问求算法设计与实践06

线性规划建模

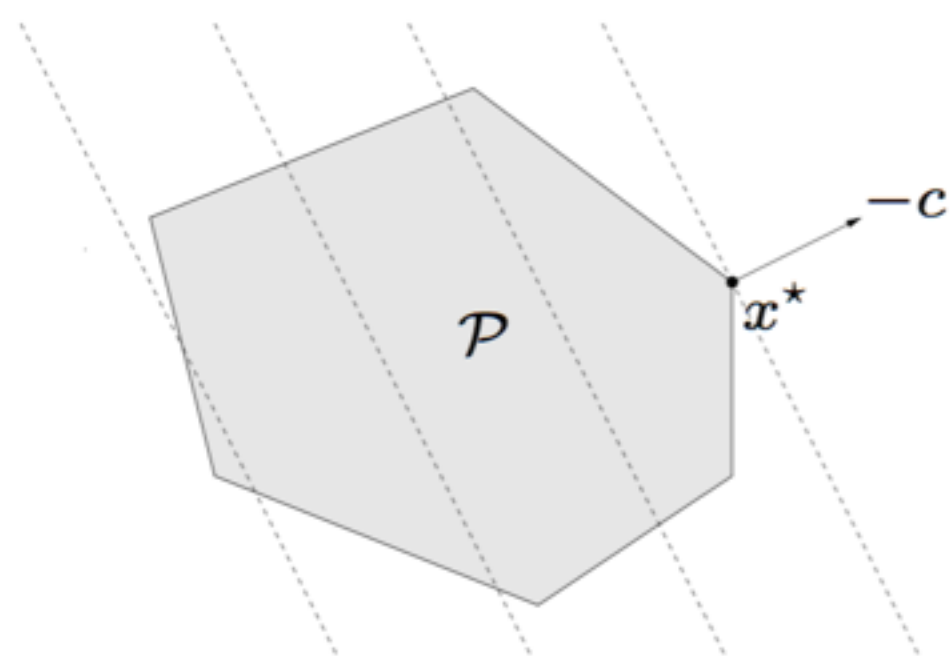
Linear Programming

- 定义
- 算法
  - Simplex Algorithm(1947)
  - Interior Point Algorithm(1984), O(N^3.5*L)
  - 软件包: linprog,lingo,cvxopt…

Linear program (LP)

$$
\begin{array}{ll}
\text{minimize} & c^T x + d \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



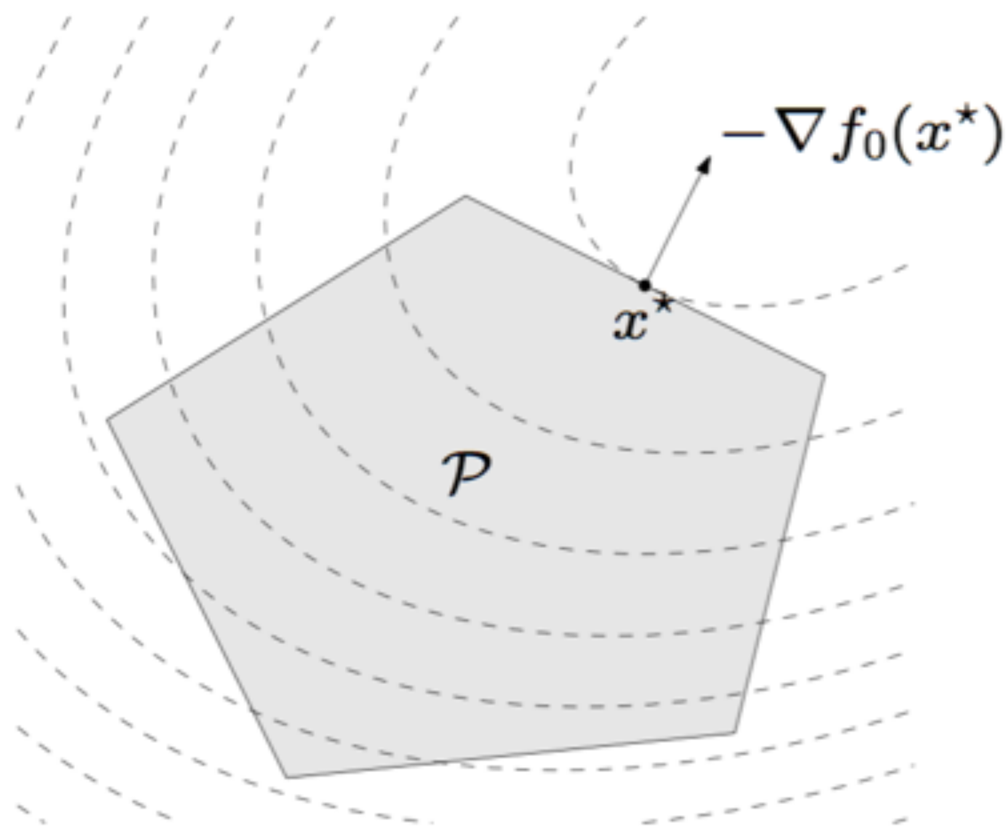
More Programmings

- Linear-fractional Programming
- Quadratic Programming (with Quadratic Constraints)
- Geometric Programming
- Semidefinite Programming
- …
- All about Convex!
- hint:dual problem

Quadratic program (QP)

$$\begin{array}{ll}\text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

- $P \in \mathbf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Quadratically constrained quadratic program (QCQP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- $P_i \in \mathbf{S}^n_+$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}^n_{++}$, feasible region is intersection of m ellipsoids and an affine set

Two-way partitioning

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

- a nonconvex problem; feasible set contains 2^n discrete points
- interpretation: partition $\{1, \ldots, n\}$ in two sets; W_{ij} is cost of assigning i, j to the same set; $-W_{ij}$ is cost of assigning to different sets

dual function

$$\begin{aligned} g(\nu) = \inf_x (x^T W x + \sum_i \nu_i (x_i^2 - 1)) &= \inf_x x^T (W + \mathbf{diag}(\nu)) x - \mathbf{1}^T \nu \\ &= \begin{cases} -\mathbf{1}^T \nu & W + \mathbf{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

lower bound property: $p^\star \geq -\mathbf{1}^T \nu$ if $W + \mathbf{diag}(\nu) \succeq 0$

example: $\nu = -\lambda_{\min}(W)\mathbf{1}$ gives bound $p^\star \geq n\lambda_{\min}(W)$

Diet Problem

A healthy diet contains m different nutrients in quantities at least equal to $b_1, \ldots, b_m$. We can compose such a diet by choosing nonnegative quantities $x_1, \ldots, x_n$ of n different foods. One unit quantity of food j contains an amount a_{ij} of nutrient i, and has a cost of c_j. We want to determine the cheapest diet that satisfies the nutritional requirements. This problem can be formulated as the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b \\ & x \succeq 0. \end{array}$$

Shortest Path Problem

- Di表示到第i个点的最短路
- w(u,v)表示u->v的边权
- 求S->T的最短路

Minimum-Cost Flow

- Consider a network of **n** nodes, with directed links connecting each pair of nodes. The variables in the problem are the flows on each link: **Xij**$_{ij}$ will denote the flow from node **i** to node **j**. The cost of the flow along the link from node **i** to node **j** is given by **Cij*Xij**, where **Cij**$_{ij}$ are given constants. The total cost across the network is

C=sigma Cij*Xij

- Each link flow **Xij**$_{ij}$ is also subject to a given lower bound **Lij** (usually assumed to be nonnegative) and an upper bound **Uij**.

- The problem is to minimize the total cost of flow through the network, subject to the constraints described above. Formulate this problem as an LP.

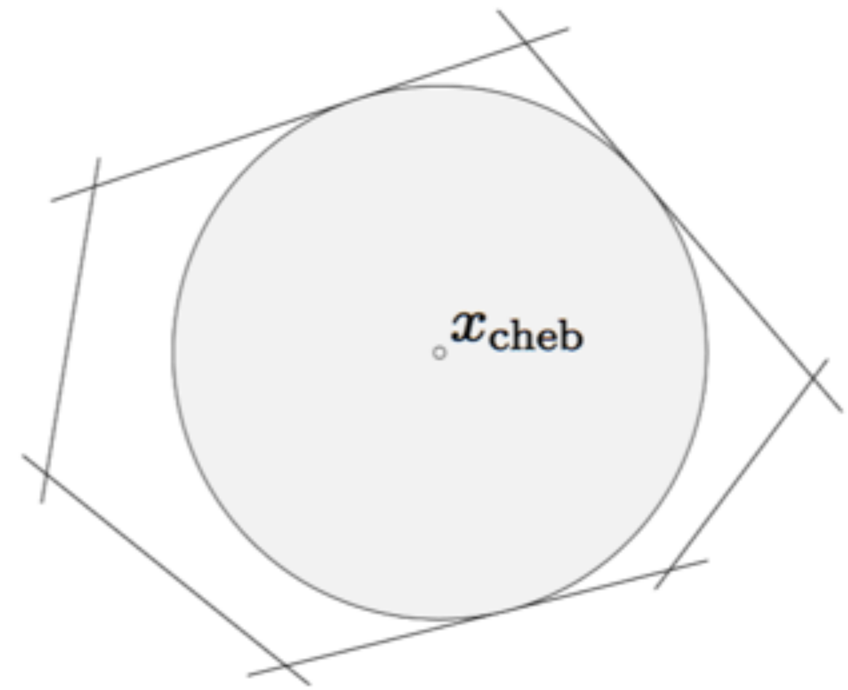
Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \leq r\}$$



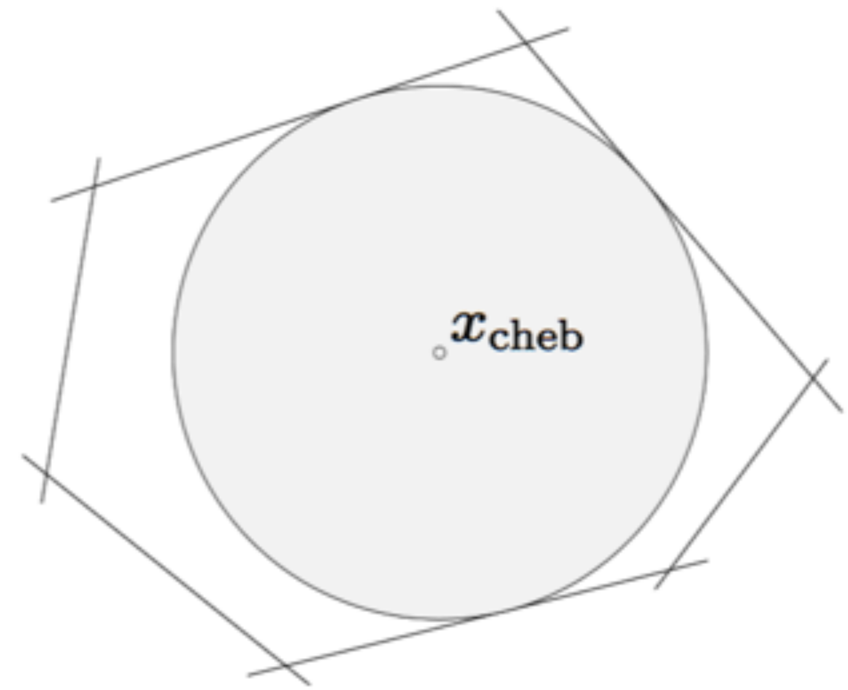
Chebyshev center of a polyhedron

Chebyshev center of

$$\mathcal{P} = \{x \mid a_i^T x \leq b_i,\ i = 1, \ldots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{x_c + u \mid \|u\|_2 \leq r\}$$



- $a_i^T x \leq b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c + u) \mid \|u\|_2 \leq r\} = a_i^T x_c + r\|a_i\|_2 \leq b_i$$

- hence, x_c, r can be determined by solving the LP

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + r\|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m \end{array}$$

Dynamic activity planning

We consider the problem of choosing, or planning, the activity levels of n activities, or sectors of an economy, over N time periods. We let $x_j(t) \geq 0$, $t = 1, \dots, N$, denote the activity level of sector j, in period t. The activities both consume and produce products or goods in proportion to their activity levels. The amount of good i produced per unit of activity j is given by a_{ij}. Similarly, the amount of good i consumed per unit of activity j is b_{ij}. The total amount of goods produced in period t is given by $Ax(t) \in \mathbf{R}^m$, and the amount of goods consumed is $Bx(t) \in \mathbf{R}^m$. (Although we refer to these products as 'goods', they can also include unwanted products such as pollutants.)

The goods consumed in a period cannot exceed those produced in the previous period: we must have $Bx(t+1) \preceq Ax(t)$ for $t = 1, \dots, N$. A vector $g_0 \in \mathbf{R}^m$ of initial goods is given, which constrains the first period activity levels: $Bx(1) \preceq g_0$. The (vectors of) excess goods not consumed by the activities are given by

$$\begin{aligned} s(0) &= g_0 - Bx(1) \\ s(t) &= Ax(t) - Bx(t+1), \quad t = 1, \dots, N-1 \\ s(N) &= Ax(N). \end{aligned}$$

The objective is to maximize a discounted total value of excess goods:

$$c^T s(0) + \gamma c^T s(1) + \cdots + \gamma^N c^T s(N),$$

where $c \in \mathbf{R}^m$ gives the values of the goods, and $\gamma > 0$ is a discount factor. (The value c_i is negative if the ith product is unwanted, *e.g.*, a pollutant; $|c_i|$ is then the cost of disposal per unit.)

Putting it all together we arrive at the LP

$$\begin{array}{ll}
\text{maximize} & c^T s(0) + \gamma c^T s(1) + \cdots + \gamma^N c^T s(N) \\
\text{subject to} & x(t) \succeq 0, \quad t = 1, \ldots, N \\
& s(t) \succeq 0, \quad t = 0, \ldots, N \\
& s(0) = g_0 - Bx(1) \\
& s(t) = Ax(t) - Bx(t+1), \quad t = 1, \ldots, N-1 \\
& s(N) = Ax(N),
\end{array}$$

with variables $x(1), \ldots, x(N)$, $s(0), \ldots, s(N)$. This problem is a standard form LP; the variables $s(t)$ are the slack variables associated with the constraints $Bx(t+1) \preceq Ax(t)$.

Reference

[1] Boyd, Stephen, and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[2]http://stanford.edu/class/ee364a/lectures/problems.pdf