Problem 5.3.1 - can derive analytic formulas starting from (5.130)

$$S_f(\omega_0; \Delta p) = \int_0^{\pi}\int_0^{2\pi} \frac{\sin\theta\, d\theta\, d\phi}{4\pi} S_0(\omega_0 : \theta, \phi)\, e^{-jk_0 a_r(\theta,\phi)^T \Delta p}$$

where

$$a_r(\theta,\phi) = -\sin\theta\cos\phi\, i_x - \sin\theta\sin\phi\, i_y - \cos\theta\, i_z$$

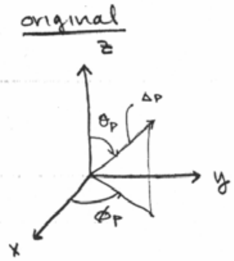
$$\Delta p = r_p \left\{ \sin\theta_p \cos\phi_p\, i_x + \sin\theta_p \sin\phi_p\, i_y + \cos\theta_p\, i_z \right\}$$

$$r_p = |\Delta p|$$

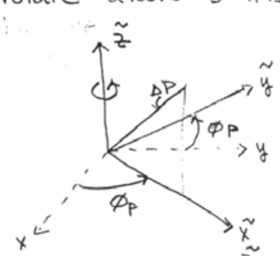
and $S_0(\omega_0 : \theta, \phi) = S_0(\omega_0)\left[1 + \alpha\cos\theta\right]$

- integral is over sphere of radius 1 centered at origin
- to solve integral, need to rotate coordinate system so $\Delta p$ lies on z-axis

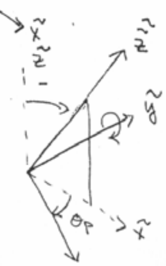
original



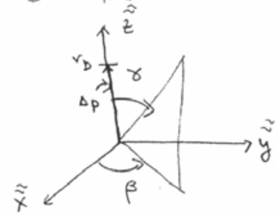

① rotate around z-axis by $\phi_p$




② rotate around $\tilde{y}$-axis by $\theta_p$



• in new coordinate system, $\underline{\Delta p}$ lies on $\tilde{\tilde{z}}$ - axis




• can rotate any vector in $x, y, z$ coordinates to $\tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{z}}$ coordinates by multiplying by the matrices

$$\begin{bmatrix} \cos\theta_p & 0 & -\sin\theta_p \\ 0 & 1 & 0 \\ \sin\theta_p & 0 & \cos\theta_p \end{bmatrix} \begin{bmatrix} \cos\phi_p & \sin\phi_p & 0 \\ -\sin\phi_p & \cos\phi_p & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} \cos\theta_p & 0 & -\sin\theta_p \\ 0 & 1 & 0 \\ \sin\theta_p & 0 & \cos\theta_p \end{bmatrix} \begin{bmatrix} \cos\phi_p & \sin\phi_p & 0 \\ -\sin\phi_p & \cos\phi_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_p \sin\theta_p \cos\phi_p \\ r_p \sin\theta_p \sin\phi_p \\ r_p \cos\theta_p \end{bmatrix}_{x,y,z}$$

$$= \begin{bmatrix} \cos\theta_p & 0 & -\sin\theta_p \\ 0 & 1 & 0 \\ \sin\theta_p & 0 & \cos\theta_p \end{bmatrix} \begin{bmatrix} r_p \sin\theta_p \\ 0 \\ r_p \cos\theta_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r_p \end{bmatrix}_{\tilde{\tilde{x}}, \tilde{\tilde{y}}, \tilde{\tilde{z}}}$$

• define new angular variables $\gamma, \beta$, integrate over sphere of radius 1 wrt these variables

$$S_f(\omega_0, \underline{\Delta p}) = \int_0^{\pi}\int_0^{2\pi} \frac{\sin\gamma \, d\gamma \, d\beta}{4\pi} S_0(\omega_0; \gamma, \beta) \, e^{-jk_0 \underline{a}_r(\gamma,\beta)\cdot\underline{\Delta p}}$$

$$\underline{a}_r(\gamma,\beta) = -\sin\gamma\cos\beta \, \underline{i}_{\tilde{\tilde{x}}} - \sin\gamma\sin\beta \, \underline{i}_{\tilde{\tilde{y}}} - \cos\gamma \, \underline{i}_{\tilde{\tilde{z}}}$$

$$\underline{a}_r(\gamma,\beta)\cdot\underline{\Delta p} = -r_p\cos\gamma$$

- need to find $S_0(\omega_0 : \gamma, \beta) = S_0(\omega_0)\, S_0(\gamma, \beta)$

- in original coordinate system

$$S_0(\theta, \phi) = 1 + \alpha \cos\theta$$

- $S_0(\theta,\phi)$ a function of z-component ($\cos\theta$) of unit vector $\underline{a}_r(\theta,\phi)$.

- express z-component of $\underline{a}_r(\theta,\phi)$ in terms of $\gamma, \beta$ by rotating $\underline{a}_r(\gamma,\beta)$ <u>back</u> to original coordinate system, take z-component

- this can be done by multiplying by inverses of original rotation matrices

$$\underline{a}_r(\theta,\phi) = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}_{x,y,z} = \begin{bmatrix} \cos\phi_p & -\sin\phi_p & 0 \\ \sin\phi_p & \cos\phi_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_p & 0 & \sin\theta_p \\ 0 & 1 & 0 \\ -\sin\theta_p & 0 & \cos\theta_p \end{bmatrix} \begin{bmatrix} \sin\gamma\cos\beta \\ \sin\gamma\sin\beta \\ \cos\gamma \end{bmatrix}_{\tilde{x},\tilde{y},\tilde{z}}$$

$$= \begin{bmatrix} \cos\phi_p & -\sin\phi_p & 0 \\ \sin\phi_p & \cos\phi_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\gamma\cos\theta_p\cos\beta + \cos\gamma\sin\theta_p \\ \sin\gamma\sin\beta \\ -\sin\gamma\sin\theta_p\cos\beta + \cos\gamma\cos\theta_p \end{bmatrix}_{\tilde{x},\tilde{y},\tilde{z}}$$

$$= \begin{bmatrix} \cdots \\ \cdots \\ \cos\gamma\cos\theta_p - \sin\gamma\sin\theta_p\cos\beta \end{bmatrix}_{x,y,z}$$

so $\boxed{\cos\theta = \cos\gamma\cos\theta_p - \sin\gamma\sin\theta_p\cos\beta}$

- now we can evaluate integrals. Let $a = k_o r_p = k_o|\Delta p|$

$$S_o(\gamma,\beta) = 1 + \alpha\cos\gamma\cos\theta_p - \alpha\sin\gamma\sin\theta_p\cos\beta$$
$$= \cos\beta\Big[\underbrace{-\alpha\sin\gamma\sin\theta_p}_{c}\Big] + \underbrace{1+\alpha\cos\gamma\cos\theta_p}_{b}$$
$$= c\cos\beta + b$$

$$S_f(\omega_o;\Delta p) = \int_0^{\pi}\int_0^{2\pi}\frac{\sin\gamma\, d\gamma\, d\beta}{4\pi}\, S_o(\omega_o)(c\cos\beta + b)\, e^{+ja\cos\gamma}$$

$$= \frac{S_o(\omega_o)}{2}\int_0^{\pi} e^{+ja\cos\gamma}\sin\gamma\, d\gamma \cdot \underbrace{\frac{1}{2\pi}\int_0^{2\pi}(c\cos\beta + b)\, d\beta}_{b}$$

$$= \frac{S_o(\omega_o)}{2}\int_0^{\pi} e^{+ja\cos\gamma}\big(1+\cos\gamma(\alpha\cos\theta_p)\big)\sin\gamma\, d\gamma$$

$$v = \cos\gamma \qquad dv = -\sin\gamma\, d\gamma,$$

$$= \frac{S_o(\omega_o)}{2}\int_{-1}^{1} e^{+jav}\,(1 + v\alpha\cos\theta_p)\, dv$$

$$= \frac{S_o(\omega_o)}{2}\int_{-1}^{1} e^{+jav}\, dv + \frac{S_o(\omega_o)}{2}\alpha\cos\theta_p\int_{-1}^{1} v e^{+jav}\, dv$$

$$= \frac{S_o(\omega_o)}{2}\left(\frac{e^{+jav}}{+ja}\Bigg|_{-1}^{1}\right) + \frac{S_o(\omega_o)}{2}\alpha\cos\theta_p\left(e^{+jav}\left[\frac{v}{+ja} - \frac{1}{a^2}\right]\Bigg|_{-1}^{1}\right)$$

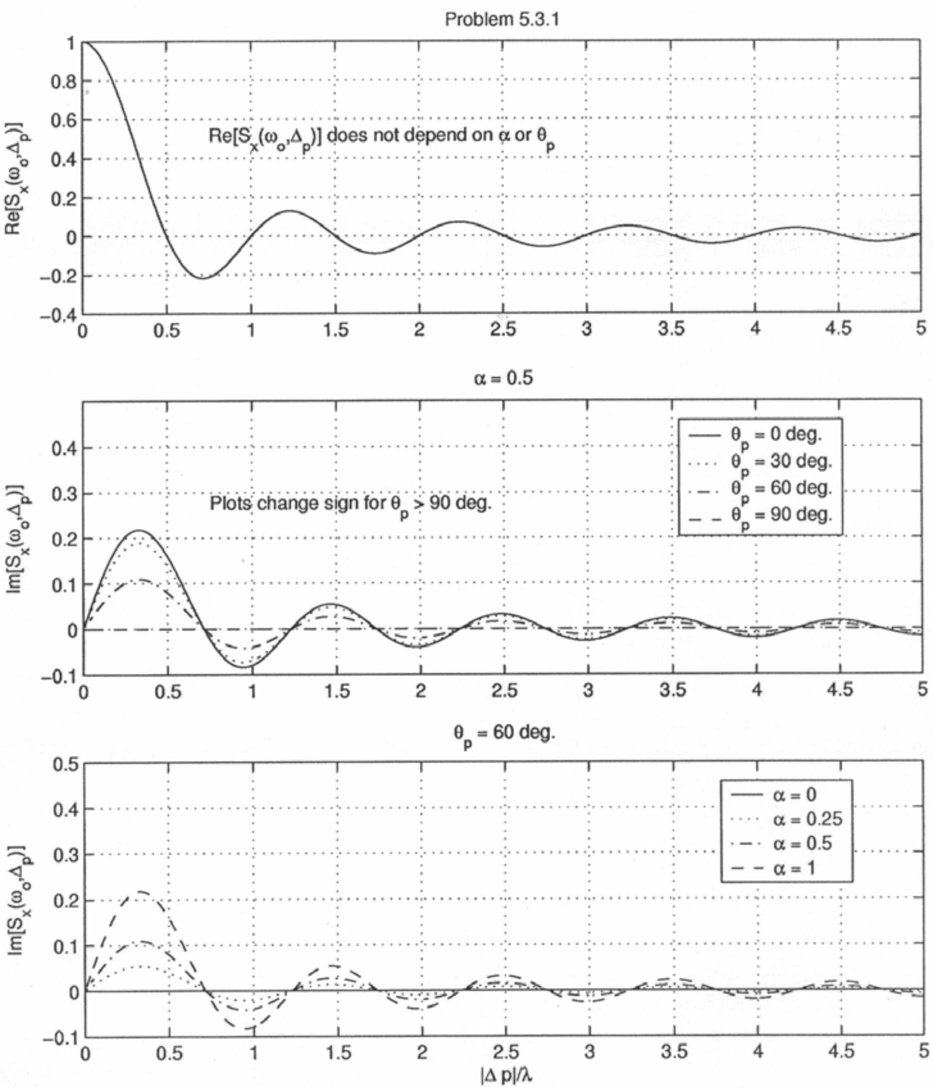
$$= S_o(\omega_o)\left[\frac{e^{+ja} - e^{-ja}}{+2ja} + \alpha\cos\theta_p\left(\frac{e^{+ja} + e^{-ja}}{+2ja} + \frac{e^{-ja} - e^{ja}}{-2a^2}\right)\left(\frac{-j}{-j}\right)\right]$$

$\boxed{5.3.1 \quad 5}$

$$= S_0(\omega_0)\left[\frac{\sin(a)}{a} - j\,\alpha\cos\theta_p\left(\frac{\cos(a)}{a} - \frac{\sin(a)}{a^2}\right)\right]$$

$$\boxed{S_f(\omega_0;\Delta\underline{p}) = S_0(\omega_0)\left\{\operatorname{sinc}\left(k_0|\Delta\underline{p}|\right) + \frac{j\,\alpha\cos\theta_p}{k_0|\Delta\underline{p}|}\left(\operatorname{sinc}\left(k_0|\Delta\underline{p}|\right) - \cos\left(k_0|\Delta\underline{p}|\right)\right)\right\}}$$

5.3.1 (6)



Problem 5.3.1
Re[S_x(ω_o,Δ_p)]
Re[S_x(ω_o,Δ_p)] does not depend on α or θ_p
α = 0.5
Im[S_x(ω_o,Δ_p)]
θ_p = 0 deg.
θ_p = 30 deg.
θ_p = 60 deg.
θ_p = 90 deg.
Plots change sign for θ_p > 90 deg.
θ_p = 60 deg.
Im[S_x(ω_o,Δ_p)]
α = 0
α = 0.25
α = 0.5
α = 1
|Δ p|/λ