Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

(1) There is a sequencen $(c_n)_{n=1}^{\infty}$ of positive numbers such that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge.

(2) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges.

Solution:

We have an "if and only if", that is, we need to prove both statements:

(1) $\Longleftarrow$ (2):

The radical expresion of (2) could be rewritten as:

$$\sqrt{\frac{a_n}{b_n}} = \sqrt{\frac{a_n}{b_n}} \cdot \sqrt{\frac{a_n}{a_n}} = \frac{a_n}{\sqrt{a_n \cdot b_n}} = \frac{a_n}{c_n}$$

$$\sqrt{\frac{a_n}{b_n}} = \sqrt{\frac{a_n}{b_n}} \cdot \sqrt{\frac{b_n}{b_n}} = \frac{\sqrt{a_n \cdot b_n}}{b_n} = \frac{c_n}{b_n}$$

If we choose $c_n = \sqrt{a_n \cdot b_n}$, and knowing that $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges, then $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ also converge; and therefore $(c_n)_{n=1}^{\infty}$ both sums converegent.

(1) $\Longrightarrow$ (2):

Using the Cauchy-Schwarz inequality:

$$\left(\sum_{k=1}^{n} \alpha_k \cdot \beta_k \right)^2 \leqslant \left(\sum_{k=1}^{n} \alpha_k^2 \right) \left(\sum_{k=1}^{n} \beta_k^2 \right)$$

And setting $\alpha_k = \sqrt{\dfrac{a_k}{c_k}}$ y $\beta_k = \sqrt{\dfrac{c_k}{b_k}}$, we obtain that

$$
\begin{aligned}
&0 < \left(\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{c_n}} \cdot \sqrt{\frac{c_n}{b_n}} \right)^2 \leqslant \left(\sum_{n=1}^{\infty} \left(\sqrt{\frac{a_n}{c_n}} \right)^2 \right) \cdot \left(\sum_{n=1}^{\infty} \left(\sqrt{\frac{c_n}{b_n}} \right)^2 \right) \\
\iff &0 < \left(\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}} \right)^2 \leqslant \left(\sum_{n=1}^{\infty} \frac{a_n}{c_n} \right) \cdot \left(\sum_{n=1}^{\infty} \frac{c_n}{b_n} \right) \\
\iff &0 < \sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}} \leqslant \sqrt{\left(\sum_{n=1}^{\infty} \frac{a_n}{c_n} \right) \cdot \left(\sum_{n=1}^{\infty} \frac{c_n}{b_n} \right)}
\end{aligned}
$$

Since the series $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ is bounded froom above by a finite value (and also from below, because it is a sequence of positive numbers), it is proven that the series converges.

And finally, it is demonstrated that (1) $\iff$ (2).