4.4 Linear Combinations and Spanning Sets

Linear Combination

A vector w is called a ***linear combination*** of the vectors $v_1, v_2, \ldots, v_r$ if it can be written in the form $w = k_1v_1 + k_2v_2 + \ldots + k_rv_r$, where $k_1, k_2, \ldots, k_r$ are scalars.

If $r = 1$, then $w = k_1v_1$.

Example 1: *Show that every vector $v = (a,b,c)$ in R^3 is expressible as a linear combination of the standard basis vectors $(1,0,0), (0,1,0)$ and $(0,0,1)$.*

Solution: Since $v = (a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$, so v can be expressible as a linear combination of the vectors $(1,0,0), (0,1,0)$ and $(0,0,1)$.

Example 2: *Write $v = (1,-2,5)$ as a linear combination of the vectors*

$$v_1 = (1,1,1), v_2 = (1,2,3), v_3 = (2,-1,1)$$

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Solution: Let a, b and c be scalars and consider

$$v = av_1 + bv_2 + cv_3 \qquad \ldots(1)$$

$$(1,-2,5) = a(1,1,1) + b(1,2,3) + c(2,-1,1) \qquad \ldots(2)$$

$$(1,-2,5) = (a,a,a) + (b,2b,3b) + (2c,-c,c)$$

$$(1,-2,5) = (a+b+2c, a+2b-c, a+3b+c)$$

$$\Rightarrow a+b+2c = 1$$

$$a+2b-c = -2$$

$$a+3b+c = 5$$

The augmented matrix of this system is

$$A_b = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right], \quad \begin{array}{l} R_2 - R_1, \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right], \quad \begin{array}{l} R_1 - R_2, \\ R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right], \quad \frac{1}{5}R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right], \quad \begin{array}{l} R_1 - 5R_3, \\ R_2 + 3R_3 \end{array}$$

This gives $a = -6,\ b = 3,\ c = 2$. Making these substitutions in (2), we have

$$(1,-2,5) = -6(1,1,1) + 3(1,2,3) + 2(2,-1,1)$$

Example 3: *Express the vector* $(9,2,7)$ *in* R^3 *as a linear combination of the vectors* $(1,2,-1)$ *and* $(6,4,2)$.

Solution: In order to express $(9,2,7)$ as a linear combination of the vectors $(1,2,-1)$ and $(6,4,2)$, we must have scalars k_1, k_2 such that

$$(9,2,7) = k_1(1,2,-1) + k_2(6,4,2) \qquad \ldots(1)$$

$$(9,2,7) = (k_1, 2k_1, -k_1) + (6k_2, 4k_2, 2k_2)$$

$$(9,2,7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$k_1 + 6k_2 = 9$$

$$2k_1 + 4k_2 = 2$$

$$-k_1 + 2k_2 = 7$$

The augmented matrix of this system is

$$A_b = \begin{bmatrix} 1 & 6 & : & 9 \\ 2 & 4 & : & 2 \\ -1 & 2 & : & 7 \end{bmatrix}$$

We reduce this matrix to echelon form as follows:

$$A_b = \begin{bmatrix} 1 & 6 & : & 9 \\ 2 & 4 & : & 2 \\ -1 & 2 & : & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & : & 9 \\ 0 & -8 & : & -16 \\ 0 & 8 & : & 16 \end{bmatrix}, \quad \begin{array}{l} R_2 - 2R_1, \\ R_3 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 6 & : & 9 \\ 0 & -8 & : & -16 \\ 0 & 0 & : & 0 \end{bmatrix}, \quad R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 6 & : & 9 \\ 0 & 1 & : & 2 \\ 0 & 0 & : & 0 \end{bmatrix}, \quad -\frac{1}{8}R_2$$

which is in echelon form. It is clear from this matrix that the rank of coefficient matrix is equal to the rank of augmented matrix, so the given system is consistent and using backward substitution, we have

$$k_1 + 6k_2 = 9,\ k_2 = 2$$

$$\Rightarrow k_1 = -3,\ k_2 = 2$$

Putting these values in (1), we have

$$(9,2,7) = -3(1,2,-1) + 2(6,4,2)$$

which is the required expression of $(9,2,7)$ as a linear combination of vectors $(1,2,-1)$ and $(6,4,2)$.

Example 4: *Show that the vector* $(4,-1,8)$ *in* R^3 *cannot be expressed as a linear combination of the vectors* $(1,2,-1)$ *and* $(6,4,2)$.

Solution: In order to express $(4,-1,8)$ as a linear combination of the vectors $(1,2,-1)$ and $(6,4,2)$, we must have scalars k_1, k_2 such that

$$(4,-1,8) = k_1(1,2,-1) + k_2(6,4,2) \qquad \ldots(1)$$

$$\Rightarrow (4,-1,8) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$\Rightarrow k_1 + 6k_2 = 4$$

$$2k_1 + 4k_2 = -1$$

$$-k_1 + 2k_2 = 8$$

The augmented matrix of this system is

$$A_b = \begin{bmatrix} 1 & 6 & \vdots & 4 \\ 2 & 4 & \vdots & -1 \\ -1 & 2 & \vdots & 8 \end{bmatrix}$$

We reduce this matrix to echelon form as follows:

$$A_b = \begin{bmatrix} 1 & 6 & \vdots & 4 \\ 2 & 4 & \vdots & -1 \\ -1 & 2 & \vdots & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & \vdots & 4 \\ 0 & -8 & \vdots & -9 \\ 0 & 8 & \vdots & 12 \end{bmatrix}, \quad \begin{matrix} R_2 - 2R_1, \\ R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 6 & \vdots & 4 \\ 0 & -8 & \vdots & -9 \\ 0 & 0 & \vdots & 3 \end{bmatrix}, \quad R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 6 & \vdots & 4 \\ 0 & 1 & \vdots & \frac{9}{8} \\ 0 & 0 & \vdots & 1 \end{bmatrix}, \quad -\frac{1}{8}R_2, \frac{1}{3}R_3$$

which is in echelon form. It is clear from this matrix that the rank of coefficient matrix is 2 and the rank of augmented matrix is 3. Since the coefficient matrix and the augmented matrix have different ranks, so the given system is inconsistent. Therefore, $(4,-1,8)$ cannot be expressed as linear combination of vectors $(1,2,-1)$ and $(6,4,2)$.

Spanned and Spanning Sets

If $S=\{v_1,v_2,\ldots,v_r\}$ is a set of vectors in vector space V, then the set of all linear combinations of the vectors $v_1,v_2,\ldots,v_r$ is called the ***spanned set*** by $v_1,v_2,\ldots,v_r$ and is denoted by $<S>$. We read $<S>$ as span of S, we say that that the vectors $v_1,v_2,\ldots,v_r$ **span** $<S>$. The set S itself is called the ***spanning set***.

> In the case where S is the empty set, it will be convenient to agree that span $(\varnothing)=\{0\}$.

Example 5: Testing for Spanning

Show that the vectors $(1,2)$ *and* $(3,5)$ *span the vector space* R^2.

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Solution: We must determine whether an arbitrary vector $v=(x,y)$ in R^2 can be expressed as a linear combination vectors $(1,2)$ and $(3,5)$, so let

$$(x,y)=k_1(1,2)+k_2(3,5) \qquad \ldots(1)$$

$$(x,y)=(k_1,2k_1)+(3k_2,5k_2)=(k_1+3k_2,2k_1+5k_2)$$

Expressing this equation in terms of components gives

$$k_1+3k_2=x$$
$$2k_1+5k_2=y$$

Augmented matrix is

$$A_b=\begin{bmatrix}1 & 3 & x\\ 2 & 5 & y\end{bmatrix}\sim\begin{bmatrix}1 & 3 & x\\ 0 & -1 & y-2x\end{bmatrix},\ R_2-2R_1$$

$$\sim\begin{bmatrix}1 & 0 & x+3(y-2x)\\ 0 & -1 & y-2x\end{bmatrix},\ R_1+3R$$

$$\sim\begin{bmatrix}1 & 0 & 3y-5x\\ 0 & 1 & 2x-y\end{bmatrix},\ -1R_2$$

This gives

$$k_1=3y-5x$$
$$k_2=2x-y$$

Putting these values of k_1,k_2 in (1), we have

$$(x,y)=(3y-5x)(1,2)+(2x-y)(3,5)$$

Since any $v=(x,y)$ in R^2 can be expressed as a linear combination vectors $(1,2)$ and $(3,5)$, so the vectors $(1,2)$ and $(3,5)$ span the vector space R^2.

4.5 Linear Independence

In this section we will consider the question of whether the vectors in a given set are interrelated in the sense that one or more of them can be expressed as a linear combination of the others. This is important to know in applications because the existence of such relationships often signals that some kind of complication is likely to occur.

Linearly Independent Set

If $S = \{v_1, v_2, \ldots, v_r\}$ is a non-empty set of vectors such the equation

$$k_1v_1 + k_2v_2 + \ldots + k_rv_r = 0$$

is satisfied only for

$$k_1 = k_2 = \ldots = k_r = 0$$

then S is called a ***linearly independent set*** and the vectors $v_1, v_2, \ldots, v_r$ are said to be ***linearly independent vectors.***

PU, 2013, Mathematics A-III, BS (Math/Stat/Chem); PU, 2017, Linear Algebra, BS (Physics)

Linearly Dependent Set

If $S = \{v_1, v_2, \ldots, v_r\}$ is a non-empty set of vectors such the equation

$$k_1v_1 + k_2v_2 + \ldots + k_rv_r = 0$$

is satisfied for at least one $k_i \neq 0$, then S is called a ***linearly dependent set*** and the vectors $v_1, v_2, \ldots, v_r$ are said to be ***linearly dependent vectors.***

<u>**Example 1**</u>: *Show that the set $\{(1,3),(2,5)\}$ is linearly independent but the set $\{(1,3),(2,6)\}$ is linearly dependent in R^2.*

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<u>**Solution**</u>: Consider $a(1,3) + b(2,5) = 0$

$$(a, 3a) + (2b, 5b) = 0$$

$$(a + 2b, 3a + 5b) = (0,0)$$

$$a + 2b = 0 \qquad \ldots(1)$$

$$3a + 5b = 0 \qquad \ldots(2)$$

Putting the value of a from (1) in (2), we have

$$3(-2b) + 5b = 0 \qquad -b = 0$$

$$-6b + 5b = 0 \qquad b = 0$$

Putting this value of b in (1), we have $a = 0$.

This shows that $\{(1,3),(2,5)\}$ is linearly independent set in R^2.

Consider $$a(1,3)+b(2,6)=0 \quad \text{...(1)}$$

$$(a,3a)+(2b,6b)=0$$

$$(a+2b,3a+6b)=(0,0)$$

$$a+2b=0 \quad \text{and} \quad 3a+6b=0$$

$$\text{or} \quad a+2b=0$$

$$\text{or} \quad a=-2b$$

This shows that (1) is satisfied for $a=-2b$ even if $b \neq 0$, so $\{(1,3),(2,6)\}$ is linearly dependent in R^2.

This is called linear independence of vectors.

(ii) **Determine whether the following vectors in R^2 are linearly dependent or linearly independent:**

(a) $(1,1),(3,1)$ **(b)** $(2,1),(1,-1)$

(c) $(3,-1),(6,-2)$ **(d)** $(2,5),(4,10)$

Sol:(a) Let k_1, k_2 be any scalars and consider

$$k_1(1,1)+k_2(3,1)=(0,0) \quad \text{...(1)}$$

$$\Rightarrow (k_1+3k_2, k_1+k_2)=(0,0)$$

$$\Rightarrow k_1+3k_2=0$$

$$k_1+k_2=0$$

$$\Rightarrow k_1=0, k_2=0$$

This shows that the vectors $(1,1),(3,1)$ are linearly independent.

(b) Let k_1, k_2 be any scalars and consider

$$k_1(2,1)+k_2(1,-1)=(0,0) \quad \text{...(1)}$$

$$\Rightarrow (2k_1+k_2, k_1-k_2)=(0,0)$$

$$\Rightarrow 2k_1+k_2=0, k_1-k_2=0$$

$$\Rightarrow k_1=0, k_2=0$$

This shows that the vectors $(2,1),(1,-1)$ are linearly independent.

(c) Let k_1, k_2 be any scalars and consider

$$k_1(3.-1)+k_2(6,-2)=(0,0) \quad \text{...(1)}$$

$$\Rightarrow (3k_1+6k_2, -k_1-2k_2)=(0,0)$$

$$\Rightarrow 3k_1+6k_2=0$$

$$-k_1-2k_2=0$$

$$\Rightarrow k_1=-2k_2$$

This shows that the equation (1) has infinite solutions, so the vectors $(3,-1),(6,-2)$ are linearly dependent.

(d) Let k_1, k_2 be any scalars and consider

$$k_1(2,5)+k_2(4,10)=(0,0) \quad \text{...(1)}$$

$$\Rightarrow (2k_1+5k_2, 4k_1+10k_2)=(0,0)$$

$$\Rightarrow 2k_1+5k_2=0$$

$$4k_1+10k_2=0$$

$$\Rightarrow k_1 = \frac{5}{2}k_2$$

This shows that the equation (1) has infinite solutions, so the vectors $(2,5),(4,10)$ are linearly dependent.

(iii) Determine whether the vectors $(8,-1,3),(4,0,1)$ are linearly dependent or linearly independent in R^3.

Sol: Let k_1, k_2 be any scalars and consider

$$k_1(8,-1,3) + k_2(4,0,1) = (0,0,0) \quad ...(1)$$

$$\Rightarrow (8k_1 + 4k_2, -k_1, 3k_1 + k_2) = (0,0,0)$$

$$\Rightarrow 8k_1 + 4k_2 = 0$$

$$-k_1 = 0$$

$$3k_1 + k_2 = 0$$

$$\Rightarrow k_1 = 0, k_2 = 0$$

This shows that the vectors $(8,-1,3),(4,0,1)$ are linearly independent.

(iv) Determine whether the following vectors in R^3 are linearly dependent or linearly independent:

(a) $(1,-1,2),(4,0,1)$ (b) $(1,0,0),(0,1,0),(0,0,1)$

Sol: Let k_1, k_2 be any scalars and consider

$$k_1(1,-1,2) + k_2(4,0,1) = (0,0,0) \quad ...(1)$$

$$\Rightarrow (k_1 + 4k_2, -k_1, 2k_1 + k_2) = (0,0,0)$$

$$\Rightarrow k_1 + 4k_2 = 0$$

$$-k_1 = 0$$

$$2k_1 + k_2 = 0$$

$$\Rightarrow k_1 = 0, k_2 = 0$$

This shows that the vectors $(1,-1,2),(4,0,1)$ are linearly independent.

Sol: Let k_1, k_2, k_3 be any scalars and consider

$$k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (0,0,0) \quad ...(1)$$

$$\Rightarrow (k_1, k_2, k_3) = (0,0,0)$$

$$\Rightarrow k_1 = 0, k_2 = 0, k_3 = 0$$

This shows that the vectors $(1,0,0),(0,1,0),(0,0,1)$ are linearly independent.

(v) Determine whether the vectors $(0,0,2,2),(3,3,0,0),(1,1,0,-1)$ in R^4 are linearly dependent or linearly independent.

Sol: Let k_1, k_2, k_3 be any scalars and consider

$$k_1(0,0,2,2) + k_2(3,3,0,0) + k_3(1,1,0,-1) = (0,0,0,0) \quad ...(1)$$

Determine whether the following vectors in R^2 are linearly dependent or linearly independent (Questions 2 – 3):

2. $(3,-1),(4,5),(-4,7)$

Sol: Let k_1, k_2, k_3 be any scalars and consider

$$k_1(3,-1)+k_2(4,5)+k_3(-4,7)=(0,0) \quad \text{...(1)}$$

$$\Rightarrow (3k_1+4k_2-4k_3, -k_1+5k_2+7k_3)=(0,0)$$

$$\Rightarrow 3k_1+4k_2-4k_3=0$$

$$-k_1+5k_2+7k_3=0 \quad \text{...(2)}$$

We reduce the augmented matrix of this system to echelon form as follows:

$$A_b=\begin{bmatrix} 3 & 4 & -4 & : & 0 \\ -1 & 5 & 7 & : & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 5 & 7 & : & 0 \\ 3 & 4 & -4 & : & 0 \end{bmatrix}, \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -5 & -7 & : & 0 \\ 3 & 4 & -4 & : & 0 \end{bmatrix}, \quad -1R_1$$

$$\sim \begin{bmatrix} 1 & -5 & -7 & : & 0 \\ 0 & 19 & 17 & : & 0 \end{bmatrix}, \quad R_2-3R_1$$

$$\sim \begin{bmatrix} 1 & -5 & -7 & : & 0 \\ 0 & 1 & \frac{17}{19} & : & 0 \end{bmatrix}, \quad \frac{1}{19}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{48}{19} & : & 0 \\ 0 & 1 & \frac{17}{19} & : & 0 \end{bmatrix}, \quad R_1+5R_2$$

$$\Rightarrow k_1 - \frac{48}{19}k_3 = 0$$

$$k_2 + \frac{17}{19}k_3 = 0$$

$$\Rightarrow k_1 = \frac{48}{19}k_3, k_2 = -\frac{17}{19}k_3$$

This shows that the system (2), and therefore, the equation (1) has infinite solutions for different values of k_3. Particularly, for $k_3 = 19$, we have $k_1 = 48$, $k_2 = -17$. Thus equation (1) is satisfied for $k_1 = 48$, $k_2 = -17$, $k_3 = 19$. Hence the vectors $(3,-1),(4,5),(-4,7)$ are linearly dependent.

3. $\mathbf{(2,-1),(4,5),(-4,7)}$

Sol: Let k_1, k_2, k_3 be any scalars and consider

$$k_1(2,-1) + k_2(4,5) + k_3(-4,7) = (0,0)$$

$$\Rightarrow (2k_1 + 4k_2 - 4k_3, -k_1 + 5k_2 + 7k_3) = (0,0) \quad \text{...(1)}$$

$$\Rightarrow 2k_1 + 4k_2 - 4k_3 = 0$$

$$-k_1 + 5k_2 + 7k_3 = 0 \quad \text{...(2)}$$

We reduce the augmented matrix of this system to echelon form as follows:

$$A_b = \begin{bmatrix} 2 & 4 & -4 & : & 0 \\ -1 & 5 & 7 & : & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 5 & 7 & : & 0 \\ 2 & 4 & -4 & : & 0 \end{bmatrix}, R_{12}$$

$$\sim \begin{bmatrix} 1 & -5 & -7 & : & 0 \\ 2 & 4 & -4 & : & 0 \end{bmatrix}, -1R_1$$

$$\sim \begin{bmatrix} 1 & -5 & -7 & : & 0 \\ 0 & 14 & 10 & : & 0 \end{bmatrix}, R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -5 & -7 & : & 0 \\ 0 & 1 & \frac{5}{7} & : & 0 \end{bmatrix}, \frac{1}{14}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{24}{7} & : & 0 \\ 0 & 1 & \frac{5}{7} & : & 0 \end{bmatrix}, R_1 + 5R_2$$

$$\Rightarrow k_1 - \frac{24}{7}k_3 = 0, \quad k_2 + \frac{5}{7}k_3 = 0$$

$$\Rightarrow k_1 = \frac{24}{7}k_3, \quad k_2 = -\frac{5}{7}k_3$$

This shows that the system (2), and therefore, the equation (1) has infinite solutions for different values of k_3. Particularly, for $k_3 = 7$, we have $k_1 = 24$, $k_2 = -5$. Thus equation (1) is satisfied for $k_1 = 24$, $k_2 = -5$, $k_3 = 7$. Hence the vectors $(2,-1),(4,5),(-4,7)$ are linearly dependent.

Determine whether the following vectors in R^3 are linearly dependent or linearly independent (Questions 4 – 15):

4. $(-1,2,4),(5,-10,20)$

Sol: Let k_1, k_2 be any scalars and consider

$$k_1(-1,2,4)+k_2(5,-10,20)=(0,0,0) \quad \ldots(1)$$

$$\Rightarrow (-k_1+5k_2, 2k_1-10k_2, 4k_1+20k_2)=(0,0,0)$$

$$\Rightarrow -k_1+5k_2=0$$

$$2k_1-10k_2=0 \quad \ldots(2)$$

$$4k_1+20k_2=0$$

We reduce the augmented matrix of this system to echelon form as follows:

$$A_b=\begin{bmatrix}-1 & 5 & : & 0\\ 2 & -10 & : & 0\\ 4 & 20 & : & 0\end{bmatrix} \sim \begin{bmatrix}1 & -5 & : & 0\\ 1 & -5 & : & 0\\ 1 & 5 & : & 0\end{bmatrix}, \; -1R_1, \frac{1}{2}R_2, \frac{1}{4}R_3$$

$$\sim \begin{bmatrix}1 & -5 & : & 0\\ 0 & 0 & : & 0\\ 0 & 10 & : & 0\end{bmatrix}, \; \begin{matrix}R_2-R_1,\\ R_3-R_1\end{matrix}$$

$$\sim \begin{bmatrix}1 & -5 & : & 0\\ 0 & 10 & : & 0\\ 0 & 0 & : & 0\end{bmatrix}, \; R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix}1 & -5 & : & 0\\ 0 & 1 & : & 0\\ 0 & 0 & : & 0\end{bmatrix}, \; \frac{1}{10}R_3$$

$$\sim \begin{bmatrix}1 & 0 & : & 0\\ 0 & 1 & : & 0\\ 0 & 0 & : & 0\end{bmatrix}, \; R_1+5R_2$$

$$\Rightarrow k_1=0, k_2=0$$

This shows that the vectors $(-1,2,4),(5,-10,20)$ are linearly independent.

5. $(4,-1,2),(-4,10,2)$

Sol: Let k_1, k_2 be any scalars and consider

$$k_1(4,-1,2)+k_2(-4,10,2)=(0,0,0) \quad \ldots(1)$$

$$\Rightarrow (4k_1-4k_2, -k_1+10k_2, 2k_1+2k_2)=(0,0,0)$$

$$\Rightarrow 4k_1-4k_2=0 \quad \ldots(2)$$

$$-k_1+10k_2=0$$

$$2k_1+2k_2=0$$

We reduce the augmented matrix of this system to echelon form as follows:

$$A_b = \begin{bmatrix} 4 & -4 & : & 0 \\ -1 & 10 & : & 0 \\ 2 & 2 & : & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & : & 0 \\ -1 & 10 & : & 0 \\ 1 & 1 & : & 0 \end{bmatrix}, \quad \frac{1}{4}R_1, \frac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & -1 & : & 0 \\ 0 & 9 & : & 0 \\ 0 & 2 & : & 0 \end{bmatrix}, \quad \begin{matrix} R_2 + R_1, \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & : & 0 \\ 0 & 1 & : & 0 \\ 0 & 1 & : & 0 \end{bmatrix}, \quad \frac{1}{9}R_2, \frac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}, \quad \begin{matrix} R_1 + R_2 \\ R_3 - R_2 \end{matrix}$$

$$\Rightarrow k_1 = 0, \ k_2 = 0$$

This shows that the vectors $(4,-1,2),(-4,10,2)$ are linearly independent.

6. $\mathbf{(1,-2,3),(5,6,-1),(3,2,1)}$

Sol: Let k_1, k_2, k_3 be any scalars and consider

$$k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = (0,0,0) \qquad \text{...(1)}$$

$$\Rightarrow (k_1 + 5k_2 + 3k_3, -2k_1 + 6k_2 + 2k_3, 3k_1 - k_2 + k_3) = (0,0,0)$$

$$\Rightarrow k_1 + 5k_2 + 3k_3 = 0$$

$$-2k_1 + 6k_2 + 2k_3 = 0$$

$$3k_1 - k_2 + k_3 = 0$$

We reduce the augmented matrix of this system to echelon form as follows:

$$A_b = \begin{bmatrix} 1 & 5 & 3 & : & 0 \\ -2 & 6 & 2 & : & 0 \\ 3 & -1 & 1 & : & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 & : & 0 \\ 0 & 16 & 8 & : & 0 \\ 0 & -16 & -8 & : & 0 \end{bmatrix}, \quad \begin{matrix} R_2 + 2R_1, \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 5 & 3 & : & 0 \\ 0 & 16 & 8 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}, \quad R_3 + R_2 \sim \begin{bmatrix} 1 & 5 & 3 & : & 0 \\ 0 & 1 & \frac{1}{2} & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}, \quad \frac{1}{16}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & : & 0 \\ 0 & 1 & \frac{1}{2} & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}, \quad R_1 - 5R_2$$

$$\Rightarrow k_1 + \frac{1}{2}k_3 = 0, \ k_2 + \frac{1}{2}k_3 = 0$$

or $\quad k_1 = -\frac{1}{2}t,\ k_2 = -\frac{1}{2}t,\ k_3 = t,\ t \in R$

This shows that (1) has infinite solutions, so the vectors $(1,-2,3)$, $(5,6,-1), (3,2,1)$ are linearly dependent.

7. $\quad$ **$(-3,0,4), (5,-1,2), (1,1,3)$**

Sol: Let k_1, k_2, k_3 be any scalars and consider

$$k_1(-3,0,4) + k_2(5,-1,2) + k_3(1,1,3) = (0,0,0) \qquad \ldots(1)$$

$$\Rightarrow (-3k_1 + 5k_2 + k_3, -k_2 + k_3, 4k_1 + 2k_2 + 3k_3) = (0,0,0)$$

$$\Rightarrow -3k_1 + 5k_2 + k_3 = 0$$

$$-k_2 + k_3 = 0$$

$$4k_1 + 2k_2 + 3k_3 = 0$$

We reduce the augmented matrix of this system to echelon form as follows:

$$A_b = \begin{bmatrix} -3 & 5 & 1 & : & 0 \\ 0 & -1 & 1 & : & 0 \\ 4 & 2 & 3 & : & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 4 & : & 0 \\ 0 & -1 & 1 & : & 0 \\ 4 & 2 & 3 & : & 0 \end{bmatrix},\ R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 7 & 4 & : & 0 \\ 0 & -1 & 1 & : & 0 \\ 0 & -26 & -13 & : & 0 \end{bmatrix},\ R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 7 & 4 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & 1 & \frac{1}{2} & : & 0 \end{bmatrix},\ -1R_2, -\frac{1}{26}R_3$$

$$\sim \begin{bmatrix} 1 & 7 & 4 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & \frac{3}{2} & : & 0 \end{bmatrix},\ R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 7 & 4 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix},\ \frac{2}{3}R_3$$

$$\sim \begin{bmatrix} 1 & 7 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix},\ \begin{matrix} R_1 - 4R_3, \\ R_2 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix},\ R_1 - 7R_2$$

$$\Rightarrow k_1 = 0, k_2 = 0, k_3 = 0$$

This shows that the vectors $(-3,0,4),(5,-1,2),(1,1,3)$ are linearly independent.

8. $\mathbf{(2,-2,0),(6,1,4),(2,0,-4)}$

<u>Sol</u>: Let k_1,k_2,k_3 be any scalars and consider

$$k_1(2,-2,0)+k_2(6,1,4)+k_3(2,0,-4)=(0,0,0) \qquad (1)$$

$$\Rightarrow (2k_1+6k_2+2k_3,-2k_1+k_2,4k_2-4k_3)=(0,0,0)$$

$$\Rightarrow 2k_1+6k_2+2k_3=0$$

$$-2k_1+k_2=0$$

$$4k_2-4k_3=0$$

We reduce the augmented matrix of this system to echelon form as follows:

$$A_b=\left[\begin{array}{ccc:c} 2 & 6 & 2 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 3 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right], \frac{1}{2}R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 1 & 0 \\ 0 & 7 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right], \quad R_2+2R_1, \frac{1}{4}R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 7 & 2 & 0 \end{array}\right], \quad R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 9 & 0 \end{array}\right], \quad R_3-7R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad \frac{1}{9}R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad R_1-R_3, \; R_2+R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad R_1+3R_2$$

$$\Rightarrow k_1=0, k_2=0, k_3=0$$

This shows that the vectors $(2,-2,0),(6,1,4),(2,0,-4)$ are linearly independent.