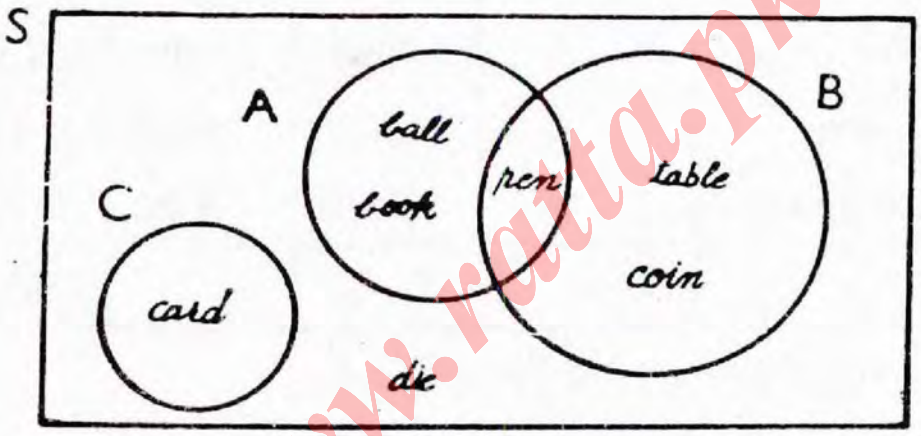
Chapter 6

PROBABILITY

6.1. Given S = {chair, student, pen}.

Proper subsets are {chair, student}, {chair, pen}, {student, pen}, {chair}, {student}, {pen}, ϕ.

6.2. The given sample space is illustrated by the following Venn Diagram:



6.3. (a) Given $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{3\}$. Then

(i) $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

(ii) $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$

(iii) $B \times A = \{(2, 1), (2, 2), (3, 1), (3, 2)\}$

(iv) Now

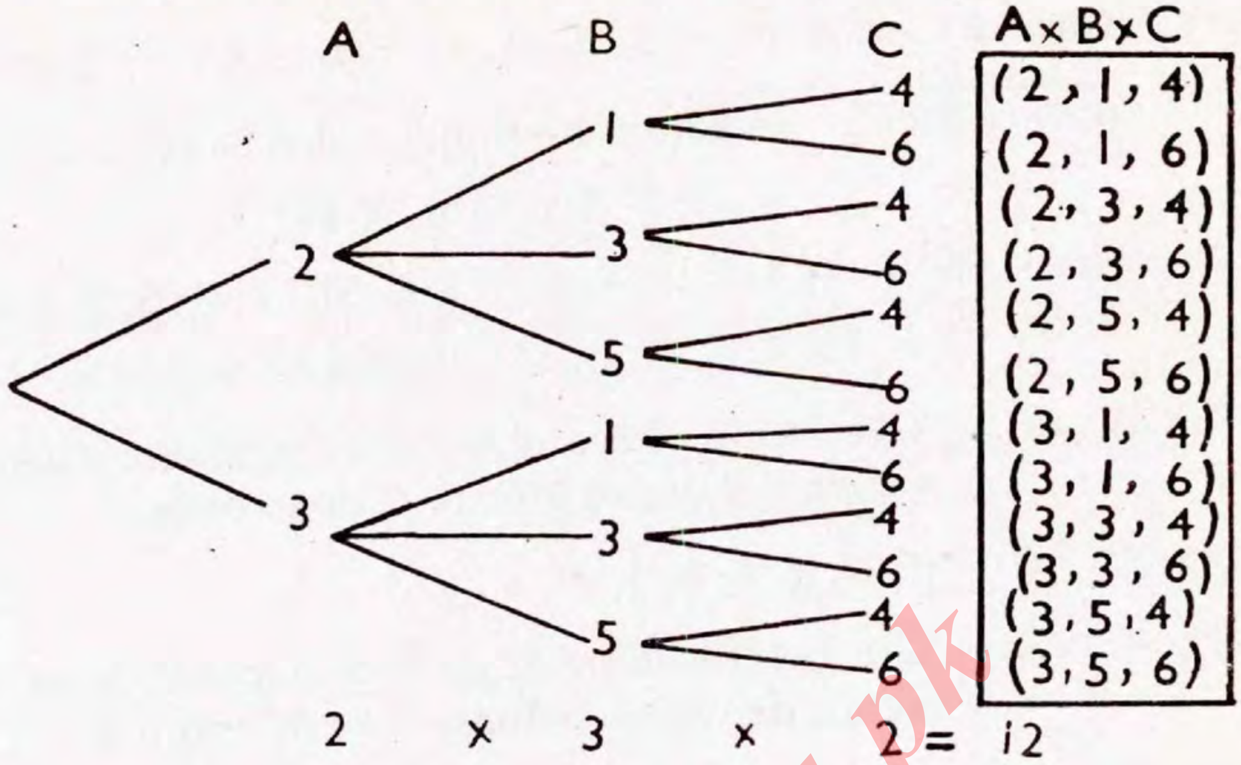
$A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$, and

$B \times C = \{(2, 3), (3, 3)$. Therefore

$(A \times B) \cup (B \times C) = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

(v) $(A \times B) \cap (B \times C) = \{(2, 3)\}$

(b) The desired "tree diagram" is constructed as below:



6.4. (b) $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{2, 3, 4\}$, $B = \{3, 4, 5\}$ and $C = \{5, 6, 7\}$

Now $\bar{A}$ is the complement of A. It consists of the members in S which are not in A. Thus

$\bar{A} = \{1, 5, 6, 7, 8, 9, 10\}$

Similarly, $\bar{B} = \{1, 2, 6, 7, 8, 9, 10\}$

(i) $\bar{A} \cap B$ = The intersection of $\bar{A}$ and B consists of those elements which belong to both $\bar{A}$ and B.

Hence $\bar{A} \cap B = \{5\}$.

(ii) $\bar{A} \cup B$ = The union of $\bar{A}$ and B consists of those elements whcih belong to $\bar{A}$ or to B or to both.

Hence $\bar{A} \cup B = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(iii) $\bar{A} \cap \bar{B}$ = The intersection of $\bar{A}$ and $\bar{B}$ consists of those elements belonging to $\bar{A}$ and $\bar{B}$

$= \{1, 6, 7, 8, 9, 10\}$

$\overline{\bar{A} \cap \bar{B}}$ = The complement of $\bar{A} \cap \bar{B}$ consists of those elements in S which are not in $\bar{A} \cap \bar{B}$

Hence $\overline{\bar{A} \cap \bar{B}} = \{2, 3, 4, 5\}$.

(iv) Now $B \cup C = \{3, 4, 5, 6, 7\}$, $A \cap (B \cup C) = \{3, 4\}$

$\therefore$ $\overline{A \cap (B \cup C)}$ = The complement of $A \cap (B \cup C)$

$= \{1, 2, 5, 6, 7, 8, 9, 10\}$.

6.5 Here $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{0, 2, 4, 6, 8\}$,

$B = \{1, 3, 5, 7, 9\}$, $C = \{2,3,4,5\}$ and $D = \{1,6,7\}$

(i) $A \cup C$ = The union of A and C consists of those elements which belong to A or to C or to both.

Hence $A \cup C = \{0, 2, 3, 4, 5, 6, 8\}$.

(ii) $A \cap B$ = The intersection of A and B consists of those elements which belong to both A and B.

Hence $A \cap B = \phi$

(iii) $\bar{C}$ = The complement of C consists of those elements in S which are not in C.

Hence $\bar{C} = \{0, 1, 6, 7, 8, 9\}$

(iv) $\bar{C} \cap D$ = The intersection of $\bar{C}$ and D consists of those elements which belong to both $\bar{C}$ and D.

$\therefore$ $\bar{C} \cap D = \{1, 6, 7\}$.

$(\bar{C} \cap D) \cup B$ = The union of $(\bar{C} \cap D)$ and B consists of those elements which belong to at least one of them.

Hence $(\bar{C} \cap D) \cup B = \{1, 3, 5, 6, 7, 9\}$

(v) $S \cap C$ = The intersection of S and C consists of those elements which belong to both S and C.

$\therefore$ $S \cap C = \{2, 3, 4, 5\}$

$\overline{S \cap C}$ = The complement of $S \cap C$ consists of elements in S which are not in C.

Hence $(\overline{S \cap C}) = \{0, 1, 6, 7, 8, 9\}$.

(vi) $A \cap C \cap \bar{D}$ = The intersection of A, C and $\bar{D}$ consists of elements which belong to A and to C and to $\bar{D}$.

Hence $A \cap C \cap \bar{D} = \{2, 4\}$.

6.6 (a) The sample space S is represented by the following array of 36 equally likely outcomes:

$$S = \begin{Bmatrix} (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & (6,1) \\ (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & (6,2) \\ (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & (6,3) \\ (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & (6,4) \\ (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & (6,5) \\ (1,6) & (2,6) & (3,6) & (4,6) & (5,6) & (6,6) \end{Bmatrix}$$

The two events are:

$A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

$B = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6)\ (6,1), (6,2), (6,3), (6,4), (6,5)\}$.

(b) Using the sample space given in (a), we see that the two events A and B consists of the following elements:

$A = \{(1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5)\}$; and

$B = \{(1,3), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,3)\ (5,3), (6,3)\}$

The complements of A and B are

$\bar{A} = \{(1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), (5,1), (5,3), (5,5), (6,2), (6,4), (6,6)\}$; and

$\bar{B} = \{(1,1), (2,1), (4,1), (5,1), (6,1), (1,2), (2,2), (4,2), (5,2), (6,2), (1,4), (2,4), (4,4), (5,4), (6,4), (1,5), (2,5), (4,5), (5,5), (6,5), (1,6), (2,6), (4,6), (5,6), (6,6)\}$

Now $A \cup B$ = The union of A and B consists of those elements which belong to A or B or both.

$\therefore \quad A \cup B = \{ (1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5), (1,3), (3,1), (3,3), (3,5), (5,3) \}$

Thus $A \cup B$ contian 23 elements.

$A \cap B$ = The intersection of A and B consists of those elements which belong to both A and B. Thus

$A \cap B = \{(2,3), (3,2), (3,4), (4,3), (3,6), (6,3)\}$, *i.e.*

$n(A \cap B) = 6,$

$A - B = A \cap \bar{B}$ = The set of all elements of A which do not belong to B.

$= \{ (1,2), (1,4), (1,6), (2,1), (2,5), (4,1), (4,5), (5,2), (5,4), (5,6), (6,1), (6,5)\}$; *i.e.* $n(A-B) = 12$.

$(A \cap \bar{B}) \cup \bar{A}$ = The union of $(A \cap \bar{B})$ and $\bar{A}$ consists of those elements which belong to $(A \cap \bar{B})$ or $\bar{A}$ or both. Thus

$(A \cap \bar{B}) \cup \bar{A} = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,4), (2,5), (2,6), (3,1), (3,3), (3,5), (4,1), (4,2), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,4), (6,5), (6,6) \}$,

i.e. $(A \cap \bar{B}) \cup \bar{A}$ consists of 30 sample points.

6.7 (a) All the possible (i) combinations and (ii) permutations of 3 letters chosen from the four letters A, B, C and D are listed in the table below:

Combinations	Permutations
ABC	*ABC, ACB, BAC, BCA, CAB, CBA*
ABD	*ABD, ADB, BAD, BDA, DAB, DBA*
ACD	*ACD, ADC, CAD, CDA, DAC, DCA*
BCD	*BCD, BDC, CBD, CDB, DBC, DCB*

Enumeration gives a total of 4 combinations and 24 permutations: These results are also obtained by using the formulas as below:

(i) The total number of combinations is

$$\binom{n}{r} = \binom{4}{3} = \frac{4!}{3!\,(4-3)!} = 4, \text{ and}$$

(ii) the total number of permutations is

$$^{n}P_{r} = {}^{4}P_{3} = \frac{4!}{(4-3)!} = 4 \times 3 \times 2 = 24.$$

(b) (i) It is evident that the order in which 3 officers: *i.e.* 1 president, 1 vice-president and 1 secretary-treasurer are to be chosen is of significance. Thus there are 15 choices for the first officer, 14 choices for the second officer and 13 choices for the third officer. Hence, the number of sample points is $15 \times 14 \times 13 = 2730$. In other words, the number of permutations is

$$^{15}P_{3} = \frac{15!}{(15-3)!} = \frac{15 \times 14 \times 13 \times 12!}{12!} = 2730$$

(ii) Since the order in which the three persons of the committee are chosen, is unimportant, it is therefore a problem involving combinations. Thus

$$\binom{15}{3} = \frac{15!}{(15-3)!\,3!} = \frac{15 \times 14 \times 13}{1 \times 2 \times 3} = 455.$$

6.8. The total number of possible bridge hands that can be selected from an ordinary deck of 52 cards is given by

$$\binom{52}{13} = \frac{52!}{(52-13)!\,13!} = \frac{52!}{39!\,13!} = 635{,}013{,}559{,}600$$

6.9. The total number of possible ways in which $n = 10$ persons can be divided into 3 groups consisting of 5, 3 and 2 persons is given by

$$\binom{10}{5\ \ 3\ \ 2} = \frac{10!}{5!\,3!\,2!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!\,3 \times 2 \times 1 \times 2 \times 1} = 2520$$

6.10. (b) The sample space consists of 4 elements a, b, c, d, i.e. S = {a, b, c, d}.

Then the possible subsets or events are

ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a, b, c, d\}$.

Thus we have $16 = 2^4$ subsets of events which include the empty set and the set itself.

(c) Let $S = \{e_1, e_2, e_3, \ldots, e_n\}$. Then all the possible subsets will consists of

(1) the subsets with no element, *i.e.* the empty set ϕ,

(2) the subsets consisting of exactly 1 elements such as $\{e_1\}$, $\{e_2\}$, ...

(3) the subsets consisting of exactly 2 elements such as $\{e_1, e_2\}$, $\{e_1, e_3\}$, ...

... ...

$(n+1)$, the set itself consisting of all n elements $\{e_1, e_2, \ldots, e_n\}$.

Now,

(1) the number of empty subsets is given by $\binom{n}{0}$;

(2) the number of subsets with exactly one element is given by $\binom{n}{1}$;

(3) the number of subsets with exactly 2 elements is given by $\binom{n}{2}$;

... ...

$(n+1)$ the number of subsets with all n elements is given by $\binom{n}{n}$.

Thus the number of all possible subsets of a set containing n elements is given by

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}.$$

Again, $$2^n = (1+1)^n = \binom{n}{0}(1)^n + \binom{n}{1}(1)^{n-1}(1) + \binom{n}{2}(1)^{n-2}(1)^2 + \ldots + \binom{n}{n}(1)^n.$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$$

Hence the result.

6.12. (a) Let A, B and C denote the events that the stock price will go up, remain unchanged and go down respectively.

Then $P(A) = 0.60$, $P(B) = 0.38$ and $P(C) = 0.25$.

Since the three events are mutually exclusive and collectively exhaustive, therefore their sum should be equal to one. But

$$P(A) + P(B) + P(C) = 0.60 + 0.38 + 0.25 = 1.23,$$

which is greater than 1.

Hence the investment counsellor's claim is wrong.

(b) When two coins are tossed once, the sample space consists of 4 equally likely sample points, *i.e.*

$$S = \{HH, HT, TH, TT\}.$$

As the sample points are equally likely, a probability of $\frac{1}{4}$ is assigned to each sample point. Thus

$P(HH) = 1/4$, $P(\text{one head and one tail}) = 2/4 = 1/2$, and

$P(2 \text{ tails}) = 1/4$.

Hence the given statement is wrong.

(c) The events "no accident", "one accident" and "two or more accidents" are mutually exclusive, therefore the sum of their probabilities should not exceed 1.

Now the sum of the given probabilities $= 0.90 + 0.02 + 0.09 = 1.01$, which exceeds 1.

Hence the statement is not correct.

(d) **As the events A, B and C are mutually exclusive, the sum of their probabilties should not exceed unity.**

Now $P(\text{A}) = \dfrac{1}{6}$,

$$\frac{2}{3}P(B) = \frac{1}{6} \text{ or } P(B) = \frac{1}{6} \times \frac{3}{2} = \frac{1}{4};$$

$$\frac{1}{4}P(C) = \frac{1}{6} \text{ or } P(C) = \frac{2}{3},$$

Adding, $P(A) + P(B) + P(C) = \frac{1}{6} + \frac{1}{4} + \frac{2}{3} = \frac{2+3+8}{12} = \frac{13}{12}$.

The sum turns out to be greater than 1, so the given statement is wrong.

6.13. (i) The sample space S contains 6 sample points, i.e.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let A denote the event that an odd number appears.

Then $A = \{1, 3, 5\}$

Hence $P(A) = \dfrac{\text{number of sample points in } A}{\text{number of sample points in } S} = \dfrac{n(A)}{n(S)} = 3/6 = 1/2.$

(ii) **The sample space S in a single toss of a pair of fair dice consists of 36 elements.**

Let A be the event that the sum 8 appears. Then

$A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

Hence $P(A) = \dfrac{\text{number of sample points in } A}{\text{number of sample points in } S} = \dfrac{5}{36}$.

(iii) **The sample space S consists of 8 elements.**

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Let A be the event that at least one head appears. Then

$A = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$.

Hence $P(A) = \frac{7}{8}$.

(iv) **Let S = sample space and A = a king, ace, jack of clubs or queen of diamonds appear. Then**

S can occur in $\binom{52}{1} =$ 52 ways, the number of ways that a single card can be drawn from 52 cards.

A can occur in $\binom{10}{1}$ = 10 ways, the number of ways that the card drawn is a king, ace, jack of clubs or queen of diamonds.

Hence $P(A) = \frac{10}{52} = \frac{5}{26}$.

6.14. (b) The box contains 10 red, 30 white, 20 blue and 15 orange, i.e. 75 marbles in all.

One marble can be drawn in $\binom{75}{1}$ = 75 ways

$\therefore$ (i) $P(\text{marble is orange or red}) = \frac{15+10}{75} = \frac{1}{3}$

(ii) $P(\text{marble is not-'red or blue'}) = \frac{45}{75} = \frac{3}{5}$

(iii) $P(\text{marble is not blue}) = \frac{55}{75} = \frac{11}{15}$

(iv) $P(\text{marble drawn is white}) = \frac{30}{75} = \frac{2}{5}$

(v) $P(\text{marble is red, white or blue}) = \frac{60}{75} = \frac{4}{5}$

6.15. (a) The sample space S is represented by the following array of 36 equally likely outcomes:

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

The various total number of dots that may turn up are, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

Let X denote the total of dots that may turn up. Then

$$P(X=2) = \frac{\text{number of sample points making a total of 2}}{\text{number of sample points in } S} = 1/36;$$

$$P(X=3) = \frac{\text{number of sample points making a total of 3}}{\text{number of sample points in } S} = 2/36;$$

Similarly, we find that

$P(X=4) = 3/36;$ $P(X=5) = 4/36;$

$P(X=6) = 5/36;$ $P(X=7) = 6/36;$

$P(X=8) = 5/36;$ $P(X=9) = 4/36;$

$P(X=10) = 3/36;$ $P(X=11) = 4/36;$

$P(X=12) = 1/36,$

Again, $P(\text{dots will total at least } 4) = P(X\leq 4) = \frac{33}{36} = \frac{11}{12}$

(b) Whenever two dice are thrown, the sample space S has 36 outcomes, which we assume, are equally likely and a probability of $\frac{1}{36}$ is attached with each outcome.

Let A be the event that a total of *more than* 7 occurs and B, the event that a total of *less than* 7 comes up. Then the event A has 15 outcomes, *i.e.*

$$A = \{(2,6), (3,6), (4,6), (5,6), (6,6), (3,5), (4,5), (5,5), (6,5), (4,4), (5,4), (6,4), (5,3), (6,3), (6,2)\}.$$

Thus $P(A) = \frac{15}{36} = \frac{5}{12}$.

Similarly, we see that the event B contains 15 sample points.

$\therefore P(B) = \frac{15}{36} = \frac{5}{12} = P(A).$

Let C be the event that *exactly 7* is thrown. Then A, B and C are mutually exclusive and collectively exhaustive events. Therefore

$$P(A) + P(B) + P(C) = 1$$

or $$P(C) = 1 - P(A) - P(B) = 1 - \frac{5}{12} - \frac{5}{12} = \frac{1}{6}.$$

6.16. (a) S consists of 36 sample points.

Let A = (sum of the upper face numbers is odd), and

B = (at least one ace). Then

$$A = \begin{Bmatrix} (1,2) & (2,1) & (3,2) & (4,1) & (5,2) & (6,1) \\ (1,4) & (2,3) & (3,4) & (4,3) & (5,4) & (6,3) \\ (1,6) & (2,5) & (3,6) & (4,5) & (5,6) & (6,5) \end{Bmatrix}$$

$B = \{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (2,1),\ (3,1),\ (4,1),\ (5,1),\ (6,1)\}$

$$\bar{B} = \begin{Bmatrix} (2,2) & (3,2) & (4,2) & (5,2) & (6,2) \\ (2,3) & (3,3) & (4,3) & (5,3) & (6,3) \\ (2,4) & (3,4) & (4,4) & (5,4) & (6,4) \\ (2,5) & (3,5) & (4,5) & (5,5) & (6,5) \\ (2,6) & (3,6) & (4,6) & (5,6) & (6,6) \end{Bmatrix}$$

Now $\quad A \cap B$ = The intersection of A and B consists of those points which belong to both A and B.

$\therefore \quad A \cap B = \{(1,2), (1,4), (1,6), (2,1), (4,1), (6,1)\}$

It thus contains 6 sample points.

And $\quad A \cup B$ = The unon of A and B consists of those points which belong to A or B or both.

$$\therefore \quad A \cup B = \begin{Bmatrix} (1,1) & (1,5) & (2,5) & (4,1) & (5,1) & (6,1) \\ (1,2) & (1,6) & (3,1) & (4,3) & (5,2) & (6,3) \\ (1,3) & (2,1) & (3,2) & (4,5) & (5,4) & (6,5) \\ (1,4) & (2,3) & (3,4) & (3,6) & (5,6) & \end{Bmatrix}$$

i.e. $A \cup B$ contains 23 sample points.

Again $\quad A \cap \bar{B}$ = The intersection of A and $\bar{B}$ consists of points which belong to both A and $\bar{B}$.

Thus $\quad A \cap \bar{B} = \begin{Bmatrix} (2,3), (2,5), (3,2), (3,4), (3,6), (4,3) \\ (4,5), (5,2), (5,4), (5,6), (6,3), (6,5) \end{Bmatrix}$

i.e. $\quad A \cap \bar{B}$ consists of 12 sample points.

Hence $\quad P(A \cap B) = \dfrac{6}{36} = \dfrac{1}{6}, \quad P(A \cup B) = \dfrac{23}{36}$, and

$$P(A \cup \bar{B}) = \frac{12}{36} = \frac{1}{3}.$$

(b) Let S = sample space, A = the sum shown is 8, and

B = the two show the same number. Then

S consists of 36 elements.

$A = \{(6,2), (5,3), (4,4), (3,5), (2,6)\}$

$B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

Hence $P(A) = \dfrac{5}{36}$; $P(B) = \dfrac{6}{36} = \dfrac{1}{6}$;

$P(A \cap B) = \dfrac{1}{36}$; as $A \cap B = \{(4,4)\}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{36} + \frac{6}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18};$$

6.17. The possible outcomes, when the numbers on two discs are drawn without replacement, are $\binom{6}{2} = 15$, which are given below:

(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5),
(2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)

Let k denote the sum of two numbers. Then

$P(k = 3) = \dfrac{1}{15}$; $P(k=4) = 1/15$; $P(k=5) = 2/15$;

$P(k = 6) = \dfrac{2}{15}$; $P(k=7) = 3/15$; $P(k=8) = 2/15$;

$P(k = 9) = \dfrac{2}{15}$; $P(k=10) = 1/15$; $P(k=11) = 1/15$;

6.18. The sample space S consists of $6^2 = 36$ sample points.

(i) Let A be the event that the product of the numbers on the dice is between 8 and 16. Then

$A = \{(4,2), (3,3), (2,4), (5,2), (4,3), (3,4), (2,5), (6,2),$
$(5,3), (4,4), (3,5), (2,6)\}$, *i.e.* $n(A) = 12$ sample points.

$\therefore \quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{12}{36} = \dfrac{1}{3}.$

(ii) Let B be the event that the product of the numbers on the dice is divisible by 4. Then

$$B = \{(4,1),\ (1,4),\ (4,2),\ (2,4),\ (6,2),\ (4,4),\ (2,6),\ (4,5),\ (4,3),\ (2,2),\ (3,4),\ (5,4),\ (6,4),\ (4,6),\ (6,6)\},$$

i.e. $n(B) = 15$ sample points.

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{15}{36}.$$

6.19. (b) Let A be the event that *at least* one 6 occurs in 4 tosses of a fair die, and $\bar{A}$ be the event that no 6 occurs. Then

$$P(A) = 1 - P(\bar{A}) = 1 - \left(\frac{5}{6}\right)^4 = 1 - \frac{625}{1296} = 0.52$$

Let B be the event that *at least* one double six occurs in 24 tosses of two fair dice, and $\bar{B}$, the event that no double six occurs. Then

$$P(B) = 1 - P(\bar{B}) = 1 - \left(\frac{35}{36}\right)^{24} = 0.49$$

(c) The sample space S consists of 216 sample points.

The sample points in different events can be conveniently obtained by combining the faces of the third die with pertinent sums of the first two dice. The sums are given below:

Ist die

		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
2nd die	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Now 9 can be made up by the following combinations:

3rd die	Sum of the 1st and 2nd die	No. of possibilities
1	8	5
2	7	6
3	6	5
4	5	4
5	4	3
6	3	2
Total		25

Hence $P(\text{getting a nine}) = \frac{25}{216}$.

Similarly, 10 can be made up in 27 ways.

Thus $P(\text{getting a 'ten'}) = \frac{27}{216} = \frac{1}{8}$.

6.20. (a) The sample space S contains $\binom{52}{2} = 1326$ sample points, the number of ways in which two cards can be drawn from 52 cards.

Let A be the event that one card drawn is a king and the other is a queen. Then

A contains $\binom{4}{1}\binom{4}{1} = 16$ sample points.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{16}{1326} = \frac{8}{663}.$$

(b) S consists of $\binom{8}{5} = 56$ sample points.

A can choose one joker and 4 other cards in $\binom{1}{1}\binom{7}{4}$, *i.e.* 35 ways.

$$\therefore \quad P(A \text{ has the Joker}) = \frac{35}{56} = \frac{5}{8}.$$

6.21 (a) There are 10 good and 2 bad eggs in the refrigerator.

S can occur in $\binom{12}{4} = 495$ ways, the number of ways in which 4 eggs can be chosen from 12 eggs.

Let A denote the event that exactly one egg is bad and B denote the event that at least one egg is bad. Then

(i) A can occur in $\binom{2}{1}\binom{10}{3} = 240$ ways.

Hence $P(A) = \frac{240}{495} = \frac{16}{33}$.

(ii) $P(B) = 1 - P(\text{no egg is bad}) = 1 - P(\bar{B})$, where

$\bar{B}$ can occur in $\binom{10}{4} = 210$ ways.

$\therefore \quad P(B) = 1 - \frac{210}{495} = \frac{285}{495} = \frac{19}{33}$.

(b) The carton contains 3 bad eggs and 9 good eggs. S can occur in $\binom{12}{3} = 220$ ways, the number of ways in which 3 eggs can be chosen from 12 eggs.

(i) Let A denote the event that no bad egg (*i.e.* all good eggs) is chosen. Then $n(A) = \binom{9}{3} = 84$.

$\therefore \quad P(A) = \frac{84}{220} = 0.38$.

(ii) *At least* one bad egg is the complement of the event that no bad egg is chosen. Therefore,

$P(\text{at least one bad egg}) = 1 - P(\text{no bad egg})$

$= 1 - 0.38 = 0.62$

(iii) Let B be the event that *exactly* 2 bad eggs are chosen. Then $n(B) = \binom{3}{2}\binom{9}{1} = 27$.

$\therefore \quad P(B) = \frac{27}{220} = 0.12$.

6.22. (a) The sample space S for this experiment is

$S = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and therefore $n(S) = 10$.

Let A be the event that the integer chosen is an even number. Then

$A = \{4, 6, 8, 10, 12\}$, *i.e.* $n(A) = 5$.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{5}{10} = \frac{1}{2}$$

Let B be the event that the integer chosen is an even number and is divisible by 3. Then

$B = \{6, 12\}$, *i.e.* $n(B) = 2$.

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{2}{10} = \frac{1}{5}.$$

(b) The sample space S contains $\binom{20}{3}$, *i.e.* 1140 sample points, the number of ways in which 3 digits can be chosen from 20 digits.

(i) There are two mutually exclusive ways of getting an even sum, *i.e.* either 3 digits are even or 2 odd and 1 even. This can occur in $\binom{10}{3} + \binom{10}{2}\binom{10}{1}$, *i.e.* $120 + 450 = 570$ ways.

$$\therefore \quad P(\text{Sum is even}) = \frac{570}{1140} = \frac{1}{2}.$$

(ii) When all 3 digits are odd, the product will not be even; and 3 odd digits from 10 odd digits can be chosen in $\binom{10}{3}$, *i.e.* 120 ways.

Thus $\quad P(\text{product is even}) = 1 - P(\text{product is not even})$

$$= 1 - \frac{120}{1140} = 1 - \frac{2}{19} = \frac{17}{19}.$$

(c) S consists of $\binom{13}{3} = 286$ sample points.

(i) Let A be the event that all balls drawn are of different colours. Then A contains $\binom{4}{1}\binom{4}{1}\binom{5}{1}$, *i.e.* $4 \times 4 \times 5 = 80$ sample points.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{80}{286} = \frac{40}{143}.$$

(ii) Let B be the event that all balls drawn are of same colour. Then B contains $\binom{4}{3} + \binom{4}{3} + \binom{5}{3}$, *i.e.* $4+4+10=18$ sample points.

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{18}{286} = \frac{9}{143}.$$

6.23 (a) We know that in one throw of a die, $P(6) = \frac{1}{6}$ and $P(\text{not } 6) = \frac{5}{6}$

(i) When 5 dice are thrown, then

$$P(\textit{at least} \text{ one six}) = 1 - P\text{ (no 6's)}$$
$$= 1 - \left(\frac{5}{6}\right)^5 = 0.598$$

(ii) When n dice are thrown, then

$$P(\textit{at least} \text{ one six}) = 1 - P\text{ (no 6's)} = 1 - \left(\frac{5}{6}\right)^n.$$

(b) We need to find n such that

$$1 - \left(\frac{5}{6}\right)^n \geq 0.99, \ \textit{i.e.} \ \left(\frac{5}{6}\right)^n \leq 0.01$$

Taking logs of both sides, we get

$$n \log\left(\frac{5}{6}\right) \leq \log 0.01$$

Dividing both sides by $\log\left(\frac{5}{6}\right)$ and reversing the inequality sign as $\log\left(\frac{5}{6}\right)$ is negative, we have

$$n \geq \frac{\log(0.01)}{\log(5/6)} \ [\because \log 0.01 = \bar{2}.0000, \ \log(5/6) = \log(0.833) = \bar{1}.9206]$$

≥ 25.3 so least $n = 26$.

Hence 26 dice must be thrown so that the probability of obtaining at least one 6 is at least 0.99.

(c) P(target is hit at least once) = $1 - P$ (target is not hit)

$$= 1 - (0.3)^n$$

Now we need to find n such that

$$1 - (0.3)^n \geq 0.995, \ \ i.e. \ (0.3)^n \leq 0.005$$

Taking log of both sides, we obtain

$$n \log (0.3) \leq \log (0.005)$$

Dividing both sides by log (0.3) and reversing the inequality sign since log (0.3) is negative, we have

$$n \geq \frac{\log (0.005)}{\log (0.3)}$$

$$\geq \frac{\bar{3}.6990}{\bar{1}.4771} \geq 4.4$$

so least $n = 5$

Hence 5 missiles should be fired so that the probability that the target is hit *at least* once is greater than 0.995.

6.24. The sample space S contains $\binom{14}{6} = 3003$ sample points, which are all equally likely, exhaustive and mutually exclusive.

(i) Let A denote the event that 3 red and 3 not-red, *i.e.* black or white balls are drawn. Then

A contains $\binom{4}{3} \times \binom{10}{3}$, *i.e.* $4 \times 120 = 480$ sample points.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{480}{3003} = \frac{160}{1001}.$$

(ii) Let B denote the event that *at least* two white balls are drawn. Then *at least* two white means 2, 3, 4 or 5 white balls. Let B_1, B_2, B_3 and B_4 denote the events that 2 white and 4 others, 3 white and 3 others, 4 white and 2 others, 5 white and 1 other ball respectively are drawn. Then

B_1 contains $\binom{5}{2}\binom{9}{4} = 1260$ sample points;

B_2 contains $\binom{5}{3}\binom{9}{3} = 840$ sample points;

B_3 contains $\binom{5}{4}\binom{9}{2} = 180$ sample points; and

B_4 contains $\binom{5}{5}\binom{9}{1} = 9$ sample points.

office.um.pk

Now $\quad B = B_1 \cup B_2 \cup B_3 \cup B_4$. Therefore

$$n(B) = n(B_1) + n(B_2) + n(B_3) + n(B_4)$$
$$= 1260 + 840 + 180 + 9 = 2289.$$

Hence $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{2289}{3003} = \dfrac{109}{143}$.

6.25. (a) The sample space S contains $\binom{10}{3} = 120$ sample points, the number of ways in which 3 people can be selected from 10 applicants.

(i) Let A represent the event that the three selected will be girls. Then A contains $\binom{6}{3} = 20$ sample points, the number of ways in which 3 girls can be selected from 6 girls.

Therefore $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{20}{120} = \dfrac{1}{6}$.

(ii) Let B denote the event that the three selected will be boys. Then B contains $\binom{4}{3} = 4$ sample points.

Therefore $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{4}{120} = \dfrac{1}{30}$.

(iii) *At least* one boy means, one, two or three boys. Let C denote the event that *at least* one boy is selected. Then

$$n(C) = \binom{4}{1}\binom{6}{2} + \binom{4}{2}\binom{6}{1} + \binom{4}{3}\binom{6}{0}$$
$$= 60 + 36 + 4 = 100 \text{ sample points.}$$

Hence $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{100}{120} = \dfrac{5}{6}$.

(b) The sample space S contains $\binom{14}{5} = 2002$ sample points.

Let A denote the event that more men are chosen than women. Then

$$n(A) = \binom{6}{3}\binom{8}{2} + \binom{6}{4}\binom{8}{1} + \binom{6}{5}$$

$$= (20 \times 28) + (15 \times 8) + 6 = 686$$

Hence $P(A) = \dfrac{686}{2002} = 0.34.$

6.26 The sample space S contains $\binom{10}{3} = 120$ sample points

(i) Let A denote the event that orders are placed with the in-state suppliers only. Then A contains $\binom{4}{3} = 4$ sample points.

$$\therefore \qquad P(A) = \frac{n(A)}{n(S)} = \frac{4}{120} = \frac{1}{30}.$$

(ii) Let B denote the event that orders are placed with the out-of-state suppliers only. Then B cointains $\binom{6}{3} = 20$ sample points

$$\therefore \qquad P(B) = \frac{n(B)}{n(S)} = \frac{20}{120} = \frac{1}{6}.$$

(iii) *At least* one in-state supplier means one, two or three in-state suppliers. Let C be the event that orders are placed with *at least* one in-state supplier. Then C contains $\binom{4}{1}\binom{6}{2} + \binom{4}{2}\binom{6}{1} + \binom{4}{3} = 60 + 36 + 4 = 100$ sample points.

Therefore $\qquad P(C) = \dfrac{n(C)}{n(S)} = \dfrac{100}{120} = \dfrac{5}{6}.$

6.27. (b) Let A be the event that the person is a man, and B be the event that the person has brown eyes. Then we need $P(A \cup B)$.

Now $\quad P(A) = \dfrac{10}{30},\ P(B) = \dfrac{15}{30}$ and $P(A \cap B) = \dfrac{5}{30},$

Hence $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{10}{30} + \frac{15}{30} - \frac{5}{30} = \frac{20}{30} = \frac{2}{3}.$$

(c) Let W be the event that the person chosen be a woman, and G be the event that the person wears glasses. Then we need P(W ∪ G)

Now $P(W) = \frac{7}{20}$, $P(G) = \frac{6}{20}$ and $P(W \cap G) = \frac{4}{20}$

Hence $P(W \cup G) = P(W) + P(G) - P(W \cap G)$

$$= \frac{7}{20} + \frac{6}{20} - \frac{4}{20} = \frac{9}{20} = 0.45.$$

6.28. (b) The sample space in this case is S = {1, 2, 3, 4, ..., 50} and therefore n(S) = 50.

Let A represent the event that the integer selected is divisible by 6, B, the event that the integer chosen is divisible by 8, and $A \cap B$, the event that the integer chosen is divisible by both 6 and 8, *i.e.* by 24. Then we seek $P(A \cup B)$.

Now $n(A) = \left[\frac{50}{6}\right] = 8$, $n(B) = \left[\frac{50}{8}\right] = 6$, and

$n(A \cap B) = \left[\frac{50}{24}\right] = 2$, where $[x]$ stands for the highest integer in x.

Hence $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{8}{50} + \frac{6}{50} - \frac{2}{50} = \frac{12}{50} = \frac{6}{25}.$$

6.29. (c) The sample space S consists of $\binom{200}{1} = 200$ sample points.

Let A be the event that the item chosen is a bolt and B, the event that the item chosen is rusted. Then we need $P(A \cup B)$.

Now A contains $\binom{50}{1} = 50$ sample points, and

B contains $\binom{100}{1} = 100$ sample points. Therefore

$$P(A) = \frac{n(A)}{n(S)} = \frac{50}{200}, \text{ and } P(B) = \frac{n(B)}{n(S)} = \frac{100}{200}.$$

The events A and B are not mutually exclusive, as an item may be both rusted and a bolt. Therefore the event

$$A \cap B \text{ contains } \binom{25}{1} = 25 \text{ sample points and}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{25}{200}.$$

Hence $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{50}{200} + \frac{100}{200} - \frac{25}{200} = \frac{125}{200} = \frac{5}{8}.$$

6.30. (b) S consists of 6^2 = 36 sample points.

Let A represent the event that the sum of dots is equal to 7 and B, be the event that the sum of dots is equal to 11. Then we need $P(A \cup B)$.

$$\text{Now} \quad P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}, \text{ and}$$

$$P(B) = \frac{n(A)}{n(S)} = \frac{2}{36} = \frac{1}{18}.$$

Since the two events A and B are mutually exclusive, therefore

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}.$$

(c) (i) Since A and B are mutually exclusive events, therefore

$$P(A \cup B) = P(A) + P(B) = 0.4 + 0.5 = 0.9.$$

(ii) Now $P(\bar{A}) = 1 - P(A) = 1 - 0.4 = 0.6.$

6.31. (a) Now $A \cap \bar{B}$, A−B or A − (A ∩ B) are the same sets.

$\therefore \quad (A \cap \bar{B}) \cup (B \cap \bar{A}) = [A - (A \cap B)] \cup [B - (B \cap A)]$; and

$$P[(A \cap \bar{B}) \cup (B \cap \bar{A})] = P[A - (A \cap B)] + P[B - (B \cap A)]$$

$$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - 2P(A \cap B)$$

(b) $P(A \cup B) = \frac{3}{4}$, $P(\bar{A}) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{4}$ (given)

$\therefore$ (i) $P(A) = 1 - P(\bar{A}) = 1 - \dfrac{2}{3} = \dfrac{1}{3}$

(ii) Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore$ $P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$$= \frac{3}{4} - \frac{1}{3} + \frac{1}{4} = \frac{9 - 4 + 3}{12} = \frac{8}{12} = \frac{2}{3}.$$

(iii) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

(c) **Given**: $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{4}$ and $P(\bar{B}) = \frac{5}{8}$.

Then $P(\bar{A}) = 1 - P(A) = \frac{1}{2}$, and $P(B) = 1 - P(\bar{B}) = \frac{3}{8}$.

$\therefore$ (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$= \frac{1}{2} + \frac{3}{8} - \frac{3}{4} = \frac{4 + 3 - 6}{8} = \frac{1}{8}$$

(ii) Using De Morgan's Law $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$, we have

$P(\bar{A} \cap \bar{B}) = P[(\overline{A \cup B})]$

$$= 1 - P(A \cup B) = 1 - \frac{3}{4} = \frac{1}{4}.$$

(iii) Using De Morgan's Law $(\overline{A \cap B}) = \bar{A} \cup \bar{B}$, we have

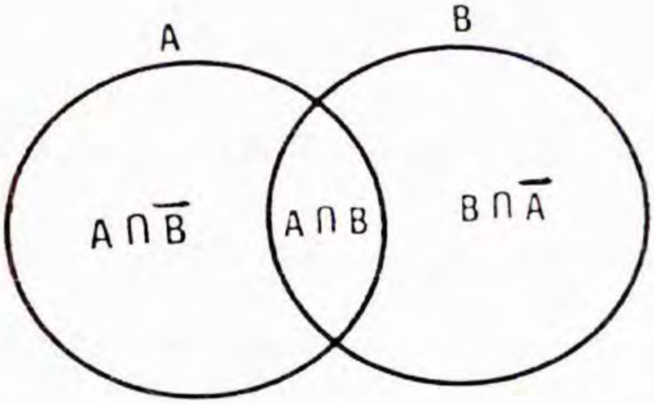
$P(\bar{A} \cup \bar{B}) = P[(\overline{A \cap B})]$

$$= 1 - P(A \cap B) = 1 - \frac{1}{8} = \frac{7}{8}.$$

(iv) $P(B \cap \bar{A}) = P(B - A \cap B) = P(B) - P(B \cap A)$

$$= \frac{3}{8} - \frac{1}{8} = \frac{1}{4}.$$

6.32. (i) Let A and B denote two sets of points with points in common represented by $A \cap B$. From the Venn diagram, we find that set A is composed of two disjoint sets $A \cap \bar{B}$ and $A \cap B$. Similarly, B is composed of $A \cap B$ and $B \cap \bar{A}$. The three mutually exclusive sets, i.e. $A \cap \bar{B}$, $A \cap B$ and $B \cap \bar{A}$ include points which are either in A or B. Thus the required probability is given by



$$P(A \cup B) = P(A \cap \bar{B}) + P(A \cap B) + P(B \cap \bar{A})$$

Now to obtain $A \cap \bar{B}$, the points in common represented by $A \cap B$ are removed from A. Thus

$$A \cap \bar{B} = A - A \cap B$$

$$\therefore \qquad P(A \cap \bar{B}) = P(A - A \cap B) = P(A) - P(A \cap B)$$

Similarly, $P(B \cap \bar{A}) = P(B) - P(A \cap B)$

Substituting these values, we get

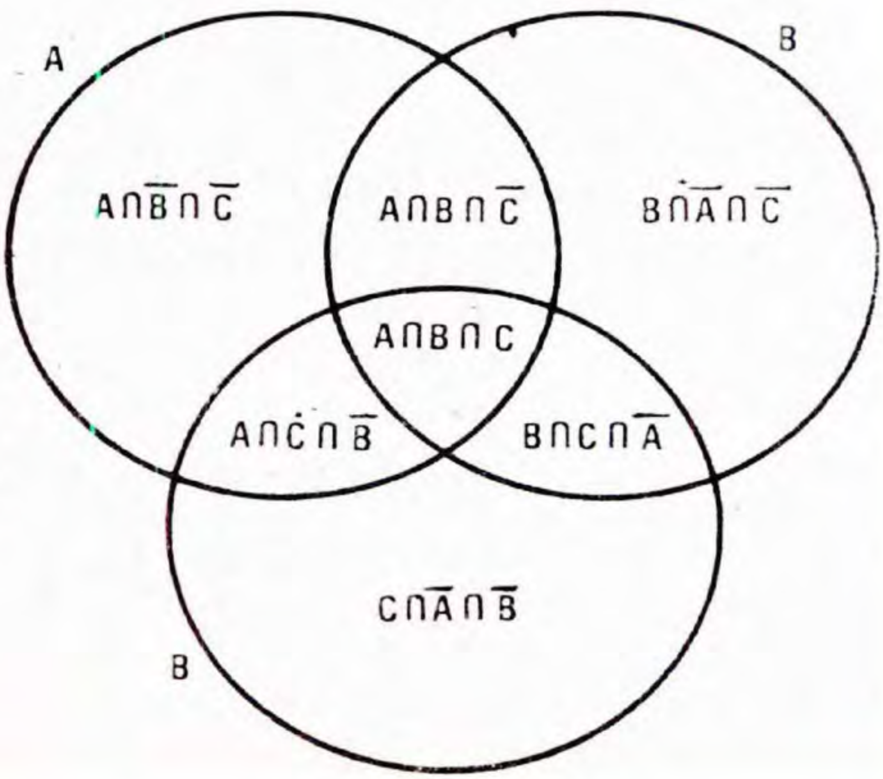
$$P(A \cup B) = P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B).$$

(ii) Let A, B and C denote three sets of points having points in common as indicated in the Venn diagram below:

The symbol $A \cap \bar{B} \cap \bar{C}$ means the set of points in A but not in B nor in C. The other symbols have similar meanings.

The points which are either in A or B or C are included in the seven mutually exclusive sets (fig.) The required probability is given by



$$P(A\cup B\cup C) = P(A\cap\bar{B}\cap\bar{C}) + P(A\cap B\cap\bar{C}) + P(B\cap\bar{A}\cap\bar{C}) +$$
$$P(B\cap C\cap\bar{A}) + P(C\cap\bar{A}\cap\bar{B}) + P(A\cap C\cap\bar{B}) + P(A\cap B\cap C)$$

In order to obtain $A\cap\bar{B}\cap\bar{C}$, we remove points common to A and B, and to A and C, but in so doing we have removed points common to A, B and C twice.

Thus $\quad A\cap\bar{B}\cap\bar{C} = A - A\cap B - A\cap C + A\cap B\cap C.$

$$\therefore\ P(A\cap\bar{B}\cap\bar{C}) = P(A) - P(A\cap B) - P(A\cap C) + P(A\cap B\cap C).$$

Similarly, we find that

$$P(A\cap B\cap\bar{C}) = P(A\cap B) - P(A\cap B\cap C)$$

$$P(B\cap\bar{A}\cap\bar{C}) = P(B) - P(B\cap C) - P(B\cap A) + P(B\cap C\cap A).$$

$$P(B\cap C\cap\bar{A}) = P(B\cap C) - P(A\cap B\cap C)$$

$$P(C\cap\bar{A}\cap\bar{B}) = P(C) - P(C\cap A) - P(C\cap B) + P(C\cap A\cap B)$$

$$P(A\cap C\cap\bar{B}) = P(A\cap C) - P(A\cap B\cap C)$$

Substituting these values, considering that $P(A\cap B) = P(B\cap A)$, etc., and simpifying, we get

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) -$$
$$P(B\cap C) - P(A\cap C) + P(A\cap B\cap C).$$

6.33. (b) The sample space in this experiment is reduced by excluding the sample points containing the same numbers.

Since there are 6 sample points which contain the same numbers, therefore the *reduced* sample space consists of $36-6=30$ sample points.

(i) Let A denote the event that the sum of 6 occurs when the numbers appearing are different. Then

$A = \{(1, 5), (2, 4), (4, 2), (5, 1)\}$, and therefore

$$P(A) = \frac{4}{30} = \frac{2}{15}.$$

(ii) Let B be the event that the sum of 4 or less occurs when the numbers appearing are different. Then

$B = \{(3,1), (1,3), (2,1), (1,2)\}$ and thus

$$P(B) = \frac{4}{30} = \frac{2}{15}.$$

Alternative Method:

The sample space S consists of $6^2 = 36$ sample points.

(i) Let A be the event that the sum of 6 appears and B, the event that two different numbers appear on the dice. Then

$$P(B) = \frac{n(B)}{n(S)} = \frac{30}{36}, \text{ and } P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{4}{36}.$$

Hence $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{4}{36} \div \dfrac{30}{36} = \dfrac{4}{30} = \dfrac{2}{15}$.

(ii) Let C be the event that the sum of 4 or less appears. Then

$$C \cap B = \{(1,3), (3,1), (2,1), (1,2)\}, \text{ and } P(C \cap B) = \frac{4}{36}.$$

Hence $P(C/B) = \dfrac{P(C \cap B)}{P(B)} = \dfrac{4}{36} \div \dfrac{30}{36} = \dfrac{4}{30} = \dfrac{2}{15}$.

6.34. (a) Let A = (first tube is good) and B = (other tube is good). Then S can occur in $\binom{10}{2} = 45$ ways, the number of ways in which 2 tubes can be drawn from 10 tubes.

Now $A \cap B$ can occur in $\binom{6}{2} = 15$ ways, the number of ways in which 2 good tubes can be drawn from 6 good tubes.

$$\therefore \quad P(A \cap B) = \frac{15}{45} = \frac{1}{3}.$$

$$P(A) = \frac{6}{10} = \frac{3}{5}$$

Hence $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/3}{3/5} = \dfrac{5}{9}$.

(b) Let A = (the sum of dots is odd) and B = (the sum of dots is 7). Then we seek $P(B/A)$.

S consists of $6^2 = 36$ sample points and A contains18 sample points. Therefore

$$P(A) = \frac{18}{36} = \frac{1}{2}.$$

The event $A \cap B$ contains 6 sample points. Thus

$$P(A \cap B) = \frac{6}{36} = \frac{1}{6}.$$

$$\text{Hence } P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}.$$

(c) Let A represent the event that the employee has accounting background and E be the event that the employee is an executive. Then

$$P(A) = 0.20 \text{ and } P(A \cap E) = 0.05$$

Hence the desired probability is

$$P(E/A) = \frac{P(A \cap E)}{P(A)} = \frac{0.05}{0.20} = 0.25.$$

6.35. (i) Let R_1, R_2 and R_3 denote the events that the first flower is red, the second flower is red and the third flower is red. Then we need $P(R_1 \cap R_2 \cap R_3)$. Since flowers are picked up at random one by one without replacement, therefore

$$P(R_1 \cap R_2 \cap R_3) = P(R_1)\, P(R_2/R_1)\, P(R_3/R_1 \cap R_2)$$
$$= \frac{10}{22} \times \frac{9}{21} \times \frac{8}{20} = \frac{6}{77} = 0.0779$$

(ii) The number of sequences in which the event 2 red and 2 white flowers in the first four picked up can occur, is given by

$$\frac{4!}{2!\,2!} = 6$$

Now $\quad P(\text{2 red and 2 white in one sequence}) = \dfrac{10 \times 9 \times 12 \times 11}{22 \times 21 \times 20 \times 19}$

$$= \frac{9}{133}$$

$\therefore$ Required probability $= \dfrac{9}{133} \times 6 = 0.406$

(iii) $P(\text{3rd is red/first 2 are white}) = \dfrac{P(\text{3rd red} \cap \text{first two white})}{P(\text{first 2 are white})}$

$$= \frac{\frac{12}{22} \times \frac{11}{21} \times \frac{10}{20}}{\frac{12}{22} \times \frac{11}{21}} = \frac{1}{2}.$$

6.36. (c) Given P(A) = 0.60, P(B) = 0.40 and P(A $\cap$ B) = 0.24. Then

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.24}{0.40} = 0.60;$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.60 + 0.40 - 0.24 = 0.76;$$

$$P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{0.60 - 0.24}{1 - 0.4} = \frac{0.36}{0.60} = 0.60;$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{0.24}{0.60} = 0.40; \text{ and}$$

$$P(\bar{B}) = 1 - P(B) = 1 - 0.40 = 0.60.$$

The events A and B are *independent* as

$$P(A/B) = 0.60 = P(A); \; P(B/A) = 0.40 = P(B) \text{ and}$$

$$P(A \cap B) = 0.24 = (0.60)\,(0.40) = P(A)\,P(B).$$

6.37. (b) (i) When A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

i.e. $0.7 = 0.4 + p$ or $p = 0.3$.

(ii) When A and B are independent, we have

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A) \,.\, P(B) \\ &= P(A) + P(B)\,[1 - P(A)] \end{aligned}$$

or $0.7 = 0.4 + p(1 - 0.4)$

or $\quad 0.3 = 0.6\,p \quad$ or $\quad p = 0.5.$

(c) (i) When A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B)$$

i.e. $\quad 0.60 = 0.50 + P(B)$

$\therefore \quad P(B) = 0.10.$

(ii) When A and B are independent, we have

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A) \,.\, P(B) \\ &= P(A) + P(B)\,[1 - P(A)] \end{aligned}$$

or $\quad 0.60 = 0.50 + P(B)\,[1 - 0.50]$

or $\quad 0.10 = 0.5\,P(B)$

or $\quad P(B) = 0.20$

(iii) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) + P(B) - P(A \cup B)}{P(B)}$

or $\quad 0.40 = \dfrac{0.50 + P(B) - 0.60}{P(B)}$

or $\quad 0.40\,P(B) = P(B) - 0.10$

or $\quad 0.10 = P(B) - 0.40\,P(B)$

or $\quad 0.10 = 0.60\,P(B)$

or $\quad P(B) = 1/6 = 0.17$

6.38 (a) The statement is true.

(b) The statement is false. If A and B are independent, then

$$P(A/B) = P(A).$$

(c) The statement is true.

(d) The statement is false "Independent" does not mean that two events have equal probabilities.

6.39. (b) Given P(A and B will occur simultaneously), **i.e.** $P(A \cap B) = \frac{1}{8}$ and P(neither of them will occur), *i.e.* $P(\bar{A} \cap \bar{B}) = \frac{3}{8}$.

$\because$ A and B are independent, therefore we have

$$\frac{1}{8} = P(A \cap B) = P(A) \,.\, P(B), \text{ and}$$

$$\begin{aligned} \frac{3}{8} &= P(\bar{A} \cap \bar{B}) = [P(\bar{A}) \,.\, P(\bar{B})] \\ &= [1 - P(A)]\ [1 - P(B)] \\ &= 1 - P(A) - P(B) + P(A)\,P(B) \end{aligned}$$

or $\qquad P(A) + P(B) = 1 - \frac{3}{8} + \frac{1}{8} = \frac{6}{8}.$

Now, we find two numbers whose sum is 6/8 and whose product is 1/8. Two such numbers are 1/2 and 1/4.

Hence $P(A) = 1/2$ and $P(B) = 1/4$ or $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{2}$

6.40 S consists of $6^2 = 36$ sample points.

Now E_1 = the event that a 6 appears on at least one die, therefore

$E_1 = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,5), (6,4), (6,3), (6,2), (6,1)\}$

i.e. $n(E_1) = 11$ sample points.

E_2 = the event that a 5 appears on exactly one die, therefore

$E_2 = \{(1,5),(2,5),(3,5),(4,5),(5,6),(6,5),(5,4),(5,3),(5,2),(5,1)\}$

It thus contains 10 sample points.

E_3 = the event that same number appears on both dice. Then

$E_3 = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

i.e. $n(E_3) = 6$ sample points.

Now $E_1 \cap E_2 = \{(5,6), (6,5)\}$, $E_2 \cap E_3 = \{\ \}$, and $E_1 \cap E_3 = \{(6,6)\}$.

Thus $n(E_1 \cap E_2) = 2$, $n(E_2 \cap E_3) = 0$ and $n(E_1 \cap E_3) = 1$.

We therefore have $P(E_1) = \frac{11}{36}$, $P(E_2) = \frac{10}{36}$, $P(E_3) = \frac{6}{36}$,

$$P(E_1 \cap E_2) = \frac{2}{36}, P(E_2 \cap E_3) = 0, \text{ and } P(E_1 \cap E_3) = \frac{1}{36}.$$

(i) As $\left(\frac{11}{36}\right)\left(\frac{10}{36}\right) \neq \frac{2}{36}$, *i.e.* $P(E_1)\,P(E_2) \neq P(E_1 \cap E_2)$,

$\therefore$ E_1 and E_2 are not independent.

(ii) As $\left(\frac{10}{36}\right)\left(\frac{6}{36}\right) \neq 0$, *i.e.* $P(E_2)\,P(E_3) \neq P(E_2 \cap E_3)$

$\therefore$ E_2 and E_3 are not independent.

(iii) As $\left(\frac{11}{36}\right)\left(\frac{6}{36}\right) \neq \frac{1}{36}$, *i.e.* $P(E_1)\,P(E_3) \neq P(E_1 \cap E_3)$

$\therefore$ E_1 and E_3 are not independent.

6.41 (a) $P(A \text{ cannot solve a problem}) = \frac{25}{100} = \frac{1}{4}$

$P(B \text{ cannot solve a problem}) = \frac{30}{100} = \frac{3}{10}$

$\therefore$ $P(\text{both } A \text{ and } B \text{ cannot solve a problem}) = \frac{1}{4} \times \frac{3}{10} = \frac{3}{40}$

Hence $P(\text{either } A \text{ or } B \text{ can solve a problem}) = 1 - \frac{3}{40} = \frac{37}{40}$.

(b) Let A_1 be the event that the first card is a red ace, A_2 be the event that the second card is a ten or jack, and A_3 be the event that the third card is greater than 3 but less than 7. Then, as the cards are drawn in succession, without replacement, we have

$$P(A_1) = \frac{2}{52},\ P(A_2/A_1) = \frac{8}{51},\ P(A_3/A_1 \cap A_2) = \frac{12}{50}$$

$$\therefore\ P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2/A_1)\,P(A_3/A_1 \cap A_2)$$

$$= \frac{2}{52} \times \frac{8}{51} \times \frac{12}{50} = \frac{8}{5525} = 0.0014$$

6.42 Here $P(A) = \frac{5}{7}$ and $P(B) = \frac{7}{9}$.

(i) P(both of them will die) $= P(\bar{A} \cap \bar{B})$

$= P(\bar{A}).P(\bar{B}) = \left(1 - \frac{5}{7}\right)\left(1 - \frac{7}{9}\right)$

$= \frac{2}{7} \times \frac{2}{9} = \frac{4}{63}$.

(ii) $P(A$ will be alive and B dead$)$ $= P(A \cap \bar{B}) = P(A) \,.\, P(\bar{B})$

$= \frac{5}{7}\left(1 - \frac{7}{9}\right) = \frac{10}{63}$

(iii) $P(B$ will be alive and A dead$)$ $= P(\bar{A} \cap B) = P(\bar{A}) \,.\, P(B)$

$= \left(1 - \frac{5}{7}\right) \times \frac{7}{9} = \frac{2}{9}$

(iv) P(both of them will be alive) $= P(A \cap B) = P(A) \,.\, P(B)$

$= \frac{5}{7} \times \frac{7}{9} = \frac{5}{9}$.

6.43 (a) Given

Bag	Red	Black	Total
1	3	5	8
2	4	7	11
Total	7	12	19

Now the probability of selecting the first bag is $\frac{1}{2}$, and if the first bag is selected, the probability that the ball drawn is red, is $\frac{3}{8}$.

Hence the probability that a red ball is drawn from the first bag is $\frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$.

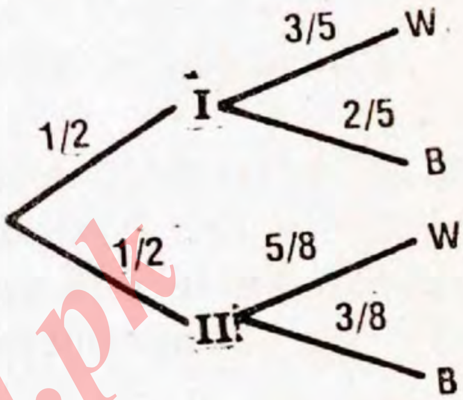
Similarly, the probability of drawing a red ball from the second bag is $\frac{1}{2} \times \frac{4}{11} = \frac{2}{11}$.

Hence the required probability $= \dfrac{3}{16} + \dfrac{2}{11}$ (the events are mutually exclusive)

$$= \frac{65}{176}.$$

(b) First we select one of the two urns and then we draw a ball which is either white (W) or black (B).

Let us construct the following tree diagram with respective probabilities to describe this process.



Now the probability of a white ball if urn I is selected is $\dfrac{1}{2} \times \dfrac{3}{5}$.

Similarly, the probability of a white ball from urn II, is $\dfrac{1}{2} \times \dfrac{5}{8}$

Hence the required probability $= \dfrac{1}{2} \times \dfrac{3}{5} + \dfrac{1}{2} \times \dfrac{5}{8}$

$$= \frac{3}{10} + \frac{5}{16} = \frac{49}{80}.$$

6.44. Let A = (the first drawing gives 3 white balls), B = (the second drawing gives 3 black balls) and S = Sample space. We seek P(A $\cap$ B).

S can occur in $\dbinom{13}{3} = 286$ ways, the number of ways in which 3 balls can be drawn from 13 balls.

A can occur in $\dbinom{5}{3} = 10$ ways, the number of ways in which 3 white balls can be drawn from 5 white balls.

B can occur in $\dbinom{8}{3} = 56$ ways.

$$\therefore \quad P(A) = \frac{10}{286}$$

To find the probability of B, when the white balls drawn in the first drawing are not being replaced, *i.e.* $P(B/A)$, the sample space S is reduced. Thus

$$S \text{ can occur in } \binom{10}{3} = 120 \text{ ways.}$$

$$\therefore \qquad P(B/A) = \frac{56}{120}.$$

$$\text{Hence } P(A \cap B) = P(A) \, . \, P(B/A) = \frac{10}{286} \times \frac{56}{120} = \frac{7}{429}.$$

6.45 (b) Let S_1 denote the event that the spotted egg is chosen in the first draw, S_2, the event that the spotted egg is chosen in the second draw, and C_3, the event that a clear egg is chosen in the third draw.

Then we need $P(S_1 \cap S_2 \cap C_3)$ which by the multiplication law, may be written as

$$P(S_1 \cap S_2 \cap C_3) = P(S_1)\, P(S_2/S_1)\, P(C_3/S_1 \cap S_2).$$

Now $\quad P(S_1) = \frac{5}{30},$

$P(S_2/S_1) = \frac{4}{29}$, as after the occurrence of S_1, ther are 29 eggs left out of which 4 are spotted; and

$P(C_3/S_1 \cap S_2) = \frac{25}{28}$, as after the second draw, there are 28 eggs left out of which 25 are clear eggs.

$$\text{Hence } P(S_1 \cap S_2 \cap C_3) = \frac{5}{30} \times \frac{4}{29} \times \frac{25}{28} = \frac{25}{1218}.$$

6.46. Let A denote the event that 1 girl and 2 boys are selected. Then A can occur in any of the following three mutually exclusive ways.

A_1 = a girl from first group, a boy from second and a boy from third group.

A_2 = a boy from first group, a girl from second and a boy from 3rd group.

A_3 = a boy from 1st group, a boy from 2nd group and a girl from 3rd group.

Thus $P(A_1) = \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{9}{32}$

$P(A_2) = \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{3}{32}$

$P(A_3) = \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{1}{32}$.

Hence $P(A) = P(A_1) + P(A_2) + P(A_3)$, ($A_1$, A_2, A_3 are mutually exclusive).

$$= \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}.$$

6.47. Let A = (only girls turn up for the party) and B = (2 girls and 1 boy turn up for the party). Then

(i) $P(A)$ = the probability of turning up of a girl from 1st family, a girl from 2nd family and a girl from 3rd family.

$$= \frac{2}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{32}.$$

(ii) B can occur in any of the following 3 mutually exclusive ways, where order corresponds to first, second and third family.

B_1 = girl – girl – boy;

B_2 = girl – boy – girl;

B_3 = boy – girl – girl.

Thus $P(B_1) = \frac{2}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{32}$,

$P(B_2) = \frac{2}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{32}$,

$P(B_3) = \frac{2}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{32}$.

Hence $P(B) = P(B_1) + P(B_2) + P(B_3)$.

$$= \frac{1}{32} + \frac{9}{32} + \frac{3}{32} = \frac{13}{32}.$$

6.48. Let A = (the book is favourably reviewed by the first reviewer), B = (the book is favourably reviewed by the second reviewer) and C = (the book is favourably reviewed by the third reviewer). Then

$$P(A) = \frac{3}{5},\ P(B) = \frac{4}{7} \text{ and } P(C) = \frac{2}{5}.$$

Since the events are independent, so $P(\bar{A}) = \frac{2}{5}$, $P(\bar{B}) = \frac{3}{7}$, and $P(\bar{C}) = \frac{3}{5}$.

A majority out of 3 reviewers will favour if two or three review the book favourably.

Let E be the event that only two favour. Then

$$E = (A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C).$$

In other words, if only two favour, then they would be either only the first and second, $A \cap B \cap \bar{C}$, or only the first and third, $A \cap \bar{B} \cap C$, or only the second and third, $\bar{A} \cap B \cap C$. Since these events are mutually exclusive, therefore

$$\begin{aligned} P(E) &= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C). \\ &= P(A).P(B).P(\bar{C}) + P(A).P(\bar{B})\ P(C) + P(\bar{A})\ P(B)\ P(C) \\ &= \frac{3}{5}\cdot\frac{4}{7}\cdot\frac{3}{5} + \frac{3}{5}\cdot\frac{3}{7}\cdot\frac{2}{5} + \frac{2}{5}\cdot\frac{4}{7}\cdot\frac{2}{5} \\ &= \frac{36}{175} + \frac{18}{175} + \frac{16}{175} = \frac{70}{175} = \frac{2}{5}. \end{aligned}$$

In case all three also favour, we have the required probability as

$$\frac{2}{5} + P(A \cap B \cap C) = \frac{2}{5} + \frac{3}{5}\cdot\frac{4}{7}\cdot\frac{2}{5} = \frac{94}{174}.$$

6.49. (a) Here $P(A) = \frac{4}{5}$, $P(B) = \frac{3}{4}$, and $P(C) = \frac{2}{3}$

At least two shots mean two or more. Thus

$$P(A \text{ and } B \text{ hit but not } C) = \frac{4}{5} \times \frac{3}{4} \times \left(1 - \frac{2}{3}\right) = \frac{1}{5}$$

$$P(A \text{ and } C \text{ hit but not } B) = \frac{4}{5}\left(1 - \frac{3}{4}\right) \cdot \frac{2}{3} = \frac{2}{15}$$

$$P(B \text{ and } C \text{ hit but not } A) = \left(1 - \frac{4}{5}\right) \cdot \frac{3}{4} \times \frac{2}{3} = \frac{1}{10}$$

$$P(A, B \text{ and } C \text{ hit}) = \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{2}{5}$$

Since these ways are mutually exclusive, therefore the required probability $= \frac{1}{5} + \frac{2}{15} + \frac{1}{10} + \frac{2}{5}$

$$= \frac{6 + 4 + 3 + 12}{30} = \frac{25}{30} = \frac{5}{6}.$$

(b) Given the probabilities of a wrong decision by each member as

$$P(A) = 0.05,\ P(B) = 0.05, \text{ and } P(C) = 0.10,$$

Let each member vote independently, then

$$P(\bar{A}) = 0.95,\ P(\bar{B}) = 0.95 \text{ and } P(\bar{C}) = 0.90.$$

Let E be the event that a wrong decision on the basis of a majority vote is made by the committee. Then

$$E = (A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C).$$

$$\therefore \quad P(E) = P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) + P(A \cap B \cap C)$$

$$= P(A) \cdot P(B) \cdot P(\bar{C}) + P(A)P(\bar{B})P(C) + P(\bar{A})P(B)P(C) + P(A)\,P(B)\,P(C)$$

$$= (0.05)\,(0.05)\,(0.90) + (0.05)\,(0.95)\,(0.10) + (0.95)\,(0.05)\,(0.10) + (0.05)\,(0.05)\,(0.10)$$

$$= 0.00225 + 0.00475 + 0.00475 + 0.00025 = 0.012$$

This indicates that the committee will be wrong in 1.2% of its decisions.

6.50. **Let A = (first man hits the target), B = (second man hits the target), and C = (third man hits the target).**

Then $P(A) = \frac{1}{6}$, $P(B) = \frac{1}{4}$ and $P(C) = \frac{1}{3}$.

Also $P(\bar{A}) = \frac{5}{6}$, $P(\bar{B}) = \frac{3}{4}$ and $P(\bar{C}) = \frac{2}{3}$ as the events are independent.

(i) Let E be the event that exactly one man hits the target.

Then $E = (A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$,

where $A \cap \bar{B} \cap \bar{C}$ implies that only the first man hits, $\bar{A} \cap B \cap \bar{C}$ means only the second man hits, etc. Since the three events are mutually exclusive, therefore

$$\begin{aligned} P(E) &= P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C) \\ &= P(A).P(\bar{B}).P(\bar{C}) + P(\bar{A}).P(B).P(\bar{C}) + P(\bar{A}).P(\bar{B}).P(C) \\ &= \frac{1}{6}\cdot\frac{3}{4}\cdot\frac{2}{3} + \frac{5}{6}\cdot\frac{1}{4}\cdot\frac{2}{3} + \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{3} \\ &= \frac{6}{72} + \frac{10}{72} + \frac{15}{72} = \frac{31}{72}. \end{aligned}$$

(ii) Here we require the probability that the first man hit the target given that only one man hit the target, *i.e.* $P(A/E)$.

Now $A \cap E = A \cap \bar{B} \cap \bar{C}$ is the event that fonly the first man hit the target. Therefore

$$\begin{aligned} P(A \cap E) = P(A \cap \bar{B} \cap \bar{C}) &= P(A) . P(\bar{B}) . P(\bar{C}) \\ &= \frac{1}{6}\cdot\frac{3}{4}\cdot\frac{2}{3} = \frac{1}{12}. \end{aligned}$$

Hence $P(A/E) = \dfrac{P(A \cap E)}{P(E)} = \dfrac{1/12}{31/72} = \dfrac{6}{31}$.

6.51. **Let A_1, A_2, A_3 represent the events that the first, the second and the third missile hit the target respectively. Then**

$P(A_1) = 0.40$, $P(A_2) = 0.50$, $P(A_3) = 0.60$, and

$P(\bar{A}_1) = 0.60$, $P(\bar{A}_2) = 0.50$, $P(\bar{A}_3) = 0.40$

(i) $P(\text{all the missiles hit the target}) = P(A_1 \cap A_2 \cap A_3)$
$= P(A_1)\, P(A_2)\, P(A_3)$
$= (0.4)\,(0.5)\,(0.6) = 0.12$

(ii) $P(\textit{at least}\text{ one of the three hits the target})$
$= 1 - P\,(\text{no missile hits the target})$
$= 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)$
$= 1 - P(\bar{A}_1)\, P(\bar{A}_2)\, P(\bar{A}_3)$
$= 1 - (0.6)\,(0.5)\,(0.4) = 0.88.$

(iii) $P(\text{exactly one hits the target})$
$= P(A_1 \cap \bar{A}_2 \cap \bar{A}_3) + P(\bar{A}_1 \cap A_2 \cap \bar{A}_3) + P(\bar{A}_1 \cap \bar{A}_2 \cap A_3)$
$= P(A_1)P(\bar{A}_2)P(\bar{A}_3) + P(\bar{A}_1)P(A_2) \times P(\bar{A}_3) + P(\bar{A}_1)\, P(\bar{A}_2)\, P(A_3)$
$= (0.40)(0.5)(0.4) + (0.6)(0.5)(0.4) + (0.6)(0.5)(0.6)$
$= 0.08 + 0.12 + 0.18 = 0.38.$

(iv) $P(\text{exactly two hit the target})$
$= P(A_1 \cap A_2 \cap \bar{A}_3) + P(A_1 \cap \bar{A}_2 \cap A_3) + P(\bar{A}_1 \cap A_2 \cap A_3)$
$= P(A_1)P(A_2)P(\bar{A}_3) + P(A_1)P(\bar{A}_2)P(A_3) + P(\bar{A}_1)P(A_2)P(A_3)$
$= (0.4)\,(0.5)\,(0.4) + (0.4)\,(0.5)\,(0.6) + (0.6)\,(0.5)\,(0.6)$
$= 0.08 + 0.12 + 0.18 = 0.38.$

6.52. Given that $P(A \text{ wins any one game}) = \frac{6}{12} = \frac{1}{2}$,

$P(B \text{ wins any one game}) = \frac{4}{12} = \frac{1}{3}$, and

$P(\text{any one game ends in a tie}) = \frac{2}{12} = \frac{1}{6}$.

Let A_1, A_2, A_3 represent the events "A wins" in first, second and third game respectively;
B_1, B_2, B_3 represent the events "B wins" in first, second and third game respectively; and
T_1, T_2, T_3 represent the events "a tie occurs" in first, second and third game respectively. Then

(a) $P(A \text{ wins all 3 games}) = P(A_1 \cap A_2 \cap A_3)$

$$= P(A_1).P(A_2).P(A_3) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

(b) $P(\text{2 games end in a tie}) = P(T_1 \cap T_2 \cap \bar{T}_3) + P(T_1 \cap \bar{T}_2 \cap T_3) + P(\bar{T}_1 \cup T_2 \cap T_3)$; (2 in tie and third not).

$$= P(T_1).P(T_2).P(\bar{T}_3) + P(T_1) + P(\bar{T}_2).P(T_3) + P(\bar{T}_1).P(T_2).P(T_3)$$

$$= \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6} + \frac{1}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}$$

$$= \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{15}{216} = \frac{5}{72}.$$

(c) $P\,(A\ \&\ B \text{ wins alternately}) = P(A_1 \cap B_2 \cap A_3) + P(B_1 \cap A_2 \cap B_3)$

$$= P(A_1)\,.\,P(B_2)\,.\,P(A_3) + P(B_1)\,.\,P(A_2)\,.\,P(B_3)$$

$$= \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{3} = \frac{1}{12} + \frac{1}{18} = \frac{5}{36}.$$

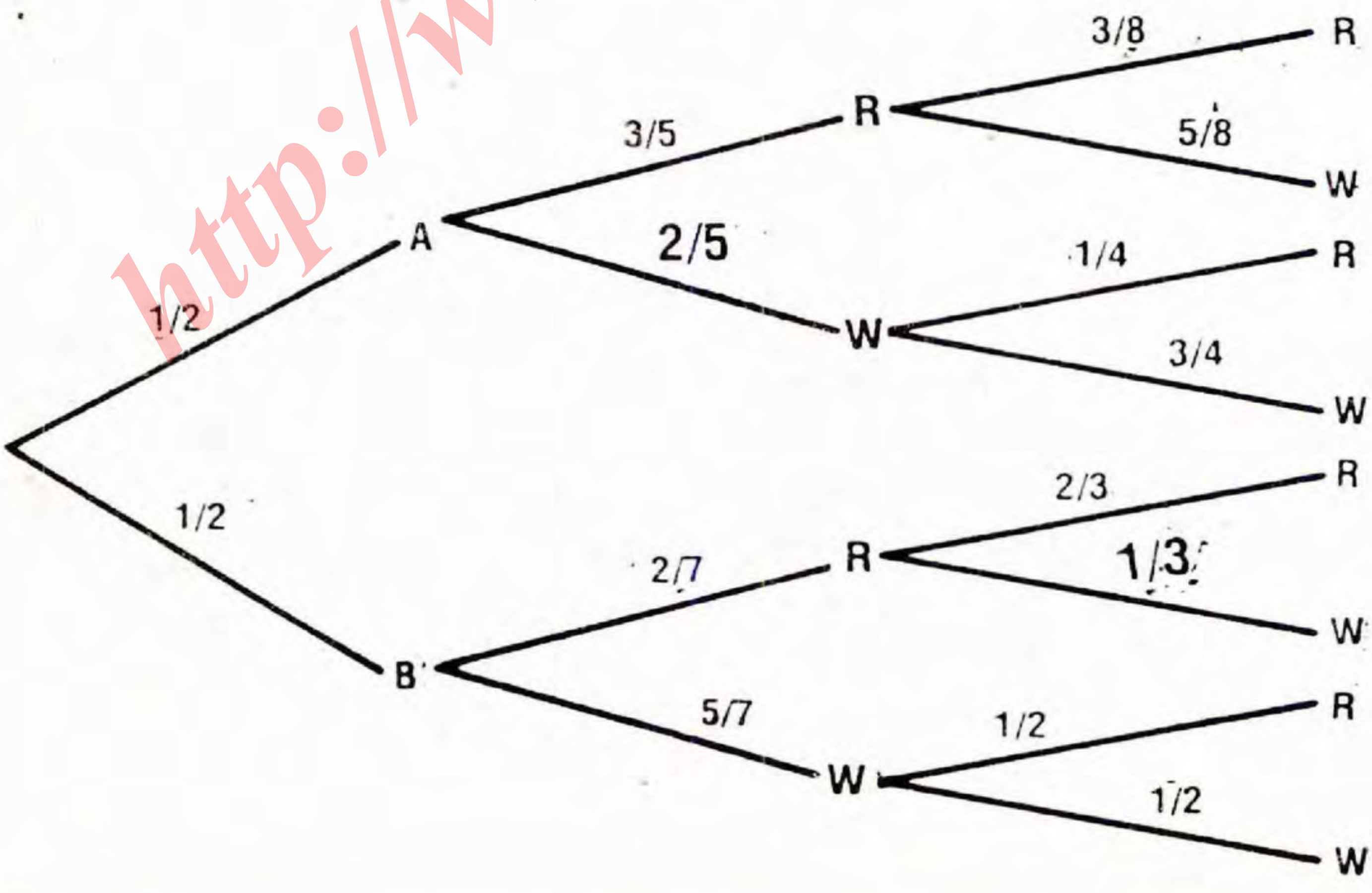
(d) $P(B \text{ wins at least one game}) = 1 - P(B \text{ wins no game})$

$$= 1 - P(\bar{B}_1 \cap \bar{B}_2 \cap \bar{B}_3)$$

$$= 1 - P(\bar{B}_1)\,.\,P(\bar{B}_2)\,.\,P(\bar{B}_3)$$

$$= 1 - \left(\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\right) = 1 - \frac{8}{27} = \frac{19}{27}$$

6.53. The tree diagram of the process is constructed as below:



Suppose urn A is selected and a red ball (R) is drawn, and put into urn B, then urn B will contain 3 red balls (R) and 5 white balls (W).

Since there are four paths leading to two balls of the same colour, we get the required probability as

$$p = \left(\frac{1}{2}\cdot\frac{3}{5}\cdot\frac{3}{8}\right) + \left(\frac{1}{2}\cdot\frac{2}{5}\cdot\frac{3}{4}\right) + \left(\frac{1}{2}\cdot\frac{2}{7}\cdot\frac{2}{3}\right) + \left(\frac{1}{2}\cdot\frac{5}{7}\cdot\frac{1}{2}\right)$$

$$= \frac{9}{80} + \frac{3}{20} + \frac{2}{21} + \frac{5}{28} = \frac{901}{1680}$$

6.54. There are 6 persons (5 friends and 1 host) and not two persons will have the same birth day.

Now, each person can have any one of the 30 days (days of the month of April) as his birth day and assuming each day of the month is equally likely, we have $(30)^6$ ways in which the 6 persons can have their birth days.

As the 6 persons will have distinct birth days, the first person can have any one of the 30 days as his birth day, the second can have any one of the remaining 29 days as his birth day, the third person can have any one of the remaining 28 days as his birth day, and so on.

Thus there are $30 \times 29 \times 28 \times 27 \times 26 \times 25$ ways in which the 6 persons can have distinct birth day.

Hence the required probability, p, is

$$p = \frac{30 \times 29 \times 28 \times 27 \times 26 \times 25}{(30)^6}$$

$$= \frac{30}{30} \times \frac{29}{30} \times \frac{28}{30} \times \frac{27}{30} \times \frac{26}{30} \times \frac{25}{30} = \frac{2639}{4500} = 0.586$$

6.55. (a) The probability of throwing a head with a coin = $\frac{1}{2}$.

A can win in the first, third, fifth, ..., toss, while B can win in the second, fourth, sixth, ..., toss.

Thus $\quad P(\text{A's winning}) = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^4 + \ldots$

($\because$ A's win in 3rd toss is associated with failure of both A and B once).

$$= \frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} = \frac{2}{3}$$

$$P(\text{B's winning}) = \frac{1}{2} \times \frac{1}{2} + \left(\frac{1}{2}\right)^3 \times \frac{1}{2} + \left(\frac{1}{2}\right)^5 \times \frac{1}{2} + \ldots$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{3}$$

Hence their respective chances of winning are $\frac{2}{3} : \frac{1}{3}$ or $2 : 1$.

(b) Let A, B and C represent the three persons.

The probability of throwing a head with a coin $= \frac{1}{2}$.

Now A can win in the first, fourth, seventh, ..., throws

$$\therefore \text{ Chance of A's winning} = \frac{1}{2} + \left(\frac{1}{2}\right)^3 \times \frac{1}{2} + \left(\frac{1}{2}\right)^6 \times \frac{1}{2} + \ldots$$

$$= \frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)^3} = \frac{4}{7}$$

B can win in the second, fifth, eighth, ..., throws

$$\therefore \text{ Chance of B's winning} = \frac{1}{2} \times \frac{1}{2} + \left(\frac{1}{2}\right)^4 \times \frac{1}{2} + \left(\frac{1}{2}\right)^7 \times \frac{1}{2} + \ldots$$

$$= \frac{\frac{1}{4}}{1-\left(\frac{1}{2}\right)^3} = \frac{2}{7}$$

C can win in the third, sixth, ninth, ..., throws

$$\therefore \text{ Chance of C's winning} = \left(\frac{1}{2}\right)^2 \times \frac{1}{2} + \left(\frac{1}{2}\right)^5 \times \frac{1}{2} + \left(\frac{1}{2}\right)^8 \times \frac{1}{2} + \ldots$$

$$= \frac{\left(\frac{1}{2}\right)^3}{1 - \left(\frac{1}{2}\right)^3} = \frac{1}{7}$$

Hence their respective chances of winning are $\frac{4}{7}$, $\frac{2}{7}$ and $\frac{1}{7}$.

6.56. (a) S contains 2^6 = 64 sample points.

Let A be the event that exactly 4 heads are obtained. Then the sample points corresponding to A are

$$\binom{6}{4} = 15$$

$$\therefore \quad P(A) = \frac{15}{64}.$$

(b) Here n = 16, p = $\frac{1}{2}$ and q = $\frac{1}{2}$

(i) P(exactly 8 heads are obtained) $= \binom{16}{8}\left(\frac{1}{2}\right)^{16}$

$$= \frac{16!}{8!\,(16-8)!} \cdot \frac{1}{65536}$$

$$= \frac{6435}{32768} = 0.196.$$

(ii) P(exactly 11 heads are obtained) $= \binom{16}{11}\left(\frac{1}{2}\right)^{16}$

$$= \frac{16!}{11!\,(16-11)!} \cdot \frac{1}{65536}$$

$$= \frac{273}{4096} = 0.067.$$

6.57. Let p denote the probability of passing an examination. Then $p = 0.4$ and $q = 1 - p = 0.6$. Here $n = 6$.

(i) P(2 candidates will pass) $= \binom{6}{2}(0.4)^2\,(0.6)^{6-2}$

$$= 15(0.4)^2(0.6)^4 = 0.31104 = 0.31.$$

(ii) P(5 candidates will pass) $= \binom{6}{5}(0.4)^5\,(0.6)^{6-5}$

$$= 6(0.4)^5(0.6) = 0.036864 = 0.04.$$

Now $\quad P$(all candidates pass) $= p^6$, and

P(all candidates fail) $= q^6$.

But $\quad p = 0.4$ and $q = 0.6$, therefore the probabilities are not equal.

The probability of all passing would be equal to the probability of all failing when $p = q = 0.5$.

6.58. (b) The probability of selecting an urn is $\frac{1}{3}$, i.e.

$$P(A) = P(B) = P(C) = \frac{1}{3}.$$

Let E be the event that the ball drawn is white. Then

$$P(E/A) = \frac{1}{3},\quad P(E/B) = \frac{2}{3} \text{ and } P(E/C) = \frac{2}{4}.$$

By Bayes' theorem, we have

$$P(C/E) = \frac{P(C)\,P(E/C)}{P(A)\,P(E/A) + P(B)\,P(E/B) + P(C)\,P(E/C)}$$

$$= \frac{\frac{1}{3}\cdot\frac{2}{4}}{\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot\frac{2}{4}} = \frac{\frac{2}{12}}{\frac{1}{9}+\frac{2}{9}+\frac{1}{6}} = \frac{\frac{2}{12}}{\frac{1}{2}} = \frac{1}{3}.$$

6.59. Let A_1 denote the event that the ideal coin is selected,

A_2 denote the event that the 2nd coin is selected and

A_3 denote the event that the 3rd coin is selected.

Then $\quad P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$

Again, let E denote the event that head appears both the times when the coin is tossed twice. Then

$$P(E/A_1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4},$$ (Probability of two heads, given that the ideal coin is tossed).

$$P(E/A_2) = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}, \text{ and}$$

$$P(E/A_3) = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}.$$

Hence by Bayes' theorem, the desired probaility is

$$P(A_1/E) = \frac{P(A_1)\,P(E/A_1)}{\sum P(A_i)\,P(E/A_i)}, \quad i = 1, 2, 3.$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{4}}{\frac{1}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{9} + \frac{1}{3}\cdot\frac{4}{9}} = \frac{1/12}{29/108} = \frac{1}{12}\times\frac{108}{29} = \frac{9}{29}.$$

6.60. Let A denote the event that the student is taller than 6 feet, *W*, the event that the student chosen is a woman, and M, the event that the student is a man. Then a need P(W/A).

By Bayes' theorem, we get the desired probability as

$$P(W/A) = \frac{P(W)\,P(A/W)}{P(W)\,P(A/W) + P(M)\,P(A/M)}$$

$$= \frac{(0.60)\,(0.01)}{(0.60)\,(0.01) + (0.40)\,(0.04)} = \frac{0.006}{0.022} = \frac{3}{11}.$$

6.61. Let A, B and C denote the events that the cake is baked by cook A, cook B and cook C respectively. Then

$$P(A) = 0.50,\ P(B) = 0.30 \text{ and } P(C) = 0.20.$$

Again, let E denote the event that the cake fails to rise. Then

$$P(E/A) = 0.02,\ P(E/B) = 0.03 \text{ and } P(E/C) = 0.05.$$

We need $P(A/E)$.

Using Bayes' theorem, we get

$$P(A/E) = \frac{P(A)\,P(E/A)}{P(A)\,P(E/A) + P(B)\,P(E/B) + P(C)\,P(E/C)}$$

$$= \frac{(0.50)\,(0.02)}{(0.50)\,(0.02) + (0.30)\,(0.03) + (0.20)\,(0.05)}$$

$$= \frac{0.010}{0.029} = \frac{10}{29}.$$

Hence the desired proportion of "failures" caused by A

$$= \left(\frac{10}{29}\right) \times 100 = \frac{1000}{29}\ \%$$

6.62. Let H, M and L denote the events that the box chosen contains high quality, medium quality and low quality bulbs respectively. Then

$$P(H) = 0.2,\ P(M) = 0.4 \text{ and } P(L) = 0.4.$$

Again, let S denote the event that the two bulbs tested are satisfactory. Then

$$P(S/H) = 1.0,\ P(S/M) = (1.0 - 0.1)^2 = 0.81, \text{ and}$$

$$P(S/L) = (1.0 - 0.2)^2 = 0.64.$$

We need (i) $P(H/S)$, (ii) $P(M/S)$ and (iii) $P(L/S)$.

Using Bayes' theorem, we get

(i) $$P(H/S) = \frac{P(H)\,P(S/H)}{P(H)\,P(S/H) + P(M)\,P(S/M) + P(L)\,P(S/L)}$$

$$= \frac{(0.2)\,(1.0)}{(0.2 \times 1.0) + (0.4 \times 0.81) + (0.4 \times 0.64)}$$

$$= \frac{0.2}{0.2 + 0.324 + 0.256} = \frac{0.2}{0.78} = 0.256.$$

Similarly,

(ii) $$P(M/S) = \frac{0.324}{0.78} = 0.415, \text{ and}$$

(iii) $$P(L/S) = \frac{0.256}{0.78} = 0.328.$$

6.63. The probabilities of three diseases A_1, A_2, A_3, under the given conditions are

$$P(A_1) = \frac{1}{2},\ P(A_2) = \frac{1}{6} \text{ and } P(A_3) = \frac{1}{3}.$$

Let E denote the test carried out to help the diagnosis. Then

$P(E/A_1) = \binom{5}{4}(0.1)^4(0.9)$. [Given disease A_1, the
$= 0.00045$. probability of 4 positive results out of 5 test]

Similarly,

$$P(E/A_2) = \binom{5}{4}(0.2)^4(0.9) = 0.00640, \text{ and}$$

$$P(E/A_3) = \binom{5}{4}(0.9)^4(0.1) = 0.32805$$

Hence by Bayes' theorem, the desired probabilities are

$$P(A_1/E) = \frac{P(A_1)\,P(E/A_1)}{\sum P(A_i)\,P(E/A_i)}, \text{ where } i = 1, 2, 3.$$

$$= \frac{\frac{1}{2} \times (0.00045)}{\frac{1}{2} \times (0.00045) + \frac{1}{6} \times (0.00640) + \frac{1}{3} \times (0.32805)}$$

$$= \frac{0.000225}{(0.000225) + (0.001067) + (0.109350)} = \frac{0.000225}{0.110642}$$

$$= 0.0020;$$

$P(A_2/E) = 0.0096$; and $P(A_3/E) = 0.9883$.

These results can be summarized as below:

Event	Prior Probabilities $P(A_i)$	Conditional Probabilities $P(E/A_i)$	Joint Probabilities $P(A_i \cap E)$	Posterior Probabilities $P(A_i/E)$
Disease A_1	1/2	0.00045	0.000225	0.0020
Disease A_2	1/6	0.00640	0.001067	0.0096
Disease A_3	1/3	0.32805	0.109350	0.9883
	1		0.110642	0.9999

The posterior probability $P(A_i/E)$ is, by Bayes' theorem,

$$P(A_i/E) = \frac{P(A_i \cap E)}{\sum P(A_i \cap E)}, \text{ where } P(A_i \cap E) = P(A_i)\,P(E/A_i).$$

●●●●●●●●●●