

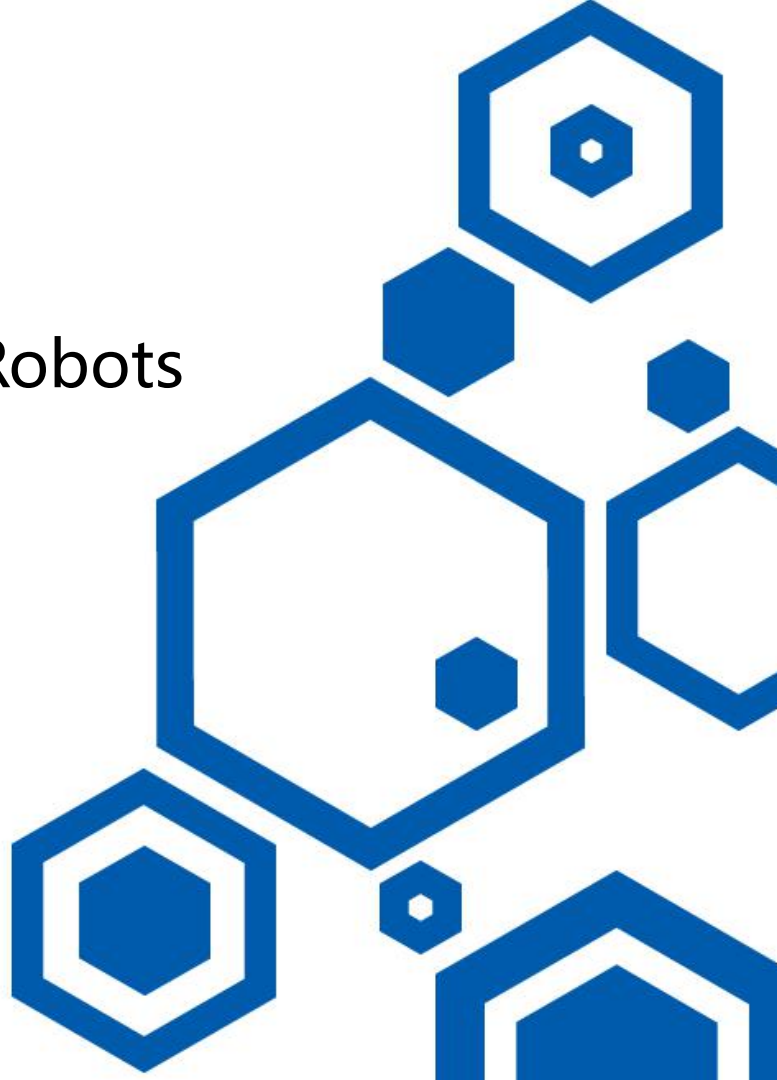


Motion Planning for Mobile Robots

第五章作业讲评



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Homework 1.1: In matlab, use the quadprog QP solver, write down a minimum snap trajectory generator

Homework 1.2: In matlab, generate minimum snap trajectory based on the closed form solution

Homework 2.1: In C++/ROS, use the OOQP solver, write down a minimum snap trajectory generator

Homework 2.2: In C++/ROS, use Eigen, generate minimum snap trajectory based on the closed form solution

HW 1

Matlab 作业

本次作业采用 `many relative timeline`，分段轨迹表达式如下：

$$f_k(t) = \sum_{i=0}^N p_{k,i} t^i, \quad 0 \leq t \leq \Delta T_k$$

- Derivative constraints : 规定起始状态 p, v, a, j 、终止状态 p, v, a, j 、中间点 p
- Continuity constraints : 规定为前后两端轨迹交点处 p, v, a, j 连续

本次作业独立求解 x 和 y 轴的多项式系数，该功能封装为函数 `MinimumSnapQPSolver()` 和 `MinimumSnapCloseformSolver()`，分别代表 QP 解法和闭式解法。

$$\begin{aligned} \min \quad & \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_M \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix} \\ \text{s.t.} \quad & \mathbf{A}_{\text{eq}} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix} = \mathbf{d}_{\text{eq}} \end{aligned}$$

It's a typical **convex optimization** program.

HW 1.1

getQ()

$$f(t) = \sum_i p_i t^i$$

$$\Rightarrow f^{(4)}(t) = \sum_{i \geq 4} i(i-1)(i-2)(i-3)t^{i-4}p_i$$

$$\Rightarrow (f^{(4)}(t))^2 = \sum_{i \geq 4, l \geq 4} i(i-1)(i-2)(i-3)l(l-1)(l-2)(l-3)t^{i+l-8}p_i p_l$$

$$\Rightarrow J(T) = \int_{T_{j-1}}^{T_j} (f^{(4)}(t))^2 dt = \sum_{i \geq 4, l \geq 4} \frac{i(i-1)(i-2)(i-3)l(l-1)(l-2)(l-3)}{i+l-7} (T_j^{i+l-7} - T_{j-1}^{i+l-7}) p_i p_l$$

$$\Rightarrow J(T) = \int_{T_{j-1}}^{T_j} (f^{(4)}(t))^2 dt$$

$$= \begin{bmatrix} \vdots \\ p_i \\ \vdots \end{bmatrix}^T \begin{bmatrix} \vdots & \vdots \\ \dots & \frac{i(i-1)(i-2)(i-3)l(l-1)(l-2)(l-3)}{i+l-7} T_j^{i+l-7} & \dots \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ p_l \\ \vdots \end{bmatrix}$$

$$\Rightarrow J_j(T) = \mathbf{p}_j^T \mathbf{Q}_j \mathbf{p}_j \quad \text{Minimize this!}$$

```
1 for i = 4:n_order
2   for l = 4:n_order
3     den = i + l - 7;
4     Q_k(i+1,l+1) = i*(i-1)*(i-2)*(i-3)*l*(l-1)*(l-2)*(l-3)/den*(t_k^den);
5   end
6 end
```

$$\min \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_M \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_M \end{bmatrix}$$

升幂

$$f(t) = [p_0, p_1, \dots, p_7] \cdot [1, t, t^2, \dots, t^7] = \mathbf{p} \cdot [1, t, t^2, \dots, t^7]$$

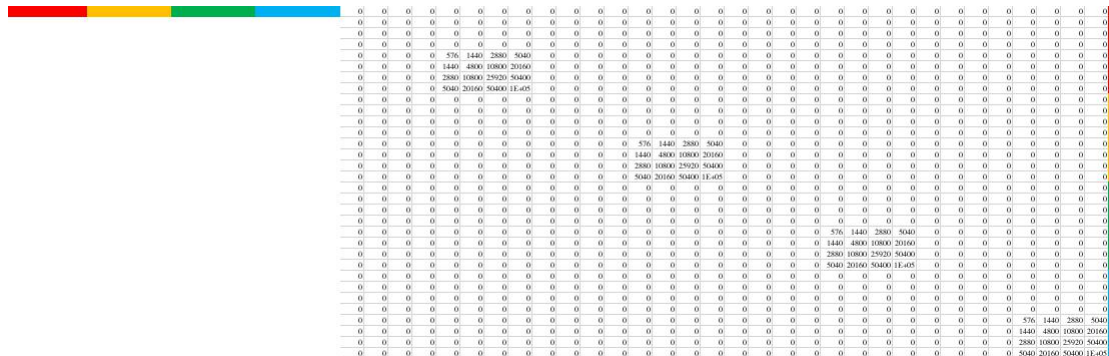
$$f'(t) = \mathbf{p} \cdot [0, 1, 2t, 3t^2, 4t^3, \dots, 7t^6]$$

$$f''(t) = \mathbf{p} \cdot [0, 0, 2, 6t, 12t^2, \dots, 42t^5]$$

$$f'''(t) = \mathbf{p} \cdot [0, 0, 0, 6, 24t, \dots, 210t^4]$$

HW 1.1

getQ()



$$\min \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_M \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_M \end{bmatrix}$$

升幂

$$f(t) = [p_0, p_1, \dots, p_7] \cdot [1, t, t^2, \dots, t^7] = \mathbf{p} \cdot [1, t, t^2, \dots, t^7]$$

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$$f''(t) = \mathbf{p} \cdot [0, 0, 2, 6t, 12t^2, \dots, 42t^5]$$

$$f'''(t) = \mathbf{p} \cdot [0, 0, 0, 6, 24t, \dots, 210t^4]$$

```

1 for i = 4:n_order
2     for l = 4:n_order
3         den = i + l - 7;
4         Q_k(i+1,l+1) = i*(i-1)*(i-2)*(i-3)*l*(l-1)*(l-2)*(l-3)/den*(t_k^den);
5     end
6 end
    
```

HW 1.1

getAbeq()

- Derivative constraint for one polynomial segment

- Also models waypoint constraint (0^{th} order derivative)

$$\begin{aligned} f_j^{(k)}(T_j) &= x_j^{(k)} \\ \Rightarrow \sum_{l \geq k} \frac{l!}{(l-k)!} T_j^{l-k} p_{j,l} &= x_{T,j}^{(k)} \\ \Rightarrow \begin{bmatrix} \dots & \frac{l!}{(l-k)!} T_j^{l-k} & \dots \end{bmatrix} \begin{bmatrix} \vdots \\ p_{j,l} \end{bmatrix} &= x_{T,j}^{(k)} \\ \Rightarrow \begin{bmatrix} \dots & \frac{l!}{(l-k)!} T_{j-1}^{l-k} & \dots \end{bmatrix} \begin{bmatrix} \vdots \\ p_{j,l} \end{bmatrix} &= \begin{bmatrix} x_{0,j}^{(k)} \\ x_{T,j}^{(k)} \end{bmatrix} \\ \Rightarrow \mathbf{A}_j \mathbf{p}_j &= \mathbf{d}_j \end{aligned}$$

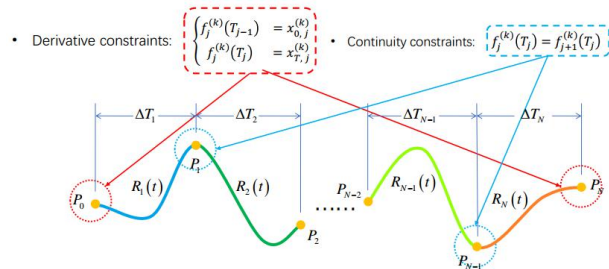
- Continuity constraint between two segments

- Ensures continuity between trajectory segments when no specific derivatives are given

$$\begin{aligned} f_j^{(k)}(T_j) &= f_{j+1}^{(k)}(T_j) \\ \Rightarrow \sum_{l \geq k} \frac{l!}{(l-k)!} T_j^{l-k} p_{j,l} - \sum_{l \geq k} \frac{l!}{(l-k)!} T_j^{l-k} p_{j+1,l} &= 0 \\ \Rightarrow \begin{bmatrix} \dots & \frac{l!}{(l-k)!} T_j^{l-k} & \dots & -\frac{l!}{(l-k)!} T_j^{l-k} & \dots \end{bmatrix} \begin{bmatrix} \vdots \\ p_{j,l} \\ \vdots \\ p_{j+1,l} \end{bmatrix} &= 0 \\ \Rightarrow [\mathbf{A}_j - \mathbf{A}_{j+1}] \begin{bmatrix} \mathbf{p}_j \\ \mathbf{p}_{j+1} \end{bmatrix} &= 0 \end{aligned}$$

这里主要根据 Derivative constraints 和 Continuity constraints 构建方程，最终整理成 \mathbf{A}_{eq} 和 \mathbf{b}_{eq}

Constraints:



HW 1.1

```
%#####  
% p,v,a,j constraint in start,  
Aeq_start = zeros(4, n_all_poly);  
T = 0;  
for k = 0 : 3 % p,v,a,j  
    for i = k : n_order % i >= k  
        Aeq_start(k+1,i+1) = factorial(i)/factorial(i-k)*(T^(i-k));  
    end  
end  
beq_start = start_cond';  
  
%#####  
% p,v,a,j constraint in end  
Aeq_end = zeros(4, n_all_poly);  
T = ts(end);  
idx = (n_seg-1)*(n_order+1);  
for k = 0 : 3 % p,v,a,j  
    for i = k : n_order % i >= k  
        Aeq_end(k+1,idx+i+1) = factorial(i)/factorial(i-k)*(T^(i-k));  
    end  
end  
beq_end = end_cond';  
  
%#####  
% position constrain in all middle waypoints  
Aeq_wp = zeros(n_seg-1, n_all_poly);  
for n = 0 : n_seg-1-1  
    T = ts(n+1);  
    idx = (n)*(n_order+1);  
    for i = 0 : n_order  
        Aeq_wp(n+1, idx+i+1) = T^i;  
    end  
end  
beq_wp = waypoints(2:end-1);
```

- Derivative constraint for one polynomial segment
 - Also models waypoint constraint (0^{th} order derivative)

$$\begin{aligned} f_j^{(k)}(T_j) &= x_j^{(k)} \\ \Rightarrow \sum_{i \geq k} \frac{i!}{(i-k)!} T_j^{i-k} p_{j,i} &= x_{T,j}^{(k)} \\ \Rightarrow \begin{bmatrix} \dots & \frac{i!}{(i-k)!} T_j^{i-k} & \dots \end{bmatrix} \begin{bmatrix} \vdots \\ p_{j,i} \\ \vdots \end{bmatrix} &= x_{T,j}^{(k)} \\ \Rightarrow \begin{bmatrix} \dots & \frac{i!}{(i-k)!} T_{j-1}^{i-k} & \dots \end{bmatrix} \begin{bmatrix} \vdots \\ p_{j,i} \\ \vdots \end{bmatrix} &= \begin{bmatrix} x_{0,j}^{(k)} \\ x_{T,j}^{(k)} \end{bmatrix} \\ \Rightarrow \mathbf{A}_j \mathbf{p}_j &= \mathbf{d}_j \end{aligned}$$

HW 1.1

```
%#####
% position continuity constrain between each 2 segments
Aeq_con_p = zeros(n_seg-1, n_all_poly);
beq_con_p = zeros(n_seg-1, 1);
% STEP 2.4: write expression of Aeq_con_p and beq_con_p
k = 0; % k = 0,1,2,3
for n = 0 : n_seg-1-1
    T = ts(n+1);
    idx = (n)*(n_order+1);
    for i = k : n_order
        Aeq_con_p(n+1, idx+i+1) = factorial(i)/factorial(i-k)*(T^(i-k));
    end
    T = 0;
    idx = (n+1)*(n_order+1);
    for i = k : n_order
        Aeq_con_p(n+1, idx+i+1) = -factorial(i)/factorial(i-k)*(T^(i-k));
    end
end
end

for i=0:n_seg-1
    %#####
    % STEP 3: get the coefficients of i-th segment of both x-axis and
    idx = (i*n_poly_perseg+1) : ((i+1)*n_poly_perseg);
    Pxi = flipud(poly_coef_x(idx));
    Pyi = flipud(poly_coef_y(idx));
    for t = 0:tstep:ts(i+1)
        X_n(k) = polyval(Pxi, t);
        Y_n(k) = polyval(Pyi, t);
        k = k + 1;
    end
end
```

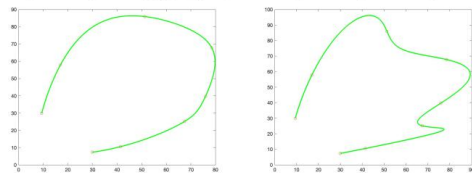
- Continuity constraint between two segments
 - Ensures continuity between trajectory segments when no specific derivatives are given

$$f_j^{(k)}(T_j) = f_{j+1}^{(k)}(T_j)$$

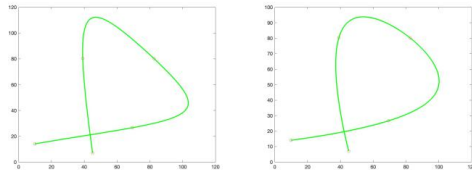
$$\Rightarrow \sum_{l \geq k} \frac{l!}{(l-k)!} T_j^{l-k} p_{j,l} - \sum_{l \geq k} \frac{l!}{(l-k)!} T_j^{l-k} p_{j+1,l} = 0$$

$$\Rightarrow \left[\dots \frac{l!}{(l-k)!} T_j^{l-k} \dots - \frac{l!}{(l-k)!} T_j^{l-k} \dots \right] \begin{bmatrix} p_{j,l} \\ \vdots \\ p_{j+1,l} \end{bmatrix} = 0$$

$$\Rightarrow [A_j \quad -A_{j+1}] \begin{bmatrix} p_j \\ p_{j+1} \end{bmatrix} = 0$$



运行截图(左: 时间间隔固定, 右: 时间间隔按距离划分)



运行截图(左: 时间间隔固定, 右: 时间间隔按距离划分)

HW 1.2

- We have $\mathbf{M}\mathbf{p}=\mathbf{d}$, where \mathbf{M} is a mapping matrix that maps polynomial coefficients to derivatives
- Use a selection matrix \mathbf{C} to separate free (\mathbf{d}_P) and constrained (\mathbf{d}_F) variables
 - Free variables : derivatives unspecified, only enforced by continuity constraints

$$\begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_M \end{bmatrix} = \mathbf{C}^T \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix} \quad J = \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}^T \underbrace{\mathbf{C}\mathbf{M}^{-T}\mathbf{Q}\mathbf{M}^{-1}\mathbf{C}^T}_{\mathbf{R}} \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix} = \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}^T \begin{bmatrix} \mathbf{R}_{FF} & \mathbf{R}_{FP} \\ \mathbf{R}_{PF} & \mathbf{R}_{PP} \end{bmatrix} \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}$$

- Turned into an unconstrained quadratic programming that can be solved in closed form:

$$J = \mathbf{d}_F^T \mathbf{R}_{FF} \mathbf{d}_F + \mathbf{d}_F^T \mathbf{R}_{FP} \mathbf{d}_P + \mathbf{d}_P^T \mathbf{R}_{PF} \mathbf{d}_F + \mathbf{d}_P^T \mathbf{R}_{PP} \mathbf{d}_P$$

$$\mathbf{d}_P^* = -\mathbf{R}_{PP}^{-1} \mathbf{R}_{FP}^T \mathbf{d}_F$$

HW 1.2

getM()

```
M = [];  
for n = 1:n_seg  
    M_k = [];  
    %#####  
    % STEP 1.1: calculate M_k of the k-th segment  
    T = 0;  
    for k = 0 : 3 % p,v,a,j at t0  
        for i = k : n_order  
            M_k(k+1,i+1) = factorial(i)/factorial(i-k)*(T^(i-k));  
        end  
    end  
    T = ts(n);  
    for k = 0 : 3 % p,v,a,j at T  
        for i = k : n_order  
            M_k(4+k+1,i+1) = factorial(i)/factorial(i-k)*(T^(i-k));  
        end  
    end  
    M = blkdiag(M, M_k);  
end
```

$$M_j \mathbf{p}_j = \mathbf{d}_j$$

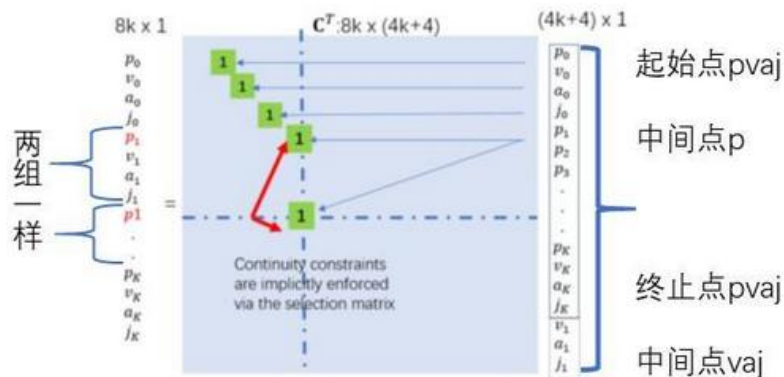
$$f_j^{(k)}(T_j) = x_j^{(k)} \\ \Rightarrow \sum_{i \geq k} \frac{i!}{(i-k)!} T_j^{i-k} p_{j,i} = x_{T,j}^{(k)}$$

HW 1.2

getCt()

%% Check Ct

```
syms p0 v0 a0 j0 p1 p2 p3 v3 a3 j3 a1 v1 j1 a2 v2 j2 real
dfdp = [p0 v0 a0 j0 p1 p2 p3 v3 a3 j3, v1 a1 j1, v2 a2 j2]';
d = [p0 v0 a0 j0 p1 v1 a1 j1 p1 v1 a1 j1 p2 v2 a2 j2 p2 v2 a2 j2 p3 v3 a3 j3]';
Ct = getCt(3, 7);
dd = Ct*dfdp;
assert(isequaln(d,dd))
```



```
#####
% STEP 2.1: finish the expression of Ct
% fixed derivatives
% p0,v0,a0,j0
for i = 1 : 4
    Ct(i,i) = 1;
end
% p1,p2,...,p(n-1)
idx_df = size(Ct,2);
for i = 1 : n_seg - 1
    idx = i*8;
    Ct(idx-4+1, idx_df+i) = 1;
    Ct(idx+1, idx_df+i) = 1;
end
% pn,vn,a0,jn
idx_df = size(Ct,2);
for i = 1 : 4
    idx = n_seg*8-4;
    Ct(idx+1, idx_df+i) = 1;
end
% free derivatives
% v1,a1,j1,...,v(n-1),a(n-1),j(n-1)
idx_df = size(Ct,2);
for i = 1 : n_seg-1
    idx = i*8;
    idxf2 = idx_df + (i-1)*3;
    Ct(idx-4+2, idxf2+1) = 1;
    Ct(idx-4+3, idxf2+2) = 1;
    Ct(idx-4+4, idxf2+3) = 1;
    Ct(idx+2, idxf2+1) = 1;
    Ct(idx+3, idxf2+2) = 1;
    Ct(idx+4, idxf2+3) = 1;
end
end
```

HW 2

```
1. double accInv = 1.0 / _Acc;
2. double velInv = 1.0 / _Vel;
3.
4. // 加速时间与距离, 减速需要耗费同样资源
5. double accTime = _Vel * accInv; // v = v0 + at, v0 = 0
6. double accDist = 0.5 * _Vel * accTime; // d = 0.5 * (v0 + v) * t, v0 = 0
7. double accTime2 = 2.0 * accTime;
8. double accDist2 = 2.0 * accDist; // 包含加速和减速的总距离
9.
10. for(int i = 0; i < Path.rows() - 1; i++)
11. {
12.     double t = 0.0;
13.     double dist = (Path.row(i + 1) - Path.row(i)).norm();
14.
15.     if(dist <= accDist2)
16.         t = 2.0 * std::sqrt(dist * accInv); // 0.5 * d = v0 * t + 0.5 * a * t * t, v0 = 0
17.     else
18.         t = accTime2 + (dist - accDist2) * velInv;
19.
20.     time(i) = t;
21. }
```

- OOPQ
- OSQP
- qpOASES

感谢各位聆听

Thanks for Listening

