V}^{-1} \begin{bmatrix} \Gamma d \\ -y_{\mathrm{c}} \end{bmatrix}, \tag{96}$$

where the disturbance d may be substituted by an estimate.

Theorem 6.3. *Consider the system (2) with the output equation (5). Let Assumptions A2 and A3 and condition (21) be met, and d be absent, $K = [K_{PD} \quad K_I]$ be given by (80), with $\tilde{F} \in \mathbb{R}^{(\varkappa+m)\times(\varkappa+m)}$ being chosen to be a stable diagonal matrix, and $\tilde{Z} \in \mathbb{R}^{r\times(\varkappa+m)}$ be an arbitrary parameter matrix satisfying the constraint (81). Further, if the initial values are chosen to satisfy*

$$\nu\left(\tilde{V}\right) \left\| \begin{bmatrix} x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(0) \\ q(0) \end{bmatrix} - \begin{bmatrix} X_{\mathrm{e}} \\ q_{\mathrm{e}} \end{bmatrix} \right\| \leq d_0, \tag{97}$$

where d_0 is the maximum number such that

$$\Omega = \{z | \ z \in \mathbb{R}^{\varkappa}, \|z - X_{\mathrm{e}}\| \leq d_0\} \subset \mathbb{F}, \tag{98}$$

then the controller (6) for the system (2) guarantees (26)-(28) and the feasibility requirement (85).

Proof. It follows from Lemma 6.2 that, under condition (97), there also holds

$$\left\| \begin{bmatrix} x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(t) \\ q(t) \end{bmatrix} - \begin{bmatrix} X_{\mathrm{e}} \\ q_{\mathrm{e}} \end{bmatrix} \right\| \leq d_0, \quad \forall t \geq 0. \tag{99}$$

Thus we have

$$\begin{aligned} & \left\| x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(t) - X_{\mathrm{e}} \right\| \\ \leq & \left\| \begin{bmatrix} x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(t) \\ q(t) \end{bmatrix} - \begin{bmatrix} X_{\mathrm{e}} \\ q_{\mathrm{e}} \end{bmatrix} \right\| \\ \leq & \ d_0, \quad \forall t \geq 0. \end{aligned} \tag{100}$$

This shows

$$x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(t) \in \Omega \subset \mathbb{F}.$$

The proof is done. $\square$

6.3.2. Case of MRT

For simplicity, let us only consider the MRT control without an integration, that is, the case of Problem 5.2, but with Assumption A1 removed, and the additional feasibility requirement (85) added. Again, the idea of solving this problem is to adopt the following outline:

Applying Lemma 5.2, we can obtain a pair of polynomial matrices $N(s)$ and $D(s)$ in the form of (61) satisfying the generalized RCF (69). Then according to the second conclusion in Lemma 5.1, the general expression of the gain matrix K_{FB} can be obtained. Finally, with the help of Lemma 6.2, the feasibility condition for MRT in sub-fully actuated systems can be obtained.

Before implementing this idea, let us first impose some assumption on the reference model (8). Let $\Omega_0 \subset \mathbb{R}^p$ be a set of admissible initial values of system (8), then we can define the following objective set:

$$\Omega = \{ G x_{\mathrm{m}}(t) \,|\, \dot{x}_{\mathrm{m}} = A_{\mathrm{m}} x_{\mathrm{m}}, t \geq 0, \quad x_{\mathrm{m}}(0) \in \Omega_0 \}. \tag{101}$$

Theoretically, the reference model (8) is allowed to be unstable. However, such a case is very rare. Thus in this subsection we restrict the reference model (8) to be stable or critically stable. As a consequence, the above set Ω defined in (101) is bounded. Furthermore, it is reasonable to require that Ω is far away enough from $\mathbb{S}$. This motivates us to propose the following assumption.

Assumption A4 There exists a $x_c \in \bar{\Omega}$ and a constant $d_c > 0$, such that

$$\bar{\Omega} \subset \mathbb{B} = \{ x \,|\, \|x - x_c\| \leq d_c \},$$

and

$$d_0 = d(\mathbb{B}, \mathbb{S}) > 0. \tag{102}$$

The notation $\bar{\Omega}$ represents the closed-cover of Ω, and $d(\mathbb{B}, \mathbb{S})$ represents the distance between the two sets $\mathbb{B}$ and $\mathbb{S}$, which is defined by

$$d(\mathbb{B}, \mathbb{S}) = \inf \{\|y - z\| \mid y \in \mathbb{B},\ z \in \mathbb{S}\},$$

With the above preparation, we can now prove the following result about the MRT control of the sub-fully actuated system (2).

Theorem 6.4. *Consider the system (2) with the output equation (5). Let Assumptions A2', A3 and A4 be met, and* $d = 0$*;* K_{FB} *be given by (78), with* $F \in \mathbb{R}^{\varkappa \times \varkappa}$ *being chosen to be a stable diagonal matrix, and* $Z \in \mathbb{R}^{r \times \varkappa}$ *be an arbitrary parameter matrix satisfying the constraint (65). Further, if the initial values are chosen to satisfy* $x_{\mathrm{m}}(0) \in \Omega_0$ *and*

$$\nu(V) \left\| x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(0) - Gx_{\mathrm{m}}(0) \right\| \leq d_0, \tag{103}$$

where V *is given by (78), then the controller (43) for the system guarantees (44)-(45) and the feasibility requirement (85).*

Proof. Recall that, under given conditions, the controller (43) results in the closed-loop system given by (40), with

$$A_E^c = VFV^{-1}.$$

Applying Lemma 6.2, we have, due to (103), the following relation:

$$\left\| x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(t) - Gx_{\mathrm{m}}(t) \right\| \leq d_0, \quad \forall t \geq 0. \tag{104}$$

On the other hand, since $x_{\mathrm{m}}(0) \in \Omega_0$, we have $Gx_{\mathrm{m}} \in \Omega$. Further, recalling $\Omega \subset \mathbb{B}$, we obviously have

$$\|Gx_{\mathrm{m}} - x_c\| \leq d_c. \tag{105}$$

Therefore, using (104) and (105), gives

$$\begin{aligned} & \left\| x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(t) - x_c \right\| \\ \leq\ & \left\| x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(t) - Gx_{\mathrm{m}} \right\| + \|Gx_{\mathrm{m}} - x_c\| \\ \leq\ & d_0 + d_c, \quad \forall t \geq 0. \end{aligned}$$

This clearly implies (85). □

Remark 3. It is easily recognized the feasibility conditions given in Theorems 6.3 and 6.4 may be conservative. In certain cases, e.g., when $\det B\left(x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}\right)$ is dependent on only some, but not all, x_k's and their derivatives, less conservative initial value ranges can be provided by applying decoupled design approaches (see Section 3.3, and also Duan (2021d)). The next section gives a demonstration of this idea with an illustrative example.

Remark 4. The results presented in this subsection are conservative but are strictly correct. While in many cases, the set of singular points is only a low-dimensional hyperplane in the space $\mathbb{R}^{\varkappa}$, in this case, as long as the steady response of the closed-loop system does not intersect with $\mathbb{S}$, the probability that the system response in the transient process intersects with $\mathbb{S}$ is very small. This is why some practical applications on control of sub-fully actuated systems simply overlook such a feasibility analysis problem. However, for applications of high-grade security or high cost, a trial and test approach is certainly not acceptable.

7. Illustrative example

7.1. The system model

The Example 6.2 in Duan (2021c) treats a classical example system proposed in Brockett (1983). In the treatment of Duan (2021c), the problem is reduced to the control of the following HOFA system:

$$\begin{bmatrix} \ddot{z} \\ \dot{y} \end{bmatrix} = B(y, \dot{z})\, u, \tag{106}$$

where

$$B(y, \dot{z}) = \begin{bmatrix} y & \frac{\dot{z}}{y} \\ 0 & 1 \end{bmatrix},$$

and

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is the control vector. Clearly, the set of singular points of the system is

$$\mathbb{S} = \left\{ \begin{bmatrix} z \\ \dot{z} \\ y \end{bmatrix} \middle| \; y = 0 \right\}.$$

In the Example 6.2 in Duan (2021c), a decoupled design is proposed, which gives a non-conservative feasibility condition. However, with decoupled designs, the design degrees of freedom is dramatically reduced. Now in this section, we will present coupled designs using the PID and MRT control.

By the following control transformation,

$$u = B^{-1}(y, \dot{z})\, v, \tag{107}$$

where v is a two-dimensional intermediate control vector, the system is turned into

$$\begin{bmatrix} \ddot{z} \\ \dot{y} \end{bmatrix} = v. \tag{108}$$

which can be converted into the following state-space form:

$$\begin{bmatrix} \dot{z}^{(0\sim1)} \\ \dot{y} \end{bmatrix} = A_E \begin{bmatrix} z^{(0\sim1)} \\ y \end{bmatrix} + B_E v, \tag{109}$$

where

$$A_E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{110}$$

The output equation of the system is taken to be

$$y = C \begin{bmatrix} z^{(0\sim1)} \\ y \end{bmatrix}, \tag{111}$$

where

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

7.2. Generalized PID design

In this section, our aim of the design is to let the system output y regulate at a constant value y_c. As a result of Theorem 3.3, the controller finally designed for the system is

$$\begin{cases} u = B^{-1}(y, \dot{z})\, v \\ v = K_{PD} \begin{bmatrix} z^{(0\sim1)} \\ y \end{bmatrix} + K_I \int_0^t [y(\sigma) - y_c]\, \mathrm{d}\sigma. \end{cases} \tag{112}$$

It follows from Theorem 3.3 that

$$K = [K_{PD} \quad K_I]$$

is determined by

$$\tilde{A}_c = \tilde{A} + \tilde{B}K = \tilde{V}\tilde{F}\tilde{V}^{-1}, \tag{113}$$

where

$$\tilde{A} = \begin{bmatrix} A_E & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{114}$$

$$\tilde{B} = \begin{bmatrix} B_E \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{115}$$

and $\tilde{F}$ is a stable real matrix, and $\tilde{V}$ a nonsingular real matrix.

7.2.1. Solution of RCFs

It follows from Lemmas 5.2 and 5.3,

$$N_1(s) = \begin{bmatrix} 1 \\ s \end{bmatrix}, \quad D_1(s) = s^2. \tag{116}$$

$$N_2(s) = 1, \quad D_2(s) = s. \tag{117}$$

and

$$N(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}, \quad D(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}. \tag{118}$$

Thus

$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad N^{\#}(s) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{119}$$

Note that

$$CN_0 = [0 \quad 1],$$

we have

$$P = 1, \quad T_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{120}$$

Thus it follows from Lemma 5.3 that

$$\tilde{N}(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & s \\ 0 & 1 \end{bmatrix}, \quad \tilde{D}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix}, \tag{121}$$

which gives

$$\tilde{N}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{N}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{D}_0 = \tilde{D}_1 = 0, \quad \tilde{D}_2 = I_2.$$

7.2.2. Solution of K_{PD} and K_I

Choose

$$\tilde{F} = \mathrm{diag}\left(-1, -2, -3, -4\right),$$

we obtain

$$\left\{ \begin{array}{l} \tilde{V} = \tilde{N}_0 \tilde{Z} + \tilde{N}_1 \tilde{Z} \tilde{F} \\ \tilde{W} = \tilde{Z} \tilde{F}^2. \end{array} \right. \tag{122}$$

By minimizing the condition number

$$\nu\left(\tilde{V}\right) = \left\|\tilde{V}\right\| \left\|\tilde{V}^{-1}\right\|$$

we obtain

$$\tilde{Z} = \left[\begin{array}{cccc} -107.4746 & -42.8741 & -110.4591 & 170.2421 \\ -12.3083 & 18.5433 & -151.2066 & -57.0688 \end{array} \right], \tag{123}$$

which corresponds to $\nu\left(\tilde{V}\right) = 9.1778$. Hence

$$\begin{array}{rcl} K & = & \tilde{W}\tilde{V}^{-1} \\ & = & \left[\begin{array}{cccc} -7.9952 & -5.9984 & 0.0868 & 0.2947 \\ 0.1093 & 0.0478 & -4.0016 & -3.0058 \end{array} \right]. \end{array}$$

Thus

$$K_{PD} = \left[\begin{array}{ccc} -7.9952 & -5.9984 & 0.0868 \\ 0.1093 & 0.0478 & -4.0016 \end{array} \right], \tag{124}$$

$$K_I = \left[\begin{array}{c} 0.2947 \\ -3.0058 \end{array} \right]. \tag{125}$$

It can be obtained that

$$\tilde{A}_c = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ -7.9952 & -5.9984 & 0.0868 & 0.2947 \\ 0.1093 & 0.0478 & -4.0016 & -3.0058 \\ 0 & 0 & 1 & 0 \end{array} \right].$$

Further choose

$$y_{\mathrm{c}} = 16, \tag{126}$$

we have

$$\begin{bmatrix} X_{\mathrm{e}} \\ q_{\mathrm{e}} \end{bmatrix} = -\tilde{A}_c^{-1} \begin{bmatrix} 0 \\ -y_{\mathrm{c}} \end{bmatrix} = \begin{bmatrix} -0.6121 \\ 0 \\ 16.0000 \\ -21.3225 \end{bmatrix}. \tag{127}$$

Note that the distance of y_{c} from $\mathbb{S}$ is $d_0 = y_{\mathrm{c}} = 16$, by (97) the set of feasible initial values is obtained as

$$\begin{aligned} & \left\| \begin{bmatrix} x_k^{(0\sim\mu_k-1)}|_{k=1\sim\eta}(0) \\ q(0) \end{bmatrix} - \begin{bmatrix} X_{\mathrm{e}} \\ q_{\mathrm{e}} \end{bmatrix} \right\| \\ \leq \; & \frac{d_0}{\nu\left(\tilde{V}\right)} = 1.74334. \end{aligned} \tag{128}$$

7.3. Model reference tracking design

Let the coefficient matrices of the reference model (8) be taken as

$$A_{\mathrm{m}} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}, \quad C_{\mathrm{m}} = [1 \quad 0].$$

In view of (118), we have

$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{129}$$

$$D_0 = 0, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{130}$$

thus Assumption A2', that is,

$$CN_0Z + CN_1ZA_{\mathrm{m}} = C_{\mathrm{m}}, \tag{131}$$

gives

$$Z = \begin{bmatrix} \alpha & \beta \\ 1 & 0 \end{bmatrix}, \tag{132}$$

where α and β are two arbitrary real scalars. Therefore, it follows from (75) and (132) that

$$G = N_0Z + N_1ZA_{\mathrm{m}} = \begin{bmatrix} \alpha & \beta \\ -4\beta & \alpha \\ 1 & 0 \end{bmatrix}, \tag{133}$$

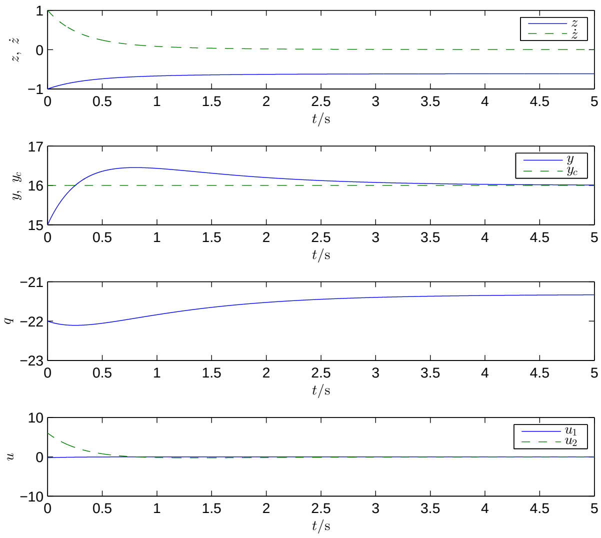

Figure 1. Generalized PID control, $d = 0.0$

$$H = D_1 Z A_{\mathrm{m}} + D_2 Z A_m^2 = \left[\begin{array}{cc} -4\alpha & -4\beta \\ 0 & 1 \end{array} \right]. \tag{134}$$

For simplicity, let us just take K_{PD} and K_I as in (124) and (125), respectively, then K_{FF} is given by (42). Through minimizing

$$J = \left\| H\left(\alpha, \beta\right) - K_{PD} G\left(\alpha, \beta\right) \right\|,$$

we obtain the optimal parameter as

$$\alpha = -0.7845, \ \ \beta = 0.0702.$$

Hence

$$K_{FF} = \left[\begin{array}{cc} -4.9050 & -4.4257 \\ 4.1008 & 1.0298 \end{array} \right].$$

7.4. Simulation results

7.4.1. PID control

When the initial values are chosen as

$$z\left(0\right) = -1, \ \ \dot{z}\left(0\right) = 1, \ \ y\left(0\right) = 15, \ \ q\left(0\right) = -22, \tag{135}$$

it can be easily verified that the condition (128) is met. Corresponding to the cases of $d = 0$ and $d = 5.0$, the corresponding simulation results are shown in Figures 1 and 2, respectively. The results clearly support the theoretical conclusions.

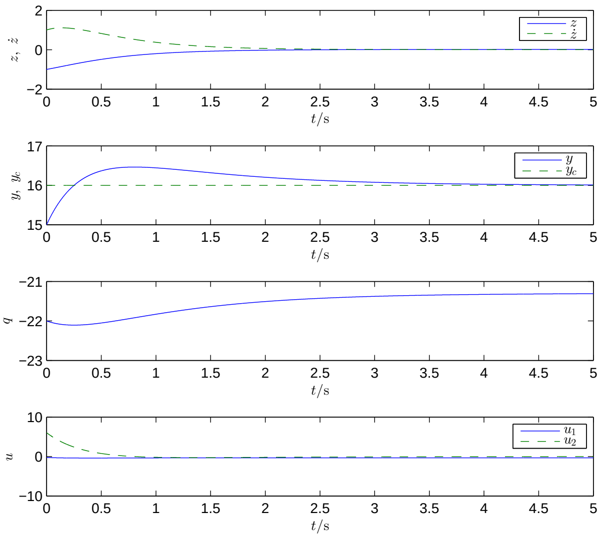

Figure 2. Generalized PID control, $d = 5.0$

7.4.2. Model reference tracking

Again, take $y_c = 16$, and the system initial values as in (135). Furthermore, take the initial value of the reference model as

$$x_m(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Then, for the case of $d = 0$ and $d = 5.0$, we also carry out the simulation and the results are shown in Figures 3 and 4, respectively. Again, the results coincide with the theories.

7.4.3. Case of slow time-varying d and y_c

In this subsection, let us further check the effect of the designed controllers when applied to systems with slow time-varying disturbance d and a slow time-varying commend signal y_c. For paper length limitation, here we only present the results of the generalized PID controller.

Case A: $d(t)$ is time-varying only

Consider the case where only the disturbance d is replaced with the following time-varying signal

$$d(t) = 5 + 2\cos(0.5t), \tag{136}$$

while keeping all the other parameters unchanged. The simulation results are shown in Figure 5. Comparison of Figure 5 with Figure 2 gives the following observation: the time-varying disturbance does affect the variables z and $\dot{z}$, but it almost has no affection on the steady response of the variable y, as desired.

Case B: Both d and y_c are time-varying

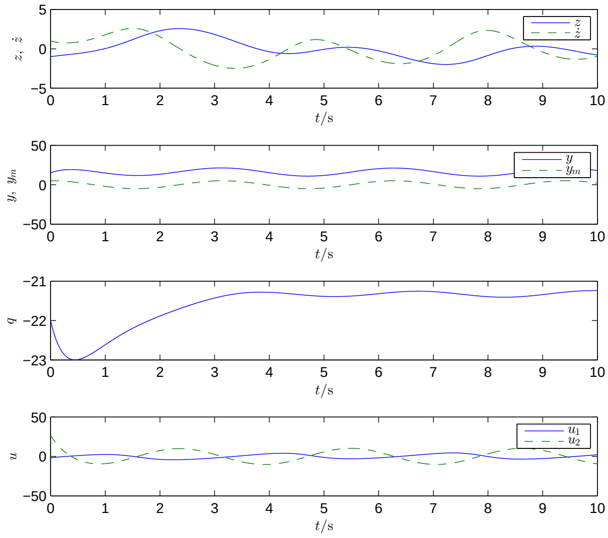

Figure 3. Model reference control, $d = 0.0$

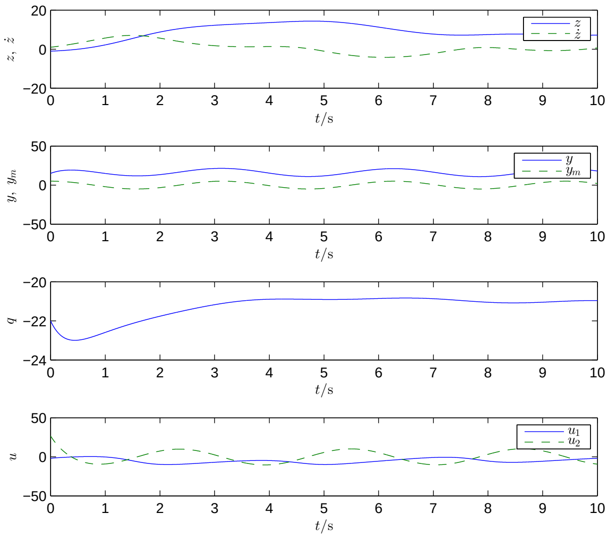

Figure 4. Model reference control, $d = 5.0$

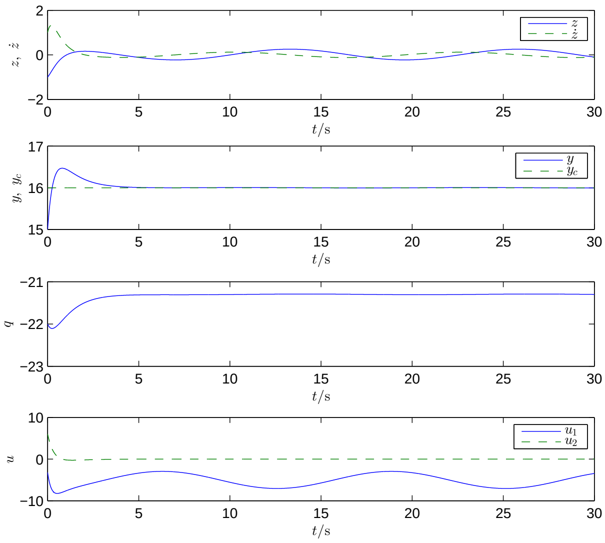

Figure 5. Generalized PID control, Case A

In the case that d is replaced with that in (136), and, simultaneously, y_{c} is replaced with

$$y_{\mathrm{c}}(t) = 16 + 0.1\sin(0.5t),$$

while keeping the initial values still the same as in (135), the simulation results are shown in Figure 6. It is observed that in this case the variable $y(t)$ still follows the signal $y_{\mathrm{c}}(t)$ closely, as desired.

8. Conclusions

Tracking and regulation is one of the most important design objectives in control systems design. For tracking and regulation in linear systems, the problem has been well solved. While for nonlinear systems, with the widely used state-space approaches the problem is only solved for some very special systems and the results are also often limited to certain local senses.

Parallel to the state-space approaches, HOFA approaches have been recently proposed and demonstrated to be much more effective in dealing with system control problems (Duan (2020a,b,c), and Duan (2020d,e,f,g, 2021a,b,c)). It is further shown in this paper that, once a general nonlinear dynamical system is represented by a HOFA model, the problem of tracking control can be solved perfectly in the sense that the closed-loop system becomes constant linear, besides realization of the required asymptotical tracking property. Concretely, the paper has shown the following:

(1) the well-known PID scheme can be easily generalized to suit a HOFA system subject to a constant or slow time-varying disturbance, and to achieve asymptotical tracking of a constant or slow time-varying signal;
(2) a MRT controller also exists for a HOFA system, which guarantees asymptotical

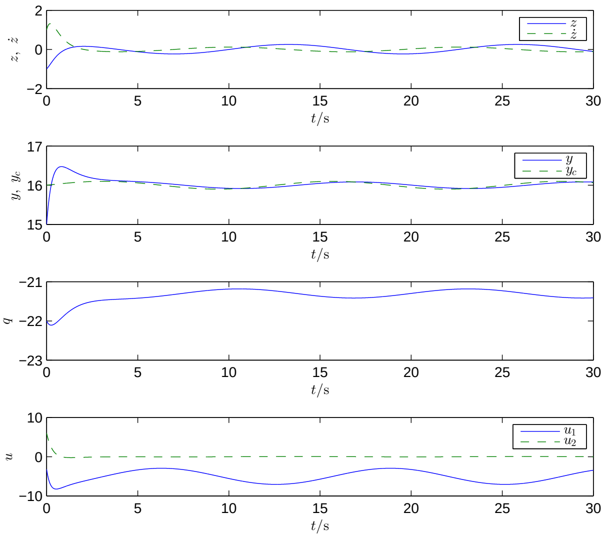

Figure 6. Generalized PID control, Case B

tracking of the signal generated by the reference model, and the tracking accuracy may not be affected when the system is subject to a constant disturbance;

(3) the closed-loop systems resulted in by both designs are constant linear, and the controllers as well as the closed-loop systems can be completely parameterized, with the degrees of freedom properly utilized to further improve the system performance; and

(4) both designs can be generalized to the case of sub-fully actuated systems, and effective feasibility conditions in terms of the system initial values are provided by using the parameterized closed-loop eigenstructure.

9. Appendix: Proof of Lemma 5.3

Since $[A_E, B_E]$ is controllable, there exists a unimodular matrix

$$Q(s) = \begin{bmatrix} * & N(s) \\ * & D(s) \end{bmatrix},$$

which satisfies

$$\begin{bmatrix} sI_{\varkappa} - A_E & B_E \end{bmatrix} Q(s) = \begin{bmatrix} I_{\varkappa} & 0 \end{bmatrix}.$$

Using the above relation, we have

$$
\begin{aligned}
& \operatorname{rank}\begin{bmatrix} sI_{\varkappa+m}-\tilde{A} & \tilde{B} \end{bmatrix} \\
= \ & \operatorname{rank}\begin{bmatrix} sI_{\varkappa}-A_E & 0 & B_E \\ -C & sI_m & 0 \end{bmatrix} \\
= \ & \operatorname{rank}\begin{bmatrix} sI_{\varkappa}-A_E & B_E & 0 \\ -C & 0 & sI_m \end{bmatrix} \\
= \ & \operatorname{rank}\left(\begin{bmatrix} sI_{\varkappa}-A_E & B_E & 0 \\ -C & 0 & sI_m \end{bmatrix}\begin{bmatrix} Q(s) & 0 \\ 0 & I_m \end{bmatrix}\right) \\
= \ & \operatorname{rank}\begin{bmatrix} I_{\varkappa} & 0 & 0 \\ * & -CN(s) & sI_m \end{bmatrix} \\
= \ & \operatorname{rank}\begin{bmatrix} I_{\varkappa} & 0 & 0 \\ 0 & -CN_0 & sI_m \end{bmatrix} \\
= \ & \varkappa + \operatorname{rank} CN_0, \quad \forall s \in \mathbb{C}.
\end{aligned}
$$

Therefore,

$$\operatorname{rank}\begin{bmatrix} sI_{\varkappa+m}-\tilde{A} & \tilde{B} \end{bmatrix} = \varkappa + m, \quad \forall s \in \mathbb{C},$$

if and only if

$$\operatorname{rank} CN_0 = m, \quad \forall s \in \mathbb{C}.$$

Thus the first conclusion of the lemma follows from the well-known PBH criterion.

From (73), we can obtain

$$PCN_0T_1 = I_m, \tag{137}$$

and

$$PCN_0T_2 = 0. \tag{138}$$

While the former is easily seen to be equivalent to

$$CN_0T_1P = I_m. \tag{139}$$

Using the above equations and the given conditions in the lemma, we have

$$\begin{aligned}
&\left(sI-\tilde{A}\right)\begin{bmatrix} N(s) \\ CN^{\#}(s) \end{bmatrix}T_2-\tilde{B}D(s)T_2 \\
=\ &\left(\begin{bmatrix} sI-A_E & 0 \\ -C & sI \end{bmatrix}\begin{bmatrix} N(s) \\ CN^{\#}(s) \end{bmatrix}-\begin{bmatrix} B_E \\ 0 \end{bmatrix}D(s)\right)T_2 \\
=\ &\begin{bmatrix} (sI-A_E)N(s)-B_ED(s) \\ -CN_0 \end{bmatrix}T_2 \\
=\ &\begin{bmatrix} 0 \\ -CN_0T_2 \end{bmatrix} \\
=\ &0,
\end{aligned}$$

and

$$\begin{aligned}
&\left(sI-\tilde{A}\right)\begin{bmatrix} sN(s) \\ CN(s) \end{bmatrix}T_1P-s\tilde{B}D(s)T_1P \\
=\ &\left(\begin{bmatrix} sI-A_E & 0 \\ -C & sI \end{bmatrix}\begin{bmatrix} sN(s) \\ CN(s) \end{bmatrix}-\begin{bmatrix} sB_ED(s) \\ 0 \end{bmatrix}\right)T_1P \\
=\ &s\begin{bmatrix} (sI-A_E)N(s)-B_ED(s) \\ 0 \end{bmatrix}T_1P \\
=\ &0.
\end{aligned}$$

Combining the above two aspects, proves that $\tilde{N}(s)$ and $\tilde{D}(s)$ satisfy the right factorization (71).

Next, using a series of row transformations, we can show

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix} \tilde{N}(s) \\ \tilde{D}(s) \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} \begin{bmatrix} N(s) \\ CN^{\#}(s) \\ D(s) \end{bmatrix}T_2 & \begin{bmatrix} sN(s) \\ CN(s) \\ sD(s) \end{bmatrix}T_1P \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} \begin{bmatrix} N(s) \\ D(s) \\ CN^{\#}(s) \end{bmatrix}T_2 & \begin{bmatrix} sN(s) \\ sD(s) \\ CN(s) \end{bmatrix}T_1P \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} \begin{bmatrix} 0 \\ I_m \\ CN^{\#}(s) \end{bmatrix}T_2 & \begin{bmatrix} 0 \\ sI_m \\ CN(s) \end{bmatrix}T_1P \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}T_2 & \begin{bmatrix} 0 \\ sI_m \\ CN_0 \end{bmatrix}T_1P \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} \begin{bmatrix} 0 \\ T_2 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ sT_1P \\ I_m \end{bmatrix} \end{bmatrix} \\
&= m+(r-m) \\
&= r,
\end{aligned}$$

thus the right coprimeness of $\tilde{N}(s)$ and $\tilde{D}(s)$ is also preserved. The whole proof is then completed.

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