# SSY345 – Sensor Fusion and Non-Linear Filtering
# Home Assignment 1 - Analysis

## Basic information

This home assignment is related to the material in lecture 1 and 2. A large part of the assignment focuses on understanding of the basic concepts that we will rely on later in the course.

In the analysis part we want you to use the toolbox that you have developed and apply it to a practical scenario. Associated with each scenario is a set of tasks that we would like you to perform.

The result of the tasks should in general be visualised and compiled together in a report (pdf-file). A template for the report can also be found on course homepage. Note that, it is sufficient to write short concise answers to the questions but they should be clearly motivate by what can be seen in the figures. Only properly referenced or captioned figures such that it is understandable what will result in POE. Also, all the technical terms and central concepts to explain an answer should be used without altering their actual meaning. The report should be uploaded on the course homepage before the deadline.

# 1 Properties of random variables

*The purpose of this question is to recall some of the important properties of random variables, specifically expected value and (co)variance.*

a) Let $x$ be a scalar Gaussian random variable $x \sim \mathcal{N}(\mu, \sigma^2)$. Using the definition of expected value (integral form), show that:

   i) $\mathbb{E}[x] = \mu$.

   ii) $\text{Var}[x] = \mathbb{E}\left[(x-\mu)^2\right] = \sigma^2$.

   *Hint: the expectation integral can be solved by, e.g., variable substitution using* $t = \frac{x-\mu}{2\sigma}$.
   *Hint-2:* $\int_{-\infty}^{\infty} e^{-tx^2} dx = \frac{\sqrt{\pi}}{\sqrt{t}}$

b) Let $\mathbf{q}$ be a multi-variate random variable with known probability density function $p(\mathbf{q})$. Further, let $\mathbf{z} = \mathbf{A}\mathbf{q}$ where $\mathbf{A}$ is a constant matrix.

   i) Using the integral form of expectation, show that $\mathbb{E}[\mathbf{z}] = \mathbf{A}\mathbb{E}[\mathbf{q}]$.

   ii) Using the result in i), show that $\text{Cov}[\mathbf{z}] = \mathbf{A}\text{Cov}[\mathbf{q}]\mathbf{A}^T$, where $\text{Cov}[\cdot]$ is the covariance operator.

c) Now, assume that $\mathbf{z} = \mathbf{A}\mathbf{q}$ where

$$p(\mathbf{q}) = \mathcal{N}\left(\mathbf{q}; \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 0.3 & 0 \\ 0 & 8 \end{bmatrix}\right)$$

and

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}.$$

Use the result from 1b) and your developed Matlab-function `sigmaEllipse2D()` to illustrate the mean and covariance of $\mathbf{z}$ and $\mathbf{q}$ in the same figure. What can you say about the way $\mathbf{A}$ effects the mean and covariance? What happens to the correlation/covariance of the individual components in $\mathbf{q}$ after the transformation? How can this be traced back to the structure of $\mathbf{A}$?

## 2 Transformation of random variables

*In this course we will consider statistical models that can be viewed as transformations of random variables. In this assignment you will investigate what happens to the properties of random variables when they are transformed through such models (functions).*

a) Let $x$ be a scalar Gaussian random variable $x \sim N(0,2)$ and that $z = 3x$. Determine $p(z)$ by assuming that it is Gaussian and calculating $\mathbb{E}[z]$ and Var[z] analytically. Compare your result to the numerical Gaussian approximation given by `approxGaussianTransform` by jointly plotting the histogram of the transformed samples, together with Gaussian pdfs having the calculated and approximated mean and covariance, respectively. Does the histogram and two pdfs match well? From what you see in the figure, what conclusions can you draw regarding the properties of $p(z)$ and the different approximations? How does the number of samples used in the approximation affect the result?

b) Now, assume that $z = x^3$. If possible, repeat the task in 2a) for this non-linear transformed random variable. It is fine to skip sub-tasks as long as you motivate why it is reasonable to do so.

c) Compare your results in 2a) and 2b). What can you conclude regarding the properties of $p(z)$ for the different transformations?

Hint: A Gaussian pdf is completely determined by it mean and variance and can be illustrated using `normpdf()` in Matlab (check whenever it accepts variance or standard deviation carefully!). To illustrate histograms, we recommend that you use the `histogram()`-function in MATLAB where the 'Normalization' property is set to 'pdf'.

# 3 Understanding the conditional density

*In this problem you will build some confidence on conditional densities.*

Let $x$ be a random variable and $y = h(x) + r$ where $h(x)$ is a deterministic, known, possibly non-linear function of $x$ and $r \sim \mathcal{N}(0, \sigma_r^2)$. (Deterministic function is a function that gives the same output for the same input in every trial. When a function is known/given, it means we can evaluate it but we avoid giving a specific form to show that our analysis works for any deterministic function.)

**Tasks:**

a) With the information given, is it possible to describe $p(y)$? If yes, what is the distribution of $y$? If no, motivate why it cannot be determined.

b) Is it possible describe $p(y|x)$? If yes, what is the distribution of $y$ given $x$? If no, motivate why it cannot be determined.

c) Now, assume that $h(x) = Hx$, where $H$ is a deterministic and known constant. Is it possible to describe $p(y)$ or $p(y|x)$? Motivate and explain your answers clearly.

d) Repeat the previous tasks for $x \sim N(\mu_x, \sigma_x^2)$. Motivate and explain your answers clearly.

e) Prepare a simulation to illustrate and verify your answers.
(In this question, we intentionally let you to decide all the parameters so you will have a chance to practice the ability to verify a theoretic analysis using a practical tool like Matlab. Note that sampling data and analysing it does NOT prove anything but it gives a great insight on how to approach many problems.)

Hint: For simulating task a) and task b), you have to choose a distribution for $x$ different than Gaussian. A potential choice would be uniform distribution. You may use the rand() function to sample from uniform distribution. As a quick reminder, don't forget to do the simulations for both $h(x)$ being a linear and non-linear function of $x$. Also, histogram() and normpdf() and unifpdf() might be useful in this question.

## 4 MMSE and MAP estimators

*The purpose of this question is to understand the fundamental difference between MMSE and MAP estimators.*

**Task:** Let $\theta$ be a random variable with discrete prior probability mass function

$$\Pr\{\theta\} = \begin{cases} 0.5 & \text{if } \theta = -1 \\ 0.5 & \text{if } \theta = 1 \end{cases} \tag{1}$$

or equivalently expressed as probability density function

$$p(\theta) = 0.5(\delta(\theta - 1) + \delta(\theta + 1)). \tag{2}$$

You receive a single noisy observation of $\theta$ as

$$y = \theta + w \tag{3}$$

where $W \sim \mathcal{N}(0, \sigma^2)$.

a) For this task only assume $\sigma^2 = 0.5^2$. To have a better understanding of the question, sample $y$ (you should sample $\theta$ and $w$, and add them up) and draw the histogram of $y$. Which distribution does it look like? Motivate your answer.

b) For this task only, let $\sigma^2 = 0.5^2$ and you are given a sample of $y = 0.7$. You are now asked to make a guess on $\theta$. What would be your guess? Why?

c) We will now work on the things we will need for the MAP and MMSE estimator. First determine $p(y|\theta)$ and then show that

$$p(y) = 0.5\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(y-1)^2}{2\sigma^2}} + 0.5\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(y+1)^2}{2\sigma^2}} \tag{4}$$

(Hint: $p(y) = \int p(y|\theta)p(\theta)d\theta$ )

d) Next, evaluate the posterior $\Pr\{\theta|y\}$ by using, $p(y)$, $p(\theta)$, $p(y|\theta)$ and Bayesian rule.
Hint: You should evaluate it separately for $\theta = 1$ and $\theta = -1$. Note that $\theta$ cannot take any other value.

e) We are now ready to evaluate the MMSE and MAP estimators. Find the MMSE estimator using

$$\hat{\theta}_{MMSE} = \sum_{\theta} \theta \Pr\{\theta|y\}. \tag{5}$$

Hint: To simplify your final result, you may use the exponential form of $tanh()$.

f) Finally, find the MAP estimator using $\hat{\theta}_{MAP} = \arg\max_\theta p(\theta|y)$.
Hint: As in the case of MMSE estimator, you should remember that $\theta$ only takes two values. The MAP estimator, in this scenario, yields a very simple and intuitive expression.

g) Reflect on the difference between MMSE and MAP estimators in this case. Does your guess at 4b) coincides with MMSE or MAP estimator? How are the different estimators forming their decisions in this particular example? Be as concise as possible.