A and B are square complex matrices of the same size and $\operatorname{rank}(AB - BA) = 1$. Show that $(AB - BA)^2 = 0$.

Solution:

Let $M = AB - BA$, and let J be the Jordan normal form of matrix M. Then, $\operatorname{rank}(M) = \operatorname{rank}(J) = 1$. This implies that all eigenvalues of M are zero except for one; or all eigenvalues of M are zero but having $\dim(\ker(M)) = n - 1$, with n being the order of matrices A and B. These alternatives are:

$$J = \begin{pmatrix} \lambda & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

It can be easily seen that the first option is not valid, because $\operatorname{tr}(M) = \operatorname{tr}(J) = \lambda \neq 0$. Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, then $\operatorname{tr}(M) = \operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0$.

Hence, the only option is the second one. It is satisfied that $J^2 = 0$, because J contains a 2×2 submatrix that is nilpotent. As a result, it has been proved that for any complex matrices A and B with $\operatorname{rank}(AB - BA) = 1$, then $(AB - BA)^2 = 0$.