Calcular:

$$\lim_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\,\mathrm{d}x$$

Solución:

Se puede utilizar la regla de L'Hôpital, ya que $\lim\limits_{A\to\infty}\int_1^A A^{\frac{1}{x}}\,\mathrm{d}x = \int_1^\infty \infty^{\frac{1}{x}}\,\mathrm{d}x = \infty$. Y, obviamente, $\lim\limits_{A\to\infty} A = \infty$. Entonces:

$$\lim_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\,\mathrm{d}x = \lim_{A\to\infty} \frac{\dfrac{\mathrm{d}}{\mathrm{d}A}\left(\displaystyle\int_1^A A^{\frac{1}{x}}\,\mathrm{d}x\right)}{\dfrac{\mathrm{d}}{\mathrm{d}A}(A)} = \lim_{A\to\infty}\frac{\mathrm{d}}{\mathrm{d}A}\left(\int_1^A A^{\frac{1}{x}}\,\mathrm{d}x\right)$$

Para derivar esta integral, es necesario utilizar la fórmula de Leibniz:

$$F(x) = \int_{g(x)}^{h(x)} f(t,x)\,\mathrm{d}t \Longrightarrow$$
$$\Longrightarrow F'(x) = \int_{g(x)}^{h(x)} \frac{\partial}{\partial x}\big(f(t,x)\big)\,\mathrm{d}t\ + f\big(h(x),x\big)\cdot h'(x) - f\big(g(x),x\big)\cdot g'(x)$$

Por lo que:

$$\frac{\mathrm{d}}{\mathrm{d}A}\left(\int_1^A A^{\frac{1}{x}}\,\mathrm{d}x\right) = \int_1^A \frac{\partial}{\partial A}\big(A^{\frac{1}{x}}\big)\,\mathrm{d}x\ + A^{\frac{1}{A}}\cdot 1 - A^{\frac{1}{1}}\cdot 0 = \int_1^A \frac{1}{x}\cdot\big(A^{\frac{1}{x}-1}\big)\,\mathrm{d}x\ + A^{\frac{1}{A}}$$

Entonces, volviendo a la expresión inicial:

$$\lim_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\,\mathrm{d}x = \lim_{A\to\infty}\left(\int_1^A \frac{1}{x}\cdot\big(A^{\frac{1}{x}-1}\big)\,\mathrm{d}x\ + A^{\frac{1}{A}}\right) =$$
$$= \lim_{A\to\infty}\int_1^A \frac{1}{x}\cdot\big(A^{\frac{1}{x}-1}\big)\,\mathrm{d}x\ + \lim_{A\to\infty} A^{\frac{1}{A}}$$

Ahora hay dos límites para calcular. Uno de ellos se puede calcular fácilmente, ya que $\lim\limits_{A\to\infty} A^{\frac{1}{A}} = \lim\limits_{A\to\infty} e^{\frac{\ln A}{A}} = e^0 = 1$, sabiendo que $\ln A \ll A$ cuando $A\to\infty$.

El otro límite se puede calcular como:

$$\lim_{A\to\infty}\int_1^A \frac{1}{x}\cdot\left(A^{\frac{1}{x}-1}\right)\mathrm{d}x = \lim_{A\to\infty}\frac{\displaystyle\int_1^A \frac{1}{x}\cdot A^{\frac{1}{x}}\,\mathrm{d}x}{A} = \lim_{A\to\infty}\frac{\dfrac{\mathrm{d}}{\mathrm{d}A}\left(\displaystyle\int_1^A \frac{1}{x}\cdot A^{\frac{1}{x}}\,\mathrm{d}x\right)}{\dfrac{\mathrm{d}}{\mathrm{d}A}(A)} =$$

$$= \lim_{A\to\infty}\frac{\mathrm{d}}{\mathrm{d}A}\left(\int_1^A \frac{1}{x}\cdot A^{\frac{1}{x}}\,\mathrm{d}x\right)$$

Hay que derivar una integral parecida a la anterior. El resultado es:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}A}\left(\int_1^A \frac{1}{x}\cdot A^{\frac{1}{x}}\,\mathrm{d}x\right) &= \int_1^A \frac{\partial}{\partial A}\left(\frac{1}{x}\cdot A^{\frac{1}{x}}\right)\mathrm{d}x \;+\frac{1}{A}\cdot A^{\frac{1}{A}}\cdot 1-\frac{1}{1}\cdot A^{\frac{1}{1}}\cdot 0\\ &= \int_1^A \frac{1}{x^2}\cdot A^{\frac{1}{x}-1}\,\mathrm{d}x \;+\frac{A^{\frac{1}{A}}}{A} = -\left[\frac{A^{\frac{1}{x}-1}}{\ln A}\right]_1^A+\frac{A^{\frac{1}{A}}}{A}\\ &= \frac{1}{\ln A}-\frac{A^{\frac{1}{A}}}{A\cdot\ln A}+\frac{A^{\frac{1}{A}}}{A}\end{aligned}$$

Utilizando el resultado de $\lim\limits_{A\to\infty} A^{\frac{1}{A}} = 1$, se llega a que:

$$\lim_{A\to\infty}\frac{\mathrm{d}}{\mathrm{d}A}\left(\int_1^A \frac{1}{x}\cdot A^{\frac{1}{x}}\,\mathrm{d}x\right) = \lim_{A\to\infty}\left(\frac{1}{\ln A}-\frac{A^{\frac{1}{A}}}{A\cdot\ln A}+\frac{A^{\frac{1}{A}}}{A}\right) = \frac{1}{\infty}-\frac{1}{\infty}+\frac{1}{\infty}=0$$

Y, finalmente:

$$\lim_{A\to\infty}\frac{1}{A}\int_1^A A^{\frac{1}{x}}\,\mathrm{d}x = \lim_{A\to\infty}\int_1^A \frac{1}{x}\cdot\left(A^{\frac{1}{x}-1}\right)\mathrm{d}x \;+\lim_{A\to\infty}A^{\frac{1}{A}} = 0+1=1$$