Let $0 < a < b$. Prove that:

$$\int_a^b (x^2+1)\, e^{-x^2}\, \mathrm{d}x \geqslant e^{-a^2} - e^{-b^2}$$

Solution:

It is easy to see that:

$$\int_{a^2}^{b^2} e^{-t}\, \mathrm{d}t = e^{-a^2} - e^{-b^2}$$

Doing this change of variable $t = x^2$:

$$\int_a^b (x^2+1)\, e^{-x^2}\, \mathrm{d}x = \int_{a^2}^{b^2} (t+1)\, e^{-t}\, \frac{\mathrm{d}t}{2\sqrt{t}}$$

Then:

$$\begin{aligned}
\int_a^b (x^2+1)\, e^{-x^2}\, \mathrm{d}x \geqslant e^{-a^2} - e^{-b^2} &\iff \int_{a^2}^{b^2} (t+1)\, e^{-t}\, \frac{\mathrm{d}t}{2\sqrt{t}} \geqslant \int_{a^2}^{b^2} e^{-t}\, \mathrm{d}t \\
&\iff \frac{t+1}{2\sqrt{t}} \geqslant 1 \\
&\iff \frac{t^2+2t+1}{4t} \geqslant 1 \\
&\iff (t-1)^2 \geqslant 0
\end{aligned}$$

Since the previous condition is satisfied for all $t \in \mathbb{R}$, the required inequality is proven.