Asymptotic Analysis-2

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Topics

- *Asymptotic notation using Limits*
- *Definitions of O, Ω, Θ, o, ω notations*
- *Analysis of Summations*

Asymptotic Analysis

Using Limits

- Use of basic definition for determining the asymptotic behavior is often awkward. It involves *ad hoc* approach or some kind of manipulation to prove algebraic relations.

- Calculus provides an alternative method for the analysis. It depends on evaluating the following limit.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$$

where *f(n)* is a *growth function for an algorithm* and *g(n)* is a *standard function*

- Depending upon the value c , the relation between *f(n)* and *g(n)* can be expressed in terms of asymptotic notations. In most cases it is easier to use limits, compared to basic method, to determine asymptotic behavior of growth functions.

➢ It will be seen that the Calculus notation $n\to\infty$ is *equivalent* to the algebraic condition *for all* $n \geq n_0$. Either of these conditions implies *large input*

O-Notation Using Limit

Definition

If $f(n)$ is running time of an algorithm and $g(n)$ is some standard growth function such that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c, \quad \text{where } c \text{ is a positive constant such that } 0 \le c < \infty$$

then $f(n) = O(g(n))$

- Note that *infinity* is excluded from the range of permissible values for the constant c

O-Notation

Examples

Example(1): $3n^2 + 5n + 20 = O(n^2)$

$$\lim_{n \to \infty} \frac{3n^2 + 5n + 20}{n^2}$$

$$= 3 + 5/n + 20/n^2$$

$$= 3+0+0 = 3 \quad (\text{positive constant})$$

Therefore, $3n^2 + 5n + 20 = O(n^2)$

Example(2): $10n^2 + 25n + 7 = O(n^3)$

$$\lim_{n \to \infty} \frac{10n^2 + 25n + 7}{n^3}$$

$$= 10/n + 25/n^2 + 7/n^3$$

$$= 0+0+0 = 0$$

Therefore, $10n^2 + 25n + 7 = O(n^3)$

O-Notation

Examples

Example(3): $lg\ n = O(n)$

$$\lim_{n \to \infty} \frac{lg\ n}{n} = \infty / \infty$$ *(Need to apply the L' Hopital Rule)*

In order to compute differential of *lg n* we first convert *binary logarithm* to *natural logarithm*. Converting *lg n (binary log)* to *ln(n) (natural log)* , by the using formula $lg\ n = ln\ n / ln\ 2$

$$\lim_{n \to \infty} \frac{lg\ n}{n} = \frac{(ln\ n)}{(ln\ 2)\ n}$$ *(Converting to natural log)*

$$\lim_{n \to \infty} \frac{1}{ln\ 2.\ n}$$ *(Differentiating numerator and denominator)*

$$= 0$$ *(Evaluating limits)*

Therefore, $lg\ n = O(n)$

O-Notation

Examples

Example(4): $n^2 = O(2^n)$

$$\lim_{n \to \infty} \frac{n^2}{2^n} = \infty / \infty$$ *(Need to apply the L' Hopital Rule)*

Since $\frac{d}{dn}(2^n) = \ln 2 . 2^n$, *(Calculus rule for differentiating the exponential functions)*

$$\lim_{n \to \infty} \frac{2n}{\ln 2\ 2^n}$$ *(Differentiating numerator and denominator)*

$$= \infty / \infty$$ *(Need again to apply the L'Hopital rule)*

$$\lim_{n \to \infty} \frac{2}{(\ln 2)^2 . 2^n}$$ *(Again differentiating numerator and denominator)*

$$= 0$$ *(Evaluating the limits)*

Therefore, $n^2 = O(2^n)$

Ω-Notation Using Limit

Definition

If $f(n)$ is running time of an algorithm and $g(n)$ is some standard growth function such that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c, \quad \text{where } c \text{ is a constant, such that } 0 < c \leq \infty$$

then $f(n) = \Omega(g(n))$.

- Note that zero value for the constant c is excluded from the permissible range, but *infinity* is included.

Ω-Notation

Examples

Example(1): $7n^2 + 12n+8= \Omega(n^2)$

$$\lim_{n \to \infty} \frac{7n^2 + 14n + 8}{n^2}$$

$$= 7 + 14/n + 8/n^2$$

$$= 7 + 0 + 0$$

$$= 7$$ *(Non-zero constant)*

Therefore, $7n^2 + 14n + 8 = \Omega(n^2)$

Example(2): $10n^3 + 5n + 2 = \Omega(n^2)$

$$\lim_{n \to \infty} \frac{10n^3 + 5n + 2}{n^2}$$

$$= 10n + 5/n + 2/n^2$$

$$= \infty + 0 + 0$$

$$= \infty$$

Therefore, $10n^3 + 5n + 2 = \Omega(n^2)$

Θ-Notation Using Limit

Definition

If $f(n)$ is running time of an algorithm and $g(n)$ is some standard growth function such that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c, \quad \text{where } c \text{ is a constant, such that } 0 < c < \infty$$

then $f(n) = \Theta(g(n))$.

- Note that *zero* value and i*nfinity* are excluded from the permissible values for the range of constant c

Θ-Notation

Examples

Example(1): $45n^3 - 3n^2 - 5n + 20 = \theta(n^3)$

$$\lim_{n \to \infty} \frac{45n^3 - 3n^2 - 5n + 20}{n^3}$$

$$= 45 - 3/n - 5/n^2 + 20/n^3$$

$$= 45 - 0 - 0 + 0$$

$$= 45 \qquad \textit{(non-zero constant)}$$

Therefore, $45n^3 - 3n^2 - 5n + 20 = \theta(n^3)$

Example(2): $n \lg n + n + n^2 = \theta(n^2)$

$$\lim_{n \to \infty} \frac{n \lg n + n + n^2}{n^2}$$

$$= \lg n / n + 1/n + 1$$

$$= 0 + 0 + 1$$

$$= 1 \qquad \textit{(non-zero constant)}$$

Thus, $n \lg n + n + n^2 = \theta(n^2)$

Θ-Notation

Examples

Example(3): $45n^3 - 3n^2 - 5n + 20 \neq \theta(n^4)$

$$\lim_{n \to \infty} \frac{45n^3 - 3n^2 - 5n + 20}{n^4}$$

$$= 45/n - 3/n^2 - 5/n^3 + 20/n^4$$
$$= 0 - 0 - 0 + 0$$
$$= 0$$ *(zero is excluded from the permissible range for Θ-notation)*

Therefore, $45n^3 - 3n^2 - 5n + 20 \neq \theta(n^3)$

Example(4): $n \lg n + n \neq \theta(n^2)$

$$\lim_{n \to \infty} \frac{n \lg n + n}{n^2}$$

$$= \lg n / n + 1/n$$ *(Using the L'Hopital Rule to evaluate the limit of lgn /n)*
$$= 0 + 0$$
$$= 0$$

Thus, $n \lg n + n \neq \theta(n^2)$

o-Notation Using Limit

Definition

If $f(n)$ is running time and $g(n)$ is some standard growth function such that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \; (\; zero \;)$$

then $f(n) = o(\, g(n) \,)$ *(Read f (n) is small-oh of g(n))*

- $o(\, g(n) \,)$ is referred to as the *loose upper bound* for $f(n)$

o-Notation

Examples

Example(1): $5n + 20 = o(n^2)$

$$\lim_{n \to \infty} \frac{5n + 20}{n^2}$$

$$= 5/n + 20/n^2$$

$$= 0 + 0$$

Therefore, $5n + 20 = o(n^2)$

Example(2): $10n^2 + 25n + 7 = o(n^3)$

$$\lim_{n \to \infty} \frac{10n^2 + 25n + 7}{n^3}$$

$$= 10/n + 25/n^2 + 7/n^3$$

$$= 0+0+0$$

$$=0$$

Therefore, $10n^2 + 25n + 7 = o(n^3)$

o-Notation

Examples

Example(3): $lg\ n = o\ (n)$

$$\lim_{n \to \infty} \frac{lg\ n}{n} = \frac{\infty}{\infty}$$ *(Need to use L'Hopital Rule)*

$$\lim_{n \to \infty} \frac{ln\ n}{n\ ln\ 2}$$ *(Converting to natural log , $lg\ n = log_2 n = log_e n / log_e 2$)*

$$\lim_{n \to \infty} \frac{1}{n\ ln 2}$$ *(Differentiating numerator and denominator)*

$$= 0$$

Therefore, $lg\ n = o\ (n)$

ω-Notation

Definition

If *f(n)* is running time and *g(n)* is some standard growth function such that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

then $f(n) = \omega(g(n))$ *(Read f(n) is small-omega of g(n))*

- $\omega(g(n))$ is referred to as the *loose lower bound for f(n)*

ω-Notation

Examples

Example(1): $5n^3 + 20n^2 + n + 10 = \omega(n^2)$

$$\lim_{n \to \infty} \frac{5n^3 + 20n^2 + n + 10}{n^2}$$

$$= 5n + 20 + 1/n + 10/n^2$$

$$= \infty + 20 + 0 + 0$$

$$= \infty$$

Therefore, by definition, $5n^3 + 20n^2 + n + 10 = \omega(n^2)$

Example(2): $10n^2 + 25n + 7 = \omega(n)$

$$\lim_{n \to \infty} \frac{10n^2 + 25n + 7}{n}$$

$$= n + 25 + 7/n$$

$$= \infty + 25 + 0$$

$$= \infty$$

Therefore, $10n^2 + 25n + 7 = \omega(n)$

ω-Notation

Examples

Example(3): $n! = \omega(2^n)$

$$\lim_{n \to \infty} \frac{n!}{2^n} = \infty / \infty$$ *(Need to use the LHpoital Rule*

The function n! cannot be differentiated directly. We first use *Stirling's approximation:*

$$n! = \sqrt{2\pi n}\,(n/e)^n \quad \text{for large } n$$

$$\lim_{n \to \infty} \frac{n!}{2^n}$$

$$= \frac{\sqrt{2\pi n}\,(n/e)^n}{2^n}$$

$$= \sqrt{2\pi n}\,(n/2e)^n$$

$$= \infty$$

Thus, $n! = \omega(2^n)$

Asymptotic Notation

Summary

Let $f(n)$ be time complexity and $g(n)$ standard function, such that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \alpha$

Table below summarizes the asymptotic behavior of $f(n)$ in terms of $g(n)$

Notation	***Using Basic Definition***	***Using Limits***	***Asymptotic Bound***
$f(n)=O(g(n))$	$f(n) \le c.g(n)$ *for some* $c>0$, *and* $n \ge n_0$	$0 < \alpha < \infty$	*upper tight*
$f(n)=o(g(n))$	$f(n) < c.g(n)$ *for all* $c>0$, *and* $n \ge n_0$	$\alpha = 0$	*upper loose*
$f(n)=\Omega(g(n))$	$f(n) \ge c.g(n)$ *for some* $c>0$ *and* $n \ge n_0$	$0 < \alpha \le \infty$	*lower tight*
$f(n)=\omega(g(n))$	$f(n) > c.g(n)$ *for all* $c>0$ *and* $n \ge n_0$	$\alpha = \infty$	*lower loose*
$f(n)=\theta(g(n))$	$c_1.g \le f(n) \le c_2.g(n)$ *for some* $c_1>0$, $c_2>0$ *and* $n \ge n_0$	$0 < \alpha < \infty$	*tight*

Asymptotic Set Notations

Definitions

- The asymptotic notation can also be expressed in Set notations by using the limits

- Let $f(n)$ be time complexity and $g(n)$ standard function, such that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$

where c is *zero, positive constant* or *infinity*.

- The set notations for the asymptotic behavior is are defined as follows

$$(i)\quad O(g(n)) = \{ f(n): \quad 0 \le c < \infty \}$$

$$(ii)\quad \Omega(g(n)) = \{ f(n) : \quad 0 < c \le \infty \}$$

$$(iii)\quad \Theta(g(n)) = \{ f(n): \; 0 < c < \infty \}$$

$$(iv)\quad o(g(n)) = \{ f(n): \quad c = 0 \}$$

$$(v)\quad \omega(g(n)) = \{ f(n): \quad c = \infty \}$$

Asymptotic Set Notations

Examples

Example (1): Let $f(n) \in O(g(n))$, $f(n) \in \Omega(f(n))$. Then

$O(g(n)) = \{ f(n): 0 \leq c < \infty \}$ *(definition of O-notation)*

$\Omega(g(n)) = \{ f(n): 0 < c \leq \infty \}$ *(definition of Ω-notation)*

$O(g(n)) \cap \Omega(g(n)) = \{ f(n) : 0 < c < \infty \}$ *(Performing Set intersection operation)*

$= \Theta(g(n))$ *(definition of Θ-notation }*

Thus, $O(g(n)) \cap \Omega(g(n)) = \Theta(g(n))$

Example (2): Suppose $f(n) \in o(g(n))$, and $f(n) \in \omega(g(n))$. Then

$o(g(n)) = \{ f(n): c = 0 \}$ *(definition of o-notation)*

$\Omega(g(n)) = \{ f(n): c = \infty \}$ *(definition of ω-notation)*

Therefore, $o(g(n)) \cap \omega(g(n)) = \varphi$ *(Empty set)*

Analysis of Summations

Arithmetic Summation

Asymptotic Behavior

- The sum of first of n terms of *arithmetic series* is :

$$1 + 2 + 3 \ldots\ldots\ldots + n = n(n+1)/2$$

Let $f(n) = n(n+1)/2$

and $g(n) = n^2$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{n(n+1)/2}{n^2} = \frac{n^2/2 + n/2}{n^2} = 1/2 + 1/2n = 1/2 + 0 = 1/2$$

- Since the limit is *non-zero and finite* it follows

$$f(n) = \theta(g(n)) = \theta(n^2)$$

Or, $1 + 2 + \ldots\ldots\ldots + n = \theta(n^2)$

Geometric Summation

Asymptotic Behavior

- The asymptotic behavior of *geometric series*

$$1 + r + r^2 + \ldots\ldots + r^n$$

depends on *geometric ratio r*. Three cases need to be considered

Case $r > 1$: It can be shown that sum $f(n)$ of first n terms is as follows

$$f(n) = 1 + r + r^2 + \ldots\ldots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Case $r = 1$: This is trivial

$$f(n) = 1 + 1 + 1 + \ldots\ldots + 1 = n = \Theta(n)$$

Case $r < 1$: It can be shown that

$$f(n) = 1 + r + r^2 + \ldots\ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

- The asymptotic behavior in first case and third case is explored by computing limits.

Geometric Summation

Case r > 1

Let $f(n) = 1 + r + r^2 + \ldots\ldots + r^n = \dfrac{r^{n+1} - 1}{r - 1}$

Let $g(n) = r^n$

Consider, the limit

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{r^{n+1} - 1}{(r - 1).r^n} = \frac{r - 1/r^n}{(r - 1)}$$

Since $r>1$, $1/r^n \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{r}{(r - 1)} > 0, \text{ since } r > 1$$

Therefore, $f(n) = \theta(r^n)$ *for* $r>1$

Or, $1 + r + r^2 + \ldots\ldots + r^n = \theta(r^n)$ *for* $r > 1$

Geometric Summation

Case r < 1

Consider $1 + r + r^2 + \dots\dots + r^n = \dfrac{r^{n+1} - 1}{r - 1}$

Let $g(n)=c$ *where c is some positive constant*

Taking the limit

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{r^{n+1} - 1}{(r - 1).c} = \frac{1 - r^{n+1}}{(1 - r)c}$$

Since $r<1$, $r^{n+1} \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{1}{(1 - r).\, c} > 0, \text{ since } r < 1$$

Therefore, $f(n) = \quad \theta(g(n)) = \theta(c) = \theta(1) \quad for\ r < 1$

Or, $1 + r + r^2 + \dots\dots + r^n = \theta(1) \quad for\ r < 1$

Geometric Summation

Asymptotic Behavior

From the preceding analysis it follows that the asymptotic behavior of geometric summation is

$$S(n) = 1 + r + r^2 + \ldots\ldots\ldots + r^n = \begin{cases} \Theta(r^n) & \text{when } r > 1 \\ \Theta(1) & \text{when } r < 1 \end{cases}$$

- When $r < 1$, the *largest* term in the geometric summation would be 1 (*the first term)* , and $S(n) = \Theta(1)$. On the other hand if $r > 1$, the *largest term* would be r^n (*the last term*), and $S(n) = \Theta(r^n)$ In the light of this observation it can said that *asymptotic behavior of geometric summation is determined by the largest term i.e* $S(n) = \Theta$ *(largest term)*.

Example(1): The geometric series

$$1 + 2^1 + 2^2 + \ldots + 2^n$$

has *geometric ratio* $r = 2 > 1$,

Thus, $1 + 2^1 + 2^2 + \ldots + 2^n = \Theta(2^n)$, ► *Θ(largest term)*

Example(2): The geometric series

$$1 + (2/3)^1 + (2/3)^2 + \ldots + (2/3)^n$$

has *geometric ratio* $r = 2/3 < 1$,

Thus, $1 + (2/3)^1 + (2/3)^2 + \ldots + (2/3)^n = \Theta(1)$, ► *Θ(largest term)*

Logarithm Summation

Asymptotic Behavior

- The *logarithmic series* has the summation

$$lg(1)+lg(2)+ lg(3)+ \ldots\ldots +lg(n)$$

Let $f(n) = lg(2)+ lg(3)+\ldots.+.lg(n) = lg(2.3\ldots\ldots.n) = lg(n!)$
and $g(n) = n\ lg\ n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{lg\ n!}{n\ lg\ n} = \frac{lg\ (\sqrt{(2\pi n)}(n/e)^n\ )}{n\ lg\ n}$$ (*Using Stirling's approximation*)

Now, $lg\ (\sqrt{(2\pi n)}(n/e)^n\ ) = (1+lg\ \pi + lg\ n)/2 + n\ lg\ n - n\ lg\ e$, therefore

$$\lim_{n\to\infty} \frac{lg\ (\sqrt{(2\pi n)}(n/e)^n\ )}{n\ lg\ n} = (\ 1+lg\ \pi + lg\ n\ )/(2\ nlg\ n) + 1 - lg\ e\ /\ lg\ n = (0+ 1- 0)=1$$

Since limit is *non-zero and finite*, it follows
$f(n)= \theta(g(n))$

Or, $lg(1)+lg(2)+ lg(3)+\ldots\ldots +lg(n) = \theta(n\ lg\ n)$ ...

Harmonic Summation

Asymptotic Behavior

The sum of first n terms of *Harmonic series* is

$$1 + 1/2 + 1/3 + \ldots\ldots + 1/n$$

Let $f(n) = 1 + 1/2 + 1/3 + \ldots\ldots + 1/n$
and $g(n) = \lg n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{1+1/2+1/3+ \ldots\ldots +1/n}{\lg n}$$

It can be shown that $1 + 1/2 + 1/3 + \ldots + 1/n = \lg(n) + \gamma + 1/2n - 1/12n^2 + \ldots$ where $\gamma \approx 0.5772$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\lg n + \gamma + 1/2n - 1/12n^2 + \ldots}{\lg n} = 1+0+0-0+0+\ldots=1$$

Since limit is *finite and non-zero,* it follows

$$f(n) = \theta(g(n))$$

Or, $1 + 1/2 + 1/3 + \ldots\ldots + 1/n = \theta(\lg n)$