EECS 545: Machine Learning

Lecture 5. Linear models of classification & generative models

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Outline

- Probabilistic generative models
  – Gaussian Discriminant Analysis
  – Naive Bayes
- Discriminant functions
  – Fisher's linear discriminant
  – Perceptron learning algorithm

Probabilistic generative models

Learning the Classifier

- Goal: Learn the distributions $p(C_k \mid \mathrm{x})$.
  (a) Discriminative models: Directly model $p(C_k|\mathbf{x})$ and learn parameters from the training set.
  (b) Generative models: Learn class densities $p(x|C_k)$ and priors $p(C_k)$

Probabilistic Generative Models

- Bayes' theorem reduces the classification problem $p(C_k \mid \mathrm{x})$ to estimating the distribution of the data…

- Density estimation problems are easy to learn from labeled training data.
  - $p(C_k)$
  - $p(\mathbf{x} \mid C_k)$
- Maximum likelihood parameter estimation.

Probabilistic Generative Models

- For two classes, Bayes' theorem says:

$$p(C_1|\mathbf{x}) = \frac{p(\mathbf{x}|C_1)p(C_1)}{p(\mathbf{x}|C_1)p(C_1) + p(\mathbf{x}|C_2)p(C_2)}$$

- Use *log odds*:

$$a = \ln \frac{p(C_1|\mathbf{x})}{p(C_2|\mathbf{x})} = \ln \frac{p(\mathbf{x}|C_1)p(C_1)}{p(\mathbf{x}|C_2)p(C_2)}$$

- To define the posterior via the *sigmoid*:

$$p(C_1|\mathbf{x}) = \frac{1}{1 + \exp(-a)} = \sigma(a)$$

Comparing the approaches: Discriminative vs. Generative

- The *generative* approach is typically model-based, and makes it possible to generate synthetic data from $p(\mathrm{x} \mid C_k)$.
  – By comparing the synthetic data and real data, we get a sense of how good the generative model is.

- The *discriminative* approach will typically have fewer parameters to estimate and have less assumptions about data distribution.
  – Linear (e.g., logistic regression) versus quadratic (e.g., Gaussian discriminant analysis) in the dimension of the input.
  – Less generative assumptions about the data (however, constructing the features may need prior knowledge)

Gaussian Discriminant Analysis

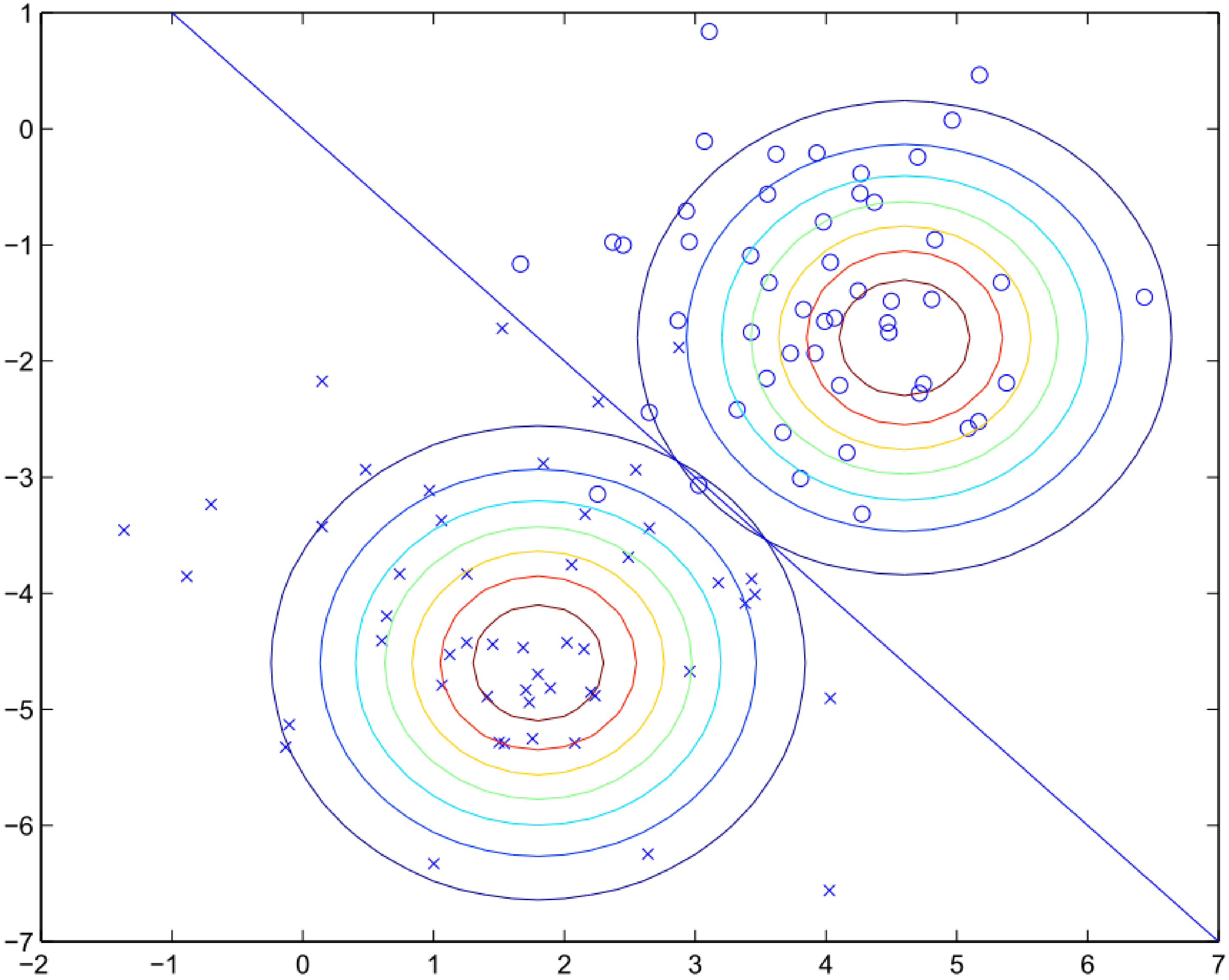
Gaussian Discriminant Analysis

- Prior distribution
  - $p(C_k)$: Constant (e.g., Bernoulli)
- Likelihood
  - $P(x|C_k)$: Gaussian distribution

$$p(\mathbf{x}|C_k) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\mathbf{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu_k)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu_k)\right\}$$

- Classification: use Bayes rule (previous slide)

Gaussian Discriminant Analysis



Class-Conditional Densities

- Suppose we model $p(\text{x} \mid C_k)$ as Gaussians with the <u>same covariance</u> matrix.

$$p(\mathbf{x}|C_k) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \mu_k)^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \mu_k)\right\}$$

- This gives us $p(C_1|\mathbf{x}) = \sigma(\mathbf{w}^T\mathbf{x} + w_0)$
  - where $\mathbf{w} = \boldsymbol{\Sigma}^{-1}(\mu_1 - \mu_2)$

    and

$$w_0 = -\frac{1}{2}\mu_1^T \boldsymbol{\Sigma}^{-1}\mu_1 + \frac{1}{2}\mu_2^T \boldsymbol{\Sigma}^{-1}\mu_2 + \ln \frac{p(C_1)}{p(C_2)}$$

Derivation

$$
\begin{aligned}
P(x, C_1) &= P(x|C_1)P(C_1) \\
&= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2}(x-\mu_1)^T \Sigma^{-1} (x-\mu_1) \right\} P(C_1) \\
P(x, C_2) &= P(x|C_2)P(C_2) \\
&= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2}(x-\mu_2)^T \Sigma^{-1} (x-\mu_2) \right\} P(C_2)
\end{aligned}
$$

$$
\log \frac{P(C_1|x)}{P(C_2|x)} = \log \frac{P(C_1|x)}{1 - P(C_1|x)}
$$

"Log-odds"

Derivation

$$
\begin{aligned}
P(x, C_1) &= P(x|C_1)P(C_1) \\
&= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1)\right\} P(C_1) \\
P(x, C_2) &= P(x|C_2)P(C_2) \\
&= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu_2)^T \Sigma^{-1}(x-\mu_2)\right\} P(C_2)
\end{aligned}
$$

$$
\log \frac{P(C_1|x)}{P(C_2|x)} = \log \frac{P(C_1|x)}{1 - P(C_1|x)}
$$

"Log-odds"

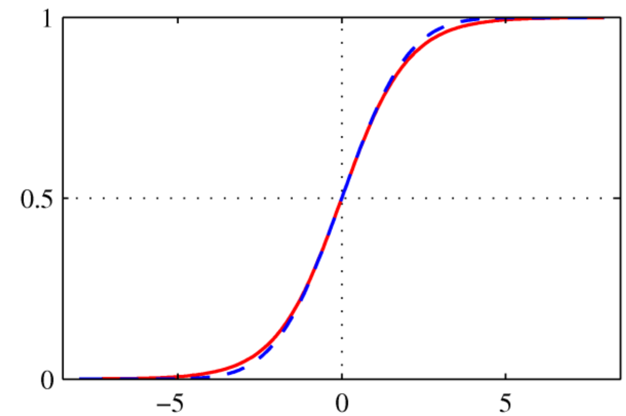
$$
\begin{aligned}
&= \log \frac{\exp\left\{-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1)\right\}}{\exp\left\{-\frac{1}{2}(x-\mu_2)^T \Sigma^{-1}(x-\mu_2)\right\}} + \log \frac{P(C_1)}{P(C_2)} \\
&= \left\{-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1)\right\} - \left\{-\frac{1}{2}(x-\mu_2)^T \Sigma^{-1}(x-\mu_2)\right\} + \log \frac{P(C_1)}{P(C_2)} \\
&= (\mu_1 - \mu_2)^T \Sigma^{-1} x - \frac{1}{2}\mu_1 \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2 \Sigma^{-1} \mu_2 + \log \frac{P(C_1)}{P(C_2)} \\
&= \left(\Sigma^{-1}(\mu_1 - \mu_2)\right)^T x + w_0
\end{aligned}
$$

$$
\text{where } w_0 = -\tfrac{1}{2}\mu_1 \Sigma^{-1} \mu_1 + \tfrac{1}{2}\mu_2 \Sigma^{-1} \mu_2 + \log \tfrac{P(C_1)}{P(C_2)}
$$

Class-Conditional Densities (for shared covariances)

- $P(C_k|x)$ is a sigmoid function:

$$\sigma(a) = \frac{1}{1+\exp(-a)}$$



  – with log-odds (*logit* function):

$$a = \log\left(\tfrac{\sigma}{1-\sigma}\right) = \left(\Sigma^{-1}(\mu_1 - \mu_2)\right)^T x + w_0$$

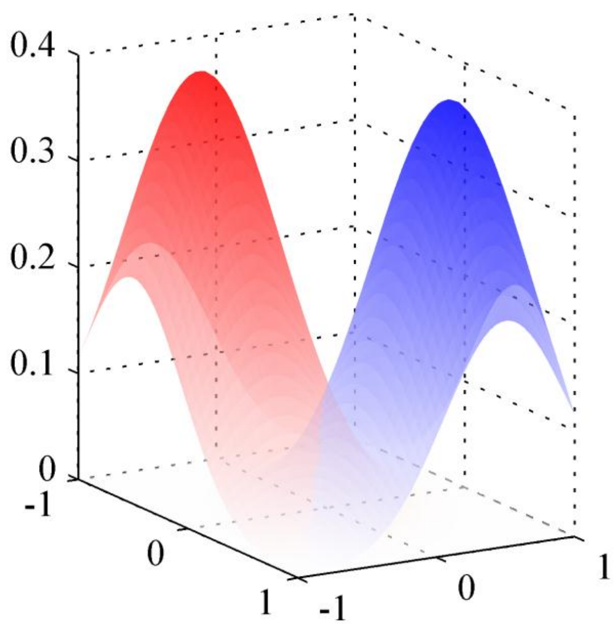
$$\text{where } w_0 = -\tfrac{1}{2}\mu_1\Sigma^{-1}\mu_1 + \tfrac{1}{2}\mu_2\Sigma^{-1}\mu_2 + \log\frac{P(C_1)}{P(C_2)}$$

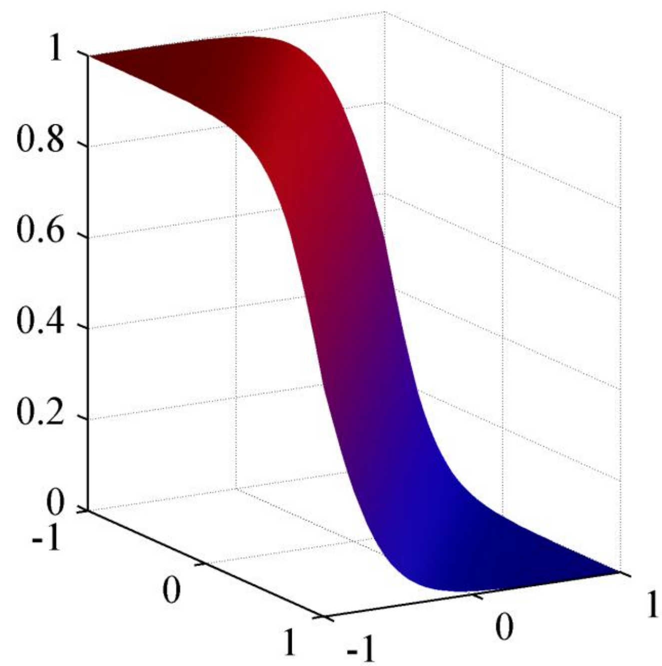
- Generalizes to *normalized exponential*, or *softmax*.

$$p_i = \frac{\exp(q_i)}{\sum_j \exp(q_j)}$$

Linear Decision Boundaries

- With the same covariance matrices, the boundary $p(C_1 \mid x) = p(C_2 \mid x)$ is linear.
  – Different priors $p(C_1)$, $p(C_2)$ just shift it around.





Learning parameters via maximum likelihood

- Given training data $\{(x^{(1)}, t^{(1)}), \ldots, (x^{(N)}, t^{(N)})$, and a generative model ("shared covariance")

$$p(t) = \phi^t (1-\phi)^{1-t}$$

$$p(\mathbf{x}|t=0) = \frac{1}{\sqrt{2\pi}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu_0)^T \Sigma^{-1} (\mathbf{x}-\mu_0)\right)$$

$$p(\mathbf{x}|t=1) = \frac{1}{\sqrt{2\pi}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu_1)^T \Sigma^{-1} (\mathbf{x}-\mu_1)\right)$$

$$\log \mathcal{L}$$

$$= \log \prod_{i=1}^{N} P\left(x^{(i)}, t^{(i)}\right)$$

$$= \sum_{i=1}^{N} \log P\left(x^{(i)}, t^{(i)}\right)$$

Learning via maximum likelihood

- Maximum likelihood estimation (homework):

$$\phi = \frac{1}{N}\sum_{i=1}^{N} 1\{t_i = 1\}$$

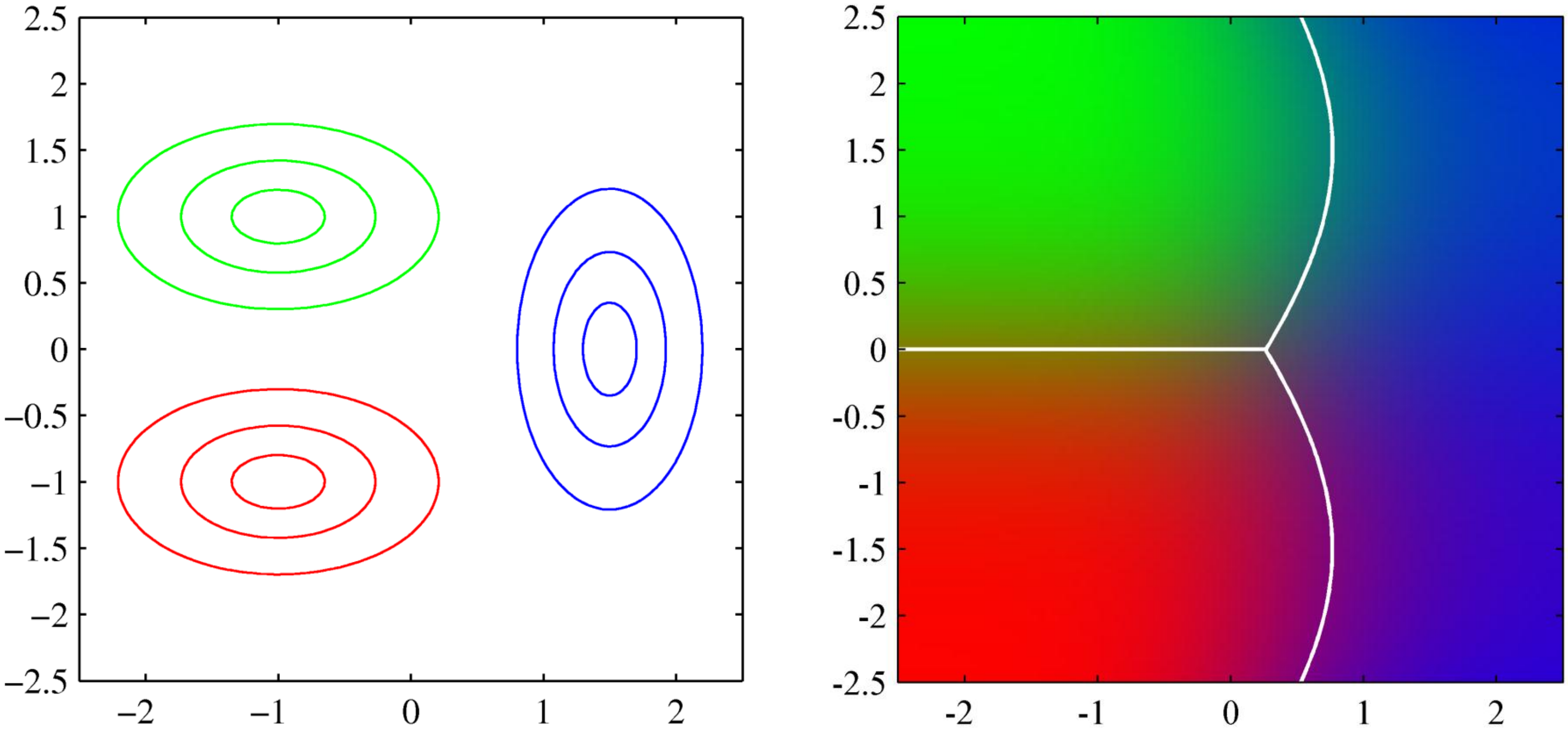
$$\mu_0 = \frac{\sum_{i=1}^{N} 1\{t_i = 0\}\mathbf{x}_i}{\sum_{i=1}^{N} 1\{t_i = 0\}}$$

$$\mu_1 = \frac{\sum_{i=1}^{N} 1\{t_i = 1\}\mathbf{x}_i}{\sum_{i=1}^{N} 1\{t_i = 1\}}$$

$$\Sigma = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i - \mu_{t_i})(\mathbf{x}_i - \mu_{t_i})^T.$$

Different Covariances

- Decision boundaries can be quadratic.



Comparison between GDA and Logistic regression

- Logistic regression:
  - For an M-dimensional feature space, this model has M parameters to fit.
- Gaussian Discriminative Analysis
  - 2M parameters for the means of $p(\mathbf{x} \mid C_1)$ and $p(\mathbf{x} \mid C_2)$
  - M(M+1)/2 parameters for the shared covariance matrix
- Logistic regression is has less parameters and is more flexible about data distribution.
- GDA has a stronger modeling assumption, and works well when the distribution follows the assumption.