CSC2549 Physics-Based Animation

Doctor Strange | Frame Store

Reminders

Assignment #3 is due Friday

https://github.com/dilevin/CSC2549-a3-finite-elements-3d

Assignment #4 is live and is due on October 25th

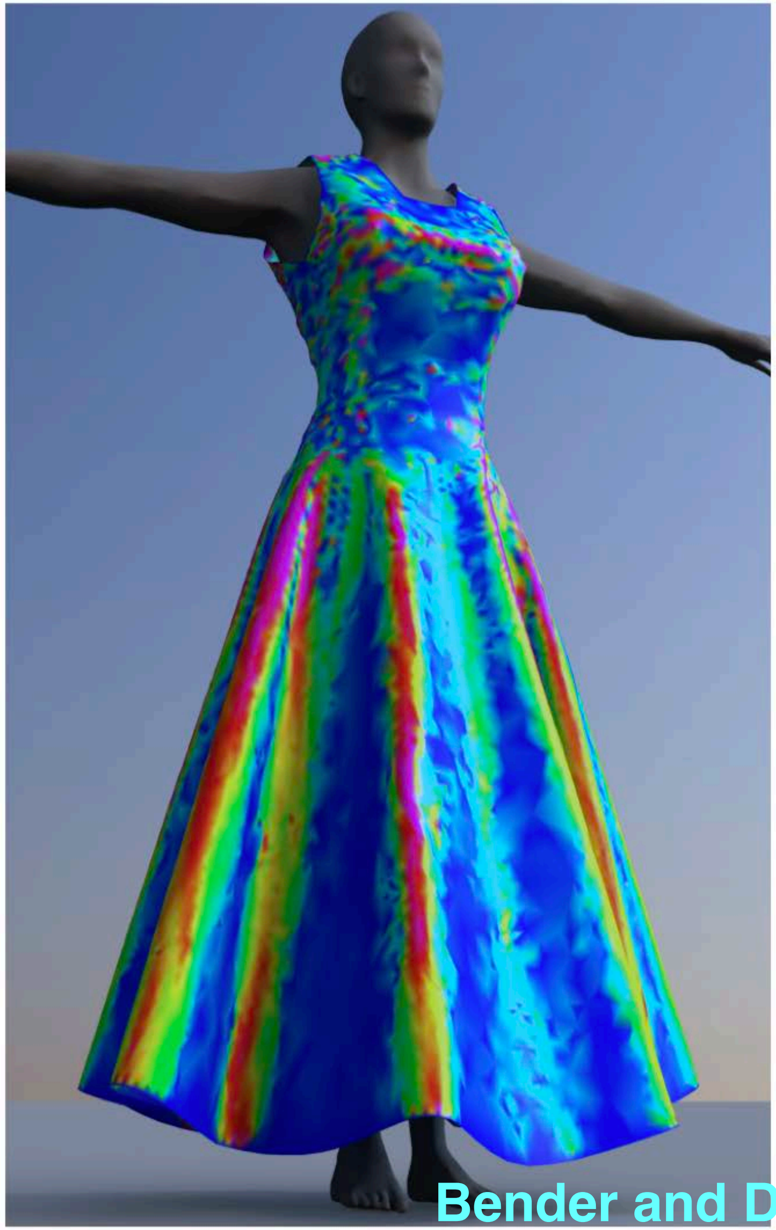
https://github.com/dilevin/CSC2549-a4-cloth-simulation

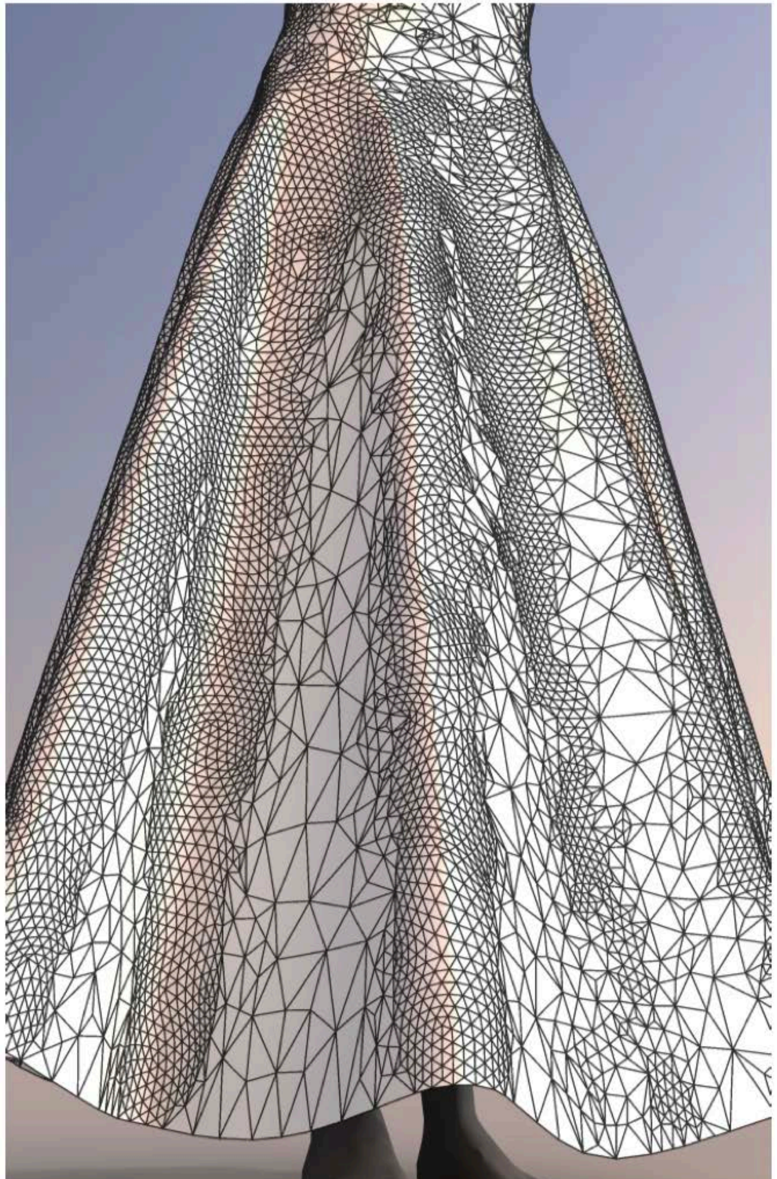
Graphics Reading Group

Seminar Room in BA5166 (Dynamic Graphics Project)

Wednesdays 11am

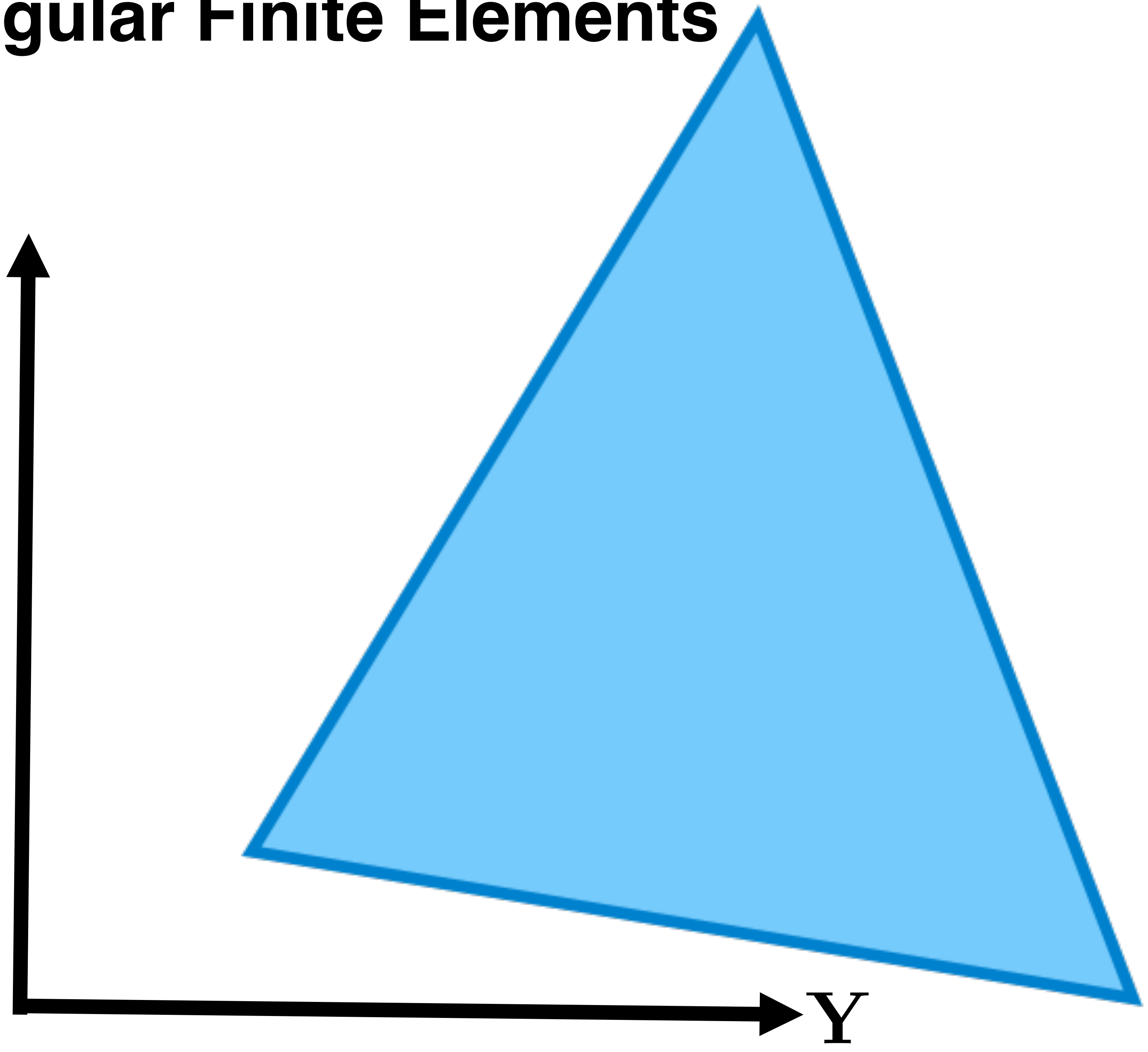
Finite Elements on Surfaces



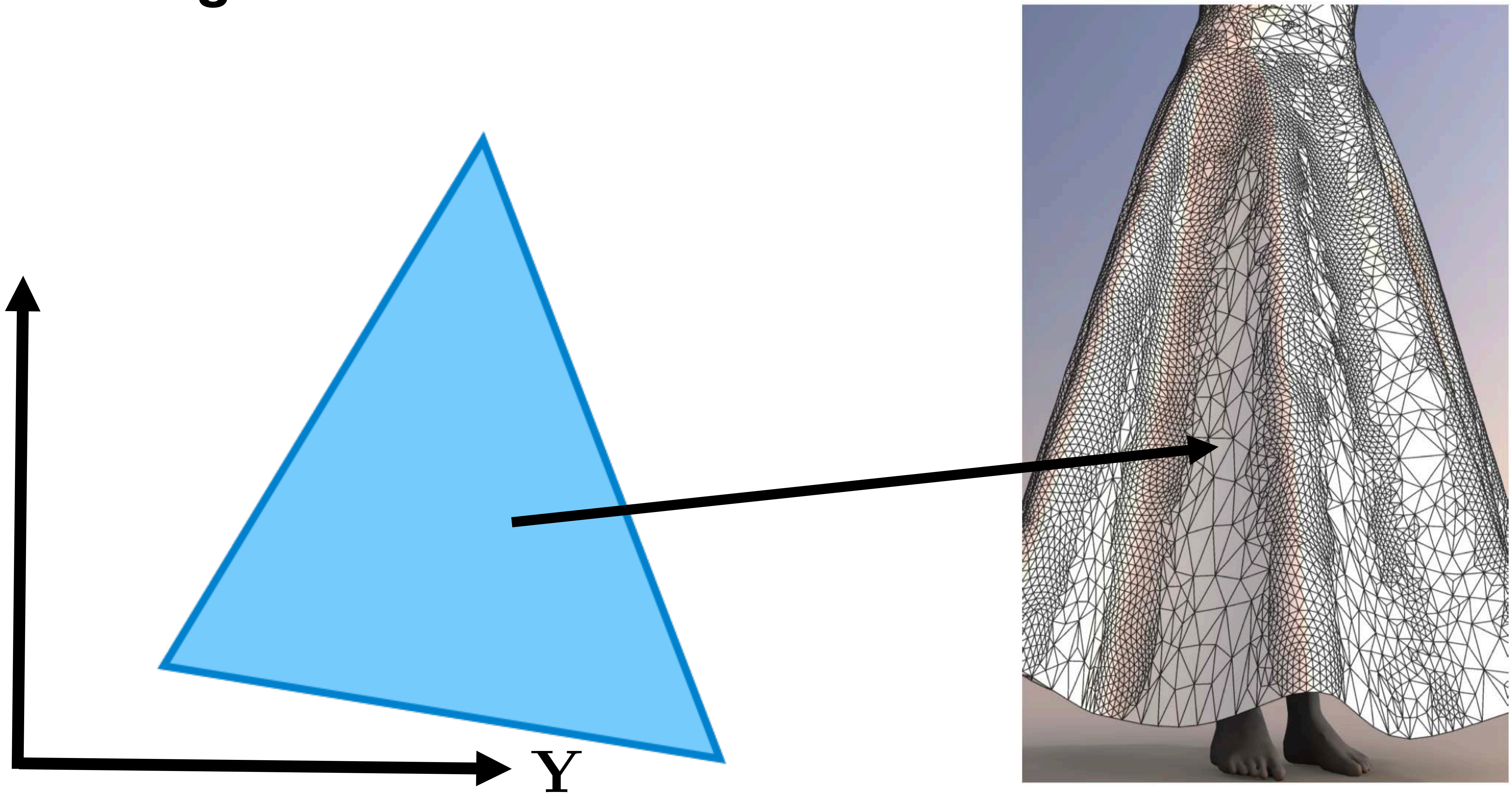


Bender and Deul | Adaptive cloth simulation using corotational finite elements

Triangular Finite Elements



Triangular Finite Elements



Barycentric Coordinates as Shape Functions

$$\bar{Y} = \bar{Y}_0 \phi_0 + \bar{Y}_1 \phi_1 + \bar{Y}_3 \phi_2$$

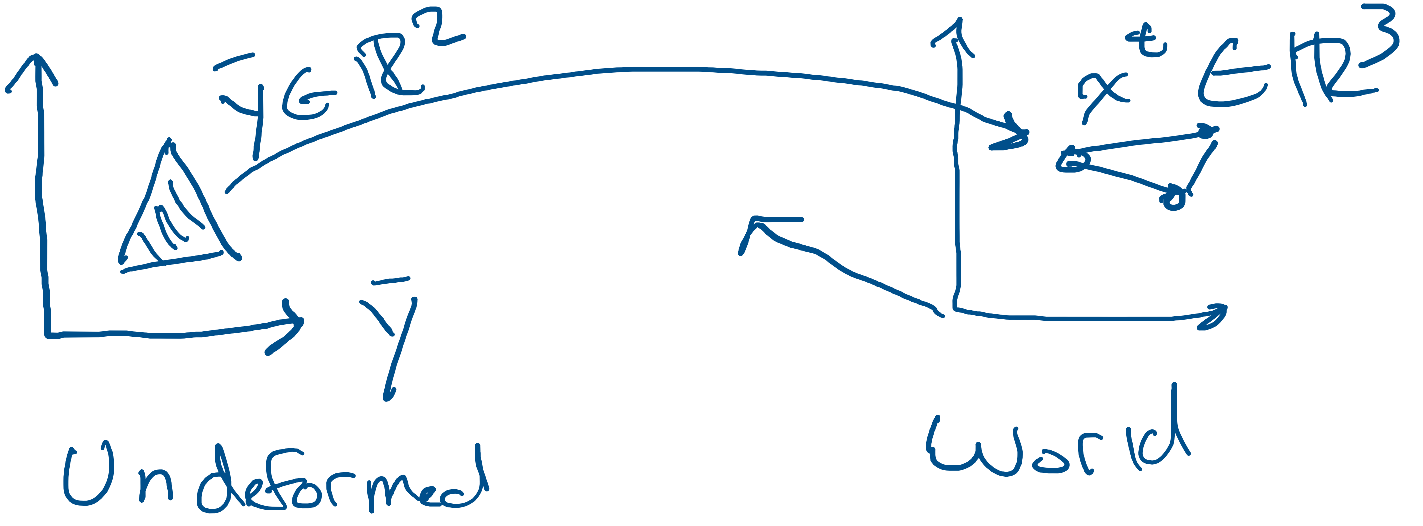
$\phi_i(\bar{Y})$

$\bar{L}$ per-vertex coordinates

$$\phi_2(\bar{Y}) = 1 - \phi_0 - \phi_1$$

Definition of Bary Coords

Generalized Coordinates and Velocities



$$x^t(\bar{Y}) = x_0^t \, \phi_0(\bar{Y}) + x_1^t \, \phi_1(\bar{Y}) + x_2^t \, \phi_2(\bar{Y})$$

3D vertex positions

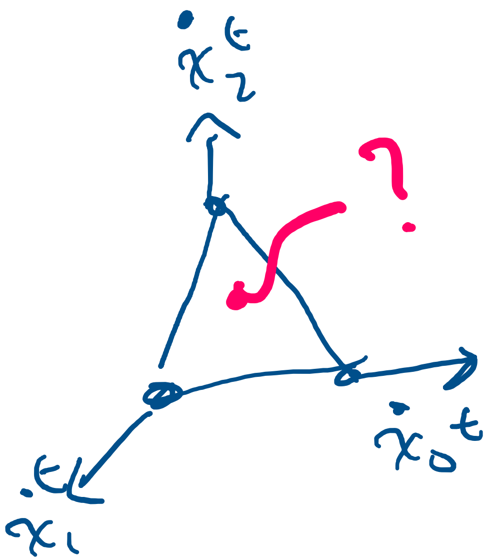
$$x^t(\bar{Y}) = \underbrace{\begin{bmatrix} \phi_0 I & \phi_1 I & \phi_2 I \end{bmatrix}}_{N(\bar{Y})} \underbrace{\begin{bmatrix} x_0^t \\ x_1^t \\ x_2^t \end{bmatrix}}_{q \in \mathbb{R}^9}$$

Generalized coordinates (q)

$$x^t = N(\bar{Y})\, q^t$$

$$v^t(\bar{Y}) = \frac{d}{dt} x^t = N(\bar{Y})\, \dot{q}^t$$

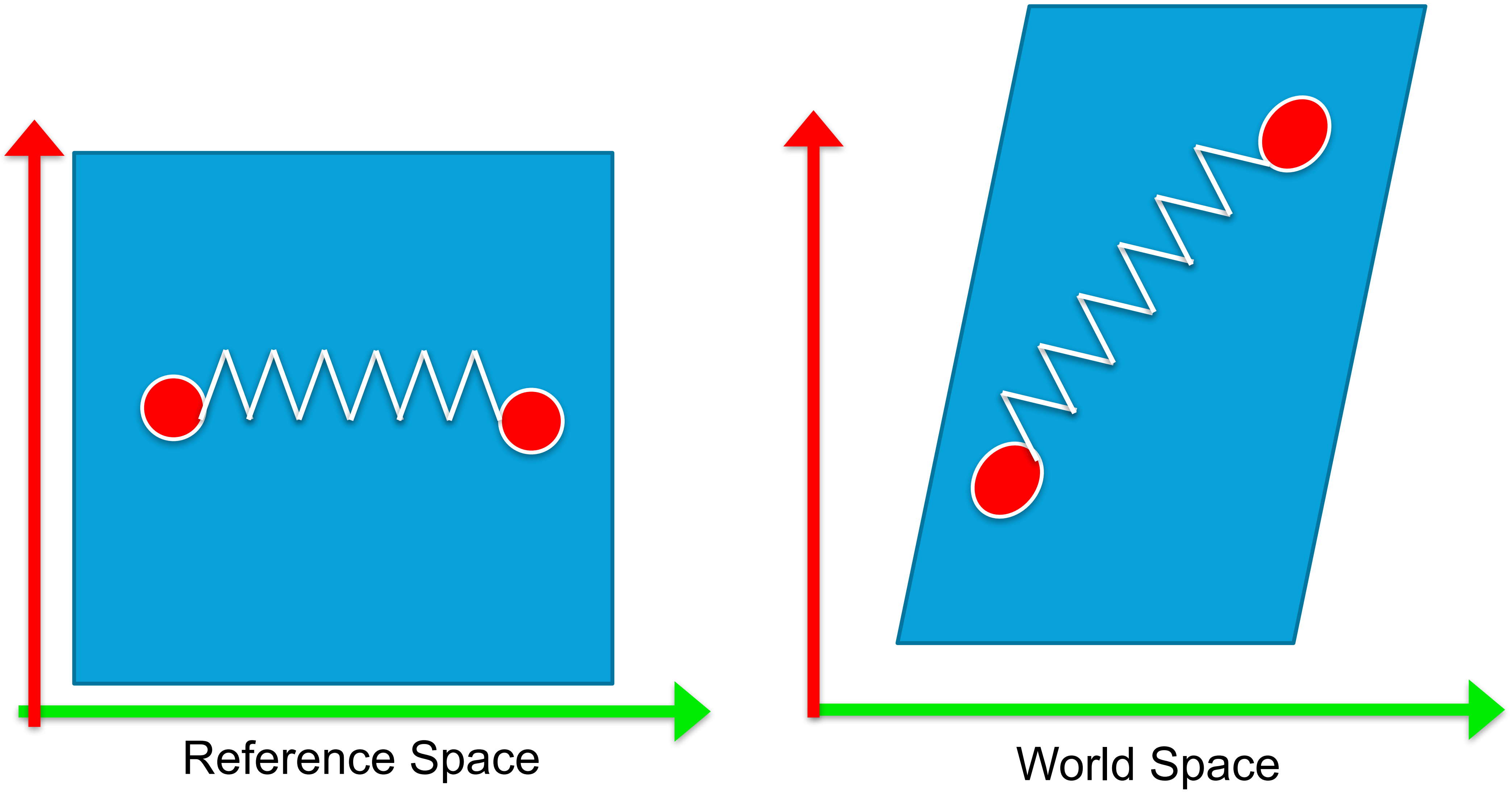
$$\dot{q}^t = \begin{bmatrix} \dot{x}_0^t \\ \dot{x}_1^t \\ \dot{x}_2^t \end{bmatrix}$$



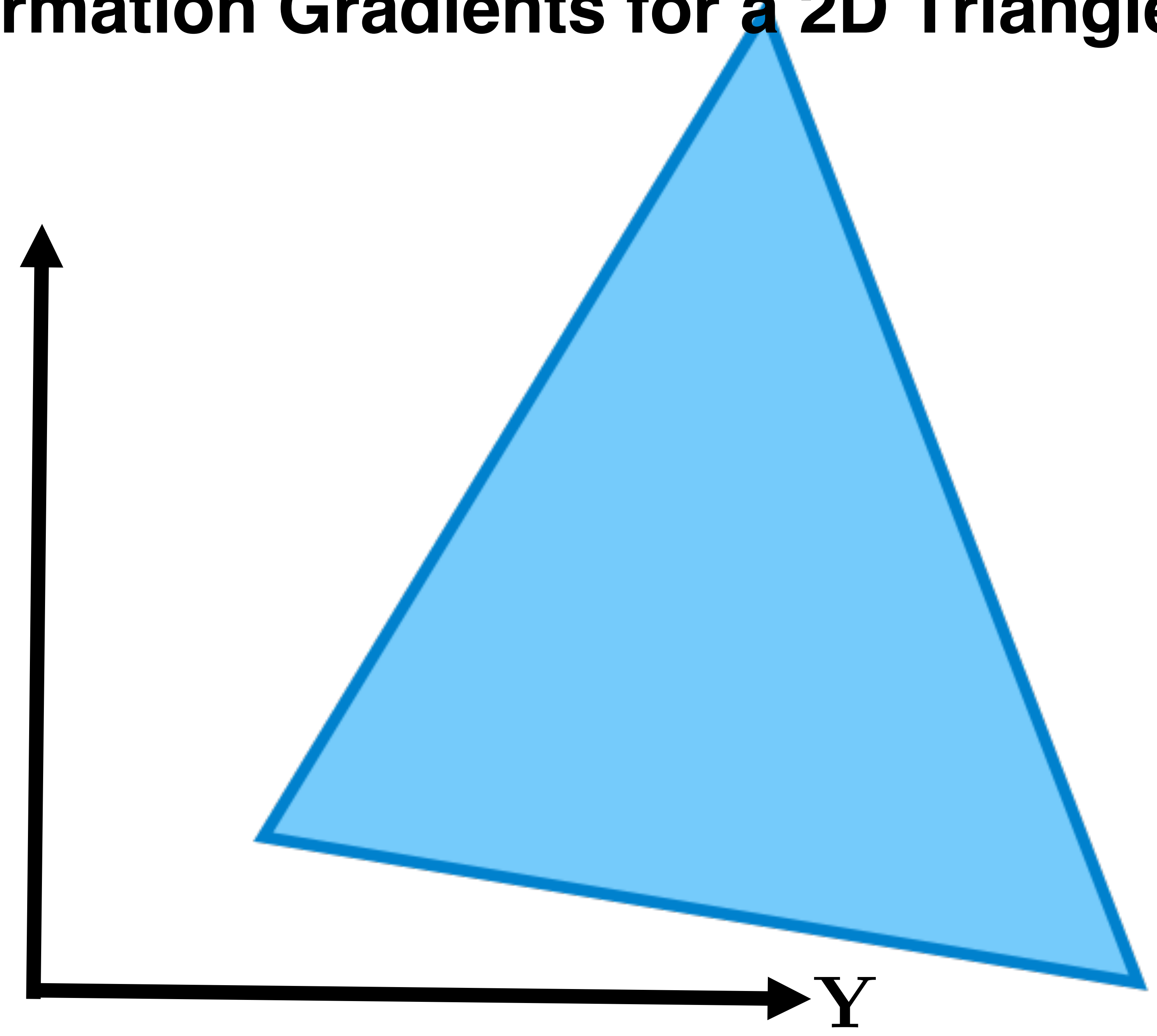
$\dot{q}^t$

q-velocity

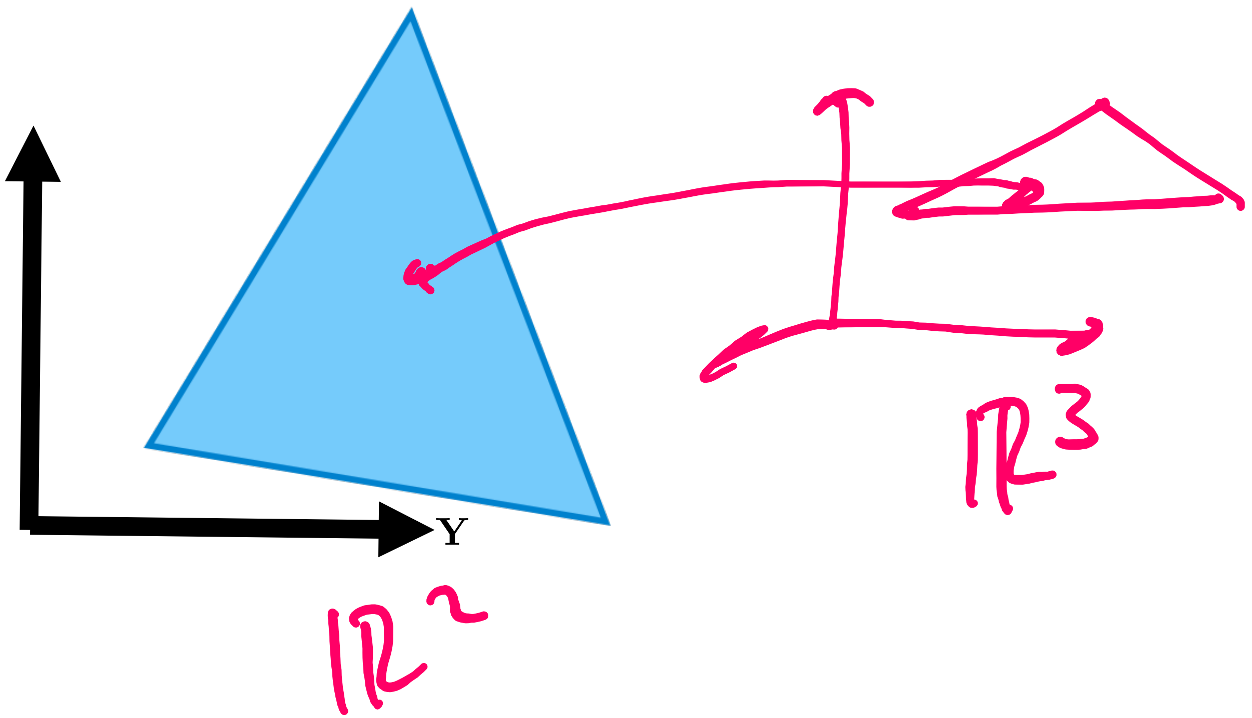
Deformation



Deformation Gradients for a 2D Triangle



Deformation Gradients for a 2D Triangle

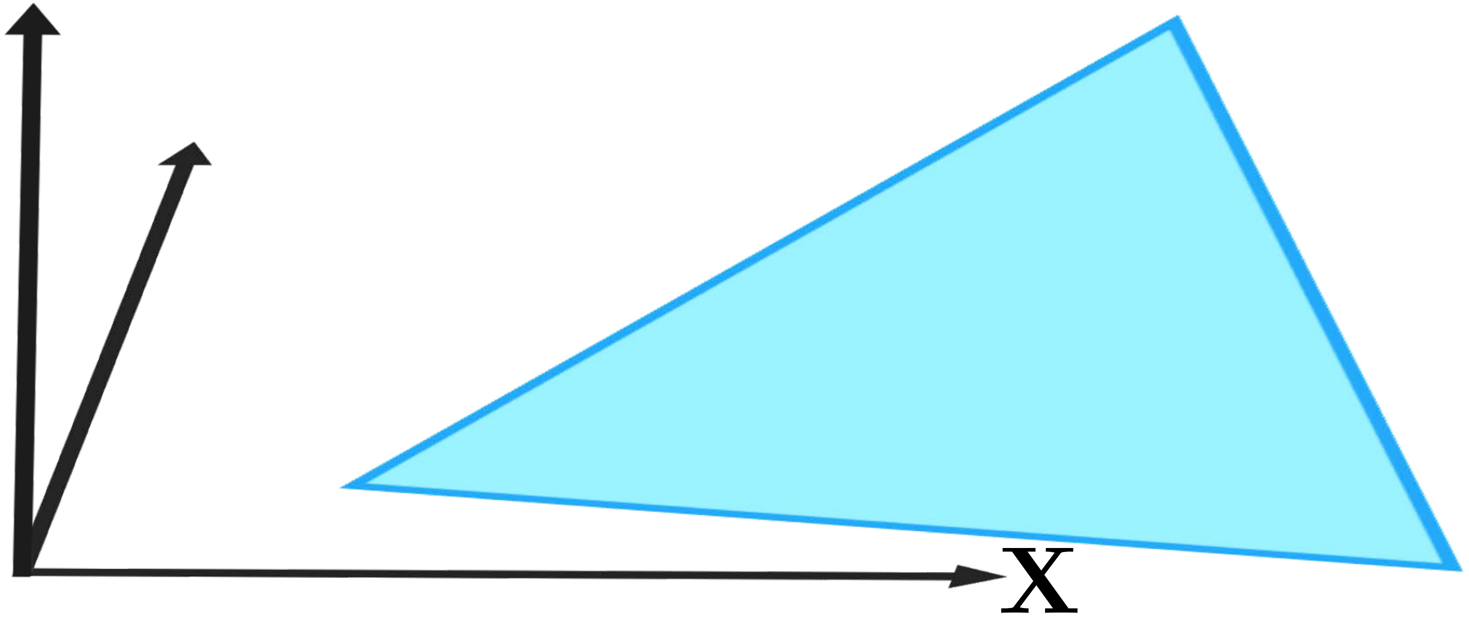


Deformation Gradient: $F = \frac{\partial x}{\partial \bar{X}} \rightarrow \in \mathbb{R}^{3\times 3}$ ($x \leftarrow$ 3D, $\bar{X} \leftarrow$ 3D)

(Volumetric)

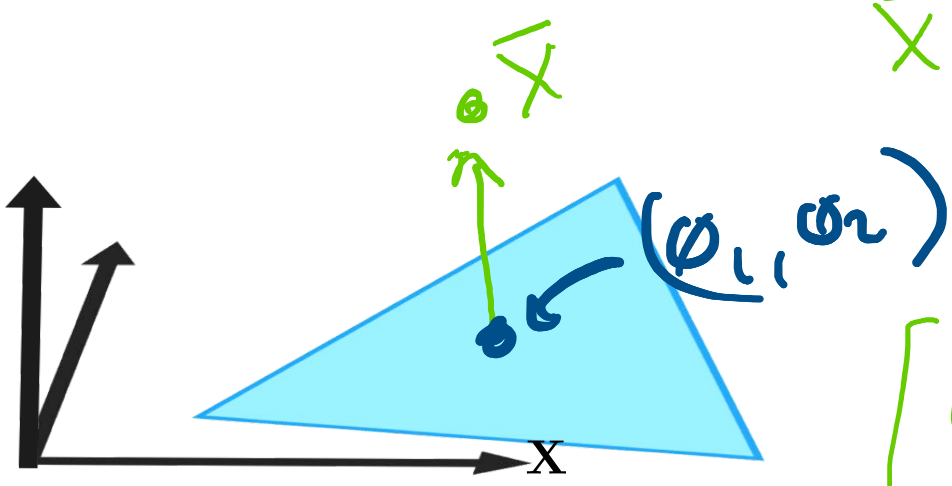
(Cloth) $\boxed{F_{cloth} = \frac{\partial x}{\partial \bar{X}} \rightarrow \in \mathbb{R}^{3\times 2}}$ ($x \leftarrow$ 3D, $\bar{X} \leftarrow$ 2D)

Deformation Gradients for a 2D Triangle in 3D



$$\bar{X} = \begin{bmatrix} \bar{Y} \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

Deformation Gradients for a 2D Triangle in 3D



$$\bar{X} = \bar{X}_0 \phi_0 + \bar{X}_1 \phi_1 + \bar{X}_2 \phi_2$$

Constraint!

$$\underbrace{\left[(\bar{X}_1 - \bar{X}_0) \quad (\bar{X}_2 - \bar{X}_0)\right]}_{D \,\in\, \mathbb{R}^{3\times 2}} \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = \bar{X} - \bar{X}_0$$

$$\underbrace{D^T D}_{\mathbb{R}^{2\times 2}} \begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = D^T(\bar{X} - \bar{X}_0)$$

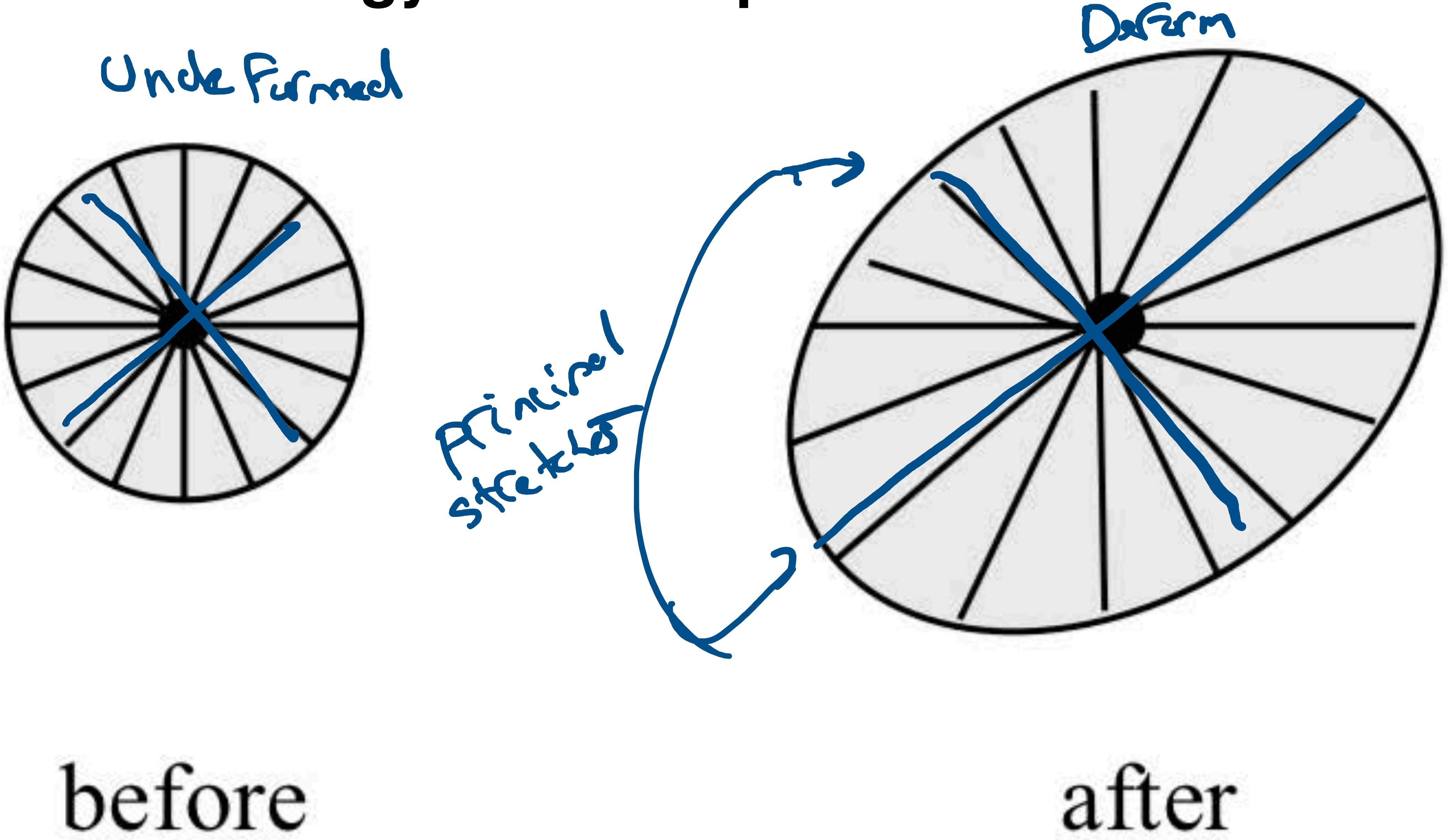
$$\frac{\partial x^t(\bar{X})}{\partial \bar{X}} \in \mathbb{R}^{3\times 3}$$

Kinetic Energy

$$T = \frac{1}{2} \int_{\Delta} \rho \, \dot{q}^T N^T N \dot{q} \, dV$$

$$T = \frac{1}{2} \dot{q}^T \Big(\underbrace{\int_{\Delta} \rho \, N^T N \, dV}_{M_e} \Big) \dot{q}$$

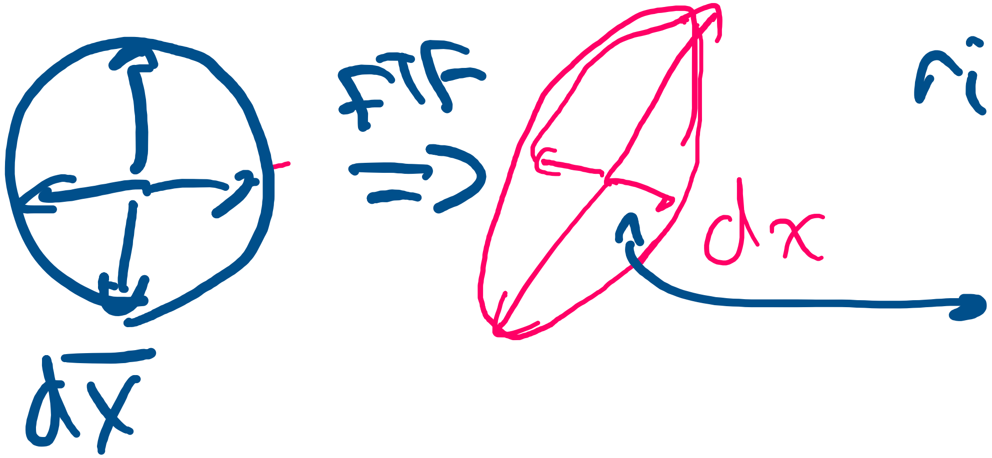
Potential Energy via Principal Stretches



Principal Stretches

Undeformed Deformed

$$|dx|^2 = d\bar{X}^T \underbrace{F^T F}_{\text{right Cauchy strain tensor}} d\bar{X}$$



$$F^T F = V \Lambda U^T$$

$$|dx|^2 = (d\bar{X}^T V)\Lambda(V^T d\bar{X})$$

* Principal Stretches:

$S = \sqrt{\Lambda}$ ← eigenvalues $F^T F$

$F = U \underbrace{S}_{\text{singular values of } F} V^T$

A Quadratic Energy Model

1. Measure deformation
2. Measure "volume" change.

$$\Psi(s_0, s_1, s_2) = \underbrace{\mu \cdot \left(\sum_{i=0}^{2}(s_i - 1)^2\right)}_{\text{Deform}} + \underbrace{\frac{\lambda}{2} \cdot (s_0 \cdot s_1 \cdot s_2 - 1)}_{\text{Volume}}$$

Volumetric

$$\Psi(s_0, s_1) = \mu \sum_{i=0}^{1}(s_i - 1)^2 + \frac{\lambda}{2} \cdot (\underbrace{s_0 \cdot s_1}_{\text{Area}} - 1)^2$$

A Quadratic Energy Model in 2D

Problem: area term isn't quadratic :(

Solution: linearize around $s_0 = 1$ $s_1 = 1$

$$f(s_0, s_1) = s_0 \cdot s_1 \xrightarrow[\text{linearization}]{} \frac{\partial f}{\partial s_0}(s_0 - 1) + \frac{\partial f}{\partial s_1}(s_1 - 1) + f(1,1)$$

$$\Rightarrow 1 + s_0 - 1 + s_1 - 1 = \underbrace{s_0 + s_1 - 1}_{\text{approximation of area term}}$$

We want this to $= 1$

Quadratic penalty: $\frac{1}{2}(s_0 + s_1 - 1 - 1)^2 = \frac{1}{2}(s_0 + s_1 - 2)^2$

Final Energy: $\mu \sum_{i=0}^{1} \underbrace{(s_i - 1)^2}_{\text{deformation}} + \frac{\lambda}{2} \underbrace{(s_0 + s_1 - 2)^2}_{\text{linear}}$

"Linear" Forces that rotate /w element

Co-rotational linearly elasticity

Gradients of Principal Stretch Materials

Forces need $\frac{\partial \Psi}{\partial F}$

$$\text{Forces} = -\underbrace{\frac{\partial F}{\partial q}^T}_{B^T} \frac{\partial V}{\partial F},$$

$$F = \begin{bmatrix} F_{11} \\ F_{12} \\ \vdots \\ F_{33} \end{bmatrix}$$

$$V(F) = \int_\Delta \Psi(F)\, d\Delta, \quad \frac{\partial V}{\partial F} = \int_\Delta \frac{\partial \Psi}{\partial F}\, d\Delta$$

YIKES

$$\Psi(s_0(F), s_1(F)),\qquad F = U\begin{bmatrix} s_0 & & \\ & s_1 & \\ & & s_2 \end{bmatrix}V^T$$

$$\Downarrow$$

$$U^T F V = \begin{bmatrix} s_0 & & \\ & s_1 & \\ & & s_2 \end{bmatrix}$$

$$\frac{\partial \Psi}{\partial F} = \sum_{s_i} \frac{\partial s_i}{\partial F}^T \underbrace{\frac{\partial \Psi}{\partial s_i}}_{\text{scalar}}$$

scalar ← $\partial\Psi$

scalar ← ∂s_i

$$\frac{\partial \Psi}{\partial F} = U \underbrace{\begin{bmatrix} \frac{\partial \Psi}{\partial s_0} & & \\ & \ddots & \\ & & \frac{\partial \Psi}{\partial s_2} \end{bmatrix}}_{\Psi_{,s}} V^T$$

Hessian of Principal Stretch Materials

$$H = \underbrace{\frac{\partial F}{\partial q}}_{B^T}^T \frac{\partial^2 V}{\partial F^2} \underbrace{\frac{\partial F}{\partial q}}_{B} \qquad \frac{\partial^2 V}{\partial F^2} = \int_\Delta \frac{\partial^2 \Psi}{\partial F^2} d\Delta$$

$$\frac{\partial}{\partial F_{ij}} \frac{\partial \Psi}{\partial F} = \frac{\partial}{\partial F_{ij}} \left(U \Psi_{,s} V^T \right)$$

$$= \frac{\partial U}{\partial F_{ij}} \Psi_{,s} V^T + U \frac{\partial \Psi_{,s}}{\partial F_{ij}} V^T + U \Psi_{,s} \frac{\partial V^T}{\partial F_{ij}}$$

$$\frac{\partial \Psi_{,s}}{\partial F_{ij}} \underset{\text{Chain Rule}}{\Longrightarrow} \sum_{i=0}^{2} \frac{\partial s_k}{\partial F_{ij}} \frac{\partial \Psi_{,s}}{\partial s_k} = \underbrace{\frac{\partial s_0}{\partial F_{ij}}}_{\text{Scalar}} \underbrace{\frac{\partial \Psi_{,s}}{\partial s_0}}_{\text{vector}} + \underbrace{\frac{\partial s_1}{\partial F_{ij}}}_{\text{scalar}} \underbrace{\frac{\partial \Psi_{,s}}{\partial s_1}}_{\text{vector}} \cdots$$

equivalent to

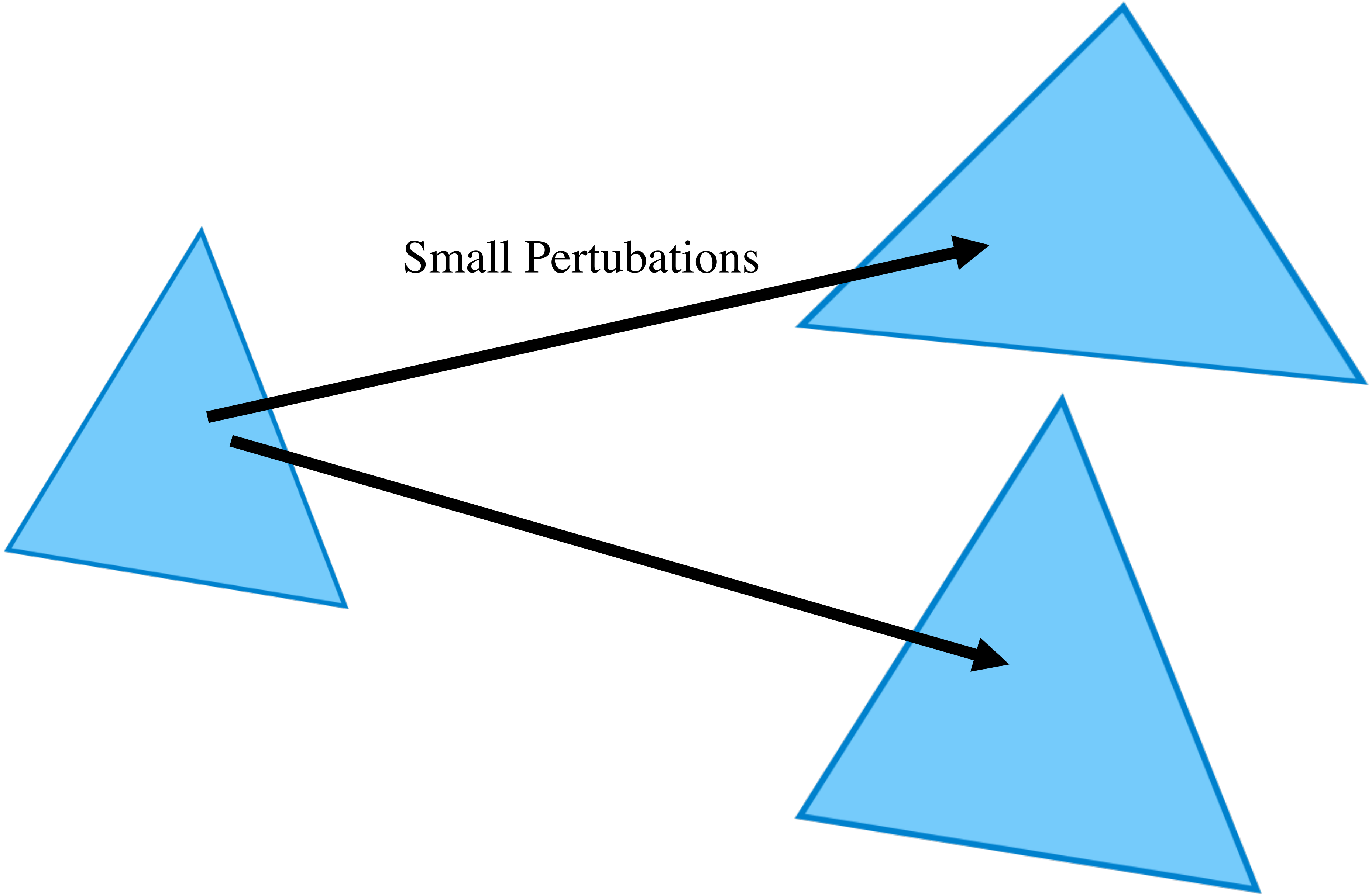
$$\underbrace{\begin{bmatrix} \frac{\partial \Psi}{\partial s_0^2} & \frac{\partial \Psi}{\partial s_1 \partial s_0} & 0 \\ \frac{\partial \Psi}{\partial s_1 \partial s_0} & \frac{\partial \Psi}{\partial s_1^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{You compute}} \cdot \begin{bmatrix} \frac{\partial s_0}{\partial F_{ij}} \\ \frac{\partial s_1}{\partial F_{ij}} \\ \frac{\partial s_2}{\partial F_{ij}} \end{bmatrix} \leftarrow \text{Given Code!}$$

Vector ds^{ij} <

$$\frac{\partial U}{\partial F_{ij}} \Psi_{,s} V^T + U \frac{\partial \Psi_{,s}}{\partial F_{ij}} V^T + U \Psi_{,s} \frac{\partial V^T}{\partial F_{ij}}$$

$$= \frac{\partial U}{\partial F_{ij}} \Psi_{,s} V^T + U \underbrace{\begin{bmatrix} ds_1^{ij} & & \\ & ds_2^{ij} & \\ & & ds_3^{ij} \end{bmatrix}}_{\text{turn ds into diagonal matrix}} V^T + U \Psi_{,s} \frac{\partial U^T}{\partial F_{ij}}$$

The Singularity



Collisions in Simulation

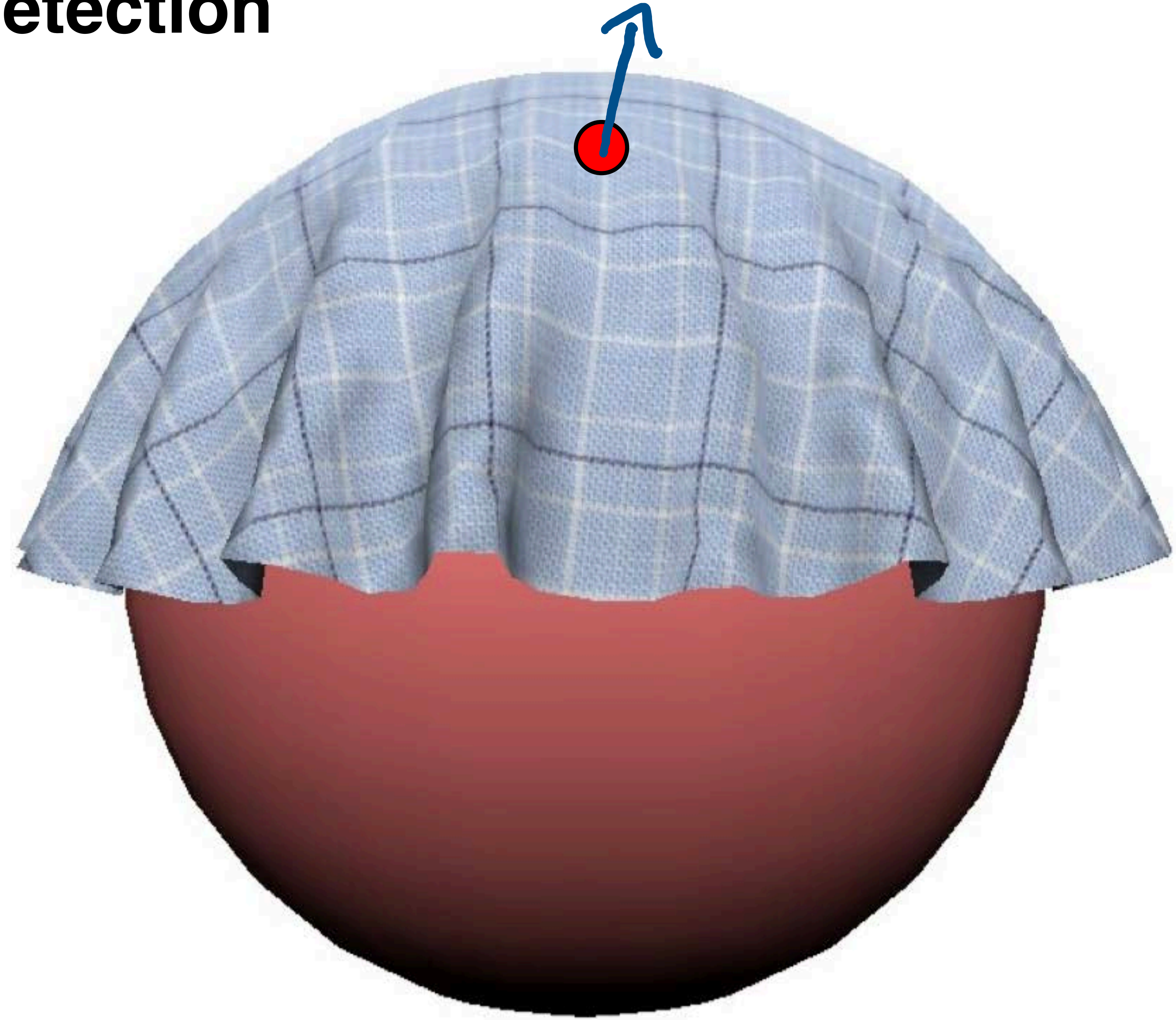
Two phases detection and response

Detection: Did I hit anything ?

Response: I hit something ! What do I do ?

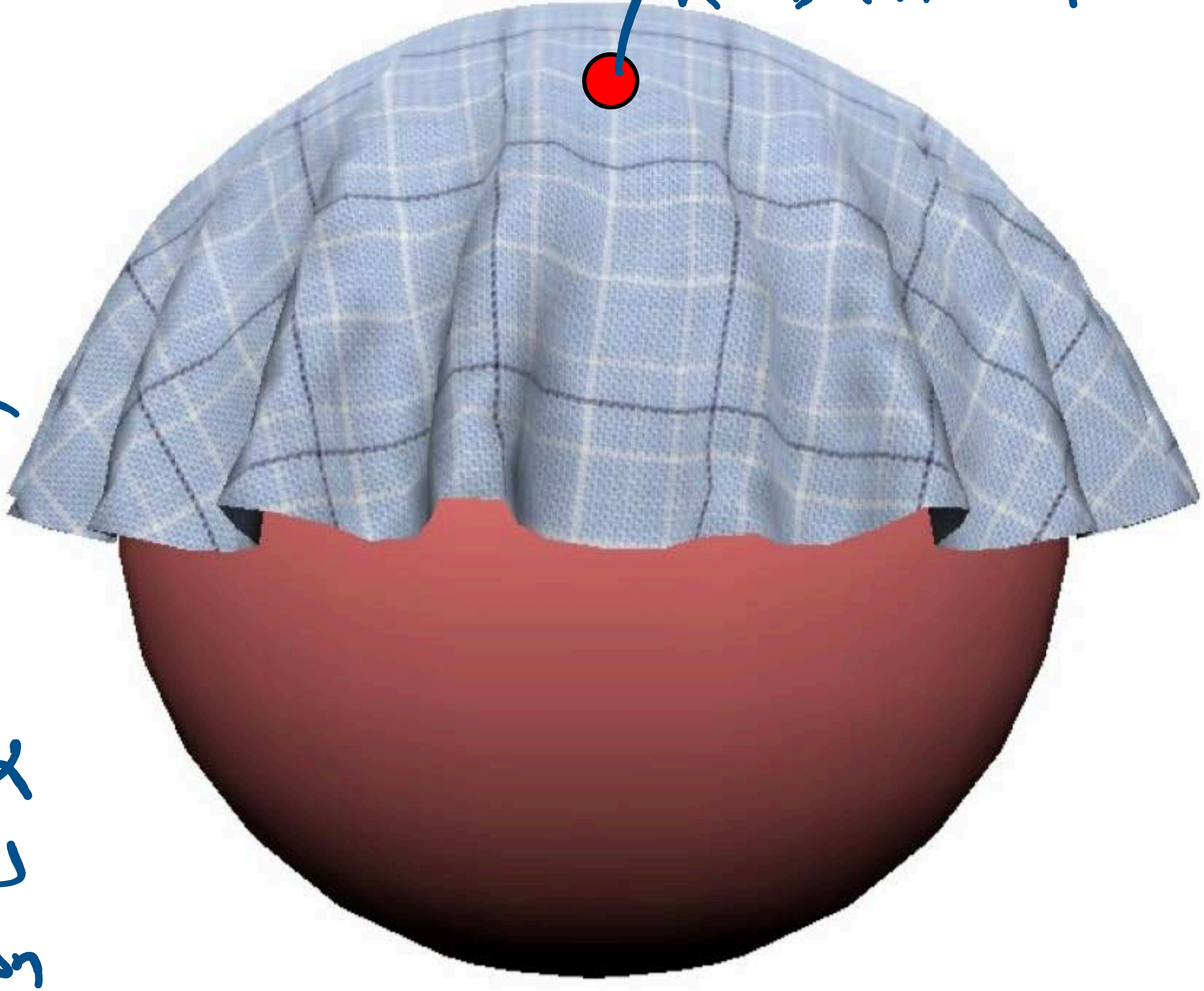
Assignment #4 collision with analytical sphere.

Collision Detection



Collision Response

$\vec{n}$, $|\vec{n}| = 1$



$\alpha = n^T \dot{q}_c$

collision index

if $\alpha < 0$

filtered

$$\underbrace{\dot{q}_i = \dot{q}_c - \vec{n}\alpha}_{\text{projection}}$$

Next Week: