# Домашнее задание к уроку № 9

① Найти $\frac{dz}{dt}$, если $z=z(x,y)$, $x=x(t)$, $y=y(t)$:

1) $z=x^2+y^2+xy$, $x=a\cdot\sin t$, $y=a\cos t$

$$\frac{dz}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$\frac{\partial z}{\partial x}=2x+y$; $\frac{\partial z}{\partial y}=2y+x$; $\frac{\partial x}{\partial t}=a\cdot\cos t$; $\frac{\partial y}{\partial t}=-a\sin t$

$$\frac{dz}{dt}=(2x+y)\cdot a\cdot\cos t-(2y+x)\cdot a\cdot\sin t$$

2) $z=x^2y^3u$, $x=t$, $y=t^2$, $u=\sin t$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t}$$

$\frac{\partial z}{\partial x}=2xy^3u$; $\frac{\partial z}{\partial y}=3y^2x^2u$; $\frac{\partial z}{\partial u}=x^2y^3$; $\frac{\partial x}{\partial t}=1$; $\frac{\partial y}{\partial t}=2t$; $\frac{\partial u}{\partial t}=\cos t$

$$\frac{dz}{dt}=2xy^3u+3y^2x^2u\cdot 2t+x^2y^3\cdot\cos t$$

② Если даны $z=f(x;y)$, $x=x(u;v)$, $y=y(u;v)$ найти $\frac{\partial z}{\partial u}$; $\frac{\partial z}{\partial v}$ и $dz$:

1) $z=x^3+y^3$, $x=uv$, $y=\frac{u}{v}$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$\frac{\partial z}{\partial x}=3x^2$; $\frac{\partial x}{\partial u}=v$; $\frac{\partial z}{\partial y}=3y^2$; $\frac{\partial y}{\partial u}=\frac{1}{v}$

$$\frac{\partial z}{\partial u}=3x^2\cdot v+3y^2\cdot\frac{1}{v}$$

• $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial v}$

$\dfrac{\partial x}{\partial v} = u \quad ; \quad \dfrac{\partial y}{\partial v} = -\dfrac{3}{v^2}$

$\dfrac{\partial z}{\partial v} = 3x^2 \cdot u + 3y^2 \cdot \left(-\dfrac{3}{v^2}\right)$

$z'_u = 3x^2 \cdot v + \dfrac{3y^2}{v} \quad ; \quad z'_v = 3x^2 u - \dfrac{9y^2}{v^2}$

подставим $u$ и $v$

$z'_u = 3(u \cdot v)^2 \cdot v + \dfrac{3}{v}\left(\dfrac{u}{v}\right)^2$

$z'_v = 3(u \cdot v)^2 \cdot u - \dfrac{9}{v^2} \cdot \left(\dfrac{u}{v}\right)^2$

• $dz = \dfrac{\partial z}{\partial x} \cdot dx + \dfrac{\partial z}{\partial y} \cdot dy$

$dx = \dfrac{\partial x}{\partial u} \cdot du + \dfrac{\partial x}{\partial v} \cdot dv = v \cdot du + u \cdot dv$

$dy = \dfrac{\partial y}{\partial u} du + \dfrac{\partial y}{\partial v} \cdot dv = \dfrac{1}{v} \cdot du + \left(-\dfrac{3}{v^2}\right) \cdot dv$

$dz = 3x^2 \cdot (v \cdot du + u \cdot dv) + 3y^2 \cdot \left(\dfrac{1}{v} du + \left(-\dfrac{3}{v^2}\right) \cdot dv\right) =$

$= 3x^2 \cdot v \cdot du + 3x^2 \cdot u \cdot dv + \dfrac{3y^2}{v} du - \dfrac{9y^2}{v^2} dv =$

$= \left(3x^2 v + \dfrac{3y^2}{v}\right) du + \left(3x^2 \cdot u - \dfrac{9y^2}{v^2}\right) dv$

д) $z = \cos xy$, $x = ue^v$, $y = v \ln u$

• $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial u}$

$\dfrac{\partial z}{\partial x} = -y \sin x \; ; \; \dfrac{\partial x}{\partial u} = e^v \; ; \; \dfrac{\partial z}{\partial y} = -x \sin y \; ; \; \dfrac{\partial y}{\partial u} = \dfrac{v}{u}$

$\dfrac{\partial z}{\partial u} = -y \sin x \cdot e^v + (-x \sin y) \cdot \dfrac{v}{u}$

• $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$

$\frac{\partial x}{\partial v} = u e^{v}$ ; $\frac{\partial y}{\partial v} = \ln u$

$\frac{\partial z}{\partial v} = -y \cdot \sin x \cdot u e^{v} + (-x \cdot \sin y) \ln u$

подставляем $u$ и $v$

$z'_u = -v \cdot \ln u \cdot \sin(u e^{v}) \cdot e^{v} + (-u e^{v} \cdot \sin(v \ln u) \cdot \frac{v}{u}$

$z'_v = -(v \ln u) \cdot \sin(u e^{v}) \cdot u e^{v} + (-u e^{v} \cdot \sin(v \ln u) \cdot \ln u$

• $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$dx = \frac{\partial x}{\partial u} \cdot du + \frac{\partial x}{\partial v} \cdot dv = e^{v} \cdot du + u e^{v} \cdot dv$

$dy = \frac{\partial y}{\partial u} \cdot du + \frac{\partial y}{\partial v} \cdot dv = \frac{v}{u} \cdot du + \ln u \cdot dv$

$dz = -y \sin x \cdot (e^{v} \cdot du + u e^{v} \cdot dv) + u e^{v} \cdot (\frac{v}{u} \cdot du + \ln u\, dv) =$

$= -y \sin x \; e^{v} du - y \sin x \cdot u e^{v} \cdot dv + \frac{\cancel{u} e^{v} \cdot v}{\cancel{u}} du +$

$+ u e^{v} \ln u\, dv =$

$= (e^{v} \cdot v - y \cdot \sin x \cdot e^{v}) du + (u e^{v} \ln u - y \cdot \sin x \cdot u e^{v}) dv =$

$= e^{v} (v - y \cdot \sin x) du + u e^{v} (\ln u - y \sin x) dv$

③ Найти производные $y'(x)$ неявных функций, заданных уравнениями:

$x e^{2y} - y \ln x = 8$

$F(x,y) = x e^{2y} - y \ln x - 8$

$F'_y = 2x e^{2y} - \ln x$ , если $2x e^{2y} - \ln x \neq 0$ , то

$F'_x = e^{2y} - \frac{y}{x}$

$y'(x) = -\frac{e^{2y} - \frac{y}{x}}{2x e^{2y} - \ln x}$

④ Составить уравнение касательной прямой и нормали к кривой $y=y(x)$, заданной уравнением $F(x,y)=0$ в точке $M_0(x_0,y_0)$:

$x^3y-y^3x=6$ , $M_0(2;1)$

$F(x,y)=x^3y-y^3x-6$

$F(2;1)=8\cdot1-1\cdot2-6=0$ – точка лежит на функции

$F'_y=x^3-3y^2x$ $\quad F'_y(2,1)=8-3\cdot1\cdot2=2$

$F'_x=3x^2y-y^3$ $\quad F'_x(2,1)=3\cdot4\cdot1-1=11$

Уравнение касательной:

$y-y_0=k(x-x_0)$ $\quad y'=\frac{F'_x}{F'_y}$

$k=y'(x_0)=-\frac{11}{2}$

$(t)$: $y-1=-\frac{11}{2}(x-2)$

Уравнение нормали:

$y-y_0=-\frac{1}{k}(x-x_0)$

$(n)$: $y-1=\frac{2}{11}(x-2)$

⑤ Для данных функций найти требуемую частную производную или дифференциал:

1) $z=\sin x\sin y$ , $d^2z$ – ?

• $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\sin y\cdot\cos x\,dx+\sin x\cos y\,dy$

• $\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(\sin y\cos x)=-\sin y\cdot\cos x$

$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(\sin y\cos x)=\cos x\cos y$

$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(\sin x\cdot\cos y)=-\sin x\sin y$

$\cdot\ d^2z = -\sin y \cos x\, dx^2 + 2\cos x \cos y - \sin x \sin y$

2) $z = xy + \sin(x+y)$, $\frac{\partial^2 z}{\partial x^2}$ - ?

$\cdot\ \frac{\partial z}{\partial x} = y + \cos(x+y)$

$\cdot\ \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(y + \cos(x+y)\right) = \left(\cos(x+y)\right)' \cdot (x+y)' = -\sin(x+y)$

3) $z = \operatorname{arctg} \frac{x+y}{1-xy}$, $\frac{\partial^2 z}{\partial x \partial y}$ - ?

$\cdot\ \frac{\partial^n z}{\partial x} = \left(\operatorname{arctg}\frac{x+y}{1-xy}\right)' = \left(\operatorname{arctg}\frac{x+y}{1-xy}\right)' \cdot \left(\frac{x+y}{1-xy}\right)' =$

$$= \frac{\frac{1}{1-yx} + \frac{y(x+y)}{(1-yx)^2}}{1 + \frac{(x+y)^2}{(1-yx)^2}} \cdot \left(\frac{x+y}{1\cdot xy}\right)' =$$

$$= \frac{\frac{1}{1-yx} + \frac{y(x+y)}{(1-yx)^2}}{1 + \frac{(x+y)^2}{(1-yx)^2}} \cdot \frac{(x+y)' \cdot (1-yx) - (x+y) \cdot (1-yx)'}{(1-yx)^2} =$$

$$= \frac{\frac{1}{1-yx} + \frac{y(x+y)}{(1-yx)^2}}{1 + \frac{(x+y)^2}{(1-yx)^2}} \cdot \cancel{\frac{(1-yx) \cdot (x+y) \cdot (-y)}{(1-yx)^2}} =$$

$$= \frac{\frac{1}{1-yx} + \frac{y(x+y)}{(1-yx)^2}}{1 + \frac{(x+y)^2}{(1-yx)^2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\frac{1}{1-yx} + y\cdot\frac{x+y}{(1-yx)^2}}{1+\frac{(x+y)^2}{(1-yx)^2}}\right)' =$$

$$= \frac{\left(\frac{1}{1-yx} + y\cdot\frac{x+y}{(1-yx)^2}\right)'\cdot\left(1+\frac{(y+x)^2}{(1-yx)^2}\right) - \left(\frac{1}{1-yx} + y\frac{x+y}{(1-yx)^2}\right)\cdot\left(1+\frac{(y+x)^2}{(1-yx)^2}\right)'}{\left(1+\frac{(y+x)^2}{(1-yx)^2}\right)^2} =$$

$$= \frac{\left(\frac{y}{(1-yx)^2} + \frac{y+x}{(1-yx)^2} + \frac{x}{(1-xy)^2}\right)\cdot\left(1+\frac{(y+x)^2}{(1-yx)^2}\right) - \left(\frac{1}{1-yx} + y\cdot\frac{x+y}{(1-yx)^2}\right)\cdot\frac{2y+2x}{(1-yx)^2}}{\left(1+\frac{(y+x)^2}{(1-yx)^2}\right)^2} =$$

$$= \frac{\frac{\frac{yx}{(1-yx)^2} + \frac{y+x}{(1-yx)^2} + \frac{x}{(1-xy)^2}}{1+\frac{(y+x)^2}{(1-yx)^2}} - (2y+2x)\cdot\frac{\frac{1}{1-yx} + y\cdot\frac{x+y}{(1-yx)^2}}{(1-yx)^2\cdot\left(1+\frac{(y+x)^2}{(1-yx)^2}\right)^2}}{}$$

$$\left(\frac{1}{1-yx} + y\cdot\frac{x+y}{(1-yx)^2}\right)' = \left(\frac{1}{1-xy}\right)' + \left(y\cdot\frac{y+x}{(1-yx)^2}\right)' = \frac{x}{(1-xy)^2} + \frac{y}{(1-yx)^2} + \frac{y+x}{(1-yx)^2} =$$

$$= \frac{y}{(1-yx)^2} + \frac{y+x}{(1-yx)^2} + \frac{x}{(1-xy)^2}$$

$$\left(\frac{1}{1-xy}\right)' = \frac{x}{(1-xy)^2} \qquad \left(\frac{y\cdot(y+x)}{1-yx}\right)' = \frac{y}{(1-yx)^2} + \frac{y+x}{(1-yx)^2}$$

$$\left(1+\frac{y+x^2}{(1-yx)^2}\right)' = \frac{2y+2x}{(1-yx)^2} \qquad \left(\frac{(y+x)^2}{(1-yx)^2}\right)' = \frac{2y+2x}{(1-yx)^2}$$

⑥ Найти $y'$, $y''$ и $y'''$ для неявной функции $y=y(x)$, заданной неявно уравнением $x^2-xy+2y^2+x-y=1$ при $x=0$, если $y(0)=1$

$x^2-xy+2y^2+x-y-1=0$

$y'=2x-y-xy'+4y\cdot y'+1-y'=0$

$y''=2-y''-y'-xy''+4\cdot(y')^2-y''=0$

$y'''=-y'''-(y')^2-y''-xy'''+4(y')^2-y'''=-2y'''-xy'''-y''+3(y')^2$

$x=0$ ; $y=1$

$y'(0)=-1-0+4\cdot0+1-0=0$

$y''(0)=2-0-0-0+4\cdot0-0=0$

$y'''(0)=0-0-0-0+0-0=0$

⑦ Для функции $z=\operatorname{arctg}\frac{y}{x}$ построить линии уровня и градиент. Сравнить их направление в точках $(1;1)$, $(1;-1)$

$z=\operatorname{arctg}\frac{y}{x}=C$ $\qquad -\frac{\pi}{2}<\operatorname{arctg}\frac{y}{x}\leq\frac{\pi}{2}$

$z'_x=-\dfrac{y}{x^2\cdot\left(1+\frac{y^2}{x^2}\right)}$ $\qquad z'_x(1;1)=-\frac{1}{2}$ $\qquad z'_x(1;-1)=\frac{1}{2}$

$z'_y(1;1)=\frac{1}{2}$ $\qquad z'_y(1;-1)=\frac{1}{2}$

$z'_y=\dfrac{1}{x\left(\frac{y^2}{x^2}+1\right)}$

$\overrightarrow{grad}\ z(1;1)=\left(-\frac{1}{2};\frac{1}{2}\right)=-\frac{1}{2}\vec{i}+\frac{1}{2}\vec{j}$

$\overrightarrow{grad}\ z(1;-1)=\left(\frac{1}{2};\frac{1}{2}\right)=\frac{1}{2}\vec{i}+\frac{1}{2}\vec{j}$