4.6 Basis and Dimension

Basis

A set $S = \{v_1, v_2, \ldots, v_n\}$ of linearly independent vectors of a vector space V is called a ***basis*** for V if S spans V.

In other words a linearly independent set $S = \{v_1, v_2, \ldots, v_n\}$ of vectors of a vector space V is said to be the ***basis*** for V if each vector of V can be expressed as linear combination of vectors $v_1, v_2, \ldots, v_n$, i.e. $V = \langle S \rangle$.

PU, 2013, Mathematics A-III, BS (Math/Stat/Chem)

Theorem 1: *If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then every vector v in V can be expressed in the form $v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ in exactly one way*

PU, 2014, Linear Algebra BS (Physics)

Example 2: *Let $v_1 = (1,2,1), v_2 = (2,9,0), v_3 = (3,3,4)$. Show that the set $S = \{v_1, v_2, v_3\}$ is a basis for R^3.*

Solution: First we show that S is a linearly independent set, for this let k_1, k_2, k_3 be any scalars and consider

$$k_1v_1 + k_2v_2 + k_3v_3 = 0 \quad ...(1)$$

$$\Rightarrow k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4) = (0,0,0)$$

$$\Rightarrow (k_1 + 2k_2 + 3k_3, 2k_1 + 9k_2 + 3k_3, k_1 + 4k_3) = (0,0,0)$$

$$\Rightarrow k_1 + 2k_2 + 3k_3 = 0$$
$$2k_1 + 9k_2 + 3k_3 = 0 \quad ...(2)$$
$$k_1 + 4k_3 = 0$$

We reduce the augmented matrix of this system to echelon form as follows:

$$A_b = \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 2 & 9 & 3 & : & 0 \\ 1 & 0 & 4 & : & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & 5 & -3 & : & 0 \\ 0 & -2 & 1 & : & 0 \end{bmatrix}, \quad \begin{matrix} R_2 - 2R_1, \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & -2 & 1 & : & 0 \end{bmatrix}, \quad R_2 + 2R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & -1 & : & 0 \end{bmatrix}, \quad R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix}, \quad -1R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix}, \quad \begin{matrix} R_1 - 3R_3, \\ R_2 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix}, \quad R_1 - 2R_2$$

$$\Rightarrow k_1 = 0, k_2 = 0, k_3 = 0$$

This shows that the set $S = \{v_1, v_2, v_3\}$ is linearly independent. To show that S spans R^3, let $(a,b,c) \in R^3$ be an arbitrary vector of R^3 and consider

$$(a,b,c) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4) \quad ...(3)$$

$$\Rightarrow (a,b,c) = (c_1 + 2c_2 + 3c_3, 2c_1 + 9c_2 + 3c_3, c_1 + 4c_3)$$

$$\Rightarrow c_1 + 2c_2 + 3c_3 = a$$
$$2c_1 + 9c_2 + 3c_3 = b \quad \ldots(4)$$
$$c_1 + 4c_3 = c$$

We reduce the augmented matrix of this system to echelon form as follows:

$$A_b = \begin{bmatrix} 1 & 2 & 3 & \vdots & a \\ 2 & 9 & 3 & \vdots & b \\ 1 & 0 & 4 & \vdots & c \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & \vdots & a \\ 0 & 5 & -3 & \vdots & b-2a \\ 0 & -2 & 1 & \vdots & c-a \end{bmatrix}, \begin{matrix} R_2 - 2R_1, \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \vdots & a \\ 0 & 1 & -1 & \vdots & b-4a+2c \\ 0 & -2 & 1 & \vdots & c-a \end{bmatrix}, R_2 + 2R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \vdots & a \\ 0 & 1 & -1 & \vdots & b-4a+2c \\ 0 & 0 & -1 & \vdots & 2b-9a+5c \end{bmatrix}, R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & \vdots & a \\ 0 & 1 & -1 & \vdots & b-4a+2c \\ 0 & 0 & 1 & \vdots & 9a-2b-5c \end{bmatrix}, -1R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & -26a+6b+15c \\ 0 & 1 & 0 & \vdots & 5a-b-3c \\ 0 & 0 & 1 & \vdots & 9a-2b-5c \end{bmatrix}, \begin{matrix} R_1 - 3R_3, \\ R_2 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \vdots & -36a+8b+21c \\ 0 & 1 & 0 & \vdots & 5a-b-3c \\ 0 & 0 & 1 & \vdots & 9a-2b-5c \end{bmatrix}, R_1 - 2R_2$$

$$\Rightarrow c_1 = -36a + 8b + 21c$$
$$c_2 = 5a - b - 3c$$
$$c_3 = 9a - 2b - 5c$$

Putting these values in (3), we have

$$(a,b,c) = (-36a+8b+21c)(1,2,1) + (5a-b-3c)(2,9,0) + (9a-2b-5c)(3,3,4)$$

This shows that the arbitrary vector (a,b,c) of R^3 can be expressed as a linear combination of vectors of S, so S spans R^3. Hence S is a basis for R^3.

Example 3: *Determine whether or not the given set of vectors* $(2,4,-3),(0,1,1)$ *and* $(0,1,-1)$ *is a basis for* R^3.

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Solution: Let $S = \{(2,4,-3),(0,1,1),(0,1,-1)\}$. First we show that S is a linearly independent set, for this let k_1, k_2, k_3 be any scalars and consider

$$k_1(2,4,-3)+k_2(0,1,1)+k_3(0,1,-1)=(0,0,0) \quad ...(1)$$
$$\Rightarrow (2k_1, 4k_1+k_2+k_3, -3k_1+k_2-k_3)=(0,0,0)$$
$$\Rightarrow 2k_1=0$$
$$4k_1+k_2+k_3=0$$
$$-3k_1+k_2-k_3=0$$
$$\Rightarrow k_1=0, k_2=0, k_3=0$$

This shows that the set S is linearly independent. To show that S spans R^3, let $(x,y,z)\in R^3$ be an arbitrary vector of R^3 and consider

$$(x,y,z)=c_1(2,4,-3)+c_2(0,1,1)+c_3(0,1,-1) \quad ...(2)$$
$$\Rightarrow (x,y,z)=(2c_1, 4c_1+c_2+c_3, -3c_1+c_2-c_3)$$
$$\Rightarrow 2c_1=x$$
$$4c_1+c_2+c_3=y \quad ...(3)$$
$$-3c_1+c_2-c_3=z$$

We reduce augmented matrix of this system to echelon form:

$$A_b=\begin{bmatrix} 2 & 0 & 0 & : & x \\ 4 & 1 & 1 & : & y \\ -3 & 1 & -1 & : & z \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & : & x \\ 0 & 1 & 1 & : & y-2x \\ -1 & 1 & -1 & : & z+x \end{bmatrix}, \quad \begin{matrix} R_2-2R_1, \\ R_3+R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 2x+z \\ 0 & 1 & 1 & : & y-2x \\ -1 & 1 & -1 & : & z+x \end{bmatrix}, \quad R_1+R_3$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 2x+z \\ 0 & 1 & 1 & : & y-2x \\ 0 & 2 & -2 & : & 3x+2z \end{bmatrix}, \quad R_3+R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 2x+z \\ 0 & 1 & 1 & : & y-2x \\ 0 & 0 & -4 & : & 7x-2y+2z \end{bmatrix}, \quad R_3-2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 2x+z \\ 0 & 1 & 1 & : & y-2x \\ 0 & 0 & 1 & : & -\frac{1}{4}(7x-2y+2z) \end{bmatrix}, \quad -\frac{1}{4}R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & : & \frac{1}{4}(x+2y+2z) \\ 0 & 1 & 0 & : & \frac{1}{4}(-x+2y+2z) \\ 0 & 0 & 1 & : & -\frac{1}{4}(7x-2y+2z) \end{bmatrix}, \quad \begin{matrix} R_1+R_3, \\ R_2-R_3 \end{matrix}$$

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$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 0 & \frac{1}{2}x \\ 0 & 1 & 0 & \frac{1}{4}(-x+2y+2z) \\ 0 & 0 & 1 & -\frac{1}{4}(7x-2y+2z) \end{array}\right], \quad R_1-R_2$$

$$\Rightarrow c_1=\frac{1}{2}x$$

$$c_2=\frac{1}{4}(-x+2y+2z)$$

$$c_3=-\frac{1}{4}(7x-2y+2z)$$

Putting these values in (2), we have

$$(x,y,z)=\frac{1}{2}x(2,4,-3)+\frac{1}{4}(-x+2y+2z)(0,1,1)-\frac{1}{4}(7x-2y+2z)(0,1,-1)$$

This shows that the arbitrary vector (x,y,z) of R^3 can be expressed as a linear combination of vectors of S, so S spans R^3. Hence S is a basis for R^3.

Example 4: *Determine whether or not the set of vectors* $\{(1,2,-1),(0,3,1),(1,-5,3)\}$ *is a basis for* R^3.

$c_1 = k_1, c_2 = k_2, \ldots$

Thus, the two expressions for v are the same.

Definition: If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, and $v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ is the expression for a vector v in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the ***coordinates*** of v relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in R^n constructed from these coordinates is called the *coordinate vector of v relative to S*; it is denoted by $(v)_s = (c_1, c_2, \ldots, c_n)$.

It should be noted that coordinate vectors depend not only on the basis S but also on the order in which the basis vectors are written; a change in the order of the basis vectors results in a corresponding change of order for the entries in the coordinate vectors.

Example 1: *If* $i = (1,0,0), j = (0,1,0), k = (0,0,1)$, *then show that* $S = \{i, j, k\}$ *is a basis for* R^3.

Solution: We have already shown that $S = \{i, j, k\}$ is linearly independent set in R^3. This set also spans R^3 since any vector (a,b,c) in R^3 can be written as

$$v = (a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1) = ai + bj + ck \qquad \ldots(1)$$

Thus, S is a basis for R^3; it is called the *standard basis* for R^3. Looking at the coefficients of i, j, k in (1), it follows that the coordinates of v relative to the standard basis are a, b, and c, so $(v)_S = (a,b,c)$. Comparing this result to (1) we see that $v = (v)_S$. The last equation holds for standard basis and it may or may not hold for other basis.

Standard Basis

If $e_1 = (1,0,\ldots,0), e_2 = (0,1,\ldots,0), \ldots, e_n = (0,0,\ldots,1)$, then $\{e_1, e_2, \ldots, e_n\}$ is a basis for R^n and is called the ***standard basis*** *for* R^n.

Example 6: *If* $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, *then show that the set* $S = \{M_1, M_2, M_3, M_4\}$ *is a basis for the vector space* M_{22} *of* 2×2 *matrices.*

Solution: First we show that S is a linearly independent set, for this let k_1, k_2, k_3, k_4 be any scalars and consider

$$k_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow k_1 = 0,\ k_2 = 0, k_3 = 0, k_4 = 0$$

This shows that S is a linearly independent set. To show that S spans M_{22}, let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any arbitrary matrix of M_{22}, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

so S spans M_{22}. Hence S is a basis for M_{22}.

The basis S in this example is called the ***standard basis*** for M_{22}. More generally, the ***standard basis*** for M_{mn} consists of the mn different matrices with a single 1 and zeros for the remaining entries.

Finite and Infinite Dimensional Spaces

A nonzero vector space V is called ***finite-dimensional*** if it contains a finite set of vectors $\{v_1, v_2, \ldots, v_n\}$ that forms a basis. If no such set exists, V is called ***infinite-dimensional***. In addition, we shall regard the zero vector space to be finite-dimensional. If the basis of a vector space V has n vectors, then we say that V is of *dimension* n, and we write $\dim V = n$. We define the zero vector space to have dimension zero.