Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Prove or disprove each of the following statements:

a) If f is continuous and $\operatorname{Im}(f) = \mathbb{R}$ then f is monotonic.

b) If f is monotonic and $\operatorname{Im}(f) = \mathbb{R}$ then f is continuous.

c) If f is monotonic and f is continuous then $\operatorname{Im}(f) = \mathbb{R}$.

Solution:

It is easy to find a counter-example for case a). A polynomial of odd degree, such as $f(x) = x^3 - x$, is continuous and also satisfies that $\operatorname{Im}(f) = \mathbb{R}$. However, it is not monotonic since $f'(x) = 3x^2 - 1 = 0 \iff x = \pm\frac{\sqrt{3}}{3}$, which implies that $f(x)$ is strictly decreasing for $x \in \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ and increasing for the rest of values of x. Therefore, it is not monotonic.

Statement b) is true because if f is monotonic, when trying to prove that f is not continuous, then there should be an essential discontinuity. But this fact contradicts that $\operatorname{Im}(f) = \mathbb{R}$. Hence, case b) is correct.

Again, it is easy to find a counter-example for case c). For instance, $f(x) = e^x$ is a continuous and monotonic function, since $f'(x) = e^x > 0, \forall x \in \mathbb{R}$. Nevertheless, $f(x) > 0$, and thus $\operatorname{Im}(f) = \mathbb{R}^+ \neq \mathbb{R}$.