4.2.7 This is a special case of a planar array with

$$P_n = \begin{bmatrix} R\cos\phi_n \\ R\sin\phi_n \\ 0 \end{bmatrix} \qquad \phi_n = n\frac{2\pi}{N}$$

$$D = \frac{|B_{max}|^2}{\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}|B(\theta,\phi)|^2 \sin\theta\, d\theta\, d\phi}$$

$$B(\theta,\phi) = \sum_{n=0}^{N-1} w_n^* e^{j\frac{2\pi}{\lambda}(P_{xn}\sin\theta\cos\phi + P_{yn}\sin\theta\sin\phi)}$$

Following the derivation in 4.1.1.2

$$|B(\theta,\phi)|^2 = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w_n^* w_m e^{j\frac{2\pi}{\lambda}\sin\theta[(P_{xn}-P_{xm})\cos\phi + (P_{yn}-P_{ym})\sin\phi]}$$

now let $\rho_{mn} = |P_n - P_m| = \sqrt{(P_{xn}-P_{xm})^2 + (P_{yn}-P_{ym})^2}$

$$= R\sqrt{(\cos\phi_n - \cos\phi_m)^2 + (\sin\phi_n - \sin\phi_m)^2}$$

$$= R\sqrt{2(1-\cos\phi_n\cos\phi_m - \sin\phi_n\sin\phi_m)}$$

$$= 2R\sqrt{\tfrac{1}{2}(1-\cos|\phi_n-\phi_m|)}$$

$$\boxed{\rho_{mn} = 2R\sin\left(\frac{|\phi_n-\phi_m|}{2}\right)}$$

$$\phi_{mn} = \arctan\frac{P_{yn}-P_{ym}}{P_{xn}-P_{xm}} =$$

$$\boxed{\phi_{mn} = \arctan\frac{\sin\phi_n - \sin\phi_m}{\cos\phi_n - \cos\phi_m}}$$

$$\Rightarrow \quad P_{xn} - P_{xm} = \rho_{mn}\cos\phi_{mn}$$

$$P_{yn} - P_{ym} = \rho_{mn}\sin\phi_{mn}$$

$$|B(\theta,\phi)|^2 = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w_n^* w_m e^{j\frac{2\pi}{\lambda}\rho_{mn}\sin\theta[\underbrace{\cos\phi\cos\phi_{mn} + \sin\phi\sin\phi_{mn}}_{\cos(\phi-\phi_{mn})}]}$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w_n^* w_m e^{j\frac{2\pi}{\lambda}\rho_{mn}\sin\theta\cos(\phi-\phi_{mn})}$$

Then $\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} |B(\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w_n^* w_m \ \frac{1}{2}\int_0^{\pi} \sin\theta \, d\theta \left\{ \frac{1}{2\pi} \underbrace{\int_0^{2\pi} e^{j\frac{2\pi}{\lambda}\rho_{mn}\sin\theta\cos(\phi-\phi_{mn})} d\phi}_{J_0\left(\frac{2\pi}{\lambda}\rho_{mn}\sin\theta\right)} \right\}$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w_n^* w_m \ \underbrace{\frac{1}{2}\int_0^{\pi} J_0\left(\frac{2\pi}{\lambda}\rho_{mn}\sin\theta\right)\sin\theta \, d\theta}_{\mathrm{sinc}\left(\frac{2\pi}{\lambda}\rho_{mn}\right)}$$

$$\boxed{D = \frac{|B(\theta_0,\phi_0)|^2}{\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w_n^* w_m \,\mathrm{sinc}\left(\frac{2\pi}{\lambda}\rho_{mn}\right)}}$$

$$\boxed{D = \frac{\left|\underline{w}^{\#}\underline{v}(\theta_0,\phi_0)\right|^2}{\underline{w}^{\#}\underline{C}\,\underline{w}} \qquad c_{k\ell} = \mathrm{sinc}\left(\frac{2\pi}{\lambda}\rho_{k\ell}\right)}$$