📘 A Layman's Read on How DFT Bins Work — Time–Frequency Intuition with Math

🎯 Objective

To build **unshakable intuition** about:

- How Discrete Fourier Transform (DFT) works
- How time and frequency domains are connected
- Why DFT formula is shaped the way it is
- How the **rotating vector/bin** concept helps us “detect” frequencies
- How **signal × cosine/sine** relates to **Fourier series** and **DFT**

This document will clarify how **bins are formed**, how **frequency is measured**, and how **sampled time-domain data** reveals its frequency content.

🧱 Part 1: What Is DFT Doing Conceptually?

The DFT takes **N samples** from a signal and asks:

> "Which sinusoids of different frequencies are present in this data, and with what strength (amplitude and phase)?"

It does this by **projecting** the signal onto a **set of rotating unit vectors** (complex sinusoids), each rotating at a different rate (i.e., different frequency). Each such unit vector corresponds to one **bin**.

⚙️ Part 2: How DFT Bin Frequencies Are Defined

You have:

- Sampling Rate = F_s samples/second
- N samples
- Frequency resolution (bin width):

$$\Delta f = \frac{F_s}{N}$$

Then DFT gives you **N frequency bins**:

$$f_k = k \cdot \frac{F_s}{N}, \quad k = 0, 1, ..., N-1$$

Each bin checks whether the input signal contains a **frequency component at** f_k.

🧠 Part 3: The DFT Formula (Time–Frequency Bridge)

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j2\pi \frac{kn}{N}}$$

Where:

- $x[n]$: your time-domain sampled signal
- $X[k]$: the frequency content at the k-th bin
- $e^{-j2\pi \frac{kn}{N}}$: complex sinusoid rotating at bin frequency f_k
- n: time index (sample number)
- $\frac{kn}{N}$: determines **how much the vector has rotated** at sample n

This is essentially **correlating** the signal against a rotating reference sinusoid — i.e., "How much of this frequency exists in the signal?"

⏳ Part 4: Understanding Time in the DFT Formula

Each sample $x[n]$ was taken at time:

$$t_n = \frac{n}{F_s}$$

So at sample n, the rotation of a vector with frequency f_k is:

$$\theta_{n,k} = 2\pi f_k t_n = 2\pi \cdot \frac{kF_s}{N} \cdot \frac{n}{F_s} = 2\pi \cdot \frac{kn}{N}$$

This shows that the DFT **knows**:

- Time sample n
- Frequency bin k
- And rotates the unit vector accordingly

✅ **Time and frequency are both baked into the DFT formula.**

🔄 Part 5: What Does the Rotating Vector Do?

For each bin k, you imagine a vector rotating at:

$$\text{Frequency} = \frac{kF_s}{N}$$

For each time sample $x[n]$, we multiply it with this rotating vector's **complex conjugate**. Why?

Because:

- **If the signal matches the rotation rate**, these multiplications add up (constructive interference).
- **If the signal doesn’t match**, they cancel out (destructive interference).

This multiplication acts like a **filter** — it highlights only the matching frequency.

🔍 Part 6: Signal × Cosine and Sine Projections (Real View)

To understand this without complex numbers:

- Multiply the signal $x[n]$ with $\cos(2\pi \frac{kn}{N})$ and $\sin(2\pi \frac{kn}{N})$
- These are the real and imaginary parts of the rotating exponential.

This is exactly what **Fourier series** does:

- Decomposes a signal into **sums of sine and cosine terms**
- The DFT is just the sampled (digital) version of this

So:

$$X[k] \approx \sum x[n]\cos(2\pi \frac{kn}{N}) + j\sum x[n]\sin(2\pi \frac{kn}{N})$$

It checks **how much your signal resembles** each sine and cosine wave in the basis.

🧠 Part 7: Why It's Called a "Bin"

Because each $X[k]$ is a **container** (or bin) that holds **how much of frequency** f_k exists in the signal.

So when you look at the DFT result:

- You don't see time anymore
- You see frequency contributions
- Each bin = **1 frequency** = $f_k = k \cdot \frac{F_s}{N}$

✅ Summary: Core Ideas to Remember

Concept	Formula / Meaning
Frequency Resolution	$\frac{F_s}{N}$ — difference between adjacent bins
Frequency of bin k	$f_k = k \cdot \frac{F_s}{N}$
Time of sample n	$t_n = \frac{n}{F_s}$
DFT formula	$X[k] = \sum x[n]e^{-j2\pi kn/N}$
Signal similarity check	Multiply by cosine and sine components
What DFT is doing	Projecting signal onto sinusoids of various frequencies
Role of rotating vector	Acts as reference sinusoid for each bin

🔁 Final Analogy: DFT as a Set of Spinning Antennas

Imagine **N spinning antennas**, each tuned to a different frequency. Your signal hits all of them. Each antenna (bin) reports:

> “How much of **my frequency** do I detect in this signal?”

Some antennas pick up strong signals (peaks in DFT). Others pick up little to nothing.

That’s the **frequency-domain view** of your signal.