Alright, let's break down the essentials of Multiple Integrals for a 1st-year undergraduate:

Multiple Integrals: Notes for a 1st Year Undergraduate

Multiple integrals extend the concept of definite integrals to functions of more than one variable over regions in higher dimensions (planes, volumes, etc.).

1. Double Integrals over Rectangular Regions

- Consider a function $f(x, y)$ defined on a rectangular region $R = [a, b] \times [c, d] = \{(x, y) \mid a \le x \le b, c \le y \le d\}$.
- The double integral of f over R is defined as the limit of a Riemann sum: $\iint_{R} f(x, y) dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$ where ΔA_{ij} is the area of a small rectangle in the partition of R, and (x_{ij}^*, y_{ij}^*) is a sample point in that rectangle.
- **Fubini's Theorem (for Rectangular Regions):** If $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$, then $\iint_{R} f(x, y) dA = \int_{a}^{b} \left(\int_{c}^{d} f(x, y) dy \right) dx = \int_{c}^{d} \left(\int_{a}^{b} f(x, y) dx \right) dy$ This means we can evaluate the double integral as an iterated integral, integrating with respect to one variable at a time. The order of integration doesn't matter for continuous functions over rectangles.

2. Double Integrals over General Regions

- Consider a function $f(x, y)$ defined on a bounded region D in the xy-plane.
- **Type I Region (Vertically Simple):** $D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$, where $g_1(x)$ and $g_2(x)$ are continuous functions. $\iint_{D} f(x, y) dA = \int_{a}^{b} \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$
- **Type II Region (Horizontally Simple):** $D = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$, where $h_1(y)$ and $h_2(y)$ are continuous functions. $\iint_{D} f(x, y) dA = \int_{c}^{d} \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$
- **Evaluating Double Integrals:** To evaluate over a general region, you need to determine the limits of integration based on the boundaries of the region. Sketching the region D is often helpful. You might need to split the region into subregions that are either Type I or Type II.
- **Properties of Double Integrals:**
  - $\iint_{D} [f(x, y) + g(x, y)] dA = \iint_{D} f(x, y) dA + \iint_{D} g(x, y) dA$ (Linearity)
  - $\iint_{D} c f(x, y) dA = c \iint_{D} f(x, y) dA$ (Constant Multiple)
  - If $f(x, y) \ge 0$ on D, then $\iint_{D} f(x, y) dA \ge 0$.
  - If $f(x, y) \le g(x, y)$ on D, then $\iint_{D} f(x, y) dA \le \iint_{D} g(x, y) dA$.
  - $\iint_{D} 1 \, dA = \text{Area}(D)$.

3. Change of Variables in Double Integrals

- Sometimes, integrating in Cartesian coordinates (x, y) is difficult. Changing to a different coordinate system (u, v) can simplify the integral.
- **Jacobian Determinant:** When changing variables, the area element $dA = dx dy$ transforms to $|J(u, v)| du dv$, where $J(u, v)$ is the Jacobian determinant: $J(u, v) =$

$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$

- **Polar Coordinates:** A common change of variables:
    - $x = r \cos \theta, y = r \sin \theta$
    - Jacobian: $\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$
    - Area element: $dA = r\, dr\, d\theta$
    - $\iint_{D} f(x, y)\, dA = \iint_{D'} f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta$, where D' is the region in the $r\theta$-plane corresponding to D. Polar coordinates are useful for regions with circular symmetry.

4. Triple Integrals

- Extend the concept to functions of three variables $f(x, y, z)$ over a region E in 3D space.
- The triple integral is defined as the limit of a Riemann sum: $\iiint_{E} f(x, y, z)\, dV = \lim \sum \sum \sum f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk}$
- **Iterated Integrals:** Similar to double integrals, triple integrals can be evaluated as iterated integrals: $\iiint_{E} f(x, y, z)\, dV = \int \int \int f(x, y, z)\, dz\, dy\, dx$ The limits of integration depend on the boundaries of the region E. The order of integration can sometimes be changed, but the limits will need to be adjusted accordingly.
- **Change of Variables in Triple Integrals:**
    - **Cylindrical Coordinates:**
        - $x = r \cos \theta, y = r \sin \theta, z = z$
        - Jacobian: $\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$
        - Volume element: $dV = r\, dr\, d\theta\, dz$
        - Useful for regions with cylindrical symmetry.
    - **Spherical Coordinates:**
        - $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$
        - Jacobian: $\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin \phi$
        - Volume element: $dV = \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta$
        - Useful for regions with spherical symmetry.

5. Applications of Multiple Integrals

- **Area of a Region:** $A = \iint_{D} 1 \, dA$.
- **Volume of a Solid:** $V = \iiint_{E} 1 \, dV$.
- **Mass of a Lamina or Solid:** $m = \iint_{D} \rho(x, y)\, dA$ or $m = \iiint_{E} \rho(x, y, z)\, dV$, where ρ is the density function.
- **Center of Mass:** $(\bar{x}, \bar{y}) = \frac{1}{m} \iint_{D} (x \rho(x, y), y \rho(x, y))\, dA$ (for a lamina), and similarly for solids.
- **Moments of Inertia:** Measure the resistance of an object to rotational motion. Examples include $I_x = \iint_{D} y^2 \rho(x, y)\, dA$, $I_y = \iint_{D} x^2 \rho(x, y)\, dA$, $I_z = \iiint_{E} (x^2 + y^2) \rho(x, y, z)\, dV$.

Understanding how to set up the limits of integration based on the region of integration and choosing the appropriate coordinate system are key skills in working with multiple integrals. Practice visualizing the regions and applying Fubini's theorem and change of variables will be essential.