4.3.1

$w_\phi(\phi) = 1$

$$w_r(r) = \begin{cases} 1-(r/R)^2 & 0 \le r \le R \\ 0 & \text{elsewhere} \end{cases}$$

For $w_\phi(\phi)$ uniform, $B(\theta,\phi)$ is constant in ϕ and $B(\theta)$ is given by (4.204)

$$B(\theta) = 2\pi \int_0^R w_r(r)\, J_0\!\left(\frac{2\pi r}{\lambda}\sin\theta\right) r\,dr$$

$$= 2\pi \int_0^R \left\{1-(r/R)^2\right\} J_0\!\left(\frac{2\pi r}{\lambda}\sin\theta\right) r\,dr$$

- we'll need the following Bessel Function identities

(i) $2m\, J_m(x) = x\left(J_{m-1}(x) + J_{m+1}(x)\right)$

(ii) $\displaystyle\int_0^z x^m J_{m-1}(x)\,dx = x^m J_m(x)\Big|_0^z = z^m J_m(z) \qquad m \ge 1$

- Let $\alpha = \frac{2\pi}{\lambda}\sin\theta \qquad x = \alpha r \qquad r = \frac{x}{\alpha} \qquad dr = \frac{dx}{\alpha}$

$$B(\theta) = 2\pi \int_0^{\alpha R} \left(1-\left(x/\alpha R\right)^2\right) J_0(x)\, \frac{x}{\alpha}\cdot\frac{dx}{\alpha}$$

$$= \frac{2\pi}{\alpha^2}\int_0^{\alpha R} x\,J_0(x)\,dx + \frac{2\pi}{\alpha^2}\int_0^{\alpha R} \frac{x^3}{(\alpha R)^2} J_0(x)\,dx$$

→ using (i), replace $J_0(x) = \frac{2J_1(x)}{x} - J_2(x)$ in 2nd integra

$$= \frac{2\pi}{\alpha^2}\int_0^{\alpha R} x\,J_0(x)\,dx + \frac{4\pi}{\alpha^2}\int_0^{\alpha R} \frac{x^2}{(\alpha R)^2} J_1(x)\,dx - \frac{2\pi}{\alpha^2}\int_0^{\alpha R}\frac{x^3}{(\alpha R)^2} J_2(x)\,dx$$

→ using (ii)

$$= \frac{2\pi}{\alpha^2}\left\{(\alpha R)\,J_1(\alpha R) + \frac{2(\alpha R)^2}{(\alpha R)^2} J_2(\alpha R) - \frac{(\alpha R)^3}{(\alpha R)^2} J_3(\alpha R)\right\}$$

4.3.1 con't'd

$$B(\theta) = \frac{2\pi}{\alpha^2}\left\{ \underbrace{(\alpha R)\left[J_1(\alpha R) + J_3(\alpha R)\right]}_{2\cdot 2\, J_2(\alpha R)} - 2J_2(\alpha R)\right]$$

$$= \frac{4\pi}{\alpha^2} J_2(\alpha R) \quad ; \quad \alpha = \frac{2\pi}{\lambda}\sin\theta$$

let $\psi_R = 2\pi \frac{R}{\lambda}\sin\theta = \alpha R$

$$\boxed{B(\theta) = 4\pi R^2 \left.\frac{J_2(\psi_R)}{\psi_R^2}\right|_{\psi_R = \frac{2\pi R}{\lambda}\sin\theta}}$$

using

$$J_n(x) = \frac{(x/2)^n}{n!}\left[1 - \frac{(x/2)^2}{(n+1)!} + \cdots\right.$$

$$\frac{J_n(x)}{x^n} = \frac{(1/2)^n}{n!}\left[1 - \frac{(x/2)^2}{(n+1)!} + \cdots\right.$$

$$\frac{J_n(0)}{0^n} \rightarrow \frac{(1/2)^n}{n!}$$

$$\Rightarrow \frac{J_2(0)}{0^2} \rightarrow \frac{(1/2)^2}{2} = \frac{1}{8}$$

$$B(0) = \frac{\pi R^2}{2}$$

- normalized Beampattern is

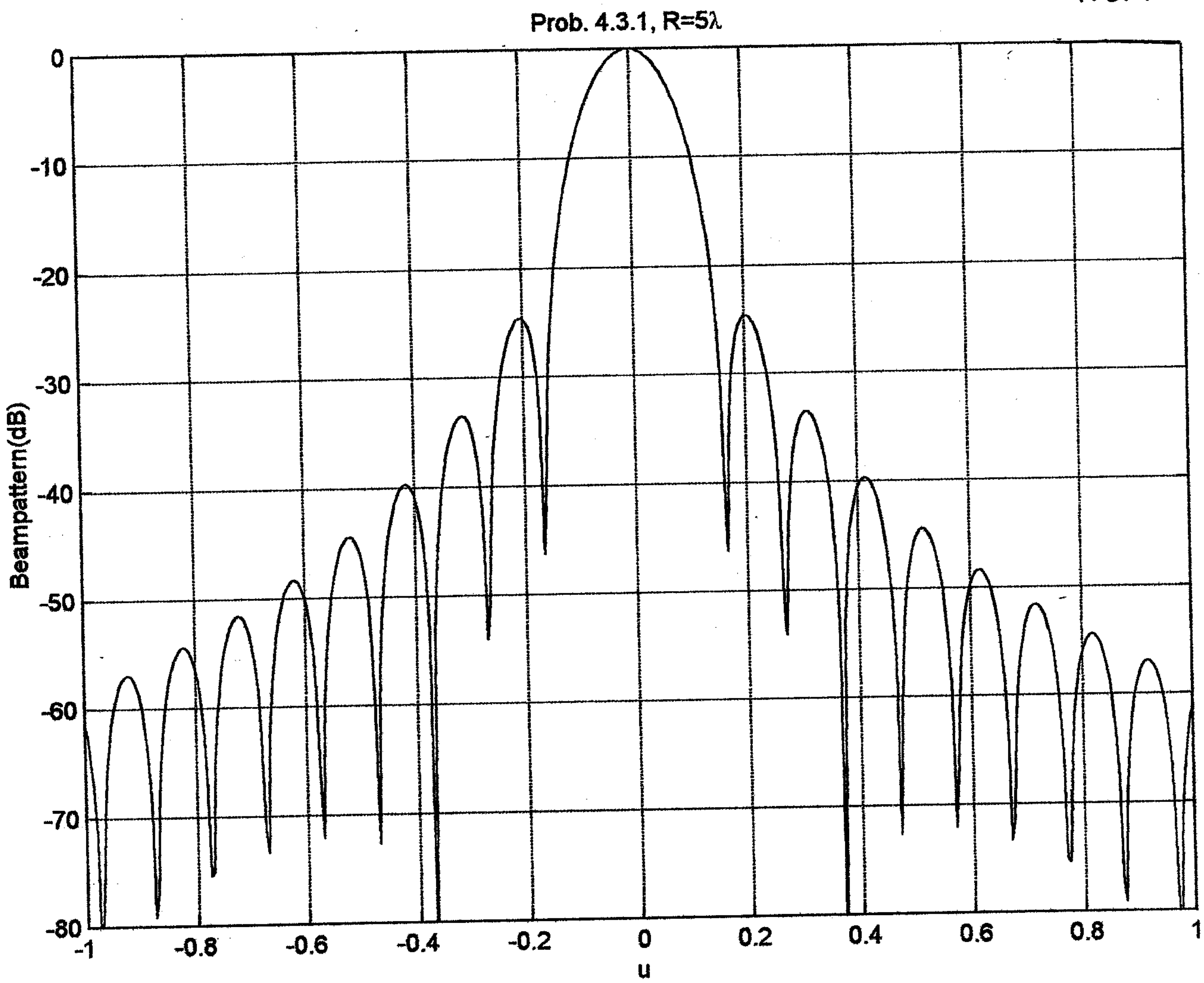
$$\boxed{B_n(\theta) = 8\, J_2(\psi_R)/\psi_R^2}$$

- to find directivity

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} |B_n(\theta)|^2 \sin\theta\, d\theta\, d\phi = \frac{1}{2}\int_0^{\pi} \frac{64 \sin\theta}{(2\pi R/\lambda \sin\theta)^4} J_2^2\left(\frac{2\pi R}{\lambda}\sin\theta\right) d\theta$$

- probably can be evaluated, given the right identity

For now, do numerical integration



Numerical Values @ $R=5\lambda$

HPBW = $36.4/(R/\lambda)$

BWNN = $93.7/(R/\lambda)$

1st sidelobe = -24.6 dB

Directivity = $0.375\left(\frac{2\pi R}{\lambda}\right)^2$