Recap

* $$\mathbb{E}\, g(X) = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx.$$

* $$\mathbb{E}[\alpha X + b] = \alpha\, \mathbb{E} X + b.$$

* $$\mathbb{E}\, g(X, Y) = \iint_{-\infty}^{\infty} g(x, y)\, f_{X,Y}(x, y)\, dx\, dy.$$

* $$\mathbb{E}[\alpha X + \beta Y + b] = \alpha\, \mathbb{E} X + \beta\, \mathbb{E} Y + b.$$

$$(X, Y, Z) \sim f_{X,Y,Z}(x, y, z).$$

* $$\mathbb{P}\{(X,Y,Z) \in \mathcal{R}\} = \iiint\limits_{(x,y,z)\in\mathcal{R}} f_{X,Y,Z}(x, y, z)\, dx\, dy\, dz$$

* $$\mathbb{E}\, g(X, Y, Z) = \int_{-\infty}^{\infty}\!\!\int\!\!\int g(x,y,z)\, f_{X,Y,Z}(x,y,z)\, dx\, d$$

* $$\mathbb{E}\left[\alpha X + \beta Y + \gamma Z + b\right] = \alpha\,\mathbb{E}X + \beta\,\mathbb{E}Y + \gamma$$
$$+ b.$$

$$* \quad X_1, X_2, \ldots X_n \sim f_{X_1 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

$$* \quad \mathbb{P}\{(X_1, X_2, \ldots X_n) \in B\} = \int \cdots \int_{(x_1 \ldots x_n) \in B} f_{X_1, \ldots, X_n}(x_1, \ldots x_n)\, dx_1 \ldots dx_n$$

$$\mathbb{E}\, g(X_1 \ldots X_n) = \int\int_{-\infty}^{\infty} \cdots \int g(x_1, x_2, \ldots, x_n)\, f_{X_1 \ldots X_n}(x_1 \ldots x_n)\, dx_1 \ldots dx_n$$

$$\mathbb{E}\left[\sum_{i=1}^{n} \alpha_i X_i + b\right] = \sum_{i=1}^{n} \alpha_i \mathbb{E} X_i + b. \text{ (Linearity)}$$

Conditional density.

* $X \sim f_X(x)$
* B is some event.

- $\mathbb{P}\{X \in A \mid B\} = \dfrac{\mathbb{P}\{X \in A, B\}}{\mathbb{P}\{B\}}.$

$=$

$$* \quad \mathbb{P}\{X \in A \mid X \in B\} = \frac{\mathbb{P}\{X \in A \cap X \in B\}}{\mathbb{P}\{X \in B\}}.$$

$$= \frac{\int_{x \in (A \cap B)} f_X(x)\, dx}{\mathbb{P}\{X \in B\}}.$$

$$= \int_{x \in A \cap B} \left[\frac{f_X(x)}{\mathbb{P}\{X \in B\}} \right] dx$$

$$= \cancel{\begin{cases} \int_{x \in A} \frac{f_X(x)}{\mathbb{P}\{X \in B\}}\, dx & \text{if } x \in B. \\ 0 & \text{o.w} \end{cases}}$$

$$= \int_{x \in A} \left[\frac{f_X(x)}{\mathbb{P}\{X \in B\}} \underbrace{\mathbb{1}\{x \in B\}}_{= \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{o.w} \end{cases}} \right] dx .$$

Analogy

$$\mathbb{P}\{X \in A\} = \int_{x \in A} f_X(x) \, dx$$

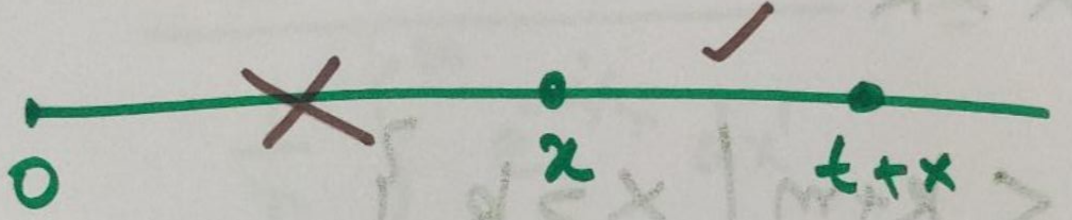
Define

$$f_{X/B}(x/B) = \frac{f_X(x)}{\mathbb{P}\{X \in B\}} \, 1\{x \in B\}.$$

$$= \begin{cases} \frac{f_X(x)}{\mathbb{P}\{X \in B\}} & \text{if } x \in B \\ 0 & \text{o.w.} \end{cases}$$

Example :-

$X \sim exp(\lambda)$



$$\mathbb{P}\{X \overset{>}{\cancel{<}} t+x \mid X \geq x\} = \frac{\mathbb{P}\{x \leq X \leq t+x\}}{\mathbb{P}\{X \geq x\}}$$

1	2	- - - .	k	k+1	

1	2	. . .	k	k+1	. . .		

$k+m$

$X > k$

$$\mathbb{P}\{X < k+m \mid X > k\}$$

$$= \frac{\mathbb{P}\{X' = k+1 \cup X' = k+2 \cup \cdots \cup X' = k+m\}}{\mathbb{P}\{X' = k+1 \cup X' = k+2 \cup \cdots \sim\}}$$

$=$

$$= \frac{\frac{1}{\tau}\int_x^{t+x} e^{-x'/\tau}\, dx'}{\frac{1}{\tau}\int_x^{\infty} e^{-x'/\tau}\, dx'} = \frac{-e^{-x'/\tau}\Big|_x^{t+x}}{-e^{-x'/\tau}\Big|_x^{\infty}}$$

$$\{X < t + x \mid X \geq x\} = \frac{e^{-x/\tau} - e^{-(x+t)/\tau}}{e^{-x/\tau}} = 1 - e^{-t/\tau}.$$

$$\mathbb{P}\{X \geq t + x \mid X \geq x\} = e^{-t/\tau}$$

$$f_{X,Y|B}(x,y|B) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{\mathbb{P}\{(X,Y) \in B\}} & \text{if } x \in B \\ 0 & \text{o.w.} \end{cases}$$

Recap:-

* $f_{X,Y}(x,y)$

$$f_{X|Y}(x/y) := \frac{f_{X,Y}(x,y)}{f_Y(y)} \implies f_{X|Y}(x/y)\, f_Y(y) = f_{X,Y}(x,y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy.$$

$$= \int_{-\infty}^{\infty} f_{X|Y}(x|y)\, f_Y(y)\, dy.$$

$$\mathbb{P}\{X \in A \mid Y = y\} = \int_A f_{X|Y}(x|y)\, dx$$

$$X, Y, Z \sim f_{X,Y,Z}(x, y, z)$$

$$f_{X,Y|Z}(x, y|z) := \frac{f_{X,Y,Z}(x, y, z)}{f_Z(z)}$$

$$\mathbb{E}\, g(X,Y) = \iint_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy.$$

$$\mathbb{E}\left[g(X,Y) \mid Y = y \right] = \int g(x,y)\, f_{X|Y}(x|y)\, dx$$

$$\mathbb{E}\, g(x) = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx$$

- $$\mathbb{E}\left[g(x) \mid A\right] = \int_{-\infty}^{\infty} g(x)\, f_{x/A}(x/A)\, dx$$

- $$\mathbb{E}\left[g(x) \mid Y = y\right] = \int_{-\infty}^{\infty} g(x)\, f_{x/Y}(x/y)\, dx.$$

<u>Defn:</u> (i) Conditional Exp of X given A is given by

$$\mathbb{E}[X/A] = \int_{-\infty}^{\infty} x \, f_{X/A}(x|A) \, dx.$$

(ii) Conditional Exp of X given $Y=y$ is defined as

$$\mathbb{E}[X|Y=y] := \int_{-\infty}^{\infty} x \, f_{X/Y}(x/y) \, dx$$

__Defn:-__ We say that $X_1, X_2, \dots, X_n$ are __ind__ r.v if $I \subseteq \{1, 2, \dots, n\}$, we have

$$f_{X_i : i \in I}(x_i : i \in I) = \prod_{i \in I} f_{X_i}(x_i).$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = f_X(x)$$

$$Y_1 = x + W_1, \quad Y_2 = x + W_2$$

$$f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = f_{X_1|X_2\dots X_n}(x_1|x_2\dots x_n)\, f_{X_2|X_3\dots X_n}(x_2|x_3\dots x_n)$$

$$\times f_{X_3|X_4\dots X_n}(x_3|x_4\dots x_n) \cdots\cdots f_{X_{n-1}|X_n}(x_{n-1}|x_n)$$

$$\times f_{X_n}(x_n)$$

Defn:- We say that X & Y are $\underline{\text{ind}}$ if

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y). \quad \forall\, x, y.$$

$$f_{X|Y,Z}(x|y,z) := \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)}$$

$$f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z)\,\underbrace{f_{Y,Z}(y,z)}_{f_{Y|Z}(y|z)\,f_Z(z)}.$$

$$f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z)\,f_{Y|Z}(y|z)\,f_Z(z).$$