Tutorial 2

PS1-6 m.v. Gaussian

$x \in \mathbb{R}^d, \ \mu \in \mathbb{R}^d, \ \Sigma \in \mathbb{S}^d_{++}$

pdf: $p(x) = \frac{1}{(2\pi)^{d/2} \underbrace{|\Sigma|^{\frac{1}{2}}}} e^{-\frac{1}{2} \underbrace{\|x-\mu\|^2_\Sigma}} = N(x|\mu, \Sigma)$

a) Σ is diagonal matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_d^2 \end{bmatrix}$$

$$|\Sigma| = \prod_{i=1}^d \sigma_i^2 \quad , \quad \Sigma^{-1} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_d^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_d^2 \end{bmatrix}$$

$$\|x-\mu\|^2_\Sigma = (x-\mu)^T \Sigma^{-1} (x-\mu)$$

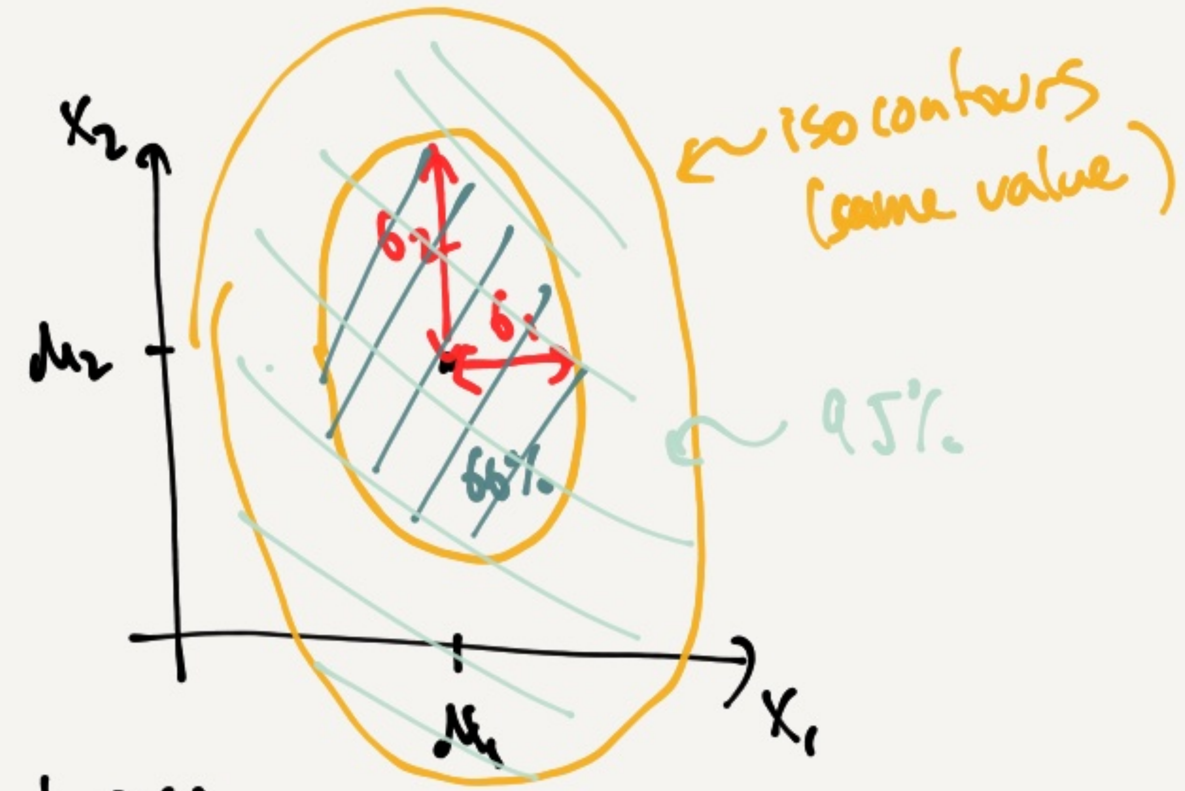
$$= (x-\mu)^T \underbrace{\begin{bmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_d^2} \end{bmatrix} (x-\mu)}_{\begin{bmatrix} \frac{1}{\sigma_1^2}(x_1-\mu_1) \\ \vdots \\ \frac{1}{\sigma_d^2}(x_d-\mu_d) \end{bmatrix}} = \sum_{i=1}^d \frac{1}{\sigma_i^2}(x_i-\mu_i)^2$$

$$\Rightarrow p(x) = \frac{1}{(2\pi)^{d/2} \left[\prod_{i=1}^d \sigma_i^2\right]^{\frac{1}{2}}} e^{-\frac{1}{2}\sum_{i=1}^d \frac{1}{\sigma_i^2}(x_i-\mu_i)^2}$$

$$= \underbrace{\prod_{i=1}^d \frac{1}{(2\pi)^{1/2}\sigma_i} e^{-\frac{1}{2\sigma_i^2}(x_i-\mu_i)^2}} = \boxed{\prod_{i=1}^d N(x_i|\mu_i, \sigma_i^2)}$$

product of univariate Gaussians of x_i

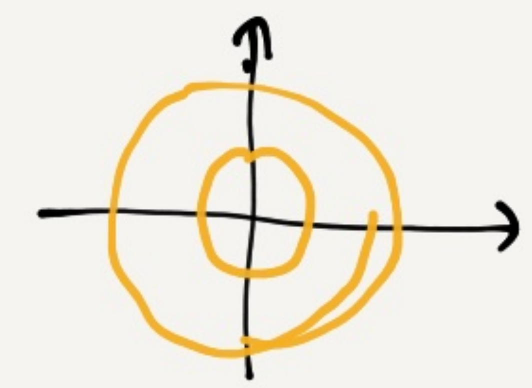
$\therefore x_i$'s are independent.



Special case

$\sigma_i^2 = \sigma^2 \ \forall i \quad , \quad \Sigma = \sigma^2 I$

"circular" Gaussians
isotropic covariance
iid covariance



d) general cov.

<u>eigenvector / eigenvalue</u>

$$\Sigma v_i = \lambda_i v_i \qquad (\Sigma - \lambda I)v = 0$$

eigenvector — scalar eigenvalue

for the d eigenpairs:
$$\Sigma \underbrace{\begin{bmatrix} v_1 & \cdots & v_d \end{bmatrix}}_{V = \text{matrix of eigenvectors}} = \begin{bmatrix} v_1 & \cdots & v_d \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix}}_{\Lambda = \text{matrix of eigenvalues}}$$

$$\Sigma V = V \Lambda$$
$*V^{-1}$ $\quad$ $*V^{-1}$

<u>eigendecomposition</u>: $\Sigma = V \Lambda V^{-1}$

for symmetric matrices: $V^T V = I \Rightarrow V^{-1} = V^T$

$$\Rightarrow \Sigma = V \Lambda V^T$$

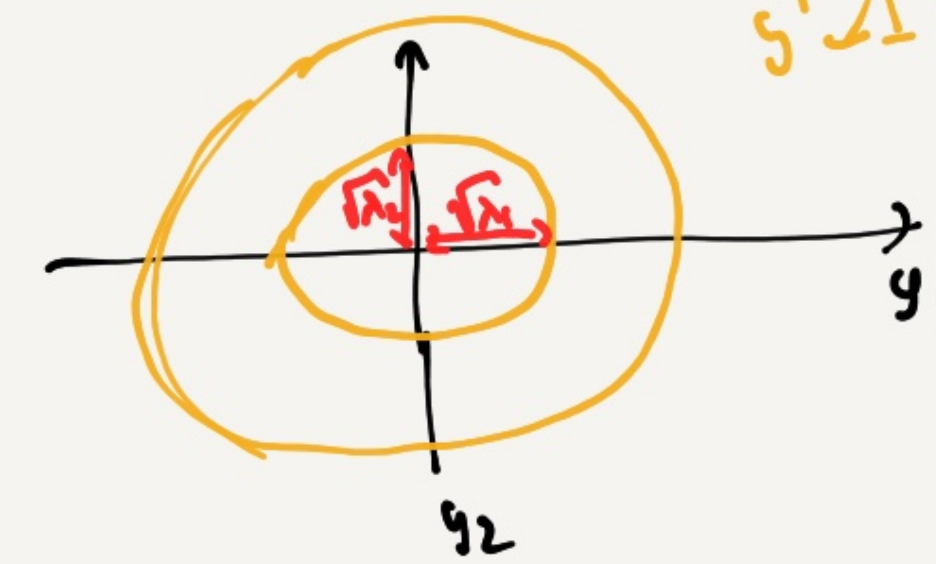
$$(AB)^{-1} = B^{-1} A^{-1}$$

<u>inverse</u>:
$$\Sigma^{-1} = (V \Lambda V^T)^{-1}$$
$$= V^{T^{-1}} \Lambda^{-1} V^{-1}$$

$$\Sigma^{-1} = V \Lambda^{-1} V^T \quad , \quad \Lambda^{-1} = \begin{bmatrix} 1/\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_d \end{bmatrix}$$

$$\|x - \mu\|^2_\Sigma = (x-\mu)^T \Sigma^{-1} (x - \mu)$$
$$= \underbrace{(x-\mu)^T V}_{y^T} \underbrace{\Lambda^{-1}}_{\text{diagonal}} \underbrace{V^T (x - \mu)}_{y} = y^T \Lambda^{-1} y$$

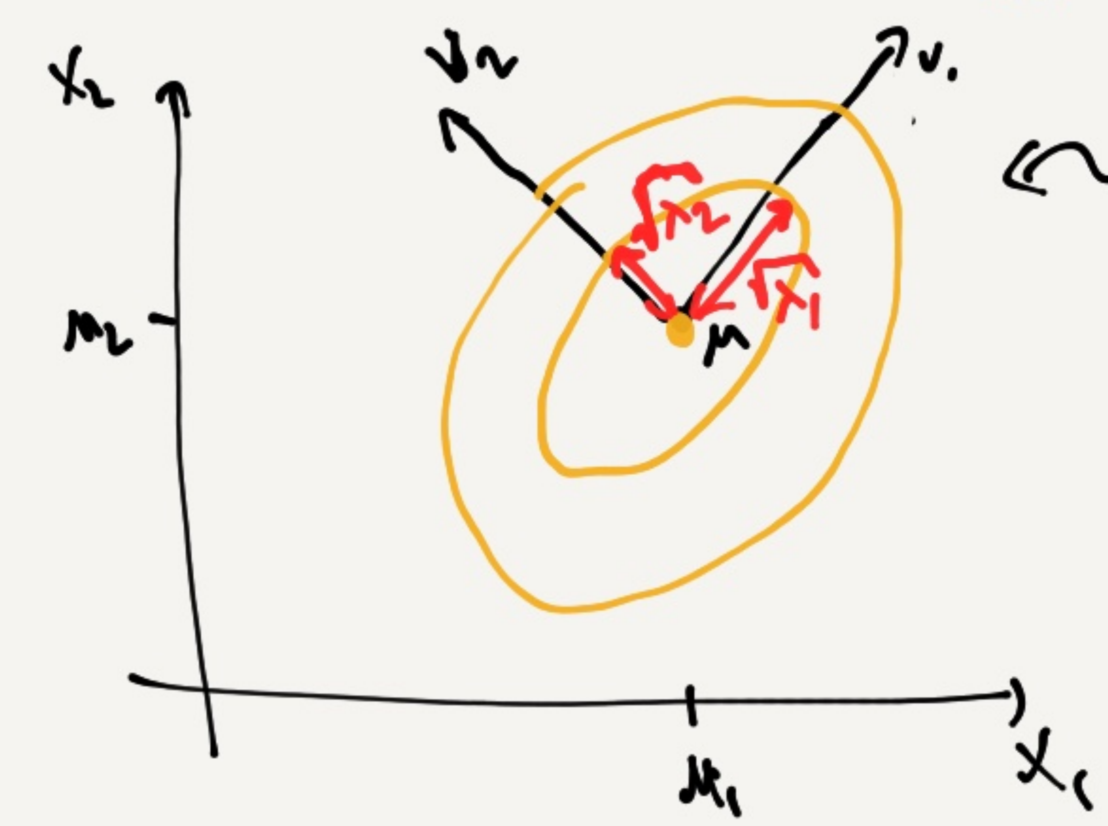
$y^T \Lambda^{-1} y$



We have $y = V^T (x - \mu)$

$Vy = x - \mu \Rightarrow x = \mu + Vy$

translation — rotation



rotated ellipse. aligned w/ eigenvectors.

$\boxed{\text{PS1-10}}$ completing the square

$f(x) = x^T A x - 2x^T b + c$ (suppose A is symmetric)

$\Rightarrow f(x) = (x-d)^T A (x-d) + e$

$= x^T A x - 2 x^T A d + d^T A d + e$

$x^T A d \leftarrow =$
$d^T A x = (x^T A d)^T$ if A is symmetric

2nd term

$x^T \underbrace{b} = x^T \underbrace{Ad}$

$\Rightarrow \boxed{d = A^{-1} b}$

3rd term

$c = d^T A d + e$

$\Rightarrow \boxed{e = c - d^T A d} = c - b^T \underbrace{\underbrace{A^{-T}}_{A^{-1}} A}_{I} A^{-1} b = \boxed{c - b^T A^{-1} b}$