Lecture No.23 **Mathematical Induction**

PRINCIPLE OF MATHEMATICAL INDUCTION:
Let P(n) be a propositional function defined for all positive integers n. P(n) is true for every positive integer n if

1.Basis Step:

The proposition P(1) is true.

2.Inductive Step:

If P(k) is true then P(k + 1) is true for all integers $k \geq 1$.
i.e. $\forall$ k $p(k) \rightarrow P(k+1)$

EXAMPLE:

Use Mathematical Induction to prove that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1$$

SOLUTION:

Let

$$P(n): 1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

1.Basis Step:

P(1) is true.

For n = 1, left hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1, so LHS = 1 and RHS is

$$R.H.S=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

so the proposition is true for n = 1.

2. Inductive Step: Suppose P(k) is true for, some integers $k \geq 1$.

(1) $$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

To prove P(k + 1) is true. That is,

(2) $$1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}$$

Consider L.H.S. of (2)

$$\begin{aligned} 1+2+3+\cdots+(k+1) &= 1+2+3+\cdots+k+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) \quad \text{using (1)} \\ &= (k+1)\left[\frac{k}{2}+1\right] \end{aligned}$$

$$= (k+1)\left[\frac{k+2}{2}\right]$$

$$= \frac{(k+1)(k+2)}{2} = \text{RHS of (2)}$$

Hence by principle of Mathematical Induction the given result true for all integers greater or equal to 1.

EXERCISE:

Use mathematical induction to prove that

$1+3+5+\ldots+(2n-1) = n^2$ for all integers $n \geq 1$.

SOLUTION:

Let P(n) be the equation $1+3+5+\ldots+(2n-1) = n^2$

1. **Basis Step**:

P(1) is true

For n = 1, L.H.S of P(1) = 1and

R.H.S = 12 = 1

Hence the equation is true for n = 1

2. **Inductive Step**:

Suppose P(k) is true for some integer $k \geq 1$. That is,

$1 + 3 + 5 + \ldots + (2k - 1) = k^2$ ....(1)

To prove P(k+1) is true; i.e.,

$1 + 3 + 5 + \ldots + [2(k+1)-1] = (k+1)^2$ ....(2)

Consider L.H.S. of (2)

$$1+3+5+\cdots+[2(k+1)-1] = 1+3+5+\cdots+(2k+1)$$
$$= 1+3+5+\cdots+(2k-1)+(2k+1)$$
$$= k^2+(2k+1) \qquad \text{using (1)}$$
$$= (k+1)^2$$
$$= \text{R.H.S. of (2)}$$

Thus P(k+1) is also true. Hence by mathematical induction, the given equation is true for all integers $n \geq 1$.

EXERCISE:

Use mathematical induction to prove that

$1+2+2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all integers $n \geq 0$

SOLUTION:

Let P(n): $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

1. **Basis Step:**

P(0) is true.

For n = 0
L.H.S of P(0) = 1
R.H.S of P(0) = $2^{0+1} - 1 = 2 - 1 = 1$
Hence P(0) is true.

2. **Inductive Step:**

Suppose P(k) is true for some integer k ≥ 0; i.e.,

$1+2+2^2+\ldots+2^k = 2^{k+1} - 1$ ....(1)

To prove P(k+1) is true, i.e.,

$1+2+2^2+\ldots+2^{k+1} = 2k+1+1 - 1$ ....(2)

Consider LHS of equation (2)

$$\begin{aligned}1+2+2^2+\ldots+2^{k+1} &= (1+2+2^2+\ldots+2^k) + 2^{k+1}\\ &= (2^{k+1} - 1) + 2^{k+1}\\ &= 2\cdot 2^{k+1} - 1\\ &= 2^{k+1+1} - 1 \quad = \text{R.H.S of (2)}\end{aligned}$$

Hence P(k+1) is true and consequently by mathematical induction the given propositional function is true for all integers n ≥ 0.

EXERCISE:

Prove by mathematical induction

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers n ≥1.

SOLUTION:

Let P(n) denotes the given equation

1. **Basis step:**

P(1) is true
For n = 1
L.H.S of P(1) = 12 = 1

$$\text{R.H.S of P(1)} = \frac{1(1+1)(2(1)+1)}{6} = \frac{(1)(2)(3)}{6} = \frac{6}{6} = 1$$

So L.H.S = R.H.S of P(1).Hence P(1) is true

2.Inductive Step:

Suppose P(k) is true for some integer k ≥1;

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \text{.........(1)}$$

To prove P(k+1) is true; i.e.;

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \qquad \text{...(2)}$$

Consider LHS of above equation (2)

$$\begin{aligned}
1^2+2^2+3^2+\cdots+(k+1)^2 &= 1^2+2^2+3^2+\cdots+k^2+(k+1)^2\\
&= \frac{k(k+1)(2k+1)}{6}+(k+1)^2\\
&= (k+1)\left[\frac{k(2k+1)}{6}+(k+1)\right]\\
&= (k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right]\\
&= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]\\
&= \frac{(k+1)(2k^2+7k+6)}{6}\\
&= \frac{(k+1)(k+2)(2k+3)}{6}\\
&= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}
\end{aligned}$$

EXERCISE:

Prove by mathematical induction

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1} \qquad \text{for all integers } n\geq 1$$

SOLUTION:

Let P(n) be the given equation.

1. Basis Step:

P(1) is true

For n = 1

L.H.S of P(1) = $\frac{1}{1\cdot 2}=\frac{1}{1\times 2}=\frac{1}{2}$

R.H.S of P(1) = $\frac{1}{1+1}=\frac{1}{2}$

Hence P(1) is true

2. Inductive Step:

Suppose P(k) is true, for some integer k≥1. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad \ldots(1)$$

To prove P(k+1) is true. That is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+1+1)}=\frac{k+1}{(k+1)+1} \quad \ldots\ldots..(2)$$

Now we will consider the L.H.S of the equation (2) and will try to get the R.H.S by using equation (1) and some simple computation.

Consider LHS of (2)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}$$
$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$
$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$
$$=\frac{(k+1)^2}{(k+1)(k+2)}$$
$$=\frac{k+1}{(k+2)}$$
= RHS of (2)

Hence P(k+1) is also true and so by Mathematical induction the given equation is true for all integers n ≥1.

EXERCISE:

Use mathematical induction to prove that

$$\sum_{i=1}^{n+1} i2^i = n\cdot 2^{n+2}+2, \qquad \text{for all integers } n\ge 0$$

SOLUTION:

1.Basis Step:

To prove the formula for n = 0, we need to show that

$$\sum_{i=1}^{0+1} i.2^i = 0\cdot 2^{0+2}+2$$

Now, L.H.S = $\sum_{i=1}^{1} i\cdot 2^i = (1)2^1 = 2$

R.H.S = $0\cdot 2^2+2=0+2=2$

Hence the formula is true for n = 0

2.Inductive Step:

Suppose for some integer n = k ≥0

$$\sum_{i=1}^{k+1} i\cdot 2^i = k\cdot 2^{k+2}+2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We must show that $\sum_{i=1}^{k+2} i\cdot 2^i = (k+1)\cdot 2^{k+1+2}+2$ ....(2)

Consider LHS of (2)

$$\sum_{i=1}^{k+2} i\cdot 2^{i} = \sum_{i=1}^{k+1} i\cdot 2^{i} + (k+2)\cdot 2^{k+2}$$
$$= (k\cdot 2^{k+2}+2)+(k+2)\cdot 2^{k+2}$$
$$= (k+k+2)2^{k+2}+2$$
$$= (2k+2)\cdot 2^{k+2}+2$$
$$= (k+1)2\cdot 2^{k+2}+2$$
$$= (k+1)\cdot 2^{k+1+2}+2$$
$$= \text{RHS of equation (2)}$$

Hence the inductive step is proved as well. Accordingly by mathematical induction the given formula is true for all integers $n\geq 0$.

EXERCISE:

Use mathematical induction to prove that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \quad \text{for all integers } n \geq 2$$

SOLUTION:

1. **Basis Step**:

For $n = 2$

$$\text{L.H.S} = 1-\frac{1}{2^2}=1-\frac{1}{4}=\frac{3}{4}$$
$$\text{R.H.S} = \frac{2+1}{2(2)}=\frac{3}{4}$$

Hence the given formula is true for $n = 2$

2. **Inductive Step**:

Suppose for some integer $k \geq 2$

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k} \quad \text{.....................(1)}$$

We must show that

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)} \quad \text{.............(2)}$$

Consider L.H.S of (2)

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left[\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\right]\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\left(\frac{1}{2k}\right)\left(\frac{k^2+2k+1-1}{(k+1)}\right)$$

$$=\frac{k^2+2k}{2k(k+1)}=\frac{k(k+2)}{2k(k+1)}$$

$$=\frac{k+1+1}{2(k+1)}= \text{ RHS of (2)}$$

Hence by mathematical induction the given equation is true

EXERCISE:

Prove by mathematical induction

$$\sum_{i=1}^{n} i(i!)=(n+1)!-1 \qquad \text{for all integers } n\geq 1$$

SOLUTION:

1. **Basis step:**

For n = 1

L.H.S $=\sum_{i=1}^{n} i(i!)=(1)(1!)=1$

R.H.,S = (1+1)! - 1 = 2! - 1

= 2 -1 = 1

Hence

$$\sum_{i=1}^{1} i(i!)=(1+1)!-1$$

which proves the basis step.

2.Inductive Step:

Suppose for any integer k ≥1

$$\sum_{i=1}^{k} i(i!)=(k+1)!-1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We need to prove that

$$\sum_{i=1}^{k+1} i(i!)=(k+1+1)!-1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$