Laplace Transforms: Notes for a 1st Year Undergraduate

The Laplace Transform is a powerful integral transform that converts a time-domain function $f(t)$ into a frequency-domain function $F(s)$. This transformation often simplifies the process of solving linear differential equations, especially those with discontinuous forcing functions.
Here's a breakdown of the key concepts:

1. Definition of the Laplace Transform

The Laplace transform of a function $f(t)$ defined for $t \ge 0$ is given by:
$F(s) = \mathcal{L}\{f(t)\}(s) = \int_{0}^{\infty} f(t) e^{-st} \, dt$
where:

- $s = \sigma + j\omega$ is a complex variable in the frequency domain (though for many basic applications, we can initially consider s to be real).
- $\mathcal{L}$ is the Laplace transform operator.
- $F(s)$ is the Laplace transform of $f(t)$.
- The integral must converge for the Laplace transform to exist. This typically requires $f(t)$ to be piecewise continuous and of exponential order.

2. Region of Convergence (ROC)

- The Laplace transform integral may converge only for certain values of s. The set of all such values of s is called the Region of Convergence (ROC).
- Specifying the ROC is crucial because different time-domain functions can have the same algebraic expression in the s-domain but with different ROCs.
- For functions that are zero for $t < 0$ (causal signals), the ROC is typically a right-half plane in the complex s-plane ($\text{Re}(s) > \sigma_0$ for some σ_0).

3. Common Laplace Transforms

It's essential to know the Laplace transforms of some basic functions:

$f(t)$ (for $t \ge 0$)	$F(s) = \mathcal{L}\{f(t)\}(s)$	ROC
1	$\frac{1}{s}$	$\text{Re}(s) > 0$
t	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
t^n ($n = 0, 1, 2, \dots$)	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
e^{at}	$\frac{1}{s-a}$	$\text{Re}(s) > \text{Re}(a)$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$\text{Re}(s) > 0$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$\text{Re}(s) > 0$
$u(t)$ (Unit Step Function)	$\frac{1}{s}$	$\text{Re}(s) > 0$
$\delta(t)$ (Dirac Delta Function)	1	All s

You'll often use these basic transforms and the properties of the Laplace transform to find the transforms of more complex functions.

4. Properties of the Laplace Transform

These properties are crucial for solving differential equations and manipulating Laplace transforms:

- **Linearity:** $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} = aF(s) + bG(s)$, where a and b are constants.
- **Time Shifting (Translation in Time):** $\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s)$ for $a \ge 0$, where u(t-a) is the unit step function shifted by a.
- **Frequency Shifting (Translation in s):** $\mathcal{L}\{e^{at}f(t)\} = F(s - a)$.
- **Differentiation in the Time Domain:** $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
  - $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
  - $\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$ (This is particularly useful for solving ODEs with initial conditions).
- **Integration in the Time Domain:** $\mathcal{L}\left\{\int_{0}^{t} f(\tau) \, d\tau\right\} = \frac{1}{s}F(s)$.
- **Multiplication by t:** $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$.
- **Division by t:** $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_{s}^{\infty} F(\sigma) \, d\sigma$ (This property has conditions on the behavior of f(t) near t=0).
- **Convolution in the Time Domain:** $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\left\{\int_{0}^{t} f(\tau)g(t - \tau) \, d\tau\right\} = F(s)G(s)$.
- **Differentiation in the Frequency Domain:** $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$.
- **Initial Value Theorem:** $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$ (if the limit exists).
- **Final Value Theorem:** $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ (if the limit exists and all poles of sF(s) are in the left-half plane).

5. Inverse Laplace Transform

The inverse Laplace transform converts a function F(s) in the s-domain back to a function f(t) in the time domain:
$f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$
Finding the inverse Laplace transform often involves:

- **Using a table of known Laplace transforms in reverse.**
- **Partial fraction expansion:** Breaking down a rational function F(s) into simpler terms whose inverse transforms are known. This is a very common technique when F(s) is a ratio of polynomials.
- **Contour integration (using complex analysis),** which is generally covered in more advanced courses.

6. Solving Linear Differential Equations with Laplace Transforms

The Laplace transform provides a powerful method for solving linear ordinary differential equations (ODEs) with constant coefficients:

1. **Take the Laplace transform of both sides of the differential equation.** Use the properties of the Laplace transform, especially the differentiation property, to convert derivatives of y(t) into algebraic expressions involving $Y(s) = \mathcal{L}\{y(t)\}$. Incorporate the initial conditions at t=0.

2. **Solve the resulting algebraic equation for Y(s).**
3. **Find the inverse Laplace transform of Y(s) to obtain the solution y(t).** This often involves using partial fraction expansion and a table of Laplace transforms.

This method is particularly advantageous for problems with:

- **Non-zero initial conditions:** These are automatically handled by the differentiation property.
- **Discontinuous forcing functions (e.g., step functions, impulse functions):** These are easily represented and manipulated in the s-domain.

7. Applications of Laplace Transforms

Beyond solving ODEs, Laplace transforms are crucial in various fields:

- **Control Systems:** Analyzing the stability and response of feedback control systems. Transfer functions are defined in the s-domain using Laplace transforms.
- **Circuit Analysis:** Analyzing transient behavior in electrical circuits with capacitors and inductors.
- **Signal Processing:** Analyzing and designing filters.
- **Communications:** Studying the response of systems to various input signals.

In Summary

The Laplace transform is a vital tool for engineers and scientists. It provides a way to move from the time domain to the frequency domain, often simplifying the solution of linear differential equations and the analysis of dynamic systems. Understanding the definition, basic transforms, key properties, and the inverse Laplace transform are fundamental steps in your first year. Mastering partial fraction expansion is particularly important for finding inverse transforms and solving ODEs.