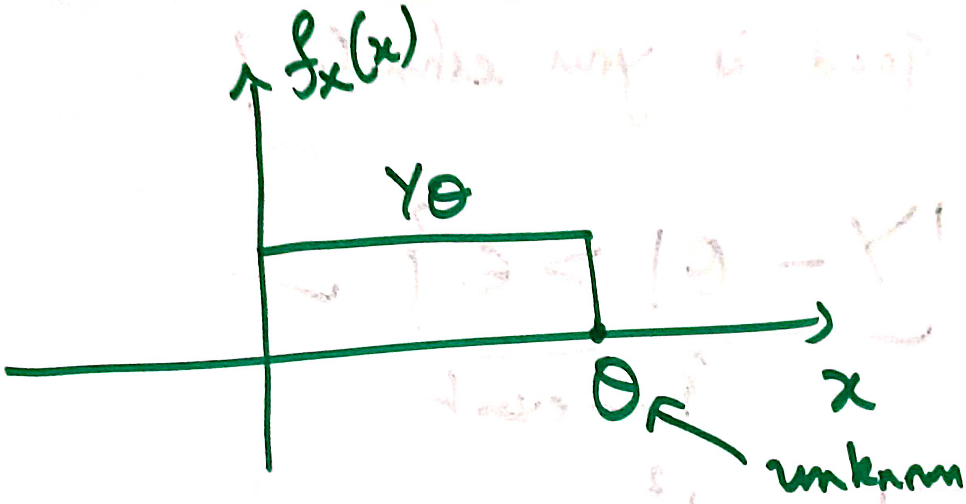
Problem:-



$$X_1, X_2, \dots, X_n \overset{iid}{\sim} f_X(x)$$

Estimator Y $\cancel{\hat{\Theta}_n} := \max\{X_1, \dots, X_n\}$

How good is your estimate ?

* $\mathbb{P}\left\{ \underbrace{|Y - \theta| > \epsilon}_{\text{Bad event}} \right\}$ ✓

* $\mathbb{E}|Y - \theta|^2$ ✓

* $\mathbb{E}|Y - \theta|$ ✓

$$\mathbb{P}\{\max\{x_1 \dots x_n\} > \theta + \epsilon\} = 0.$$

$$- \mathbb{P}\{\max\{x_1 \dots x_n\} < \theta - \epsilon\}$$

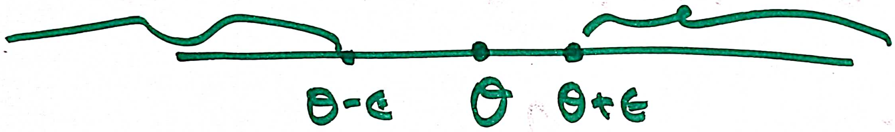
$$= \mathbb{P}\{x_1 < \theta - \epsilon \cap x_2 < \theta - \epsilon \cap \dots \cap x_n < \theta - \epsilon\}.$$

$$= \prod_{i=1}^{n} \mathbb{P}\{x_i < \theta - \epsilon\}.$$

$$= \left(\mathbb{P}\{x_1 < \theta - \epsilon\}\right)^n.$$

$$\mathbb{P}\{|\max\{X_1 \dots X_n\} - \theta| > \epsilon\}$$

$$= \mathbb{P}\{\max\{X_1 \dots X_n\} > \epsilon + \theta \cup \max\{X_1 \dots X_n\} < -\epsilon + \theta\}$$

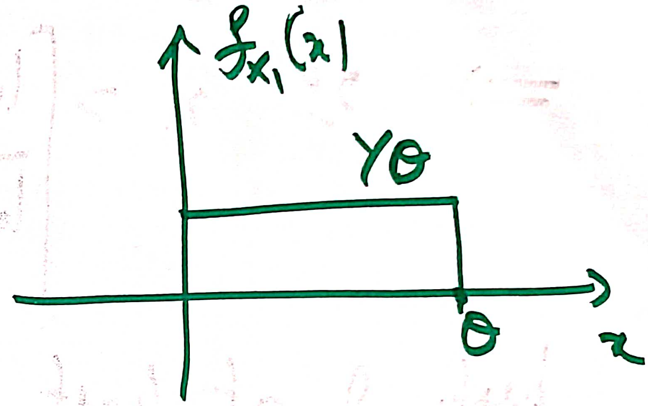


$$= \mathbb{P}\{\cancel{\max}\, Y > \epsilon + \theta\} + \mathbb{P}\{Y < \theta - \epsilon\}.$$

$$\mathbb{P}\{X_1 < \theta - \epsilon\}$$

$$= \int_0^{\theta-\epsilon} \frac{1}{\theta}\, dx = \frac{\theta-\epsilon}{\theta}$$

$$= \left(1 - \frac{\epsilon}{\theta}\right)$$



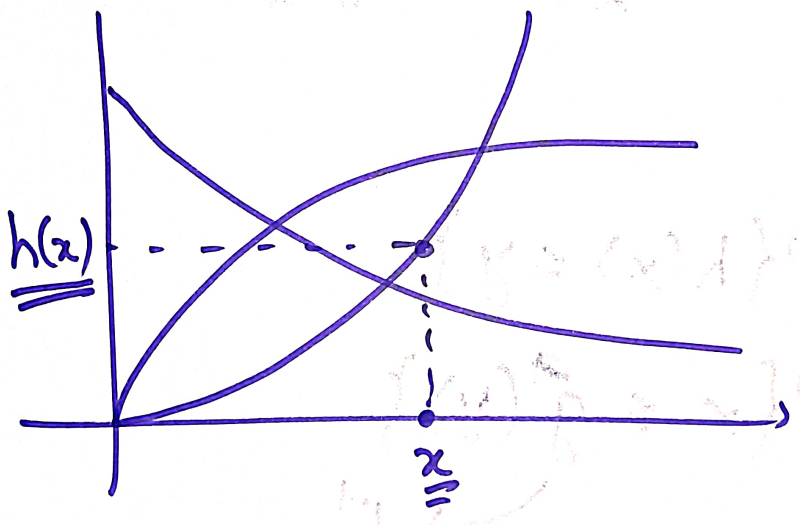
$$\mathbb{P}\{|\max\{X_1 \dots X_n\} - \theta| > \epsilon\} = \left(1 - \frac{\epsilon}{\theta}\right)^n$$

$$< \delta$$

$\Longrightarrow$ If $n > \left\lceil \dfrac{\log \frac{1}{\delta}}{\log \frac{1}{1-\epsilon/\theta}} \right\rceil$, then with a prob. of at least $1-\delta$, we have

$$\left| \max\{X_1, \dots, X_n\} - \theta \right| < \epsilon .$$

<u>Fact</u> :- If $h(x)$ is monotonically increasing (decreasing), then $\exists$ an inverse fn. $g(x)$.



$$X \sim f_X(x)$$

$$Y = h(X) \sim \ ? \qquad f_X(g(y))\, g'(y)$$

monotonic

$$F_Y(y) = \mathbb{P}\{h(X) \le y\}$$

$$= \mathbb{P}\{X \le \underbrace{g}_{h^{-1}.}(y)\}$$

$$= F_X(g(y)).$$

Suppose $X \sim f_X(x)$ & $Y \sim f_Y(y)$ ind.

$$Z = X + Y.$$

$$F_Z(z) = \mathbb{P}\{X + Y \le z\}$$

$$= \int_{-\infty}^{\infty} \mathbb{P}\{X + Y \le z \mid Y = y\} f_Y(y)\, dy.$$

$$= \int_{-\infty}^{\infty} \mathbb{P}\{X \le z - y \mid Y = y\} f_Y(y)\, dy.$$

* $X \sim U[0,1]$

* $Y = h(X)$.

  - $F_Y(y) = \mathbb{P}\{h(x) \leq y\}$

$$= \mathbb{P}\{x \leq g(y)\}.$$

$$= \begin{cases} 0 & \text{if } g(y) > 1 \text{ or } g(y) < 0 \\ g(y) & \text{o.w.} \end{cases}$$

$\Rightarrow \quad f_Y(y) = g'(y)$ under certain cond.

→ make it look like function

* $X \sim U[0,1]$

* $Y = h(X)$.

  - $F_Y(y) = \mathbb{P}\{h(X) \le y\}$

    $= \mathbb{P}\{X \le g(y)\}$.

    $= \begin{cases} 0 & \text{if } g(y) > 1 \text{ or } g(y) < 0 \\ g(y) & \text{o.w.} \end{cases}$

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make it look like function!

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$$= \int_{-\infty}^{\infty} \mathbb{P}\{X \le z - y \mid Y=y\} f_Y(y)\, dy.$$

$$= \int_{-\infty}^{\infty} \underbrace{\mathbb{P}\{X \leq z-y\}}_{F_X(z-y).} f_Y(y)\, dy.$$

$$\Rightarrow \boxed{f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy}$$

Convolution.

* $M_x(0) = 1$

* $$\frac{dM_x(s)}{ds} = \mathbb{E}\left[x e^{sx}\right].$$

$$- \left.\frac{dM_x(s)}{ds}\right|_{s=0} = \mu = \mathbb{E}x.$$

* $$\frac{d^2M_x(s)}{ds^2} = \mathbb{E}\left[x^2 e^{sx}\right]$$

$$- \left.\frac{d^2M_x(s)}{ds^2}\right|_{s=0} = \mathbb{E}x^2.$$

$$\longrightarrow \left.\frac{d^n M_x(s)}{ds^n}\right|_{s=0} = E x^n.$$

Example :- $X \sim pois(\lambda)$

$$M_x(s) = \sum_{k=0}^{\infty} e^{sk} \frac{\lambda^k}{k!} e^{-\lambda}.$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^s \lambda)^k}{k!} = e^{-\lambda}\left[e^{e^s \lambda}\right]$$

$$= e^{-\lambda(1-e^s)} = e^{\lambda(e^s \cdots)}$$

d $\quad M_x^{(1)}(s) = e^{\lambda(e^s - 1)} \lambda e^s$

$$M_x^{(1)}(s)\Big|_{s=0} = \lambda .$$

$$M_x^{(2)}(s) = e^{\lambda(e^s-1)} \lambda e^s + e^{2s} \lambda^2 e^{\lambda(e^s-1)}$$

$$M_x^{(2)}(0) = \lambda + \lambda^2 .$$

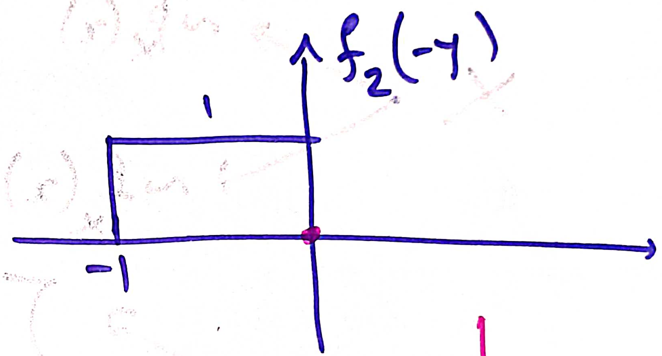
Moment Generating Function

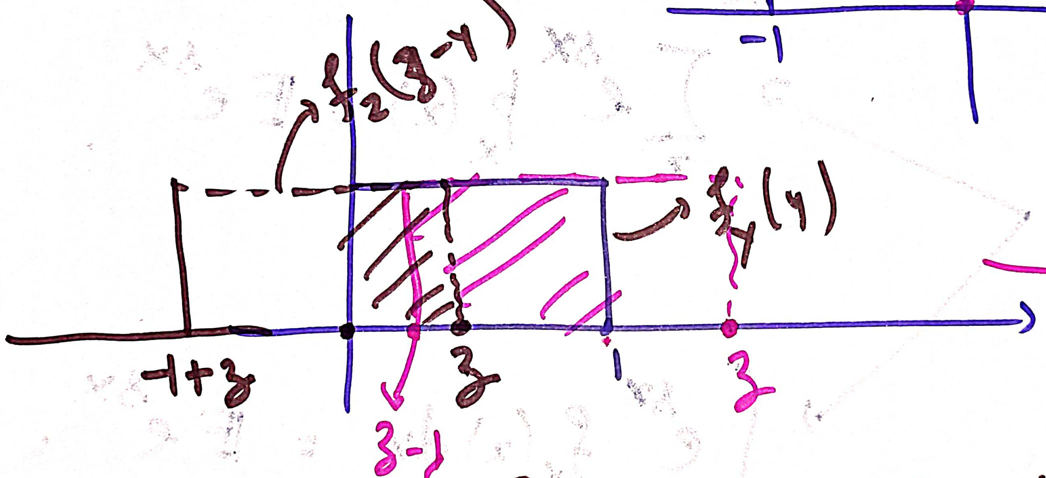
$$X \begin{cases} \to \sim p_X(x) \\ \to \sim f_X(x) \end{cases}$$

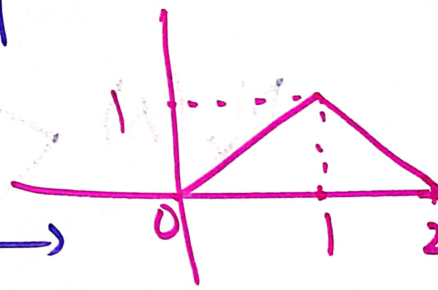
$$M_X(s) \begin{cases} \to \sum_x e^{sx} p_X(x) = \mathbb{E}\, e^{sX}. \\ \to \int e^{sx} f_X(x)\, dx = \mathbb{E}\, e^{sX}. \end{cases}$$

$X \sim U[0,1]$, $Y \sim U[0,1]$ $\quad X \perp Y$

$Z = X + Y.$

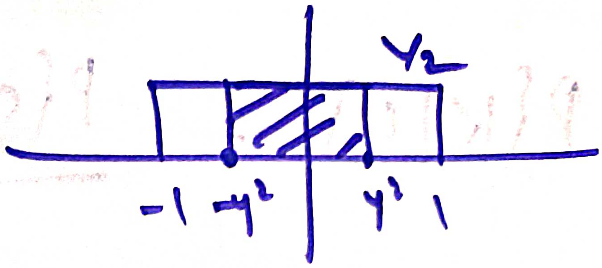






$$f_Z(z) = \begin{cases} z & 0 \le z \le 1 \\ 2 - z & z > 1 \;\&\; z < 2. \end{cases}$$

$X \sim \text{Unif}[-1, 1].$



$Y = \sqrt{|X|}.$

$$\mathbb{P}\{Y \le y\} = \mathbb{P}\{\sqrt{|X|} \le y\} \quad ; \; 0 \le y \le 1$$

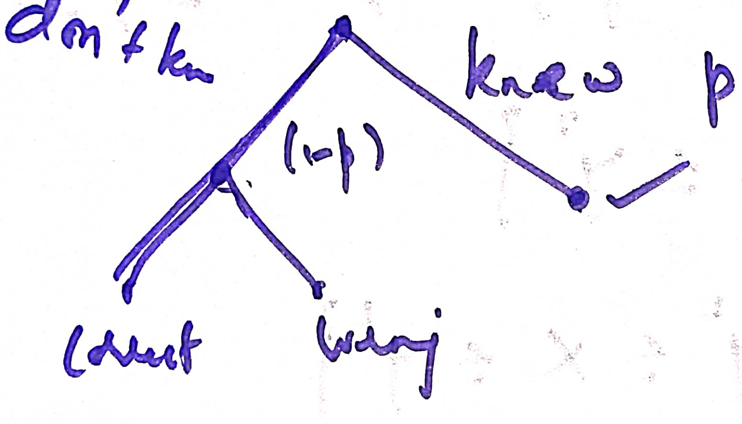
$$= \mathbb{P}\{|X| \le y^2\}$$

$$= \mathbb{P}\{-y^2 \le X \le y^2\}$$

$$= y^2$$

$f_Y(y) = 2y, \quad 0 \le y \le 1,$

$$\mathbb{P}\{K \mid C\} = \frac{\mathbb{P}\{C \mid K\}\,\mathbb{P}\{K\}}{\mathbb{P}\{C\}}.$$



$$\underline{\underline{P(C) = p + \frac{1-p}{c}}}$$

Last prob :- Let $X \geq 0$ with pmf $p_X(x)$.

Show that

$$\mathbb{E}X = \sum_x \mathbb{P}\{X > x\}.$$

$$\sum_x \mathbb{P}\{X > x\} = \mathbb{P}\{X > 0\} + \mathbb{P}\{X > 1\} + \cdots$$

$$= p_X(1) + p_X(2) + \cdots + p_X(N) + p_X(2) + p_X(3) + \cdots$$

$$\cdots + p_X(N) + p_X(3) + p_X(4) + \cdots$$

$$+ p_X(N)$$