5.5.23

$$\bar{S}_o(\omega:u) = S_x(\omega)\frac{2}{\sqrt{2\pi\sigma_o^2}}\exp\left\{-\frac{(u-u_s)^2}{2\sigma_o^2}\right\}$$

a) $$S_f(\omega:\Delta p_z) = \frac{1}{2}\int_{-1}^{1} e^{+jk_o\Delta p_z u}\,\bar{S}_o(\omega:u)\,du$$

$$= S_x(\omega)\int_{-1}^{1} e^{+jk_o\Delta p_z u}\frac{1}{\sqrt{2\pi\sigma_o^2}}\exp\left\{-\frac{(u-u_s)^2}{2\sigma_o^2}\right\}du$$

let $v = u-u_s$

$$= S_x(\omega)\,e^{+jk_o\Delta p_z u_s}\int_{-1-u_s}^{1-u_s}\frac{1}{\sqrt{2\pi\sigma_o^2}}e^{-v^2/2\sigma_o^2}e^{+jk_o\Delta p_z v}\,dv$$

- assume $\sigma_o \ll 1$, u_s not too close to ± 1, then Gaussian distribution narrow wrt integration interval, practically the same as integrating over $(-\infty,\infty)$. Completing the square in the exponent we have

$$S_f(\omega:\Delta p_z) = S_x(\omega)\,e^{+jk_o\Delta p_z u_s}\int_{-\infty}^{\infty}\frac{dv}{\sqrt{2\pi\sigma_o^2}}\exp\left\{-\frac{1}{2\sigma_o^2}\left[v^2 - j2\sigma_o^2 k_o\Delta p_z v + (j\sigma_o^2 k_o\Delta p_z)^2 - (j\sigma_o^2 k_o\Delta p_z)^2\right]\right\}$$

$$= S_x(\omega)\,e^{+jk_o\Delta p_z u_s}\,e^{(j\sigma_o^2 k_o\Delta p_z)^2/2\sigma_o^2}\underbrace{\int_{-\infty}^{\infty}\frac{dv}{\sqrt{2\pi\sigma_o^2}}\exp\left\{-\frac{1}{2\sigma_o^2}\left[v - j2\sigma_o^2 k_o\Delta p_z\right]^2\right\}}_{1}$$

$k_o = 2\pi/\lambda \Rightarrow$

$$\boxed{S_f(\omega:\Delta p_z) = S_x(\omega)\,e^{+j\frac{2\pi}{\lambda}u_s\Delta p_z}\exp\left\{-\frac{1}{2}\left(\frac{2\pi}{\lambda}\sigma_o\Delta p_z\right)^2\right\}}$$

b) N-element standard linear array $d = \lambda/2$

$$\underline{S}_x(\omega)_{n,m} = S_f(\omega : p_{z_n} - p_{z_m}) = S_f\left(\omega : (n-m)\frac{\lambda_0}{2}\right)$$

$$= S_x(\omega)\, e^{j\pi \frac{\lambda_0}{\lambda}(n-m)u_s} \exp\left\{-\frac{1}{2}\left(\pi \frac{\lambda_0}{\lambda}(n-m)\sigma_0\right)^2\right\}$$

$$= S_x(\omega)\, e^{+j\pi\frac{\lambda_0}{\lambda} n u_s}\, e^{-j\pi\frac{\lambda_0}{\lambda} m u_s} \exp\left\{-\frac{1}{2}\left(\pi\frac{\lambda_0}{\lambda}(n-m)\sigma_0\right)^2\right\}$$

This can be written in matrix notation as

$$\underline{S}_x(\omega) = S_x(\omega)\, \underline{v}(\omega : u_s)\, v(\omega : u_s)^{\#} \odot \underline{B}(\omega)$$

$\odot$ Hadamard Product

$$\underline{v}(\omega : u_s) = \begin{bmatrix} e^{-j\left(\frac{N-1}{2}\right)\frac{\lambda_0}{\lambda} u_s} \\ \vdots \\ e^{j\left(\frac{N-1}{2}\right)\frac{\lambda_0}{\lambda} u_s} \end{bmatrix}$$

$$B_{nm}(\omega) = \exp\left\{-\frac{1}{2}\left(\pi\frac{\lambda_0}{\lambda}(n-m)\sigma_0\right)^2\right\}$$

c) Let $N = 10$, $S_x(\omega) = 1$, $\omega = \omega_0$. see plots

As the spreading factor (σ_0) increases, more eigenvalues become significant. The steering direction only produces a shift in eigenbeams, with no change in eigenvalues.

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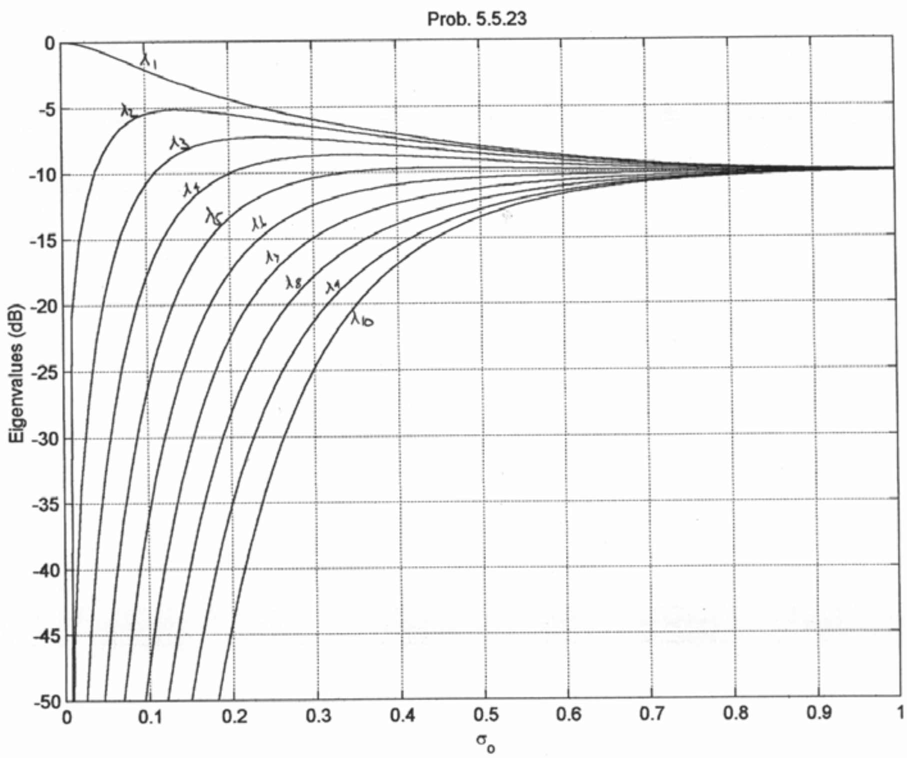
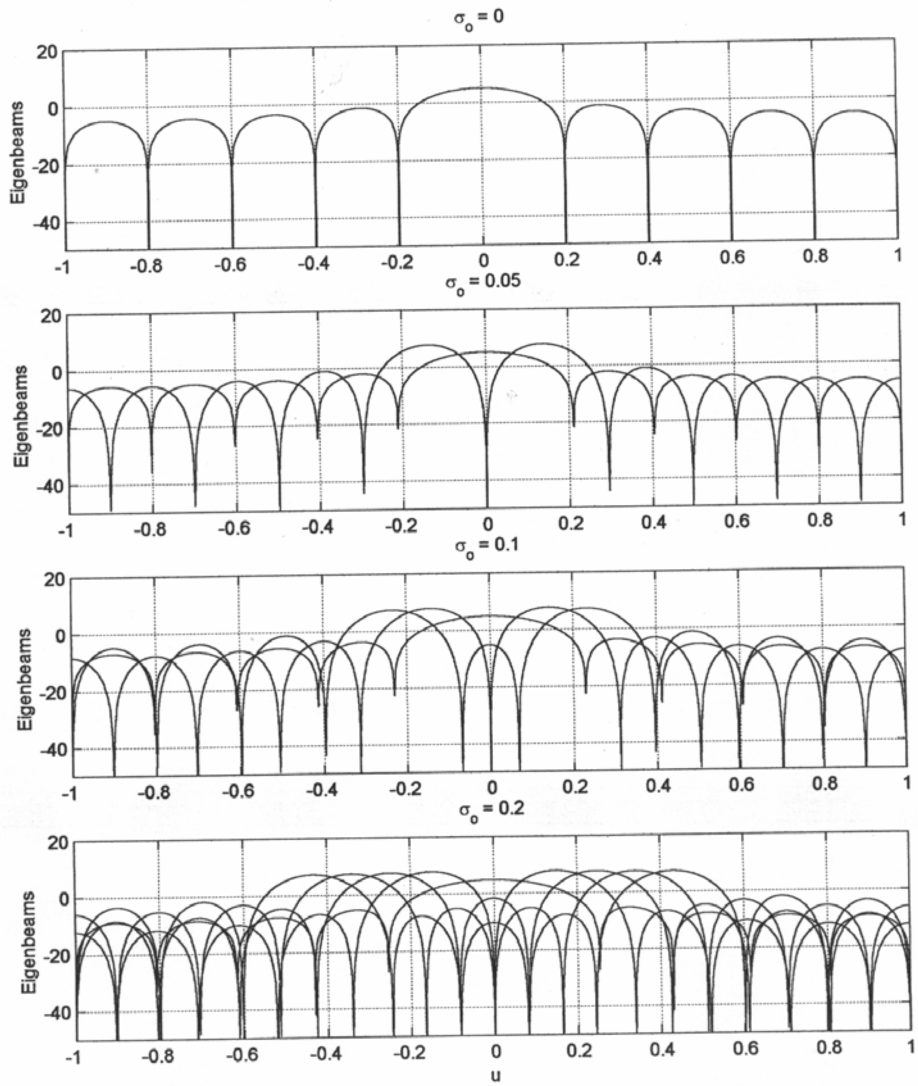
Eigenvalues vs. σ_0. Same for all u_s.



Significant eigenbeams for $u_s = 0$, various σ_o



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Significant eigenbeams for $u_s = 0.3$, various σ_o.