Q1: Suppose 100 people throw a ball into a box & then proceed to randomly pick out a ball. What is the expected number of people to get their own ball back.

Soln: Let X be the number of people who get their own ball.

Let X_j represent whether person j gets their own ball. That is, $X_j = 1$ if a person gets their ball & 0 if not.

We have, $X = \sum_{j=1}^{100} X_j \Rightarrow \mathbb{E}[X] = \sum_{j=1}^{100} \mathbb{E}[X_j]$

Since person j is equally likely to get any ball, we have $P(X_j) = \frac{1}{100}$.

Thus $X_j \sim \text{Bernoulli}\left(\frac{1}{100}\right) \Rightarrow \mathbb{E}[X_j] = \frac{1}{100}$

$\therefore \mathbb{E}[X] = 1$.

2. Let X be a discrete random variable with pmf p given by:

x	-2	-1	0	1	2
$p_X(x)$	$1/15$	$2/15$	$3/15$	$4/15$	$5/15$

a) Let $Y = X^2$. Find the pmf of Y.

b) Find the value of the cdf of X at $-1/2, 3/4, 7/8, 1, 1.5, 5$.

c) " " " " " " " Y at " " " " " "

Soln: $Y=1$ when $X=1$ or $X=-1$ so
$$P(Y=1) = P(X=1) + P(X=-1)$$

Values y of Y	0	1	4
pmf $p_Y(y)$	3/15	6/15	6/15

a	-1.5	$3/4$	$7/8$	1	1.5	5
$F_X(a)$	1/15	6/15	6/15	10/15	10/15	1
$F_Y(a)$	0	3/15	3/15	9/15	9/15	1

③ Let X be a random variable that takes values from 0 to 9 with equal probability $1/10$.

(a) Find the PMF of the random variable $Y = X \bmod (3)$.

(b) " " " " " " " $Y = 5 \bmod (X+1)$

Soln:
$$p_Y(y) = \sum_{\{x \mid x \bmod (3) = y\}} p_X(x)$$

$$p_Y(0) = p_X(0) + p_X(3) + p_X(6) + p_X(9) = \frac{4}{10}$$

$$p_Y(1) = p_X(1) + p_X(4) + p_X(7) = 3/10$$

$$p_Y(2) = p_X(2) + p_X(5) + p_X(8) = 3/10$$

$$p_Y(y) = 0 \quad \text{if } y \notin \{0,1,2\}$$

(b) $\|^{ly}$, using the formula $p_Y(y) = \sum_{\{x \mid 5 \bmod (x+1) = y\}} p_X(x)$

$$p_Y(y) = \begin{cases} 2/10, & \text{if } y=0, \\ 2/10, & \text{if } y=1, \\ 1/10, & \text{if } y=2, \\ 5/10, & \text{if } y=5, \\ 0, & \text{o.w.} \end{cases}$$

④ If X is a random variable that is uniformly distributed between -1 & 1. Find the pdf of $\sqrt{|X|}$ & the pdf of $-\ln|X|$.

Soln: Let $Y = \sqrt{|X|}$, we have, for $0 \le y \le 1$

$$F_Y(y) = P(Y \le y) = P(\sqrt{|X|} \le y) = P(-y^2 \le X \le y^2) = y^2$$

& therefore by differentiation,

$$f_Y(y) = 2y, \quad \text{for } 0 \le y \le 1.$$

Let $Y = -\ln|X|$. We have, for $y \geq 0$,

$$F_Y(y) = P(Y \le y) = P(\ln|X| \geq -y) = P(X \geq e^{-y}) + P(X \le -e^{-y})$$

$$= 1 - e^{-y},$$

& ∴ by differentiation,

$$f_Y(y) = e^{-y} \quad \text{for } y \geq 0.$$

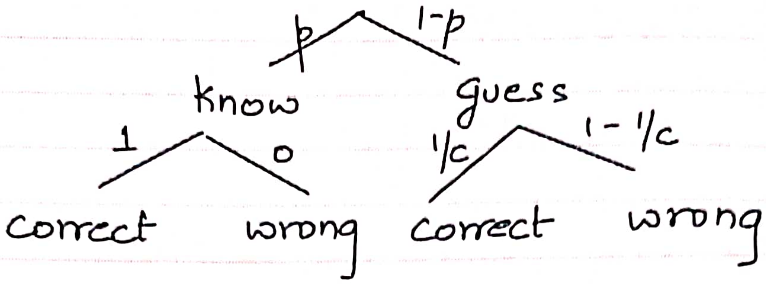
(5) You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly & also approximately by using Poisson PMF.

Soln: The no of guests that have the same birthday as you is binomial with $p = 1/365$ & $n = 499$. Thus the probability that exactly one other guest has the same birthday is,

$$^{499}C_1 \left(\frac{1}{365}\right)\left(\frac{364}{365}\right)^{498} \approx 0.3486.$$

Let $\lambda = np = \frac{499}{365} \approx 1.367$. The poisson approximation is $e^{-\lambda}\lambda = e^{-1.367} \times 1.367 \approx 0.3483$, which closely agrees with the correct probability based on the binomial.

(6) Suppose you are taking a multiple choice test with c choices for each question. In answering a question on this test, the probability that you know the answer is p. If you don't the answer, you choose one at random. What is the probability that you knew the answer to a question, given that you answered it correctly?



Let C be the event that you answer the question correctly. Let K be the event that you actually know the answer. Let G be the event that you are guessing

$P(C|K) = 1 \quad P(K) = p$

Using law of total probability

$$P(C) = P(C|K)\,p(K) + P(C/G)\,p(G)$$

$$= 1 \times p + (1-p)\frac{1}{c} = p + \frac{(1-p)}{c}$$

$$\therefore \quad P(K|C) = \frac{P(C|K)\,P(K)}{P(C)} = \frac{p}{p + (1-p)/c}$$

⑦ ⓐ Suppose that X has pdf $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ compute the cdf, $F_X(x)$.

ⓑ If $Y = X^2$, compute the pdf & cdf of Y.

Soln: ⓐ $F_X(x) = \int_0^x \lambda e^{-\lambda y}\,dy = -\frac{\lambda\left[e^{-y}\right]_0^x}{\lambda} = 1 - e^{-\lambda x}$

(6) $F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X^2 \le y) = P(X \le \sqrt{y})$

$= 1 - e^{-\lambda\sqrt{y}}$

Differentiating $F_Y(y)$ w.r.t y, we have

$$f_Y(y) = \frac{\lambda}{2} y^{-1/2} e^{-\lambda\sqrt{y}}$$

(8) Let X be a random variable that takes on any of the values -1, 0 and 1 with

$P\{X=-1\} = 0.2$, $P\{X=0\} = 0.5$, $P\{X=1\} = 0.3$

Compute $E[X^2]$

Soln: Let $Y = X^2$ $\therefore$ pmf of Y,

$P\{Y=1\} = P\{X^2=1\} = P\{X=-1\} + P\{X=1\}$

$= 0.2 + 0.3 = 0.5$

$P\{Y=0\} = P\{X=0\} = 0.5$

$\therefore E[X^2] = E[Y] = 1 \times (0.5) + 0(0.5) = 0.5$

(9) Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Soln: If we let X equal the number of heads (successes) that appear, then X is a binomial r.v. with parameters $(n=5,\ p=1/2)$

$$P\{X=0\} = {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$P\{X=1\} = {}^5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 = \frac{5}{32}$$

$$P\{X=2\} = {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = \frac{10}{32}$$

$$P\{X=3\} = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = \frac{10}{32}$$

$$P\{X=4\} = {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) = \frac{5}{32}$$

$$P\{X=5\} = {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 = \frac{1}{32}$$