Problem Set 2

Problem 2.

(a) Since $D = \{k_1, k_2 \cdots k_N\}$ and muber of bomb hits fix the Poisson Distribution

To estimate the λ. the data log-likelyhood : $l(\lambda) = \log p(D|\lambda)$.

$$\log p(D|\lambda) = \log \prod_{i=1}^{N} p(x=k_i|\lambda) = \sum_{i=1}^{N} \log P(x=k_i|\lambda)$$

$$= \sum_{i=1}^{N} \log\left(\frac{1}{k_i!} e^{-\lambda} \lambda^{k_i}\right) = \sum_{i=1}^{N}\left[\log \frac{1}{k_i!} + (-\lambda) + k_i \log\lambda\right]$$

For $$\frac{\partial l(\lambda)}{\partial \lambda} = \frac{\partial \left[-\sum_{i=1}^{N} \log k_i! - N\lambda + \log\lambda \sum_{i=1}^{N} k_i\right]}{\partial \lambda}$$

$$= -N + \frac{\sum_{i=1}^{N} k_i}{\lambda} \quad \text{let } \frac{\partial l(\lambda)}{\partial \lambda} = 0$$

Thus, $\hat{\lambda} = \frac{\sum_{i=1}^{N} k_i}{N}$ what' more $\frac{\partial^2 l(\lambda)}{\partial \lambda^2} = -\frac{\sum_{i=1}^{N} k_i}{\lambda^2} < 0$

$\hat{\lambda} = \frac{\sum_{i=1}^{N} k_i}{N}$ $\quad l(\theta)$ is maximum.

(b) Sin $\hat{\lambda} = \frac{\sum_{i=1}^{N} k_i}{N}$ To get $\text{Bias}(\hat{\lambda}) = E(\hat{\lambda}) - \lambda$

$$E(\hat{\lambda}) = E\left(\frac{\sum_{i=1}^{N} k_i}{N}\right) = \frac{1}{N} E\left(\sum_{i=1}^{N} k_i\right) = \frac{\sum_{i=1}^{N} E(k_i)}{N}$$

$= \frac{N\lambda}{N} = \lambda$ Thus $\text{Bias}(\hat{\lambda}) = 0$, ML estimator is unbiased

For estimator variance $\text{Var}(\hat{\lambda}) = E[(\hat{\lambda} - \lambda)^2]$

$$= \frac{1}{N^2} E\left[\left(\sum_{i=1}^{N} (k_i - \lambda)\right)^2\right] = \frac{1}{N^2} E\left(\sum_{i=1}^{N}\sum_{j=1}^{N} (k_i - \lambda)(k_j - \lambda)\right)$$

when $i=j$ $E = \text{Var}(x) = \lambda$ else $E = 0$

Thus, $$\text{Var}(\hat{\lambda}) = \frac{N\lambda}{N^2} = \frac{\lambda}{N}$$

(c) According to the above data

$$\hat{\lambda} = \frac{\sum_{i=1}^{N} k_i}{N} = \frac{211 + 2\times 93 + 3\times 35 + 4\times 7 + 5}{576} = 0.929$$

(d) For the expected number of cell with k hits

$k=0$ Expected $= 576 \times P(x=0|\hat{\lambda}) = 576 \times e^{-\hat{\lambda}} \approx 227$

$k=1$ Expected $= 576 \times P(x=1|\hat{\lambda}) = 576 \times e^{-\hat{\lambda}} \hat{\lambda} \approx 211$

$k=2$ Expected $= 576 \times P(x=2|\hat{\lambda}) = 576 \times \frac{e^{-\hat{\lambda}}}{2} \hat{\lambda}^2 \approx 98$

$k=3$ Expected $= 576 \times P(x=3|\hat{\lambda}) = 576 \times \frac{e^{-\hat{\lambda}}}{6} \hat{\lambda}^3 \approx 30$

$k=4$ Expected $= 576 \times P(x=4|\hat{\lambda}) = 576 \times \frac{e^{-\hat{\lambda}}}{24} \hat{\lambda}^4 \approx 7$

$k=5+$ Expected $= 576 \times P(x=5|\hat{\lambda}) = 576 \times \frac{e^{-\hat{\lambda}}}{120} \hat{\lambda}^5 \approx 1$

Only one cell was hit 5 and over,

So the most of cells just were hit randomly

The Germans targeting purely chance.

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