(Exercise 6.3)

Q1 Check which of the following define linear transformations from R^3 to R^2 ?

(i) $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$

(i) then we prove $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now $T(u_1+u_2) = T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big)$

$= T(x_1+y_1, x_2+y_2, x_3+y_3)$

$= \big((x_1+y_1)-(x_2+y_2), (x_1+y_1)-(x_3+y_3)\big)$

$= (x_1+y_1-x_2-y_2, x_1+y_1-x_3-y_3)$

$= (x_1-x_2+y_1-y_2, x_1-x_3+y_1-y_3)$

$= (x_1-x_2, x_1-x_3) + (y_1-y_2, y_1-y_3)$

$= T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$

$= T(u_1) + T(u_2)$

(ii) Let $a \in R$ & $u_1 = (x_1, x_2, x_3) \in R^3$

then we prove $T(au_1) = aT(u_1)$

Now $T(au_1) = T\big(a(x_1, x_2, x_3)\big)$

$= T(ax_1, ax_2, ax_3)$

$= (ax_1 - ax_2, ax_1 - ax_3)$

$= a(x_1 - x_2, x_1 - x_3)$

$= aT(x_1, x_2, x_3)$

$= aT(u_1)$

Hence, T is a linear transformation from R^3 to R^2

(ii) $T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$

let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (|x_1+y_1|, (x_2+y_2) - (x_3+y_3))$$
$$\therefore T(u_1+u_2) = (|x_1+y_1|, x_2+y_2-x_3-y_3) \quad \text{——— (1)}$$

Now

$$T(u_1)+T(u_2) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$
$$= (|x_1|, x_2-x_3) + (|y_1|, y_2-y_3)$$
$$= (|x_1|+|y_1|, x_2-x_3+y_2-y_3)$$
$$\therefore T(u_1)+T(u_2) = (|x_1|+|y_1|, x_2+y_2-x_3-y_3) \quad \text{——— (2)}$$

From (1) & (2)

$$T(u_1+u_2) \neq T(u_1)+T(u_2)$$

Hence T is not a linear transformation from R^3 to R^2.

(iii) $T(x_1, x_2, x_3) = (x_1+1, x_2+x_3)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (x_1+1, x_2+x_3)$

let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$

$$T(u_1+u_2) = T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_1+y_1+1, x_2+y_2+x_3+y_3)$$
$$T(u_1+u_2) = (x_1+y_1+1, x_2+x_3+y_2+y_3) \quad \text{--- (1)}$$

Now

$$T(u_1)+T(u_2) = T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= (x_1+1, x_2+x_3) + (y_1+1, y_2+y_3)$$
$$= (x_1+1+y_1+1, x_2+x_3+y_2+y_3)$$
$$T(u_1)+T(u_2) = (x_1+y_1+2, x_2+x_3+y_2+y_3) \quad \text{--- (2)}$$

From (1) & (2)

$$T(u_1+u_2) \neq T(u_1)+T(u_2)$$

Hence T is not a linear transformation from R^3 to R^2.

(iv) $T(x_1,x_2,x_3) = (0, x_3)$

<u>Sol.</u> Given transformation is

$$T(x_1,x_2,x_3) = (0, x_3)$$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove.

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (0, x_3+y_3)$$
$$= (0,x_3)+(0,y_3)$$
$$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$\therefore T(u_1+u_2) = T(u_1)+T(u_2)$$

(ii) Let $\alpha \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove

$$T(\alpha u_1) = \alpha T(u_1)$$

Now $T(\alpha u_1) = T(\alpha(x_1,x_2,x_3))$

$$T(au_1) = T(ax_1, ax_2, ax_3)$$
$$= (0, ax_3)$$
$$= a(0, x_3)$$
$$= aT(x_1, x_2, x_3)$$
$$T(au_1) = aT(u_1)$$

Hence T is a linear transformation from R^3 to R^2.

(v) $T(x_1, x_2, x_3) = \left(\frac{x_1+x_2}{x_3}, x_3\right)$

Sol. Given transformation is

$$T(x_1, x_2, x_3) = \left(\frac{x_1+x_2}{x_3}, x_3\right)$$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big)$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= \left(\frac{x_1+y_1+x_2+y_2}{x_3+y_3}, x_3+y_3\right) \quad \text{——— (1)}$$

Now

$$T(u_1) + T(u_2) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$
$$= \left(\frac{x_1+x_2}{x_3}, x_3\right) + \left(\frac{y_1+y_2}{y_3}, y_3\right)$$
$$= \left(\frac{x_1+x_2}{x_3} + \frac{y_1+y_2}{y_3}, x_3+y_3\right)$$
$$= \left(\frac{y_3(x_1+x_2)+x_3(y_1+y_2)}{x_3y_3}, x_3+y_3\right) \quad \text{——— (2)}$$

From (1) & (2)

$$T(u_1+u_2) \neq T(u_1) + T(u_2)$$

Hence T is not a linear transformation from R^3 to R^2.

(vi) $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3,\ x_3 - 3x_2 - 2x_1)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3,\ x_3 - 3x_2 - 2x_1)$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$
\begin{aligned}
T(u_1 + u_2) &= T((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\
&= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= (3(x_1 + y_1) - 2(x_2 + y_2) + (x_3 + y_3),\ (x_3 + y_3) - 3(x_2 + y_2) - 2(x_1 + y_1)) \\
&= (3x_1 - 2x_2 + x_3 + 3y_1 - 2y_2 + y_3,\ x_3 - 3x_2 - 2x_1 + y_3 - 3y_2 - 2y_1) \\
&= (3x_1 - 2x_2 + x_3,\ x_3 - 3x_2 - 2x_1) + (3y_1 - 2y_2 + y_3,\ y_3 - 3y_2 - 2y_1) \\
&= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
&= T(u_1) + T(u_2)
\end{aligned}
$$

(ii) Let $a \in R$ & $u_1 = (x_1, x_2, x_3) \in R^3$ then we prove

$T(au_1) = aT(u_1)$

$$
\begin{aligned}
\text{Now } T(au_1) &= T(a(x_1, x_2, x_3)) \\
&= T(ax_1, ax_2, ax_3) \\
&= (3ax_1 - 2ax_2 + ax_3,\ ax_3 - 3ax_2 - 2ax_1) \\
&= a(3x_1 - 2x_2 + x_3,\ x_3 - 3x_2 - 2x_1) \\
&= aT(x_1, x_2, x_3) \\
&= aT(u_1)
\end{aligned}
$$

Hence T is a linear transformation from R^3 to R^2.

Q2 Show that each of the following defines linear transformation from R^3 to R^3.

(i) $T(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 - x_3,\ x_1)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1)$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned}
T(u_1+u_2) &= T\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big) \\
&= T(x_1+y_1, x_2+y_2, x_3+y_3) \\
&= \big((x_1+y_1) - (x_2+y_2), (x_2+y_2) - (x_3+y_3), x_1+y_1\big) \\
&= (x_1+y_1-x_2-y_2, x_2+y_2-x_3-y_3, x_1+y_1) \\
&= (x_1-x_2+y_1-y_2, x_2-x_3+y_2-y_3, x_1+y_1) \\
&= (x_1-x_2, x_2-x_3, x_1) + (y_1-y_2, y_2-y_3, y_1) \\
&= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
&= T(u_1) + T(u_2)
\end{aligned}$$

Now

$$\begin{aligned}
T(au_1) &= T\big(a(x_1, x_2, x_3)\big) \\
&= T(ax_1, ax_2, ax_3) \\
&= (ax_1 - ax_2, ax_2 - ax_3, ax_1) \\
&= a(x_1 - x_2, x_2 - x_3, x_1) \\
&= aT(x_1, x_2, x_3) \\
&= aT(u_1)
\end{aligned}$$

Hence T is a linear transformation from R^3 to R^3

(ii) $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

Sol. Given transformation is

$T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_1+y_1+x_2+y_2, \ -(x_1+y_1)-(x_2+y_2), \ x_3+y_3)$$
$$= (x_1+y_1+x_2+y_2, \ -x_1-y_1-x_2-y_2, \ x_3+y_3)$$
$$= (x_1+x_2+y_1+y_2, \ -x_1-x_2-y_1-y_2, \ x_3+y_3)$$
$$= (x_1+x_2, -x_1-x_2, x_3) + (y_1+y_2, -y_1-y_2, y_3)$$
$$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= T(u_1) + T(u_2)$$

(ii) Let $a \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove $T(au_1) = aT(u_1)$

Now

$$T(au_1) = T(a(x_1,x_2,x_3))$$
$$= T(ax_1, ax_2, ax_3)$$
$$= (ax_1+ax_2, \ -ax_1-ax_2, \ ax_3)$$
$$= a(x_1+x_2, -x_1-x_2, x_3)$$
$$= aT(x_1,x_2,x_3)$$
$$= aT(u_1)$$

Hence T is a <u>linear</u> transformation from R^3 to R^3

(iii) $T(x_1,x_2,x_3) = (x_2, -x_1, -x_3)$

Sol. Given transformation is

$T(x_1,x_2,x_3) = (x_2, -x_1, -x_3)$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$

$$T(u_1+u_2) = T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= (x_2+y_2, -(x_1+y_1), -(x_3+y_3))$$
$$= (x_2+y_2, -x_1-y_1, -x_3-y_3).$$
$$= (x_2, -x_1, -x_3) + (y_2, -y_1, -y_3)$$
$$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= T(u_1) + T(u_2)$$

(ii) Let $a \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove

$$T(au_1) = aT(u_1)$$

Now

$$T(au_1) = T(a(x_1,x_2,x_3))$$
$$= T(ax_1, ax_2, ax_3)$$
$$= (ax_2, -ax_1, -ax_3)$$
$$= a(x_2, -x_1, -x_3)$$
$$= aT(x_1,x_2,x_3)$$
$$= aT(u_1)$$

Hence T is a <u>linear</u> transformation from R^3 to R^3.

(iv) $T(x_1,x_2,x_3) = (x_1-3x_2-2x_3, x_2-4x_3, x_3)$

Sol. Given transformation is

$$T(x_1,x_2,x_3) = (x_1-3x_2-2x_3, x_2-4x_3, x_3)$$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$T(u_1+u_2) = T((x_1,x_2,x_3)+(y_1,y_2,y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= ((x_1+y_1)-3(x_2+y_2)-2(x_3+y_3), (x_2+y_2)-4(x_3+y_3), x_3+y_3)$$
$$= (x_1-3x_2-2x_3+y_1-3y_2-2y_3, x_2-4x_3+y_2-4y_3, x_3+y_3)$$

$$T(u_1+u_2) = (x_1-3x_2-2x_3\,,\, x_2-4x_3\,,\, x_3) + (y_1-3y_2-2y_3\,,\, y_2-4y_3\,,\, y_3)$$
$$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= T(u_1) + T(u_2)$$

(ii) let $\alpha \in R$ & $u_1 = (x_1,x_2,x_3) \in R^3$ then we prove
$$T(\alpha u_1) = \alpha T(u_1)$$

Now

$$T(\alpha u_1) = T\big(\alpha(x_1,x_2,x_3)\big)$$
$$= T(\alpha x_1, \alpha x_2, \alpha x_3)$$
$$= (3\alpha x_1 - 3\alpha x_2 - 2\alpha x_3\,,\, \alpha x_2 - 4\alpha x_3\,,\, \alpha x_3)$$
$$= \alpha(3x_1 - 3x_2 - 2x_3\,,\, x_2 - 4x_3\,,\, x_3)$$
$$= \alpha T(x_1,x_2,x_3)$$
$$= \alpha T(u_1)$$

Hence T is a linear transformation from R^3 to R^3.

(v) $T(x_1,x_2,x_3) = (x_1+x_3\,,\, x_1-x_3\,,\, x_2)$

Sol. Given Transformation is
$$T(x_1,x_2,x_3) = (x_1+x_3\,,\, x_1-x_3\,,\, x_2)$$

Let $u_1 = (x_1,x_2,x_3)$

& $u_2 = (y_1,y_2,y_3) \in R^3$ then we prove
$$T(u_1+u_2) = T(u_1) + T(u_2)$$

Now
$$T(u_1+u_2) = T\big((x_1,x_2,x_3)+(y_1,y_2,y_3)\big)$$
$$= T(x_1+y_1\,,\, x_2+y_2\,,\, x_3+y_3)$$
$$= \big((x_1+y_1)+(x_3+y_3)\,,\, (x_1+y_1)-(x_3+y_3)\,,\, x_2+y_2\big)$$
$$= (x_1+x_3+y_1+y_3\,,\, x_1-x_3+y_1-y_3\,,\, x_2+y_2)$$
$$= (x_1+x_3\,,\, x_1-x_3\,,\, x_2) + (y_1+y_3\,,\, y_1-y_3\,,\, y_2)$$
$$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$$
$$= T(u_1) + T(u_2)$$

(ii) Let $a \in R$ & $u_1 = (x_1, x_2, x_3) \in R^3$ then we prove

$T(au_1) = aT(u_1)$

Now

$$\begin{aligned} T(au_1) &= T(a(x_1, x_2, x_3)) \\ &= T(ax_1, ax_2, ax_3) \\ &= (ax_1 + ax_3, ax_1 - ax_3, ax_2) \\ &= a(x_1 + x_3, x_1 - x_3, x_2) \\ &= aT(x_1, x_2, x_3) \\ &= aT(u_1) \end{aligned}$$

Hence T is a linear transformation from R^3 to R^3.

Q3 Show that each of the following transformations is not linear.

(i) $T : R^2 \longrightarrow R$ defined by $T(x_1, x_2) = x_1 x_2$

Sol. Given transformation is

$T(x_1, x_2) = x_1 x_2$

let $u_1 = (x_1, x_2)$

& $u_2 = (y_1, y_2) \in R^2$ then we prove

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1 + u_2) &= T((x_1, x_2) + (y_1, y_2)) \\ &= T(x_1 + y_1, x_2 + y_2) \\ &= (x_1 + y_1)(x_2 + y_2) \quad \text{——— (1)} \end{aligned}$$

&

$$\begin{aligned} T(u_1) + T(u_2) &= T(x_1, x_2) + T(y_1, y_2) \\ &= x_1 x_2 + y_1 y_2 \quad \text{——— (2)} \end{aligned}$$

from (1) & (2) $T(u_1 + u_2) \neq T(u_1) + T(u_2)$.

Hence T is not a linear transformation from R^2 to R

(ii) $T: R^2 \longrightarrow R^3$ defined by $T(x_1, x_2) = (x_1+1, 2x_2, x_1+x_2)$

Sol: Given transformation is

$$T(x_1, x_2) = (x_1+1, 2x_2, x_1+x_2)$$

Let $u_1 = (x_1, x_2)$

& $u_2 = (y_1, y_2) \in R^2$ then we prove

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$$\begin{aligned} T(u_1+u_2) &= T\big((x_1, x_2)+(y_1, y_2)\big) \\ &= T(x_1+y_1, x_2+y_2) \\ &= \big(x_1+y_1+1, 2(x_2+y_2), (x_1+y_1)+(x_2+y_2)\big) \\ &= (x_1+y_1+1, 2x_2+2y_2, x_1+x_2+y_1+y_2) \quad \text{--- (1)} \end{aligned}$$

$$\begin{aligned} T(u_1)+T(u_2) &= T(x_1, x_2) + T(y_1, y_2) \\ &= (x_1+1, 2x_2, x_1+x_2) + (y_1+1, 2y_2, y_1+y_2) \\ &= (x_1+y_1+2, 2x_2+2y_2, x_1+x_2+y_1+y_2) \quad \text{--- (2)} \end{aligned}$$

from (1) & (2)

$$T(u_1+u_2) \neq T(u_1)+T(u_2)$$

Hence T is not a linear transformation from R^2 to R^3.

(iii) $T: R^3 \longrightarrow R^2$ defined by $T(x_1, x_2, x_3) = (|x_1|, 0)$

Sol: Given transformation is

$$T(x_1, x_2, x_3) = (|x_1|, 0)$$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$$\begin{aligned} T(u_1+u_2) &= T\big((x_1, x_2, x_3)+(y_1, y_2, y_3)\big) \\ &= T(x_1+y_1, x_2+y_2, x_3+y_3) \end{aligned}$$

$T(u_1+u_2) = (|x_1+y_1|, 0)$ ——— ①

&

$T(u_1)+T(u_2) = T(x_1,x_2,x_3) + T(y_1,y_2,y_3)$

$= (|x_1|, 0) + (|y_1|, 0)$

$= (|x_1|+|y_1|, 0)$ ——— ②

From ① & ②

$T(u_1+u_2) \neq T(u_1)+T(u_2)$

Hence T is not a linear transformation from R^3 to R^2.

(iv) $T: R^2 \to R^2$ defined by $T(x_1,x_2) = (x_1^2, x_2^2)$

Sol. Given transformation is

$T(x_1,x_2) = (x_1^2, x_2^2)$

Let $u_1 = (x_1,x_2)$

& $u_2 = (y_1,y_2) \in R^2$ then we prove

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$

Now

$T(u_1+u_2) = T((x_1,x_2)+(y_1,y_2))$

$= T(x_1+y_1, x_2+y_2)$

$= ((x_1+y_1)^2, (x_2+y_2)^2)$ ———

&

$T(u_1)+T(u_2) = T(x_1,x_2) + T(y_1,y_2)$

$= (x_1^2, x_2^2) + (y_1^2, y_2^2)$

$= (x_1^2+y_1^2, x_2^2+y_2^2)$ ——— ②

from ① & ②

$T(u_1+u_2) \neq T(u_1)+T(u_2)$

Hence T is not a linear transformation from R^2 to R^2

(v) $T: R^3 \longrightarrow R^3$ defined by $T(x_1, x_2, x_3) = (x_1, x_2, x_3) + (1,1,1)$

Sol: Given transformation is

$T(x_1, x_2, x_3) = (x_1, x_2, x_3) + (1,1,1)$

Let $u_1 = (x_1, x_2, x_3)$

& $u_2 = (y_1, y_2, y_3) \in R^3$ then we prove

(i) $T(u_1+u_2) = T(u_1) + T(u_2)$

Now

$$\begin{aligned} T(u_1+u_2) &= T\big((x_1,x_2,x_3)+(y_1,y_2,y_3)\big) \\ &= T(x_1+y_1,\ x_2+y_2,\ x_3+y_3) \\ &= (x_1+y_1,\ x_2+y_2,\ x_3+y_3) + (1,1,1) \\ &= (x_1+y_1+1,\ x_2+y_2+1,\ x_3+y_3+1) \quad \text{——— ①} \end{aligned}$$

&

$$\begin{aligned} T(u_1)+T(u_2) &= T(x_1,x_2,x_3) + T(y_1,y_2,y_3) \\ &= (x_1,x_2,x_3)+(1,1,1)+(y_1,y_2,y_3)+(1,1,1) \\ &= (x_1+1,\ x_2+1,\ x_3+1) + (y_1+1,\ y_2+1,\ y_3+1) \\ &= (x_1+y_1+2,\ x_2+y_2+2,\ x_3+y_3+2) \quad \text{——— ②} \end{aligned}$$

from ① & ②

$T(u_1+u_2) \neq T(u_1) + T(u_2)$

Hence T is not a linear transformation from R^3 to R^3.

Q3 Determine which of the following transformations are linear:

(a) $T: M_{22} \longrightarrow R$ defined by

(i) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d$

Sol: Given transformation is

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d$ ...

Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

& $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_{22}$ then we prove

(i) $T(A_1+A_2) = T(A_1)+T(A_2)$

Now

$$T(A_1+A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right)$$

$$= a_1+a_2+d_1+d_2 \quad \text{———} \quad (1)$$

&

$$T(A_1)+T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= a_1+d_1 + a_2+d_2$$

$$= a_1+a_2+d_1+d_2 \quad \text{———} \quad (2)$$

from (1) & (2)

$$T(A_1+A_2) = T(A_1)+T(A_2)$$

(ii) Let $a \in R$ & $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in M_{22}$ then we prove

$$T(aA_1) = aT(A_1)$$

Now

$$T(aA_1) = T\left(a\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} aa_1 & ab_1 \\ ac_1 & ad_1 \end{bmatrix}\right)$$

$$= aa_1 + ad_1$$

$$= a(a_1+d_1)$$

$$= aT\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$$

$$= aT(A_1)$$

Hence T is a linear transformation from M_{22} to R

(ii) $T: M_{22} \longrightarrow R$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Sol. Given transformation is

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

& $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

then we prove

$$T(A_1 + A_2) = T(A_1) + T(A_2)$$

Now

$$T(A_1 + A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right)$$

$$= (a_1+a_2)(d_1+d_2) - (b_1+b_2)(c_1+c_2) \text{ ——— } (1)$$

Now

$$T(A_1) + T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= a_1 d_1 - b_1 c_1 + a_2 d_2 - b_2 c_2 \text{ ——— } (2)$$

from (1) & (2)

$$T(A_1 + A_2) \neq T(A_1) + T(A_2)$$

Hence T is a linear transformation from M_{22} to R.

(b) $T: P_2(x) \longrightarrow P_2(x)$ defined by

(i) $T(a + bx + cx^2) = a + (b+c)x + (2a - 3b)x^2$

Sol. Given transformation is

$$T(a + bx + cx^2) = a + (b+c)x + (2a - 3b)x^2$$

Let $u = a+bx+cx^2$

& $v = p+qx+\lambda x^2 \in P_2(x)$ then we prove

(i) $T(u+v) = T(u)+T(v)$

Now

$$
\begin{aligned}
T(u+v) &= T\left((a+bx+cx^2)+(p+qx+\lambda x^2)\right) \\
&= T\left((a+p)+(b+q)x+(c+\lambda)x^2\right) \\
&= (a+p)+(b+q+c+\lambda)x+(2a+2p-3b-3q)x^2 \\
&= (a+p)+(b+c+q+\lambda)x+(2a-3b+2p-3q)x^2 \\
&= \left(a+(b+c)x+(2a-3b)x^2\right)+\left(p+(q+\lambda)x+(2p-3q)x^2\right) \\
&= T(a+bx+cx^2)+T(p+qx+\lambda x^2) \\
&= T(u)+T(v)
\end{aligned}
$$

(ii) Let $k \in R$ & $u = a+bx+cx^2$ then we prove

$T(ku) = kT(u)$

Now

$$
\begin{aligned}
T(ku) &= T\left(k(a+bx+cx^2)\right) \\
&= T(ka+kbx+kcx^2) \\
&= ka+(kb+kc)x+(2ka-3kb)x^2 \\
&= k\left(a+(b+c)x+(2a-3b)x^2\right) \\
&= kT(a+bx+cx^2) \\
&= kT(u)
\end{aligned}
$$

Hence T is a linear transformation from $P_2(x)$ to $P_2(x)$.

(iii) $T: P_2(x) \longrightarrow P_2(x)$ defined by

$T(a+bx+cx^2) = (a+1)+bx+cx^2$

Sol. Given transformation is

$T(a+bx+cx^2) = (a+1)+bx+cx^2$

Let $u = a+bx+cx^2$

& $v = p+qx+\lambda x^2 \in P_2(x)$ then we prove

(i) $T(u+v) = T(u) + T(v)$

Now

$$T(u+v) = T\big((a+bx+cx^2)+(p+qx+rx^2)\big)$$
$$= T\big((a+p)+(b+q)x+(c+r)x^2\big)$$
$$= (a+p+1)+(b+q)x+(c+r)x^2 \quad \text{———} \ (1)$$

&

$$T(u)+T(v) = T(a+bx+cx^2) + T(p+qx+rx^2)$$
$$= (a+1)+bx+cx^2 + (p+1)+qx+rx^2$$
$$= (a+p+2)+(b+q)x+(c+r)x^2 \quad \text{———} \ (2)$$

From (1) & (2)

$$T(u+v) \neq T(u) + T(v)$$

Hence T is not a linear transformation from $P_2(x)$ to $P_2(x)$.

Q5. If A is an $m\times n$ matrix, show that $T(x) = Ax$ is a linear transformation from R^n to R^m where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \in R^n$$

Sol: Given transformation is

$$T(x) = Ax$$

Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \quad \& \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix} \in R^n$$

Then we prove

$$T(x+y) = T(x)+T(y)$$

Now

$T(x+y) =$