Alright, let's dive into the essentials of Vector Calculus for a 1st-year undergraduate:

Vector Calculus: Notes for a 1st Year Undergraduate

Vector Calculus extends the concepts of calculus (differentiation and integration) to vector fields. It's crucial for understanding physics and engineering in multiple dimensions.

1. Vectors in 3D Space

- **Representation:** A vector $\mathbf{a}$ in three dimensions can be represented in component form as $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$, where a_x, a_y, a_z are the scalar components along the x, y, z axes, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors in these directions. Alternatively, it can be written as $\mathbf{a} = \langle a_x, a_y, a_z \rangle$.
- **Magnitude (Length):** The magnitude or length of a vector $\mathbf{a}$ is given by $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$.
- **Unit Vector:** A vector with a magnitude of 1. The unit vector in the direction of a non-zero vector $\mathbf{a}$ is $\hat{\mathbf{u}} = \frac{\mathbf{a}}{|\mathbf{a}|}$.
- **Vector Operations:**
  - **Addition/Subtraction:** $\mathbf{a} \pm \mathbf{b} = \langle a_x \pm b_x, a_y \pm b_y, a_z \pm b_z \rangle$.
  - **Scalar Multiplication:** $c\mathbf{a} = \langle ca_x, ca_y, ca_z \rangle$, where c is a scalar.
- **Dot Product (Scalar Product):** For two vectors $\mathbf{a}$ and $\mathbf{b}$, their dot product is a scalar defined as:
  - $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ ($0 \le \theta \le \pi$).
  - In component form: $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$.
  - **Properties:**
    - $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (Commutative)
    - $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (Distributive)
    - $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
    - $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (perpendicular), provided neither is the zero vector.
  - **Applications:** Finding the angle between vectors, projection of one vector onto another.
- **Cross Product (Vector Product):** For two vectors $\mathbf{a}$ and $\mathbf{b}$, their cross product is a vector defined as:
  - $\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \mathbf{n}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ in the direction given by the right-hand rule.
  - In component form: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y) \mathbf{i} - (a_x b_z - a_z b_x) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}$.

  - **Properties:**
    - $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ (Anti-commutative)
    - $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (Distributive)
    - $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
    - $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel (or one of them is the zero vector).
    - $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.
  - **Applications:** Finding a vector perpendicular to two given vectors, calculating torque and angular momentum.

2. Vector Functions and Space Curves

- **Vector Function:** A function that maps a scalar variable (often time t) to a vector: $\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} = \langle x(t), y(t), z(t) \rangle$. As t varies, the terminal point of $\mathbf{r}(t)$ traces a curve in space.
- **Derivative of a Vector Function:** $\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h} = x'(t) \mathbf{i} + y'(t) \mathbf{j} + z'(t) \mathbf{k}$, provided these derivatives exist. $\mathbf{r}'(t)$ is the tangent vector to the curve at the point corresponding to t, pointing in the direction of increasing t.
- **Unit Tangent Vector:** $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, gives the direction of the curve at a point.
- **Arc Length:** The length of the curve traced by $\mathbf{r}(t)$ from $t = a$ to $t = b$ is $s = \int_{a}^{b} |\mathbf{r}'(t)| dt$.

3. Partial Derivatives

- For a scalar function of several variables, $f(x, y, z)$, the partial derivative with respect to one variable (e.g., x) is found by treating the other variables as constants and differentiating with respect to that variable: $\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$. Similarly for $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.
- **Higher-Order Partial Derivatives:** We can take partial derivatives of partial derivatives (e.g., $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$).
- **Clairaut's Theorem (Equality of Mixed Partials):** If the second partial derivatives are continuous, then $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

4. Gradient, Divergence, and Curl

- **The Gradient of a Scalar Function:** For a scalar function $f(x, y, z)$, the gradient is a vector field defined as:
  - $\nabla f = \text{grad } f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$.
  - The gradient vector at a point points in the direction of the greatest rate of increase

of the function at that point, and its magnitude is the value of that greatest rate of increase.
    - ∇f is normal (perpendicular) to the level surfaces of $f(x, y, z) = c$.
- **The Divergence of a Vector Field:** For a vector field $\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$, the divergence is a scalar function defined as:
    - $\nabla \cdot \mathbf{F} = \text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.
    - The divergence measures the "outflow" or "source strength" of the vector field at a point. A positive divergence indicates a source, a negative divergence indicates a sink, and zero divergence indicates that the field is incompressible.
- **The Curl of a Vector Field:** For a vector field $\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$, the curl is a vector field defined as:
    - $\nabla \times \mathbf{F} = \text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathbf{k}$.
    - The curl measures the "rotation" or "vorticity" of the vector field at a point. A non-zero curl indicates that the field has a tendency to rotate.

5. Line Integrals

- **Line Integral of a Scalar Function:** The integral of a scalar function $f(x, y, z)$ along a curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \le t \le b$ is:
    - $\int_{C} f(x, y, z) ds = \int_{a}^{b} f(x(t), y(t), z(t)) |\mathbf{r}'(t)| dt$, where $ds = |\mathbf{r}'(t)| dt$ is the arc length element.
- **Line Integral of a Vector Field:** The integral of a vector field $\mathbf{F}$ along a curve C parameterized by $\mathbf{r}(t)$ from a to b is:
    - $\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_{C} (P dx + Q dy + R dz)$, where $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ and $d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} = \mathbf{r}'(t) dt$.
    - This integral represents the work done by a force field $\mathbf{F}$ in moving a particle along the curve C.
- **Fundamental Theorem for Line Integrals:** If $\mathbf{F} = \nabla f$ (i.e., $\mathbf{F}$ is a conservative vector field), then the line integral of $\mathbf{F}$ along a curve C from point A to point B is independent of the path and is given by:
    - $\int_{C} \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$, where f is a potential function for $\mathbf{F}$.
    - A vector field $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$ in a simply connected region.

This overview covers the fundamental concepts of vector calculus you'll likely encounter in your first year. Remember to practice applying these definitions and theorems through various examples.