4.9 Matrix of Linear Transformation

Coordinate Vector

Let V and W be finite dimensional vector spaces over the same field F with $\dim V = n$ and $\dim W = m$. Let $B = \{v_1, v_2, \ldots, v_n\}$ and $E = \{w_1, w_2, \ldots, w_n\}$ be any bases for V and W respectively. Any vector v in V can be expressed in a unique way as a linear combination of $v_1, v_2, \ldots, v_n$, i.e.

$$v = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n, \quad x_i \in F \quad (1 \le i \le n)$$

We call $(x_1, x_2, \ldots, x_n)$ the *coordinate vector* of v relative to the basis B.

If $T : V \to W$ is a linear transformation, then the images $T(v_1), \ldots, T(v_n)$ are elements of W and each can be expressed uniquely as a linear combination of the basis vectors $w_1, w_2, \ldots, w_n$, i.e.

$$\begin{aligned}
T(v_1) &= a_{11} w_1 + a_{21} w_2 + \ldots + a_{m1} w_m \\
T(v_2) &= a_{12} w_1 + a_{22} w_2 + \ldots + a_{m2} w_m \\
&\vdots \\
T(v_j) &= a_{1j} w_1 + a_{2j} w_2 + \ldots + a_{mj} w_m \\
&\vdots \\
T(v_n) &= a_{1n} w_1 + a_{2n} w_2 + \ldots + a_{mn} w_m
\end{aligned}$$

where $a_{ij} \in F$. The $m \times n$ matrix whose jth column is the coordinate vector of $T(v_j)$ is called the *matrix of linear transformation* T with respect to the bases B and E. Thus, the matrix of $T : V \to W$ relative to the bases B and E is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix},$$

Conversely, if $A = [a_{ij}]$ is an $m \times n$ matrix with entries $a_{ij} \in F$, then A represents a linear transformation $T : F^n \to F^m$ defined by the equation $T(x) = Ax$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is column vector of F^n.

Example 1: *Find the matrix of the linear transformation $T : R^3 \to R^4$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_3, x_1)$ with respect to the standard bases for R^3 and R^4.*

Solution: $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_3, x_1)$ ...(1)

Standard basis for R^3 is $B = \{(1,0,0), (0,1,0), (0,0,1)\}$, so by (1)

$$T(1,0,0) = (1,0,0,1) = 1(1,0,0,0) + 0(0,1,0,0) + 1(0,0,1,0) + 1(0,0,0,1)$$
$$T(0,1,0) = (1,1,0,0) = 1(1,0,0,0) + 1(0,1,0,0) + 0(0,0,1,0) + 0(0,0,0,1)$$
$$T(0,0,1) = (0,1,1,0) = 0(1,0,0,0) + 1(0,1,0,0) + 1(0,0,1,0) + 0(0,0,0,1)$$

Thus, the matrix of linear transformation T is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Example 2: *Find the determinant of the matrix obtained from the linear transformation $T : R^3 \to R^3$ defined by $T(x, y, z) = (3x - 2z, 5y + 7z, x + y + z)$.*

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Solution: $T(x, y, z) = (3x - 2z, 5y + 7z, x + y + z)$ ...(1)

Standard basis for R^3 is $B = \{(1,0,0), (0,1,0), (0,0,1)\}$, so by (1)

$$T(1,0,0) = (3,0,1) = 3(1,0,0) + 0(0,1,0) + 1(0,0,1)$$
$$T(0,1,0) = (0,5,1) = 0(1,0,0) + 5(0,1,0) + 1(0,0,1)$$
$$T(0,0,1) = (-2,7,1) = -2(1,0,0) + 7(0,1,0) + 1(0,0,1)$$

Matrix of linear transformation T is $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 5 & 7 \\ 1 & 1 & 1 \end{bmatrix}$.

Determinant of linear transformation T is

$$\det A = \begin{vmatrix} 3 & 0 & -2 \\ 0 & 5 & 7 \\ 1 & 1 & 1 \end{vmatrix} = 3(5-7) - 0 - 2(0-5)$$

$$= 3(-2) - 2(-5) = -6 + 10 = 4$$

Example 3: *A linear transformation $T: R^2 \to R^3$ maps the vector $(1,1)$ into $(0,1,2)$ and the vector $(-1,1)$ into $(2,1,0)$. What matrix does T represent with respect to the standard bases for R^2 and R^3?*

Solution: Here $T(1,1) = (0,1,2)$ and $T(-1,1) = (2,1,0)$. Let $x = (x_1, x_2) \in R^2$ and $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ be the matrix of linear transformation $T: R^2 \to R^3$, then

$$T(x) = Ax = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \\ ex_1 + fx_2 \end{bmatrix}$$

$$\Rightarrow T(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2, ex_1 + fx_2)$$

$$\Rightarrow T(1,1) = (a+b, c+d, e+f)$$

$$\Rightarrow (0,1,2) = (a+b, c+d, e+f)$$

$$\Rightarrow a+b = 0, c+d = 1, e+f = 2 \qquad \ldots(1)$$

Similarly, $T(-1,1) = (2,1,0)$

$$\Rightarrow (-a+b, -c+d, -e+f) = (2,1,0)$$

$$\Rightarrow -a+b = 2,\ -c+d = 1,\ -e+f = 0 \qquad \ldots(2)$$

Solving (1) and (2), we have $a = -1, b = 1, c = 0, d = 1, e = 1, f = 1$. Putting these values in matrix A, we have $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Example 4: *Find the matrix of linear transformation $T: R \to R^2$ defined by $T(x) = (3x, 5x)$ with respect to the standard bases of R and R^2.*

Solution: The standard bases for R and R^2 are $\{e\}, \{e_1, e_2\}$ respectively, where $e = (1), e_1 = (1,0), e_2 = (0,1)$. Now

$$T(e) = T(1) = (3,5) = (3,0) + (0,5) = 3(1,0) + 5(0,1) = 3e_1 + 5e_2$$

This shows that the matrix of linear transformation is $A = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Example 5: *Find the matrix of linear transformation $T: R^3 \to R^2$ defined by $T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3, 5x_1 + 3x_2 - 2x_3)$ with respect to the standard bases of R^3 and R^2.*

Solution: The standard bases for R^3 and R^2 are $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{(1,0),(0,1)\}$ respectively. Now

$$T(1,0,0) = (3,5) = (3,0) + (0,5) = 3(1,0) + 5(0,1)$$
$$T(0,1,0) = (-4,3) = (-4,0) + (0,3) = -4(1,0) + 3(0,1)$$
$$T(0,0,1) = (9,-2) = (9,0) + (0,-2) = 9(1,0) - 2(0,1)$$

Thus, the matrix of linear transformation is $A = \begin{bmatrix} 3 & -4 & 9 \\ 5 & 3 & -2 \end{bmatrix}$.

Example 6: *Find the matrix of linear transformation $T: R^4 \to R$ defined by $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 - 7x_3 + x_4$ with respect to the standard bases of R^4 and R.*

Solution: $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ and $\{(1)\}$ are the standard bases for R^4 and R respectively. Now

$$T(1,0,0,0) = 2 = 2(1)$$
$$T(0,1,0,0) = 3 = 3(1)$$
$$T(0,0,1,0) = -7 = -7(1)$$
$$T(0,0,0,1) = 1 = 1(1)$$

Thus, the matrix of linear transformation is $A = [2 \quad 3 \quad -7 \quad 1]$.

Example 7: *Find the matrix of linear transformation $T: R^2 \to R^4$ defined by $T(x_1, x_2) = (3x_1 + 4x_2, 5x_1 - 2x_2, x_1 + 7x_2, 4x_1)$ with respect to the standard bases of R^2 and R^4.*

Solution: $\{(1,0),(0,1)\}$ and $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ are the standard bases for R^2 and R^4 respectively. Now

$$T(1,0) = (3,5,1,4) = 3(1,0,0,0) + 5(0,1,0,0) + 1(0,0,1,0) + 4(0,0,0,1)$$
$$T(0,1) = (4,-2,7,0) = 4(1,0,0,0) - 2(0,1,0,0) + 7(0,0,1,0) + 0(0,0,0,1)$$

Thus, the matrix of linear transformation is $A = \begin{bmatrix} 3 & 4 \\ 5 & -2 \\ 1 & 7 \\ 4 & 0 \end{bmatrix}$.

Example 8: *The matrix of a linear transformation $T: R^3 \to R^3$ is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.*