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$T(x_1,x_2,x_3)=(x_1,x_2,0)$ gives

$$T(1,0,0)=(1,0,0)=1(1,0,0)+0(0,1,0)+0(0,0,1)$$
$$T(0,1,0)=(0,1,0)=0(1,0,0)+1(0,1,0)+0(0,0,1)$$
$$T(0,0,1)=(0,0,0)=0(1,0,0)+0(0,1,0)+0(0,0,1)$$

Thus, the matrix of linear transformation is $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

(ii) Find the matrix of the linear transformation $T:R^3\to R^3$ defined by $T(x_1,x_2,x_3)=(2x_1,x_2,0)$ with respect to the standard basis for R^3.

Sol: The standard bases for R^3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$. Now $T(x_1,x_2,x_3)=(2x_1,x_2,0)$ gives

$$T(1,0,0)=(2,0,0)=2(1,0,0)+0(0,1,0)+0(0,0,1)$$
$$T(0,1,0)=(0,1,0)=0(1,0,0)+1(0,1,0)+0(0,0,1)$$
$$T(0,0,1)=(0,0,0)=0(1,0,0)+0(0,1,0)+0(0,0,1)$$

Thus, the matrix of linear transformation is $A=\begin{bmatrix}2&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

(iii) Find the matrix of the linear transformation $T:R^3\to R^3$ defined by $T(x_1,x_2,x_3)=(x_1,2x_2,x_3)$ with respect to the standard basis for R^3.

Sol: The standard bases for R^3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$. Now $T(x_1,x_2,x_3)=(x_1,2x_2,x_3)$ gives

$$T(1,0,0)=(1,0,0)=1(1,0,0)+0(0,1,0)+0(0,0,1)$$
$$T(0,1,0)=(0,2,0)=0(1,0,0)+2(0,1,0)+0(0,0,1)$$
$$T(0,0,1)=(0,0,1)=0(1,0,0)+0(0,1,0)+1(0,0,1)$$

Thus, the matrix of linear transformation is $A=\begin{bmatrix}1&0&0\\0&2&\\0&0&1\end{bmatrix}$.

(iv) Find the matrix of the linear transformation $T:R^3\to R^3$ defined by $T(x_1,x_2,x_3)=(x_1+x_2,-x_1-x_2,x_3)$ with respect to the standard basis for R^3.

Sol: The standard bases for R^3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$. Now $T(x_1,x_2,x_3)=(x_1+x_2,-x_1-x_2,x_3)$ gives

$$T(1,0,0)=(1,-1,0)=1(1,0,0)-1(0,1,0)+0(0,0,1)$$
$$T(0,1,0)=(1,-1,0)=1(1,0,0)-1(0,1,0)+0(0,0,1)$$

$$T(0,0,1)=(0,0,1)=0(1,0,0)+0(0,1,0)+1(0,0,1)$$

Thus, the matrix of linear transformation is $A=\begin{bmatrix}1 & 1 & 0\\ -1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

(v) **Find the matrix of linear transformation $T:R\to R^2$ defined by $T(x)=(2x,5x)$ with respect to the standard bases of R and R^2.**

Sol: Standard bases for R and R^2 are $\{e\},\{e_1,e_2\}$ respectively, where $e=(1),e_1=(1,0),e_2=(0,1)$. Now $T(x)=(2x,5x)$ gives

$$T(e)=T(1)=(2,5)=(2,0)+(0,5)=2(1,0)+5(0,1)=2e_1+5e_2$$

This shows that the matrix of linear transformation is $A=\begin{bmatrix}2\\ 5\end{bmatrix}$.

Long Questions

2. **Find the matrix of the linear transformation $T:R^3\to R^4$ defined by $T(x_1,x_2,x_3)=(x_1+x_2,x_2+x_3,x_1-x_3,x_1)$ with respect to the standard bases for R^3 and R^4.**

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Sol: $T(x_1,x_2,x_3)=(x_1+x_2,x_2+x_3,x_1-x_3,x_1)$ ...(1)

Standard basis for R^3 is $B=\{(1,0,0),(0,1,0),(0,0,1)\}$, so by (1)

$$T(1,0,0)=(1,0,1,1)=1(1,0,0,0)+0(0,1,0,0)+1(0,0,1,0)+1(0,0,0,1)$$
$$T(0,1,0)=(1,1,0,0)=1(1,0,0,0)+1(0,1,0,0)+0(0,0,1,0)+0(0,0,0,1)$$
$$T(0,0,1)=(0,1,-1,0)=0(1,0,0,0)+1(0,1,0,0)-1(0,0,1,0)+0(0,0,0,1)$$

Thus, the matrix of linear transformation T is $A=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & -1\\ 1 & 0 & 0\end{bmatrix}$.

3. **Find the matrix of the linear transformation $T:R^3\to R^4$ defined by $T(x_1,x_2,x_3)=(2x_1,x_1+x_2+x_3,x_1+2x_2-x_3,x_1+3x_3)$ with respect to the standard bases for R^3 and R^4.**

Sol: $T(x_1,x_2,x_3)=(2x_1,x_1+x_2+x_3,x_1+2x_2-x_3,x_1+3x_3)$ ...(1)

Standard basis for R^3 is $B=\{(1,0,0),(0,1,0),(0,0,1)\}$, so by (1)

$$T(1,0,0)=(2,1,1,1)=2(1,0,0,0)+1(0,1,0,0)+1(0,0,1,0)+1(0,0,0,1)$$
$$T(0,1,0)=(0,1,2,0)=0(1,0,0,0)+1(0,1,0,0)+2(0,0,1,0)+0(0,0,0,1)$$
$$T(0,0,1)=(0,1,-1,3)=0(1,0,0,0)+1(0,1,0,0)-1(0,0,1,0)+3(0,0,0,1)$$

Thus, the matrix of linear transformation T is

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$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 3 \end{bmatrix}$$

4. Find the matrix of the linear transformation $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, -x_2, x_3)$ with respect to the standard basis for R^3.

Sol: The standard bases for R^3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$.

Now $T(x_1, x_2, x_3) = (x_1 + x_2, -x_2, x_3)$ gives

$$T(1,0,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$
$$T(0,1,0) = (1,-1,0) = 1(1,0,0) - 1(0,1,0) + 0(0,0,1)$$
$$T(0,0,1) = (0,0,1) = 0(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

Thus, the matrix of linear transformation is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

5. Find the matrix of the linear transformation $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$ with respect to the standard basis for R^3.

Sol: The standard bases for R^3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$.

Now $T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$ gives

$$T(1,0,0) = (0,-1,0) = 0(1,0,0) - 1(0,1,0) + 0(0,0,1)$$
$$T(0,1,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$
$$T(0,0,1) = (0,0,-1) = 0(1,0,0) + 0(0,1,0) - 1(0,0,1)$$

Thus, the matrix of linear transformation is $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

6. Find the matrix of the linear transformation $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_2, x_1, x_3)$ with respect to the standard basis for R^3.

Sol: The standard bases for R^3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$.

Now $T(x_1, x_2, x_3) = (x_2, x_1, x_3)$ gives

$$T(1,0,0) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$$
$$T(0,1,0) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$$
$$T(0,0,1) = (0,0,-1) = 0(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

Thus, the matrix of linear transformation is $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7. **Find the matrix of the linear transformation $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_1, x_2 + x_3, x_1 + x_2 + x_3)$ with respect to the standard basis for R^3.**

Sol: The standard bases for R^3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$.

Now $T(x_1,x_2,x_3) = (x_1,x_2+x_3,x_1+x_2+x_3)$ gives

$$T(1,0,0) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$
$$T(0,1,0) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$$
$$T(0,0,1) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

Thus, the matrix of linear transformation is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

8. **A linear transformation $T: R^2 \to R^3$ maps the vector $(1,1)$ into $(0,1,3)$ and the vector $(-1,1)$ into $(2,1,0)$. What matrix does T represent with respect to the standard bases for R^2 and R^3?**

Sol: Here $T(1,1) = (0,1,3)$ and $T(-1,1) = (2,1,0)$. Let $x = (x_1,x_2) \in R^2$ and $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

be the matrix of linear transformation $T: R^2 \to R^3$, then

$$T(x) = Ax = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \\ ex_1 + fx_2 \end{bmatrix}$$

$$\Rightarrow T(x_1,x_2) = (ax_1 + bx_2, cx_1 + dx_2, ex_1 + fx_2)$$
$$\Rightarrow T(1,1) = (a+b, c+d, e+f)$$
$$\Rightarrow (0,1,3) = (a+b, c+d, e+f)$$
$$\Rightarrow a+b = 0,\ c+d = 1,\ e+f = 3 \qquad ...(1)$$

And $T(-1,1) = (2,1,0)$

$$\Rightarrow (-a+b, -c+d, -e+f) = (2,1,0)$$
$$\Rightarrow -a+b = 2, -c+d = 1, -e+f = 0 \qquad ...(2)$$

Solving (1) and (2), we have

$$a = -1, b = 1, c = 0, d = 1, e = \tfrac{3}{2}, f = \tfrac{3}{2}$$

Putting these values in matrix A, we have $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$.

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9. **A linear transformation $T:R^2 \to R^3$ maps the vector $(1,1)$ into $(0,1,1)$ and the vector $(-1,0)$ into $(1,1,0)$. What matrix does T represent with respect to the standard bases for R^2 and R^3?**

Sol: Here $T(1,1)=(0,1,1)$ and $T(-1,0)=(1,1,0)$. Let $x=(x_1,x_2)\in R^2$ and $A=\begin{bmatrix} a & b \\ c & d \\ e & f\end{bmatrix}$ be the matrix of linear transformation $T:R^2\to R^3$, then

$$T(x)=Ax=\begin{bmatrix} a & b \\ c & d \\ e & f\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix}=\begin{bmatrix} ax_1+bx_2 \\ cx_1+dx_2 \\ ex_1+fx_2\end{bmatrix}$$

$$\Rightarrow T(x_1,x_2)=(ax_1+bx_2,cx_1+dx_2,ex_1+fx_2)$$

$$\Rightarrow T(1,1)=(a+b,c+d,e+f)$$

$$\Rightarrow (0,1,1)=(a+b,c+d,e+f)$$

$$\Rightarrow a+b=0,c+d=1,e+f=1 \qquad \ldots(1)$$

And $\quad T(-1,0)=(1,1,0)\Rightarrow(-a+b,-c+d,-e+f)=(1,1,0)$

$$\Rightarrow -a+b=1,-c+d=1,-e+f=0 \qquad \ldots(2)$$

Solving (1) and (2), we have

$$a=-\frac{1}{2},b=\frac{1}{2},c=0,d=1,e=\frac{1}{2},f=\frac{1}{2}$$

Putting these values in matrix A, we have $A=\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$.

10. **Find the matrix of linear transformation $T:R\to R^2$ defined by $T(x)=(x,4x)$ with respect to the standard bases of R and R^2.**

Sol: Standard bases for R and R^2 are $\{e\},\{e_1,e_2\}$ respectively, where $e=(1),e_1=(1,0),e_2=(0,1)$. Now $T(x)=(x,4x)$ gives

$$T(e)=T(1)=(1,4)=(1,0)+(0,4)=1(1,0)+4(0,1)=1e_1+4e_2$$

This shows that the matrix of linear transformation is $A=\begin{bmatrix}1\\4\end{bmatrix}$.

11. **Find the matrix of linear transformation $T:R^3\to R^2$ defined by $T(x_1,x_2,x_3)=(x_1-4x_2+9x_3,5x_1+3x_2-2x_3)$ with respect to the standard bases of R^3 and R^2.**

Sol: $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{(1,0),(0,1)\}$ are the standard bases for R^3 and R^2 are respectively. Now

$T(x_1,x_2,x_3)=(x_1-4x_2+9x_3,5x_1+3x_2-2x_3)$ gives

$\lfloor 4 \quad 0 \rfloor$

17. **The matrix of linear transformation $T: R^3 \to R^3$ is $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.**

Determine T in terms of coordinates of a vector $x \in R^3$.

<u>Sol</u>: Let $x = (x_1, x_2, x_3) \in R^3$, then $T(x) = Ax$ gives

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 + 2x_3 \\ 0x_1 + x_2 + x_3 \\ -x_1 + x_2 + x_3 \end{bmatrix}$$

$$\Rightarrow T(x_1, x_2, x_3) = (3x_1 + x_2 + 2x_3, x_2 + x_3, -x_1 + x_2 + x_3)$$

18. **The matrix of linear transformation $T: R^3 \to R^3$ is $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.**

Determine T in terms of coordinates of a vector $x \in R^3$.

Sol: Let $x = (x_1, x_2, x_3) \in R^3$, then $T(x) = Ax$ gives

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 + 2x_3 \\ 0x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$$

$$\Rightarrow T(x_1, x_2, x_3) = (3x_1 + x_2 + 2x_3, x_2 + x_3, x_1 + x_2 + x_3)$$

19. **The matrix $\begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 \end{bmatrix}$ is the matrix of linear transformation**

$T: R^n \to R^m$ Determine m, n and express T in terms of coordinates.

Sol: Since the number of columns of A determines n, so $n = 5$, and the number of rows of A determines m, so $m = 3$, therefore, we take $x = (x_1, x_2, x_3, x_4, x_5)$.
Now $T(x) = Ax$ gives

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 + 2x_4 \\ x_1 + x_3 + x_5 \\ -x_2 + x_4 + x_5 \end{bmatrix}$$

or $T(x_1, x_2, x_3, x_4, x_5) = (3x_1 + x_2 + 2x_4, x_1 + x_3 + x_5, -x_2 + x_4 + x_5)$

20. **The matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ is the matrix of linear transformation**