$\LaTeX$ command declarations here.

In [1]:
```python
# plotting
%matplotlib inline
from matplotlib import pyplot as plt;
if "bmh" in plt.style.available: plt.style.use("bmh");

# scientific
import numpy as np;

from scipy import linalg
import matplotlib as mpl
from matplotlib import colors

from sklearn.discriminant_analysis import LinearDiscriminantAnalysis
from sklearn.discriminant_analysis import QuadraticDiscriminantAnalysi

# rise config
from notebook.services.config import ConfigManager
cm = ConfigManager()
cm.update('livereveal', {
              'theme': 'simple',
              'start_slideshow_at': 'selected',
})
```

Out[1]:
```
{u'start_slideshow_at': 'selected', u'theme': 'simple', u'transitio
n': u'none'}
```

EECS 545: Machine Learning

Lecture 06: Probability Models and Logistic Regression

- Instructor: **Jacob Abernethy**
- Date: January 27, 2016

Lecture Exposition Credit: Benjamin Bray, Saket Dewangan

HW2 + Kaggle

- Check out http://kaggle.com (http://kaggle.com), an online platform for machine learning competitive challenges

learning competitive challenges
- Homework 2 requires not one but two submissions to "in-class" Kaggle challenges
- There will be a performance requirement (i.e. need your_score > score_benchmark)
- Winners will get bragging rights and a prize in lecture

Small Aside: check out the Michigan Data Science Team

- http://mdst.eecs.umich.edu (http://mdst.eecs.umich.edu)
- Like doing Kaggle competitions for fun? We give prizes!
- We got a little mention in this morning's University Record (http://record.umich.edu/articles/program-gives-undergraduates-use-high-performance-computing)

New Location for Lecture Content on Github

- EECS 545 Lectures now Open Source!
- Henceforth, lectures will be hosted at https://github.com/thejakeyboy/umich-eecs545-lectures (https://github.com/thejakeyboy/umich-eecs545-lectures)
- Will soon be using Github, not Canvas, for lecture notes
- We have an impressive lecture development team:
  - Ben Bray
  - Saket Dewangan
  - Valli Chockalingam
  - me

Extra Credit Opportunity #1: Lecture development

- Imagine that you:
  1. Discover some bugs in the slides
  2. Find a way to make a better visualization
  3. Have some nice content that you want to contribute as additional material, etc.
- If so: clone the repository, make a contribution, and submit it as a *patch request*!
- Contributions that we deem are great receive 0-2pts extra credit on the final exam (up to a total of 10pts)

(up to a total of 10pts)

Extra Credit Opportunity #2: HW Solutions

- Starting now, **you** can help us develop the HW solutions. **IF**
  - you write up your HW in $\LaTeX$ AND
  - your solutions are really simple and beautiful AND
  - you submit your homework early (by Friday, 3 days before due date)
- Then we may contact you to contribute your .tex file
- GSIs will compile the solutions, and for each contributed answer we'll give you roughly 1pt on your final exam (up to 10pts)

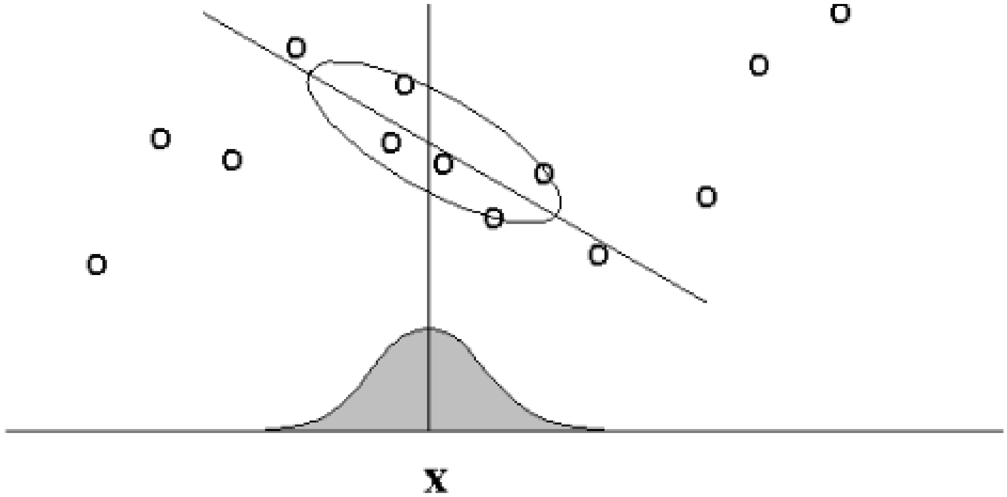
Outline

- Locally-Weighted Linear Regression
- Probabilistic Models
  - Generative Models
  - Discriminative Models
- Logistic Regression
  - Intuition, Motivation
  - Newton's Method

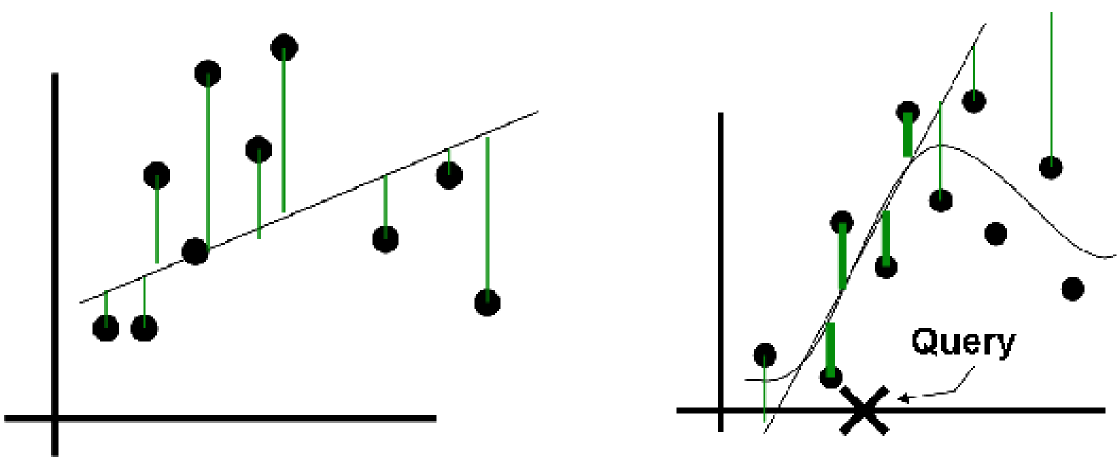
Locally-Weighted Linear Regression

Locally-Weighted Linear Regression

Main Idea: When predicting $f(x)$, give high weights for *neighbors* of x.



Regular vs. Locally-Weighted Linear Regression



Regular linear regression

Locally weighted linear regression

Regular vs. Locally-Weighted Linear Regression

Linear Regression

1. Fit w to minimize $\sum_k (t_k - w^T \phi(x_k))^2$
2. Output $w^T \phi(x_k)$

Locally-weighted Linear Regression

1. Fit w to minimize $\sum_k r_k(t_k - w^T \phi(x_k))^2$ for some weights r_k
2. Output $w^T \phi(x_k)$

Locally-Weighted Linear Regression

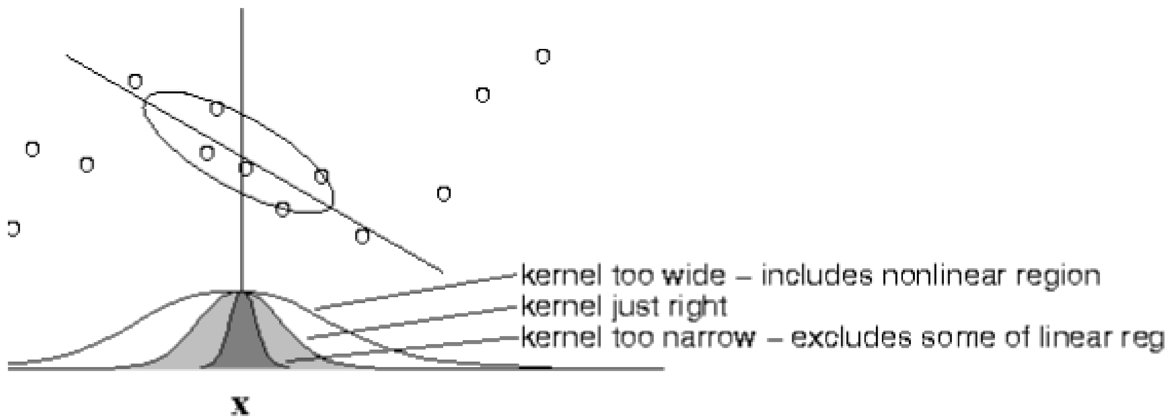
- The standard choice for weights r uses the **Gaussian Kernel**, with **kernel width** τ

$$r_k = \exp\left(-\frac{||x_k - x||^2}{2\tau^2}\right)$$

- Note r_k depends on both x (query point); must solve linear regression for each query point x.
- Can be reformulated as a modified version of least squares problem.

Locally-Weighted Linear Regression

- Choice of kernel width matters.
  - (requires hyperparameter tuning!)



> The estimator is minimized when kernel includes as many training points as can be accomodated by the model. Too large a kernel includes points that degrade the fit; too small a kernel neglects points that increase confidence in the fit.

Let's back up a little...

Probabilistic Models & Bayesian Statistics

> *Disclaimer:* These slides were written by a Bayesian :)

Probabilistic Models & Bayesian Statistics

Last time, there were several questions about **priors**.

- Represent prior beliefs about acceptable values for model parameters.
- *Example:* In linear regression, L_2 regularization can be interpreted as placing a **Gaussian Prior** on the regression coefficients.

Probabilistic Models & Bayesian Statistics

All statistical models and machine learning algorithms make assumptions.

- All reasoning is based on implicit assumptions.
- A **Bayesian** will tell you that his prior is a way of explicitly stating those assumptions.

Probabilistic Models & Bayesian Statistics

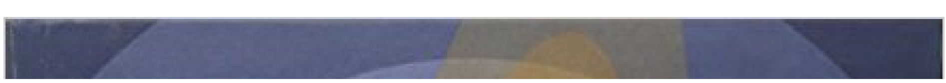
This can all get very philosophical, but...

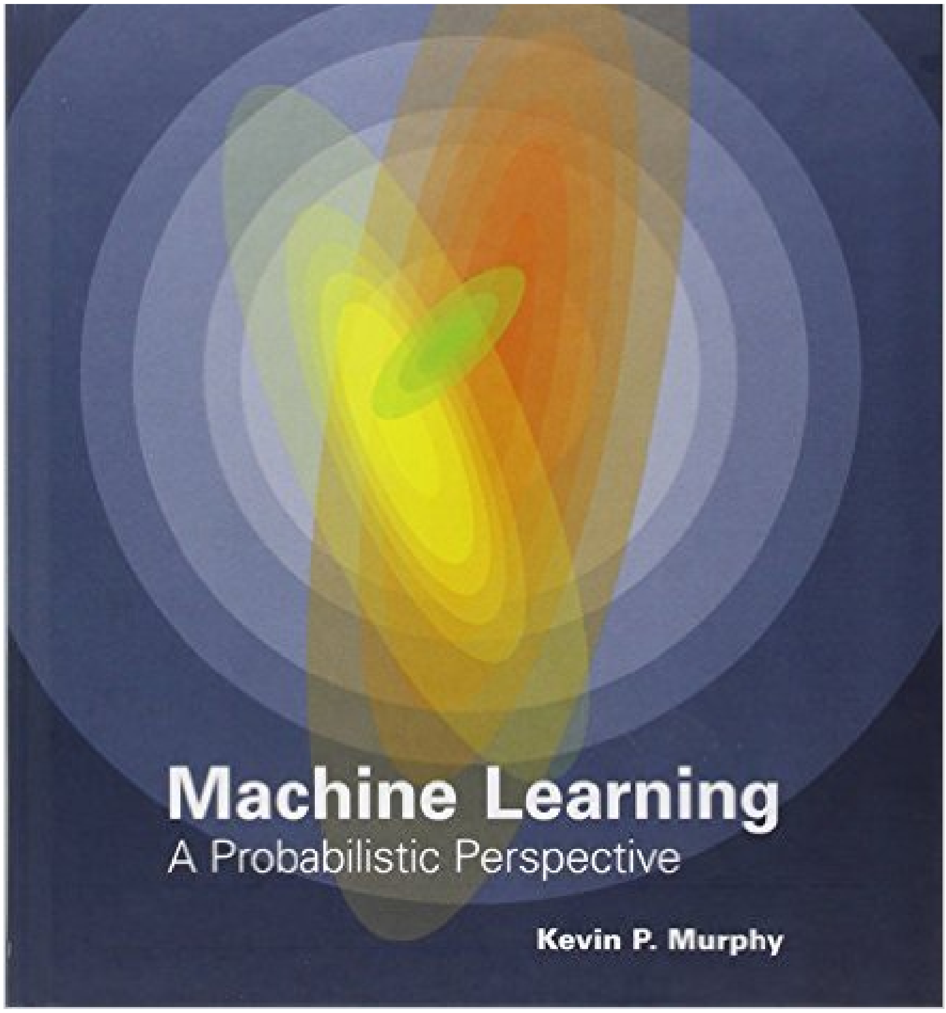
- Bayesian reasoning is best seen as a useful tool.
- Many concepts in machine learning have Bayesian interpretations.
  - Choice of loss / error function, regularization, etc.

We'll mention these things as they come up.

Probabilistic Models & Bayesian Statistics

For a fully Bayesian take on machine learning, check out the **Murphy** textbook:





Review: Classification

- **Goal:** Assign each feature vector x to one of K distinct classes C_k, where $k = 1, \dots, K$.
  - Data X
  - Labels Y
- The case $K = 2$ is **Binary Classification**
  - $t = 1$ means $x \in C_1$
  - $t = 0$ means $x \in C_2$ (or sometimes $t = -1$
- For the case $K > 2$, use **one-hot encoding**,

$$t = (0, 1, 0, \dots, 0, 0)^T \implies x \in C_2$$

Generative Models

A **generative model** learns a joint model $P(Y, X) = P(X \mid Y)P(Y)$.

- Perform inference using the **posterior**, via Bayes' Rule:

$$P(Y \mid X) = \frac{P(X \mid Y)P(Y)}{P(X)}$$

$P(X)$

- Specifies how to generate observed features X if labels Y are known
- By comparing the synthetic data and real data, we get a sense of how good the generative model is.

Generative Models: Examples

Simple examples:

- Naive Bayes (Later)
- Gaussian Discriminant Analysis (Later)

More abstract examples:

- Linear Regression
- Most Bayesian models

Discriminative Models

Conversely, a **discriminative model** fits $P(Y \mid X)$ directly from data.

- Goal: select a hypothesis to *discriminates* between class labels
- Does not (necessarily) provide the ability to *generate* new random examples
- allows us to focus purely on the classification task

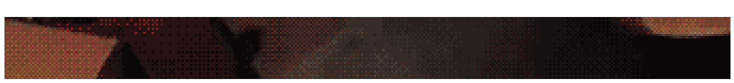
We will discuss the pros and cons of each method later.

Discriminative Models

The discriminative approach will typically

- have fewer parameters to estimate
- make fewer assumptions about data distribution
  - Linear (logistic regression) vs quadratic (GDA) in the input dimension
- make fewer generative assumptions about the data
  - However, reconstruction features from labels may require prior knowledge

Break Time!



> Thanks to Bryan for the GIF!

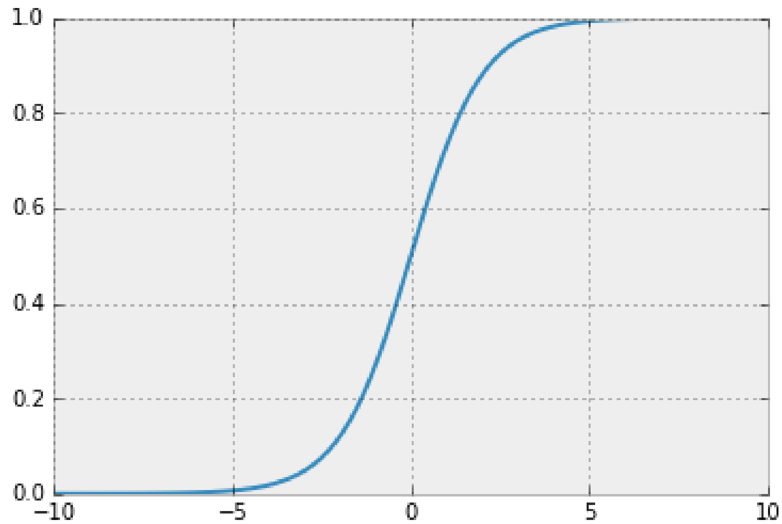
Logistic Regression

Sigmoid and Logit Functions

The **logistic sigmoid function** is

$$\sigma(a) = \frac{1}{1 + \exp(-a)} = \frac{\exp(a)}{1 + \exp(a)}$$

In [10]:
```
def sigmoid(a):
    return 1 / (1 + np.exp(-a));

xvals = np.linspace(-10,10,100);
plt.plot(xvals, sigmoid(xvals));
```

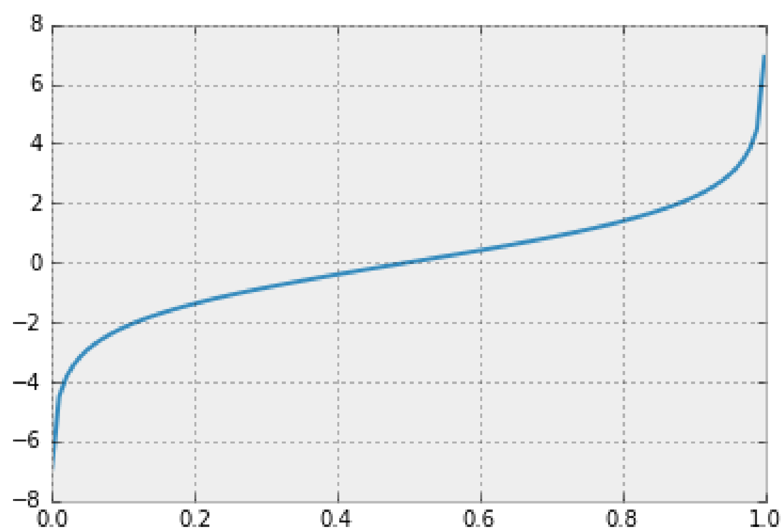


Sigmoid and Logit Functions

Its inverse is the **logit function** or the log-odds ratio,

$$a = \ln\left(\frac{\sigma}{1-\sigma}\right)$$

In [11]:
```
def logit(sigma):
    return np.log(sigma / (1-sigma));

xvals = np.linspace(0.001, 0.999, 100);
plt.plot(xvals, logit(xvals));
```



Sigmoid and Logit Functions

Sigmoid and Logit Functions

The sigmoid function generalizes to the **normalized exponential** or **softmax** function

Given any real numbers $q_1, \ldots, q_n$, we can generate a distribution on n objects using:

$$p_k = \frac{\exp(q_k)}{\sum_j \exp(q_j)}$$

Logistic Regression

- Simpest discriminative model that is **linear** in the parameters.
- Models the **class posterior** using a sigmoid applied to a linear function of the feature vector:
$$y \sim \text{Bernoulli}[\sigma(w^T \phi(x))]$$
$$P(y|\phi(x)) = y(\phi(x)) = \sigma(w^T \phi(x))$$
- We can solve the paramter w by maximizing the likelihood of the training data.

Logistic Regression: Why Sigmoid?

- For two classes, **Bayes' theorem** says:
$$p(C_1|x) = \frac{p(x|C_1) \cdot p(C_1)}{p(x|C_1) \cdot p(C_1) + p(x|C_2) \cdot p(C_2)}$$
- The **log odds** is defined to be:
$$a = \ln \frac{p(C_1|x)}{p(C_2|x)} = \ln \frac{p(x|C_1) \cdot p(C_1)}{p(x|C_2) \cdot p(C_2)}$$
- In terms of the log odds, the posterior is defined as:
$$p(C_1|x) = \frac{1}{1 + \exp(-a)} = \sigma(a)$$

Logistic Regression: Intuition

- Given data x and learned weights w, pick the label with the largest **posterior probability**
$$P(t = 1|x, w) = \sigma(w^T \phi(x))$$
$$P(t = 0|x, w) = 1 - \sigma(w^T \phi(x))$$
- This is equivalent to setting a threshold at $p = 0.5$.
  - Classify x as positive ($y = 1$) if $\sigma(w^T \phi(x)) > 0.5$
  - This creates a **linear decision boundary** in the feature space! (for $\phi(x) \in \mathbb{R}^d$)

Logistic Regression: Intuition

- Classify x as positive if $\sigma(w^T \phi(x)) > 0.5$.

$$\sigma(w^T \phi(x)) = \frac{\exp(w^T \phi(x))}{1 + \exp(w^T \phi(x))} > 0.5$$
$$\implies \boxed{w^T \phi(x) > 0}$$

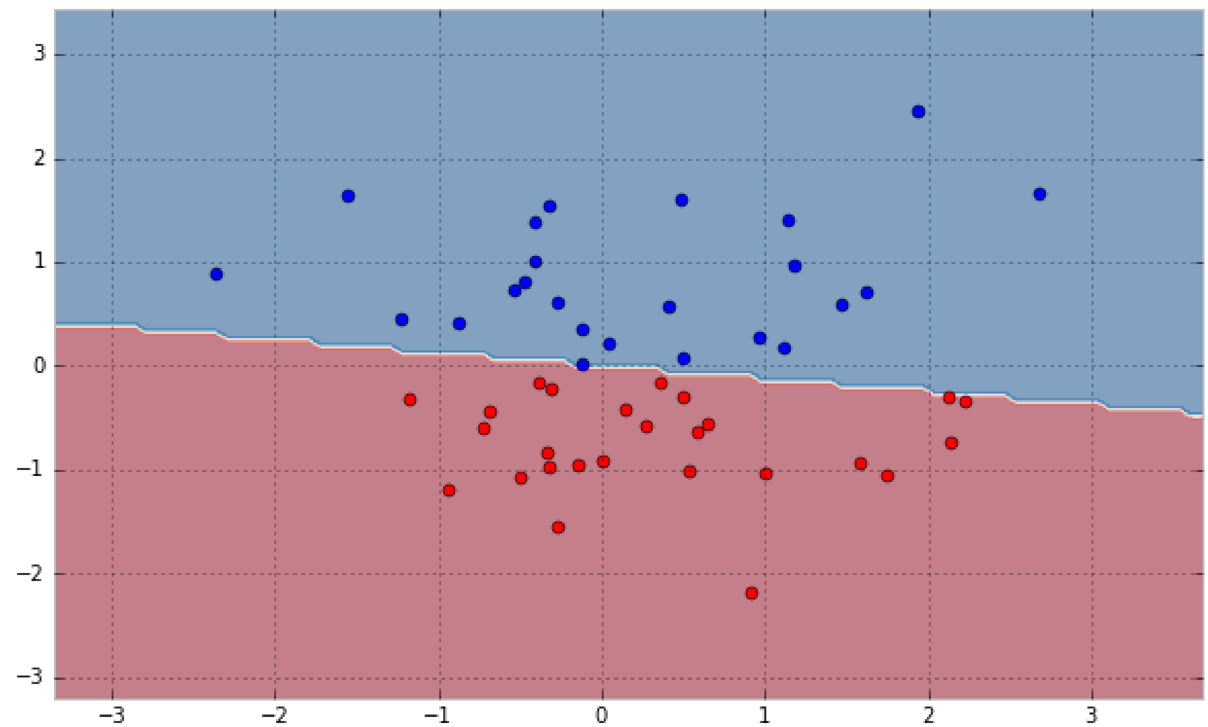
- This is the equation for a half-plane in $\mathbb{R}^d$, with **normal vector** w!

Logistic Regression: Linear Decision

In [4]:
```
# source code for plot on NEXT SLIDE!
def plot_linear_boundary():
    # random data + normal
    x = np.random.randn(2,50);
    w = np.random.randn(2);
    # classify based on w
    labels = np.dot(w.T, x) > 0;
    blue, red = x[:,labels==0], x[:,labels==1];

    # grid over plot window
    xx = np.linspace( min(x[0])-1, max(x[0])+1, 100);
    yy = np.linspace( min(x[1])-1, max(x[1])+1, 100);
    X,Y = np.meshgrid(xx, yy);

    # compute w.T*x for each point on grid
    Z = np.array([X.ravel(), Y.ravel()]);
    Z = np.dot(w.T, Z).reshape(X.shape) < 0;

    plt.contourf(X, Y, Z, cmap="RdBu", alpha=0.5);
    plt.plot(blue[0], blue[1], 'ob', red[0], red[1], 'or');
```

Logistic Regression: Linear Decision

In [5]:
```
plt.figure(figsize=(10,6))
plot_linear_boundary();
```



Logistic Regression: Likelihood

- We saw before that the **likelihood** for each binary label is:
$$P(t=1|x,w) = \sigma(w^T\phi(x))$$
$$P(t=0|x,w) = 1-\sigma(w^T\phi(x))$$
- With a clever trick, this
$$P(t|x,w) = \sigma(w^T\phi(x))^t \cdot (1-\sigma(w^T\phi(x)))^{1-t}$$

Logistic Regression

- For a data set $\{(\phi(x_n), t_n)\}$ where $t_n \in \{0,1\}$, the **likelihood function** is
$$P(t|w) = \prod_{n=1}^{N} y_n^{t_n}(1-y_n)^{1-t_n}$$
  - where $y_n = P(C_1|\phi(x_n)) = \sigma(w^T\phi(x_n))$
- Minimize the **loss** or **negative log-likelihood**, $E(w) = -\ln P(t|w)$
  - maximizes the likelihood

Derivation: $\nabla_w \ln P(t|w)$

$= \sum_{n=1} \nabla_w \left[t_n \ln \sigma(w^T\phi(x_n)) + (1-t_n)\ln(1-\sigma(w^T\phi(x_n))\right]$

$$= \sum_{n=1}^{N} \left(t_n \frac{y_n(1-y_n)}{y_n} - (1-t_n)\frac{y_n(1-y_n)}{1-y_n} \right) \nabla_w \left[w^T \phi(x_n) \right]$$

$$= \sum_{n=1}^{N} \left(t_n(1-y_n) - (1-t_n)y_n \right) \nabla_w \left[w^T \phi(x_n) \right]$$

$$= \sum_{n=1}^{N} (t_n - y_n)\phi(x_n) = \sum_{n=1}^{N} \left[t_n - \sigma(w^T \phi(x_n)) \right] \phi(x_n)$$

Logistic Regression: Gradient Descent

We have just shown that the gradient of the loss is

$$\nabla_w E(w) = \sum_{n=1}^{N} (y_n - t_n)\phi(x_n)$$
$$y_n = P(C_1|\phi(x_n)) = \sigma(w^T \phi(x_n))$$

- This resembles the gradient expression from linear regression with least squares!

$$\begin{array}{rl} \text{Linear} & y_n - t_n = \sigma(w^T \phi(x_n)) - t_n \\ \text{Logistic} & y_n - t_n = w^T \phi(x_n) - t_n \end{array}$$

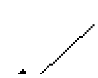
Newton's Method: Overview

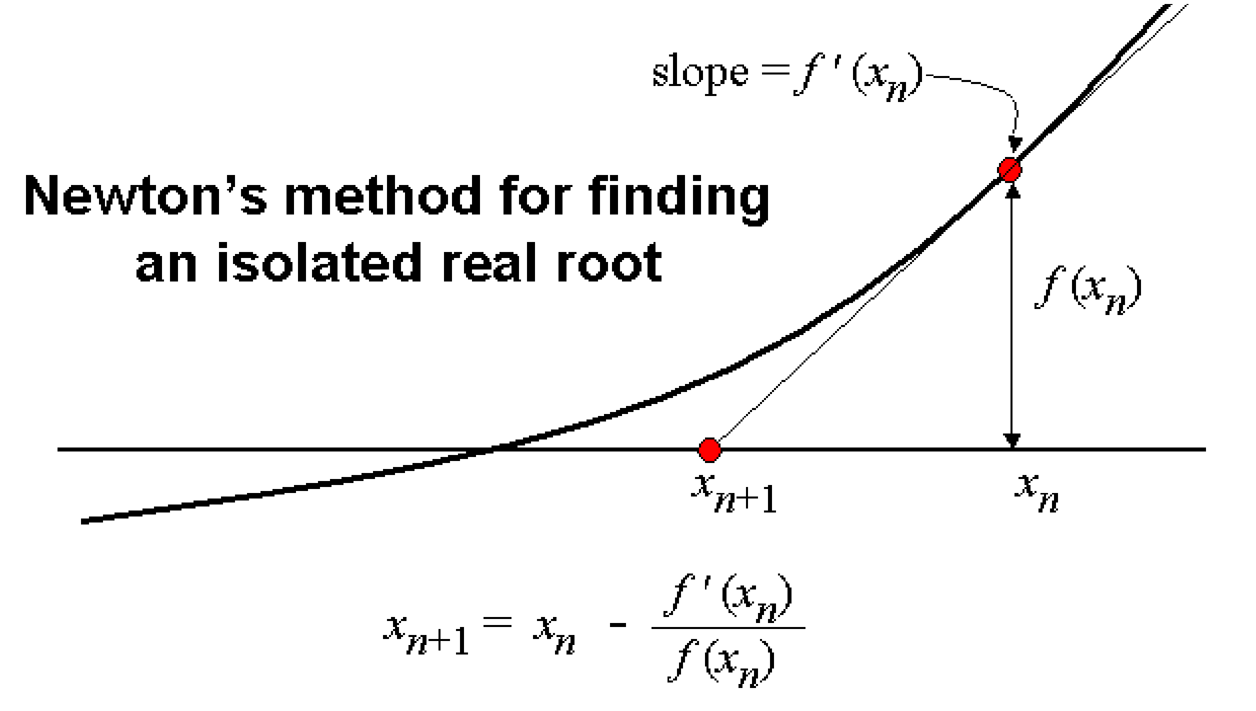
- **Goal:** Minimize a general function $F(w)$ in one dimension by solving for
$$f(w) = \frac{\partial F}{\partial w} = 0$$
- **Newton's Method:** To find roots of f, Repeat until convergence:
$$w \leftarrow w - \frac{f(w)}{f'(w)}$$

Newton's Method: Geometric Intuition

- Find the roots of $f(w)$ by following its **tangent lines**. The tangent line to f at w_{k-1} has equation
$$\ell(w) = f(w_{k-1}) + (w - w_{k-1})f'(w_{k-1})$$
- Set next iterate w_{k+1} to be **root** of tangent line:
$$f(w_{k-1}) + (w - w_{k-1})f'(w_{k-1}) = 0$$
$$\implies \boxed{w = w_{k-1} - \frac{f(w_{k-1})}{f'(w_{k-1})}}$$

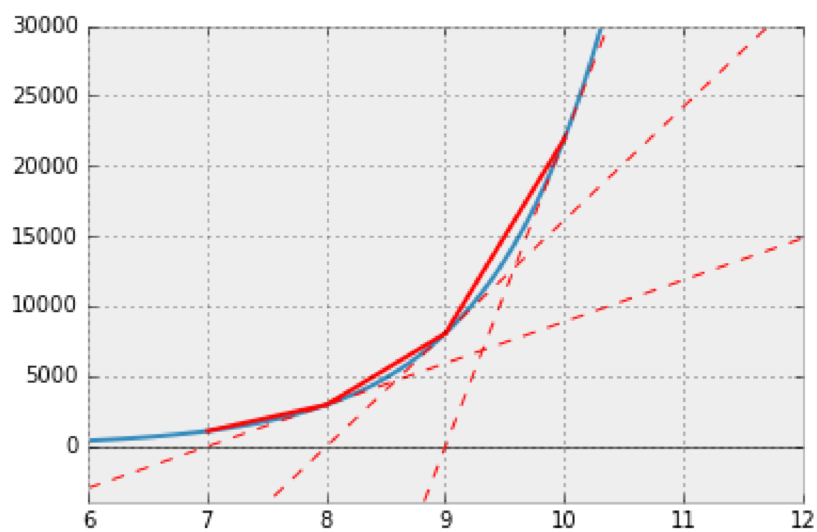
Newton's Method: Geometric Intuition





In [6]:
```
# custom newton's method -- see Canvas
from newton_plot import *;

def fn(x): return np.exp(x) - x**2;
def d1(x): return np.exp(x) - 2*x;
def d2(x): return np.exp(x) - 2;

lst = [];
print("Newton's Method:", newton_exact(d1, d2, 10, lst=lst, maxn=4));
plot_optimization(plt.gca(), fn, d1, lst, xlim=(6,12), ylim=(-4000, 30
```

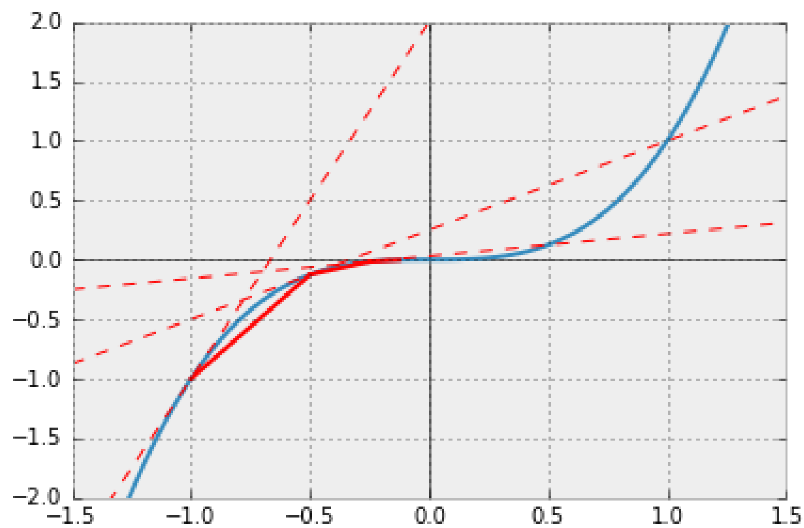
```
Newton's Method did not converge.
("Newton's Method:", 6.018373602193873)
```



In [7]:

```python
# custom newton's method -- see Canvas
from newton_plot import *;

def fn(x): return x**3;
def d1(x): return 3 * x**2;
def d2(x): return 6 * x;

lst = [];
print("Newton's Method:", newton_exact(d1, d2, -1, lst=lst, maxn=4));
plot_optimization(plt.gca(), fn, d1, lst, xlim=(-1.5,1.5), ylim=(-2,2)
```

```
Newton's Method did not converge.
("Newton's Method:", -0.0625)
```



Newton's Method: Recap

To minimize $F(w)$, find roots of $F'(w)$ via Newton's Method.

Repeat until convergence:

$$\boxed{w \leftarrow w - \frac{F'(w)}{F''(w)}}$$

Newton's Method: Multivariate Case

Replace second derivative with the **Hessian Matrix**,

$$H_{ij}(w) = \frac{\partial^2 F}{\partial w_i \partial w_j}$$

Newton update becomes:

$$\boxed{w \leftarrow w - H^{-1} \nabla_w F}$$

Recall: Linear Regression

- For linear regression, least squares has a **closed-form solution**:
$$w_{ML} = (\Phi^T \Phi)^{-1} \Phi^T t$$
- This generalizes to weighted least squares, with diagonal weight matrix R,
$$w_{WLS} = (\Phi^T R \Phi)^{-1} \Phi^T R t$$

Logistic Regression: Newton's Method

- For logistic regression, however, $\nabla_w E(w) = 0$ is **nonlinear**, and no closed-form solution exists.

We must iterate!

- Newton's method is a good choice in many cases.

Iterative Solution

- Apply Newton's method to solve $\nabla_w E(w) = 0$
- This involves least squares with weights $R_{nn} = y_n(1 - y_n)$
- Since R depends on w, and vice-versa, we get...

Iteratively-Reweighted Least Squares (IRLS)

Repeat Until Convergence:
1. $w^{(new)} = w_{WLS} = (\Phi^T R \Phi)^{-1} \Phi^T R z$
2. $z = \Phi w^{(old)} - R^{-1}(y - t)$

Some additional material (possibly for next lecture) to follow

Gaussian Discriminant Analysis

Gaussian Discriminant Analysis

- **Generative** probabilistic classifier, model $P(Y, X) = P(X \mid Y) P(Y)$
- Viewed as a Bayesian model, we have
  - **Prior** $p(y = C_k)$: Constant (eg. Bernoulli)
  - **Likelihood** $p(x|y = C_k)$: Gaussian distribution

$$p(x|y = C_k) = \frac{1}{\cdots} exp\left\{ -\frac{1}{2}(x - \mu_k)^T \Sigma^{-1} (x - \mu_k) \right\}$$

$$(2\pi)^{\frac{D}{2}}|\Sigma|^{\frac{1}{2}}$$

Gaussian Discriminant Analysis

Use Bayes' Rule as before for classification. For any class i,

$$p(C_i|x) = \frac{p(x|C_i) \cdot p(C_i)}{\Sigma_k p(x|C_k) \cdot p(C_k)}$$

In [8]:

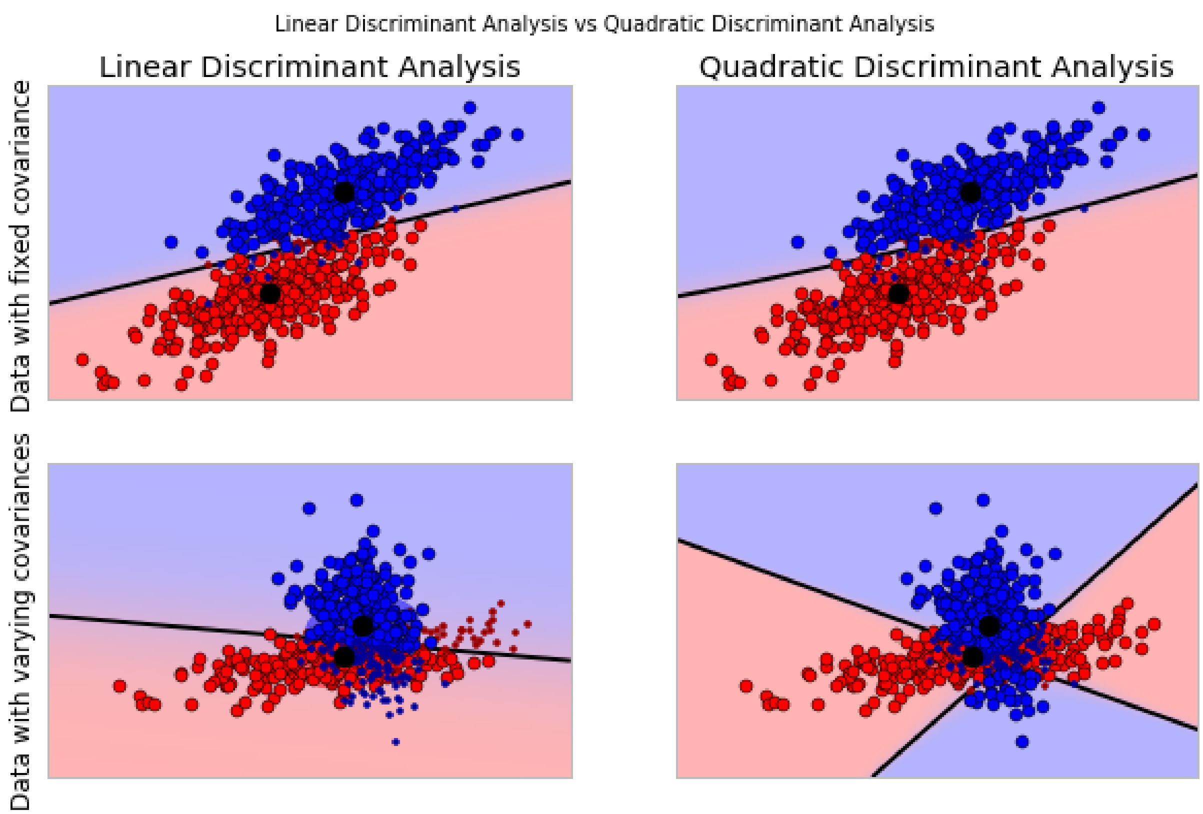
```
# Credits: Scikit-learn example on Linear and Quadratic Discriminant A


# colormap
cmap = colors.LinearSegmentedColormap(
    'red_blue_classes',
    {'red': [(0, 1, 1), (1, 0.7, 0.7)],
     'green': [(0, 0.7, 0.7), (1, 0.7, 0.7)],
     'blue': [(0, 0.7, 0.7), (1, 1, 1)]})
plt.cm.register_cmap(cmap=cmap)


###########################################################################
# generate datasets
def dataset_fixed_cov():
    '''Generate 2 Gaussians samples with the same covariance matrix'''
    n, dim = 300, 2
    np.random.seed(0)
    C = np.array([[0., -0.23], [0.83, .23]])
    X = np.r_[np.dot(np.random.randn(n, dim), C),
              np.dot(np.random.randn(n, dim), C) + np.array([1, 1])]
    y = np.hstack((np.zeros(n), np.ones(n)))
    return X, y


def dataset_cov():
    '''Generate 2 Gaussians samples with different covariance matrices
    n, dim = 300, 2
    np.random.seed(0)
    C = np.array([[0., -1.], [2.5, .7]]) * 2.
    X = np.r_[np.dot(np.random.randn(n, dim), C),
              np.dot(np.random.randn(n, dim), C.T) + np.array([1, 4])]
    y = np.hstack((np.zeros(n), np.ones(n)))
    return X, y


###########################################################################
# plot functions
def plot_data(lda, X, y, y_pred, fig_index):
    splot = plt.subplot(2, 2, fig_index)
    if fig_index == 1:
        plt.title('Linear Discriminant Analysis')
        plt.ylabel('Data with fixed covariance')
    elif fig_index == 2:
        plt.title('Quadratic Discriminant Analysis')
    elif fig_index == 3:
        plt.ylabel('Data with varying covariances')

    tp = (y == y_pred)  # True Positive
    tp0, tp1 = tp[y == 0], tp[y == 1]
    X0, X1 = X[y == 0], X[y == 1]
    X0_tp, X0_fp = X0[tp0], X0[~tp0]
    X1_tp, X1_fp = X1[tp1], X1[~tp1]
```

```
    # class 0: dots
    plt.plot(X0_tp[:, 0], X0_tp[:, 1], 'o', color='red')
    plt.plot(X0_fp[:, 0], X0_fp[:, 1], '.', color='#990000')  # dark r

    # class 1: dots
    plt.plot(X1_tp[:, 0], X1_tp[:, 1], 'o', color='blue')
    plt.plot(X1_fp[:, 0], X1_fp[:, 1], '.', color='#000099')  # dark b

    # class 0 and 1 : areas
    nx, ny = 200, 100
    x_min, x_max = plt.xlim()
    y_min, y_max = plt.ylim()
    xx, yy = np.meshgrid(np.linspace(x_min, x_max, nx),
                         np.linspace(y_min, y_max, ny))
    Z = lda.predict_proba(np.c_[xx.ravel(), yy.ravel()])
    Z = Z[:, 1].reshape(xx.shape)
    plt.pcolormesh(xx, yy, Z, cmap='red_blue_classes',
                   norm=colors.Normalize(0., 1.))
    plt.contour(xx, yy, Z, [0.5], linewidths=2., colors='k')

    # means
    plt.plot(lda.means_[0][0], lda.means_[0][1],
             'o', color='black', markersize=10)
    plt.plot(lda.means_[1][0], lda.means_[1][1],
             'o', color='black', markersize=10)

    return splot


def plot_ellipse(splot, mean, cov, color):
    v, w = linalg.eigh(cov)
    u = w[0] / linalg.norm(w[0])
    angle = np.arctan(u[1] / u[0])
    angle = 180 * angle / np.pi  # convert to degrees
    # filled Gaussian at 2 standard deviation
    ell = mpl.patches.Ellipse(mean, 2 * v[0] ** 0.5, 2 * v[1] ** 0.5,
                              180 + angle, color=color)
    ell.set_clip_box(splot.bbox)
    ell.set_alpha(0.5)
    splot.add_artist(ell)
    splot.set_xticks(())
    splot.set_yticks(())


def plot_lda_cov(lda, splot):
    plot_ellipse(splot, lda.means_[0], lda.covariance_, 'red')
    plot_ellipse(splot, lda.means_[1], lda.covariance_, 'blue')


def plot_qda_cov(qda, splot):
    plot_ellipse(splot, qda.means_[0], qda.covariances_[0], 'red')
    plot_ellipse(splot, qda.means_[1], qda.covariances_[1], 'blue')
```

```
##########################################################################
```

GDA Example

In [9]:
```
plt.figure(figsize=(10,6))

for i, (X, y) in enumerate([dataset_fixed_cov(), dataset_cov()]):
    # Linear Discriminant Analysis
    lda = LinearDiscriminantAnalysis(solver="svd", store_covariance=Tr
    y_pred = lda.fit(X, y).predict(X)
    splot = plot_data(lda, X, y, y_pred, fig_index=2 * i + 1)
    plot_lda_cov(lda, splot)
    plt.axis('tight')

    # Quadratic Discriminant Analysis
    qda = QuadraticDiscriminantAnalysis(store_covariances=True)
    y_pred = qda.fit(X, y).predict(X)
    splot = plot_data(qda, X, y, y_pred, fig_index=2 * i + 2)
    plot_qda_cov(qda, splot)
    plt.axis('tight')
plt.suptitle('Linear Discriminant Analysis vs Quadratic Discriminant A
plt.show()
```



Class Conditional Densities

Suppose we model $p(x|C_k)$ as Gaussians with the *same covariance* matrix:

$$p(x|C_k) = \frac{1}{(2\pi)^{\frac{D}{2}}|\Sigma|^{\frac{1}{2}}} exp\left\{-\frac{1}{2}(x-\mu_k)^T\Sigma^{-1}(x-\mu_k)\right\}$$

$(2\pi)^{\frac{D}{2}} |\Sigma|^{\frac{1}{2}}$

This gives us $p(C_1|x) = \sigma(w^T x + w_0)$

where $w = \Sigma^{-1}(\mu_1 - \mu_2)$

and $w_0 = -\frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2^T \Sigma^{-1} \mu_2 + ln\frac{p(C_1)}{p(C_2)}$

Derivation

$$\begin{aligned}
P(x, C_1) &= P(x|C_1) \cdot P(C_1) \\
&= \frac{1}{(2\pi)^{\frac{D}{2}} |\Sigma|^{\frac{1}{2}}} exp\left\{ -\frac{1}{2}(x - \mu_1)^T \Sigma^{-1} (x - \mu_1) \right\} \cdot P(C_1) \\
P(x, C_2) &= P(x|C_2) \cdot P(C_2) \\
&= \frac{1}{(2\pi)^{\frac{D}{2}} |\Sigma|^{\frac{1}{2}}} exp\left\{ -\frac{1}{2}(x - \mu_2)^T \Sigma^{-1} (x - \mu_2) \right\} \cdot P(C_2)
\end{aligned}$$

"Log Odds"

$$\begin{aligned}
log\frac{p(C_1|x)}{p(C_2|x)} &= log\frac{p(C_1|x)}{1 - p(C_1|x)} \\
&= log\frac{exp\left\{-\frac{1}{2}(x - \mu_1)^T \Sigma^{-1} (x - \mu_1)\right\}}{exp\left\{-\frac{1}{2}(x - \mu_2)^T \Sigma^{-1} (x - \mu_2)\right\}} + log\frac{P(C_1)}{P(C_2)} \\
&= \left\{ -\frac{1}{2}(x - \mu_1)^T \Sigma^{-1} (x - \mu_1) \right\} - \left\{ -\frac{1}{2}(x - \mu_2)^T \Sigma^{-1} (x - \mu_2) \right\} + log\frac{P(C}{P(C} \\
&= (\mu_1 - \mu_2)^T \Sigma^{-1} x - \frac{1}{2}\mu_1 \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2 \Sigma^{-1} \mu_2 + log\frac{P(C_1)}{P(C_2)} \\
&= (\Sigma^{-1}(\mu_1 - \mu_2))^T x + w_0
\end{aligned}$$

where $w_0 = -\frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2^T \Sigma^{-1} \mu_2 + log\frac{p(C_1)}{p(C_2)}$

Class-Conditional Densities

(for shared covariances)

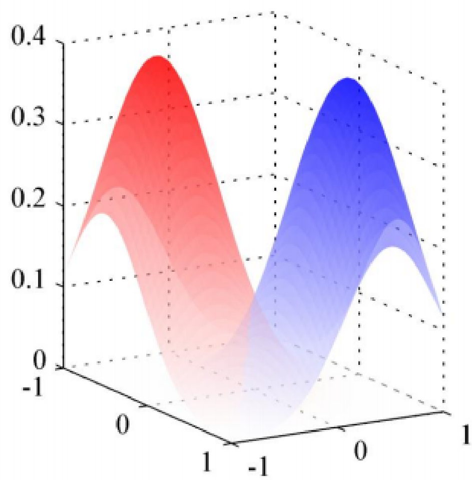
- $P(C_k|x)$ is a sigmoid function: $\sigma(a) = \frac{1}{1+exp(-a)}$
- with logg-odds (*logit function*): $a = log(\frac{\sigma}{1-\sigma} = (\Sigma^{-1}(\mu_1 - \mu_2))^T x + w_0$

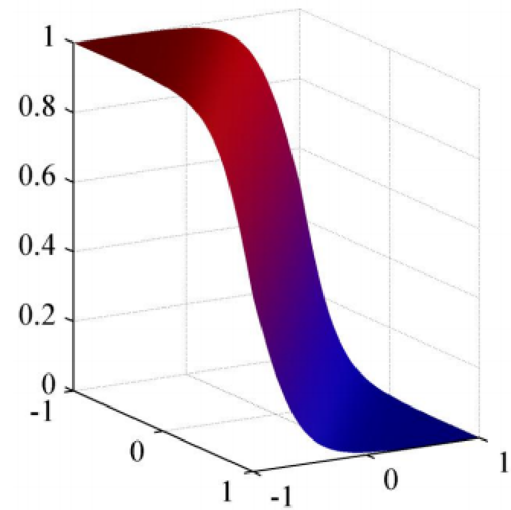
where $w_0 = -\frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2^T \Sigma^{-1} \mu_2 + log\frac{p(C_1)}{p(C_2)}$

- Generalizes to normalized exponential or *softmax*: $p_i = \frac{exp(q_i)}{\Sigma_j exp(q_j)}$

Linear Decision Boundaries

- With the same covariance matrices, the boundary $p(C_1|x) = p(C_2|x)$ is linear.
  - Different priors $p(C_1), P(C_2)$ just shift it around.





Learning parameters via maximum likelihood

- Given training data $\{(x^{(1)}, t^{(1)}), \ldots, (x^{(N)}, t^{(N)})\}$ and a generative model (*shared covariance*)

$$p(t) = \phi^t (1-\phi)^{(1-t)}$$

$$p(x|t=0) = \frac{1}{\sqrt{2\pi}|\Sigma|^{\frac{1}{2}}} exp(-\frac{1}{2}(x-\mu_0)^T \Sigma^{-1}(x-\mu_0))$$

$$p(x|t=1) = \frac{1}{\sqrt{2\pi}|\Sigma|^{\frac{1}{2}}} exp(-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1))$$

log likelihood

$$\begin{aligned} log(L) &= \prod_{i=1}^{N} P(X^{(i)}, t^{(i)}) \\ &= \sum_{i=1}^{N} logP(X^{(i)}, t^{(i)}) \end{aligned}$$

Learning via maximum likelihood

Maximum likelihood estimation solutions

Maximum likelihood estimation solutions

$$\phi = \frac{1}{N} \sum_{i=1}^{N} 1\{t_i = 1\}$$

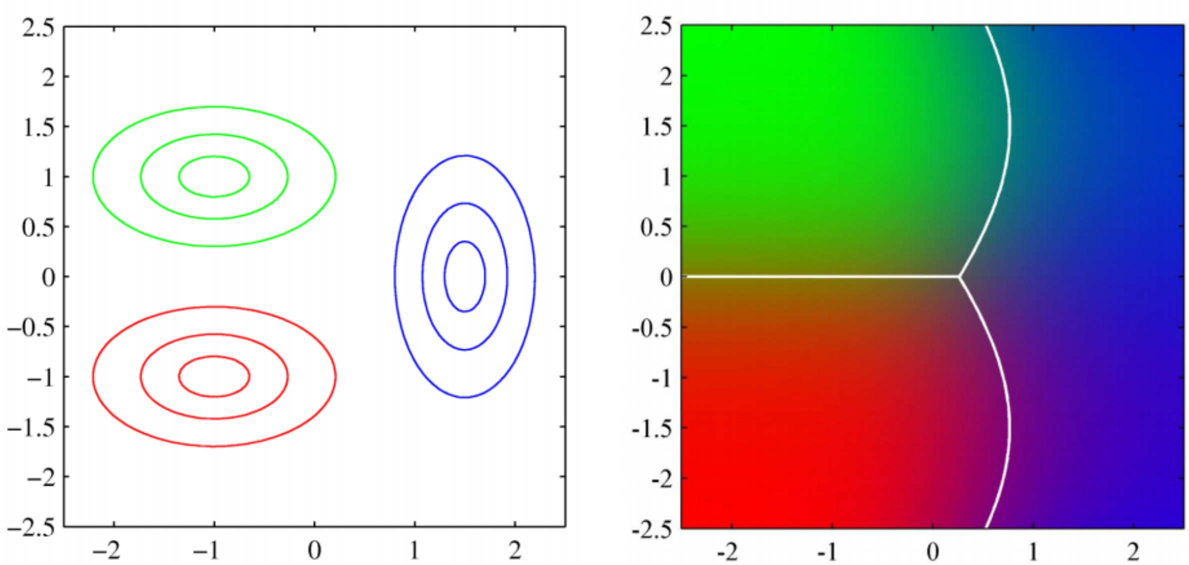
$$\mu_0 = \frac{\sum_{i=1}^{N} 1\{t_i = 0\} x_i}{\sum_{i=1}^{N} 1\{t_i = 0\}}$$

$$\mu_1 = \frac{\sum_{i=1}^{N} 1\{t_i = 1\} x_i}{\sum_{i=1}^{N} 1\{t_i = 1\}}$$

$$\Sigma = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu_{t_i})(x_i - \mu_{t_i})^T$$

The case of Different Covariances

- Decision boundaries can be quadratic.



GDA v. Logistic Regression

- For an M-dimensional feature space,
  - Logistic regression has to fit M parameters.
  - GDA has to fit
    - 2M parameters for means of $p(x|C_1)$ and $p(x|C_2)$
    - M(M+1)/2 parameters for the shared covariance matrix.

- Logistic regression has less parameters and is more flexible about data distribution.
- GDA has a stronger modeling assumption, and works well when the distribution follows the assumption.