📘 Chapter: Interpreting DFT Output and Understanding Fundamental Frequency

🎯 Chapter Objective

To clearly understand:

- What each DFT/FFT output value means
- How to calculate and interpret frequency bins
- What "fundamental frequency" is and why it matters
- How sampling rate and FFT size affect frequency resolution
- How to read FFT plots with confidence

✅ 1. What Does the DFT Actually Do?

The Discrete Fourier Transform (DFT) takes a time-domain signal and decomposes it into a set of **sinusoids** of different frequencies, amplitudes, and phases.
The DFT is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j2\pi kn/N}$$

Where:

- $x[n]$: input signal (time domain)
- $X[k]$: output spectrum (frequency domain)
- N: number of samples
- k: index of the DFT bin
  Each $X[k]$ tells you how much of frequency f_k is present in the original signal.

📏 2. Frequency of Each Bin — The Key Formula

To map a DFT bin k to its real-world frequency:

$$f_k = \frac{k}{N} \cdot f_s$$

Where:

- f_k: frequency represented by bin k
- f_s: sampling rate (Hz)
- N: number of samples (FFT size)

  This tells us that:
- $X[0]$ = DC (0 Hz)
- $X[1]$ = fundamental frequency
- $X[k]$ = contains the strength (amplitude & phase) of f_k

🔁 3. What Is the Fundamental Frequency?

The **fundamental frequency** is the smallest frequency that your DFT can detect — it's the spacing between each frequency bin:

$$f_{\text{fundamental}} = \frac{f_s}{N}$$

It sets the **frequency resolution** of your FFT.

🔢 Example:

If:

- $f_s = 800$ Hz
- $N = 8$

  Then:
- $f_{\text{fundamental}} = \frac{800}{8} = \boxed{100 \text{ Hz}}$
- Bin $k = 1$ represents 100 Hz
- Bin $k = 2 \rightarrow$ 200 Hz, and so on.

  This means: **You can't detect anything in between these bins!** If your signal has a frequency at 125 Hz, it won't appear clearly unless you increase N.

📊 4. Interpreting DFT Output: Magnitude and Phase

Each $X[k]$ is a **complex number**:

$$X[k] = \text{Re}(X[k]) + j \cdot \text{Im}(X[k])$$

From this you can derive:

- **Amplitude** (strength of that frequency):

$$|X[k]| = \sqrt{\text{Re}^2 + \text{Im}^2}$$

- **Phase** (angle/offset of that frequency):

$$\angle X[k] = \tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\right)$$

Use these to:

- Visualize the **spectrum**
- Reconstruct or modify specific frequency components

🧠 5. Symmetry of DFT for Real Signals

If your input $x[n]$ is real (not complex), then:

$$X[N-k] = \overline{X[k]} \quad \text{(complex conjugate)}$$

This means:

- You only need to look at the **first N/2+1 bins**
- The second half of the spectrum mirrors the first

  That's why NumPy's `np.fft.rfft()` (real FFT) gives only the meaningful part.

📈 6. How to Read FFT Plots

import numpy as np

import matplotlib.pyplot as plt

fs = 800

N = 8

t = np.arange(N) / fs

x = np.cos(2 * np.pi * 100 * t) # Cosine at 100 Hz

X = np.fft.fft(x)

freqs = np.fft.fftfreq(N, d=1/fs)

plt.stem(freqs, np.abs(X), use_line_collection=True)

plt.title("DFT Magnitude Spectrum")

plt.xlabel("Frequency (Hz)")

plt.ylabel("Amplitude")

plt.grid(True)

plt.show()

🔍 Interpreting the Plot

Bin k	**Frequency f_k**	**Meaning**
0	0 Hz (DC)	Average value (bias)
1	100 Hz	Main peak — cosine frequency
2	200 Hz	No content here
4	400 Hz (Nyquist)	Half the sampling rate
5–7	Mirror of bins 1–3	Symmetric, usually ignored for real signals
✏️ Quick Practice Example		
Given:		

- $f_s = 1000$ Hz
- $N = 500$

  Then:
- Fundamental frequency $f_{\text{fund}} = 1000/500 = 2$ Hz
- Bin $k = 100$ = $2 \times 100 = 200$ Hz
- You can only resolve multiples of 2 Hz

🎓 Key Takeaways

Concept	Key Point
DFT output $X[k]$	Complex value for frequency f_k
Frequency per bin	$f_k = \frac{k}{N} f_s$
Fundamental frequency	Smallest detectable frequency = f_s/N
FFT resolution	Increase N for finer resolution
Real signal DFT	Symmetric — use first $N/2+1$ bins
FFT plots	Show magnitude (amplitude) of each frequency

📘 What's Next?

In the next section, we'll:

- Apply these fundamentals to **real noisy signals**
- See how **FIR filters** change the spectrum
- Compare **before/after FFT plots**