

Let A be the $n \times n$ matrix whose $(i, j)^{\text{th}}$ entry is $i + j$ for all $i, j = 1, 2, \dots, n$. What is the rank of A ?

Solution:

The matrix A can be rewritten as

$$\begin{pmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & \dots & n+n \end{pmatrix}$$

Subtracting the first row from the rest, we obtain another matrix but with the same rank:

$$\begin{pmatrix} 2 & 3 & \dots & n+1 \\ 1 & 1 & \dots & 1 \\ 2 & 2 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & n-1 & \dots & n-1 \end{pmatrix}$$

And this matrix has several rows that are proportional to the second row, so it is equivalent to the matrix

$$\begin{pmatrix} 2 & 3 & \dots & n+1 \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

And this last matrix has rank equal to 2. Since it is equivalent to matrix A , then the rank of A is also 2.