Lecture No.35 Probability

INTRODUCTION TO PROBABILITY

INTRODUCTION:

Combinatorics and probability theory share common origins. The theory of probability was first developed in the seventeenth century when certain gambling games were analyzed by the French mathematician Blaise Pascal. It was in these studies that Pascal discovered various properties of the binomial coefficients. In the eighteenth century, the French mathematician Laplace, who also studied gambling, gave definition of the probability as the number of successful outcomes divided by the number of total outcomes.

DEFINITIONS:

An **experiment** is a procedure that yields a given set of possible outcomes.

The **sample space** of the experiment is the set of possible outcomes.

An **event** is a subset of the sample space.

EXAMPLE:

When a die is tossed the sample space S of the experiment have the following six outcomes. $S = \{1, 2, 3, 4, 5, 6\}$

Let E_1 be the event that a 6 occurs,

E_2 be the event that an even number occurs,

E_3 be the event that an odd number occurs,

E_4 be the event that a prime number occurs,

E_5 be the event that a number less than 5 occurs, and

E_6 be the event that a number greater than 6 occurs.

Then

$E_1 = \{6\}$ $E_2 = \{2, 4, 6\}$

$E_3 = \{1, 3, 5\}$ $E_4 = \{2, 3, 5\}$

$E_5 = \{1, 2, 3, 4\}$ $E_6 = \Phi$

EXAMPLE:

When a pair of dice is tossed, the sample space S of the experiment has the following thirty-six outcomes

$$
\begin{aligned}
S = \{&(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\\
&(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)\\
&(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\\
&(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\\
&(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\\
&(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}
\end{aligned}
$$

or more compactly,

$\{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26,$
$31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46,$
$51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$

Let E be the event in which the sum of the numbers is ten.
Then
$E = \{(4, 6), (5, 5), (6, 4)\}$

DEFINITION:
Let S be a finite sample space such that all the outcomes are equally likely to occur.
The **probability** of an event E, which is a subset of sample space S, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the numbr of total outcomes in } S} = \frac{n(E)}{n(S)}$$

REMARK:
Since $\varphi \subseteq E \subseteq S$ therefore, $0 \leq n(E) \leq n(S)$. It follows that the probability of an event is always between 0 and 1. (Since $n(\phi) = 0, \; n(S) = 1$)

EXAMPLE:
What is the probability of getting a number greater than 4 when a dice is tossed?

SOLUTION:
When a dice is rolled its sample space is S={1,2,3,4,5,6}
Let E be the event that a number greater than 4 occurs. Then $E = \{5, 6\}$
Hence,

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

EXAMPLE:
What is the probability of getting a total of eight or nine when a pair of dice is tossed?

SOLUTION:
When a pair of dice is tossed,its sample space S has the 36 outcomes which are as fellows:

$$\begin{aligned} S = \{&(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\\ &(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)\\ &(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\\ &(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\\ &(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\\ &(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}\end{aligned}$$

Let E be the event that the sum of the numbers is eight or nine. Then
$E = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (3, 6), (4, 5), (5, 4), (6, 3)\}$
Hence,

$$P(E) = \frac{n(E)}{n(S)} = \frac{9}{36} = \frac{1}{4}$$

EXAMPLE:

An urn contains four red and five blue balls. What is the probability that a ball chosen from the urn is blue?

SOLUTION:

Since there are four red balls and five blue balls so if we take out one ball from the urn then there is possibility that it may be one of from four red and one of from five blue balls hence there are total of nine possibilities. Thus we have
The total number of possible outcomes = 4 + 5 = 9
Now our favourable event is that we get the blue ball when we choose a ball from the urn.
So we have

The total number of favorable outcomes = 5

Now we have Favorable outcomes 5 and our sample space has total outcomes 9 .Thus we have

The probability that a ball chosen = 5/9

EXERCISE:

Two cards are drawn at random from an ordinary pack of 52 cards. Find the probability **p** that (i) both are spades, (ii) one is a spade and one is a heart.

SOLUTION:

There are $\binom{52}{2} = 1326$ ways to draw 2 cards from 52 card

(i) There are $\binom{13}{2} = 78$ ways to draw 2 spades from 13 spades (as spades are 13 in 52 cards); hence

$$p = \frac{\text{number of ways 2 spades can be drawn}}{\text{number of ways 2 cards can be drawn}} = \frac{78}{1326} = \frac{1}{17}$$

ii) Since there are 13 spades and 13 hearts, there are $\binom{13}{1}\binom{13}{1} = 13.13 = 169$ ways to draw a spade and a heart; hence $p = \frac{169}{1326} = \frac{13}{102}$

EXAMPLE:

In a lottery, players win the first prize when they pick three digits that match, in the correct order, three digits kept secret. A second prize is won if only two digits match. What is the probability of winning (a) the first prize, (b) the second prize?

SOLUTION:

Using the product rule, there are $10^3 = 1000$ ways to choose three digits.

(a) There is only one way to choose all three digits correctly. Hence the probability that a player wins the first prize is 1/1000 = 0.001.

(b) There are three possible cases:

(i)The first digit is incorrect and the other two digits are correct
(ii)The second digit is incorrect and the other two digits are correct
(iii)The thirds digit is incorrect and the other digits are correct

To count the number of successes with the first digit incorrect, note that there are nine choices for the first digit to be incorrect, and one each for the other two digits to be correct. Hence, there are nine ways to choose three digits where the first digit is incorrect, but the other two are correct. Similarly, there are nine ways for the other two cases. Hence, there are $9 + 9 + 9 = 27$ ways to choose three digits with two of the three digits correct.

It follows that the probability that a player wins the second prize is $27/1000 = 0.027$.

EXAMPLE:
What is the probability that a hand of five cards contains four cards of one kind?

SOLUTION:

(i) For determining the favorable outcomes we note that
The number of ways to pick one kind = $C(13, 1)$
The number of ways to pick the four of this kind out of the four of this kind in the deck = $C(4, 4)$

The number of ways to pick the fifth card from the remaining 48 cards = $C(48, 1)$
Hence, using the product rule the number of hands of five cards with four cards of one kind = $C(13, 1) \times C(4, 4) \times C(48, 1)$

$$\frac{C(13,1)\cdot C(4,4)\cdot C(48,1)}{C(52,5)} = \frac{13\cdot 1\cdot 48}{2,598,960} \approx 0.0024$$

(ii) The total number of different hands of five cards = $C(52, 5)$.

From (i) and (ii) it follows that the probability that a hand of five cards contains four cards of one kind is

EXAMPLE:
Find the probability that a hand of five cards contains three cards of one kind and two of another kind.

SOLUTION:
(i) For determining the favorable outcomes we note that
The number of ways to pick two kinds = $C(13, 2)$
The number of ways to pick three out four of the first kind = $C(4, 3)$
The number of ways to pick two out four of the second kind = $C(4, 2)$
Hence, using the product rule the number of hands of five cards with three cards of one kind and two of another kind = $C(13, 2) \times C(4, 3) \times C(4, 2)$

(ii) The total number of different hands of five cards = $C(52, 5)$.

From (i) and (ii) it follows that the probability that a hand of five cards contains three cards of one kind and two of another kind is

$$\frac{C(13,2)\cdot C(4,3)\cdot C(4,2)}{C(52,5)} = \frac{3744}{2,598,960} \approx 0.0014$$

EXAMPLE:

What is the probability that a randomly chosen positive two-digit number is a multiple of 6?

SOLUTION:

1. There are $\left\lfloor \frac{99}{6} \right\rfloor = \left\lfloor 16 + \frac{1}{2} \right\rfloor = 16$ positive integers from 1 to 99 that are divisible by 6. Out of these $16 - 1 = 15$ are two-digit numbers(as 6 is a multiple of 6 but not a two-digit number).
2. There are $99 - 9 = 90$ positive two-digit numbers in all.

Hence, the probability that a randomly chosen positive two-digit number is a multiple of 6 = 15/90 = 1/6 ≈ 0.166667

DEFINITION:

Let E be an event in a sample space S, the complement of E is the event that occurs if E does not occur. It is denoted by E^c. Note that $E^c = S\backslash E$

EXAMPLE:

Let E be the event that an even number occurs when a die is tossed. Then E^c is the event that an odd number occurs.

THEOREM:

Let E be an event in a sample space S. The probability of the complementary event E^c of E is given by

$$P(E^c) = 1 - P(E).$$

EXAMPLE:

Let 2 items be chosen at random from a lot containing 12 items of which 4 are defective. What is the probability that (i) none of the items chosen are defective, (ii) at least one item is defective?

SOLUTION:

The number of ways 2 items can be chosen from 12 items = $C(12, 2) = 66$.

(i) Let A be the event that none of the items chosen are defective.
The number of favorable outcomes for A = The number of ways 2 items can be chosen from 8 non-defective items = $C(8, 2) = 28$.
Hence, P(A) = 28/66 = 14/33.

(ii) Let B be the event that at least one item chosen is defective.

Then clearly, $B = A^c$

It follows that

$P(B) = P(A^C)$
$=1 - P(A) = 1 - 14/33 = 19/33.$

EXERCISE:

Three light bulbs are chosen at random from 15 bulbs of which 5 are defective. Find the probability p that (i) none is defective, (ii) exactly one is defective, (iii) at least one is defective.

SOLUTION:

There are $\binom{15}{3} = 455$ ways to choose 3 bulbs from the 15 bulbs.

(i) Since there are 15 - 5 = 10 non-defective bulbs, there are $\binom{10}{3} = 120$ ways to choose 3 non-defective bulbs.

Thus

$$p = \frac{120}{455} = \frac{24}{91}$$

(ii) There are 5 defective bulbs and $\binom{10}{2} = 45$ different pairs of non-defective bulbs; hence there are $\binom{5}{1}\binom{10}{2} = 5.45 = 225$ ways to choose 3 bulbs of which one is defective.

Thus

$$P = \frac{225}{455} = \frac{45}{91}.$$

(iii) The event that at least one is defective is the complement of the event that none are defective which has by (i), probability $\frac{24}{91}$.

Hence $p(\text{atleast one is defective}) = 1 - p(\text{none is defective}) = 1 - \frac{24}{91} = \frac{67}{91}$

Lecture No.36 **Laws of Probability**

ADDITION LAW OF PROBABILITY

THEOREM:

If A and B are two disjoint (mutually exclusive) events of a sample space S, then

$$P(A \cup B) = P(A) + P(B)$$

In words, the probability of the happening of an event A or an event B or both is equal to the sum of the probabilities of event A and event B provided the events have nothing in common.

PROOF:

By inclusion - exclusion principle for mutually disjoint sets,

$$n(A \cup B) = n(A) + n(B)$$

Dividing both sides by n(S), we get

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)+n(B)}{n(S)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)}$$

$$\Rightarrow \quad P(A \cup B) = p(A) \quad +P(B)$$

EXAMPLE:

Suppose a die is rolled. Let A be the event that 1 appears & B be the event that some even number appears on the die. Then

$S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1\}$ & $B = \{2, 4, 6\}$

Clearly A & B are disjoint events and

$$P(A) = \frac{1}{6}, \qquad P(B) = \frac{3}{6}$$

Hence the probability that a 1 appears or some even number appears is given by

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{1}{6} + \frac{3}{6}$$

$$= \frac{4}{6} = \frac{2}{3} \qquad Ans.$$

EXERCISE:

A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting a white or a black ball in a single draw.

SOLUTION:

Let A be the event of getting a white ball and B be the event of getting a black ball.

Total number of balls = 6 + 5 + 4 = 15

Since the two events are disjoint (mutually exclusive),therefore

$$P(A) = \frac{6}{15}, \qquad P(B) = \frac{5}{15}$$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{6}{15} + \frac{5}{15} \\ &= \frac{11}{15} \qquad \textit{Ans} \end{aligned}$$

EXERCISE:

A pair of dice is thrown. Find the probability of getting a total of 5 or 11.

SOLUTION:

When two dice are thrown, the sample space has 6 * 6 = 36 outcomes. Let A be the event that a total of 5 occurs and B be the event that a total of 11 occurs. Then

A = {(1,4), (2,3), (3,2), (4,1)} and B = {(5,6), (6,5)}

Clearly, the events A and B are disjoint (mutually exclusive) with probabilities given by

Now by using the sum Rule for Mutually Exclusive events we get

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{4}{36} + \frac{2}{36} = \frac{6}{36} = \frac{1}{6} \qquad \textit{Ans} \end{aligned}$$

EXERCISE:

For any two event A and B of a sample space S. Prove that

$P(A\backslash B) = P(A \cap B') = P(A) - P(A \cap B)$

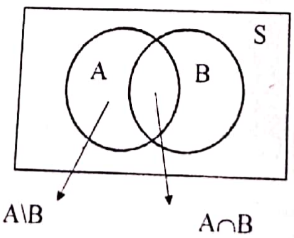
SOLUTION:

The event A can be written as the union of two disjoint events A\B and $A \cap B$. i.e. $A = (A\backslash B) \cup (A \cap B)$

Hence, by addition law of probability

$P(A) = P(A\backslash B) + P(A \cap B)$

$\Rightarrow P(A\backslash B) = P(A) - P(A \cap B)$



GENERAL ADDITION LAW OF PROBABILITY

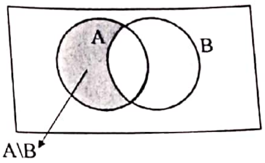
THEOREM

If A and B are any two events of a sample space S, then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

PROOF:

The event $A \cup B$ may be written as the union of two disjoint events A\B and B.
i.e., $A \cup B = (A \backslash B) \cup B$
Hence, by addition law of probability (for disjoint events)

$$P(A \cup B) = P(A \backslash B) + P(B)$$
$$= [P(A) - P(A \cap B)] + P(B)$$
$$= P(A) + P(B) - P(A \cap B) \quad \text{(proved)}$$



REMARK:

By inclusion - exclusion principle

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ (where A and B are finite)

Dividing both sides by n(S) and denoting the ratios as respective probabilities we get the

Generalized Addition Law of probability.

i.e $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

EXERCISE:

Let A and B be events in a sample space S, and let
$P(A) = 0.65$, $P(B) = 0.30$ and $P(A \cap B) = 0.15$
Determine the probability of the events

(a) $A \cap B'$ (b) $A \cup B$ (c) $A' \cap B'$

SOLUTION:

(a) As we know that

$$P(A \cap B') = P(A) - P(A \cap B) \quad (\text{as } A-B = A \cap B')$$
$$= 0.65 - 0.15$$
$$= 0.50$$

(b) By addition Law of probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (\text{as } A \cap B \neq \phi)$$
$$= 0.65 + 0.30 - 0.15$$
$$= 0.80$$

(c) By DeMorgan's Law

$A' \cap B' = (A \cup B)'$

$\therefore P(A' \cap B') = P(A \cup B)'$

$= 1 - P(A \cup B)$

$= 1 - 0.80$

$= 0.20$ Ans.

EXERCISE:

Let A, B, C and D be events which form a partition of a sample space S. If $P(A) = P(B)$, $P(C) = 2\,P(A)$ and $P(D) = 2\,P(C)$. Determine each of the following probabilities.

(a) $P(A)$ (b) $P(A \cup B)$ (c) $P(A \cup C \cup D)$

SOLUTION:

(a) Since A, B, C and D form a partition of S, therefore $S = A \cup B \cup C \cup D$ and A, B, C, D are pair wise disjoint. Hence, by addition law of probability.

$P(S) = P(A) + P(B) + P(C) + P(D)$

$\Rightarrow 1 = P(A) + P(A) + 2P(A) + 2\,P(C)$

$\Rightarrow 1 = 4P(A) + 2\,(2P(A))$

$\Rightarrow 1 = 8\,P(A)$

$\Rightarrow P(A) = \frac{1}{8}$

(b) $P(A \cup B) = P(A) + P(B)$

$= P(A) + P(A)$ $[\because P(B) = P(A)]$

$= 2\,P(A)$

$= 2\left(\frac{1}{8}\right) = \frac{1}{4}$

(c) $P(A \cup C \cup D) = P(A) + P(C) + P(D)$

$= P(A) + 2\,P(A) + 2\,(2P(A))$ $[\because P(C) = 2\,P(A)\ \&\ P(D) = 2\,P(C)]$

$= 7\,P(A)$

$= 7\left(\frac{1}{8}\right) = \frac{7}{8}$ Ans.

EXERCISE:

A card is drawn from a well-shuffled pack of playing card. What is the probability that it is either a spade or an ace?

SOLUTION:

Let A be the event of drawing a spade and B be the event of drawing an ace. Now A and B are not disjoint events. $A \cap B$ represents the event of drawing an ace of spades.

Now

$$P(A) = \frac{13}{52}; \qquad P(B) = \frac{4}{52}$$

$$P(A \cap B) = \frac{1}{52}$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$$

$$= \frac{16}{52} = \frac{4}{13}$$

EXERCISE:

A class contains 10 boys and 20 girls of which half the boys and half the girls have brown eyes. Find the probability that a student chosen at random is a boy or has brown eyes.

SOLUTION:

Let A be the event that a boy is chosen and B be the event that a student with brown eyes is chosen.Then A and B are not disjoint events. Infact, $A \cap B$ represents the event that a boy with brown eyes is chosen.

$$P(A) = \frac{10}{10+20} = \frac{10}{30}$$

$$P(B) = \frac{5+10}{10+20} = \frac{15}{30}$$

and

$$P(A \cap B) = \frac{5}{10+20} = \frac{5}{30} \quad (\textit{as some boys also have brown eyes})$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{10}{30} + \frac{15}{30} - \frac{5}{30}$$

$$= \frac{20}{30} = \frac{2}{3} \qquad Ans.$$

EXERCISE:

An integer is chosen at random from the first 100 positive integers. What is the probability that the integer chosen is divisible by 6 or by 8?

SOLUTION:

Let A be the event that the integer chosen is divisible by 6, and B be the event that the integer chosen is divisible by 8.

$A \cap B$ is the event that the integer is divisible by both 6 and 8 (i.e. as their L.C.M. is 24)

Now

$$n(A)=\left\lfloor \frac{100}{6} \right\rfloor = 16; \qquad n(B)=\left\lfloor \frac{100}{8} \right\rfloor = 12$$

and

$$n(A\cap B)=\left\lfloor \frac{100}{24} \right\rfloor = 4$$

Hence $P(A\cup B) = P(A)+P(B)-P(A\cap B)$

$$= \frac{16}{100}+\frac{12}{100}-\frac{4}{100}$$

$$= \frac{24}{100} = \frac{6}{25} \qquad Ans$$

OR

Let A denote the event that the integer chosen is divisible by 6, and B denote the event that the integer chosen is divisible by 8 i.e

A={6,12,18,24,...,90,96} $\Rightarrow$ n(A)=16 $\Rightarrow P(A)=\frac{16}{100}$

B={8,16,24,40,...,88,96} $\Rightarrow$ n(B)=12 $\Rightarrow P(B)=\frac{12}{100}$

$A\cap B$={24,48,72,96} $\Rightarrow$ n($A\cap B$)=12 $\Rightarrow P(A\cap B)=\frac{4}{100}$

EXERCISE:

A student attends mathematics class with probability 0.7 skips accounting class with probability 0.4, and attends both with probability 0.5. Find the probability that

(1) he attends at least one class
(2) he attends exactly one class

SOLUTION:

(1) Let A be the event that the student attends mathematics class and B be the event that the student attends accounting class.

Then given

P(A) = 0.7; P(B) = 1 - 0.4 = 0.6

And $P(A\cap B) = 0.5$, $P(A\cup B) = ?$

By addition law of probability

$P(A \cup B) = P(A) + P(B) - P(A\cap B)$

$= 0.7 + 0.6 - 0.5$

$= 0.8$

(2) Students can attend exactly one class in two ways

(a)He attends mathematics class but not accounting i.e., event $A \cap B^c$ or

(b)He does not attend mathematics class and attends accounting class i.e., event $A^c \cap B$

Since the two event $A\cap B^c$ and $A^c \cap B$ are disjoint, hence required probability is

$P(A \cap B^c) + P(A^c \cap B)$
Now
$P(A \cap B^c) = P(A \backslash B)$
$= P(A) - P(A \cap B)$
$= 0.7 - 0.5$
$= 0.2$
and
$P(A^c \cap B) = P(B \backslash A)$
$= P(B) - P(A \cap B)$
$= 0.6 - 0.5$
$= 0.1$
Hence required probability is
$P(A \cap B^c) + P(A^c \cap B) = 0.2 + 0.1 = 0.3$

PROBABILITY OF SUB EVENT

THEOREM:

If A and B are two events such that $A \subseteq B$, then $P(A) \leq P(B)$

PROOF:

Suppose $A \subseteq B$. The event B may be written as the union of disjoint events $B \cap A$ and $B \cap \bar{A}$

i.e., $B = (B \cap A) \cup (B \cap \bar{A})$

But $B \cap A = A$ (as $A \subseteq B$)

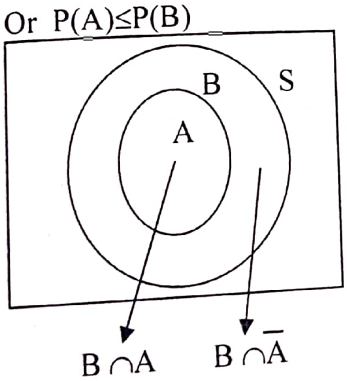
So $B = A \cup (B \cap \bar{A})$

$\therefore P(B) = P(A) + P(B \cap \bar{A})$

But $P(B \cap \bar{A}) \geq 0$

Hence $P(B) \geq P(A)$

Or $P(A) \leq P(B)$



EXERCISE:
Let A and B be subsets of a sample space S with $P(A) = 0.7$ and $P(B) = 0.5$. What are the maximum and minimum possible values of $P(A \cup B)$.

SOLUTION:

By addition law of probabilities
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.7 + 0.5 - P(A \cap B)$

= 1.2 - P(A ∩B)

Since probability of any event is always less than or equal to 1, therefore

max P(A ∪B) = 1, for which P(A ∩B) = 0.2

Next to find the minimum value, we note

$A \cap B \subseteq B$

$\Rightarrow P(A \cap B) \le P(B) = 0.5$

Thus for min P(A∪B) we take maximum possible value of P(A∩B) which is 0.5. Hence

min P(A ∪B) = 1.2 - max P(A ∩B)

= 1.2 - 0.5

= 0.7 is the required minimum value.

ADDITION LAW OF PROBABILITY FOR THREE EVENTS:

If A, B and C are any three events, then

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

REMARK:

If A, B, C are mutually disjoint events, then P(A∪B∪C)= P(A) + P(B) + P(C)

EXERCISE:

Three newspapers A, B, C are published in a city and a survey of readers indicates the following:

20% read A, 16% read B, 14% read C

8% read both A and B, 5% read both A and C

4% read both B and C, 2% read all the three

For a person chosen at random, find the probability that he reads none of the papers.

SOLUTION:

Given

$$P(A) = 20\% = \frac{20}{100} = 0.2; \qquad P(B) = 16\% = \frac{16}{100} = 0.16$$

$$P(C) = 14\% = \frac{14}{100} = 0.14; \quad P(A\cap B) = 8\% = \frac{8}{100} = 0.08$$

$$P(A\cap C) = 5\% = \frac{5}{100} = 0.05; P(B\cap C) = 4\% = \frac{4}{100} = 0.04$$

and

$$P(A\cap B \cap C) = 2\% = \frac{2}{100} = 0.02$$

Now the probability that person reads A or B or C = P(A∪B∪C)

$$= P(A) + P(B) + P(C) - P(A\cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= \frac{20}{100} + \frac{16}{100} + \frac{14}{100} - \frac{8}{100} - \frac{5}{100} - \frac{4}{100} + \frac{2}{100}$$

$$= \frac{35}{100}$$

Hence, the probability that he reads none of the papers

$$= P((A \cup B \cup C)^c)$$
$$= 1 - P(A \cup B \cup C)$$
$$= 1 - \frac{35}{100}$$
$$= \frac{65}{100}$$
$$= 65\%$$

EXERCISE:

Let A, B and C be events in a sample space S, with $A \cup B \cup C = S$, $A \cap (B \cup C) = \phi$, $P(A) = 0.2$, $P(B) = 0.5$ and $P(C) = 0.7$. Find $P(A^C)$, $P(B \cup C)$, $P(B \cap C)$.

SOLUTION:

$$P(A^c) = 1 - P(A)$$
$$= 1 - 0.2$$
$$P(A^c) = 0.8$$

Next, given that the events A and $B \cup C$ are disjoint, since $A \cap (B \cup C) = \phi$, therefore

$$P(A \cup (B \cup C)) = P(A) + P(B \cup C)$$
$$\Rightarrow \quad P(S) = 0.2 + P(B \cup C)$$
$$\Rightarrow \quad 1 = 0.2 + P(B \cup C)$$
$$\Rightarrow \quad P(B \cup C) = 1 - 0.2 = 0.8$$

Finally, by addition law of probability

$$P(B \cup C) = P(B) + P(C) - P(B \cap C)$$
$$\Rightarrow \quad 0.8 = 0.5 + 0.7 - P(B \cap C)$$

$\Rightarrow P(B \cap C) = 0.4$ is the required probability.

Lecture# 37 Conditional probability

CONDITIONAL PROBABILITY
MULTIPLICATION THEOREM
INDEPENDENT EVENTS

EXAMPLE:

a. What is the probability of getting a 2 when a dice is tossed?
b. An even number appears on tossing a die.
   (i) What is the probability that the number is 2?
   (ii) What is the probability that the number is 3?

SOLUTION:

When a dice is tossed, the sample space is $S=\{1,2,3,4,5,6\}$

$n(S)=6$

a. Let "A" denote the event of getting a 2 i.e $A=\{2\}$, $n(A)=1$

$$P(\text{2 appears when the die is tossed}) = \frac{n(A)}{n(S)} = \frac{1}{6}$$

b. (i) Let "S_1" denote the total number of even numbers from a sample space S, when a dice is tossed (i.e $S_1 \subseteq S$)

$S_1=\{2,4,6\}$, $n(S_1)=3$

Let "B" denote the event of getting a 2 from total number of even number i.e $B=\{2\}$

$n(B) = 1$

$$P(\text{2 appears; given that the number is even}) = P(B) = \frac{n(B)}{n(S_1)} = \frac{1}{3}$$

(ii) Let "C" denote the event of getting a 3 in S_1(among the even numbers)i.e $C=\{\ \}$

$n(C)=0$

$$P(\text{3 appears; given that the number is even}) = P(C) = \frac{n(C)}{n(S_1)} = \frac{0}{3} = 0$$

EXAMPLE:

Suppose that an urn contains 3 red balls, 2 blue balls, and 4 white balls, and that a ball is selected at random.

Let E be the event that the ball selected is red.

Then $P(E) = 3/9$ (as there are 3 red balls out of total 9 balls)

Let F be the event that the ball selected is not white.

Then the probability of E if it is already known that the selected ball is not white would be

P(red ball selected; given that the selected ball is not white) = 3/5 (as we count no white ball so there are total 9 balls(i.e 2 blue and 3 red balls))

This is called the conditional probability of E given F and is denoted by $P(E|F)$.

DEFINITION:

Let E and F be two events in the sample space of an experiment with $P(F) \neq 0$. The **conditional probability** of E given F, denoted by $P(E|F)$, is defined as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

EXAMPLE:

Let A and B be events of an experiment such that $P(B) = 1/4$ and $P(A \cap B) = 1/6$.

What is the conditional probability $P(A|B)$?

SOLUTION:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/4} = \frac{4}{6} = \frac{2}{3}$$

EXERCISE:

Let A and B be events with $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$

Find

(*i*) $P(A|B)$

(*ii*) $P(B|A)$

(*iii*) $P(A \cup B)$

(*iv*) $P(A^c|B^c)$

SOLUTION:

Using the formula of the conditional Probability we can write

(*i*) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{1/4}{1/3} = \frac{3}{4}$

(*ii*) $P(B|A) = \frac{P(B \cap A)}{P(A)}$

$= \frac{1/4}{1/2} = \frac{2}{4} = \frac{1}{2}$ (As $P(B \cap A) = P(A \cap B) = 1/4$)

(*iii*) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$= \frac{7}{12}$

(iv) $P(A^c \mid B^c) = \dfrac{P(A^c \cap B^c)}{P(B^c)}$

$= \dfrac{P((A \cup B)^c)}{P(B^c)}$ (By using DeMorgan's Law)

$= \dfrac{1 - P(A \cup B)}{1 - P(B)}$ $[P(E^c) = 1 - P(E)]$

$= \dfrac{1 - 7/12}{1 - 1/3} = \dfrac{5/12}{2/3} = \dfrac{5}{12} \times \dfrac{3}{2} = \dfrac{5}{8}$

EXERCISE:

Find P(B|A) if

(i) A is a subset of B

(ii) A and B are mutually exclusive

SOLUTION:

(i) When $A \subseteq B$, then $B \cap A = A$ (As $A \cap A \subseteq B \cap A$ □□ $\subseteq B \cap A$ ……………..(i) also we know that $B \cap A \subseteq A$ ……………..(ii) , From (i) and (ii) clearly $B \cap A = A$)

$$\therefore \quad P(B \mid A) = \frac{P(B \cap A)}{P(A)} \quad (as\, B \cap A = A \Rightarrow P(B \cap A) = P(A))$$

$$= \frac{P(A)}{P(A)} = 1$$

(ii) When A and B are mutually exclusive, then $B \cap A = \varnothing$

$$\therefore \quad P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{P(\phi)}{P(A)} \qquad (Since\ B \cap A = \phi \Rightarrow P(B \cap A) = 0)$$

$$= \frac{0}{P(A)} = 0 \qquad (\ as\ P(\phi) = 0\ )$$

EXAMPLE:

Suppose that an urn contains **three** red balls marked 1, 2, 3, one blue ball marked 4, and **four** white balls marked 5, 6, 7, 8.

A ball is selected at random and its color and number noted.

(i) What is the probability that it is red?

(ii) What is the probability that it has an even number marked on it?

(iii) What is the probability that it is red, if it is known that the ball selected has an even number marked on it?

(iv) What is the probability that it has an even number marked on it, if it is known that the ball selected is red?

SOLUTION:

Let E be the event that the ball selected is red .

(i) P(E) = 3/8

let F be the event that the ball selected has an even number marked on it.

(ii) P(F) = 4/8 (as there are four even numbers 2,4,6 & 8 out of total eight numbers).

(iii) $E\cap F$ is the event that the ball selected is red and has an even number marked on it. Clearly $P(E\cap F)$ =1/8 (as there is only one ball which is red and marked an even number "2" out of total eight balls).

Hence,

P(Selected ball is red, given that the ball selected has an even number marked on it.) = P(E|F)

$$= \frac{P(E \cap F)}{P(F)} = \frac{1/8}{4/8} = 1/4$$

(iv) P(Selected ball has an even number marked on it, given that the ball selected is red) = P(F|E)

$$= \frac{P(E \cap F)}{P(E)} = \frac{1/8}{3/8} = \frac{1}{3}$$

EXAMPLE:

Let a pair of dice be tossed. If the sum is 7, find the probability that one of the dice is 2.

SOLUTION:

Let E be the event that a 2 appears on at least one of the two dice, and F be the event that the sum is 7.

Then

E = {(1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6) ,(3, 2) (4, 2), (5, 2), ,(6, 2)}

F = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

$E\cap F$ = {(2, 5), (5, 2)}

P(F) = 6/36 and $P(E\cap F)$ = 2/36.

Hence,

P(Probability that one of the dice is 2, given that the sum is 7)

= P(E|F)

$$= \frac{P(E \cap F)}{P(E)} = \frac{2/36}{6/36} = \frac{1}{3}$$

EXAMPLE:

A man visits a family who has two children. One of the children, a boy, comes into the room.

Find the probability that the other child is also a boy if

(i) The other child is known to be elder,

(ii) Nothing is known about the other child.

SOLUTION:

The sample space of the experiment is S = $\{bb, bg, gb, gg\}$
(The outcome bg specifies that younger is a boy and elder is a girl, etc.)
Let A be the event that both the children are boys.
Then, A = $\{bb\}$.
(i) Let B be the event that the younger is a boy. Then, B = $\{bb, bg\}$,and $A \cap B = \{bb\}$.
Hence, the required probability is
P(Probability that the other child is also a boy, given that the other child is elder) =

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2}$$

(ii) Let C be the event that one of the children is a boy.
Then $C = \{bb, bg, gb\}$, and $A \cap C = \{bb\}$.
Hence, the required probability is
P(Probability that both the children are boys, given that one of the children is a boy)
= P(A|C)

$$= \frac{P(A \cap C)}{P(C)} = \frac{1/4}{3/4} = \frac{1}{3}$$

MULTIPLICATION THEOREM

Let E and F be two events in the sample space of an experiment, then
$P(E \cap F) = P(F)P(E \mid F)$
Or $P(E \cap F) = P(E)P(F \mid E)$

Let $E_1, E_2, \ldots, E_n$ be events in the sample space of an experiment, then
$P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1)P(E_2 \mid E_1)\, P(E_3 \mid E_1 \cap E_2) \ldots P(E_n \mid E_1 \cap E_2 \cap \ldots \cap E_n)$

EXAMPLE:

A lot contains 12 items of which 4 are defective. Three items are drawn at random from the lot one after the other. What is the probability that all three are non-defective?

SOLUTION:

Let A_1 be the event that the first item is not defective.
Let A_2 be the event that the second item is not defective.
Let A_3 be the event that the third item is not defective.
Then $P(A_1) = 8/12$, $P(A_2|A_1) = 7/11$, and $P(A_3|A_1 \cap A_2) = 6/10$
Hence, by multiplication theorem, the probability that all three are non-defective is
$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2)$

$$= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55}$$

INDEPENDENCE:

An event A is said to be independent of an event B if the probability that A occurs is not influenced by whether B has or has not occurred. That is, $P(A|B) = P(A)$.

It follows then from the Multiplication Theorem that,
$P(A \cap B) = P(B)P(A|B) = P(B)P(A)$
We also know that,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$
$$= \frac{P(A)P(B)}{P(A)} \quad \textit{Because } P(A \cap B) = P(A)P(B) \text{ ,due to independence}$$
$$= P(B)$$

EXAMPLE:

Let A be the event that a randomly generated bit string of length four begins with a 1 and let B be the event that a randomly generated bit string of length four contains an even number of 0s.
Are A and B independent events?

SOLUTION:

A = {1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}
B = {0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111}

Since there are 16 bit strings of length four, we have
P(A) = 8/16 = 1/2, P(B) = 8/16 = 1/2
Also,
$A \cap B$ = {1001, 1010, 1100, 1111} so that $P(A \cap B)$ = 4/16 =1/4
We note that,

$$P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)$$

Hence A and B are independent events.

EXAMPLE:

Let a fair coin be tossed three times. Let A be the event that first toss is heads, B be the even that the second toss is a heads, and C be the event that exactly two heads are tossed in a row. Examine pair wise independence of the three events.

SOLUTION:

The sample space of the experiment is
S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} and the events are
A = {HHH, HHT, HTH, HTT}
B = {HHH, HHT, THH, THT}
C = {HHT, THH}
$A \cap B$ = {HHH, HHT}, $A \cap C$ = {HHT}, $B \cap C$ = {HHT, THH}

It follows that
P(A) = 4/8=1/2
P(B) = 4/8= 1/2
P(C) = 2/8=1/4
and
$P(A \cap B)$=2/8=1/4
$P(A \cap C)$= 1/8

$P(B \cap C) = 2/8 = 1/4$

Note that,

$$P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cap B), \text{ so that A and B are independent.}$$

$$P(A)P(C) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} = P(A \cap C), \text{ so that A and C are independent.}$$

$$P(B)P(C) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \neq P(B \cap C), \text{ so that B and C are dependent.}$$

EXAMPLE:

The probability that A hits a target is 1/3 and the probability that B hits the target is 2/5. What is the probability that target will be hit if A and B each shoot at the target?

SOLUTION:

It is clear from the nature of the experiment that the two events are independent.

Hence,

$P(A \cap B) = P(A)P(B)$

It follows that,

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= P(A) + P(B) - P(A)P(B) \quad \text{(due to independence)} \\
&= \frac{1}{3} + \frac{2}{5} - \frac{1}{3} \cdot \frac{2}{5} \\
&= \frac{3}{5}
\end{aligned}
$$

Lecture# 38 Random variable

RANDOM VARIABLE
PROBABILITY DISTRIBUTION
EXPECTATION AND VARIANCE

INTRODUCTION:

Suppose S is the sample space of some experiment. The outcomes of the experiment, or the points in S, need not be numbers. For example in tossing a coin, the outcomes are H (heads) or T(tails), and in tossing a pair of dice the outcomes are pairs of integers.

However, we frequently wish to assign a specific number to each outcome of the experiment. For example, in coin tossing, it may be convenient to assign 1 to H and 0 to T; or in the tossing of a pair of dice, we may want to assign the sum of the two integers to the outcome. Such an assignment of numerical values is called a random variable.

RANDOM VARIABLE:
A random variable X is a rule that assigns a numerical value to each outcome in a sample Space S. **OR** It is a function which maps each outcome of the sample space into the set of real numbers.

We shall let X(S) denote the set of numbers assigned by a random variable X, and refer to X(S) as the range space.
In formal terminology, X is a function from S(sample space) to the set of real numbers R, and X(S) is the range of X.

REMARK:
1. A random variable is also called a chance variable, or a stochastic variable(not called simply a variable, because it is a function).
2. Random variables are usually denoted by capital letters such as X, Y, Z; and the values taken by them are represented by the corresponding small letters.

EXAMPLE:
A pair of fair dice is tossed. The sample space S consists of the 36 ordered pairs i.e

$$S = \{(1,1), (1,2), (1,3), \ldots, (6,6)\}$$

Let X assign to each point in S the sum of the numbers; then X is a random variable with range space i.e

$$X(S) = \{2,3,4,5,6,7,8,9,10,11,12\}$$

Let Y assign to each point in S the maximum of the two numbers in the outcomes; then Y is a random variable with range space.

$$Y(S) = \{1,2,3,4,5,6\}$$

PROBABILITY DISTRIBUTION OF A RANDOM VARIABLE:
Let $X(S) = \{x_1, x_2, \ldots, x_n\}$ be the range space of a random variable X defined on a finite sample space S.
Define a function f on X(S) as follows:

$f(x_i) = P(X = x_i)$
= sum of probabilities of points in S whose image is x_i.

This function f is called the probability distribution or the probability function of X.
The probability distribution f of X is usually given in the form of a table.

x_1	x_2	...	x_n
$f(x_1)$	$f(x_2)$	...	$f(x_n)$

The distribution f satisfies the conditions.

$(i) \quad f(x_i) \geq 0$ and $(ii) \quad \sum_{i=1}^{n} f(x_i) = 1$

EXAMPLE:

A pair of fair dice is tossed. Let X assign to each point (a, b) in S = {(1,1), (1,2), …, (6,6)}, the sum of its number, i.e., X (a,b) = a+b. Compute the distribution f of X.

SOLUTION:

X is clearly a random variable with range space
X(S) = {2,3,4,5,6,7,8,9,10, 11, 12}
(because X (a,b) = a+b ⇒ X (1,1) = 1+1=2, X (1,2) = 1+2=3, X (1,3) = 1+3=4 etc).

The distribution f of X may be computed as:

$$f(2) = P(X=2)=P(\{(1,1)\}) = \frac{1}{36}$$

$$f(3) = P(X=3) = P(\{(1,2), (2,1)\}) = \frac{2}{36}$$

$$f(4) = P(X=4) = P(\{(1,3), (2,2),(3,1)\}) = \frac{3}{36}$$

$$f(5) = P(X=5) = P(\{(1,4), (2,3),(3,2),(4,1)\}) = \frac{4}{36}$$

$$f(6) = P(X=6) = P(\{(1,5),(2,4),(3,3),(4,2),(5,1)\}) = \frac{5}{36}$$

$$f(7) = P(X=7) = P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,2)\}) = \frac{6}{36}$$

$$f(8) = P(X=8) = P(\{(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{5}{36}$$

$$f(9) = P(X=9) = P(\{(3,6),(4,5),(5,4),(6,3)\}) = \frac{4}{36}$$

$$f(10) = P(X=10) = P(\{(4,6),(5,5),(6,4)\}) = \frac{3}{36}$$

$$f(11) = P(X = 11) = P(\{(5,6),(6,5)\}) = \frac{2}{36}$$

$$f(12) = P(X = 12) = P(\{(6,6)\}) = \frac{1}{36}$$

The distribution of X consists of the points in X(S) with their respective probabilities.

x_i	2	3	4	5	6	7	8	9	10	11	12
$f(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

EXAMPLE:

A box contains 12 items of which three are defective. A sample of three items is selected from the box.
If X denotes the number of defective items in the sample; find the distribution of X.

SOLUTION:

The sample space S consists of $\binom{12}{3} = 220$ that is 220 different samples of size 3.
The random variable X, denoting the number of defective items has the range space X(S) = {0,1,2,3}
There are $\binom{3}{0}\binom{9}{3} = 84$ samples of size 3 with no defective items;

hence

$$p_0 = P(X = 0) = \frac{84}{220}$$

There are $\binom{3}{1}\binom{9}{2} = 108$ samples of size 3 containing one defective item;
hence

$$p_1 = P(X = 1) = \frac{108}{220}$$

There are $\binom{3}{2}\binom{9}{1} = 27$ samples of size 3 containing two defective items;

hence

$$p_2 = P(X = 2) = \frac{27}{220}$$

Finally, there is $\binom{3}{3}\binom{9}{0} = 27$, only one sample of size 3 containing three defective items;
hence

$$p_3 = P(X = 3) = \frac{1}{220}$$

The distribution of X follows:

x_i	0	1	2	3
p_i	84/220	108/220	27/220	1/220

EXPECTATION OF A RANDOM VARIABLE

Let X be a random variable with probability distribution

x_i	x_1	x_2	x_3	...	x_n
$f(x_i)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	...	$f(x_n)$

The mean (denoted μ) or the expectation of X (written E(X)) is defined by

$$\mu = E(X) = x_1\, f(x_1) + x_2\, f(x_2) + \ldots + x_n f(x_n)$$

$$= \sum_{i=1}^{n} x_i f(x_i)$$

EXAMPLE:

What is the expectation of the number of heads when three fair coins are tossed?

SOLUTION:

The sample space of the experiment is:

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

Let the random variable X represents the number of heads (i.e 0,1,2,3) when three fair coins are tossed. Then X has the probability distribution.

x_i	$x_0=0$	$x_1=1$	$x_2=2$	$x_3=3$
$f(x_i)$	1/8	3/8	3/8	1/8

Hence, expectation of X is

$$E(X) = x_0\, f(x_0) + x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3)$$

$$= 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8} = \frac{12}{8} = 1\cdot 5$$

EXERCISE:

A player tosses two fair coins. He wins Rs. 1 if one head appears, Rs.2 if two heads appear. On the other hand, he loses Rs.5 if no heads appear. Determine the expected value E of the game and if it is favourable to be player.

SOLUTION:

The sample space of the experiment is S = {HH, HT, TH, TT}

Now

$P(\text{Two heads}) = P(HH) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$P(\text{One head}) = P(HT, TH) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$

$P(\text{No heads}) = P(TT) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

Thus, the probability of winning Rs.2 is $\frac{1}{4}$, of winning Rs 1 is $\frac{1}{2}$ and of losing Rs 5 is $\frac{1}{4}$

Hence,

$$E = 2\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) - 5\left(\frac{1}{4}\right)$$
$$= -\frac{1}{4} = -0.25$$

Since, the expected value of the game is negative, so it is unfavorable to the player.

EXAMPLE:

A coin is weighted so that and $P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$
The coin is tossed three times.
Let X denotes the number of heads that appear.
(a) Find the distribution of X
(b) Find the expectation of E(X)

SOLUTION:

(a) The sample space is S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

The probabilities of the points in sample space are

$$p(HHH) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{27}{64} \qquad p(HHT) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{9}{64}$$

$$p(HTH) = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{9}{64} \qquad p(THH) = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4} = \frac{9}{64}$$

$$p(HTT) = \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{3}{64} \qquad p(THT) = \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{3}{64}$$

$$p(TTH) = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{3}{64} \qquad p(TTT) = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{1}{64}$$

The random variable X denoting the number of heads assumes the values 0,1,2,3 with the probabilities:

$$P(0) = P(TTT) = \frac{1}{64}$$

$$P(1) = P(HTT, THT, TTH) = \frac{3}{64} + \frac{3}{64} + \frac{3}{64} = \frac{9}{64}$$

$$P(2) = P(HHT, HTH, THH) = \frac{9}{64} + \frac{9}{64} + \frac{9}{64} = \frac{27}{64}$$

$$P(3) = P(HHH) = \frac{27}{64}$$

Hence, the distribution of X is

x_i	0	1	2	3
$P(x_i)$	1/64	9/64	27/64	27/64

(b) The expected value E(X) is obtained by multiplying each value of X by its probability and taking the sum.
The distribution of X is

x_i	0	1	2	3
$P(x_i)$	1/64	9/64	27/64	27/64

Hence

$$E(X) = 0\left(\frac{1}{64}\right) + 1\left(\frac{9}{64}\right) + 2\left(\frac{27}{64}\right) + 3\left(\frac{27}{64}\right)$$

$$= \frac{144}{64}$$

$$= 2 \cdot 25$$

VARIANCE AND STANDARD DEVIATION OF A RANDOM VARIABLE:

Let X be a random variable with mean μ and the probability distribution

x_1	x_2	x_3	…	x_n
$f(x_1)$	$f(x_2)$	$f(x_3)$	…	$f(x_n)$

The variance of X, measures the "spread" or "dispersion" of X from the mean μ and is denoted and defined as

$$\sigma_x^2 = \text{Var}(X) = \sum_{i=1}^{n}(x_i - \mu)^2 f(x_i)$$
$$= E((X-\mu)^2)$$
$$= E(X^2) - \mu^2$$
$$= \sum x_i^2 f(x_i) - \mu^2$$

The last expression is a more convenient form for computing Var(X).
The standard derivation of X, denoted by σ_X , is the non-negative square root of Var(X):

Where $\sigma_X = \sqrt{\text{Var}(X)}$

EXERCISE:

Find the expectation μ, variance σ^2 and standard deviation σ of the distribution given in the following table.

x_i	1	3	4	5
$f(x_i)$	0.4	0.1	0.2	0.3

SOLUTION:

$\mu = E(X) = \sum x_i f(x_i)$
$= 1(0.4) + 3(0.1) + 4(0.2) + 5(0.3)$
$= 0.4 + 0.3 + 0.8 + 1.5$
$= 3.0$

Next
$E(X^2) = \sum x_i^2 f(x_i)$
$= 1^2 (0.4) + 3^2 (0.1) + 4^2 (0.2) + 5^2 (0.3)$
$= 0.4 + 0.9 + 3.2 + 7.5$
$= 12.0$

Hence
$\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2$
$= 12.0 - (3.0)^2 = 3.0$

and $\sigma = \sqrt{\text{Var}(X)} = \sqrt{3.0} \approx 1.7$

EXERCISE:

A pair of fair dice is thrown. Let X denote the maximum of the two numbers which appears.
(a) Find the distribution of X
(b) Find the μ, variance σ_x^2 = Var(X), and standard deviation σx of X

SOLUTION:

(a) The sample space S consist of the 36 pairs of integers (a,b) where a and b range from 1 to 6;
that is S = {(1,1), (1,2), …, (6,6)}
Since X assigns to each pair in S the larger of the two integers, the value of X are the integers from 1 to 6.

Note that:

$$f(1) = P(X = 1) = P(\{(1,1)\}) = \frac{1}{36}$$

$$f(2) = P(X = 2) = P(\{(2,1),(2,2),(1,2)\}) = \frac{3}{36}$$

$$f(3) = P(X = 3) = P(\{(3,1),(3,2),(3,3),(2,3),(1,3)\}) = \frac{5}{36}$$

$$f(4) = P(X = 4) = P(\{(4,1),(4,2),(4,3),(4,4),(3,4),(2,4),(1,4)\}) = \frac{7}{36}$$

Similarly

$$f(5) = P(X = 5) = \{(1,5),(2,5),(3,5),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$$
$$= \frac{9}{36}$$

$$f(6) = P(X = 6) = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$
$$= \frac{11}{36}$$

Hence, the probability distribution of x is:

x_i	1	2	3	4	5	6
$f(x_i)$	1/36	3/36	5/36	7/36	9/36	11/36

(b) We find the expectation (mean) of X as

$$\mu = E(X) = \sum x_i f(x_i)$$
$$= 1\cdot\frac{1}{36} + 2\cdot\frac{1}{36} + 3\cdot\frac{5}{36} + 4\cdot\frac{7}{36} + 5\cdot\frac{9}{36} + 6\cdot\frac{11}{36}$$
$$= \frac{161}{36} \approx 4.5$$

Next

$$E(X^2) = \sum x_i^2 f(x_i)$$
$$= 1^2\cdot\frac{1}{36} + 2^2\cdot\frac{1}{36} + 3^2\cdot\frac{1}{36} + 4^2\cdot\frac{7}{36} + 5^2\cdot\frac{9}{36} + 6^2\cdot\frac{11}{36}$$
$$= \frac{791}{36} \approx 22.0$$

Then

$$\sigma_x^2 = \text{Var}(X) = E(X^2) - \mu^2$$
$$= 22.0 - (4.5)^2$$
$$= 17.5$$

and

$$\sigma_x = \sqrt{17.5} \approx 1.3$$