6.5.5

- D planewave signals plus white noise

$$x = Vf + n \qquad (N\times 1)$$

where

$$V = [v_1 \cdots v_D] \qquad (N\times D)$$
$$\underline{f} = [f_1 \cdots f_D]^T \qquad (D\times 1)$$

- $S_x = V S_f V^\# + S_n$

- MMSE estimate of all D signals

$$\hat{f} = \begin{bmatrix} \hat{f}_1 \\ \vdots \\ \hat{f}_D \end{bmatrix} = H_0 x$$

where H_0 is the $D\times N$ filter

$$H_0 = S_{fx^\#} S_x^{-1}$$

- assume noise uncorrelated with signals

$$S_{fx^\#} = E\{fx^\#\} = E\{ff^\#\} V^\# = S_f V^\#$$

$$\Rightarrow H_0 = S_f V^\# S_x^{-1} = (I + S_f V^\# S_n^{-1} V)^{-1} S_f V^\# S_n^{-1}$$

- assuming S_f^{-1} exists, this can also be written as

$$H_0 = \underbrace{(I + S_f V^\# S_n^{-1} V)^{-1} (S_f^{-1})^{-1}} V^\# S_n^{-1}$$

$A^{-1}B^{-1} = (BA)^{-1}$

$$= (S_f^{-1} + V^\# S_n^{-1} V)^{-1} V^\# S_n^{-1} = S_f \underbrace{S_f^{-1} (S_f^{-1} + V^\# S_n^{-1} V)^{-1}} V^\# S_n^{-1}$$

$$H_0 = S_f (I + V^\# S_n^{-1} V S_f)^{-1} V^\# S_n^{-1}$$

- when $S_n = \sigma_W^2 I \Rightarrow$ $\boxed{H_0 = \frac{S_f}{\sigma_W^2} \left(I + V^\# V \frac{S_f}{\sigma_W^2}\right)^{-1} V^\#}$

- MMSE estimate of 1st signal only

$\hat{f}_1 = H_1 x$ where H_1 is the $1 \times N$ filter

$H_1 = S_{f_1 x^\#} S_x^{-1}$

$S_{f_1 x^\#} = E\{f_1 x^\#\} = E\{f_1 f^\#\} V^\# = \Sigma_1 V^\#$

where Σ_1 is first raw of $S_f = \begin{bmatrix} \Sigma_1 \\ \hline \Sigma_f \end{bmatrix}$

$$\Rightarrow \underline{\underline{H_1 = \Sigma_1 V^\# S_x^{-1}}}$$

comparing to $H_0 = S_f V^\# S_x^{-1} = \begin{bmatrix} \Sigma_1 \\ \hline \Sigma_f \end{bmatrix} V^\# S_x^{-1}$

$$= \begin{bmatrix} \Sigma_1 V^\# S_x^{-1} \\ \hline \Sigma_f V^\# S_x^{-1} \end{bmatrix}$$

we see that $\underline{\underline{H_1 \text{ is first raw of } H_0}}$, thus estimating f_1 as one of D desired signals, or by itself with remaining signals as interferers results in the same processor.

for white noise $H_0 = \frac{S_f}{\sigma_W^2} (I + V^\# V \frac{S_f}{\sigma_W^2})^{-1} V^\#$

$$= \frac{1}{\sigma_W^2} \begin{bmatrix} \Sigma_1 (I + V^\# V S_f/\sigma_W^2)^{-1} V^\# \\ \Sigma_f (I + V^\# V S_f/\sigma_W^2)^{-1} V^\# \end{bmatrix}$$

$$\Rightarrow \boxed{H_1 = \frac{\Sigma_1}{\sigma_W^2} (I + V^\# V \frac{S_f}{\sigma_W^2})^{-1} V^\#, \quad \Sigma_1 = \text{first raw of } S_f}$$

The result $H_1 = \frac{\Sigma_1}{\sigma_W^2}\left(I + V^{\#} V \frac{S_f}{\sigma_W^2}\right)^{-1} V^{\#}$ holds for

arbitrary $D < N-1$ and does not require S_f diagonal

expanding this for $D=2$ does not lead to any enlightening expressions for the general case.

- However if $S_f = \left[\begin{array}{c|c} \sigma_1^2 & 0 \\ \hline 0 & S_I \end{array}\right]$ (signal 1 uncorrelated with rest)

$$\Rightarrow \Sigma_1 = \left[\begin{array}{c|c} \sigma_1^2 & 0 \end{array}\right]$$

we can get an interesting expression. Note that

$$\frac{\Sigma_1}{\sigma_W^2}\left(I + V^{\#} V S_f/\sigma_W^2\right)^{-1} = \frac{\sigma_1^2}{\sigma_W^2} \cdot \text{first row of } \left(I + V^{\#} V S_f/\sigma_W^2\right)$$

partition $V = [v_1 \,\vdots\, V_I] \Rightarrow V^{\#} V = \begin{bmatrix} v_1^{\#} v_1 & v_1^{\#} V_I \\ V_I^{\#} v_1 & V_I^{\#} V_I \end{bmatrix}$

define $\rho_{1I} = \frac{1}{N} v_1^{\#} V_I \Rightarrow$

$$V^{\#} V = \begin{bmatrix} N & N\rho_{1I} \\ N\rho_{1I}^{\#} & V_I^{\#} V_I \end{bmatrix}$$

$$I + V^{\#} V \frac{S_f}{\sigma_W^2} = \begin{bmatrix} 1 + N\sigma_1^2/\sigma_W^2 & N\rho_{1I} S_I/\sigma_W^2 \\ N\rho_{1I}^{\#} \sigma_1^2/\sigma_W^2 & I + V_I^{\#} V_I S_I/\sigma_W^2 \end{bmatrix}$$

1st row of inverse of Partitioned Matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \left[\begin{array}{c|c} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} & -(A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} A_{12} A_{22}^{-1} \\ \hline \cdot & \cdot \end{array}\right]$$

$$(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1} = \left[1+N\sigma_1^2/\sigma_W^2 - N\rho_{1I}S_I/\sigma_W^2\,(I+V_I^{\#}V_I S_I/\sigma_W^2)^{-1}N\rho_{1I}^{\#}\sigma_1^2/\sigma_{\dots}\right.$$

$$A_{12}A_{22}^{-1} = N\rho_{1I}S_I/\sigma_W^2\,(I+V_I^{\#}V_I S_I/\sigma_W^2)^{-1} = N\rho_{1I}\left(I+\frac{S_I}{\sigma_W^2}V_I^{\#}V_I\right)^{-1}\frac{S}{\sigma}$$

$$\Rightarrow H_1 = \frac{\sigma_1^2}{\sigma_W^2}(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1}\left[v_1^{\#} - A_{12}A_{22}^{-1}V_2^{\#}\right]$$

letting $S_n \equiv \sigma_W^2 I + V_I S_I V_I^{\#}$, we see that

$$(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1} = (1+\sigma_1^2 v_1^{\#}S_n^{-1}V_1)^{-1} = \frac{\Lambda}{\sigma_1^2+\Lambda}$$

where $\Lambda = (v_1^{\#}S_n^{-1}v_1)^{-1}$

$$\Rightarrow H_1 = \frac{\sigma_1^2\Lambda}{\sigma_1^2+\Lambda}\cdot\frac{N}{\sigma_W^2}\left[\frac{V_1}{N}^{\#} - N\rho_{1I}\left(I+\frac{S_I}{\sigma_W^2}V_I^{\#}V_I\right)^{-1}\frac{S_I}{\sigma_W^2}\cdot\frac{V_I}{N}^{\#}\right]$$

which corresponds to the processor shown in Fig. 6.34.