6.2.8

$$\underline{W}_{dq} = \underline{w}_{dq} / \| \underline{w}_{dq} \|$$

a) $\min \quad \underline{w}^{\#} \underline{S}_n \underline{w} \quad \text{s.t} \quad \bar{\underline{w}}_{dq}^{\#} \underline{w} = 1$

using Lagrange multiplier as in (6.15), we have

$$\min \; J(\underline{w}) = \underline{w}^{\#} \underline{S}_n \underline{w} - \lambda_q [\bar{\underline{w}}_{dq}^{\#} \underline{w} - 1 + \underline{w}^{\#} \bar{\underline{w}}_{dq} - 1]$$

$$\frac{\partial}{\partial w^{\#}} J(\underline{w}) = \underline{S}_n \underline{w} - \lambda \bar{\underline{w}}_{dq} = 0$$

$$\Rightarrow \underline{w} = \lambda_q \underline{S}_n^{-1} \bar{\underline{w}}_{dq}$$

applying the constraint,

$$\bar{\underline{w}}_{dq}^{\#} \underline{w} = \lambda_q \bar{\underline{w}}_{dq}^{\#} \underline{S}_n^{-1} \bar{\underline{w}}_{dq} = 1$$

$$\Rightarrow \lambda_q = (\bar{\underline{w}}_{dq}^{\#} \underline{S}_n^{-1} \bar{\underline{w}}_{dq})$$

$$\Rightarrow \boxed{\underline{w} = \frac{\underline{S}_n^{-1} \bar{\underline{w}}_{dq}}{\bar{\underline{w}}_{dq}^{\#} \underline{S}_n^{-1} \bar{\underline{w}}_{dq}}}$$

b) $AG = \frac{| \underline{w}^{\#} \underline{v}_s |^2}{\underline{w}^{\#} \underline{\rho}_n \underline{w}}$, where $\underline{\rho}_n \equiv \underline{S}_n / S_n(\omega)$

$$= \frac{| \bar{\underline{w}}_{dq}^{\#} \underline{S}_n^{-1} \underline{v}_s |^2 / \cancel{(\bar{\underline{w}}_{dq}^{\#} \underline{S}_n^{-1} \bar{\underline{w}}_{dq})^2}}{(\bar{\underline{w}}_{dq}^{\#} \underline{S}_n^{-1} \underline{\rho}_n \underline{S}_n^{-1} \bar{\underline{w}}_{dq}) / \cancel{(\bar{\underline{w}}_{dq}^{\#} \underline{S}_n^{-1} \bar{\underline{w}}_{dq})^2}}$$

$$S_n = \rho_n S_n(\omega)$$
$$S_n^{-1} = \rho_n^{-1} S_n(\omega)^{-1}$$

$$AG = \frac{|\bar{w}_{dq}^{\#} \rho_n^{-1} v_s|^2 / \cancel{S_n(\omega)^2}}{(\bar{w}_{dq}^{\#} \rho_n^{-1} \bar{w}_{dq}) / \cancel{S_n(\omega)^2}}$$

$$\boxed{AG = \frac{|\bar{\underline{w}}_{dq}^{\#} \rho_n^{-1} \underline{v}_s|^2}{(\bar{\underline{w}}_{dq}^{\#} \rho_n^{-1} \bar{\underline{w}}_{dq})}}$$

c) $\underline{S}_n = \sigma_w^2 \underline{I}$, $S_n(\omega) = \sigma_w^2$

$\rho_n = \underline{I}$

$$\Rightarrow \quad AG = \frac{|\bar{\underline{w}}_{dq}^{\#} \underline{v}_s|^2}{\|\bar{\underline{w}}_{dq}\|^2} = \boxed{|\bar{\underline{w}}_{dq}^{\#} v_s|^2}$$

since $\bar{\underline{w}}_{dq} = \dfrac{\underline{w}_{dq}}{\|w_{dq}\|} \Rightarrow \|\bar{w}_{dq}\| = 1$