Evaluar el producto
$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}$$

Solución:

Primeramente, se puede ver que

$$\frac{(n^3+3n)^2}{n^6-64} = \frac{n^2}{n^2-4} \cdot \frac{(n^2+3)^2}{(n^2-2n+4)\cdot(n^2+2n+4)}$$

Sea $A_n = n^2+3$, se cumple que

$$A_{n+1} = (n+1)^2+3 = n^2+2n+4$$

$$A_{n-1} = (n-1)^2+3 = n^2-2n+4$$

Sea $P_k = \prod_{n=3}^{k} \frac{(n^3+3n)^2}{n^6-64} = \prod_{n=3}^{k} \left(\frac{n^2}{n^2-4} \cdot \frac{A_n^2}{A_{n-1}\cdot A_{n+1}} \right) = \prod_{n=3}^{k} \frac{n^2}{n^2-4} \cdot \prod_{n=3}^{k} \frac{A_n^2}{A_{n-1}\cdot A_{n+1}}$.

Por un lado, se ve que

$$\begin{aligned}
\prod_{n=3}^{k} \frac{n^2}{n^2-4} &= \prod_{n=3}^{k} \frac{n}{n-2} \cdot \prod_{n=3}^{k} \frac{n}{n+2} \\
&= \left(\frac{\cancel{3}}{1} \cdot \frac{\cancel{4}}{2} \cdot \frac{\cancel{5}}{\cancel{3}} \cdot \frac{\cancel{6}}{\cancel{4}} \cdots \frac{\cancel{k-2}}{\cancel{k-4}} \cdot \frac{k-1}{\cancel{k-3}} \cdot \frac{k}{\cancel{k-2}} \right) \cdot \prod_{n=3}^{k} \frac{n}{n+2} \\
&= \frac{(k-1)\cdot k}{1\cdot 2} \cdot \left(\frac{3}{\cancel{5}} \cdot \frac{4}{\cancel{6}} \cdot \frac{\cancel{5}}{\cancel{7}} \cdot \frac{\cancel{6}}{\cancel{8}} \cdots \frac{\cancel{k-2}}{\cancel{k}} \cdot \frac{\cancel{k-1}}{k+1} \cdot \frac{\cancel{k}}{k+2} \right) \\
&= \frac{(k-1)\cdot k}{1\cdot 2} \cdot \frac{3\cdot 4}{(k+1)\cdot(k+2)} \\
&= 6 \cdot \frac{k^2-k}{k^2+3k+2}
\end{aligned}$$

Y, por otro lado

$$\prod_{n=3}^{k} \frac{A_n^2}{A_{n-1}\cdot A_{n+1}} = \frac{A_3\cdot \cancel{A_3}}{A_2\cdot \cancel{A_4}}\cdot\frac{\cancel{A_4}\cdot \cancel{A_4}}{\cancel{A_3}\cdot \cancel{A_5}}\cdot\frac{\cancel{A_5}\cdot \cancel{A_5}}{\cancel{A_4}\cdot \cancel{A_6}}\cdots\frac{\cancel{A_{k-1}}\cdot \cancel{A_{k-1}}}{\cancel{A_{k-2}}\cdot \cancel{A_k}}\cdot\frac{\cancel{A_k}\cdot A_k}{\cancel{A_{k-1}}\cdot A_{k+1}}$$

$$= \frac{A_3}{A_2}\cdot\frac{A_k}{A_{k+1}} = \frac{3^2+3}{3^2-2\cdot 3+4}\cdot\frac{k^2+3}{k^2+2k+4}$$

$$= \frac{12}{7}\cdot\frac{k^2+3}{k^2+2k+4}$$

Y entonces, $P_k = 6\cdot\frac{k^2-k}{k^2+3k+2}\cdot\frac{12}{7}\cdot\frac{k^2+3}{k^2+2k+4} = \frac{72}{7}\cdot\frac{k^2-k}{k^2+3k+2}\cdot\frac{k^2+3}{k^2+2k+4}$.

De esto, se sigue que $\lim_{k\to\infty} P_k = \lim_{k\to\infty}\left(\frac{72}{7}\cdot\frac{k^2-k}{k^2+3k+2}\cdot\frac{k^2+3}{k^2+2k+4}\right) = \frac{72}{7}$.

Y finalmente:

$$\prod_{n=3}^{\infty}\frac{(n^3+3n)^2}{n^6-64} = \frac{72}{7}$$