Fourier Series: Notes for a 1st Year Undergraduate

The Fourier Series is a powerful tool in mathematics and engineering that allows us to represent periodic functions as an infinite sum of simple sine and cosine waves. Essentially, it decomposes a complex, repeating signal into its fundamental frequencies and their corresponding amplitudes.
Here's a breakdown of the key concepts:

1. Periodic Functions

- A function $f(x)$ is periodic with period T if $f(x + T) = f(x)$ for all x. This means the function repeats its values at regular intervals of T.
- Examples include $\sin(x)$, $\cos(x)$ (period 2π), and $\tan(x)$ (period π).

2. The Idea Behind Fourier Series

- Imagine a complex periodic waveform. Fourier had the insight that this waveform could be constructed by adding together simpler sine and cosine waves of different frequencies and amplitudes.
- The Fourier Series aims to find the specific combination of these sine and cosine waves that perfectly reconstruct the original periodic function.

3. The Fourier Series Formula (for a function with period 2π)

If a function $f(x)$ is periodic with period 2π and satisfies certain conditions (Dirichlet conditions, which are usually met for functions encountered in engineering and physics), it can be represented by the Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where:

- $\frac{a_0}{2}$ is the DC component or the average value of the function over one period.
- The terms $a_n \cos(nx)$ represent cosine waves with frequencies that are integer multiples of the fundamental frequency ($1/2\pi$) and amplitudes a_n.
- The terms $b_n \sin(nx)$ represent sine waves with frequencies that are integer multiples of the fundamental frequency and amplitudes b_n.
- n is a positive integer representing the harmonic number (1st harmonic, 2nd harmonic, etc.).

4. Fourier Coefficients: Euler's Formulas

The coefficients a_0, a_n, and b_n are called the Fourier coefficients and can be calculated using Euler's formulas:

- $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$
- $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx$ for $n = 1, 2, 3, \dots$
- $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$ for $n = 1, 2, 3, \dots$

These integrals essentially find the "amount" of each cosine and sine wave component present in the function $f(x)$.

5. Fourier Series for a Function with Period 2L

If a function $f(x)$ is periodic with period $2L$, the Fourier series becomes:
$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$
And the Fourier coefficients are calculated as:

- $a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx$
- $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$ for $n = 1, 2, 3, \dots$
- $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$ for $n = 1, 2, 3, \dots$

This is a generalization for any arbitrary period $2L$. You can always scale a function with period T to have a period of 2π by a change of variables.

6. Even and Odd Functions and Fourier Series

The symmetry of a periodic function can simplify the calculation of Fourier coefficients:

- **Even Function:** If $f(-x) = f(x)$, then $b_n = 0$ for all $n \ge 1$. The Fourier series will only contain cosine terms (and the constant term $a_0/2$).
- **Odd Function:** If $f(-x) = -f(x)$, then $a_n = 0$ for all $n \ge 0$. The Fourier series will only contain sine terms.

7. Convergence of Fourier Series

- The Fourier series of a periodic function does not always converge to the function at every point.
- The **Dirichlet conditions** provide sufficient (but not necessary) conditions for convergence:
    - $f(x)$ must be periodic.
    - $f(x)$ must be absolutely integrable over one period.
    - $f(x)$ must have a finite number of discontinuities within one period.
    - $f(x)$ must have a finite number of maxima and minima within one period.
- At points of continuity, the Fourier series converges to $f(x)$.
- At points of discontinuity, the Fourier series converges to the average of the left-hand and right-hand limits: $\frac{f(x^-) + f(x^+)}{2}$.
- **Gibbs Phenomenon:** Near jump discontinuities, the partial sums of the Fourier series exhibit overshoot and undershoot, which doesn't disappear even with more terms.

8. Applications of Fourier Series

Fourier series have a vast range of applications across various fields, including:

- **Signal Processing:** Analyzing and filtering signals (audio, radio waves, etc.) by decomposing them into their frequency components. MP3 compression, noise reduction, and equalization rely on Fourier analysis.
- **Electrical Engineering:** Analyzing AC circuits, power systems, and designing filters.
- **Vibration Analysis:** Studying the vibrations of mechanical structures.
- **Acoustics:** Sound synthesis and analysis.
- **Heat Transfer:** Solving the heat equation (Fourier's original motivation).

- **Image Processing:** Image compression and analysis.
- **Quantum Mechanics:** Analyzing wave functions.
- **Telecommunications:** Modulation and demodulation of signals.

In Summary

The Fourier Series is a fundamental concept that allows us to represent periodic functions as a sum of sines and cosines. By calculating the Fourier coefficients, we can determine the amplitude of each frequency component in the function. This decomposition is incredibly useful for analyzing and manipulating periodic signals in a wide array of scientific and engineering applications. Understanding the basic formulas, the role of symmetry, and the concept of convergence are crucial for your first year of undergraduate studies.