Let z be a complex number with $|z+1|>2$. Prove that $|z^3+1|>1$.

Solution:

The expression $|z^3+1|$ can be written as:

$$|z^3+1| = |z+1|\cdot|z^2-z+1| > 1 \iff |z^2-z+1| \geqslant \frac{1}{2}$$

If $r\,e^{\mathrm{i}\phi} = z+1$ is defined, then $r=|z+1|>2$. It is needed to calculate the modulus of z^2-z+1. It can be computed as follows:

$$\begin{aligned}
|z^2-z+1|^2 &= (z^2-z+1)\cdot\overline{(z^2-z+1)} = \\
&= (r^2\,e^{\mathrm{i}2\phi} - 2r\,e^{\mathrm{i}\phi} + 1 - r\,e^{\mathrm{i}\phi} + 1 + 1)\cdot\overline{(r^2\,e^{\mathrm{i}2\phi} - 2r\,e^{\mathrm{i}\phi} + 1 - r\,e^{\mathrm{i}\phi} + 1 + 1)} = \\
&= (r^2\,e^{\mathrm{i}2\phi} - 3r\,e^{\mathrm{i}\phi} + 3)\cdot(r^2\,e^{-\mathrm{i}2\phi} - 3r\,e^{-\mathrm{i}\phi} + 3) = \\
&= r^4 + 9r^2 + 9 + 3r^2\cdot(e^{\mathrm{i}2\phi} + e^{-\mathrm{i}2\phi}) - (3r^3 - 9r)\cdot(e^{\mathrm{i}\phi} + e^{-\mathrm{i}\phi}) = \\
&= r^4 + 9r^2 + 9 + 6r^2\cos 2\phi - (6r^3 - 18r)\cdot\cos\phi = \\
&= r^4 + 9r^2 + 9 + 6r^2\cdot(2\cos^2\phi - 1) - (6r^3 - 18r)\cdot\cos\phi = \\
&= r^4 + 3r^2 + 9 + 12r^2\cos^2\phi - (6r^3 - 18r)\cdot\cos\phi = \\
&= 12\left(r\cos\phi - \frac{r^2+3}{4}\right)^2 + \left(\frac{r^2-3}{2}\right)^2
\end{aligned}$$

Since $r>2$, $|z^2-z+1|^2 \geqslant \left(\frac{r^2-3}{2}\right)^2 > \frac{1}{4} \implies |z^2-z+1| > \frac{1}{2}$. Finally, it is demonstrated that if $|z+1|>2$, then $|z^3+1|>1$.