Alright, focusing solely on Differential Equations for a 1st-year undergraduate, here are the key notes:

# Differential Equations: Notes for a 1st Year Undergraduate

Differential Equations (DEs) are equations that relate a function to its derivatives. They are fundamental in modeling phenomena involving rates of change.

## 1. Basic Concepts

- **Definition:** An equation involving an unknown function and one or more of its derivatives with respect to one or more independent variables.
- **Ordinary Differential Equation (ODE):** Contains derivatives with respect to only *one* independent variable (e.g., $\frac{dy}{dx} + y = x$).
- **Partial Differential Equation (PDE):** Contains partial derivatives with respect to *two or more* independent variables (e.g., $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$). (We'll primarily focus on ODEs here).
- **Order:** The order of the highest derivative present in the equation.
  - Example: $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = \sin(x)$ is a second-order ODE.
- **Degree:** The power to which the highest order derivative is raised after the equation has been made polynomial in the derivatives.
  - Example: $\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + y = 0$ is of degree 2.
  - Example: $\sqrt{\frac{dy}{dx}} = y$ needs to be squared to get $\frac{dy}{dx} = y^2$, so it's of degree 1.
- **Linear vs. Nonlinear ODE:**
  - **Linear:** The dependent variable (y) and its derivatives appear linearly – no products of y and its derivatives, no powers of y or its derivatives other than 1, and no functions of y or its derivatives (e.g., $\sin(y)$ or $e^{y'}$). The coefficients can be functions of the independent variable (x).
    - General form of a linear n-th order ODE: $a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$.
  - **Nonlinear:** Any ODE that is not linear. These can be much harder to solve.
- **Solution:** A function $y = \phi(x)$ that, when substituted into the DE, reduces it to an identity.
- **General Solution:** Contains arbitrary constants equal in number to the order of the ODE. This represents a family of solutions.
- **Particular Solution:** Obtained by assigning specific values to the arbitrary constants in the general solution using given initial or boundary conditions.
- **Initial Conditions:** Values of the dependent variable and its derivatives at a specific value of the independent variable (e.g., $y(x_0) = y_0$, $y'(x_0) = y_1$). Used to find particular solutions for initial value problems.

## 2. First-Order Differential Equations

- **Separable Equations:** Can be rearranged so that terms involving y and dy are on one side, and terms involving x and dx are on the other: $g(y)\,dy = h(x)\,dx$. Solve by integrating both sides: $\int g(y)\,dy = \int h(x)\,dx + C$.
- **Homogeneous Equations:** Can be written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. Solve using the substitution $v = \frac{y}{x}$ (so $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$). This transforms the equation into a separable equation in terms of v and x.
- **Linear First-Order Equations:** Of the form $\frac{dy}{dx} + P(x)y = Q(x)$. Solve using an integrating factor $\mu(x) = e^{\int P(x)\,dx}$. Multiplying the entire equation by $\mu(x)$ makes the left side the derivative of the product $\mu(x)y$. The solution is then found by integrating: $y(x) = \frac{1}{\mu(x)} \int \mu(x) Q(x)\,dx + \frac{C}{\mu(x)}$.
- **Exact Equations:** Of the form $M(x, y)\,dx + N(x, y)\,dy = 0$. This equation is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If it's exact, there exists a function $\psi(x, y)$ such that $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$. The general solution is $\psi(x, y) = C$. To find $\psi$, integrate M with respect to x (treating y as constant) and then differentiate the result with respect to y to find the remaining terms by comparing with N.
- **Integrating Factors (for Non-Exact Equations):** Sometimes a non-exact equation can be made exact by multiplying by a suitable integrating factor $\mu(x, y)$. Finding such a factor can be tricky, but there are special cases where $\mu$ depends only on x or only on y.
- **Applications:** Modeling population growth, radioactive decay, Newton's Law of Cooling, mixing problems, simple circuits.

## 3. Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

- Form: $ay'' + by' + cy = 0$, where a, b, c are constants ($a \neq 0$).
- Solve by assuming a solution of the form $y = e^{rx}$, where r is a constant. Substituting this into the DE yields the characteristic equation (or auxiliary equation): $ar^2 + br + c = 0$.
- The roots of the characteristic equation determine the form of the general solution:
    - **Case 1: Distinct Real Roots ($r_1 \neq r_2$):** The general solution is $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$, where $c_1$ and $c_2$ are arbitrary constants.
    - **Case 2: Repeated Real Roots ($r_1 = r_2 = r$):** The general solution is $y(x) = (c_1 + c_2 x) e^{rx}$. We need a second linearly independent solution, which turns out to be $xe^{rx}$.
    - **Case 3: Complex Conjugate Roots ($r = \alpha \pm i\beta$, where $\beta \neq 0$):** Using Euler's formula ($e^{i\theta} = \cos \theta + i \sin \theta$), the general solution is $y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$.

## 4. Second-Order Linear Non-Homogeneous Differential Equations with Constant Coefficients

- Form: $ay'' + by' + cy = f(x)$, where $f(x) \neq 0$.
- The general solution is the sum of the complementary function ($y_c(x)$) and a particular solution ($y_p(x)$): $y(x) = y_c(x) + y_p(x)$.
    - $y_c(x)$ is the general solution of the associated homogeneous equation ($ay'' + by' + cy = 0$).

    - $y_p(x)$ is any particular solution that satisfies the non-homogeneous equation.
- **Methods for finding a particular solution $y_p(x)$:**
    - **Method of Undetermined Coefficients:** If $f(x)$ has a form such as a polynomial, exponential, sine/cosine, or a sum/product of these, we can guess a particular solution of a similar form with unknown coefficients. These coefficients are then determined by substituting the guess into the non-homogeneous DE and equating coefficients. Special care needs to be taken if the form of $f(x)$ overlaps with the terms in the complementary function (in which case, we multiply our guess by powers of $x$).
    - **Method of Variation of Parameters:** A more general method that can be used even when the form of $f(x)$ is not suitable for undetermined coefficients. If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the homogeneous equation, a particular solution is given by $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where $u_1'(x) = -\frac{y_2(x) f(x)}{a W(y_1, y_2)}$ and $u_2'(x) = \frac{y_1(x) f(x)}{a W(y_1, y_2)}$, and $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ is the Wronskian of $y_1$ and $y_2$.

## 5. Applications of Second-Order DEs

- **Simple Harmonic Motion:** Modeling oscillations without damping ($y'' + \omega^2 y = 0$).
- **Damped Oscillations:** Modeling oscillations with resistive forces ($my'' + cy' + ky = 0$). Cases include overdamping, critical damping, and underdamping.
- **Forced Oscillations and Resonance:** Modeling oscillations under the influence of an external periodic force ($my'' + cy' + ky = F_0 \cos(\omega t)$). Resonance occurs when the forcing frequency is close to the natural frequency.
- **Electrical Circuits (RLC circuits).**
- **Beam Deflection.**

This provides a solid foundation for your first year of studying differential equations. Make sure to practice solving various types of problems to solidify your understanding of the methods and concepts. Good luck!