2.3.3 (i) N even

$$w \text{ real, symmetric} \Longrightarrow w = \begin{bmatrix} w_1 \\ \hline Jw_1 \end{bmatrix}$$

where w_1 is real and contains the first $\frac{N}{2}$ elements of w.

$$v(\varphi) = \begin{bmatrix} v_1(\varphi) \\ \hline Jv_1^*(\varphi) \end{bmatrix} = \begin{bmatrix} e^{-j\left(\frac{N-1}{2}\right)\varphi} \\ e^{-j\left(\frac{N-3}{2}\right)\varphi} \\ \vdots \\ \hline \vdots \\ e^{j\left(\frac{N-3}{2}\right)\varphi} \\ e^{j\left(\frac{N-1}{2}\right)\varphi} \end{bmatrix} \begin{matrix} \left.\right\} v_1(\varphi) \\ \\ \left.\right\} Jv_1^*(\varphi) \end{matrix}$$

$$B_\varphi(\varphi) = w^{\#} v(\varphi) = \left[w_1^T \mid w_1^T J^T \right] \begin{bmatrix} v_1(\varphi) \\ Jv_1^*(\varphi) \end{bmatrix} = w_1^T v_1(\varphi) + w_1^T \underbrace{J^T J}_{I} v_1^*(\varphi)$$

$$= w_1^T \left\{ v_1(\varphi) + v_1^*(\varphi) \right\}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} w_n \left\{ e^{j\left(n - \frac{N-1}{2}\right)\varphi} + e^{-j\left(n - \frac{N-1}{2}\right)\varphi} \right\}$$

$$\boxed{B_\varphi(\varphi) = \sum_{n=0}^{\frac{N}{2}-1} w_n \, 2\cos\left[\left(n - \frac{N-1}{2}\right)\varphi\right] \quad \text{which is real and symmetric i.e. } B(\varphi) = B(-\varphi).}$$

(ii) N odd

$$w = \begin{bmatrix} w_1 \\ w_0 \\ Jw_1 \end{bmatrix}$$

where w_1 is real and contains the first $\frac{N-1}{2}$ elements of w. w_0 is the real middle element of w.

$$v(\varphi) = \begin{bmatrix} v_1(\varphi) \\ 1 \\ Jv_1^*(\varphi) \end{bmatrix}$$

$$B_\varphi(\varphi) = w^{\#} v(\varphi) = \left[w_1^T \;\; w_0 \;\; w_1^T J^T \right] \begin{bmatrix} v_1(\varphi) \\ 1 \\ Jv_1^*(\varphi) \end{bmatrix} = w_1^T v_1(\varphi) + w_0 + w_1^J v_1^*(\varphi)$$

$$= w_0 + w_1^T \left\{ v_1(\varphi) + v_1^*(\varphi) \right\} = w_0 + \sum_{n=0}^{\frac{N-1}{2}} w_n \left\{ e^{j\left(n - \frac{N-1}{2}\right)\varphi} + e^{-j\left(n - \frac{N-1}{2}\right)\varphi} \right\}$$

$$\boxed{B_\varphi(\varphi) = w_0 + \sum_{n=0}^{\frac{N-1}{2}} w_n \, 2\cos\left[\left(n - \frac{N-1}{2}\right)\varphi\right] \quad \text{which is real and symmetric}}$$