📖 The Secret Life of Your Signal — How Time Meets Frequency

1️⃣ Meet the Signal

Imagine you've just taken a **snapshot** of a sound wave.
It's wiggly, not straight, and your oscilloscope shows it dancing up and down.
To your eyes, it's *just a curve*.
But hidden inside that curve is a **cast of characters** — pure tones — all playing together.

> Think of it like hearing a whole orchestra but wanting to know:
> *Who's playing? How loud? And where are they in the song?*

2️⃣ We Steal Some Samples

We **sample** this curve at regular intervals:

- Every Δt seconds, we take a reading of the amplitude.
- This gives us a sequence: `x[0], x[1], x[2]... x[N-1]`.

To a mathematician, this sequence is our **time-domain representation**.

3️⃣ The Frequency Detective — Fourier

Now, Fourier's idea was:

> "I can find the hidden notes if I compare your signal to a set of *perfectly tuned sine and cosine waves*."

Each of these perfect waves has:

- **A specific frequency** (think: bin number).
- **A specific phase** (where it starts in time).
- **A constant amplitude** (1 before scaling).

4️⃣ Enter the Twiddle Factor

The **twiddle factor** is our translator between time and frequency:

$$W_N^{kn} = e^{-j2\pi kn/N}$$

What it does:

- Spins around the **complex plane** exactly at the bin's frequency.
- Moves sample-by-sample (n steps in time).
- Aligns itself to see **how much of that bin's pure tone is hiding in the signal**.

5️⃣ How the Lock Happens

For **Bin k**:

1. The twiddle factor for that bin spins **in sync** with a wave of frequency $k/N \times f_s$.
2. We multiply each time sample by this spinning vector.
3. If the signal contains this frequency:
    - The spins *align* — adding produces a **big magnitude**.
4. If not:
    - The spins *cancel each other out* — sum is small.

💡 **This is the lock** — the bin's twiddle rotates in such a way that the frequency from time domain "stands still" in the bin's viewpoint, letting us sum it cleanly.

6️⃣ From Time to Frequency

- In **time domain**: we just have amplitudes at moments in time.
- After FFT: we have **magnitudes** and **phases** for each frequency bin.

Magnitude → how loud that note is in the signal.
Phase → where that note *starts* in the original time snapshot.

If phase = 0 → the wave starts at its peak at t=0.
If phase = π/2 → the wave starts a quarter-cycle later.

7️⃣ Why It's Beautiful

To the naked eye, a curve is just a curve.
FFT is like special glasses — suddenly you see:

- **Who** (which frequencies) are playing.
- **How much** each is contributing.
- **When** in the snapshot each started (phase).

And it does this just by **spinning vectors at the right speed** and summing them.