Math 55 - Section
Homework 3
Thursday, September 22, 2016

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

1 Prove the identity: For $m \geq 0$ and $n \geq 1$,

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$$

by induction on m. (Note: It can also be proved by induction on n, but induction on m is simpler.)

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2 Combinatorially prove for $m \geq 0, n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k}{m} = \binom{n}{m} 2^{n-m}.$$

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3 Prove that for $n \geq 0$,

$$1 + \binom{n}{1} 2 + \binom{n}{2} 4 + \cdots + \binom{n}{n-1} 2^{n-1} + \binom{n}{n} 2^n = 3^n$$

(a) combinatorially, and

(b) by binomial theorem.

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4 For $n \geq 1$, conjecture and prove a formula for the n odd-indexed Fibonacci numbers $f_1 + f_3 + f_5 + \cdots + f_{2n-1}$

(a) by induction, and

(b) combinatorially (using tilings)

(Note: $f_1 = 1$, $f_3 = 3$, $f_5 = 8$, etc.)

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5 Use the binomial theorem with Binet's formula to prove that for $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}$$

Optional BONUS for problem 5 (turn in this solution to me, and do not work on this problem with others): Prove problem 5 combinatorially.

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6 Use strong induction to prove that every integer n greater than or equal to 2 is the product of prime numbers. (A prime is a number p greater than or equal to 2 whose only positive divisors are p and 1.)

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7 Prove for $n \geq 2$,

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = n2^{n-1} + n(n-1)2^{n-2}$$

(a) combinatorially and

(b) using the binomial theorem.

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8

(a) How many n-digit positive numbers have their digits in strictly increasing order?

(b) in Non-decreasing order?

(Clarification: an n-digit positive number cannot have a leading digit of zero. In base 4, there are 10 non-decreasing 3-digit numbers: 111, 112, 113, 122, 123, 133, 222, 223, 233, and 333.)

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9

(a) How many ways are there to pick a collection of 10 balls from a large pile of red balls, blue balls and purple balls, (more than 10 of each color, and balls of the same color are indistinguishable).

(b) same problem but you must pick at least 5 red balls.

(c) same, but instead must pick at most 5 red balls.

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10 Combinatorially prove that

$$\sum_{k=0}^{m} \left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n+1}{m}\!\!\right).$$

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