Math 55 - Section
Homework 9
Tuesday, November 22, 2016

1, 2, 3, 4, 5

1

(a) Show that if $a \geq 2$ and $b \geq 2$, then $R(a,b) \leq R(a-1,b) + R(a,b-1)$.

(b) Show that
$$R(a,b) \leq \binom{a+b-2}{a-1}$$
for $a \geq 1$ and $b \geq 1$.

(c) Show that $R(a,a) \leq 2^{2(a-1)}$ for $a \geq 1$.

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2 Show that $R(4, 4) \leq 18$. (You may use facts stated in the preceding problem, even if you have not proved them.)

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3

(a) For how many graphs on vertices $\{1,\ldots,n\}$ is the graph induced by vertices $\{1,\ldots,a\}$ complete (that is, isomorphic to K_a)?

(b) Show that the number of graphs on vertices $\{1,\ldots,n\}$ that contain an induced K_a or an induced $\overline{K_a}$ is at most

$$2\binom{n}{a}2^{\binom{n}{2}-\binom{a}{2}}.$$

(c) Show that if $n < 2^{(a-1)/2}$, then the number of graphs on vertices $\{1,\ldots,n\}$ that contain an induced K_a or an induced $\overline{K_a}$ is strictly less than $2^{\binom{n}{2}}$.

(d) Conclude that $R(a,a) \geq 2^{(a-1)/2}$.

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4 Suppose p is a prime, and that r is relatively prime to p. Say that r is a *residue* mod p if there exists x such that $x^2 \equiv r \pmod{p}$, and say that r is a *non-residue* otherwise.

(a) Show that the product of two residues is a residue.

(b) Show that the product of a residue and a non-residue is a non-residue.

(c) Show that the product of two non-residues is a residue.

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5 The goal of this problem is to show that $R(4,4) > 17$ (so by the result of Problem 2 we have $R(4,4) = 18$). (You may use facts stated in the preceding problem, even if you have not proved them.) Let $G = (V, E)$ be the graph with $V = \{0, 1, \ldots, 16\}$ and edges $\{v, w\}$ whenever $v \neq w$ and $v - w$ is a residue mod 17. We'll show that this graph contains no K_4 or $\overline{K_4}$.

(a) Show that this actually defines a graph; that is, that adjacency is indeed a symmetric relation.

(b) Show that the function $f(v) = v + c$ mod 17 is an automorphism of G. Thus if G contains a K_4 or $\overline{K_4}$, it contains one that includes the vertex 0.

(c) Show that if a is a residue, then $f(v) = va^{-1}$ mod 17 is an automorphism of G. Thus if G contains a K_4 that includes vertices 0 and a, it contains one that includes vertices 0 and 1.

(d) Show that if a is a non-residue, then $f(v) = va^{-1}$ mod 17 is an isomorphism from G to its complement. Thus if G contains a $\overline{K_4}$ that includes vertices 0 and a, it contains a K_4 that includes vertices 0 and 1.

(e) Complete the proof by showing that contains no K_4 that includes the vertices 0 and 1. (What would the other two vertices have to be?)

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