Math 65
HW 4
Friday, May 20, 2016

Additional Problem #1, 6.5.{33, 35}, 6.6.{4, 12, 22}, 6.4.30, 6.6.32

Additional Problem #1 Consider the vector space C^n, the set of all real-valued functions $f(x)$ for which $f', f'', ..., f^{(n)}$ exist and are continuous, over $\mathbb{R}$. Show the differential operator

$$\mathcal{L}[y(x)] = a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x)$$

is a linear transformation, where $a_0(x), ..., a_n(x)$ are also C^n functions.

■

6.5.33 Let $S : V \to W$ and $T : U \to V$ be linear transformations.

(a) Prove that if S and T are both one-to-one, so is $S \circ T$.

(b) Prove that if S and T are both onto, so it $S \circ T$.

$\blacksquare$

6.5.35 Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces.

(a) Prove that if $\dim V < \dim W$, then T cannot be onto.

(b) Prove that if $\dim V > \dim W$, then T cannot be one-to-one.

$\blacksquare$

6.6.4 In Exercises 1-12, find the matrix $[T]_{C \leftarrow B}$ of the linear transformation $T : V \to W$ with respect to the bases B and C of V and W, respectively. Verify Theorem 6.26 for the vector $\mathbf{v}$ by computing $T(\mathbf{v})$ directly and using the theorem.

$$\begin{aligned}
&T : \mathscr{P}_2 \to \mathscr{P}_2 \text{ defined by } T(p(x)) = p(x+2),\\
&\mathcal{B} = \{1, x+2, (x+2)^2\},\\
&\mathcal{C} = \{1, x, x^2\},\\
&\mathbf{v} = p(x) = a + bx + cx^2
\end{aligned}$$

■

6.6.12 In Exercises 1-12, find the matrix $[T]_{C \leftarrow B}$ of the linear transformation $T : V \to W$ with respect to the bases B and C of V and W, respectively. Verify Theorem 6.26 for the vector $\mathbf{v}$ by computing $T(\mathbf{v})$ directly and using the theorem.

$$T : M_{22} \to M_{22} \text{ defined by } T(A) = A - A^T,$$
$$\mathcal{B} = \mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\},$$
$$\mathbf{v} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

■

6.6.22 In Exercises 19-26, determine whether the linear transformation T is invertible by considering its matrix with respect to the standard bases. If T is invertible, use Theorem 6.28 and the method of Example 6.82 to find T^{-1}.

$$T : \mathscr{P}_2 \to \mathscr{P}_2 \text{ defined by } T(p(x)) = p'(x)$$

■

6.4.30 In Exercises 29 and 30, verify that S and T are inverses.

$$S : \mathscr{P}_1 \to \mathscr{P}_1 \text{ defined by } S(a+bx) = (-4a+b) + 2ax \quad \text{and}$$
$$T : \mathscr{P}_1 \to \mathscr{P}_1 \text{ defined by } T(a+bx) = b/2 + (a+2b)x$$

In addition, calculate $[S]_B$ and $[T]_B$ for some basis $\mathcal{B}$ (of your choice) for the vector space in question. Then show that the matrices are the inverses of each other.

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6.6.32 In Exercises 31-36, a linear transformation $T : V \to V$ is given. If possible, find a basis C for V such that the matrix $[T]_C$ of T with respect to C is diagonal.

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ defined by } T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ a + b \end{bmatrix}$$

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