Math 55 - Section
Homework 4
Thursday, October 6, 2016

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13

1 Find the number of ways of arranging the letters A,E,M,O,U,Y in a sequence in such a way that the neither or the words ME or YOU occurs. (For example, the word YUMEAO would not be allowed since contains the word ME, but YUMOEA would be allowed.)

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2 How many numbers between 1 and 250 are not divisible by 2, 3 or 5?

3

a) Prove that the number of ways to assign m distinct students to n distinct classrooms such that no classroom is empty is $\sum_{k=0}^{n} \binom{n}{k} (n-k)^m (-1)^k$.

b) Combinatorially prove that if $n > m$, then $\sum_{k=0}^{n} \binom{n}{k} (n-k)^m (-1)^k = 0$.

c) Without doing any algebra, simplify $\sum_{k=0}^{n} \binom{n}{k} (n-k)^m (-1)^k$ when $m = n$.

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4 Delegations A, B, and C have respectively, 3, 3, and 4 distinct members. How many ways can the delegates be seated in a row such that:

a) All delegates from A are together.

b) All delegates are together from at least one delegation.

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5 Prove the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^{n-k} = 9^n$$

a) by binomial theorem, and

b) combinatorially with inclusion-exclusion.

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6

a) Using the polynomial method from class, find a closed form for a_n, defined recursively as follows:

$$a_0 = 2, a_1 = 11, \text{ and for } n \geq 2,$$
$$a_n = 3a_{n-1} + 10a_{n-2}.$$

b) Confirm your solution by an inductive proof.

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7

a) Using the polynomial method from class, find a closed form for a_n, defined recursively as follows:

$$a_0 = 2, a_1 = 3, \text{ and for } n \geq 2,$$
$$a_n = 6a_{n-1} - 9a_{n-2}.$$

b) Confirm your solution by an inductive proof.

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8 Find the recurrence which produces the solution

$$a_n = 3 \cdot 2^n + n2^{n+1} - 2 \cdot 5^n,$$

for all $n \geq 0$.

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9 The polynomial method works even when the roots are imaginary or complex! Solve the recurrence

$$\begin{aligned} &a_0 = 1, a_1 = 8, a_2 = 4, \text{ and for } n \geq 3, \\ &a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3}. \end{aligned}$$

(Hint: to save you from some tedious algebra, the *coefficient* of the real root is 1, and the coefficients of the other roots are imaginary.) Simplify your formula so that it doesn't use complex numbers at all. (You may have to break the formula into cases, depending on the parity of n.)

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10 Find and solve a recurrence relation for the number of ways to tile a board of length n with squares and dominoes where squares have two colors (black and white) but dominoes have just one color (white).

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11 The *Lucas numbers* are close companions with the Fibonacci numbers. They are defined by $L_0 = 2$, $L_1 = 1$ and the recurrence $L_n = L_{n-1} + L_{n-2}$.

a) Find their closed form;

b) Using their closed form and Binet's formula, prove that $F_{2n} = F_n L_n$;

c) Using their closed form and Binet's formula, prove that $2F_{k+n} = F_k L_n + F_n L_k$.

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12 Consider a sequence of numbers $b_0, b_1, b_2, \ldots$ such that $b_0 = 0, b_1 = 1$ and $b_2, b_3, \ldots$ are defined by the recurrence

$$b_{k+1} = 3b_k - b_{k-1}.$$

Use the polynomial method to find a closed form the value of b_n. Also, express you answer in terms of Fibonacci numbers and prove it correct.

$\blacksquare$

13 In this problem, we explore the Fibonacci numbers with negative indices. Given that $F_0 = 0$ and $F_1 = 1$, we can generate $F_{-1} = 1, F_{-2} = -1, F_{-3} = 2$, and so on. For $n \geq 0$, define $a_n = F_{-n}$.

a) List $a_0, a_1, \ldots, a_{10}$

b) Conjecture and prove (by induction) a formula for a_n in terms of F. Hint: Create a recurrence satisfied by a_n.

c) Use the recurrence from b) and use the polynomial method to find a closed form for a_n.

d) Using c), show that Binet's formula for F_n is valid, even when n is negative.

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