

From Time to Frequency — The Lock Mechanism

1. What We Start With

- We take a **snapshot** of a real-world signal for a fixed time window T seconds.
- We sample it into N equally spaced samples.
- To the naked eye, those samples are just **wiggly amplitudes** — no obvious “frequency tags” attached.

2. What We Want

- A list of **how much** of each possible frequency (magnitude) exists in that time window.
- The **phase** of each frequency — *when* in time each frequency’s wave starts, relative to the window.

3. The Problem

Your sampled data is just a *mixture* of waves.

Nothing in the sample array says:

“Hey, I’m the 50 Hz part” or “I’m the 100 Hz part.”

We need a way to **scan for** each possible frequency in turn, like tuning a radio.

4. Where the Twiddle Factor Comes In

The **twiddle factor** in the DFT/FFT is:

$$W_N^{nk} = e^{-j2\pi nk/N}$$

This is a **perfect, clean sinusoid** at a frequency corresponding to **bin k** .

It is our *search key* for that frequency.

5. How the Lock Works

For bin k :

1. Multiply your time samples $x[n]$ by the twiddle factor's sinusoid for that bin's frequency.
2. This is like sliding your signal under a **magnifying glass** shaped exactly like that frequency.
3. If your signal contains a wave **in sync** with this sinusoid, the multiplication produces a constant-phase, steady accumulation → **big magnitude**.
4. If your signal doesn't match, the multiplication produces alternating positive and negative contributions → they **cancel out** → **small magnitude**.

This is the “lock” — the twiddle factor is in **phase alignment** only with waves of its own frequency.

6. Where Time Connects to Frequency

- The **phase** we get after accumulation tells us **where in time** the first crest of that frequency appears *within our snapshot window*.
- In other words:
 - **Magnitude** → how strong that frequency is in the whole snapshot.
 - **Phase** → how far along its cycle that frequency was **at sample 0**.

7. Complex Plane as the Meeting Point

The FFT's complex numbers are a *meeting ground*:

- The **real axis**: cosine (in-phase component)
- The **imag axis**: sine (quadrature component)
- The vector's **angle** in this plane is the phase.
- This angle tells you:

“If I rebuild the sine wave for this frequency, I should start at this offset in time so it lines up with the original signal.”

8. Why This Is Beautiful

- A human eye sees only “time wiggles” — even an oscilloscope shows only amplitudes over time.
- Fourier’s trick is like handing every possible frequency a **secret key (twiddle factor)** and saying:

“If this key unlocks your signal with a strong magnitude, you’re in there.”

- The FFT does this for all bins simultaneously in a brilliantly optimized way — turning a hidden soup of time-domain samples into a **clear spectrum**.