ARTICLE TEMPLATE

High-order fully-actuated system approaches: Part X. Basics of discrete-time systems

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ABSTRACT
A basic introduction to high-order fully actuated (HOFA) discrete-time system approaches is given. Firstly, it is shown that, different from the continuous-time systems, general dynamical discrete-time systems can be represented by two types of discrete-time HOFA models, namely, the step forward HOFA models and the step backward HOFA models. Secondly, controllers for both models are designed, which result in constant linear closed-loop systems with arbitrarily assignable eigenstructure. The related problem of feasibility is also discussed. The well-known discrete-time feedback linearizable systems and strict-feedback systems are shown to be equivalent to both the step forward and step backward HOFA models. Finally, a generalized step backward HOFA model containing control vectors of different time instants is proposed and investigated, and demonstrated with a type of proposed pseudo feed-forward systems. The contribution in this paper has laid a fundamental basis for discrete-time HOFA approaches. Further analysis and design problems can be naturally established parallel to the continuous-time system case.

KEYWORDS
Discrete-time systems; fully actuated systems; step forward models; step backward models; feasibility

1. Introduction

With the advent and rapid updating of computers and microprocessors in the 80's, the implementations of control systems are more and more dependent on the digital hardware. This has greatly promoted the development of the digital control theory. In practice, there are mainly two ways to obtain a digital controller. One way is to first synthesize the controller in the continuous-time domain, and then to discretize the controller yielding a discrete-time approximation version of the controller. The other one is to first discretize the system to yield a discrete-time system, and then to design the discrete-time controller for the discrete-time system. The study of the latter one has produced the complete discrete-time control system theory.

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1.1. Discrete-time systems

Parallel to the continuous-time system case, researches on the linear and the nonlinear discrete-time systems have received considerable attention and fruitful results have been obtained during the past few decades.

For the general linear discrete-time systems, relatively complete and systematical analysis and design methods have been developed. These include response analysis, internal and input-output stability, controllability and observability, observer design (Rugh, 1996), pole assignment and eigenstructure assignment (Duan, Liu, & Thompson, 2003), linear quadratic regulation (Duan, 2016), robust control (Amato, 2006; Duan & Yu, 2013; Zhou, Doyle, & Glover, 1996), identification and adaptive control (Landau & Zito, 2007), and input constrained control (Zhou, Lin, & Duan, 2009).

As for the nonlinear discrete-time systems, due to their complexity and diversity, it is hard to establish a universal control approach for all systems, instead, different control methods are proposed for different kinds of systems. In general, these control methods can be roughly classified into several categories:

- Optimization-based approaches, such as the discrete-time nonlinear optimal and inverse optimal control approach (Haddad & Chellaboina, 2011; Lewis, Vrabie, & Syrmos, 2012), the discrete-time SDRE approach (Cimen, 2012), the model predictive approach (Pannek & Grüne, 2011), and the adaptive dynamic programming approach (Zhang, Liu, Luo, & Wang, 2012).
- Lyapunov-based constructive approaches, of which the core idea is to find an energy-like (Lyapunov) function and construct a control law such that the "energy" of the closed-loop system is dissipative. This type of approaches mainly include the discrete-time sliding mode approach (Argha, Su, Li, Nguyen, & Celler, 2018), the discrete-time backstepping approach (Zhang, Wen, & Soh, 2000), the discrete-time dynamic surface approach (Yoshimura, 2018), and the feedback passivation approach (Zhao & Gupta, 2016).
- Linearization-based approaches, including the feedback linearization approach (Grizzle & Kokotovic, 1988), the fuzzy T-S model approach (Kruszewski, Wang, & Guerra, 2010), the gain scheduling approach, etc.
- Intelligence-based approaches, for example, the iterative learning approach (Xu, 2009), and the fuzzy logic approach (Abidi & Xu, 2015).

Some other important and interesting topics on the control of discrete-time nonlinear systems are also discussed in the literature. An interesting one is the problem how much uncertainty can be dealt with by the feedback mechanism. It is deeply discussed by Guo and his coauthors, see for instance, Guo (1997) and Xie & Guo (2000).

1.2. Fully actuated system approaches

Nonlinear control system theories based on state-space approaches have appeared more than half a century, and yet systematic and effective methods are still lacking for the various analysis and design problems. To a certain extent, discrete-time system theories are parallel to the continuous-time system theories. Therefore, technical obstacles encountered in continuous-time systems with state-space approaches generally exist in discrete-time systems.

In the recent two series of papers, Duan (2020a,b,c), and Duan (2020d,e,f,g, 2021a,b,c,d,e), a different approach other than the state-space one is proposed for continuous-time dynamical systems, which is termed as the high-order fully actuated

(HOFA) system approach. It has been sufficiently shown that the proposed HOFA system approach is extremely effective and simple in dealing with the control problems of general dynamical nonlinear systems.

As the last one in this HOFA approach series, this paper gives a basic introduction to the HOFA approaches for discrete-time systems. Two main topics are covered, namely, HOFA models for general discrete-time dynamical systems, and control of discrete-time HOFA systems.

For HOFA models of general dynamical discrete-time systems, the linear system case is firstly discussed. Different from the continuous-time case, two types of discrete-time HOFA models are proposed, one is the step forward HOFA model, and the other is the step backward HOFA model. An important fact for continuous-time systems is extended to the discrete-time system case, that is, any controllable linear discrete-time system is equivalent to both a step forward and a step backward HOFA model. Such a fact clearly reflects two aspects, one is the generality of HOFA models, the other is that HOFA model representation reflects in fact the nature of controllability. Secondly, HOFA models for nonlinear systems are also discussed, both affine and non-affine discrete-time HOFA models are proposed. As in the linear case, the models are composed of step forward and step backward HOFA models. The two types of models are essentially different in that the former is, in nature, a time-delay system with a delay in the control input, while the latter is not. A common aspect of the two models is that they coincide with each other in the special case that the orders of the systems are both of 1, which corresponds to a normal state-space model.

For control of discrete-time HOFA models, simple controllers in state feedback form are designed, which result in constant linear closed-loop systems with arbitrarily assignable eigenstructures. The controller proposed for step backward systems is a simple static feedback control law. However, the one for a step forward discrete-time HOFA model turns out to be more complicated. The reason lies in the delay in the control input. In order to make the controller realizable, the states of certain steps ahead have to be predicted. The prediction is realized based on the open-loop system, and the controller eventually turns out to be a dynamical one.

As examples of the proposed HOFA models, the well-known discrete-time feedback linearizable systems and strict-feedback systems are considered. As a result, these two types of discrete-time systems are shown to possess both a step forward and a step backward HOFA model.

Besides the above mentioned discrete-time HOFA models, a generalized discrete-time step backward HOFA model is also proposed, which takes into consideration of the effect of the control vector at more time instants. Control of this type of models can be also very easily realized and the closed-loop system resulted in is also constant and linear with desired eigenstructures. As an example, a type of pseudo feed-forward systems are proposed and are shown to be representable by this generalized HOFA model.

State-space models concentrate on the state variables, and integrate all the independent state variables together. They are a kind of models with the intention and possibility of solving out all the state variables, and are therefore convenient for the problems of state solution (response analysis) and estimation (observation, filtering and prediction). Although control problems are also tackled with the state-space approaches, the results are not as satisfactory as desired in the nonlinear system case.

On the contrary, HOFA models, proposed in Duan (2020a,d, 2021c) and this paper, concentrate on the control variables, and have the property of being able to solve out the control vectors. Therefore, they are consequently convenient for control problems.

The full-actuation property of a HOFA model allows one to cancel the nonlinearities in the system, and eventually obtain a constant linear system with arbitrarily assignable eigenstructures. Technically, this control feature of discrete-time HOFA models proposed in this paper has adequately laid a solid basis for the development of a general HOFA system framework for discrete-time systems.

Firstly, since the closed-loop system is already constant and linear, response analysis and stability analysis no longer stand as a problem. Secondly, arbitrary assignability of the closed-loop eigenstructure reflects exactly the meaning of controllability, hence a general theory on controllability and stabilizability of general dynamical discrete-time systems can be invented parallel to those of the continuous-time systems, as in Duan (2020b, 2021c). Finally, this important control feature also allows one to convert many control design problems into corresponding ones for linear systems. Hence problems of robust control, adaptive control, disturbance rejection, asymptotical signal tracking, and optimal control can all be effectively solved as the continuous-time case (Duan, 2020f,g, 2021a,b,d,e).

This paper is organized into 8 sections. The next section gives certain symbols and notions used in the paper. In Sections 3 and 4 the discrete-time HOFA models for linear systems and nonlinear systems are proposed, respectively. Section 5 presents the designs of controllers for the proposed discrete-time HOFA systems. As examples, the discrete-time feedback linearizable systems and strict-feedback systems are treated in Section 6, and in Section 7 the proposed HOFA models are further generalized and a type of pseudo feed-forward systems are proposed and analyzed, followed by a brief concluding remark in Section 8.

2. Notations

In this section, certain notations used in the paper are explained.

2.1. General notations

In the sequential sections, I_n denotes the identity matrix, $\varnothing$ denotes the null set, $\mathbb{N}$ is the set of natural numbers, and $\Omega\backslash\Theta$ represents the complement of the set Θ in set Ω. Furthermore, $\det(A)$ and A^{-1} denote the determinant and the inverse of a matrix A, respectively.

For $x_i \in \mathbb{R}^m$, $i = 1, 2, ..., n$, we denote

$$x_{i\sim j}(k) = \begin{bmatrix} x_i(k) \\ x_{i+1}(k) \\ \vdots \\ x_j(k) \end{bmatrix}, i \leq j \leq n.$$

For $A_i \in \mathbb{R}^{m\times m}$, $i = 1, 2, ..., n$, as in Duan (2020f), the following symbols are used:

$$A_{0\sim n} = \begin{bmatrix} A_0 & A_1 & \cdots & A_n \end{bmatrix},$$

$$\Phi(A_{0\sim n}) = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & -A_1 & \cdots & -A_n \end{bmatrix}.$$

2.2. Step forward operations

For $x(k) \in \mathbb{R}^m$, it is well-known that the one-step forward operator is denoted by q, which operates in the following way:

$$\begin{aligned} qx(k) &= x(k+1), \\ q^i x(k) &= x(k+i). \end{aligned}$$

For convenience, in this paper we denote the above operation by the following notation:

$$x^{\lfloor i \rfloor}(k) = x(k+i).$$

For $x, x_i \in \mathbb{R}^m$, $n_0, n_i \in \mathbb{N}$, $n_0 < n_i$, $i = 1, 2, ..., n$, as in Duan (2020f), the following symbols are used in the paper:

$$x^{\lfloor 0\sim n \rfloor}(k) = \begin{bmatrix} x(k) \\ x(k+1) \\ \vdots \\ x(k+n) \end{bmatrix},$$

$$x^{\lfloor n_1\sim n_2 \rfloor}(k) = \begin{bmatrix} x^{\lfloor n_1 \rfloor}(k) \\ x^{\lfloor n_1+1 \rfloor}(k) \\ \vdots \\ x^{\lfloor n_2 \rfloor}(k) \end{bmatrix}, \quad n_1 \leq n_2,$$

$$x_{i\sim j}^{\lfloor n_1\sim n_2 \rfloor}(k) = \begin{bmatrix} x_i^{\lfloor n_1\sim n_2 \rfloor}(k) \\ x_{i+1}^{\lfloor n_1\sim n_2 \rfloor}(k) \\ \vdots \\ x_j^{\lfloor n_1\sim n_2 \rfloor}(k) \end{bmatrix}, \quad i \leq j, n_1 \leq n_2,$$

$$x_p^{\lfloor n_p \rfloor}(k)\,|_{p=i\sim j} = \begin{bmatrix} x_i^{\lfloor n_i \rfloor}(k) \\ x_{i+1}^{\lfloor n_{i+1} \rfloor}(k) \\ \vdots \\ x_j^{\lfloor n_j \rfloor}(k) \end{bmatrix}, \quad i \leq j,$$

$$x_p^{\lfloor n_0 \sim n_p \rfloor}(k)\,|_{p=i\sim j} = \begin{bmatrix} x_i^{\lfloor n_0 \sim n_i \rfloor}(k) \\ x_{i+1}^{\lfloor n_0 \sim n_{i+1} \rfloor}(k) \\ \vdots \\ x_j^{\lfloor n_0 \sim n_j \rfloor}(k) \end{bmatrix}, \quad i \leq j.$$

2.3. Step backward operations

For $x(k) \in \mathbb{R}^m$, it is well-known that the one-step backward operator is denoted by q^{-1}, which operates in the following way:

$$\begin{aligned} q^{-1}x(k) &= x(k-1), \\ q^{-i}x(k) &= x(k-i). \end{aligned}$$

For convenience, in this paper we denote the above operation by the following notation:

$$x^{\lceil i \rceil}(k) = x(k-i).$$

Based on this notation, for $x, x_i \in \mathbb{R}^m$, $n_0, n_i \in \mathbb{N}$, $n_0 < n_i$, $i = 1, 2, ..., n$, the following symbols are used in the paper:

$$x^{\lceil 0 \sim n \rceil}(k) = \begin{bmatrix} x(k-n) \\ \vdots \\ x(k-1) \\ x(k) \end{bmatrix},$$

$$x^{\lceil n_1 \sim n_2 \rceil}(k) = \begin{bmatrix} x^{\lceil n_2 \rceil}(k) \\ x^{\lceil n_2 - 1 \rceil}(k) \\ \vdots \\ x^{\lceil n_1 \rceil}(k) \end{bmatrix}, \quad n_1 \leq n_2,$$

$$x_{i\sim j}^{\lceil n_1 \sim n_2 \rceil}(k) = \begin{bmatrix} x_i^{\lceil n_1 \sim n_2 \rceil}(k) \\ x_{i+1}^{\lceil n_1 \sim n_2 \rceil}(k) \\ \vdots \\ x_j^{\lceil n_1 \sim n_2 \rceil}(k) \end{bmatrix}, \quad i \leq j, n_1 \leq n_2,$$

$$x_p^{\lceil n_p \rceil}(k)\,|_{p=i\sim j} = \begin{bmatrix} x_i^{\lceil n_i \rceil}(k) \\ x_{i+1}^{\lceil n_{i+1} \rceil}(k) \\ \vdots \\ x_j^{\lceil n_j \rceil}(k) \end{bmatrix}, \quad i \leq j,$$

$$x_p^{\lceil n_0 \sim n_p \rceil}(k)|_{p=i\sim j} = \begin{bmatrix} x_i^{\lceil n_0 \sim n_i \rceil}(k) \\ x_{i+1}^{\lceil n_0 \sim n_{i+1} \rceil}(k) \\ \vdots \\ x_j^{\lceil n_0 \sim n_j \rceil}(k) \end{bmatrix}, \quad i \leq j.$$

By the above notations, we clearly have the following basic relations:

$$x^{\lfloor i \rfloor}(k-i) = x^{\lceil i \rceil}(k+i) = x(k),$$

$$x^{\lceil 0\sim n \rceil}(k+n) = x^{\lfloor 0\sim n \rfloor}(k).$$

3. Linear HOFA models

Corresponding to the following continuous-time linear state-space model

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

a discrete-time linear state-space model is in the form of

$$x(k+1) = Ax(k) + Bu(k), \tag{2}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and input vector, respectively, and $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times r}$ are the coefficient matrices.

For the above first-order state-space systems (1) and (2), it is generally not realistic to require $\mathrm{rank}B = r = n$. However, the following proposition shows that, for continuous-time high-order systems, such a condition is not a restriction at all.

Proposition 3.1. *The linear system (1) is controllable if and only if it can be converted equivalently into a high-order system in the form of*

$$\begin{bmatrix} x_1^{(\mu_1)} \\ x_2^{(\mu_2)} \\ \vdots \\ x_\eta^{(\mu_\eta)} \end{bmatrix} = \begin{bmatrix} L_1\left(x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta}\right) \\ L_2\left(x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta}\right) \\ \vdots \\ L_\eta\left(x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta}\right) \end{bmatrix} + \hat{B}u, \tag{3}$$

where $\mu_p, p = 1,2,...,\eta$, are a set of integers, $x_p, p = 1,2,...,\eta$, are a set of vectors of proper dimensions, $L_p(\cdot), p = 1,2,...,\eta$, are a set of linear functions, and $\hat{B}$ is a square upper triangular matrix with diagonal elements all being 1.

The above result is in fact a modified form of the Theorem 2 in Duan (2020b). It can be proven simply following the proof therein.

A system in the form of (3) is called a linear HOFA system if

$$\mathrm{rank}\hat{B} = r = n. \tag{4}$$

A typical special continuous-time linear HOFA system is the following:

$$x^{(n)} = \sum_{i=0}^{n-1} A_i x^{(i)} + Bu, \tag{5}$$

where $A_i \in \mathbb{R}^{r\times r}, i = 1, 2, ..., n,$ are a set of matrices, and the matrix $B \in \mathbb{R}^{r\times r}$ is nonsingular.

3.1. Definition

Corresponding to the continuous-time linear HOFA model (5), we have the following discrete-time linear HOFA model:

$$x\left(k+n\right) = \sum_{i=0}^{n-1} A_i x\left(k+i\right) + Bu\left(k\right), \tag{6}$$

which can be also written, using our notations, in the form of

$$x^{\lfloor n \rfloor}\left(k\right) = \sum_{i=0}^{n-1} A_i x^{\lfloor i \rfloor}\left(k\right) + Bu\left(k\right), \tag{7}$$

where the matrix $B \in \mathbb{R}^{r\times r}$ is nonsingular. This type of models are called step forward HOFA models.

Again, corresponding to the continuous-time linear HOFA model (5), we also have another type of discrete-time model which appears as follows:

$$x\left(k+1\right) = \sum_{i=0}^{n-1} A_{n-i-1} x\left(k-i\right) + Bu\left(k\right), \tag{8}$$

where, again, $A_i \in \mathbb{R}^{r\times r}, i = 0, 1, ..., n-1,$ are a set of matrices, and the matrix $B \in \mathbb{R}^{r\times r}$ is nonsingular. This model is called a step backward HOFA model, and can be also written in the form of

$$x\left(k+1\right) = \sum_{i=0}^{n-1} A_{n-i-1} x^{\lceil i \rceil}\left(k\right) + Bu\left(k\right). \tag{9}$$

The two types of HOFA systems (7) and (9) are essentially different. As a matter of fact, shifting the time index by $n-1$ steps in the step forward HOFA system (7), gives

$$x\left(k+1\right) = \sum_{i=0}^{n-1} A_{n-i-1} x^{\lceil i \rceil}\left(k\right) + Bu\left(k-\left(n-1\right)\right). \tag{10}$$

This differs with the step backward HOFA system (9) by $n-1$ steps delay in the control vector.

In general, corresponding to the continuous-time linear HOFA model (3), we have two types of discrete-time linear HOFA models. The first type is the following step forward HOFA model:

$$\begin{bmatrix} x_1^{\lfloor \mu_1 - 1 \rfloor}(k+1) \\ x_2^{\lfloor \mu_2 - 1 \rfloor}(k+1) \\ \vdots \\ x_\eta^{\lfloor \mu_\eta - 1 \rfloor}(k+1) \end{bmatrix} = \begin{bmatrix} L_1\left(x_p^{\lfloor 0 \sim \mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}\right) \\ L_2\left(x_p^{\lfloor 0 \sim \mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}\right) \\ \vdots \\ L_\eta\left(x_p^{\lfloor 0 \sim \mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}\right) \end{bmatrix} + \hat{B}u(k), \tag{11}$$

while the second type is the following step backward HOFA model:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_\eta(k+1) \end{bmatrix} = \begin{bmatrix} L_1\left(x_p^{\lceil 0 \sim \mu_p - 1 \rceil}(k)\,|_{p=1\sim\eta}\right) \\ L_2\left(x_p^{\lceil 0 \sim \mu_p - 1 \rceil}(k)\,|_{p=1\sim\eta}\right) \\ \vdots \\ L_\eta\left(x_p^{\lceil 0 \sim \mu_p - 1 \rceil}(k)\,|_{p=1\sim\eta}\right) \end{bmatrix} + \hat{B}u(k), \tag{12}$$

where μ_p, x_p, $L_p(\cdot)\,, p = 1, 2, ..., \eta$, and $\hat{B}$ are as stated in Proposition 3.1.

If we denote

$$L(\cdot) = \begin{bmatrix} L_1(\cdot) \\ L_2(\cdot) \\ \vdots \\ L_\eta(\cdot) \end{bmatrix},$$

then the above step forward and step backward HOFA models (11) and (12) can be compactly written as

$$x_p^{\lfloor \mu_p - 1 \rfloor}(k+1)\,|_{p=1\sim\eta} = L\left(x_p^{\lfloor 0 \sim \mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}\right) + \hat{B}u(k), \tag{13}$$

and

$$x_{1\sim\eta}(k+1) = L\left(x_p^{\lceil 0 \sim \mu_p - 1 \rceil}(k)\,|_{p=1\sim\eta}\right) + \hat{B}u(k), \tag{14}$$

respectively.

By our notations, the step forward system (7) can be also written as

$$x^{\lfloor n-1 \rfloor}(k+1) = A_{0\sim n-1}x^{\lfloor 0 \sim n-1 \rfloor}(k) + Bu(k). \tag{15}$$

If we define

$$\Gamma_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_r \end{bmatrix},$$

then system (15), hence (7), can be also written in the following state-space form:

$$x^{\lfloor 0\sim n-1\rfloor}(k+1)=\Phi(-A_{0\sim n-1})\,x^{\lfloor 0\sim n-1\rfloor}(k)+\Gamma_c Bu(k).$$

Parallelly, the backward HOFA system (9) can be written in the form of

$$x(k+1)=A_{0\sim n-1}x^{\lceil 0\sim n-1\rceil}(k)+Bu(k), \tag{16}$$

which has a state-space model as

$$x^{\lceil 0\sim n-1\rceil}(k+1)=\Phi(-A_{0\sim n-1})\,x^{\lceil 0\sim n-1\rceil}(k)+\Gamma_c Bu(k).$$

3.2. Controllability canonical forms

Corresponding to the above Proposition 3.1, we can also prove that the discrete-time controllable state-space system (2) can be represented by a HOFA model. To do this, we need the following lemma which can be easily proven using the controllability canonical form of linear systems (see, also the proof of the Theorem 2 in Duan (2020b)).

Lemma 3.2. *Let $[A,B]$ be controllable. Then, there exists a state transformation $z=T(x)$, under which the above system (2) can be transformed into*

$$\begin{cases} z_{i1}(k+1)=z_{i2}(k) \\ z_{i2}(k+1)=z_{i3}(k) \\ \qquad\vdots \\ z_{i,\mu_i-1}(k+1)=z_{i\mu_i}(k) \\ z_{i\mu_i}(k+1)=\sum\limits_{l=1}^{r}\sum\limits_{j=1}^{\mu_l}a_{lj}^{(i)}z_{lj}(k)+\sum\limits_{j=i}^{r}b_{ij}u_j(k), \\ i=1,2,...,r, \end{cases} \tag{17}$$

where $a_{lj}^{(i)}, b_{ij}$, with $b_{ii}=1$, are some real numbers, and $\mu_i, i=1,2,...,r$ are the controllability indices of $[A,B]$.

Based on the above lemma, the following result can be proven.

Theorem 3.3. *The linear system (2) is controllable if and only if it can be converted equivalently into both a step forward HOFA system in the form of (11), and a step backward HOFA system in the form of (12).*

Proof. Using the first $r-1$ equations in (17), we can obtain

$$\begin{array}{rcl} z_{ij}(k) & = & z_{i1}(k+j-1), \\ j & = & 1,2,...,\mu_i,\ i=1,2,...,r, \end{array} \tag{18}$$

substituting these into the last one in system (17), yields

$$\begin{aligned}
& z_{i1}\left(k+\mu_i\right) \\
= & \sum_{l=1}^{r}\sum_{j=1}^{\mu_l} a_{lj}^{(i)} z_{lj}\left(k\right)+\sum_{j=i}^{r} b_{ij}u_j\left(k\right) \\
= & \sum_{l=1}^{r}\sum_{j=1}^{\mu_l} a_{lj}^{(i)} z_{l1}\left(k+j-1\right)+\sum_{j=i}^{r} b_{ij}u_j\left(k\right), \\
i = & \; 1,2,...,r,
\end{aligned}$$

which can be written in the form of

$$\begin{aligned}
& z_{p1}^{\lfloor\mu_p-1\rfloor}\left(k+1\right) \\
= & \; L_i\left(z_{p1}^{\lfloor 0\sim\mu_p-1\rfloor}\left(k\right), p=1,2,...,r\right)+\sum_{j=i}^{r} b_{ij}u_j\left(k\right), \\
& \qquad p=1,2,...,r,
\end{aligned}$$

where $L_i\left(\cdot\right)$ is a linear function. Clearly, the above system can be written into the form of the following step forward HOFA model:

$$z_{p1}^{\lfloor\mu_p-1\rfloor}\left(k+1\right)\Big|_{p=1\sim r}=L\left(z_{p1}^{\lfloor 0\sim\mu_p-1\rfloor}\left(k\right)\Big|_{p=1\sim r}\right)+\hat{B}u\left(k\right), \tag{19}$$

where $L\left(\cdot\right)$ is an extended vector of linear functions, and $\hat{B}$ is an upper triangular matrix with diagonal elements all being 1.

On the other side, we also have, from the first $r-1$ equations in (17),

$$\begin{aligned}
z_{ij}\left(k\right) &= z_{i\mu_i}\left(k-\mu_i+j\right), \\
j &= 1,2,...,\mu_i,\ i=1,2,...,r,
\end{aligned} \tag{20}$$

substituting these into the last one in (17), gives

$$\begin{aligned}
& z_{i\mu_i}\left(k+1\right) \\
= & \sum_{l=1}^{r}\sum_{j=1}^{\mu_l} a_{lj}^{(i)} z_{lj}\left(k\right)+\sum_{j=i}^{r} b_{ij}u_j\left(k\right) \\
= & \sum_{l=1}^{r}\sum_{j=1}^{\mu_l} a_{lj}^{(i)} z_{l\mu_l}\left(k-\mu_l+j\right)+\sum_{j=i}^{r} b_{ij}u_j\left(k\right) \\
= & \sum_{l=1}^{r}\sum_{q=0}^{\mu_{l-1}} a_{l,\mu_l-q}^{(i)} z_{l,\mu_l}\left(k-q\right)+\sum_{j=i}^{r} b_{ij}u_j\left(k\right), \\
& \qquad i=1,2,...,r,
\end{aligned}$$

which can be written as

$$\begin{aligned} z_{i\mu_i}(k+1) &= L_i\left(z_{p\mu_p}^{\lceil 0\sim\mu_p-1\rceil}(k)\Big|_{p=1\sim r}\right) + \sum_{j=i}^{r} b_{ij}u_j(k), \\ i &= 1,2,...,r, \end{aligned}$$

where $L_i(\cdot)$ is a linear function. While this can clearly be written in the form of the following step backward HOFA model:

$$z_{p\mu_p}(k+1)\big|_{p=1\sim r} = L\left(z_{p\mu_p}^{\lceil 0\sim\mu_p-1\rceil}(k)\Big|_{p=1\sim r}\right) + \hat{B}u(k), \tag{21}$$

where $L(\cdot)$ is an extended vector of linear function, and $\hat{B}$ is an upper triangular matrix with diagonal elements all being 1.

Finally, note that the above two processes are both invertible, the whole proof is done. □

The above theorem indicates that the step forward and the step backward HOFA models are in fact the controllability canonical forms of the state-space system (2).

Remark 1. The above Theorem 3.3 obviously indicates that a step forward HOFA system in the form of (11) is equivalent to a step backward HOFA system in the form of (12). We point out that such a fact does not generally hold taking account of the Law of Cause and Effect. The reason lies in the availability of the state vector $x(k)$ of the controllable linear system (2). Once $x(k)$ is available, so is $z = T(x)$, hence so is $z_{p1}^{\lfloor 0\sim\mu_p-1\rfloor}(k)\Big|_{p=1\sim r}$. It is this hidden hypothesis that has made this fact holds true.

3.3. Further generalizations

In linear system theory, the following high-order discrete-time system (self-recursive model)

$$x(k+n) = \sum_{i=0}^{n-1} A_i x(k+i) + \sum_{i=0}^{m} B_i u(k+i), \tag{22}$$

or

$$x(k+1) = \sum_{i=0}^{n-1} A_{n-i-1} x(k-i) + \sum_{i=0}^{m} B_{m-i} u(k-i), \tag{23}$$

is often encountered. It can be easily reasoned that, when $B = B_m$ is a square non-singular matrix, the system (22) can be converted equivalently into

$$x(k+n) = \sum_{i=0}^{n-1} A_i x(k+i) + Bu'(k+m),$$

where

$$\begin{aligned} u'(k+m) &= B^{-1}\sum_{i=0}^{m} B_i u(k+i) \\ &= u(k+m) + \sum_{i=0}^{m-1} B^{-1}B_i u(k+i), \end{aligned} \tag{24}$$

while the system (23) can be converted equivalently into

$$x(k+1) = \sum_{i=0}^{n-1} A_{n-i-1}x(k-i) + Bu''(k),$$

where

$$\begin{aligned} u''(k) &= B^{-1}\sum_{i=0}^{m} B_{m-i}u(k-i) \\ &= u(k) + \sum_{i=1}^{m} B^{-1}B_{m-i}u(k-i). \end{aligned} \tag{25}$$

In both cases, the original control vectors can be uniquely solved through (24) and (25).

More generally, let us consider the following system, represented by the step backward operator, as

$$A\left(q^{-1}\right)x(k) = q^{-d}B\left(q^{-1}\right)u(k), \tag{26}$$

where $d \geq 1$, and

$$\begin{aligned} A\left(q^{-1}\right) &= \sum_{i=0}^{n} A_i q^{-i}, \quad A_0 = I_r, \\ B\left(q^{-1}\right) &= \sum_{i=0}^{m} B_i q^{-i}, \quad B_0 \neq 0. \end{aligned}$$

Such a system is called a step backward HOFA system if $B\left(q^{-1}\right)$ is a unimodular matrix, that is,

$$\det B(q) \neq 0, \quad \forall q \in \mathbb{C}. \tag{27}$$

In this case, we can introduce the new control

$$u'(k) = B\left(q^{-1}\right)u(k), \tag{28}$$

which gives

$$u(k) = B^{-1}\left(q^{-1}\right)u'(k), \tag{29}$$

and the system (26) can be reduced to the following simple HOFA form:

$$A\left(q^{-1}\right)x\left(k\right)=q^{-d}u'\left(k\right). \tag{30}$$

In the case that the above condition (27) does not hold, we can first convert the systems (26) into its minimal state-space model, and then follow Theorem 3.3 to represent the system in a HOFA model.

4. Nonlinear HOFA models

4.1. Affine HOFA models

The state-space representation of a general nonlinear (affine) continuous-time system is

$$\dot{x}\left(t\right)=f\left(x\left(t\right),t\right)+B\left(x\left(t\right),t\right)u\left(t\right), \tag{31}$$

and that of a general nonlinear (affine) discrete-time system is

$$x\left(k+1\right)=f\left(x\left(k\right),k\right)+B\left(x\left(k\right),k\right)u\left(k\right), \tag{32}$$

where $f(\cdot)\in\mathbb{R}^n$ and $B(\cdot)\in\mathbb{R}^{n\times r}$ are proper functions. For such general state-space systems, we have $r\leq n$, but with the equality case being seldom valid. However, in the continuous-time case, it has been shown that, under very mild conditions, the first-order system (31) can be converted equivalently to a HOFA system.

Concretely, in Duan (2020d, 2021c,d,e), the following general continuous-time affine HOFA model is proposed and investigated:

$$\begin{bmatrix} x_1^{(\mu_1)} \\ x_2^{(\mu_2)} \\ \vdots \\ x_\eta^{(\mu_\eta)} \end{bmatrix} = \begin{bmatrix} f_1\left(x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta},t\right) \\ f_2\left(x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta},t\right) \\ \vdots \\ f_\eta\left(x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta},t\right) \end{bmatrix} +B\left(x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta},t\right)u, \tag{33}$$

where $\mu_p, p=1,2,...,\eta$, are a set of integers, $x_p\in\mathbb{R}^{r_p}, p=1,2,...,\eta$, are a set of vectors of proper dimensions, with $r_p, p=1,2,...,\eta$, being a set of distinct integers satisfying

$$r_1+r_2+\cdots+r_\eta=r. \tag{34}$$

Further, $f_p\left(\cdot\right)\in\mathbb{R}^{r_p}, p=1,2,...,\eta$, are a set of nonlinear vector functions, and $B\left(\cdot\right)\in\mathbb{R}^{r\times r}$ is a nonlinear matrix function which is nonsingular for some $t\geq 0$ and $x_p^{(0\sim\mu_p-1)}|_{p=1\sim\eta}\in\Omega$, with Ω being a set with dimension not less than 1.

Clearly, in the case of $\eta=1$, the above HOFA model (33) reduces to the form of

$$x^{(n)}=f\left(x^{(0\sim n-1)},t\right)+B\left(x^{(0\sim n-1)},t\right)u, \tag{35}$$

which is proposed in Duan (2020a) (see, also Duan (2020d)). When nonlinear uncertainties, unknown parameters and disturbances are added to this basic HOFA model, robust and adaptive control as well as disturbance rejection control have been investigated in Duan (2020f,g, 2021a,b). While with the general HOFA model (33), the problems of optimal control, generalized PID control and model reference tracking control are investigated in Duan (2021d) and Duan (2021e).

Corresponding to the continuous-time HOFA system (35), we have the following discrete-time step forward HOFA model

$$x^{\lfloor n-1\rfloor}(k+1) = f\left(x^{\lfloor 0\sim n-1\rfloor}(k), k\right) + B\left(x^{\lfloor 0\sim n-1\rfloor}(k), k\right) u(k), \tag{36}$$

and the following step backward HOFA model

$$x(k+1) = f\left(x^{\lceil 0\sim n-1\rceil}(k), k\right) + B\left(x^{\lceil 0\sim n-1\rceil}(k), k\right) u(k). \tag{37}$$

Shifting the time index in the step forward HOFA system (36) back by $n-1$ steps, gives

$$\begin{aligned} x(k+1) &= f\left(x^{\lceil 0\sim n-1\rceil}(k), k-(n-1)\right) \\ &\quad + B\left(x^{\lceil 0\sim n-1\rceil}(k), k-(n-1)\right) u(k-(n-1)). \end{aligned} \tag{38}$$

Comparing this with (37) we can clearly tell the great difference between the step forward and the step backward models.

Corresponding to the general continuous-time HOFA system (33), we can also propose the following discrete-time step forward HOFA system

$$\begin{aligned} \begin{bmatrix} x_1^{\lfloor \mu_1-1\rfloor}(k+1) \\ x_2^{\lfloor \mu_2-1\rfloor}(k+1) \\ \vdots \\ x_\eta^{\lfloor \mu_\eta-1\rfloor}(k+1) \end{bmatrix} &= \begin{bmatrix} f_1\left(x_p^{\lfloor 0\sim \mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}, k\right) \\ f_2\left(x_p^{\lfloor 0\sim \mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}, k\right) \\ \vdots \\ f_\eta\left(x_p^{\lfloor 0\sim \mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}, k\right) \end{bmatrix} \\ &\quad + B\left(x_p^{\lfloor 0\sim \mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}, k\right) u(k), \end{aligned} \tag{39}$$

and the following step backward HOFA system

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_\eta(k+1) \end{bmatrix} &= \begin{bmatrix} f_1\left(x_p^{\lceil 0\sim \mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, k\right) \\ f_2\left(x_p^{\lceil 0\sim \mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, k\right) \\ \vdots \\ f_\eta\left(x_p^{\lceil 0\sim \mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, k\right) \end{bmatrix} \\ &\quad + B\left(x_p^{\lceil 0\sim \mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, k\right) u(k), \end{aligned} \tag{40}$$

where $\mu_p, p = 1, 2, ..., \eta$, are a set of integers, $x_p \in \mathbb{R}^{r_p}, p = 1, 2, ..., \eta$, are a set of vectors, with $r_p, p = 1, 2, ..., \eta$, being a set of distinct integers satisfying (34). Further,

$f_p(\cdot) \in \mathbb{R}^{r_p}, p = 1, 2, ..., \eta$, are a set of nonlinear vector functions, and $B(\cdot) \in \mathbb{R}^{r \times r}$ is a nonlinear matrix function of its variables.

If we denote

$$f(\cdot, k) = \begin{bmatrix} f_1(\cdot, k) \\ f_2(\cdot, k) \\ \vdots \\ f_\eta(\cdot, k) \end{bmatrix}, \tag{41}$$

then the step forward HOFA system (39) and the step backward HOFA system (40) can be compactly written as

$$\begin{aligned} x_p^{\lfloor \mu_p - 1 \rfloor}(k+1)|_{p=1\sim\eta} &= f\left(x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)|_{p=1\sim\eta}, k\right) \\ &\quad + B\left(x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)|_{p=1\sim\eta}, k\right) u(k). \end{aligned} \tag{42}$$

and

$$\begin{aligned} x_{1\sim\eta}(k+1) &= f\left(x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)|_{p=1\sim\eta}, k\right) \\ &\quad + B\left(x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)|_{p=1\sim\eta}, k\right) u(k), \end{aligned} \tag{43}$$

respectively.

Let $\Omega_{pi} \subset \mathbb{R}^{r_p},\ i = 0, 1, \cdots, \mu_p - 1,\ p = 1, 2, ..., \eta$, be a set of open sets, and define

$$\begin{cases} \Omega_p = \Omega_{p0} \times \Omega_{p1} \times \cdots \times \Omega_{p,\mu_p-1} \subset \mathbb{R}^{r_p\mu_p} \\ \Omega = \Omega_0 \times \Omega_1 \times \cdots \times \Omega_\eta \subset \mathbb{R}^{\varkappa}, \end{cases} \tag{44}$$

where

$$\varkappa = \sum_{i=1}^{\eta} r_p \mu_p. \tag{45}$$

Then the following three relations are obviously equivalent:

(1) $x_p^{\lfloor i \rfloor}(k), x_p^{\lceil i \rceil}(k) \in \Omega_{pi} \subset \mathbb{R}^{r_p},\ i = 0, 1, \cdots, \mu_p - 1, p = 1, 2, ..., \eta$;
(2) $x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k), x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k) \in \Omega_p \subset \mathbb{R}^{r_p\mu_p},\ p = 1, 2, ..., \eta$; and
(3) $x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)|_{p=1\sim\eta}, x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)|_{p=1\sim\eta} \in \Omega \subset \mathbb{R}^{\varkappa}$.

With the above preparation, let us introduce the following definition.

Definition 4.1. If $x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)|_{p=1\sim\eta} \in \mathbb{R}^{\varkappa}$ or $x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)|_{p=1\sim\eta} \in \mathbb{R}^{\varkappa}$ satisfies

$$\det B\left(x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)|_{p=1\sim\eta}, k\right) = 0 \text{ or } \infty, \exists k \geq 0, \tag{46}$$

or

$$\det B\left(x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)|_{p=1\sim\eta}, k\right) = 0 \text{ or } \infty, \exists k \geq 0, \tag{47}$$

then it is called a singular point of system (39) or (40).

Let $\mathbb{S}$ be the set of all singular points of system (39) or (40), that is,

$$\mathbb{S}=\left\{ x_p^{\lfloor 0\sim\mu_p-1 \rfloor}(k)\mid_{p=1\sim\eta} \middle| (46) \text{ holds} \right\}.$$

or

$$\mathbb{S}=\left\{ x_p^{\lceil 0\sim\mu_p-1 \rceil}(k)\mid_{p=1\sim\eta} \middle| (47) \text{ holds} \right\}.$$

Then we call

$$\mathbb{F}=\mathbb{R}^{\varkappa}\backslash\mathbb{S}$$

the set of feasible points of system (39) or (40).

Corresponding to the Definition 2.1 in Duan (2021c), we can impose the following definition for discrete-time fully actuated systems.

Definition 4.2. Given system (39) or (40) and the above sets $\Omega_{pi}\subset\mathbb{R}^{r_p}$, $i=0,1,\cdots,\mu_p-1$, $p=1,2,...,\eta$, and Ω defined by (44),

- if $\mathbb{S}\cap\Omega=\varnothing$, that is, all points in Ω are feasible ones, then the system (39) or (40) is called completely fully actuated on Ω;
- if $\mathbb{S}\cap\Omega$ is only a hyperplane in Ω, then the system (39) or (40) is called sub-fully actuated on Ω;
- if $\mathbb{S}\cap\Omega$ is only a set of isolated points in Ω, then the system (39) or (40) is called almost fully actuated on Ω;
- if $\mathbb{S}\cap\Omega$ is only a finite number of points in Ω, then the system (39) or (40) is called basically fully actuated on Ω;
- if $\mathbb{S}=\varnothing$, that is, the system (39) or (40) does not have a singular point, then it is called (globally) fully actuated.

Furthermore, the system (39) or (40) is generally said to be fully actuated, if it is fully actuated on a certain set with dimension greater than 1.

In the case of $\mathbb{S}=\varnothing$, we have

$$\det B(X,k)\neq 0 \text{ or } \infty, \forall X\in\mathbb{R}^{\varkappa}, k\geq 0. \tag{48}$$

Therefore, we can introduce the control vector transformation

$$B\left(x_p^{\lceil 0\sim\mu_p-1 \rceil}(k)\mid_{p=1\sim\eta}, k\right)u(k)=\tilde{u}(k),$$

and now the (globally) step backward fully actuated system (40) can be written as

$$\begin{aligned} x_i(k+1) &= f_i\left(x_p^{\lceil 0\sim\mu_p-1 \rceil}(k)\mid_{p=1\sim\eta}, k\right)+\tilde{u}_i(k), \\ & \qquad i=1,2,...,\eta \ , \end{aligned} \tag{49}$$

where $\tilde{u}_i, i = 1, 2, ..., \eta$, are defined by

$$\tilde{u} = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_\eta \end{bmatrix}, \quad \tilde{u}_i \in \mathbb{R}^{r_i}. \tag{50}$$

4.2. Non-affine HOFA models

Parallel to the step forward affine HOFA system (39), we can also define the following non-affine one (refer to the Definition 2 in Duan (2020b)):

$$\begin{aligned} x_p^{\lfloor \mu_p - 1 \rfloor}(k+1)\,|_{p=1\sim\eta} &= f\left(x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}, k\right) \\ &\quad + g\left(x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}, k, u(k)\right). \end{aligned} \tag{51}$$

Parallel to the step backward affine HOFA system (40), we can define the following non-affine one:

$$\begin{aligned} x_{1\sim\eta}(k+1) &= f\left(x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)\,|_{p=1\sim\eta}, k\right) \\ &\quad + g\left(x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)\,|_{p=1\sim\eta}, k, u(k)\right), \end{aligned} \tag{52}$$

where $f(\cdot), g(\cdot) \in \mathbb{R}^r$ are two proper nonlinear functions. Particularly, $g(\cdot, k, u) = \tilde{u}$ forms a differential homeomorphism from u to $\tilde{u}$ for some of its variables.

If we partition $f(\cdot)$ as in (41), and $g(\cdot)$ as follows

$$g(\cdot, k, u) = \begin{bmatrix} g_1(\cdot, k, u) \\ g_2(\cdot, k, u) \\ \vdots \\ g_\eta(\cdot, k, u) \end{bmatrix}, \tag{53}$$

with $f_i(\cdot), g_i(\cdot) \in \mathbb{R}^{r_i}$, $i = 1, 2, ..., \eta$, then the step forward HOFA system (51) and the step backward HOFA system (52) can be written as

$$\begin{aligned} x_i^{\lfloor \mu_i - 1 \rfloor}(k+1) &= f_i\left(x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}, k\right) \\ &\quad + g_i\left(x_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}, k, u(k)\right), \\ &\qquad i = 1, 2, ..., \eta\ , \end{aligned} \tag{54}$$

and

$$\begin{aligned} x_i(k+1) &= f_i\left(x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)\,|_{p=1\sim\eta}, k\right) \\ &\quad + g_i\left(x_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)\,|_{p=1\sim\eta}, k, u(k)\right), \\ &\qquad i = 1, 2, ..., \eta\ , \end{aligned} \tag{55}$$

respectively.

Parallel to Definition 4.1, we can also give the following one.

Definition 4.3. Let $\mathbb{F} \subset \mathbb{R}^{\varkappa}$ be the largest set such that the following mapping

$$\tilde{u} = g\left(X, k, u\right) \tag{56}$$

forms a differential homeomorphism from u to $\tilde{u}$ for all $X \in \mathbb{F}$, and $k \geq 0$. Then the set $\mathbb{F}$ is called the set of feasible points of system (51) or (52), and any $X \in \mathbb{F}$ is called a feasible point of system (51) or (52). Furthermore, the set

$$\mathbb{S} = \mathbb{R}^{\varkappa} \backslash \mathbb{F}$$

is called the set of singular points of system (51) or (52), and any $X \in \mathbb{S}$ is called a singular point of system (51) or (52).

With $\mathbb{F}$ and $\mathbb{S}$ well-defined above, the definitions of full-actuation of system (51) or (52) can be immediately given simply by replacing the system (39) or (40) in Definition 4.2 by system (51) or (52).

Similarly, when the system (51) or (52) is globally fully actuated, we have a differential homeomorphism from u to $\tilde{u}$, $\forall X \in \mathbb{R}^{\varkappa}$, and $k \geq 0$, as

$$\tilde{u} = g\left(X, k, u\right).$$

In this case, the set of systems in (55) can be also written as in (49).

To conclude this section, let us finally point out that the definition of full-actuation can be slightly modified to define over-actuated systems. In the high-order system representations, over-actuated systems may be occasionally encountered. Such systems can be similarly treated, to an extent, as fully actuated ones in terms of control (refer to the Remark 2.1 in Duan (2020d)).

5. Controller designs

5.1. Step backward HOFA models

The following important fact about the control of the step backward HOFA system (40) or (43) can be easily verified.

Theorem 5.1. *Let $\mathbb{F} \subset \mathbb{R}^{\varkappa}$ be the set of feasible points of the system (40), whose dimension is greater than 1. Further, let $[A_i]_{0\sim\mu_i-1} \in \mathbb{R}^{r_i \times \mu_i r_i}$, $i = 1, 2, \cdots, \eta$, be a set of arbitrarily given matrices. Then the following controller*

$$\left\{\begin{array}{l} u\left(k\right) = -B^{-1}\left(\cdot\right)\left(\left[\begin{array}{c} [A_1]_{0\sim\mu_1-1}\, x_1^{\lceil 0\sim\mu_1-1\rceil}\left(k\right) \\ [A_2]_{0\sim\mu_2-1}\, x_2^{\lceil 0\sim\mu_2-1\rceil}\left(k\right) \\ \vdots \\ [A_\eta]_{0\sim\mu_\eta-1}\, x_\eta^{\lceil 0\sim\mu_\eta-1\rceil}\left(k\right) \end{array}\right] + u^*\left(k\right)\right) \\ u^*\left(k\right) = f\left(x_p^{\lceil 0\sim\mu_p-1\rceil}\left(k\right)|_{p=1\sim\eta}, k\right) - v\left(k\right), \end{array}\right. \tag{57}$$

for system (40) produces the following constant linear closed-loop system

$$\begin{aligned} x_p(k+1) &= -[A_p]_{0\sim\mu_p-1} x_p^{\lceil 0\sim\mu_p-1\rceil}(k) + v_p, \\ & p = 1,2,...,\eta \ , \end{aligned} \tag{58}$$

with v_p, $p = 1,2,...,\eta$, being defined by

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\eta \end{bmatrix}, \quad v_p \in \mathbb{R}^{r_p}, \tag{59}$$

provided that the states of the above closed-loop system (58) satisfy

$$x_p^{\lceil 0\sim\mu_p-1\rceil}|_{p=1\sim\eta} \in \mathbb{F}. \tag{60}$$

Define

$$A_E = \text{blockdiag}\left([A_p]_{0\sim\mu_p-1}, p = 1,2,...,\eta\right), \tag{61}$$

then, obviously, the controller (57) can be more compactly written as

$$\begin{cases} u(k) = -B^{-1}(\cdot)\left(A_E x_p^{\lceil 0\sim\mu_p-1\rceil}(k)|_{p=1\sim\eta} + u^*(k)\right) \\ u^*(k) = f\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)|_{p=1\sim\eta}, k\right) - v(k). \end{cases} \tag{62}$$

Furthermore, if written in state-space form, the closed-loop system (58) can be written as

$$\begin{aligned} x_p^{\lceil 0\sim\mu_p-1\rceil}(k+1) &= \Phi\left([A_p]_{0\sim\mu_p-1}\right) x_p^{\lceil 0\sim\mu_p-1\rceil}(k) + \Gamma_{cp} v_p, \\ & p = 1,2,...,\eta \ , \end{aligned} \tag{63}$$

where

$$\Gamma_{cp} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{r_p} \end{bmatrix}, \quad p = 1,2,...,\eta. \tag{64}$$

Remark 2. For the more general non-affine step backward HOFA system (51), a result similar to Theorem 5.1 still holds. The controller can be obtained using the differential homeomorphism property of the mapping $\tilde{u} = g\left(x_p^{\lceil 0\sim\mu_p-1\rceil}|_{p=1\sim\eta}, k, u\right)$, as

$$\begin{cases} u(k) = g^{-1}\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, k, \tilde{u}(k)\right) \\ \tilde{u}(k) = -\left(\begin{bmatrix} [A_1]_{0\sim\mu_1-1}\, x_1^{\lceil 0\sim\mu_1-1\rceil}(k) \\ [A_2]_{0\sim\mu_2-1}\, x_2^{\lceil 0\sim\mu_2-1\rceil}(k) \\ \vdots \\ [A_\eta]_{0\sim\mu_\eta-1}\, x_\eta^{\lceil 0\sim\mu_\eta-1\rceil}(k) \end{bmatrix} + u^*(k)\right) \\ u^*(k) = f\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, k\right) - v(k), \end{cases} \tag{65}$$

and the same closed-loop system as in (58) is achieved.

5.2. Step forward HOFA models

Now let us consider the control of the general step forward HOFA system (39). Without loss of generality, let us remove the variable k in the functions $f(\cdot)$ and $B(\cdot)$, and then the system appears as

$$\begin{bmatrix} x_1^{\lfloor \mu_1-1\rfloor}(k+1) \\ x_2^{\lfloor \mu_2-1\rfloor}(k+1) \\ \vdots \\ x_\eta^{\lfloor \mu_\eta-1\rfloor}(k+1) \end{bmatrix} = \begin{bmatrix} f_1\left(x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}\right) \\ f_2\left(x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}\right) \\ \vdots \\ f_\eta\left(x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}\right) \end{bmatrix} + B\left(x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}\right)u(k), \tag{66}$$

or, written compactly, as

$$\begin{aligned} x_p^{\lfloor \mu_p-1\rfloor}(k+1)\,|_{p=1\sim\eta} &= f\left(x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}\right) \\ &\quad + B\left(x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}\right)u(k). \end{aligned} \tag{67}$$

If $x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}$ is measurable (see, Remark 1), a controller for the above system can be chosen as

$$\begin{cases} u(k) = -B^{-1}(\cdot)\left(\begin{bmatrix} [A_1]_{0\sim\mu_1-1}\, x_1^{\lfloor 0\sim\mu_1-1\rfloor}(k) \\ [A_2]_{0\sim\mu_2-1}\, x_2^{\lfloor 0\sim\mu_2-1\rfloor}(k) \\ \vdots \\ [A_\eta]_{0\sim\mu_\eta-1}\, x_\eta^{\lfloor 0\sim\mu_\eta-1\rfloor}(k) \end{bmatrix} + u^*(k)\right) \\ u^*(k) = f\left(x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\,|_{p=1\sim\eta}\right), \end{cases} \tag{68}$$

or, equivalently, as

$$\begin{cases} u(k) = -B^{-1}(\cdot)\left(A_E x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\mid_{p=1\sim\eta} + u^*(k)\right) \\ u^*(k) = f\left(x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)\mid_{p=1\sim\eta}\right), \end{cases} \tag{69}$$

and the closed-loop system is ideally obtained as

$$\begin{aligned} x_p^{\lfloor \mu_p-1\rfloor}(k+1) &= -\left[A_p\right]_{0\sim\mu_p-1} x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k), \\ & \quad p = 1, 2, ..., \eta \ , \end{aligned} \tag{70}$$

which can be written in a state-space form as

$$\begin{aligned} x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k+1) &= \Phi\left(\left[A_p\right]_{0\sim\mu_p-1}\right) x_p^{\lfloor 0\sim\mu_p-1\rfloor}(k), \\ & \quad p = 1, 2, ..., \eta \ . \end{aligned} \tag{71}$$

Clearly, the series of matrices $\left[A_p\right]_{0\sim\mu_p-1}$, $p = 1, 2, ..., \eta$, should be so chosen that the state-space systems in (71) are asymptotically stable.

Nevertheless, when the HOFA system (66) is treated as an independent physical system, the above controller (68) depends on future states of the system, and is often not realizable. To make the controller realizable, we need to give the prediction of $x_p^{\lfloor 1\sim\mu_p-1\rfloor}(k)\mid_{p=1\sim\eta}$. This can be done through two ways, one is to build the prediction using the open-loop system (66), and the other is to give the prediction based on the closed-loop system (71). For paper length limitation, in the following we only consider the case of $\eta = 1$, that is, the HOFA system (36), the general case will be addressed in a separete paper.

The HOFA system (36), with the variable k in the functions $f(\cdot)$ and $B(\cdot)$ removed, appears as

$$x^{\lfloor n-1\rfloor}(k+1) = f\left(x^{\lfloor 0\sim n-1\rfloor}(k)\right) + B\left(x^{\lfloor 0\sim n-1\rfloor}(k)\right) u(k). \tag{72}$$

For the special system (72), the following result gives the prediction of $x^{\lfloor 0\sim n-1\rfloor}(k)$.

Proposition 5.2. *For the HOFA system (72), there exist a series of nonlinear functions $h_i(\cdot,\cdot) \in \mathbb{R}^r, i = 1, 2, ..., n-1$, such that*

$$\begin{aligned} x(k+i) &= h_i\left(x^{\lceil 0\sim n-1\rceil}, u^{\lfloor 1\sim i\rfloor}(k-n)\right), \\ i &= 1, 2, ..., n-1. \end{aligned} \tag{73}$$

$$\begin{aligned} x(k+i) &= h_i\left(x^{\lceil 0\sim n-1\rceil}, u^{\lceil n-i\sim n-1\rceil}(k)\right), \\ i &= 1, 2, ..., n-1. \end{aligned} \tag{74}$$

Proof. Following the system (72), we have

$$\begin{aligned} x(k+1) &= f\left(x^{\lfloor 0\sim n-1\rfloor}(k-(n-1))\right) \\ &\quad +B\left(x^{\lfloor 0\sim n-1\rfloor}(k-(n-1))\right)u(k-(n-1)) \\ &\triangleq h_1\left(x^{\lceil 0\sim n-1\rceil}(k), u^{\lceil n-1\rceil}(k)\right). \end{aligned} \tag{75}$$

Using (72) again, gives

$$\begin{aligned} x(k+2) &= f\left(x^{\lfloor 0\sim n-1\rfloor}(k-(n-2))\right) \\ &\quad +B\left(x^{\lfloor 0\sim n-1\rfloor}(k-(n-2))\right)u(k-(n-2)) \\ &= f\left(x(k+1), x^{\lceil 0\sim n-2\rceil}(k)\right) \\ &\quad +B\left(x(k+1), x^{\lceil 0\sim n-2\rceil}(k)\right)u(k-(n-2)). \end{aligned} \tag{76}$$

Further, substituting (75) into (76), gives

$$x(k+2) \triangleq h_2\left(x^{\lceil 0\sim n-1\rceil}, u^{\lceil n-2\sim n-1\rceil}(k)\right).$$

Continuing this process, yields the set of equations in (74). The proof is finished. □

Based on the above result, we can give the following result about control of system (72).

Theorem 5.3. *Let $\mathbb{F} \subset \mathbb{R}^{nr}$ be the set of feasible points of the system (72), whose dimension is greater than 1, and $h_i(\cdot), i = 1, 2, ..., n-1$ be the set of functions determined by (74). Further, let $A_{0\sim n-1} \in \mathbb{R}^{r\times nr}$ be an arbitrarily given matrix. Then the following controller*

$$\begin{cases} u(k) = -B^{-1}(\cdot)\left(A_{0\sim n-1}x^{\lfloor 0\sim n-1\rfloor}(k) + u^*(k)\right) \\ u^*(k) = f\left(x^{\lfloor 0\sim n-1\rfloor}(k)\right) - v(k) \\ x(k+i) = h_i\left(x^{\lceil 0\sim n-1\rceil}, u^{\lceil n-i\sim n-1\rceil}(k)\right) \\ \qquad i = 1, 2, ..., n-1, \end{cases} \tag{77}$$

for system (72) produces the constant linear closed-loop system

$$x^{\lfloor n-1\rfloor}(k+1) = -A_{0\sim n-1}x^{\lfloor 0\sim n-1\rfloor}(k) + v(k), \tag{78}$$

povided that the states of the above closed-loop system satisfy the following feasibility requirement

$$x^{\lfloor 0\sim n-1\rfloor}(k) \in \mathbb{F}, \forall k \geq 0. \tag{79}$$

To finish this section, let us make a few remarks.

Remark 3. Note that

$$\Phi\left(A_{0\sim n-1}\right)=\Phi\left(0\right)-\Gamma_{c}A_{0\sim n-1},$$

where

$$\Phi\left(0\right)=\begin{bmatrix}0 & I_r & & \\ 0 & \ddots & \ddots & \\ \vdots & \vdots & \ddots & I_r \\ 0 & 0 & \cdots & 0\end{bmatrix},\quad \Gamma_c=\begin{bmatrix}0\\ \vdots\\ 0\\ I_r\end{bmatrix},\tag{80}$$

and $[\Phi\left(0\right),\Gamma_c]$ is controllable, the solution to the feedback gains in the proposed controllers really reduces to a problem of finding a real matrix K such that $A+BK$ is Schur for some controllable matrix pair $[A,B]$. This can be completely solved according to the Lemma 5.2 in Duan (2021c) (see also Duan (1992, 1993, 1998, 2004, 2005)), where the matrix F needs to be chosen Schur. Particularly, to find an $A_{0\sim n-1}$ to make $\Phi\left(A_{0\sim n-1}\right)$ Schur, the Theorem 3.4 in Duan (2021b) can be readily used, with the matrix F chosen Schur.

Remark 4. Response analysis and stability analysis are two very fundamental and important aspects of system analysis. Generally speaking, these two problems are very hard to treat with the state-space approaches for general nonlinear dynamical control systems. However, as it is clearly seen in the above Theorems 5.1 and 5.3, with the proposed HOFA approaches these two problems really vanish since they are both converted into corresponding problems for linear systems. Bearing in mind the important rule that system analysis is often developed for system designs, we really do not need to provide response analysis and stability analysis results for the open-loop nonlinear system (39) and (40).

Remark 5. It is again clearly observed in the above Theorems 5.1 and 5.3 that the linear closed-loop systems are obtained with desired coefficient matrices, or equivalently, with desired eigenstructures. Such a property of HOFA systems totally reveals the meaning of controllability of dynamical control systems. Therefore, following the general line in Duan (2021c), controllability and stabilizability of general discrete-time systems can be also readily defined. Specifically, a dynamical system is called controllable if it can be represented by either a step forward or a step backward HOFA model. While a stabilizable dynamical system can be represented by either a step forward or a step backward HOFA model, together with a stable autonomous dynamical system.

Remark 6. The feasibility requirements (60) and (79) are essential for general HOFA systems. They vanish only for complete fully actuated systems. For continuous-time HOFA systems, two types of feasibility conditions are provided, one is dependent on a Lyapunov matrix equation (see the Lemma 5.2 in Duan (2021d)), the other is based on the closed-loop eigenstructure (see the Lemma 6.1 in Duan (2021e)). Both types of conditions turn out to be constraints on the system initial values, and can be both parallelly extended to the discrete-time system case.

6. Feedback linearizable systems and strict-feedback systems

As examples of the proposed discrete-time step forward HOFA systems and discrete-time step backward HOFA systems, in this section we treat two types of state-space systems, namely, the feedback linearizable systems and the strict-feedback systems.

6.1. Feedback linearizable systems

Consider the discrete-time control system

$$x(k+1) = f(x(k), u(k)), \tag{81}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^r$, $f(\cdot)$ is a sufficiently differentiable function, and $f(x_0, u_0) = x_0$. We are interested in the problem when system (81) is locally equivalent to a controllable linear system (Jakubczyk, 1987).

Definition 6.1. The system (81) is called locally feedback linearizable at (x_0, u_0) via the differential feedback:

$$u = \xi(x, v), \; \xi(x_0, 0) = u_0, \; \operatorname{rank} \frac{\partial}{\partial v} \xi(x_0, 0) = r, \tag{82}$$

if there exists a sufficiently differentiable transformation of coordinates in the state space:

$$x = T_0(y), \; T_0(0) = x_0, \; \operatorname{rank} \frac{\partial}{\partial y} T_0(0) = n; \tag{83}$$

such that the system (81) is equivalently transformed, under the control transformation (82) and the state transformation (83), into the form of

$$y(k+1) = Ay(k) + Bv(k), \; y(k) \in \mathbb{R}^n, \; v(k) \in \mathbb{R}^r, \tag{84}$$

with $[A, B]$ controllable.

The following result indicates that feedback linearizable systems can be represented by a HOFA model.

Proposition 6.2. *Let system (81) be locally feedback linearizable at (x_0, u_0) via the feedback (82). Then the system (81) is equivalent to both a step forward HOFA system in the form of (51) within a neighborhood of (x_0, u_0), and a step backward HOFA system in the form of (52) within a neighborhood of (x_0, u_0).*

Proof. Since system (81) is locally feedback linearizable at (x_0, u_0) via the feedback (82), by definition, there exists a neighborhood of (x_0, u_0), denoted by $\Omega(x_0, u_0)$, such that the system is equivalent to the linear controllable system (84) within $\Omega(x_0, u_0)$.

On the other side, it follows from Theorem 3.3 that there exists a one-to-one transformation

$$y = T_1\left(z_p^{\lfloor 0 \sim \mu_p - 1 \rfloor}|_{p=1\sim\eta}\right), \tag{85}$$

under which the linear system (84) is equivalent to the following step forward HOFA model:

$$z_p^{\lfloor \mu_p - 1 \rfloor}(k+1)\,|_{p=1\sim\eta} = L\left(z_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}\right) + \hat{B}v(k), \tag{86}$$

where $L(\cdot) \in \mathbb{R}^r$ is a linear function, and $\hat{B} \in \mathbb{R}^{r\times r}$ is a nonsingular matrix.

Next, it follows from (82) that the inverse of $\xi(x, v)$ with respect to v locally exists, that is,

$$v = \xi^{-1}(x, u), \quad (x, u) \in \Omega(x_0, u_0). \tag{87}$$

Using the transformation (83), turns the above equation into

$$v = \xi^{-1}(T_0(y), u), \quad (y, u) \in \Omega(0, u_0). \tag{88}$$

Further applying the transformation T_1 in (85), gives

$$\begin{aligned} v \;=\; & \xi^{-1}\left(T_0\left(T_1\left(z_p^{\lfloor 0\sim\mu_p - 1 \rfloor}|_{p=1\sim\eta}\right)\right), u\right), \\ & \left(z_p^{\lfloor 0\sim\mu_p - 1 \rfloor}|_{p=1\sim\eta}, u\right) \in \Omega(0, u_0). \end{aligned} \tag{89}$$

Finally, substituting the above equation into (86), produces

$$\begin{aligned} z_p^{\lfloor \mu_p - 1 \rfloor}(k+1)\,|_{p=1\sim\eta} \;=\; & L\left(z_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}\right) \\ & + \hat{g}\left(z_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}, u(k)\right), \end{aligned} \tag{90}$$

where

$$\hat{g}(\cdot) = \hat{B}\xi^{-1}\left(T_0\left(T_1\left(z_p^{\lfloor 0\sim\mu_p - 1 \rfloor}(k)\,|_{p=1\sim\eta}\right)\right), u(k)\right).$$

It is easily recognized that, on $\Omega(0, u_0)$, the function $\hat{g}(\cdot)$ is one-to-one with respect to $u(k)$. Therefore, the above system (90) is clearly a step forward HOFA model in the form of (51).

Again, it follows from Theorem 3.3 that there exists a one-to-one transformation

$$y = \tilde{T}_1\left(z_p^{\lceil 0\sim\mu_p - 1 \rceil}|_{p=1\sim\eta}\right), \tag{91}$$

under which the linear system (84) is equivalent to the following step backward HOFA model

$$z_{1\sim\eta}(k+1) = \tilde{L}\left(z_p^{\lceil 0\sim\mu_p - 1 \rceil}(k)\,|_{p=1\sim\eta}\right) + \tilde{B}v(k), \tag{92}$$

where $\tilde{L}(\cdot) \in \mathbb{R}^r$ is a linear function, and $\tilde{B} \in \mathbb{R}^{r\times r}$ is a nonsingular matrix. By a similar procedure, it can be shown that the above system (92) is equivalent to a step backward HOFA model in the form of (52). The proof is then completed. □

Particularly, when the system (81) is locally feedback linearizable via the following feedback

$$u = \xi(x, v) = \alpha(x) + \beta(x) v, \tag{93}$$

where $\alpha(x)$ and $\beta(x)$ satisfy

$$\alpha(x_0) = u_0, \ \operatorname{rank}\beta(x) = r. \tag{94}$$

Then we have, instead of (87), the following

$$v = \beta^{-1}(x) u - \beta^{-1}(x) \alpha(x), \quad (x, u) \in \Omega(x_0, u_0). \tag{95}$$

Further, using the transformations T_0 and T_1, we have, for the step forward case,

$$\begin{aligned} v &= \beta^{-1}\left(T_0\left(T_1\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}|_{p=1\sim\eta}\right)\right)\right) u \\ &\quad -\beta^{-1}\left(T_0\left(T_1\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}|_{p=1\sim\eta}\right)\right)\right) \\ &\quad \times\alpha\left(T_0\left(T_1\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}|_{p=1\sim\eta}\right)\right)\right), \end{aligned} \tag{96}$$

for $\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}|_{p=1\sim\eta}, u\right) \in \Omega(0, u_0)$. Substituting the above (96) into (86), produces

$$\begin{aligned} z_p^{\lfloor \mu_p-1\rfloor}(k+1)|_{p=1\sim\eta} &= \breve{f}\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)|_{p=1\sim\eta}\right) \\ &\quad +\breve{B}\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)|_{p=1\sim\eta}\right) u(k), \end{aligned} \tag{97}$$

where

$$\begin{aligned} \breve{f}(\cdot) &= L\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}(k)|_{p=1\sim\eta}\right) \\ &\quad -\hat{B}\beta^{-1}\left(T_0\left(T_1\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}|_{p=1\sim\eta}\right)\right)\right) \\ &\quad \times\alpha\left(T_0\left(T_1\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}|_{p=1\sim\eta}\right)\right)\right), \end{aligned}$$

and

$$\breve{B}(\cdot) = \hat{B}\beta^{-1}\left(T_0\left(T_1\left(z_p^{\lfloor 0\sim\mu_p-1\rfloor}|_{p=1\sim\eta}\right)\right)\right).$$

The case of step backward systems can be carried out similarly.

The above process obviously proves the following result.

Corollary 6.3. *Let system (81) be locally feedback linearizable at* (x_0, u_0) *via the feedback (93)-(94). Then the system (81) is equivalent to both a step forward HOFA system in the affine form of (39) within a neighborhood of* (x_0, u_0)*, and a step backward HOFA system in the affine form of (40) within a neighborhood of* (x_0, u_0)*.*

6.2. Strict feedback systems

Parallel to the continuous-time strict-feedback systems (see, Duan (2020e)), the discrete-time strict-feedback systems possess the following general form (Ge, Li, & Lee, 2003; Ge, Yang, Dai, Jiao, & Lee, 2009; Ge, Yang, & Lee, 2008; Xu, Liu, Wang, & Zhou, 2021):

$$\begin{cases} x_1(k+1) = f_1(x_1(k)) + g_1(x_1(k))\, x_2(k) \\ x_2(k+1) = f_2(x_{1\sim 2}(k)) + g_2(x_{1\sim 2}(k))\, x_3(k) \\ \qquad\quad \vdots \\ x_{n-1}(k+1) = f_{n-1}(x_{1\sim n-1}(k)) + g_{n-1}(x_{1\sim n-1}(k))\, x_n(k) \\ x_n(k+1) = f_n(x_{1\sim n}(k)) + g_n(x_{1\sim n}(k))\, u(k), \end{cases} \tag{98}$$

where $u, x_i \in \mathbb{R}^r$, $i = 1, 2, ..., n$, are the system input vector and state vectors, respectively, $f_i(\cdot) \in \mathbb{R}^r$, and $g_i(\cdot) \in \mathbb{R}^{r\times r}$, $i = 1, 2, ..., n$, are continuous functions, and particularly, $g_i(\cdot)$ satisfies the following assumption:

Assumption A2 $\det g_i(x_{1\sim i}) \neq 0, \quad \forall x_{1\sim i} \in \mathbb{R}^{ir}, i = 1, 2, ..., n.$

Proposition 6.4. *The above system (98) with Assumption A2 satisfied can be equivalently converted into both a step forward HOFA system in the form of (36), and a step backward HOFA system in the form of (37).*

Proof. Let us first consider the case of $n = 2$, where the system is

$$\begin{cases} x_1(k+1) = f_1(x_1(k)) + g_1(x_1(k))\, x_2(k) \\ x_2(k+1) = f_2(x_{1\sim 2}(k)) + g_2(x_{1\sim 2}(k))\, u(k). \end{cases} \tag{99}$$

From the first equation, we get

$$x_2(k) = g_1^{-1}(x_1(k))\left[x_1(k+1) - f_1(x_1(k))\right].$$

Substituting this into the second equation in (99), gives

$$x_2(k+1) = \tilde{f}_2\left(x_1^{\lfloor 0\sim 1\rfloor}(k)\right) + \tilde{g}_2\left(x_1^{\lfloor 0\sim 1\rfloor}(k)\right) u(k), \tag{100}$$

where $\tilde{f}_2(\cdot)$ and $\tilde{g}_2(\cdot)$ are some nonlinear functions.

Next, using (99) again, yields

$$x_1(k+2) = f_1(x_1(k+1)) + g_1(x_1(k+1))\, x_2(k+1).$$

Substituting (100) into the above equation, produces

$$\begin{aligned} x_1(k+2) \;\; = \;\; & f_1(x_1(k+1)) + g_1(x_1(k+1))\, \tilde{f}_2\left(x_1^{\lfloor 0\sim 1\rfloor}(k)\right) \\ & + g_1(x_1(k+1))\, \tilde{g}_2\left(x_1^{\lfloor 0\sim 1\rfloor}(k)\right) u(k), \end{aligned}$$

which can be written as

$$x^{\lfloor 1\rfloor}(k+1) = f\left(x^{\lfloor 0\sim 1\rfloor}(k)\right) + B\left(x^{\lfloor 0\sim 1\rfloor}(k)\right) u(k), \tag{101}$$

where

$$\begin{aligned} f\left(x^{\lfloor 0\sim 1\rfloor}(k)\right) &= f_1\left(x_1(k+1)\right)+g_1\left(x_1(k+1)\right)\tilde{f}_2\left(x_1^{\lfloor 0\sim 1\rfloor}(k)\right), \\ B\left(x^{\lfloor 0\sim 1\rfloor}(k)\right) &= g_1\left(x_1(k+1)\right)\tilde{g}_2\left(x_1^{\lfloor 0\sim 1\rfloor}(k)\right). \end{aligned}$$

This is obviously in the form of (36) with $n=2$.

For the case of $n>2$, the proof can be also fulfilled with the help of the method of mathematical induction, as done for the continuous-time case in Duan (2020e).

To show that system (98) can be equivalently converted into a step backward HOFA system in the form of (37), without loss of generality, let us treat the case of $n=3$, in which the system is

$$\begin{cases} x_1(k+1)=f_1(x_1(k))+g_1(x_1(k))x_2(k), \\ x_2(k+1)=f_2(x_{1\sim 2}(k))+g_2(x_{1\sim 2}(k))x_3(k), \\ x_3(k+1)=f_3(x_{1\sim 3}(k))+g_3(x_{1\sim 3}(k))u(k). \end{cases} \tag{102}$$

From the first equation in (102) we have

$$\begin{aligned} x_1(k+1) &= f_1(x_1(k))+g_1(x_1(k))x_2(k) \\ &\triangleq \overline{x}_2(k), \end{aligned} \tag{103}$$

by which and the second equation of (102), we can obtain

$$\begin{aligned} & \overline{x}_2(k+1) \\ =\ & f_1(x_1(k+1))+g_1(x_1(k+1))x_2(k+1) \\ =\ & f_1(x_1(k+1)) \\ & +g_1(x_1(k+1))\left(f_2(x_{1\sim 2}(k))+g_2(x_{1\sim 2}(k))x_3(k)\right) \\ \triangleq\ & \overline{f}_2\left(x_{1\sim 2}(k)\right)+\overline{g}_2(x_{1\sim 2}(k))x_3(k) \\ \triangleq\ & \overline{x}_3(k), \end{aligned} \tag{104}$$

where $\overline{g}_2(\cdot)=g_1g_2$ is clearly nonsingular.

Similarly, from the third equation in (102) we get

$$\begin{aligned} & \overline{x}_3(k+1) \\ =\ & \overline{f}_2\left(x_{1\sim 2}(k+1)\right)+\overline{g}_2(x_{1\sim 2}(k+1))x_3(k+1) \\ =\ & \overline{f}_2\left(x_{1\sim 2}(k+1)\right) \\ & +\overline{g}_2(x_{1\sim 2}(k+1))\left(f_3(x_{1\sim 3}(k))+g_3(x_{1\sim 3}(k))u(k)\right) \\ \triangleq\ & \overline{f}_3\left(x_{1\sim 3}(k)\right)+\overline{g}_3(x_{1\sim 3}(k))u(k), \end{aligned} \tag{105}$$

where $\overline{g}_3(\cdot)=\overline{g}_2g_3$ is also nonsingular.

Notice that

$$\begin{bmatrix} \overline{x}_1(k) \\ \overline{x}_2(k) \\ \overline{x}_3(k) \end{bmatrix} = \begin{bmatrix} x_1(k) \\ f_1(x_1(k))+g_1(x_1(k))x_2(k) \\ \overline{f}_2\left(x_{1\sim 2}(k)\right)+\overline{g}_2(x_{1\sim 2}(k))x_3(k) \end{bmatrix},$$

from which we can solve recursively

$$\begin{aligned}
x_1(k) &= \overline{x}_1(k), \\
x_2(k) &= g_1^{-1}(\overline{x}_1(k))\left(\overline{x}_2(k) - f_1(\overline{x}_1(k))\right) \\
&\triangleq h_2\left(\overline{x}_{1\sim 2}(k)\right), \\
x_3(k) &= \overline{g}_2^{-1}(x_{1\sim 2}(k))\left(\overline{x}_3(k) - \overline{f}_2\left(x_{1\sim 2}(k)\right)\right) \\
&\triangleq h_3\left(\overline{x}_{1\sim 3}(k)\right).
\end{aligned}$$

Inserting the above into (105) and copying (103)-(104), give the following system (which is equivalent to (102))

$$\begin{cases}
\overline{x}_1(k+1) &= \overline{x}_2(k), \\
\overline{x}_2(k+1) &= \overline{x}_3(k), \\
\overline{x}_3(k+1) &= \widetilde{f}_3\left(\overline{x}_{1\sim 3}(k)\right) + \widetilde{g}_3(\overline{x}_{1\sim 3}(k))u(k),
\end{cases} \tag{106}$$

where $\widetilde{g}_3(\cdot)$ is nonzero.

Finally, inserting the first two equations of (106) into the third one, gives the backward HOFA system

$$\begin{aligned}
\overline{x}_3(k+1) \quad = \quad & \widetilde{f}_3\left(\overline{x}_3(k-2), \overline{x}_3(k-1), \overline{x}_3(k)\right) \\
& + \widetilde{g}_3\left(\overline{x}_3(k-2), \overline{x}_3(k-1), \overline{x}_3(k)\right)u(k).
\end{aligned}$$

The proof is then complete. □

The importance of the above result lies in that, once a strict-feedback system is converted into a HOFA model, the system can be easily controlled in view of Theorems 5.1 and 5.3, with the closed-loop system being a constant linear one with an arbitrarily assignable eigenstructure. It should be noted that such an advantage is not generally achievable with the well-known method of backstepping. We point out that the converted HOFA model is a complete fully actuated one due to Assumption A2. Therefore, the thorny problem of singularity vanishes.

To finish this subsection, let us make some further remarks.

Remark 7. As treated in Duan (2020e), the strict-feedback system (98) can be also generalized to the second-order case and the mixed-order case. Particularly, a second-order strict-feedback system in step backward form can be represented as follows:

$$\begin{cases}
x_1(k+1) = f_1\left(x_1^{\lceil 0\sim 1\rceil}(k)\right) + g_1\left(x_1^{\lceil 0\sim 1\rceil}(k)\right)x_2(k) \\
x_2(k+1) = f_2\left(x_{1\sim 2}^{\lceil 0\sim 1\rceil}(k)\right) + g_2\left(x_{1\sim 2}^{\lceil 0\sim 1\rceil}(k)\right)x_3(k) \\
\qquad\vdots \\
x_{n-1}(k+1) = f_{n-1}\left(x_{1\sim n-1}^{\lceil 0\sim 1\rceil}(k)\right) + g_{n-1}\left(x_{1\sim n-1}^{\lceil 0\sim 1\rceil}(k)\right)x_n(k) \\
x_n(k+1) = f_n\left(x_{1\sim n}^{\lceil 0\sim 1\rceil}(k)\right) + g_n\left(x_{1\sim n}^{\lceil 0\sim 1\rceil}(k)\right)u(k),
\end{cases} \tag{107}$$

which can be also shown to be representable by a HOFA model. Moreover, as in Duan (2020e), a second-order backstepping design for the above second-order strict-feedback system can be also proposed.

Remark 8. It follows from Theorem 3.3 that a discrete-time controllable linear system can be equivalently converted into both a step forward and a step backward HOFA system. Now in this section, it happens that both the feedback linearizable systems and strict-feedback systems can be equivalently converted into both step forward and step backward HOFA systems. It is clearly seen from Section 5 that a step backward model is much more preferable since for which a simple static feedback controller is sufficient. Furthermore, the control strategy for step backward systems can be easily generalized to systems with disturbances and uncertainties, as done in the continuous-time case, while in general the one for a step forward system can not since the predictions of needed states can be not longer accurately obtained in the presence of disturbances or uncertainties.

7. Further generalizations

In Remark 1 we pointed out that, for controllable linear systems, a general input-output representation (22) or (23) can be equivalently converted into both a step forward and a step backward HOFA one. However, for nonlinear systems, the problem may be more complicated. Due to this consideration, in this section we extend our HOFA models to include the effect of the control vector at more time instants. Due to paper length, here only the backward model is considered.

7.1. Generalized HOFA models

As a further generalization of the discrete-time step backward HOFA system (40), we can also propose the following discrete-time step backward HOFA system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_\eta(k+1) \end{bmatrix} = \begin{bmatrix} f_1\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m_1\rceil}(k), k\right) \\ f_2\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m_2\rceil}(k), k\right) \\ \vdots \\ f_\eta\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m_\eta\rceil}(k), k\right) \end{bmatrix} + B\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m_\eta\rceil}(k), k\right)u(k), \quad (108)$$

where m_p, $p = 1, 2, ..., \eta$, are a set of integers, while the other variables are as stated before. Remember that the matrix function $B(\cdot) \in \mathbb{R}^{r\times r}$ is nonsingular for some of its variables.

If we denote

$$f(\cdot,\cdot,k) = \begin{bmatrix} f_1(\cdot,\cdot,k) \\ f_2(\cdot,\cdot,k) \\ \vdots \\ f_\eta(\cdot,\cdot,k) \end{bmatrix}, \quad (109)$$

then the step backward HOFA system (108) can be compactly written as

$$\begin{aligned} x_{1\sim\eta}(k+1) &= f\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}(k), k\right) \\ &\quad + B\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}(k), k\right) u(k). \end{aligned} \tag{110}$$

where

$$m = \max(m_i, i = 1, 2, ..., \eta).$$

For this extended HOFA system (110), Definition 4.1 becomes the following.

Definition 7.1. If $X \in \mathbb{R}^{\varkappa}$ and $U \in \mathbb{R}^{mr}$ satisfy

$$\det B(X, U, k) = 0 \text{ or } \infty, \exists k \geq 0, \tag{111}$$

then (X, U) is called a singular point of system (108).

Let $\mathbb{S}$ be the set of all singular points of system (108), that is,

$$\mathbb{S} = \{(X, U) |\, (111) \text{ holds}\}.$$

Then we call

$$\mathbb{F} = \mathbb{R}^{\varkappa} \backslash \mathbb{S}$$

the set of feasible points of system (108). The system (108) is generally said to be HOFA if $\mathbb{F}$ is a set with dimension greater than 1. A strict definition can be given parallel to Definition 4.2.

In the case of $\mathbb{S} = \varnothing$, we have

$$\det B(X, U, k) \neq 0 \text{ or } \infty, \forall X \in \mathbb{R}^{\varkappa}, U \in \mathbb{R}^{mr}, k \geq 0. \tag{112}$$

Therefore, we can introduce the control vector transformation

$$B\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}(k), k\right) u(k) = \tilde{u}(k),$$

and now the (globally) fully actuated system (108) can be written as

$$\begin{aligned} x_i(k+1) &= f_i\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m_i\rceil}(k), k\right) + \tilde{u}_i(k), \\ & \qquad p = 1, 2, ..., \eta\ , \end{aligned} \tag{113}$$

where $\tilde{u}_i, i = 1, 2, ..., \eta$, are defined by

$$\tilde{u} = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_\eta \end{bmatrix}, \quad \tilde{u}_i \in \mathbb{R}^{r_i}. \tag{114}$$

For the control of the extended step backward HOFA system (108) or (110), a straight forward extension of Theorem 5.1 can be given as follows.

Theorem 7.2. *Let $\mathbb{F} \subset \mathbb{R}^{\varkappa+mr}$ be the set of feasible points of the system (108), whose dimension is greater than 1. Further, let $K_{1\sim\eta} \in \mathbb{R}^{r\times\varkappa}$ be a designed matrix. Then the following controller*

$$\begin{cases} u(k) = -B^{-1}(\cdot)\left(K_{1\sim\eta} x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta} + u^*(k)\right) \\ u^*(k) = f\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}(k), k\right) - v(k), \end{cases} \tag{115}$$

for system (108) produces the following constant linear closed-loop system

$$x_p(k+1)\,|_{p=1\sim\eta} = -K_{1\sim\eta} x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta} + v(k), \tag{116}$$

provided that the following feasibility requirement is satisfied:

$$\left(x_p^{\lceil 0\sim\mu_p-1\rceil}|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}(k)\right) \in \mathbb{F}. \tag{117}$$

As a linear system, the closed-loop system (116) can be easily made stable by properly selecting the feedback gain $K_{1\sim\eta}$. For a systematic method, one can refer to Duan (2021c). Particularly, when $K_{1\sim\eta}$ is chosen as

$$K_{1\sim\eta} = \mathrm{blockdiag}\left(\left[A_p\right]_{0\sim\mu_p-1}, p = 1, 2, ..., \eta\right),$$

the closed-loop system (116) reduces to (58), or equivalently the state-space form (63).

Parallel to the extended step backward affine HOFA system (108), we can also define the following extended non-affine one:

$$\begin{aligned} x_{1\sim\eta}(k+1) &= f\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}(k), k\right) \\ &\quad + g\left(x_p^{\lceil 0\sim\mu_p-1\rceil}(k)\,|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}(k), k, u(k)\right), \end{aligned} \tag{118}$$

where $f(\cdot), g(\cdot) \in \mathbb{R}^r$ are two proper nonlinear functions.

Parallel to Definition 7.1, we can also give the following one.

Definition 7.3. Let $\mathbb{F} \subset \mathbb{R}^{\varkappa+mr}$ be the largest set such that the following mapping

$$\tilde{u} = g(X, U, k, u), \tag{119}$$

forms a differential homeomorphism from u to $\tilde{u}$ for all $(X, U) \in \mathbb{F}$, and $k \geq 0$. Then the set $\mathbb{F}$ is called the set of feasible points of system (118), and any $(X, U) \in \mathbb{F}$ is called a feasible point of system (118). Furthermore, the set

$$\mathbb{S} = \mathbb{R}^{\varkappa} \backslash \mathbb{F},$$

is called the set of singular points of system (118), and any $(X, U) \in \mathbb{S}$ is called a singular point of system (118).

With $\mathbb{F}$ and $\mathbb{S}$ well-defined above, the definitions of full-actuation of system (118) can be also immediately given parallel to Definition 4.2. Similarly, when the system (118) is globally fully actuated, we have a differential homeomorphism,

$$\tilde{u} = g\left(X, U, k, u\right),$$

from u to $\tilde{u}$, $\forall X \in \mathbb{R}^{\varkappa}$, $U \in \mathbb{R}^{mr}$, and $k \geq 0$. In this case, the set of systems in (118) can be also written as in (113).

Furthermore, for the more general non-affine step backward HOFA system (118), a result similar to Theorem 7.2 still holds. The controller can be obtained using the differential homeomorphism property of the mapping $\tilde{u} = g\left(x_p^{\lceil 0\sim\mu_p-1\rceil}|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}\left(k\right), k, u\right)$, as

$$\begin{cases} u\left(k\right) = g^{-1}\left(x_p^{\lceil 0\sim\mu_p-1\rceil}|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}\left(k\right), k, \tilde{u}\left(k\right)\right) \\ \tilde{u}\left(k\right) = -\left(K_{1\sim\eta}x_p^{\lceil 0\sim\mu_p-1\rceil}\left(k\right)|_{p=1\sim\eta} + u^*\left(k\right)\right) \\ u^*\left(k\right) = f\left(x_p^{\lceil 0\sim\mu_p-1\rceil}\left(k\right)|_{p=1\sim\eta}, u^{\lceil 1\sim m\rceil}\left(k\right), k\right) - v\left(k\right), \end{cases} \tag{120}$$

and the closed-loop system is the same as that in (116).

7.2. Pseudo feed-forward systems

As an example, let us consider the following pseudo feed-forward system

$$\begin{cases} x_1\left(k+1\right) = f_1\left(x_n\left(k\right), u\left(k\right)\right) \\ x_2\left(k+1\right) = f_2\left(x_1\left(k\right), x_n\left(k\right), u\left(k\right)\right) \\ x_3\left(k+1\right) = f_3\left(x_{1\sim 2}\left(k\right), x_n\left(k\right), u\left(k\right)\right) \\ \qquad\vdots \\ x_{n-1}\left(k+1\right) = f_{n-1}\left(x_{1\sim n-1}\left(k\right), x_n\left(k\right), u\left(k\right)\right) \\ x_n\left(k+1\right) = f_n\left(x_{1\sim n}\left(k\right)\right) + B\left(x_{1\sim n}\left(k\right)\right)u\left(k\right), \end{cases} \tag{121}$$

where $x_i \in \mathbb{R}^{r_i}$, $i = 1, 2, ..., n$, are the system state vectors, $u \in \mathbb{R}^r, r = r_n$, is the system control vector, $f_i\left(\cdot\right) \in \mathbb{R}^{r_i}$, $i = 1, 2, ..., n$, are a series of vector functions, and $B\left(\cdot\right) \in \mathbb{R}^{r\times r}$ is a continuous matrix function satisfying the following assumption.

Assumption A3 $\det B\left(x_{1\sim n}\left(k\right)\right) \neq 0, \forall x_{1\sim n}\left(k\right) \in \mathbb{R}^{nr}$.

Proposition 7.4. *The above system (121) satisfying Assumption A3 can be equivalently converted into the step backward HOFA system (108).*

Proof. Rewriting the first equation in (121) as

$$x_1\left(k\right) = f_1\left(x_n\left(k-1\right), u\left(k-1\right)\right),$$

and substituting it into the rest ones in (121), yield

$$\begin{cases} x_2(k+1) = f_2\left(x_n^{\lceil 0\sim 1\rceil}(k), u^{\lceil 1\rceil}(k)\right) \\ \qquad\vdots \\ x_{n-1}(k+1) = f_{n-1}\left(x_{2\sim n-2}(k), x_n^{\lceil 0\sim 1\rceil}(k), u^{\lceil 1\rceil}(k)\right) \\ x_n(k+1) = f_n\left(x_{2\sim n-1}(k), x_n^{\lceil 0\sim 1\rceil}(k), u^{\lceil 1\rceil}(k)\right) \\ \qquad\qquad +B\left(x_{2\sim n-1}(k), x_n^{\lceil 0\sim 1\rceil}(k), u^{\lceil 1\rceil}(k)\right)u(k). \end{cases} \tag{122}$$

Again, rewriting the first equation in (122) as

$$x_2(k) = f_2\left(x_n^{\lceil 0\sim 1\rceil}(k-1), u^{\lceil 1\rceil}(k-1)\right),$$

and substituting it into the rest equations in (122), give

$$\begin{cases} x_3(k+1) = f_3\left(x_n^{\lceil 0\sim 2\rceil}(k), u^{\lceil 1\sim 2\rceil}(k)\right) \\ \qquad\vdots \\ x_{n-1}(k+1) = f_{n-1}\left(x_{3\sim n-1}(k), x_n^{\lceil 0\sim 2\rceil}(k), u^{\lceil 1\sim 2\rceil}(k)\right) \\ x_n(k+1) = f_n\left(x_{3\sim n-1}(k), x_n^{\lceil 0\sim 2\rceil}(k), u^{\lceil 1\sim 2\rceil}(k)\right) \\ \qquad\qquad +B\left(x_{3\sim n-1}(k), x_n^{\lceil 0\sim 2\rceil}(k), u^{\lceil 1\sim 2\rceil}(k)\right)u(k). \end{cases} \tag{123}$$

Continuing this process, after $n-2$ times of operation, we obtain

$$\begin{cases} x_{n-1}(k+1) = f_{n-1}\left(x_n^{\lceil 0\sim n-2\rceil}(k), u^{\lceil 1\sim n-2\rceil}(k)\right) \\ x_n(k+1) = f_n\left(x_{n-1}(k), x_n^{\lceil 0\sim n-2\rceil}(k), u^{\lceil 1\sim n-2\rceil}(k)\right) \\ \qquad\qquad +B\left(x_{n-1}(k), x_n^{\lceil 0\sim n-2\rceil}(k), u^{\lceil 1\sim n-2\rceil}(k)\right)u(k). \end{cases} \tag{124}$$

Finally, substituting the first equation in (124) into its second one, gives

$$\begin{aligned} x_n(k+1) &= f_n\left(x_n^{\lceil 0\sim n-1\rceil}(k), u^{\lceil 1\sim n-1\rceil}(k)\right) \\ &\quad +B\left(x_n^{\lceil 0\sim n-1\rceil}(k), u^{\lceil 1\sim n-1\rceil}(k)\right)u(k). \end{aligned} \tag{125}$$

This is in the form of (108). □

Consider a specific pseudo feed-forward system, with $n=3$,

$$\begin{cases} x_1(k+1) = x_3(k) \\ x_2(k+1) = \left[1 + x_1^2(k)\right]\left[x_3(k) + \sigma u(k)\right] \\ x_3(k+1) = x_1(k)\,x_2(k) + \left[1 + x_3(k)\right]u(k). \end{cases} \tag{126}$$

Following the procedure given in the above proof of Proposition 7.4, we can obtain its following equivalent HOFA model:

$$\begin{aligned} x_3(k+1) &= f\left(x_3^{\lceil 1\sim 2\rceil}(k)\right) + [1 + x_3(k)]\,u(k) \\ &\quad + b_1\left(x_3^{\lceil 1\sim 2\rceil}(k)\right) u(k-1), \end{aligned}$$

where

$$\begin{aligned} f\left(x_3^{\lceil 1\sim 2\rceil}(k)\right) &= x_3^2(k-1)\left[1 + x_3^2(k-2)\right], \\ b_1\left(x_3^{\lceil 1\sim 2\rceil}(k)\right) &= \sigma x_3(k-1)\left[1 + x_3^2(k-2)\right]. \end{aligned}$$

The feasibility condition of this system is clearly

$$x_3(k) \neq -1. \tag{127}$$

Choose the following controller for the system:

$$\begin{cases} u(k) = -\frac{1}{1+x_3(k)}\left(\sum\limits_{i=0}^{3} a_i x_3(k-i) + u^*(k) - v(k)\right) \\ u^*(k) = f\left(x_3^{\lceil 1\sim 2\rceil}(k)\right) + b_1\left(x_3^{\lceil 1\sim 2\rceil}(k)\right) u(k-1), \end{cases}$$

then the following closed-loop system is obtained:

$$x_3(k+1) = -\sum_{i=0}^{2} a_i x_3(k-i) + v(k),$$

which has the state-space form of

$$x_3^{\lceil 0\sim 2\rceil}(k+1) = \Phi(a_{0\sim 2})\, x_3^{\lceil 0\sim 2\rceil}(k) + \Gamma_c v(k), \tag{128}$$

where, by our notations,

$$\Phi(a_{0\sim 2}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad \Gamma_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Without loss of generality, let us assume that $\Phi(a_{0\sim 2})$ is similar to a real diagonal Schur matrix

$$F = \operatorname{diag}(s_i, i = 1, 2, 3),$$

that is,

$$\Phi(a_{0\sim 2}) = \Phi(0) - \Gamma_c a_{0\sim 2} = VFV^{-1},$$

for some nonsingular V. Solution to $a_{0\sim 2}$ satisfying the above equation can be parametrically solved according to the Lemma 5.2 in Duan (2021c).

In the case of $v = 0$, we eventually have

$$x_3(k) \to 0, \text{ as } t \to \infty.$$

When the initial values are properly chosen, the feasibility requirement (127) can be satisfied. For a theoretical guarantee of the feasibility requirement (127), we can either apply the feasibility condition based on a Lyapunov matrix equation (see the Lemma 5.2 in Duan (2021d)), or the one based on the closed-loop eigenstructure (see the Lemma 6.1 in Duan (2021e)). In certain cases, we can also choose the signal v to make the steady state of $x_3(k)$ to be further away from the singularity set.

Once $x_3(k), u(k), k = 0, 1, 2, ...$ are obtained, by the first equation in (126) we can get $x_1(k), k = 0, 1, 2, ...$, and then $x_2(k), k = 0, 1, 2, ...$ by the second one in (126).

8. Concluding remarks

As the last one in this HOFA approach series, this paper is concerned with the discrete-time HOFA systems.

Regarding models of discrete-time HOFA systems, different from the continuous-time case, there are two types of general HOFA models, one is the type of step forward HOFA models, and the other is the type of step backward HOFA models. It is shown that the two types of models are essentially different in the sense that, once the step forward model is represented in the step backward form, it eventually becomes a system with a time-delay in the control vector.

As a generalization of the proposed step backward HOFA models, an extended form of the step backward HOFA model is also proposed, which takes consideration of the effect of the control vector at some following time instants in the functions $f(\cdot)$ and $B(\cdot)$.

It is proven that controllers for all types of HOFA models can be easily designed such that the closed-loop systems are constant linear ones with arbitrarily assignable eigenstructures. However, due to the delay in the control vector, the designed controller for a step forward HOFA system turns out to be a dynamical one instead of a static one as in the case for a step backward HOFA system.

The results proposed in the paper are very fundamental and are also vitally important since they lay a solid basis for discrete-time HOFA approaches. Specifically, we give the following brief comments on discrete-time HOFA system approaches.

Control systems analysis Firstly, the problems of response analysis and stability analysis are not as necessary as they are for state-space approaches since the closed-loop systems designed via HOFA approaches are constant and linear, or with a constant and linear main part (when uncertainties are added). Secondly, parallel to the continuous-time case, a general discrete-time dynamical system can be defined to be controllable if it can be equivalently converted into a step forward or step backward HOFA model. Furthermore, an uncontrollable discrete-time system is generally composed of a HOFA model and an autonomous subsystem, and the system is stabilizable if the autonomous subsystem does not exist or is stable (Duan, 2020b, 2021c).

Control systems design Thanks to the full-actuation property of the proposed discrete-time HOFA models, important control features are revealed as shown in Theorems 5.1 and 5.3. As in the continuous-time system case, these basic results allow us to extend the results to many other design problems, such as robust control (Duan, 2020f, 2021a), adaptive control (Duan, 2020g, 2021a), disturbance rejection (Duan, 2021b), optimal control (Duan, 2021d), signal tracking control (Duan, 2021e). All such control problems, as done in the continuous-time case, can be converted into corresponding ones for linear systems, hence can be effectively solved by adopting existing methods for linear systems. Another aspect associated with the discrete-time HOFA system designs is the complete parameterization of the designs. Since constant linear closed-loop systems are obtained, as done in the continuous-time case, complete parametric expressions for the feedback gains in the controllers as well as the closed-loop systems can be established, which provide all the design degrees of freedom to be used for further improving the system performance.

Finally, it is also mentioned that, like the case of discrete-time systems, HOFA approaches can be also applied to other types of systems, such as stochastic systems, and time-delay systems.

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