

CS5487 Problem Set 1

Probability Theory and Linear Algebra Review

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Probability Theory

Problem 1.1 Linear Transformation of a Random Variable

Let x be a random variable on \mathbb{R} , and $a, b \in \mathbb{R}$. Let $y = ax + b$ be the linear transformation of x . Show the following properties:

$$\mathbb{E}[y] = a\mathbb{E}[x] + b, \quad (1.1)$$

$$\text{var}(y) = a^2 \text{var}(x). \quad (1.2)$$

Now, let x be a vector r.v. on \mathbb{R}^d , and $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$. Let $y = Ax + b$ be the linear transformation of x . Show the following properties:

$$\mathbb{E}[y] = A\mathbb{E}[x] + b, \quad (1.3)$$

$$\text{cov}(y) = A\text{cov}(x)A^T. \quad (1.4)$$

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Problem 1.2 Properties of Independence

Let x and y be statistically independent random variables ($x \perp y$). Show the following properties:

$$\mathbb{E}[xy] = \mathbb{E}[x]\mathbb{E}[y], \quad (1.5)$$

$$\text{cov}(x, y) = 0. \quad (1.6)$$

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Problem 1.3 Uncorrelated vs Independence

Two random variables x and y are said to be *uncorrelated* if their covariance is 0, i.e., $\text{cov}(x, y) = 0$. For statistically independent random variables, their covariance is always 0 (see [Problem 1.2](#)), and hence independent random variables are always uncorrelated. However, the converse is generally not true; uncorrelated random variables are not necessarily independent.

Consider the following example. Let the pair of random variables (x, y) take values $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$, each with equal probability $(1/4)$.

- (a) Show that $\text{cov}(x, y) = 0$, and hence x and y are uncorrelated.
- (b) Calculate the marginal distributions, $p(x)$ and $p(y)$. Show that the $p(x, y) \neq p(x)p(y)$ and thus x and y are not independent.

(c) Now consider a more general example. Assume that x and y satisfy

$$\mathbb{E}[x|y] = \mathbb{E}[x], \quad (1.7)$$

i.e., the mean of x is the same regardless of whether y is known or not (the above example satisfies this property). Show that x and y are uncorrelated.

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Problem 1.4 Sum of Random Variables

Let x and y be random variables (possibly dependent), show the following property:

$$\mathbb{E}[x + y] = \mathbb{E}[x] + \mathbb{E}[y]. \quad (1.8)$$

Furthermore, if x and y are statistically independent ($x \perp y$), show that

$$\text{var}(x + y) = \text{var}(x) + \text{var}(y). \quad (1.9)$$

However, in general this is not the case when x and y are dependent.

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Problem 1.5 Expectation of an Indicator Variable

Let x be an indicator variable on $\{0, 1\}$. Show that

$$\mathbb{E}[x] = p(x = 1), \quad (1.10)$$

$$\text{var}(x) = p(x = 0)p(x = 1). \quad (1.11)$$

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Problem 1.6 Multivariate Gaussian

The multivariate Gaussian is a probability density over real vectors, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} \in \mathbb{R}^d$, which is parameterized by a mean vector $\mu \in \mathbb{R}^d$ and a covariance matrix $\Sigma \in \mathbb{S}_+^d$ (i.e., a d -dimensional positive-definite symmetric matrix). The density function is

$$p(x) = \mathcal{N}(x|\mu, \Sigma) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}\|x-\mu\|_{\Sigma}^2}, \quad (1.12)$$

where $|\Sigma|$ is the determinant of Σ , and

$$\|x - \mu\|_{\Sigma}^2 = (x - \mu)^T \Sigma^{-1} (x - \mu) \quad (1.13)$$

is the *Mahalanobis distance*. In this problem, we will look at how different covariance matrices affect the shape of the density.

First, consider the case where Σ is a *diagonal matrix*, i.e., the off-diagonal entries are 0,

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_d^2 \end{bmatrix}. \quad (1.14)$$

- (a) Show that with a diagonal covariance matrix, the multivariate Gaussian is equivalent to assuming that the elements of the vector are independent, and each is distributed as a univariate Gaussian, i.e.,

$$\mathcal{N}(x|\mu, \Sigma) = \prod_{i=1}^d \mathcal{N}(x_i|\mu_i, \sigma_i^2). \quad (1.15)$$

Hint: the following properties of diagonal matrices will be useful:

$$|\Sigma| = \prod_{i=1}^d \sigma_i^2, \quad \Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_d^2} \end{bmatrix}. \quad (1.16)$$

- (b) Plot the Mahalanobis distance term and probability density function for a 2-dimensional Gaussian with $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}$. How is the shape of the density affected by the diagonal terms?
- (c) Plot the Mahalanobis distance term and pdf when the variances of each dimension are the same, e.g., $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This is sometimes called an i.i.d. (independently and identically distributed) covariance matrix, isotropic covariance matrix, or circular covariance matrix.

Next, we will consider the general case for the covariance matrix.

- (d) Let $\{\lambda_i, v_i\}$ be the eigenvalue/eigenvector pairs of Σ , i.e.,

$$\Sigma v_i = \lambda_i v_i, \quad i \in \{1, \dots, d\}. \quad (1.17)$$

Show that Σ can be written as

$$\Sigma = V \Lambda V^T, \quad (1.18)$$

where $V = [v_1, \dots, v_d]$ is the matrix of eigenvectors, and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_d)$ is a diagonal matrix of the eigenvalues.

- (e) Let $y = V^T(x - \mu)$. Show that the Mahalanobis distance $\|x - \mu\|_{\Sigma}^2$ can be rewritten as $\|y\|_{\Lambda}^2$, i.e., a Mahalanobis distance with a diagonal covariance matrix. Hence, in the space of y , the multivariate Gaussian has a diagonal covariance matrix. (Hint: use [Problem 1.12](#))
- (f) Consider the transformation from y to x : $x = Vy + \mu$. What is the effect of V and μ ?
- (g) Plot the Mahalanobis distance term and probability density function for a 2-dimensional Gaussian with $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\Sigma = \begin{bmatrix} 0.625 & 0.375 \\ 0.375 & 0.625 \end{bmatrix}$. How is the shape of the density affected by the eigenvectors and eigenvalues of Σ ?

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Problem 1.7 Product of Gaussian Distributions

Show that the product of two Gaussian distributions, $\mathcal{N}(x|\mu_1, \sigma_1^2)$ and $\mathcal{N}(x|\mu_2, \sigma_2^2)$, is a scaled Gaussian,

$$\mathcal{N}(x|\mu_1, \sigma_1^2)\mathcal{N}(x|\mu_2, \sigma_2^2) = Z\mathcal{N}(x|\mu_3, \sigma_3^2), \quad (1.19)$$

where

$$\mu_3 = \sigma_3^2(\sigma_1^{-2}\mu_1 + \sigma_2^{-2}\mu_2), \quad (1.20)$$

$$\sigma_3^2 = \frac{1}{\sigma_1^{-2} + \sigma_2^{-2}}, \quad (1.21)$$

$$Z = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} e^{\frac{-1}{2(\sigma_1^2 + \sigma_2^2)}(\mu_1 - \mu_2)^2} = \mathcal{N}(\mu_1|\mu_2, \sigma_1^2 + \sigma_2^2). \quad (1.22)$$

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Problem 1.8 Product of Multivariate Gaussian Distributions

Show that the product of two d -dimensional multivariate Gaussians distributions, $\mathcal{N}(x|a, A)$ and $\mathcal{N}(x|b, B)$, is a scaled multivariate Gaussian,

$$\mathcal{N}(x|a, A)\mathcal{N}(x|b, B) = Z\mathcal{N}(x|c, C), \quad (1.23)$$

where

$$c = C(A^{-1}a + B^{-1}b), \quad (1.24)$$

$$C = (A^{-1} + B^{-1})^{-1}, \quad (1.25)$$

$$Z = \frac{1}{(2\pi)^{\frac{d}{2}} |A + B|^{\frac{1}{2}}} e^{-\frac{1}{2}(a-b)^T(A+B)^{-1}(a-b)} = \mathcal{N}(a|b, A + B). \quad (1.26)$$

Hint: after expanding the exponent term, apply the result from Problem 1.10 and (1.35).

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Problem 1.9 Correlation between Gaussian Distributions

Using the result from Problem 1.8, show that the correlation between two multivariate Gaussian distributions is

$$\int \mathcal{N}(x|a, A)\mathcal{N}(x|b, B)dx = \mathcal{N}(a|b, A + B). \quad (1.27)$$

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Problem 1.10 Completing the square

Let $x, b \in \mathbb{R}^n$, $A \in \mathbb{S}^n$, $c \in \mathbb{R}$, and let $f(x)$ be a quadratic function of x ,

$$f(x) = x^T A x - 2x^T b + c. \quad (1.28)$$

Show that $f(x)$ can be rewritten in the form

$$f(x) = (x - d)^T A (x - d) + e, \quad (1.29)$$

where

$$d = A^{-1}b, \quad (1.30)$$

$$e = c - d^T A d = c - b^T A^{-1}b. \quad (1.31)$$

Rewriting the quadratic function in (1.28) as (1.29) is a procedure known as “completing the square”, which is very useful when dealing with products of Gaussian distributions.

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Problem 1.11 Eigenvalues

Let $\{\lambda_i\}$ be the eigenvalues of $A \in \mathbb{R}^{n \times n}$. Derive the following properties:

$$\text{tr}(A) = \sum_{i=1}^n \lambda_i, \quad |A| = \prod_{i=1}^n \lambda_i. \quad (1.32)$$

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Problem 1.12 Eigenvalues of an inverse matrix

Let $\{\lambda_i, x_i\}$ be the eigenvalues/eigenvectors of $A \in \mathbb{R}^{n \times n}$. Show that $\{\frac{1}{\lambda_i}, x_i\}$ are the eigenvalues/eigenvectors of A^{-1} .

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Problem 1.13 Positive definiteness

Derive the following properties:

1. A symmetric matrix $A \in \mathbb{S}^n$ is positive definite if all its eigenvalues are greater than zero.
2. For any matrix $A \in \mathbb{R}^{m \times n}$, $G = A^T A$ is positive semidefinite.

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Problem 1.14 Positive definiteness of inner product and outer product matrices

Let $X = [x_1, \dots, x_n] \in \mathbb{R}^{d \times n}$ be a matrix of n column vectors $x_i \in \mathbb{R}^d$. We can think of each column vector x_i as a sample in our dataset X .

- (a) outer-product: Prove that $\Sigma = XX^T$ is always *positive semi-definite* ($\Sigma \succeq 0$). When will Σ be strictly *positive definite*?
- (b) inner-product: Prove that $G = X^T X$ is always *positive semi-definite*. When will G be strictly *positive definite*?

Note: If $\{x_1, \dots, x_n\}$ are zero mean samples, then Σ is n times the sample covariance matrix. G is sometimes called a *Gram matrix* or *kernel matrix*.

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Problem 1.15 Useful Matrix Inverse Identities

Show that the following identities are true:

$$(P^{-1} + B^T R^{-1} B)^{-1} B^T R^{-1} = P B^T (B P B^T + R)^{-1}, \quad (1.33)$$

$$(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1} B = B(A + B)^{-1} A, \quad (1.34)$$

$$(A^{-1} + B^{-1})^{-1} = A - A(A + B)^{-1} A = B - B(A + B)^{-1} B, \quad (1.35)$$

$$(A^{-1} + U C^{-1} V^T)^{-1} = A - A U (C + V^T A U)^{-1} V^T A \quad (1.36)$$

The last one is called the Matrix Inversion Lemma (or Sherman-Morrison-Woodbury formula) Hint: these can be verified by multiplying each side by an appropriate matrix term.

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Problem 1.16 Useful Matrix Determinant Identities

Verify that the following identities are true:

$$|I + AB^T| = |I + B^T A| \quad (1.37)$$

$$|I + ab^T| = 1 + b^T a \quad (1.38)$$

$$|A^{-1} + UV^T| = |I + V^T A U| |A^{-1}| \quad (1.39)$$

The last one is called the Matrix Determinant Lemma.

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Problem 1.17 Singular Value Decomposition (SVD)

The singular value decomposition (SVD) of a real $n \times m$ matrix A is a set of three matrices $\{U, S, V\}$, such that

$$A = U S V^T, \quad (1.40)$$

where

- $U \in \mathbb{R}^{n \times m}$ is an orthonormal matrix of left-singular vectors (columns of U), i.e., $U^T U = I$.
- $S \in \mathbb{R}^{m \times m}$ is a diagonal matrix of *singular values*, i.e. $S = \text{diag}(s_1, \dots, s_m)$. The singular values are usually ordered $s_1 \geq s_2 \geq \dots \geq s_m \geq 0$
- $V \in \mathbb{R}^{m \times m}$ is an orthonormal matrix of right-singular vectors (columns of V), i.e., $V^T V = I$.

The SVD has an intuitive interpretation, which shows how the matrix A acts on a vector $x \in \mathbb{R}^m$. Consider the matrix-vector product

$$z = Ax = USV^T x. \quad (1.41)$$

This shows that the matrix A performs 3 operations on x , namely rotation by V^T , scaling along the axis via S , and another rotation by U . The SVD is also closely related to the eigen-decomposition, matrix inverse, and pseudoinverses.

- Show that the singular values of A are the square roots of the eigenvalues of the matrix $B = AA^T$, and that the left-singular vectors (columns of U) are the associated eigenvectors.
- Show that the singular values of A are the square roots of the eigenvalues of the matrix $C = A^T A$, and that the right-singular vectors (columns of V) are the associated eigenvectors.
- Suppose $A \in \mathbb{R}^{n \times n}$ is a square matrix of rank n . Show that the inverse of A can be calculated from the SVD,

$$A^{-1} = VS^{-1}U^T. \quad (1.42)$$

- Suppose $A \in \mathbb{R}^{n \times m}$ is a “fat” matrix ($n < m$) of rank n . The Moore-Penrose pseudoinverse of A is given by

$$A^\dagger = A^T(AA^T)^{-1}. \quad (1.43)$$

Likewise, for a “tall” matrix ($n > m$) of rank m , the pseudoinverse is

$$A^\dagger = (A^T A)^{-1} A^T. \quad (1.44)$$

Show that in both cases the pseudoinverse can also be calculated using the SVD,

$$A^\dagger = VS^{-1}U^T. \quad (1.45)$$

Note: Properties in (c) and (d) also apply for lower rank matrices. In these cases, some of the singular values will be 0, and S^{-1} is hence replaced with S^\dagger , where S^\dagger replaces all non-zero diagonal elements by its reciprocal.

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