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Mock Midterm Exam - Solutions

1 Subgradient Descent [20pts]

Derive the (sub)gradient descent update rule for a one-parameter linear model using the Mean Absolute Error,

$$\mathcal{L}_{\mathsf{MAE}}(\mathbf{X}, \mathbf{y}, w) = \frac{1}{N} \sum_{n=1}^{N} |wx_n - y_n|.$$

Hint: The function $f(x) = |ax|$ is a composition of two simpler function. Use the chain rule!

Solution: Our cost function is $\mathcal{L}(\mathbf{X}, \mathbf{y}, w) = \sum \mathcal{L}_n(x_n, y_n, w)$ where $\mathcal{L}_n(x_n, y_n, w) = |wx_n - y|$. We have that

$$\frac{\partial \mathcal{L}(\mathbf{X}, \mathbf{y}, w)}{\partial w} = \sum_{n=1}^{N} \frac{\partial \mathcal{L}_n(x_n, y_n, w)}{\partial w}.$$

Let a, e be two functions such that $a(x) = |x|$ and $e(x, y, w) = wx - y$. We can rewrite $\mathcal{L}_n$ as $a \circ e$. Let us find the derivative of $\mathcal{L}_n$ using the chain rule.

$$\begin{aligned} \frac{\partial \mathcal{L}_n(x_n, y_n, w)}{\partial w} &= \frac{\partial a(e(x_n, y_n, w))}{\partial w} \\ &= \frac{\partial a(e(x_n, y_n, w))}{\partial e} \frac{\partial e(x_n, y_n, w)}{\partial w} \end{aligned}$$

We have that $\frac{\partial e}{\partial w}(x_n, y_n, w) = x_n$, but $|x|$ is not differentiable at $x = 0$. We will have to use subgradients.

Remember that a vector $\mathbf{g}$ is a subgradient of the function f in $\mathbf{x}$ if $f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{g}^\top(\mathbf{y} - \mathbf{x})$, for all $\mathbf{y}$. Note that in our one dimensional case, for a subgradient of $|x|$ in $x = 0$, we need to find g such that

$$|y| \geq gy, \ \forall y.$$

Any $g : |g| \leq 1$ will do, but since we want the error to go to 0 and stay here, we will use $g = 0$. We therefore have that

$$\frac{\partial a(x)}{\partial x} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \text{ and } \frac{\partial e(x, y, w)}{\partial w} = x$$

Which gives us the following expression for the gradient

$$\frac{\partial \mathcal{L}_n(x_n, y_n, w)}{\partial w} = \frac{1}{N} \sum_{n=1}^{N} \begin{cases} x_n & \text{if } x_n > 0 \\ 0 & \text{if } x_n = 0 \\ -x_n & \text{if } x_n < 0 \end{cases}$$

Therefore, one step of gradient descent with step size γ is given by $w^{(i+1)} = w^{(i)} + \gamma \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$

2 Multiple-Output Regression [20pts]

Let $S = \{(\mathbf{y}_n, \mathbf{x}_n)\}_{n=1}^N$ be our training set for a regression problem with $\mathbf{x}_n \in \mathbb{R}^D$ as usual. But now $\mathbf{y}_n \in \mathbb{R}^K$, i.e., we have K outputs for each input. We want to fit a linear model for each of the K outputs, i.e., we now have K regressors $f_k(\cdot)$ of the form

$$f_k(\mathbf{x}) = \mathbf{x}^\top \mathbf{w}_k,$$

where $\mathbf{w}_k^\top = (\mathbf{w}_{k1}, \cdots, \mathbf{w}_{kD})$ is the weight vector corresponding to the k-th regressor. Let $\mathbf{W}$ be the $D \times K$ matrix whose columns are the vectors $\mathbf{w}_k$.

Our goal is to minimize the following cost function $\mathcal{L}$:

$$\mathcal{L}(\mathbf{W}) = \sum_{k=1}^K \sum_{n=1}^N \frac{1}{2\sigma_k^2}(y_{nk} - \mathbf{x}_n^\top \mathbf{w}_k)^2 + \frac{1}{2}\sum_{k=1}^K \|\mathbf{w}_k\|_2^2,$$

where the σ_k are known real-valued scalars. Let $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_K)$.

For the solution, let $\mathbf{X}$ be the $D \times N$ matrix whose columns are the feature vectors $\mathbf{x}_n$.

1. (4pts) Write down the normal equations for $\mathbf{W}^\star$, the minimizer of the cost function. I.e., what is the first-order condition that $\mathbf{W}^\star$ has to fulfill in order to minimize $\mathcal{L}(\mathbf{W})$.

   Solution: Note that the cost function $\mathcal{L}(\mathbf{W})$ is the sum of K cost functions, $\mathcal{L}(\mathbf{w}_k)$, each of which only depends on its own parameter $\mathbf{w}_k$. So if we compute the gradient with respect to $\mathbf{w}_k$ then this only involves the term $\mathcal{L}(\mathbf{w}_k)$ and we get

   $$\frac{1}{\sigma_k^2}\mathbf{X}(\mathbf{X}^\top \mathbf{w}_k - \mathbf{y}) + \mathbf{w}_k = 0.$$

   This is essentially the expression we had for ridge regression.

2. (8pts) Is the minimum $\mathbf{W}^\star$ unique? Assuming it is, write down an expression for this unique solution.

   Solution: Note that as long as we have the regularization terms the problem is strictly convex and so has a unique minimizer. We have the solution

   $$\mathbf{w}_k^\star = \left(\frac{1}{\sigma_k^2}\mathbf{X}\mathbf{X}^\top + \mathbf{I}_D\right)^{-1} \frac{1}{\sigma_k^2}\mathbf{X}\mathbf{y}.$$

3. (8pts) Write down a probabilistic model, so that the MAP solution for this model coincides with minimizing the above cost function. Note that this will involve specifying the the likelihoods as well as a suitable prior (which will give you the regression term).

   Solution: You are asked to derive a probabilistic model under which is the maximum a posteriori estimate. However, since "Posterior probability $\propto$ Likelihood $\times$ Prior probability", the question asks specifically for the prior and the likelihood only. Knowing that the maximization over a Gaussian is equivalent to minimizing the mean square error, one can check that $\mathbf{w}_{\mathsf{MAP}}^\star = \arg\max_{\mathbf{w}} p(\mathbf{y} \mid \mathbf{X}, \mathbf{w})p(\mathbf{w})$ is equivalent to the above cost minimization $\mathbf{w}_{normal}^\star = \arg\min_{\mathbf{w}} \mathcal{L}(\mathbf{w})$ if:

   $$p(\mathbf{y} \mid \mathbf{X}, \mathbf{w}) = \prod_{n=1}^N \prod_{k=1}^K \mathcal{N}(y_{nk} \mid \mathbf{w}_k^\top \mathbf{x}_n, \sigma_k^2)$$

   and

   $$p(\mathbf{w}) = \prod_{k=1}^K \mathcal{N}(\mathbf{w}_k \mid \mathbf{0}, \mathbf{I}_D)$$

3 Proportional Hazard Model [20pts]

Let $S = \{(y_n, \mathbf{x}_n)\}_{n=1}^N$ be our training set for a regression problem with $\mathbf{x}_n \in \mathbb{R}^D$ as usual. We assume that the output y_n is *ordered*, i.e., takes values in the set $\{1, 2, \ldots, K\}$ where we think of these numbers as *ordered* by the natural ordering. We wish to fit a linear model.

In the *proportional hazard* model we use the following probability distribution,

$$p(y_n = k \mid \mathbf{x}_n, \mathbf{w}, \boldsymbol{\Theta}) = \frac{e^{\eta_{nk}}}{\sum_{j=1}^K e^{\eta_{nj}}},$$

where $\eta_{nk} = \Theta_k + \mathbf{x}_n^\top \mathbf{w}$. The scalars Θ_k are assumed to be ordered, i.e., $\Theta_1 > \Theta_2 \cdots > \Theta_K$. Let $\boldsymbol{\Theta} = (\Theta_1, \cdots, \Theta_K)$.

1. (4pts) Show that $p(y_n \,|\, \mathbf{x}_n, \mathbf{w}, \boldsymbol{\Theta})$ (and therefore also $p(\mathbf{y} \mid \mathbf{X}, \mathbf{w}, \boldsymbol{\Theta})$) is a valid distribution.

   Hint: What are the *two* conditions that you need to verify?

   Solution: We need to verify that the expression is non-negative and sums up (as a function of k) to 1. The first property is trivially true (the exponential function is always non-negative for real-valued arguments). The second one is true by construction (see denominator).

2. (8pts) Derive the log-likelihood for this model.

   Solution: We proceed in our standard fashion. Let $\tilde{\mathbf{y}}_n$ be a vector that is equal to the all-zero vector of length K except that $\tilde{\mathbf{y}}_{nk} = 1$ if $\mathbf{y}_n = k$. Recall that all samples are assumed to be independent so that the joint distribution is equal to the product of the individual distributions. We get

$$\begin{aligned} \ln \prod_{n=1}^N \prod_{k=1}^K p(y_n = k \mid \mathbf{x}_n, \mathbf{w}, \boldsymbol{\Theta}) &= \ln \prod_{n=1}^N \prod_{k=1}^K \frac{e^{\tilde{\mathbf{y}}_{nk}\eta_{nk}}}{(\sum_{j=1}^K e^{\eta_{nj}})^{\tilde{\mathbf{y}}_{nk}}} \\ &= \sum_{n=1}^N \sum_{k=1}^K \tilde{\mathbf{y}}_{nk}\eta_{nk} - \sum_{n=1}^N \ln\Big[\sum_{j=1}^K e^{\eta_{nj}}\Big]. \end{aligned}$$

3. (8pts) Show that the negative of the log-likelihood is convex with respect to $\boldsymbol{\Theta}$ and $\mathbf{w}$.

   Solution: We know that the sum of convex functions is convex. Therefore it suffices to show that each of the N terms

$$\sum_{n=1}^N \sum_{k=1}^K (-\tilde{\mathbf{y}}_{nk}\eta_{nk}) + \sum_{n=1}^N \ln[\sum_{j=1}^K e^{\eta_{nj}}].$$

   is convex.

   The term

$$-\sum_{k=1}^K \tilde{\mathbf{y}}_{nk}\eta_{nk} = -\sum_{k=1}^K \tilde{\mathbf{y}}_{nk}(\Theta_k + \mathbf{x}_n^\top \mathbf{w}_k)$$

   is linear in the parameters and hence convex.

   The second term is the composition of a linear function with the function $\ln(e^{t_1} + \cdots + e^{t_K})$ which we can assume to be convex. Hence the composed function is convex as well.

4 Multiple Choice Questions and Simple Problems [40pts]

Mark the correct **answer(s)**. More than one answer can be correct!

Solution: Correct solutions are marked in bold face.

- In regression, "complex" models tend to
  1. (1 pt) **overfit**
  2. (1 pt) have large bias
  3. (1 pt) **have large variance**
- In regression, "simple" models tend to
  1. (1 pt) overfit
  2. (1 pt) **have large bias**
  3. (1 pt) have large variance
- We sometimes add a regularization term because
  1. (1 pt) **this sometimes renders the minimization problem of the cost function into a strictly convex/concave problem**
  2. (1 pt) **this tends to avoid overfitting**
  3. (1 pt) this converts a regression problem into a classification problem
- The k-nearest neighbor classifier
  1. (1 pt) typically works the better the larger the dimension of the feature space
  2. (1 pt) can classify up to k classes
  3. (1 pt) **typically works the worse the larger the dimension of the feature space**
  4. (1 pt) can only be applied if the data can be linearly separated
  5. (1 pt) **has a misclassification rate of at most two times the one of the Bayes classifier if we have lots of data**
  6. (1 pt) has a misclassification rate that is two times better than the one of the Bayes classifier
- A real-valued scalar Gaussian distribution
  1. (1 pt) is a member of the exponential family with one scalar parameter
  2. (1 pt) **is a member of the exponential family with two scalar parameters**
  3. (1 pt) is not a member of the exponential family
- Which of the following statements is correct, where we assume that all the stated minima and maxima are in fact taken on in the domain of relevance. **All correct!**
  1. (1 pt) $\max\{0, x\} = \max_{\alpha \in [0,1]} \alpha x$
  2. (1 pt) $\min\{0, x\} = \min_{\alpha \in [0,1]} \alpha x$
  3. (1 pt) Let $g(x) := \min_y f(x, y)$. Then $g(x) \leq f(x, y)$
  4. (1 pt) $\max_x g(x) \leq \max_x f(x, y)$
  5. (1 pt) $\max_x \min_y f(x, y) \leq \min_y \max_x f(x, y)$
- Which of the following statements are correct?
  1. **(1 pt)** The training error is typically smaller than the test error.
  2. **(1 pt)** The original SVM formulation can be optimized using SGD.
  3. (1 pt) One iteration of SGD for ridge regression costs roughly $\Theta(ND)$.
  4. **(1 pt)** The original logistic regression formulation can be optimized using SGD.
  5. (1 pt) We discussed coordinate descent to optimize the original SVM formulation.

- The following functions are convex:
  1. **(1 pt)** $f(x) = x^2$, $x \in \mathbb{R}$
  2. (1 pt) $f(x) = x^3$, $x \in [-1, 1]$
  3. **(1 pt)** $f(x) = -x^3$, $x \in [-1, 0]$
  4. **(1 pt)** $f(x) = e^{-x}$, $x \in \mathbb{R}$
  5. (1 pt) $f(x) = e^{-x^2/2}$, $x \in \mathbb{R}$
  6. **(1 pt)** $f(x) = \ln(1/x)$, $x \in [0, \infty)$
  7. **(1 pt)** $f(x) = g(h(x))$, $x \in \mathbb{R}$, g, h are convex and increasing over $\mathbb{R}$
- (5 pts) Let $f : \mathbb{R}^D \to \mathbb{R}$ be the function $f(\mathbf{x}) := \exp(\mathbf{x}^\top \mathbf{w})$, where $\mathbf{w} \in \mathbb{R}^D$. What is $\nabla_{\mathbf{w}} f$?

$$\nabla_{\mathbf{w}} f(\mathbf{w}) = f(\mathbf{w})\, \mathbf{x}$$