Problem 2.4.2

a) $$\Upsilon_\varphi(\varphi) = \sum_{n=0}^{\frac{N}{2}-1} \left(-\frac{1}{N}\right) e^{j\left(n-\frac{N-1}{2}\right)\varphi} + \sum_{n=\frac{N}{2}}^{N-1} \left(\frac{1}{N}\right) e^{j\left(n-\frac{N-1}{2}\right)\varphi}$$

$$= \frac{1}{N}\left\{ -e^{-j\frac{N-1}{2}\varphi} \sum_{n=0}^{\frac{N}{2}-1} e^{jn\varphi} + e^{j\varphi/2} \sum_{n=0}^{\frac{N}{2}-1} e^{jn\varphi} \right\}$$

$$= \frac{1}{N}\left\{1 - e^{-j\frac{N}{2}\varphi}\right\} e^{j\frac{\varphi}{2}} \frac{1-e^{j N/2 \varphi}}{1-e^{j\varphi}}$$

$$= \frac{1}{N}\left(\frac{e^{jN/4\varphi} - e^{-j\frac{N}{4}\varphi}}{2j}\right)\left(\frac{e^{jN/4\varphi} - e^{-jN/4\varphi}}{2j}\right)\left(\frac{2j}{e^{j\varphi/2} - e^{-j\varphi/2}}\right)(2j)$$

$$\boxed{\Upsilon_\varphi(\varphi) = j\frac{2}{N}\frac{\sin^2\left(\frac{N}{4}\varphi\right)}{\sin(\varphi/2)}}$$

In the visible region:

$$\boxed{B(\varphi) = j\frac{2}{N}\frac{\sin^2\left(\frac{N}{4}\varphi\right)}{\sin(\varphi/2)} \qquad -\frac{2\pi d}{\lambda} \le \varphi \le \frac{2\pi d}{\lambda}}$$

- when $d = \lambda/2$ VR $\Rightarrow$ $-\pi \le \varphi \le \pi$, Plots on p.3 for $N=20$

Notes: Beampattern is purely imaginary with $B(\varphi) = -B(-\varphi)$ because weights are <u>conjugate asymmetric</u>

b) $$B(\varphi) = j\frac{2}{N}\frac{\sin^2(N/4\,\varphi)}{\sin(\varphi/2)} = j\frac{1}{N}\frac{\left(1-\cos\left(\frac{N}{2}\varphi\right)\right)}{\sin(\varphi/2)}$$

$$B'(\varphi) = \frac{j}{N}\;\frac{\frac{N}{2}\sin\left(\frac{N}{2}\varphi\right)\sin(\varphi/2) - \frac{1}{2}\left(1-\cos\left(\frac{N}{2}\varphi\right)\right)\cos(\varphi/2)}{\sin^2(\varphi/2)} = \frac{0}{0}$$

at $\varphi = 0$

Use L'Hospital's Rule

$$B'(\varphi) = \frac{j}{4N}\frac{T_1(\varphi)}{T_2(\varphi)}\bigg|_{\varphi=0} = \frac{j}{4N}\frac{T_1'(\varphi)}{T_2'(\varphi)}\bigg|_{\varphi=0}$$

$$T_1(\varphi) = 2\left\{N \sin\left(\tfrac{N}{2}\varphi\right)\sin(\varphi/2) - \cos(\varphi/2) + \cos\left(\tfrac{N}{2}\varphi\right)\cos(\varphi/2)\right\}$$

$$= N\cos\left(\tfrac{N-1}{2}\varphi\right) - N\cos\left(\tfrac{N+1}{2}\varphi\right) - 2\cos(\varphi/2) + \cos\left(\tfrac{N-1}{2}\varphi\right) + \cos\left(\tfrac{N+1}{2}\varphi\right)$$

$$= (N+1)\cos\left(\tfrac{N-1}{2}\varphi\right) - (N-1)\cos\left(\tfrac{N+1}{2}\varphi\right) - 2\cos(\varphi/2)$$

$$T_1'(\varphi) = -(N+1)\left(\tfrac{N-1}{2}\right)\sin\left(\tfrac{N-1}{2}\varphi\right) + (N-1)\left(\tfrac{N+1}{2}\right)\sin\left(\tfrac{N+1}{2}\varphi\right) + \sin(\varphi/2)$$

$$T_1'(0) = 0$$

$$T_2(\varphi) = \sin^2(\varphi/2) = \tfrac{1}{2}(1-\cos\varphi)$$

$$T_2'(\varphi) = \tfrac{1}{2}\sin(\varphi)$$

$$T_2'(\varphi) = 0$$

• $$B'(0) = \frac{j}{4N}\frac{T_1'(0)}{T_2'(0)} = \frac{0}{0} = \frac{j}{4N}\frac{T_1''(\varphi)}{T_2''(\varphi)}\bigg|_{\varphi=0}$$ Use L'Hospital's Rule again

$$T_1''(\varphi) = -\frac{(N+1)(N-1)(N-1)}{4}\cos\left(\tfrac{N-1}{2}\varphi\right) + \frac{(N-1)(N+1)(N+1)}{4}\cos\left(\tfrac{N+1}{2}\varphi\right) + \tfrac{1}{2}\cos(\varphi/2)$$

$$T_1''(0) = \frac{(N+1)(N-1)}{4}\left\{-(N-1)+(N+1)\right\} + \frac{1}{2} = \frac{N^2-1}{2} + \frac{1}{2} = \frac{N^2}{2}$$

$$T_2''(\varphi) = \tfrac{1}{2}\cos(\varphi)$$

$$T_2''(0) = 1/2$$

$$B'(0) = \frac{j}{4N}\frac{(N^2/2)}{(1/2)} = \boxed{j\frac{N}{4} = B'(0)}$$

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