Find all twice continuously differentiable functions $f : \mathbb{R} \to (0, +\infty)$ satisfying

$$f''(x)f(x) \geqslant 2(f'(x))^2$$

for all $x \in \mathbb{R}$.

Solution:

It can be seen that the inequality in the statement is part of a derivative. For example, by differentiating the following function:

$$g = \frac{1}{f} \Longrightarrow g'' = \left(\frac{1}{f}\right)'' = \left(\frac{-f'}{f^2}\right)' = \frac{-f''f^2 + 2f(f')^2}{f^4} = \frac{2(f')^2 - f''f}{f^3}$$

Knowing that $2(f'(x))^2 - f''(x)f(x) \leqslant 0$, we obtain

$$\frac{2(f')^2 - f''f}{f^3} \leqslant 0 \Longrightarrow g'' \leqslant 0$$

Therefore, the function g is concave for all $x \in \mathbb{R}$. Then, for any set of values $a < b$, $u < a$ and $b < v$, it holds that

$$\frac{g(a) - g(u)}{a - u} \geqslant \frac{g(b) - g(a)}{b - a} \geqslant \frac{g(v) - g(b)}{v - b}$$

Taking limits in the previous expression as $u \to -\infty$ and $v \to +\infty$, it follows that

$$0 \geqslant \frac{g(b) - g(a)}{b - a} \geqslant 0$$

Since $g = \dfrac{1}{f}$ is a strictly positive function for all $x \in \mathbb{R}$, the only possibility is that $g(a) = g(b)$ for any $a, b \in \mathbb{R}$. Then g is constant, and therefore f is also constant. From this it follows that the only functions that satisfy the inequality in the statement are the positive constant functions, of the form $f(x) = C$, with $C > 0$.