Math 60
HW 12
Thursday, June 2, 2016

7.2.{6, 14, 22, 28}, 7.3.{4, 6}, 7.4.{T/F pg.522, 6, 10, 18}

Colley 7.2.6 Find $\iint_S (x^2 + y^2)\, dS$, where S is the lateral surface of the cylinder of radius a and height h whose axis is the z-axis.

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Colley 7.2.14 In Exercises 10-18, let S denote the closed cylinder with bottom given by $z = 0$, top given by $z = 4$, and lateral surface given by the equation $x^2 + y^2 = 9$. Orient S with outward normals. Determine the indicated scalar and vector surface integrals.

$$\iint_S (x\,\mathbf{i} + y\,\mathbf{j}) \cdot d\mathbf{S}$$

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Colley 7.2.22 In Exercises 19-22, find the flux of the given vector field $\mathbf{F}$ across the upper hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$. Orient the hemisphere with an upward-pointing normal.

$$\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$$

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Colley 7.2.28 The glass dome of a futuristic greenhouse is shaped like the surface $z = 8 - 2x^2 - 2y^2$. The greenhouse has a flat dirt floor at $z = 0$. Suppose that the temperature T, at points in and around the greenhouse, varies as

$$T(x, y, z) = x^2 + y^2 + 3(z - 2)^2.$$

Then the temperature gives rise to a **heat flux density field H** given by $\mathbf{H} = -k\nabla T$. (Here k is a positive constant that depends on the insulating properties of the particular medium.) Find the total heat flux outward across the dome and the surface of the ground if $k = 1$ on the glass and $k = 3$ on the ground.

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Colley 7.3.4 In Exercises 1-4, verify Stokes's theorem for the given surface and vector field.

$$S \text{ is defined by } x^2 + y^2 + z^2 = 4,\ z \leq 0, \text{ oriented by downward normal;}$$
$$\mathbf{F} = (2y - z)\,\mathbf{i} + (x + y^2 - z)\,\mathbf{j} + (4y - 3x)\,\mathbf{k}$$

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Colley 7.3.6 In Exercises 6-9, verify Gauss's theorem for the given three-dimensional region D and vector field $\mathbf{F}$.

$$\begin{aligned} \mathbf{F} &= x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}, \\ D &= \left\{ (x, y, z) \mid 0 \le z \le 9 - x^2 - y^2 \right\} \end{aligned}$$

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Colley True/False page 522

1. The function $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\mathbf{X}(s,t) = (2s+3t+1, 4s-t, s+2t-7)$ parametrizes the plane $9x - y - 14z = 107$.

2. The function $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\mathbf{X}(s,t) = (s^2+3t-1, s^2+3, -2s^2+t)$ parametrizes the plane $x - 7y - 3z + 22 = 0$.

3. The function $\mathbf{X} : (-\infty,\infty) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3$ given by $\mathbf{X}(s,t) = (s^3 + 3\tan t - 1, s^3 + 3, -2s^3 + \tan t)$ parametrizes the plane $x - 7y - 3z + 22 = 0$.

4. The surface $\mathbf{X}(s,t) = (s^2 t, st^2, st)$ is smooth.

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Colley 7.4.6 Exercises 6-10 concern these notions of temperature, heat, and heat flux density.

Use Gauss's theorem to derive the **heat equation**,

$$\sigma\rho\frac{\partial T}{\partial t} = k\nabla^2 T.$$

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Colley 7.4.10 Exercises 6-10 concern these notions of temperature, heat, and heat flux density.

Consider the three-dimensional heat equation

$$\nabla^2 u = \frac{\partial u}{\partial t} \tag{30}$$

for functions $u(x, y, z, t)$. (Here $\nabla^2 u$ denotes the Laplacian $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2$.) In this exercise, show that any solution $T(x, y, z, t)$ to the heat equation is unique in the following sense: Let D be a bounded solid region in $\mathbb{R}^3$ and suppose that the functions $\alpha(x, y, z)$ and $\phi(x, y, z, t)$ are given. Then there exists a unique solution $T(x, y, z, t)$ to equation (30) that satisfies the conditions

$$\begin{aligned} T(x, y, z, 0) &= \alpha(x, y, z) \quad \text{for } (x, y, z) \in D, \\ &\text{and} \\ T(x, y, z, t) &= \phi(x, y, z, t) \quad \text{for } (x, y, z) \in \partial D \text{ and } t \geq 0. \end{aligned} \tag{31}$$

To establish uniqueness, let T_1 and T_2 be two solutions to equation (30) satisfying the conditions in (31) and set $w = T_1 - T_2$.

(a) Show that w must also satisfy equation (30), plus the conditions that

$$\begin{aligned} w(x, y, z, 0) &= 0 \quad \text{for all } (x, y, z) \in D, \\ &\text{and} \\ w(x, y, z, t) &= 0 \quad \text{for all } (x, y, z) \in \partial D \text{ and } t \geq 0. \end{aligned}$$

(b) For $t \geq 0$, define the "energy function"

$$E(t) = \frac{1}{2} \iiint_D [w(x, y, z, t)]^2 \, dV.$$

Use Green's first formula in Theorem 4.1 to show that $E'(t) \leq 0$ (i.e., that E does not increase with time).

(c) Show that $E(t) = 0$ for all $t \geq 0$. (Hint: Show that $E(0) = 0$ and use part (b).)

(d) Show that $w(x, y, z, t) = 0$ for all $t \geq 0$ and $(x, y, z) \in D$, and thereby conclude the uniqueness of solutions to equation (30) that satisfy the conditions in (31).

Hint: think about what the field is doing.

Colley 7.4.18 Suppose that $\mathbf{J} = \sigma\mathbf{E}$. (This is a version of Ohm's law that obtains in some electric conductors–here σ is a positive constant known as the **conductivity**.) If $\rho = 0$, show that $\mathbf{E}$ and $\mathbf{B}$ satisfy the so-called **telegrapher's equation**,

$$\nabla^2\mathbf{F} = \mu_0\sigma\frac{\partial\mathbf{F}}{\partial t} + \mu_0\epsilon_0\frac{\partial^2\mathbf{F}}{\partial t^2}.$$

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