Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy the following three properties:

1. w consists of n letters, all of them are from the alphabet $\{\texttt{a}, \texttt{b}, \texttt{c}, \texttt{d}\}$.
2. w contains an even number of letters `a`.
3. w contains an even number of letters `b`.

(For example, for $n = 2$, there are 6 such words: `aa`, `bb`, `cc`, `dd`, `cd` and `dc`)

Solution:

First, it is easy to see that the number of words that can be formed with the alphabet $\{\texttt{a}, \texttt{b}, \texttt{c}, \texttt{d}\}$ as a function of n is 4^n. The following sets are defined:

- A_n: Set of words with an even number of letters `a` and `b`.
- B_n: Set of words with an odd number of letters `a` and `b`.
- C_n: Set of words with an even number of letters `a` and an odd number of letters `b`.
- D_n: Set of words with an odd number of letters `a` and an even number of letters `b`.

Since these sets are disjoint, obviously, we know that

$$|A_n| + |B_n| + |C_n| + |D_n| = 4^n$$

On the other hand, starting from the set A_n, if we remove the last letter of each word and it is `c` or `d`, the resulting words will be in the set A_{n-1}. If the last letter is an `a`, the resulting words will be in the set D_{n-1}. And if it is a `b`, the words will belong to C_{n-1}. Therefore:

$$|A_n| = 2|A_{n-1}| + |C_{n-1}| + |D_{n-1}|$$

It can be seen that there is a bijection between the sets C_n and D_n, which consists of changing every letter `a` to `b` and vice versa, so that $|C_n| = |D_n|$. Then

$$|A_n| = 2|A_{n-1}| + |C_{n-1}| + |D_{n-1}| = 2|A_{n-1}| + 2|C_{n-1}|$$

Continuing with the same strategy, if we remove the last letter from each word in the set B_n, if it is an $\texttt{a}$, the resulting word would be in the set C_{n-1}; if it is a $\texttt{b}$, the word would be in the set D_{n-1}; and if it is a $\texttt{c}$ or a $\texttt{d}$, the word would be in the set B_{n-1}, so that

$$|B_n| = 2|B_{n-1}| + |C_{n-1}| + |D_{n-1}| = 2|B_{n-1}| + 2|C_{n-1}|$$

If we remove the last letter from the words in the set C_n, if it is a letter $\texttt{a}$, the resulting word will be in the set B_{n-1}; if it is a $\texttt{b}$, the word will be in A_{n-1}; and if it is a $\texttt{c}$ or a $\texttt{d}$, the word will be in C_{n-1}, so that

$$|C_n| = |A_{n-1}| + |B_{n-1}| + 2|C_{n-1}|$$

Summarizing, we have the following relations:

$$|A_n| + |B_n| + 2|C_n| = 4^n$$

$$\begin{aligned} |A_n| &= 2|A_{n-1}| + 2|C_{n-1}| \\ |B_n| &= 2|B_{n-1}| + 2|C_{n-1}| \\ |C_n| &= |A_{n-1}| + |B_{n-1}| + 2|C_{n-1}| \end{aligned}$$

It is easy to see that $|C_n| = 4^{n-1}$, so the relations reduce to

$$|A_n| + |B_n| + 2 \cdot 4^{n-1} = 4^n \Longrightarrow |A_n| + |B_n| = 2 \cdot 4^{n-1}$$

$$\begin{aligned} |A_n| &= 2|A_{n-1}| + 2 \cdot 4^{n-2} \\ |B_n| &= 2|B_{n-1}| + 2 \cdot 4^{n-2} \end{aligned}$$

Subtracting the last two relations, we have that

$$|A_n| - |B_n| = 2\left(|A_{n-1}| - |B_{n-1}|\right)$$

We know that $A_1 = \{\texttt{c}, \texttt{d}\}$ and that $B_1 = \varnothing$, so $|A_1| - |B_1| = 2$, and then, $|A_2| - |B_2| = 4$, and inductively

$$|A_n| - |B_n| = 2^n$$

Combining this relation with $|A_n| + |B_n| = 2 \cdot 4^{n-1}$, we have that

$$|A_n| = 2^{n-1} + 4^{n-1}$$

This is the number of words w of n letters with the required properties.