Computar la suma de la siguiente serie:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + \dots$$

Solución:

En primer lugar, es necesario descomponer la fracción en fracciones simples:

$$\frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{A}{4k+1} + \frac{B}{4k+2} + \frac{C}{4k+3} + \frac{D}{4k+4} \iff$$
$$\iff A(4k+2)(4k+3)(4k+4) + B(4k+1)(4k+3)(4k+4) +$$
$$+ C(4k+1)(4k+2)(4k+4) + D(4k+1)(4k+2)(4k+3) = 1$$

$k = \frac{-1}{4} \implies A(-1+2)(-1+3)(-1+4) = 1 \iff A = \frac{1}{6}$ $k = \frac{-1}{2} \implies B(-2+1)(-2+3)(-2+4) = 1 \iff B = \frac{-1}{2}$ $k = \frac{-3}{4} \implies C(-3+1)(-3+2)(-3+4) = 1 \iff C = \frac{1}{2}$ $k = -1 \implies D(-4+1)(-4+2)(-4+3) = 1 \iff D = \frac{-1}{6}$

Y entonces:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{1}{4k+1} - \frac{3}{4k+2} + \frac{3}{4k+3} - \frac{1}{4k+4}\right)$$

Y estas sumas se pueden reordenar de manera que se obtienen algunas series conocidas:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} =$$
$$= \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{1}{4k+1} - \frac{1}{4k+2} + \frac{1}{4k+3} - \frac{1}{4k+4}\right) -$$
$$- \frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{1}{4k+1} - \frac{3}{4k+3}\right) - \frac{1}{12}\sum_{k=0}^{\infty}\left(\frac{1}{2k+1} - \frac{2}{2k+2}\right)$$

Se sabe que: $\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}-\frac{1}{4k+2}+\frac{1}{4k+3}-\frac{1}{4k+4}\right)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=\ln(2)$

$\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}-\frac{3}{4k+3}\right)=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$

$\sum_{k=0}^{\infty}\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=\ln(2)$

Y de esta manera:

$$\sum_{k=0}^{\infty}\frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}=\frac{\ln(2)}{3}-\frac{1}{6}\cdot\frac{\pi}{4}-\frac{\ln(2)}{12}=\frac{\ln(2)}{4}-\frac{\pi}{24}$$