二分法

1.1 根据中值定理找根：

第一章 二分法

1 (a) $f(x) = x^3 - 9$ 是连续函数

$f(2) = -1 < 0$ ， $f(3) = 18 > 0$

∴ 根据中值定理， $[2, 3]$ 满足要求.

(b) $f(x) = 3x^3 + x^2 - x - 5$ 是连续函数.

$f(1) = -2 < 0$ ， $f(2) = 21 > 0$

∴ 根据中值定理， $[1, 2]$ 满足要求.

(c) $f(x) = \cos^2 x + 6 - x$ 是连续函数.

$f(6) = \cos^2 6 > 0$ 　 $f(7) = \cos^2 7 - 1 < 0$

∴ 根据中值定理， $[6, 7]$ 满足要求.

1.1 书写模板

(c) $f(x) = \cos^2 x + 6 - x$

第0步: $a_0 = 6 \quad f(a_0) < 0$

$b_0 = 7 \quad f(b_0) > 0$

$x_0 = \frac{a_0 + b_0}{2} = \frac{13}{2} \quad f(x_0) > 0$

第1步: $a_1 = x_0 = \frac{13}{2} \quad f(a_1) > 0$

$b_1 = b_0 = 7 \quad f(b_1) < 0$

$x_1 = \frac{a_1 + b_1}{2} = \frac{27}{4} \quad f(x_1) > 0$

第2步: $a_2 = x_1 = \frac{27}{4} \quad f(a_2) > 0$

$b_2 = b_1 = 7 \quad f(b_2) < 0$

$x_2 = \frac{a_2 + b_2}{2} = \frac{55}{8}$

5. (a) $f(x) = x^4 - x^3 - 10$ 是连续函数

$f(2) = 16 - 8 - 10 < 0$

$f(3) = 81 - 27 - 10 > 0$

∴ 根据中值定理, $[2, 3]$ 满足要求

(b) ∵ $|x_n - x^*| < \frac{b-a}{2^{n+1}} = \frac{1}{2^{n+1}} < 10^{-10}$

∴ $2^n > 5 \times 10^9$

$n > \log_2 5 \times 10^9$

∴ n 至少需为33, 即至少需迭代33次.

不动点迭代法

看作业本就好

牛顿法

第一章 牛顿法

2. (1) $f(x)=x^3+x^2-1$

$f'(x)=3x^2+2x$

$$g(x)=x-\frac{f(x)}{f'(x)}=x-\frac{x^3+x^2-1}{3x^2+2x}=\frac{2x^3+x^2+1}{3x^2+2x}$$

$x_0=1$

$x_1=g(x_0)=g(1)=\frac{4}{5}=0.8$

$$x_2=g(x_1)=g\left(\frac{4}{5}\right)=\frac{2\cdot\frac{64}{125}+\frac{16}{25}+1}{3\cdot\frac{16}{25}+\frac{8}{5}}=\frac{333}{440}\approx 0.7568$$

(2) $f(x)=x^2+\frac{1}{x+1}-3x$

$f'(x)=2x-\frac{1}{(x+1)^2}-3$

$$g(x)=x-\frac{f(x)}{f'(x)}=x-\frac{x^2+\frac{1}{x+1}-3x}{2x-\frac{1}{(x+1)^2}-3}$$

$x_0=1$

$$x_1=g(x_0)=g(1)=1-\frac{1+\frac{1}{2}-3}{2-\frac{1}{4}-3}=-\frac{1}{5}=-0.2$$

$$x_2=g(x_1)=g\left(-\frac{1}{5}\right)=-\frac{1}{5}-\frac{\frac{1}{25}+\frac{5}{4}+\frac{3}{5}}{-\frac{2}{5}-\frac{25}{16}-3}=\frac{359}{1985}\approx 0.1809$$

(3) $f(x)=5x-10$

$f'(x)=5$

$$g(x)=x-\frac{5x-10}{5}=2$$

$x_0=1$

$x_1=g(x_0)=g(1)=2$

$x_2=g(x_1)=g(2)=2$

第五章 - 牛顿法

3. (a) $f(x) = x^5 - 2x^4 + 2x^2 - x$

$f'(x) = 5x^4 - 8x^3 + 4x - 1$

$f''(x) = 20x^3 - 24x^2 + 4$

$f'''(x) = 60x^2 - 48x$

$r = -1$:

$f(-1) = 0$, $f'(-1) = 5 + 8 - 4 - 1 = 8 \neq 0$

∴ $r = 1$ 是1重根，牛顿法2次收敛.

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i^2} = \left| \frac{f''(-1)}{2f'(-1)} \right| = \left| \frac{-20 - 24 + 4}{2 \times 8} \right| = 2.5$$

$r = 0$:

$f(0) = 0$, $f'(0) = -1 \neq 0$

∴ $r = 0$ 是1重根，牛顿法2次收敛

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i^2} = \left| \frac{f''(0)}{2f'(0)} \right| = \left| \frac{4}{2 \times (-1)} \right| = 2$$

$r = 1$:

$f(1) = 0$, $f'(1) = 5 - 8 + 4 - 1 = 0$,

$f''(1) = 20 - 24 + 4 = 0$, $f'''(1) = 60 - 48 = 12 \neq 0$

∴ $r = 1$ 是3重根，牛顿法线性收敛.

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = \frac{m-1}{m} = \frac{2}{3}$$

朴素高斯消元法

(b) $\left[\begin{array}{ccc|c} 2 & 1 & -4 & -7 \\ 1 & -1 & 1 & -2 \\ -1 & 3 & -2 & 6 \end{array}\right] \xrightarrow{行2-\frac{1}{2}\times行1} \left[\begin{array}{ccc|c} 2 & 1 & -4 & -7 \\ 0 & -\frac{3}{2} & 3 & \frac{3}{2} \\ -1 & 3 & -2 & 6 \end{array}\right]$

$\xrightarrow{行3+\frac{1}{2}\times行1} \left[\begin{array}{ccc|c} 2 & 1 & -4 & -7 \\ 0 & -\frac{3}{2} & 3 & \frac{3}{2} \\ 0 & \frac{7}{2} & -4 & \frac{5}{2} \end{array}\right]$

$\xrightarrow{行3+\frac{7}{3}\times行2} \left[\begin{array}{ccc|c} 2 & 1 & -4 & -7 \\ 0 & -\frac{3}{2} & 3 & \frac{3}{2} \\ 0 & 0 & 3 & 6 \end{array}\right]$

$x_3 = 6/3 = 2$

$x_2 = (\frac{3}{2} - 3\cdot 2)/(-\frac{3}{2}) = 3$

$x_1 = ((-7) + 4\cdot 2 - 3)/2 = -1$

6. $\frac{0.0055}{5000^2} = \frac{x}{\frac{2}{3}\cdot 5000^3}$

$x \approx 17\,s$

LU 分解

第二章 LU分解

4. (b) $\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow{行2-1\times行1} \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

$\xrightarrow{行3-\frac{1}{2}\times行1} \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 2 \\ \frac{1}{2} & 1 & 3 \end{bmatrix}$

$\xrightarrow{行3-\frac{1}{2}\times行2} \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 2 \\ \frac{1}{2} & \frac{1}{2} & 2 \end{bmatrix}$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

$\therefore$ 解 $Lc = b$，即

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$，得：$c_1 = 2 \quad c_2 = 2 \quad c_3 = 4$

解 $Ux = c$，即

$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$，得：$x_3 = 2, x_2 = -1, x_1 = 1$

6. $\dfrac{60}{\frac{2}{3}\times 1000^3 + 500\times 2\times 1000^2} = \dfrac{x}{\frac{2}{3}\times 1000^3}$ 解得：$x = 24s$

误差来源

第2章 误差的来源

2(a)

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{行2-3\times行1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow{行2\times(-\frac{1}{2})} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{行1-2\times行2} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\|A\|_\infty = \max\{|1|+|2|, |3|+|4|\} = 7$$

$$\|A^{-1}\|_\infty = \max\{|-2|+|1|, |\tfrac{3}{2}|+|-\tfrac{1}{2}|\} = 3$$

$$\therefore cond(A) = \|A\|_\infty \|A^{-1}\|_\infty = 7 \times 3 = 21$$

(b)

$$\begin{bmatrix} 1 & 2.01 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{bmatrix} \xrightarrow{行2-3\times行1} \begin{bmatrix} 1 & 2.01 & 1 & 0 \\ 0 & -0.03 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow{行3\times(-\frac{100}{3})} \begin{bmatrix} 1 & 2.01 & 1 & 0 \\ 0 & 1 & 100 & -\frac{100}{3} \end{bmatrix}$$

$$\xrightarrow{行1+(-2.01)\times行2} \begin{bmatrix} 1 & 0 & -200 & \frac{201}{3} \\ 0 & 1 & 100 & -\frac{100}{3} \end{bmatrix}$$

PA=LU 分解

3(b). $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow{\text{行1,2交换}} \begin{bmatrix} 6 & 3 & 4 \\ 3 & 1 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

$\xrightarrow{\text{行2}-\frac{1}{2}\times\text{行1}} \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 3 & 1 & 5 \end{bmatrix}$

$\xrightarrow{\text{行3}-\frac{1}{2}\times\text{行1}} \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 3 \end{bmatrix}$

$\xrightarrow{\text{行3}-1\times\text{行2}} \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \overset{1}{\cancel{0}} & 3 \end{bmatrix}$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 6 & 3 & 4 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{行1,2交换}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P.$

$\therefore PAx = Pb, \quad LUx = Pb$

$\therefore \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ 解得: $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 3 \end{bmatrix}$

$\therefore \begin{bmatrix} 6 & 3 & 4 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ 解得: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

雅可比和高斯赛德尔迭代

(2) 通过行变换，将A变为严格对角占优阵 $A' = \begin{bmatrix} 3 & -1 & 1 \\ 1 & -8 & -2 \\ 1 & 1 & 5 \end{bmatrix}$

相应地，b变为 $b' = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

$\therefore D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 5 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ 雅可比法：$x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x^{(1)} = D^{-1}(b' - (L+U)x^{(0)}) = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{8} \\ \frac{4}{5} \end{bmatrix} \approx \begin{bmatrix} -0.6667 \\ -0.1250 \\ 0.8000 \end{bmatrix}$$

$$x^{(2)} = D^{-1}(b' - (L+U)x^{(1)}) = \begin{bmatrix} -39/40 \\ -49/120 \\ 23/24 \end{bmatrix} \approx \begin{bmatrix} -0.9750 \\ -0.4083 \\ 0.9583 \end{bmatrix}$$

高斯塞德尔法：$x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x^{(1)} = (D+L)^{-1}(b' - Ux^{(0)})$$

$$= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{24} & -\frac{1}{8} & 0 \\ -\frac{3}{40} & \frac{1}{40} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{5}{24} \\ \frac{39}{40} \end{bmatrix} \approx \begin{bmatrix} -0.6667 \\ -0.2083 \\ 0.9750 \end{bmatrix}$$

$$x^{(2)} = (D+L)^{-1}(b' - Ux^{(1)})$$

$$= \begin{bmatrix} -191/180 \\ -361/720 \\ 89/80 \end{bmatrix} \approx \begin{bmatrix} -1.0611 \\ -0.5014 \\ 1.1125 \end{bmatrix}$$

拉格朗日插值和牛顿差商

1(c) $l_1(x) = \dfrac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \dfrac{(x-2)(x-4)}{(0-2)(0-4)} = \dfrac{(x-2)(x-4)}{8}$

$l_2(x) = \dfrac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \dfrac{(x-0)(x-4)}{(2-0)(2-4)} = \dfrac{x(x-4)}{-4}$

$l_3(x) = \dfrac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \dfrac{(x-0)(x-2)}{(4-0)(4-2)} = \dfrac{x(x-2)}{8}$

$$p(x) = \sum_{i=1}^{3} y_i l_i(x)$$

$$= -2\,l_1(x) + l_2(x) + 4\,l_3(x)$$

$$= \frac{3}{2}x - 2$$

2(a)

$$\begin{array}{c|cccc}
0 & 1 & & \\
 & & 1 & \\
2 & 3 & & -\frac{4}{3} \\
 & & -3 & \\
3 & 0 & &
\end{array}$$

$\therefore\ p(x) = 1 + x + \left(-\frac{4}{3}\right)x(x-2) = -\frac{4}{3}x^2 + \frac{11}{3}x + 1$

(b)

$$\begin{array}{c|cccc}
-1 & 0 & & & \\
 & & \frac{1}{3} & & \\
2 & 1 & & -\frac{1}{12} & \\
 & & 0 & & \frac{1}{24} \\
3 & 1 & & \frac{1}{6} & \\
 & & \frac{1}{2} & & \\
5 & 2 & & &
\end{array}$$

$\therefore\ p(x) = \frac{1}{3}(x+1) - \frac{1}{12}(x+1)(x-2) + \frac{1}{24}(x+1)(x-2)(x-3)$

$$= \frac{1}{24}x^3 - \frac{1}{4}x^2 + \frac{11}{24}x + \frac{3}{4}$$

三次样条

第三章－三次样条

1. (a). $S_1(x) = x^3 + x - 1$

$S_1'(x) = 3x^2 + 1$

$S_1''(x) = 6x$

$S_2(x) = -(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$

$S_2'(x) = -3(x-1)^2 + 6(x-1) + 3$

$S_2''(x) = -6(x-1) + 6$

检验三次样条的性质：

1° $S_1(1) = 1$ ， $S_2(1) = 1$

2° $S_1'(1) = 4$ ， $S_2'(1) = 3$

3° $S_1''(1) = 6$ ， $S_2''(1) = 6$

性质2°不符合，$S(x)$ 不是三次样条函数.

(b) $S_1(x) = 2x^3 + x^2 + 4x + 5$

$S_1'(x) = 6x^2 + 2x + 4$

$S_1''(x) = 12x + 2$

$S_2(x) = (x-1)^3 + 7(x-1)^2 + 12(x-1) + 12$

$S_2'(x) = 3(x-1)^2 + 14(x-1) + 12$

$S_2''(x) = 6(x-1) + 14$

检验三次样条的性质：

1° $S_1(1) = 12$， $S_2(1) = 12$

2° $S_1'(1) = 12$， $S_2'(1) = 12$

3° $S_1''(1) = 14$， $S_2''(1) = 14$

三条性质均符合，$S(x)$ 是三次样条函数.

第三章—三次样条

7. (a) $\delta_1 = 1-0 = 1$，$\delta_2 = 2-1 = 1$

$\Delta_1 = 1-0 = 1$，$\Delta_2 = 4-1 = 3$

自然三次样条对应的方程组：

$$\begin{bmatrix} 1 & 0 & 0 \\ \delta_1 & 2\delta_1+2\delta_2 & \delta_2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3\left(\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right) \\ 0 \end{bmatrix}$$

$\therefore c_1 = 0$，$c_3 = 0$

$$c_2 = 3\left(\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right) \Big/ 2\delta_1 + 2\delta_2 = \frac{3}{2}$$

$$\therefore d_1 = \frac{c_2 - c_1}{3\delta_1} = \frac{1}{2}$$

$$d_2 = \frac{c_3 - c_2}{3\delta_2} = -\frac{1}{2}$$

$$b_1 = \frac{\Delta_1}{\delta_1} - \frac{\delta_1}{3}(2c_1 + c_2) = \frac{1}{2}$$

$$b_2 = \frac{\Delta_2}{\delta_2} - \frac{\delta_2}{3}(2c_2 + c_3) = 2$$

$$\therefore S(x) = \begin{cases} S_1(x) = \frac{1}{2}x + \frac{1}{2}x^3 & x \in [0,1] \\ S_2(x) = 1 + 2(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{2}(x-1)^3 & x \in [1,2] \end{cases}$$

最小二乘法

第四章 - 最小二乘

8.(a) 设 $\hat{y} = c_1 + c_2 x$

$$\therefore \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 6 \end{bmatrix}$$

$\therefore$ 法线方程为 $\begin{bmatrix} 4 & 8 \\ 8 & 30 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 39 \end{bmatrix}$

$\therefore c_1 = \frac{6}{7} \quad c_2 = \frac{15}{14}$ 即最优直线为 $\hat{y} = \frac{6}{7} + \frac{15}{14} x$

$$\therefore \text{RMSE} = \sqrt{\frac{\sum_{i=1}^{4} (y_i - \hat{y}_i)^2}{4}} = \sqrt{\frac{(-\frac{6}{7})^2 + (\frac{15}{14})^2 + 0^2 + (-\frac{3}{14})^2}{4}}$$

$$= \frac{3\sqrt{42}}{28} \approx 0.6944$$

(b) 设 $\hat{y} = c_1 + c_2 x$

$$\therefore \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

$\therefore$ 法线方程为 $\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$

$\therefore c_1 = \frac{24}{5} \quad c_2 = -\frac{6}{5}$ 即最优直线为 $\hat{y} = \frac{24}{5} - \frac{6}{5} x$

$$\therefore \text{RMSE} = \sqrt{\frac{\sum_{i=1}^{4} (y_i - \hat{y}_i)^2}{4}} = \sqrt{\frac{(\frac{1}{5})^2 + (-\frac{3}{5})^2 + (\frac{3}{5})^2 + (\frac{1}{5})^2}{4}}$$

$$= \frac{\sqrt{5}}{5} \approx 0.4472.$$