Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $(f(x))^n$ is a polynomial for every $n \geqslant 2$. Does it follow that f is a polynomial?

Solution:

The answer is: yes.

Let $\dfrac{f^3}{f^2} = \dfrac{p}{q}$ be an irreducible rational function with som polynomials p and q. Then $\dfrac{p^2}{q^2}$ is also irreducible, and it is satisfied that $\dfrac{p^2}{q^2} = \left(\dfrac{f^3}{f^2}\right)^2 = f^2$. Since f^2 is a polynomial, then q is constant (due to the fact that q^2 is constant too).

Hence, $f = \dfrac{f^3}{f^2} = \dfrac{p}{q}$ is a polynomial p divided by a constant q, and then it is clear that f is a polynomial.