A decorative fence

Author(s) of the problem: **Michal Forišek**
Contest-related materials by: **Richard Kráľovič, Jana Gajdošíková, Marián Dvorský**

This problem could be solved using dynamic programming. Let $T^{up}_{N,i}$ be the number of permutations that describe some cute fence of length N (let's call them *fence permutations*) starting with the element i and having the second element greater than the first one. Analogically, $T^{down}_{N,i}$ will be the number of fence permutations of length N starting with the element i and having the second element less than i. If the second element of a permutation is greater than the first one, we will say that the permutation is *going up*, otherwise it is *going down*.

We can compute the values T for all N from 1 to 20, for all appropriate i and for both *up* and *down* in advance from the following facts:

- $T^{down}_{N,1} = 0$
- $T^{down}_{N,j+1} = \sum_{k=1}^{j} T^{up}_{N-1,k} = T^{down}_{N,j} + T^{up}_{N-1,j}$
- $T^{up}_{N,i} = T^{down}_{N,N+1-i}$

We get the second equation as follows: Take all the fence permutations of length N, starting with $j+1$ and going down. The second element k of this permutation is a number between 1 and j. If we take our permutation without its first element $(j+1)$ and decrease all elements greater than j by 1, we get some fence permutation of length $N-1$, starting with k and going up. It is easy to verify, that this is a bijection between the set of fence permutations of length N, starting with $j+1$ and going down, and the set of fence permutations of length $N-1$, starting with an element less than $j+1$ and going up. Thus the two sets have the same cardinality.

Values of T could be also hardwired as constants into the solution, but it does not make much sense for $N \leq 20$ as computing them doesn't take much time and we do it only once. These values will help us find the fence permutation with the catalogue number C.

There are $T^{up}_{N,i}+T^{down}_{N,i}$ fence permutations starting with i. Now we can easily determine the first element a_1 in the C-th fence permutation. Fence permutations starting with 1 have the catalogue numbers from 1 to $T^{up}_{N,1}+T^{down}_{N,1}$, those starting with 2 have the catalogue numbers from $T^{up}_{N,1} + T^{down}_{N,1} + 1$ to $T^{up}_{N,1} + T^{down}_{N,1} + T^{up}_{N,2} + T^{down}_{N,2}$, and so on. And so a_1 is the smallest number, for which $\sum_{i=1}^{a_1} \left(T^{down}_{N,i} + T^{up}_{N,i}\right) \geq C$.

Next, we will find the second element a_2. Suppose we decrease all elements of the permutation we seek, that are greater than a_1, by 1. Then the rest of the permutation will be a fence permutation of length $N-1$. We know that if its first element is less than a_1, it has to go up, otherwise it has to go down. We also know that the fence permutation we seek is the $C_1 = \left(C - \sum_{i=1}^{a_1-1} \left(T^{down}_{N,i} + T^{up}_{N,i}\right)\right)$-th of all such fence permutations.

For x from 1 to $a_1 - 1$ there are $T^{up}_{N-1,x}$ such permutations, starting with x, for greater x there are $T^{down}_{N-1,x}$ such permutations. So we find the first element y of this permutation in the same way as above. If $y < a_1$ then $a_2 = y$ else $a_2 = y + 1$.

Now we already know whether the permutation we seek goes up or down and we know its first two elements. There probably are more such permutations and we know the catalogue number C_2 of our permutation between all possible ones. From now on determining the next element will be done in the same way.

Suppose we know the first k ($k \geq 2$) elements of our permutation and its catalogue number C_k between all such permutations. How to determine the $(k+1)$-th element?

We will view the rest of the permutation as a permutation of a smaller length (with appropriately decreased elements). Last two known elements determine whether this permutation is going up or down. We then find the first element of the modified permutation in a similar way than above (we count 'good' permutations starting with x until we reach C_k) and then appropriately increase it to get the next element of our permutation. Finally we compute the new value C_{k+1} by subtracting the count of permutations starting with a lower element from C_k.

In our implementation, the program always works with permutations of numbers $0..m-1$ for some m, instead of permutations of some m numbers from range $1..N$. It is also necessary to 'decode' these permutations. Number i will be decoded to $i+1$-th smallest number from $1..N$, that was not used before.

Both determining the elements of the permutation and decoding the permutation takes $O(N^2)$ time. Precomputing the values of T takes the same time. Memory needed for the values of T is also $O(N^2)$.