a) Show that for any $m \in \mathbb{N}$ there exists a real $m \times m$ matrix A such that $A^3 = A + I$, where I is the $m \times m$ identity matrix.

b) Show that $\det(A) > 0$ for every real $m \times m$ matrices satisfying $A^3 = A + I$.

Solution:

The first statement is easy to prove, because we can consider a diagonal matrix whose entries are roots of the polynomial $p(\lambda) = \lambda^3 - \lambda - 1$. Since this polynomial has an odd degree, it must have at least a real solution, because all complex roots are conjugated pairs.

Let $\lambda_1 \mid p(\lambda_1) = 0$, then the matrix $A = \lambda_1 I$ satisfies that $A^3 = A + I$ for all $m \in \mathbb{N}$.

To prove the second statement, we can use the fact that the determinant of A equals the product of its eigenvalues. Moreover, all possible eigenvalues of A are roots of $p(\lambda)$.

We can compute the maximums and minimums of the polynomial, so that $p'(x) = 3x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{3}}$, being $x = -\frac{1}{\sqrt{3}}$ the relative maximum and $x = \frac{1}{\sqrt{3}}$ the relative minimum. It happens that $p\left(-\frac{1}{\sqrt{3}}\right) < 0$ y $p\left(\frac{1}{\sqrt{3}}\right) < 0$, so $p(\lambda)$ has a unique real root (λ_1) and two complex solutions (λ_2 and λ_3).

Using Bolzano's theorem, we can bound λ_1 and get that $\lambda_1 \in (1, 2)$, due to the fact that $p(1) = -1 < 0$ y $p(2) = 5 > 0$. Then $\lambda_1 > 0$.

On the other hand, let $\lambda_2 = r\, e^{\mathrm{i}\phi}$ y $\lambda_3 = r\, e^{-\mathrm{i}\phi}$ be the complex roots of $p(\lambda)$. Hence, $\det(A) = \lambda_1^\alpha \cdot (\lambda_2\, \lambda_3)^\beta$, being α and β the multiplicity of the eigenvalues. It happens that $\lambda_1^\alpha > 0$ due to $\lambda_1 > 0$. An furthermore, $(\lambda_2\, \lambda_3)^\beta = \left(r\, e^{\mathrm{i}\phi} \cdot r\, e^{-\mathrm{i}\phi}\right)^\beta = (r^2)^\beta = \left(r^\beta\right)^2 > 0$. Since all factors are strictly positive, it follows that $\det(A) > 0$.