2D

LINEAR EQUATIONS

29. Definitions

Consider a system of m linear equations in n unknowns $x_1, x_2, \ldots\ldots, x_n$

i.e.,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \quad \ldots(1)$$

in which the coefficients a_{ij} and the constant terms b_i are from a fixed number field F. (This F will be the field of rationals, reals or complex numbers). *By a solution of the system (1), we mean an ordered n—tuple $\alpha = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$ of elements of F if on substituting $x_1 = \alpha_1$, $x_2 = \alpha_2$, ..., $x_n = \alpha_n$, all the m equations in (1) are simultaneously satisfied. When the system has a solution, it is said to be* ***consistent*** *otherwise the system is said to be* ***inconsistent.*** *The set of all solutions of a consistant system is called the solution set of the system.*

If b_i's are zero, then the system (1) is called ***homogeneous*** *system, otherwise (i.e. Even if at least one b_i is non-zero) a* ***non-homogeneous*** *system. Observe that a homogeneous system always has $x_1 = 0, x_2 = 0, \ldots\ldots, x_n = 0$ as a solution. So, a homogeneous system is always consistent. The solution $x_1 = x_2 = \ldots\ldots = x_n = 0$ is called the trivial solution of the system.*

We can write the system (1) in the matrix notation as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots\ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$$

or

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ is the **coefficient matrix,**

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

For the system (1), we may consider the **augmented matrix**

$$[A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$$

each row of which corresponds to an equation of (1).

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Following examples show that a system of linear equations may have no solution, unique solution or has infinitely many solutions.

Example 1. *Consider the system $2x + y = 1$, $x - 2y = -7$, $x + 3y = 6$ of three equations in two unknowns. This system has no solution.*

For, if $x = x_1$ and $y = y_1$ satisfy the first two equations, then

$$2x_1 + y_1 = 1$$
$$x_1 - 2y_1 = -7$$

Putting $x = x_1$ and $y = y_1$ in L.H.S. of third equation

$$x_1 + 3y_1 = (2x_1 + y_1) - (x_1 - 2y_1) = 1 - (-7) = 8 \neq 6$$

$\therefore$ $x = x_1, y = y_1$ does not satisfy the third equation. Hence, the system has no solution.

Example 2. *Consider the system $2x - 3y = 1$, $x + y = 3$.*

On solving for x and y

This system has (2, 1) as the **only solution.**

Example 3. *Consider the system*

$$x + y = 3$$
$$2x + 2y = 6.$$

In this case, any pair (x_1, y_1) which satisfies the first equation *i.e.* $x_1 + y_1 = 3$, automatically satisfies the second equation $\because$ $(2x_1 + 2y_1 = 2(x_1 + y_1) = 2 \times 3 = 6)$. Therefore, it is sufficient to solve the first equation. To solve the first equation take y arbitrarily, say $y = y_1$. Then, $x = 3 - y_1$, $y = y_1$ is a solution of the system for every value of y_1. Hence, the system has infinitely many solutions.

We now, observe some important results about the solution set of a homogeneous and a non-homogeneous system.

30. Theorem 21

If $AX = O$ is a homogeneous system of equations in n unknowns and $X_1 = (x_1, x_2, \ldots\ldots, x_n)$ and $X_2 = (y_1, y_2, \ldots\ldots, y_n)$ are two solutions of this system, then $X_1 + X_2 = (x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n)$ is also a solution. Also, if λ is a scalar, then $\lambda X_1 = (\lambda x_1, \lambda x_2, \ldots\ldots, \lambda x_n)$ is also a solution. 21

Proof. Since X_1 and X_2 are solutions of $AX = O$, then

$$AX_1 = O \text{ and } AX_2 = O.$$

Now $A(X_1 + X_2) = AX_1 + AX_2 = O + O = O$

$\Rightarrow$ $X_1 + X_2$ is a solution.

Again, X_1 is also a solution.

$$\therefore \qquad AX_1 = O$$

Now $A(\lambda X_1) = \lambda(AX_1) = \lambda O = O$

$\Rightarrow$ λX_1 is also a solution.

22 **Cor. 1.** *Any linear combination of solutions of $AX = O$ is also a solution of $AX = O$.*

Cor. 2. ***A system of homogeneous equations either has only the trivial solution or infinitely many solutions.*** 22

For, if $AX = O$ has a non-trivial solution X_1 then λX_1, for $\lambda \in F$ are different solutions of $AX = O$.

Now, consider the non-homogeneous system of equations $AX = B$ and associate with the system of homogeneous equations $AX = O$. The following theorem gives the connections between the solutions of the system $AX = B$ and the solutions of $AX = O$.

31. Theorem 23

If X_1 is a solution of $AX = B$ and X_2 is any solution of the associated system $AX = O$, the $X_1 + X_2$ is a solution of $AX = B$. Further if Y is a solution of $AX = B$, then $Y - X_2$ is a solution $AX = O$. 23

Proof. Since X_1 is a solution of $AX = B$ and X_2 is a solution of $AX = O$.

$\therefore$ $$AX_1 = B \text{ and } AX_2 = O$$

Now $$A(X_1 + X_2) = AX_1 + AX_2 = B + O = B$$

$\Rightarrow$ $X_1 + X_2$ is a solution of $AX = B$.

Again, since Y is a solution of $AX = B$, therefore $AY = B$

Also, $$AX_1 = B$$

Now $$A(Y - X_1) = AY - AX_1 = B - B = O$$

Hence, $Y - X_1$ is a solution of $AX = O$.

24 **Cor. A system of non-homogeneous equations $AX = B$ either has no solution, unique solution or infinitely many solutions.** 24

Proof. Let $AX = B$ be a non-homogeneous system. If it has no solution, then we ha nothing to prove.

Let X_1 be a solution of this system. If $AX = O$ has only the trivial solution, then X_1 is t unique solution of $AX = B$; otherwise corresponding to every solution Y of $AX = O$, we have solution $X_1 + Y$ of $AX = B$ and hence an infinity of solutions.

32. Equivalent Systems

Two systems of linear equations over the field F and in the same number of unknow are said to be equivalent if every solution of either system is a solution of the other, i.e., th have the same solution set.

A system of equations equivalent to (1) may be obtained from it by applying one or mo transformations :

(*i*) interchanging any two equations, (*ii*) multiplying any equation by some non-ze constant in F or (*iii*) adding to any equation a constant multiple of another equation.

To solve a given system of equations, it is sufficient to solve any system equivalent to So, we replace the given system by an equivalent one which is easily solvable. That is, we to reduce the system to such a system from which the solution can be easily determined. instance, if after successive reduction, we arrive at a system which has its co-efficient in the diagonal form, then the solution if exists, can be immediately written down.

Similarly, if the reduced system has its co-efficient matrix in the triangular form w non-zero diagonal entries, then the solution can be determined by successively solving th equations.

Consider an example of the system of **non-homogeneous** linear equations :

$$\left.\begin{aligned} x + 2y - z &= 3 \quad &...(i)\\ 3x - y + 2z &= 1 \quad &...(ii)\\ 2x - 2y + 3z &= 2 \quad &...(iii) \end{aligned}\right\}$$

How do we solve it ? We first try to eliminate one of the variables, say x. To do this, add to equation (*ii*), (– 3) times equation (*i*) and add to (*iii*), (– 2) times (*i*). The resultant syst is

$$\left.\begin{aligned} x + 2y - z &= 3\\ -7y + 5z &= -8\\ -6y + 5z &= -4 \end{aligned}\right\}$$

Clearly, the systems (1) and (2) are equivalent. Let us eliminate y between the last two equations of (2). To do this, we add to the third equation. $\left(\frac{-6}{7}\right)$ times the second equation. Then, the system (2) reduces to

$$\left.\begin{aligned} x + 2y - z &= 3 \\ -7y + 5z &= -8 \\ \frac{5}{7}z &= \frac{20}{7} \end{aligned}\right\} \quad ...(3)$$

To avoid fraction, we multiply the third equation by 7 and get the equivalent system

$$\left.\begin{aligned} x + 2y - z &= 3 \\ -7y + 5z &= -8 \\ 5z &= 20 \end{aligned}\right\} \quad ...(4)$$

All the above systems (1) to (4) have the same solution set. We obtain the solution set easily from the system (5) by solving for z, y and x successively ; namely $z = 4, y = 4, x = -1$. Thus, the solution is $x = -1, y = 4, z = 4$.

In matrix notation the system (1) is

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

and the system (4) is $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 20 \end{bmatrix}.$$

What we have done is that we have reduced the **augmented matrix**

$$\begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 3 & -1 & 2 & : & 1 \\ 2 & -2 & 3 & : & 2 \end{bmatrix}$$

of system (1) to a row equivalent matrix

$$\begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 5 & : & -8 \\ 0 & 0 & 5 & : & 20 \end{bmatrix} \text{ of system (4).}$$

Thus, we arrive at the general method : *We successively perform suitable elementary row operations on the augmented matrix [A : B] and obtain equivalent systems of linear equations. We manipulate the row operations so that the co-efficient matrix is transformed either to a triangular matrix or row-echelon form, from which the consistency or otherwise of the system can be easily tested and solution can be easily written.*

Let us consider some examples.

Example 1. *Consider the system of equations* $x_1 + x_2 + x_3 = 4, 2x_1 + 5x_2 - 2x_3 = 3, x_1 + 7x_2 - 7x_3 = 5$.

Sol. The given system can be written as AX = B where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

We apply row operations to the augmented matrix [A : B]

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 2 & 5 & -2 & : & 3 \\ 1 & 7 & -7 & : & 5 \end{bmatrix}$$

$$\underset{R_{3,1}(-1)}{\overset{R_{2,1}(-2)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 3 & -4 & : & -5 \\ 0 & 6 & -8 & : & 1 \end{bmatrix}$$

$$\overset{R_{3,2}(-2)}{\sim} \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 3 & -4 & : & -5 \\ 0 & 6 & 0 & : & 11 \end{bmatrix}$$

$$\underset{R_3\left(\frac{1}{11}\right)}{\overset{R_2\left(\frac{1}{3}\right)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 1 & -\frac{4}{3} & : & -\frac{5}{3} \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$$

Thus, the system AX = B is equivalent to

$$x_1 + x_2 + x_3 = 4$$

$$x_2 - \frac{4}{3}x_3 = -\frac{5}{3}$$

$$0x_1 + 0x_2 + 0x_3 = 1 \quad \text{or} \quad 0 = 1 \text{ which is impossible.}$$

Thus, the system is inconsistent as the third equation is never satisfied.

Here, $\rho(A) = \rho_R(A) = 2$ (the number of non-zero rows) and

$$\rho([A : B]) = 3. \text{ (the number of non-zero rows)}$$

So, $\rho(A) \neq \rho[A : B]$.

Example 2. *Consider the system*

$$5x_1 + 3x_2 + 14x_3 = 4,\ x_2 + 2x_3 = 1,\ x_1 - x_2 + 2x_3 = 0,\ 2x_1 + x_2 + 6x_3 = 2.$$

Sol. The augmented matrix [A : B] is $\begin{bmatrix} 5 & 3 & 14 & : & 4 \\ 0 & 1 & 2 & : & 1 \\ 1 & -1 & 2 & : & 0 \\ 2 & 1 & 6 & : & 2 \end{bmatrix}$

$$\overset{R_{1,3}}{\sim} \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 5 & 3 & 14 & : & 4 \\ 2 & 1 & 6 & : & 2 \end{bmatrix}$$

$$\underset{R_{4,1}(-2)}{\overset{R_{3,1}(-5)}{\sim}} \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 8 & 4 & : & 4 \\ 0 & 3 & 2 & : & 2 \end{bmatrix}$$

$$\underset{R_{4,2}(-3)}{\overset{R_{3,2}(-8)}{\sim}}\begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & -12 & : & -4 \\ 0 & 0 & -4 & : & -1 \end{bmatrix}$$

$$\overset{R_3\left(-\frac{1}{12}\right)}{\sim}\begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 1 & : & \frac{1}{3} \\ 0 & 0 & -4 & : & -1 \end{bmatrix}$$

$$\overset{R_{4,3}(4)}{\sim}\begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 1 & : & \frac{1}{3} \\ 0 & 0 & 0 & : & \frac{1}{3} \end{bmatrix}, \quad \overset{R_4(3)}{\sim}\begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 1 & : & \frac{1}{3} \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$$

The given system is equivalent to

$$x_1 - x_2 + 2x_3 = 0$$

$$x_2 + 2x_3 = 1$$

$$x_3 = \frac{1}{3}$$

$$0x_1 + 0x_2 + 0x_3 = 1 \quad \text{or } 0 = 1 \text{ which is impossible.}$$

Thus the system is inconsistent as the fourth equation is never satisfied.

Here $\rho(A) = 3$, $\rho([A : B]) = 4$ so $\rho(A) \neq \rho([A : B])$.

Example 3. *Consider the equations* $x_1 + 2x_2 - x_3 = 6$, $3x_1 - x_2 - 2x_3 = 3$, $4x_1 + 3x_2 + x_3 = 9$.

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 2 & -1 & : & 6 \\ 3 & -1 & -2 & : & 3 \\ 4 & 3 & 1 & : & 9 \end{bmatrix}$$

$$\underset{R_{3,1}(-4)}{\overset{R_{2,1}(-3)}{\sim}}\begin{bmatrix} 1 & 2 & -1 & : & 6 \\ 0 & -7 & 1 & : & -15 \\ 0 & -5 & 5 & : & -15 \end{bmatrix}$$

$$\overset{R_2(2)}{\sim}\begin{bmatrix} 1 & 2 & -1 & : & 6 \\ 0 & -14 & 2 & : & -30 \\ 0 & -5 & 5 & : & -15 \end{bmatrix}$$

$$\overset{R_{2,3}(-3)}{\sim}\begin{bmatrix} 1 & 2 & -1 & : & 6 \\ 0 & 1 & -13 & : & 15 \\ 0 & -5 & 5 & : & -15 \end{bmatrix}$$

$$\underset{\sim}{R_{3,2}(5)} \begin{bmatrix} 1 & 2 & -1 & : & 6 \\ 0 & 1 & -13 & : & 15 \\ 0 & 0 & -60 & : & 60 \end{bmatrix}$$

$$\underset{\sim}{R_3\left(-\frac{1}{60}\right)} \begin{bmatrix} 1 & 2 & -1 & : & 6 \\ 0 & 1 & -13 & : & 15 \\ 0 & 0 & 1 & : & -1 \end{bmatrix}$$

The given system is equivalent to

$$x_1 + 2x_2 - x_3 = 6$$
$$x_2 - 13x_2 = 15$$
$$x_3 = -1.$$

Solving these equations successively, we have

$$x_3 = -1, x_2 = 15 + 13x_3 = 2$$

and $\quad x_1 = 6 - 2x_2 + x_3 = 6 - 4 - 1 = 1$

Thus, $\quad x_1 = 1, x_2 = 2, x_3 = -1$ is the only (unique) solution.

Here, $\quad \rho(A) = 3 = \rho([A : B])$. (number of non-zero rows)

Example 4. *Consider the system of equations* $x_1 + 2x_2 + 2x_3 = 1$, $2x_1 + x_2 + x_3 = 2$, $3x_1 +$ $+ 2x_3 = 3$, $\quad x_2 + x_3 = 0$.

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 2 & 1 & 1 & : & 2 \\ 3 & 2 & 2 & : & 3 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$$\underset{R_{3,1}(-3)}{\overset{R_{2,1}(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & -3 & -3 & : & 0 \\ 0 & -4 & -4 & : & 0 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$$\underset{R_3\left(-\frac{1}{4}\right)}{\overset{R_2\left(-\frac{1}{3}\right)}{\sim}} \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$$\underset{R_{4,2}(-1)}{\overset{R_{1,2}(-2),\, R_{3,2}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

which is in row-echelon form.

Thus, the system is equivalent to $\quad x_1 = 1$

$$x_2 + x_3 = 0$$

Let $\quad x_2 = \lambda \quad \therefore \quad \lambda + x_3 = 0 \quad \text{or} \quad x_3 = -\lambda$

The solution set of this system is $x_1 = 1$, $x_2 = \lambda$, $x_3 = -\lambda$, where λ is any real number. Hence the system has infinitely many solutions for different values of λ.

Here, $\rho(A) = \rho([A : B]) = 2 (<$ number of unknowns).

The above examples clearly show that a given system of equations AX = B is consistent if $\rho(A) = \rho([A : B])$. The result is true, in general and is given in the following theorem due to Kronecker and Capalli.

33. Theorem 25

A system AX = B of m linear equations in n unknowns is consistent if and only if the coefficient matrix A and the augmented matrix [A : B] of the system have the same rank. 25

Proof. Since A is a submatrix of the augmented matrix

$$[A : B],\ \rho(A) \le \rho([A : B]).$$

Now, by applying elementary row operations to the augmented matrix [A : B], we reduce it to row reduced echelon matrix

$$\begin{array}{r} \\ \\ \\ \\ r\text{th row} \\ \\ \\ \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 & : & \alpha_1 \\ 0 & 1 & & 0 & : & \alpha_2 \\ 0 & 0 & & 0 & : & \alpha_3 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & & 1 & : & \alpha_r \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & & 0 & : & \alpha_m \end{bmatrix}$$

If at least one of $\alpha_{r+1}, \alpha_{r+2}, \ldots\ldots, \alpha_m$ is not zero, then the number of non-zero rows in the coefficient matrix is not equal to the number of non-zero rows in the augmented matrix.

Hence A and [A : B] do not have the same rank,

i.e., $$\rho(A) < \rho([A : B]).$$

But, this shows that at least one of the equations is $\alpha_j = 0$, $(j = r + 1, r + 2, \ldots, m)$ which is false. Hence, the system is inconsistent.

If $\alpha_{r+1} = \alpha_{r+2} = \ldots = \alpha_m = 0$, then $\rho(A) = \rho([A : B])$ and the system has one or more solutions.

34. Theorem 26

In a consistent system AX = B of rank r < n (number of unknowns) n − r of the unknowns can be assigned values arbitrarily and the other r unknowns can be expressed uniquely in terms of the n − r arbitrarily chosen unknowns. 26

Proof. Proof follows from the fact that when [A : B] is reduced to the row-echelon form $[A_R : C]$, then $[A_R : C]$ has exactly r non-zero rows.

35. Theorem

27 *A system AX = B of n non-homogeneous equations in n unknowns has a unique solution provided A is non-singular, i.e.* $\rho(A) = n$. 27

Proof. Let A be non-singular.

Then, using elementary row operations, [A : B] can be reduced to [I : C], where I is the identity matrix of order n. Then, X = C is a solution of the system.

Let, if possible, $X = D$ be a second solution of the system.

Then, $$AD = B$$

Also, $$AC = B$$

$$\Rightarrow \quad AD = AC$$

Since A is non-singular, A^{-1} exists.

Pre-multiplying both sides of (1) by A^{-1}, we have

$$A^{-1}(AD) = A^{-1}(AC)$$

$$\Rightarrow \quad (A^{-1}A)D = (A^{-1}A)C$$

$$\Rightarrow \quad ID = IC \Rightarrow D = C$$

So, the solution is unique.

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36. Finally, we **summarise :**

The system AX = B where $B \neq 0$ of m linear non-homogeneous equations in n unknowns has

(*i*) **no solution if $\rho(A) \neq \rho([A : B])$**

(*ii*) **a unique solution if $\rho(A) = \rho([A : B]) = n$**

(*iii*) **an infinite number of solutions**

if $\rho(A) = \rho([A : B]) = r < n$

In this case, the solution set involves $n—r$ parameters. 28

Example 1. *Examine the consistency of the following equations and if consistent, find the complete solution.* $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$.

Sol. Augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 14 \\ 1 & 4 & 7 & : & 30 \end{bmatrix}$$

$$\underset{R_{3,1}(-1)}{\overset{R_{2,1}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 8 \\ 0 & 3 & 6 & : & 24 \end{bmatrix}$$

$$\overset{R_{3,2}(-3)}{\sim} \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 8 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$ which is row-Echelon form.

Clearly, $$\rho(A) = \rho([A : B]) = 2.$$

Hence, the system is consistent. (By Art. 36)

Since $\rho(A) < 3$ (number of unknowns), the system has an **infinite** number of solutions. (By Art. 36)

Now the system reduces to

$$x + y + z = 6$$
$$y + 2z = 8$$

$$\Rightarrow \quad y = 8 - 2z, x = 6 - y - z = 6 - (8 - 2z) - z = z - 2$$

Taking $z = \lambda$, we have

$x = \lambda - 2, y = 8 - 2\lambda, z = \lambda$ (where λ is arbitrary) as the general solution.

Example 2. *Discuss the consistency of the system of equations*

$$x - y + z + 1 = 0$$
$$x - y + z - 1 = 0$$
$$x - y - z + 1 = 0.$$

Sol. The equations can be written as

$$x - y + z = -1$$
$$x - y + z = 1$$
$$x - y - z = -1.$$

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & -1 & 1 & : & -1 \\ 1 & -1 & 1 & : & 1 \\ 1 & -1 & -1 & : & -1 \end{bmatrix}$$

$$\underset{R_{3,1}(-1)}{\overset{R_{2,1}(-1)}{\sim}} \begin{bmatrix} 1 & -1 & 1 & : & -1 \\ 0 & 0 & 0 & : & 2 \\ 0 & 0 & -2 & : & 0 \end{bmatrix}$$

$$\overset{R_{3,2}}{\sim} \begin{bmatrix} 1 & -1 & 1 & : & -1 \\ 0 & 0 & -2 & : & 0 \\ 0 & 0 & 0 & : & 2 \end{bmatrix}$$

$$\overset{R_2(-1/2)}{\sim} \begin{bmatrix} 1 & -1 & 1 & : & -1 \\ 0 & 0 & 1 & : & 0 \\ 0 & 0 & 0 & : & 2 \end{bmatrix} \text{ which is row-echelon form.}$$

Clearly $\rho(A) = 2$ and $\rho([A : B]) = 3$. (number of non-zero rows)

Since $\rho(A) \neq \rho([A : B])$, so the system is not consistent. (By Art. 36)

Example 3. *Solve the equations* $2x_1 - x_2 + 3x_3 = 3,\quad x_1 + 2x_2 - x_3 - 5x_4 = 4,\quad x_1 + 3x_2 - 2x_3 - 7x_4 = 5.$

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix} 2 & -1 & 3 & 0 & : & 3 \\ 1 & 2 & -1 & -5 & : & 4 \\ 1 & 3 & -2 & -7 & : & 5 \end{bmatrix}$$

$$\overset{R_{1,2}}{\sim} \begin{bmatrix} 1 & 2 & -1 & -5 & : & 4 \\ 2 & -1 & 3 & 0 & : & 3 \\ 1 & 3 & -2 & -7 & : & 5 \end{bmatrix}$$

$$\overset{R_{2,1}(-2)}{\sim} \begin{bmatrix} 1 & 2 & -1 & -5 & : & 4 \\ 0 & -5 & 5 & 10 & : & -5 \\ 0 & 1 & -1 & -2 & : & 1 \end{bmatrix}$$

$$\underset{\sim}{\overset{R_{3,1}(-1)}{R_2(-1/5)}} \begin{bmatrix} 1 & 2 & -1 & -5 & : & 4 \\ 0 & 1 & -1 & -2 & : & 1 \\ 0 & 1 & -1 & -2 & : & 1 \end{bmatrix}$$

$$\overset{R_{32}(-1)}{\sim}\begin{bmatrix}1 & 2 & -1 & -5 & : & 4\\0 & 1 & -1 & -2 & : & 1\\0 & 0 & 0 & 0 & : & 0\end{bmatrix} \text{ which is row-echelon form.}$$

$\therefore$ $\rho(A) = \rho([A : B]) = 2$ (number of non-zero rows)

$\therefore$ the system is consistent. (By Art. 36)

Since $\rho(A) = 2 < 4$ (number of unknowns), the number of solutions is infinite (By Art 36) and the solutions will contain $4 - 2 = 2$ arbitrary constants.

The system reduces to $x_1 + 2x_2 - x_2 - 5x_4 = 4$

and $x_2 - x_3 - 2x_4 = 1$

Taking $x_3 = a$ and $x_4 = b$, we have

$x_2 = 1 + x_3 + 2x_4 = 1 + a + 2b$

and $x_1 = 4 - 2x_2 + x_3 + 5x_4 = 4 - 2(1 + a + 2b) + a + 5b = 2 - a + b$

Here, $x_1 = 2 - a + b$, $x_2 = 1 + a + 2b$, $x_3 = a$, $x_4 = b$ (where a, b are arbitrary constants) is the general solution.

Example 4. *For what value of λ, the system* $\begin{bmatrix}1 & 2\\3 & \lambda\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$ *has (i) no solution (ii) unique solution (iii) more than one solution ?*

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix}1 & 2 & : & 1\\3 & \lambda & : & 3\end{bmatrix}$$

$$\overset{R_{21}(-3)}{\sim}\begin{bmatrix}1 & 2 & : & 1\\0 & \lambda-6 & : & 0\end{bmatrix} \qquad \ldots(1)$$

Case I. $\lambda - 6 \neq 0$ i.e. $\lambda \neq 6$

Operating $R_2\left(\dfrac{1}{\lambda-6}\right)$ in (1), from $[A : B] \sim \begin{bmatrix}1 & 2 & : & 1\\0 & 1 & : & 0\end{bmatrix}$ which is row-echelon form.

Here $\rho(A) = \rho[A : B] = 2$ (= number of unknowns).

$\therefore$ The given system has a unique solution. (By Art. 36)

The system reduces to $x + 2y = 1$ and $y = 0$

Putting $y = 0$, we have $x = 1$

$\therefore$ The unique solution is $x = 1, y = 0$.

Case II. $\lambda - 6 = 0$ i.e. $\lambda = 6$

$\therefore$ From (1) $[A : B] \sim \begin{bmatrix}1 & 2 & : & 1\\0 & 0 & : & 0\end{bmatrix}$ which is row-echelon form.

Here $\rho(A) = \rho[A : B] = 1$ (Number of non-zero rows) < 2 (number of unknowns).

Therefore the given system of equations is consistent and has infinitely many solutions.

The reduced equations are $x + 2y = 1$.

Let $y = k$ $\therefore$ $x + 2k = 1$ $\therefore$ $x = 1 - 2k$ let $y = k$

Hence the general solution is $x = 1 - 2k, y = k$

where k is an arbitary constant.

Hence the system is consistent for all values of k.

S.S = $\{(x, y)\} = \{(1-2k, k)\}$

Example 5. *Discuss the solution of the system* $\begin{bmatrix} 1 & 2 \\ 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *for different values of* λ.

Sol. Augmented matrix is $[A : B] = \begin{bmatrix} 1 & 2 & : & 1 \\ 3 & \lambda & : & 1 \end{bmatrix}$

$$\overset{R_{2,1}(-3)}{\sim} \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & \lambda - 6 & : & -2 \end{bmatrix} \quad \text{...(1)}$$

Case I. $\lambda - 6 \neq 0$ *i.e.* $\lambda \neq 6$

operating $R_2\left(\frac{1}{\lambda - 6}\right)$ in (1),

$$[A : B] \sim \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & 1 & : & \frac{-2}{\lambda - 6} \end{bmatrix} \text{ which is row-echelon form.}$$

Here $\rho(A) = \rho(A : B) = 2$ (= Number of unknowns).

$\therefore$ The given system has a unique solution. (By Art. 36)

The given system of equations reduces to $x + 2y = 1$ and $y = \frac{-2}{\lambda - 6}$

$$\therefore \qquad x = 1 - 2y = 1 + \frac{4}{\lambda - 6} = \frac{\lambda - 6 + 4}{\lambda - 6} = \frac{\lambda - 2}{\lambda - 6}$$

Hence $x = \frac{\lambda - 2}{\lambda - 6}$ and $y = \frac{-2}{\lambda - 6}$ is the unique solution for each $\lambda \neq 6$.

Case II. $\lambda - 6 = 0$ *i.e.* $\lambda = 6$

$\therefore$ From (1) $\qquad [A : B] \sim \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & 0 & : & -2 \end{bmatrix}$

operating $\qquad R_3\left(\frac{-1}{2}\right)$, $[A : B] \sim \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & 0 & : & 1 \end{bmatrix}$

which is row-echelon form.

Here $\qquad \rho(A) (= 1) \neq \rho(A : B) (= 2)$

$\therefore$ The system is inconsistent if $\lambda = 6$.

$\therefore$ For no λ, the system has infinitely many of solutions.

Example 6. *For what* λ, *the equations* $x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$ *have a solution and solve completely in each case.* (K.U. 2003)

Sol. The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & \lambda \\ 1 & 4 & 10 & : & \lambda^2 \end{bmatrix}$$

$$\underset{R_{3,1}(-1)}{\overset{R_{2,1}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda - 1 \\ 0 & 3 & 9 & : & \lambda^2 - 1 \end{bmatrix}$$