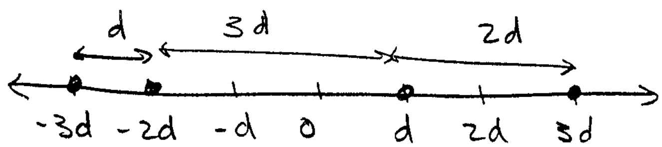
2.6.2 The array of interest is a 4-element non-uniformly spaced array with spacind $d, 3d, 2d$. Centering the array at the origin, the element positions are $p_0 = -3d$, $p_1 = -2d$, $p_2 = d$, $p_3 = 3d$,



The weighting is uniform $w_n = 1/4$ and $d = \lambda/2$.

(a) Directivity - direct calculation. For a linear array use (2.148)

$$D = \left\{ \frac{1}{2} \int_{-1}^{1} |B_u(u)|^2 du \right\}^{-1} = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |B_\psi(\psi)|^2 d\psi \right\}^{-1}$$

$$\psi = 2\pi \frac{d}{\lambda} u = \pi u$$

$$B(\psi) = \frac{1}{4} \left\{ e^{-j3\psi} + e^{-j2\psi} + e^{j\psi} + e^{j3\psi} \right\}$$

$$= \frac{1}{2} \left\{ \frac{e^{-j3\psi} + e^{j3\psi}}{2} + e^{-j\psi/2} \frac{\left(e^{-j3\psi/2} + e^{j3\psi/2}\right)}{2} \right\}$$

$$= \frac{1}{2}\cos(3\psi) + \frac{1}{2} e^{-j\psi/2} \cos(3\psi/2)$$

$$|B(\psi)|^2 = \frac{1}{4}\cos^2(3\psi) + \frac{1}{4}\cos^2(3\psi/2) + \frac{1}{2}\cos(\psi/2)\cos\left(\frac{3\psi}{2}\right)\cos(3\psi)$$

$$= \frac{1}{8} \left\{ 2 + \cos(6\psi) + \cos(3\psi) + \cos(5\psi) + \cos(4\psi) + \cos(2\psi) + \cos(\psi) \right\}$$

$$D = \left\{ \frac{1}{16\pi} \int_{-\pi}^{\pi} \left[2 + \cos(\psi) + \cos(2\psi) + \cos(3\psi) + \cos(4\psi) + \cos(5\psi) + \cos(6\psi) \right] d\psi \right\}^{-1}$$

$$= \left\{ \frac{1}{16\pi} \cdot 4\pi \right\}^{-1} = 4$$

$D = 4$

(a) Directivity - by inspection - The 4 element array with non-uniform spacing that are multiples of d is equivalent to a 7-element uniformly spaced array with weights of missing elements set to zero, ie

$$W = \left[\tfrac{1}{4}\ \tfrac{1}{4}\ 0\ 0\ \tfrac{1}{4}\ 0\ \tfrac{1}{4}\right]^T$$

Then, using (2.157) for standard linear arrays

$$D = \|w\|^{-2} = \left(\sum_{n=0}^{6} |w_n|^2\right)^{-1} = \left[4\left(\tfrac{1}{16}\right)\right]^{-1} = \underline{\underline{4}}$$

(b) General case - elements separated by integer multiples of $d = \lambda/2$. Using the same argument as above, suppose the grid spanning the N elements of the array contains M points. Then

$$D = \left\{\sum_{m=0}^{M-1} |\tilde{w}_m|^2\right\}^{-1} \qquad \left(\text{assuming} \sum_{m=0}^{M-1} \tilde{w}_m = 1\right)$$

where $\tilde{w}_m = \begin{cases} w_n & \text{when grid contains element} \\ 0 & \text{"missing" element} \end{cases}$

then $$D = \left\{\sum_{\substack{m=0 \\ \tilde{w}_m \neq 0}}^{M-1} |\tilde{w}_m|^2\right\}^{-1} = \left\{\sum_{n=0}^{N-1} |w_n|^2\right\}^{-1}$$

$$\left(\text{assuming} \sum_{m=0}^{M-1} \tilde{w}_m = \sum_{n=0}^{N-1} w_n = 1\right)$$

When $w_n = \frac{1}{N} \Rightarrow \underline{\underline{D = N}}$