$$x_1 = t,\ x_2 = \frac{t}{3},\ x_3 = \frac{t}{3},\ x_4 = t$$

where t is arbitrary. The smallest positive integer values for the unknowns occur when we let $t = 3$, so the equation can be balanced by letting

$$\therefore x_1 = 3,\ x_2 = 1,\ x_3 = 1,\ x_4 = 3$$

Making these substitutions in (1), we have

$$3\,(HCl) + 1\,(Na_3PO_4) \rightarrow 1\,(H_3PO_4) + 3\,(NaCl)$$

or $$3HCl + Na_3PO_4 \rightarrow H_3PO_4 + 3NaCl$$

Polynomial Interpolation

An important problem in various applications is to find a polynomial whose graph passes through a specified set of points in the plane; this is called an ***interpolating polynomial*** for the points. The simplest example of such a problem is to find a linear polynomial

$$p(x) = ax + b \qquad \text{...(1)}$$

whose graph passes through two known distinct points, (x_1, y_1) and (x_2, y_2), in the xy-plane. You have probably encountered various methods in analytic geometry for finding the equation of a line through two points, but here we will give a method based on linear systems that can be adapted to general polynomial interpolation.

The graph of (1) is the line $y = ax + b$, and for this line to pass through the points (x_1, y_1) and (x_2, y_2), we must have

$$y_1 = ax_1 + b \text{ and } y_2 = ax_2 + b$$

Therefore, the unknown coefficients a and b can be obtained by solving the linear system

$$ax_1 + b = y_1$$
$$ax_2 + b = y_2$$

We don't need any fancy methods to solve this system—the value of a can be obtained by subtracting the equations to eliminate b, and then the value of a can be substituted into either equation to find b.

Now let us consider the more general problem of finding a polynomial whose graph passes through n points with distinct x-coordinates

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n) \qquad \ldots(1)$$

Since there are n conditions to be satisfied, intuition suggests that we should begin by looking for a polynomial of the form

$$p(x) = a_0 + a_1x + a_2x_2 + \cdots + a_{n-1}x_{n-1} \qquad \ldots(2)$$

since a polynomial of this form has n coefficients that are at our disposal to satisfy the n conditions. However, we want to allow for cases where the points may lie on a line or have some other configuration that would make it possible to use a polynomial whose degree is less than $n - 1$; thus, we allow for the possibility that a_{n-1} and other coefficients in (2) may be zero.

The following theorem is the basic result on polynomial interpolation.

Theorem 1 Polynomial Interpolation

Given any n points in the xy-plane that have distinct x-coordinates, there is a unique polynomial of degree $n - 1$ or less whose graph passes through those points.

Let us now consider how we might go about finding the interpolating polynomial (2) whose graph passes through the points in (1). Since the graph of this polynomial is the graph of the equation

$$y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} \qquad \ldots(3)$$

it follows that the coordinates of the points must satisfy

$$\begin{aligned} a_0 + a_1x_1 + a_2x_1^2 + \ldots + a_{n-1}x_1^{n-1} &= y_1 \\ a_0 + a_1x_2 + a_2x_2^2 + \ldots + a_{n-1}x_2^{n-1} &= y_2 \\ \vdots \quad \vdots \quad \vdots \quad \quad \vdots \quad &\quad \vdots \\ a_0 + a_1x_n + a_2x_n^2 + \ldots + a_{n-1}x_n^{n-1} &= y_n \end{aligned} \qquad \ldots(4)$$

In these equations the values of x's and y's are assumed to be known, so we can view this as a linear system in the unknowns $a_0, a_1, \ldots, a_{n-1}$. From this point of view the augmented matrix for the system is

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 2 & 14 & 4 \end{bmatrix}, \begin{matrix} R_3 - R_2, \\ R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 7 & 2 \end{bmatrix}, \frac{1}{2}R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 0 & -12 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{matrix} R_1 - R_4, \\ R_2 - 7R_4, \\ R_3 - 6R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{matrix} R_1 - R_3, \\ R_2 - 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, R_1 - R_2$$

This gives

$$a_0 = 4,\ a_1 = 3,\ a_2 = -5,\ a_3 = 1$$

Putting these values in (1), we have

$$p(x) = 4 + 3x - 5x^2 + x^3$$

Electrical Circuits

Next we will show how network analysis can be used to analyze electrical circuits consisting of batteries and resistors. A ***battery*** is a source of electric energy, and a ***resistor***, such as a lightbulb, is an element that dissipates electric energy.

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{bmatrix} \quad ...(5)$$

and hence the interpolating polynomial can be found by reducing this matrix to reduced row echelon form (Gauss–Jordan elimination).

Example 6: Polynomial Interpolation by Gauss–Jordan Elimination

Find a cubic polynomial whose graph passes through the points

$$(1, 3), (2, -2), (3, -5), (4, 0)$$

Solution: Let the required cubic polynomial be

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad ...(1)$$

Since (1) passes through the points (1, 3), (2, −2), (3, −5), (4, 0), so these points must satisfy (1), i.e.

$$3 = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3$$
$$-2 = a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3$$
$$-5 = a_0 + a_1(3) + a_2(3)^2 + a_3(3)^3$$
$$0 = a_0 + a_1(4) + a_2(4)^2 + a_3(4)^3$$

Above system can be written as

$$a_0 + a_1 + a_2 + a_3 = 3$$
$$a_0 + 2a_1 + 4a_2 + 8a_3 = -2$$
$$a_0 + 3a_1 + 9a_2 + 27a_3 = -5$$
$$a_0 + 4a_1 + 16a_2 + 64a_3 = 0$$

The augmented matrix of the above system is

$$A_b = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 2 & 8 & 26 & -8 \\ 0 & 3 & 15 & 63 & -3 \end{bmatrix}, \begin{matrix} R_2 - R_1, \\ R_3 - R_1, \\ R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 1 & 4 & 13 & -4 \\ 0 & 1 & 5 & 21 & -1 \end{bmatrix}, \frac{1}{2}R_3, \frac{1}{3}R_4$$