$$\overset{R_{3,2}(-3)}{\sim}\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & \lambda - 1 \\ 0 & 0 & 0 & : & \lambda^2 - 3\lambda + 2 \end{bmatrix}$$

Thus, $\rho(A) = 2.$

$\therefore$ The system is consistent if $\rho([A : B]) = \rho(A) = 2.$

if $\lambda^2 - 3\lambda + 2 = 0$ which implies $\lambda = 1, 2.$

Case (*i*) $\lambda = 1$, then from (1), the system of equations reduces to $x + y + z = 1, y + 3z =$

Taking $z = a$(some real number), $y = -3a, x = 1 - y - z = 1 + 3a - a = 1 + 2a$

Thus, the general solution is given by $x = 1 + 2a, y = -3a, z = a$, where a is arbitrary

Case (*ii*) $\lambda = 2$, then the system reduces to $x + y + z = 1, y + 3z = 2 - 1 = 1$

Taking $z = a, y = 1 - 3a, x = 1 - y - z = 1 - (1 - 3a) - a = 2a.$

Thus, the general solution is given by $x = 2a, y = 1 - 3a, z = a$, a is arbitrary.

Example 7. *For what values of* λ, μ *the system of equations :*

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

has (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions.

Sol. The augmented matrix of the system is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$$\underset{R_{3,1}(-1)}{\overset{R_{2,1}(-1)}{\sim}}\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

$$\overset{R_{3,2}(-1)}{\sim}\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix} \quad ...($$

Case I. $\lambda - 3 \neq 0$ ***i.e.*** $\lambda \neq 3$

operating $R_3\left(\frac{1}{\lambda - 3}\right)$ in (1), $[A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & 1 & : & \frac{\mu - 10}{\lambda - 3} \end{bmatrix}$ which is row-echelon form.

Here $\rho(A) = \rho[A : B] = 3$ for all values of μ ($\because$ Number of non-zero rows is

= Number of unknowns.

$\therefore$ The given system is constent and has unique solution for $\lambda \neq 3$ (and for all values o

Case II. $\lambda - 3 = 0$ **i.e.** $\lambda = 3$

$\therefore$ From (1) $\quad [A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & 0 & : & \mu - 10 \end{bmatrix}$ ...

$$\rho(A) = 2 \qquad \text{(Number of non-zero rows)}$$

for all values of μ.

Sub-Case I. $\mu - 10 \neq 0$ *i.e.* $\mu \neq 10$

operating $R_3\left(\frac{1}{\mu - 10}\right)$ in (2) $\quad [A : B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$

which is row-echelon form of [A : B].

$$\rho[A : B] = 3 \qquad \text{(Number of non-zero rows)}$$
$$\neq \rho(A) \; (= 2)$$

$\therefore$ The given system is inconsistent *i.e.* has no solution.

for $\qquad \lambda = 3$ and $\mu \neq 10$.

Sub-Case II. $\mu - 10 = 10$ *i.e.* $\mu = 10$

$\therefore$ From Eqn. (2) $\quad [A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$ which is row-echelon form.

$$\rho\,[A : B] = 2 \qquad \text{(Number of non-zero rows)}$$
$$= \rho(A) = 2 = \text{Number of unknowns.}$$

$\therefore$ The given system has an infinite number of solutions for $\lambda = 3$ and $\mu = 10$.

Example 8. *Solve the equations :* $\lambda x + 2y - 2z = 1$, $\quad 4x + 2\lambda y - z = 2$, $\quad 6x + 6y + \lambda z = 3$ *considering specially the case when* $\lambda = 2$.

Sol. The matrix form of these equations is AX = B

where $\quad A = \begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Because a_{11} = unknown λ here.

So we shall find it difficult to reduce [A : B] to row-Echelon form.

So let us find $\quad |A| = \begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix}$

Expanding by first row

$$|A| = \lambda(2\lambda^2 + 6) - 2(4\lambda + 6) - 2(24 - 12\lambda)$$
$$= 2\lambda^3 + 6\lambda - 8\lambda - 12 - 48 + 24\lambda$$
$$= 2\lambda^3 + 22\lambda - 60 = 2[\lambda^3 + 11\lambda - 30]$$
$$= 2[\lambda^3 - 4\lambda + 15\lambda - 30] = 2[\lambda(\lambda^2 - 4) + 15(\lambda - 2)]$$
$$= 2[\lambda(\lambda - 2)(\lambda + 2) + 15(\lambda - 2)]$$

or $\qquad |A| = 2(\lambda - 2)(\lambda^2 + 2\lambda + 15) \qquad \text{...(1)}$

Now on solving $\quad \lambda^2 + 2\lambda + 15 = 0$, $\quad$ we have

$$\lambda = \frac{-2 \pm \sqrt{4 - 60}}{2} = \frac{-2 \pm \sqrt{-56}}{2} = \frac{-2 \pm 2\sqrt{-14}}{2} = -1 \pm i\sqrt{14}$$

These values of λ are imaginary and hence rejected.

We know by Art. 35 that a system AX = B of n non-homogeneous equations in n unknowns has a **unique** solution if A is non-singular

i.e. if $|A| \neq 0$ *i.e.* if $\lambda \neq 2$

if $\lambda = 2$; then from (1) $|A| = 0$ (from (1))

Solution for $\lambda = 2$

Putting $\lambda = 2$ in the given equations, the system reduces to :

$$2x + 2y - 2z = 1, \quad 4x + 4y - z = 2, \quad 6x + 6y + 2z = 3$$

Augmented matrix is $[A : B] = \begin{bmatrix} 2 & 2 & -2 & : & 1 \\ 4 & 4 & -1 & : & 2 \\ 6 & 6 & 2 & : & 3 \end{bmatrix}$

$$\underset{R_{3,1}(-3)}{\overset{R_{2,1}(-2)}{\sim}} \begin{bmatrix} 2 & 2 & -2 & : & 1 \\ 0 & 0 & 3 & : & 0 \\ 0 & 0 & 8 & : & 0 \end{bmatrix}$$

$$\overset{R_{3,2}(-8/3)}{\sim} \begin{bmatrix} 2 & 2 & -2 & : & 1 \\ 0 & 0 & 3 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$ which is Row-Echelon form.

Here $\rho(A) = \rho(A : B) = 2$ (Number of non-zero rows) < (3, the number of unknowns)

$\therefore$ An infinite number of solutions.

The reduced system of equations is $2x + 2y - 2z = 1, 3z = 0$ or $z = 0$

Taking $y = k, 2x + 2k - 0 = 1$ or $2x = 1 - 2k$

$\therefore$ $x = \frac{1}{2} - k, y = k, z = 0$

Thus the system is consistent for $\lambda = 2$ and there exists an infinite number of solutions $x = \frac{1}{2} - k, y = k, z = 0.$

EXERCISE 5

1. Discuss the consistency of the following system of equations. Find the solution set, if consistent.

(i) $x_1 - 2x_2 + x_3 = 1$

(ii) $x_1 + x_2 + x_3 = 4$
$2x_1 + 5x_2 - 2x_3 = 3$

(iii) $x - 3y + z = 2$
$2x + y + 3z = 3$
$x + 5y + 5z = 2$

(iv) $x_1 + 3x_2 - x_3 = 4$
$2x_1 + x_2 + x_3 = 7$
$2x_1 - 4x_2 + 4x_3 = 6$
$3x_1 + 4x_2 = 11$

(v) $x + 2y - 5z + 9 = 0$
$3x - y + 2z + 5 = 0$
$2x + 3y - z - 3 = 0$
$4x - 5y + z + 3 = 0$ (M.D.U. 2000)

(vi) $x + 2y + 3z + 4t = 0$
$2x + 3y + 4z - 1 = 0$
$3x + 4y + t = 2$
$4x + z + 2t = 3$

(vii) $x + y + 2z + t = 5$
$2x + 3y - z - 2t = 2$
$4x + 5y + 3z = 7.$ (K.U. 2000)

2. Show that the following equations are consistent and solve the same :

$$x + 2y - 5z = -9$$
$$3x - y + 2z = 5$$
$$2x + 3y - z = 3$$
$$4x - 5y + z = -3.$$

3. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then show that the system $AX = B$ is consistent if and only if $b_3 - 2b_1 + b_2 = 0$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

4. For what value of λ, does the system $\begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B$ have (*a*) no solution (*b*) unique solution (*c*) more that one solution if

(*i*) $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (*ii*) $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$?

5. Discuss for all values of λ, the system of equations $x + y + 4z = 6$, $x + 2y - 2z = 6$, $\lambda x + y + z = 6$ as regards the existence and nature of solutions.

6. For what value of r, the system of equations $x - 2y - z + 1 = 0$, $3x - y + 2z - 1 = 0$, $y + rz = 1$ has (*i*) no solution (*ii*) unique solution (*iii*) more than one solution.

7. For what values of a and b, the equations $x + y + 5z = 0$, $x + 2y + 3az - b = 0$, $x + 3y + az = 1$ have (*i*) no solution (*ii*) unique solution (*iii*) infinitely many solutions.

8. Show that the equations $-2x + y + z = a$, $x - 2y + z = b$, $x + y - 2z = c$ have no solution unless $a + b + c = 0$, in which case, they have infinitely many solutions. Find these solutions when $a = 1$, $b = 1$, $c = -2$.

$$\left[\textbf{Hint. } [A : B] \sim \begin{bmatrix} 1 & -2 & 1 & : & b \\ 0 & -3 & 3 & : & a + 2b \\ 0 & 0 & 0 & : & a + b + c \end{bmatrix}\right]$$

ANSWERS

1. (*i*) Consistent ; $x_1 = 1 + 2a - b$, $x_2 = a$, $x_3 = b$

(*ii*) Consistent ; $x_1 = -\frac{7a}{3} + \frac{17}{3}$, $x_2 = \frac{4a}{5} - \frac{5}{3}$, $x_3 = a$

(*iii*) Consistent ; $\left(1 - \frac{1}{5}, \frac{2}{5}\right)$

(*iv*) Consistent ; $x_1 = \frac{17}{5} - \frac{4}{5}x_3$, $x_2 = \frac{1}{5} + \frac{3}{5}x_3$

(*v*) Inconsistent

(*vi*) Consistent ; $\left(\frac{9}{11}, \frac{-1}{11}, \frac{-1}{11}, \frac{-1}{11}\right)$

(*vii*) Inconsistent

2. $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

4. (*i*) (*a*) $\lambda = 1$ (*b*) $\lambda \neq 1$, unique solution (*c*) never
(*ii*) (*a*) $\lambda = 1$, (*b*) $\lambda \neq 1$, (*c*) never

5. Unique solution if $\lambda \neq \dfrac{7}{10}$, otherwise inconsistent

6. (*i*) $r = 1$ (*ii*) $r \neq 1$ (*iii*) never

7. (*i*) $a = 1, b \neq \dfrac{1}{2}$ (*ii*) $a \neq 1$ and for all real b (*iii*) $a = 1, b = \dfrac{1}{2}$

8. $x = -1 + k, y = -1 + k, z = k$ where k is any real number.

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37. System of Homogeneous Equations

Let AX = O be a system of m homogeneous equations in n unknowns. Since the rank the coefficient matrix A and the augmented matrix [A : O] are same, the system is alw consistent. This is also otherwise obvious, since $x_1 = x_2 = \ldots = x_n = 0$ is always a solutio However, we are interested in finding all the solutions.

Case (*i*) $\rho(A) = r = n$. In this case (By Art. 36) the system has X = O as the on solution. i.e. unique solution

Case (*ii*) $\rho(A) = r < n$. Hence By Art. 36, an infinite number of solutions, one them zero solution and all the remaining non-zero solutions.

Hence $n-r$ variables can be selected and assigned arbitrary values and hence there infinite number of solutions.

If $\rho(A) = r$ and $r \leq m < n$, $n-r$ variables can be selected and assigned arbitrary value

Therefore, when the number of equations is less than the number of variables, the equ tions will always have an infinite number of solutions. 29

Example 1. *Solve the equations* $x + 2y + 3z + 4t = 0$, $8x + 5y + z + 4t = 0$, $5x + 6y +$ $+ t = 0$, $8x + 3y + 7z + 2t = 0$.

Sol. These given equations are homogeneous and their matrix form is AX = O.

Here,
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 5 & 1 & 4 \\ 5 & 6 & 8 & 1 \\ 8 & 3 & 7 & 2 \end{bmatrix}$$

$$\underset{R_{4,1}(-8)}{\overset{R_{2,1}(-8)}{\sim}}\ \underset{R_{3,1}(-5)}{} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -11 & -23 & -28 \\ 0 & -4 & -7 & -19 \\ 0 & -13 & -17 & -30 \end{bmatrix}$$

$$\underset{R_{4,3}(-3)}{\overset{R_{2,3}(-3)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 29 \\ 0 & -4 & -7 & -19 \\ 0 & -1 & 4 & 27 \end{bmatrix}$$

$$\underset{R_{4,2}(1)}{\overset{R_{3,2}(4)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 29 \\ 0 & 0 & -15 & 97 \\ 0 & 0 & 2 & 56 \end{bmatrix}$$

$$\overset{R_2\left(-\frac{1}{15}\right)}{\sim} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & -2 & 29 \\ 0 & 0 & 1 & -97/15 \\ 0 & 0 & 2 & 56 \end{bmatrix}$$

$$\overset{R_{4,3}(-2)}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 29 \\ 0 & 0 & 1 & -97/15 \\ 0 & 0 & 0 & 1034/15 \end{bmatrix}$$

$$\overset{R_{4,}\left(-\frac{1}{20}\right)}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 29 \\ 0 & 0 & 1 & -97/15 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\rho(A) = 4 =$ number of variables.

Hence, by Art. 37 (*i*) the equations have a unique solution which is the trivial solution

$$x = 0, y = 0, z = 0, t = 0.$$

Example 2. *Solve the equations* $x + 3y - 2z = 0$, $2x - y + 4z = 0$, $x - 11y + 14z = 0$.

Sol. These given equations are homogeneous and their matrix form is AX = O.

Here,
$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$$\underset{R_{3,1}(-1)}{\overset{R_{2,1}(-2)}{\sim}} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$$\overset{R_{3,2}(-2)}{\sim} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\rho(A) = 2 < 3$ (the number of variables)

$\therefore$ By Art. 37 (*ii*) the system has infinitely many solutions.

The system reduces to $x + 3y - 2z = 0, -7y + 8z = 0$

$$\Rightarrow \quad y = \frac{8}{7}z, \text{ and hence } x + 3\left(\frac{8}{7}z\right) - 2z = 0 \quad \text{or} \quad x = \frac{-24}{7}z + 2z = \frac{10}{7}z$$

Taking $z = k$, any scalar, the general solution is given by $x = -\dfrac{10}{7}k, y = \dfrac{8}{7}k, z = k.$

Example 3. *Solve the equations* $x_1 + x_2 + x_3 + x_4 = 0,\quad x_1 + 3x_2 + 2x_3 + 4x_4,$ $2x_1 + x_3 - x_4 = 0.$

Sol. These given equations are homogeneous and their matrix form is $AX = O$.

Here,
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & -1 \end{bmatrix}$$

$$\underset{R_{3,1}(-2)}{\overset{R_{2,1}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & -2 & -1 & -3 \end{bmatrix}$$

$$\underset{R_{3,2}(1)}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_{3,2}\left(\tfrac{1}{2}\right) \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, $\rho(A) = \rho_R(A) = 2$ and number of variables is 4.

$\therefore$ The system has infinitely many solutions and the general solution contains $4 - 2$, arbitrary constants.

The system is equivalent to

$$x_1 + \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \quad i.e. \quad x_1 = \frac{-1}{2}x_3 + \frac{1}{2}x_4$$

$$x_2 + \frac{1}{2}x_3 + \frac{3}{2}x_4 = 0 \quad i.e. \quad x_2 = \frac{-1}{2}x_3 - \frac{-3}{2}x_4$$

Let $\quad x_3 = a, x_4 = b,$ then $x_2 = -\dfrac{1}{2}a - \dfrac{3}{2}b$ and $x_1 = -\dfrac{1}{2}a + \dfrac{1}{2}b$

Thus, the complete solution is

$$x_1 = -\frac{1}{2}a + \frac{1}{2}b, x_2 = -\frac{1}{2}a - \frac{3}{2}b \quad \text{and} \quad x_3 = a, x_4 = b.$$

Example 4. *For what value of* λ, *the system* $\begin{bmatrix} 1 & 2 \\ 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *has (i) a unique solu*

(ii) more than one solution ?

Sol. These given equations are homogeneous and their matrix form is $AX = O$.

Here,
$$A = \begin{bmatrix} 1 & 2 \\ 3 & \lambda \end{bmatrix}$$

operate
$$\overset{R_{2,1}(-3)}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & \lambda - 6 \end{bmatrix}$$

Case I. $\lambda - 6 \neq 0$ **i.e.** $\lambda \neq 6$

Operating $R_2\left(\frac{1}{\lambda-6}\right)$ in (1), $A \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ which is row-echelon form.

$$\rho(A) = 2 \quad \text{(number of non-zero rows)}$$
$$= n \text{ (Number of unknowns } x \text{ and } y)$$

$\therefore$ By Art. 37(i), the system has unique solution and that is trival solution $(x = 0, y = 0)$

Case II. $\lambda - 6 = 0$ **i.e.** $\lambda = 6$

Putting $\lambda = 6$ in (1), $A \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ which is row-echelon form.

$\rho(A) = 1 < 2$, number of unknowns

$\therefore$ By Art. 37 (ii), the system has an infinite number of solutions and hence more than one solution.

The reduced system of equations is $x + 2y = 0$.

Let $y = k$. Therefore $x + 2k = 0$ or $x = -2k$

$\therefore$ $x = -2k$ and $y = k$ is the general solution (k being arbitrary)

Example 5. *Find the value of k such that the system of equations $x + ky + 3z = 0$, $4x + 3y + kz = 0$, $2x + y + 2z = 0$ has a nontrivial solution.*

Sol. The given equations are homogeneous and their matrix form is AX = O.

Here
$$A = \begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix}$$

Let us operate $R_{1,3}$ to bring the unknown k to a lower place in the matrix.

$$A \sim \begin{bmatrix} 2 & 1 & 2 \\ 4 & 3 & k \\ 1 & k & 3 \end{bmatrix} \underset{R_{3,1}\left(-\frac{1}{2}\right)}{\overset{R_{2,1}(-2)}{\sim}} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & k-4 \\ 0 & k-\frac{1}{2} & 2 \end{bmatrix}$$

$$\overset{R_{3,1}(-(k-\frac{1}{2}))}{\sim} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & k-4 \\ 0 & 0 & 2-(k-4)(k-\frac{1}{2}) \end{bmatrix}$$

$\therefore$ for non-trivial solution, $\rho(A) < 3$ (number of unknowns) (clearly $\rho(A) \geq 2$)

$$\Rightarrow \quad 2 - (k-4)(k - \tfrac{1}{2}) = 0.$$

$$\Rightarrow \quad 2k^2 - 9k = 0 \Rightarrow k = 0, \frac{9}{2}.$$

EXERCISE

1. Solve the following systems of homogeneous equations :

(i) $3x - y + z = 0$
$-15x + 6y - 5z = 0$
$5x - 2y + 2z = 0$

(ii) $x - 2y + 3z = 0$
$2x + 5y + 6z = 0$

(iii) $2x - y + 3z = 0$
$3x + 2y + z = 0$
$x - 4y + 5z = 0$

(iv) $x + y + 3z = 0$
$x - y + z = 0$
$x - 2y = 0$
$x - 2y + z = 0$

(v) $x + y + z + t = 0$
$x + 3y - 2z + t = 0$
$2x - 3z + 2t = 0$
$x + y + t = 0$

(vi) $x + y - z + t = 0$
$x - y + 2z - t = 0$
$3x + y + t = 0.$

2. For what value of λ, does the system $\begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 0 & 1 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$ have (*i*) a unique solution (*ii*) more than one solution.

3. Show that the only real value of λ for which the following equations have non-zero solutions is 6
$x + 2y + 3z = \lambda x,\ 3x + y + 2z = \lambda y,\ 2x + 3y + z = \lambda z.$

$$\left[\textbf{Hint. } |A| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = (6-\lambda)(\lambda^2 + 3\lambda + 3) = 0 \Rightarrow \lambda = 6, \frac{-3 \pm i\sqrt{3}}{2}.\right]$$

4. Discuss for all values of λ the system of equations :

$$2x + 3\lambda y + (3\lambda + 4)z = 0$$
$$x + (\lambda + 4)y + (4\lambda + 2)z = 0$$
$$x + 2(\lambda + 1)y + (3\lambda + 4)z = 0.$$

ANSWERS

1. (*i*) $x = 0, y = 0, z = 0$
(*ii*) $x = -3a, y = 0, z = a$ where $a \in \mathbb{R}$
(*iii*) $x = -y = -z = a$
(*iv*) $x = 0, y = 0, z = 0$
(*v*) $x = -a, y = 0, z = 0, t = a$
(*vi*) $x = -\frac{1}{2}b, y = \frac{3}{2}b - a, z = b, t = a$

2. (*i*) Unique solution if $\lambda \neq 1$
(*ii*) More than 1 solution if $\lambda = 1$

4. $\lambda \neq \pm 2$ null sol.
If $\lambda = 2$, then $x = 0, y = -5t, z = 3t$
If $\lambda = -2$, then $x = 4t, y = t, z = t$.

38. Linear Independence of Row and Column Matrices

Recall that a three dimensional vector $\vec{a} = a_1\hat{i} + a_2\hat{i} + a_3\hat{k}$ is also written as an ordered triplet (a_1, a_2, a_3) and vice versa. Generalization of this to n-dimension is an ordered n-triple $(a_1, a_2, ..., a_n)$ which is called an n-dimensional vector. Likewise, a matrix $X = [a_1, a_2 ... a_n]_{1 \times n}$ (called a row-matrix) can be considered an n-dimensional vector and is written as $X = (a_1, a_2, ..., a_n)$

$$\text{A matrix } X = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n \times 1} \qquad \text{(called a column-matrix)}$$

is also an n-dimensional vector and is wirtten as $[a_1, a_2, ..., a_n]$.

When vectors are considered as row matrices or column matrices, the operations of addition of vectors will have the same properties as addition of matrices.

39. Linear Independence and Dependence of row matrices *n-row matrices $X_1, X_2, ..., X_r$ (each of type $1 \times n$) are said to be linearly dependent if there exists r scalars $\alpha_1, \alpha_2, ..., \alpha_r$ not all zero (i.e. at least one of $\alpha_1, \alpha_2 ... \alpha_r$ is non-zero) such that*

$$\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_r X_r = O.$$

If $X_1, X_2, ..., X_r$ are not linearly dependent, we call $X_1, X_2, ..., X_r$ as linearly independent.

Note that if $X_1, X_2 ... X_r$ are linearly independent and for some scalars $\alpha_1, \alpha_2, ..., \alpha_r$,

$$\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_r X_r = O$$

then $\alpha_1 = 0, \alpha_2 = 0, ..., \alpha_r = 0$ is the only possible set of values of $\alpha_1, \alpha_2, ... \alpha_r$.

Likewise one can define the linear independence (L.I.) and linear dependence (L.D.) of column matrices.

Example 1. *Examine the linear independence of the following sets of row-matrices :*

(i) (1, 0), (0, 1) *(ii) (1, 1, 1), (1, 2, 3), (0, 1, 2)*

(iii) (1, 1, 1), (1, 2, 3), (3, 3, 4) *(iv) (1, 2, 3), (1, 0, 0), (0, 2, 3)*

Sol. (*i*) Consider $a(1, 0) + b(0, 1) = O$ for some scalars a, b.

$$\Rightarrow \quad (a, 0) + (0, b) = (0, 0) \Rightarrow a = 0, b = 0$$

$\therefore$ (1, 0) and (0, 1) are linearly independent.

(*ii*) Consider $a(1, 1, 1) + b(1, 2, 3) + c(0, 1, 2) = O$ ...(1)

for some scalars a, b, c

$$\Rightarrow \quad (a, a, a) + (b, 2b, 3b) + (o, c, 2c) = 0$$

$$\Rightarrow \quad (a + b, a + 2b + c, a + 3b + 2c) = (0, 0, 0)$$

Equating corresponding entries

$$\Rightarrow \quad a + b = 0 \quad ...(1)$$

$$a + 2b + c = 0 \quad ...(2)$$

$$a + 3b + 2c = 0 \quad ...(3)$$

Let us solve (1), (2), (3) simultaneously for a, b and c

From (1), $b = -a$,

Putting $b = -a$ in Eqn (2), $a - 2a + c = 0$ or $c = a$.

Putting $b = -a$ and $c = a$ in Eqn. (3), we have

$$a - 3a + 2a = 0 \text{ or } 0 = 0 \text{ which is satisfied for each } a$$

$\therefore$ $b = -a, c = a$ is a solution for each a.

(In particular, $a = 1, b = -1, c = 1$ is a solution)

Hence, By Art. 39 given vectors (row matrices) are linearly dependent.

Second Method. Matrix form of Homogeneous Equations (1), (2) and (3) in three unknowns a, b, c is $AX = O$

where
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

Expanding by first row $= 1(4-3) - 1(2-1) + 0(3-2) = 1 - 1 = 0$

$\therefore$ Matrix A is singular

Hence
$$\rho(A) < n\ (= 3)$$

$\therefore$ By Art. 37 (*ii*), an infinite number of sets of values of a, b, c (both zero and non-zero) exist

$\therefore$ The given row matrices are linearly dependent. (By Art. 39)

(*iii*) Consider $a(1, 1, 1) + b(1, 2, 3) + c(3, 3, 4) = O$ for some scalars a, b, c.
$$a + b + 3c = 0,\ a + 2b + 3c = 0 \quad \text{and} \quad a + 3b + 4c = 0$$
which are 3 homogeneous equations in 3 unknowns a, b, c. Matrix form is $AX = O$.

Now,
$$|A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1(8-9) - 1(4-3) + 3(3-2) = 1 \neq 0$$

$\therefore$ Matrix A is non-singular

$\therefore$
$$\rho(A) = n (= 3)$$

$\therefore$
$$a = 0, b = 0, c = 0 \text{ is the only solution.}$$

Hence, the given vectors are linearly independent.

(*iv*) Consider $a(1, 2, 3) + b(1, 0, 0) + c(0, 2, 3) = O$ for some scalars a, b, c.

or
$$(a, 2a, 3a) + (b, 0, 0) + (0, 2c, 3c) = O$$
or
$$(a + b, 2a + 2c, 3a + 3c) = O = (0, 0, 0)$$

Equating corresponding entries, we have
$$a + b = 0 \quad \text{or} \quad a + b + oc = 0$$
$$2a + 2c = 0 \quad \text{or} \quad 2a + ob + 2c = 0$$
$$3a + 3c = 0 \quad \text{or} \quad 3a + ob + 3c = 0$$

Matrix form of these three homogeneous equations in three unknowns a, b, c is $AX = O$

where
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad |A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{vmatrix}$$

Expanding by first row $= 1(0-0) - 1(6-6) + 0 = 0$

∴ *Matrix A is singular.*

∴ $\rho(A) < 3\ (= n)$

∴ By Art. 37 (*ii*) ∃ non-zero values (also) of a, b, c.

∴ The given row matrices are linearly dependent.

Example 2. *Find a if the column matrices* $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ *and* $\begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix}$ *are linearly dependent.*

Sol. Consider $\alpha \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \gamma \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix} = O$ for some scalars α, β, γ.

or $$\begin{bmatrix} \alpha \\ -\alpha \\ 3\alpha \end{bmatrix} + \begin{bmatrix} \beta \\ 2\beta \\ -3\beta \end{bmatrix} + \begin{bmatrix} a\gamma \\ 0\gamma \\ \gamma \end{bmatrix} = O \quad \text{or} \quad \begin{bmatrix} \alpha + \beta + a\gamma \\ -\alpha + 2\beta + 0\gamma \\ 3\alpha - 3\beta + \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equating corresponding entries, we have

$$\alpha + \beta + a\gamma = 0, -\alpha + 2\beta + 0\gamma = 0, \quad 3\alpha - 3\beta + \gamma = 0$$

Matrix form of these equations is AX = O

where $$A = \begin{bmatrix} 1 & 1 & a \\ -1 & 2 & 0 \\ 3 & -3 & 1 \end{bmatrix} \text{ and } X = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Because the given column matrices are linearly dependent, therefore, there exist non-zero values of α, β, γ.

∴ By Art 37 (*ii*) $\rho(A) < n\ (= 3)$.

∴ Matrix A is singular *i.e.* $|A| = 0$

i.e. $$\begin{vmatrix} 1 & 1 & a \\ -1 & 2 & 0 \\ 3 & -3 & 1 \end{vmatrix} = 0$$

Expanding by first row, $1(2-0) - 1(-1-0) + a(3-6) = 0$ or $3 - 3a = 0$ or $3a = 3$

∴ $a = 1$.

40. The k n-vectors $X_1, X_2, ..., X_k$ are linearly dependent if and only if the rank of the matrix $A = [X_1\ X_2\ ...\ X_n]$ with given vectors as columns is less than k.

Proof. Consider the relation $\alpha_1 X_1 + \alpha_2 X_2 + \ ...\ + \alpha_k X_k = O$ ...(1)

for some scalars $\alpha_1, \alpha_2, ..., \alpha_k$.

If $$X_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}, i = 1, 2, ..., k, \text{ then}$$

(1) becomes
$$\alpha_1\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{n1}\end{bmatrix}+\alpha_2\begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{n2}\end{bmatrix}+\ldots+\alpha_k\begin{bmatrix}a_{1k}\\a_{2k}\\\vdots\\a_{nk}\end{bmatrix}=O$$

$$\Rightarrow \begin{bmatrix}a_{11}\alpha_1+a_{12}\alpha_2+\ldots+a_{1k}\alpha_k\\a_{21}\alpha_1+a_{22}\alpha_2+\ldots+a_{2k}\alpha_k\\\ldots\ldots\ldots\ldots\ldots\ldots\\a_{n1}\alpha_1+a_{n2}\alpha_2+\ldots+a_{nk}\alpha_k\end{bmatrix}=O$$

$$\Rightarrow \begin{bmatrix}a_{11}&a_{12}&\ldots&a_{1k}\\a_{21}&a_{22}&\ldots&a_{2k}\\\ldots&\ldots&\ldots&\ldots\\a_{n1}&a_{n2}&\ldots&a_{nk}\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_k\end{bmatrix}=O \Rightarrow AX=O \quad \ldots(2)$$

(*i*) **Let $X_1, X_2, \ldots, X_k$ be linearly dependent.** Then, from (1) scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ are not all zero.

$\therefore$ the homogeneous linear equations given by (2) have a non-null solution. $\therefore$ $\rho(A)<k$.

(*ii*) **Conversely, let $\rho(A) < k$**

Then, the homogeneous equations given by (2) have a non-null solution.

$\Rightarrow X \neq O \Rightarrow \alpha_1, \alpha_2, \ldots, \alpha_k$ are not all zero.

$\therefore$ from (1), vectors $X_1, X_2, \ldots, X_k$ are linearly dependent.

Cor. The k n-vectors $X_1, X_2, \ldots, X_k$ are linearly independent if the rank of the matrix $A = [X_1, X_2, \ldots, X_k]$ is equal to k.

(Proceed as in the above theorem and the result follows from (2))

41. Theorem.

A square matrix A is singular if and only if its columns (rows) are linearly dependent.

Proof. Let A be an n-square matrix having $X_1, X_2, \ldots, X_n$ as its columns.

i.e., $$A = [X_1 X_2 \ldots X_n]$$

Consider the relation
$$\alpha_1X_1+\alpha_2X_2+\ldots+\alpha_nX_n = O. \quad \ldots(1)$$
for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$

Proceeding as in the theorem 40, we have
$$\begin{bmatrix}a_{11}&a_{12}&\ldots&a_{1n}\\a_{21}&a_{22}&\ldots&a_{2n}\\\vdots&\vdots&\vdots&\vdots\\a_{n1}&a_{n2}&\ldots&a_{nn}\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{bmatrix}=O$$
or
$$AX=O \quad \ldots(2)$$

(*i*) *Let the vectors $X_1, X_2, \ldots, X_n$ be linearly dependent.*

$\therefore$ from (1) scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not all zero. Therefore the homogeneous equations given by (2) have a non-null solution.

$\therefore \qquad \rho(A) < n.$

$\therefore \qquad |A| = 0 \quad \Rightarrow \quad$ A is singular.

(*ii*) *Conversely, let A be singular*

$\Rightarrow \quad |A| = 0$

$\Rightarrow$ from (2), the equations have a non-null solution.

$\Rightarrow \quad X \neq O.$

$\therefore$ the scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not all zero.

Hence, from (1), $X_1, X_2, \ldots, X_n$ are linearly dependent.

EXERCISE 7

1. Examine for linear independence or dependence of the following sets of vectors :

(*i*) $\{(1, 2, 3), (2, -2, 0)\}$ (*ii*) $\{(1, 2, 3), (3, -2, 1), (1, -6, -5)\}$

(*iii*) $\{(1, 3, 2), (5, -2, 1), (-7, 13, 14)\}$ (*iv*) $\{(1, 1, 1), (1, 2, 3), (2, 3, 8)\}$

(*v*) $\{(3, 0, -3), (-1, 1, 2), (4, 2, -2), (2, 1, 1)\}$.

2. Show that three row vectors as well as the three column vectors of the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 7 & -6 & 17 \\ 5 & 2 & 7 \end{bmatrix}$ are linearly dependent.

ANSWERS

1. (*i*) L.I. (*ii*) L.D. (*iii*) L.D. (*iv*) L.I. (*v*) L.D.