Let A and B be $n \times n$ real matrices such that

$$\operatorname{rank}(AB - BA + I) = 1$$

where I is the $n \times n$ identity matrix. Prove that

$$\operatorname{tr}(ABAB) - \operatorname{tr}(A^2B^2) = \frac{1}{2}n(n-1)$$

Solution:

Using some properties of the trace of the product of two matrices, for instance $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ and $\operatorname{tr}(X) + \operatorname{tr}(Y) = \operatorname{tr}(X+Y)$, the expression to be proven is the same as:

$$\begin{aligned}
&\operatorname{tr}(ABAB) - \operatorname{tr}(A^2B^2) = \\
&\quad = \frac{1}{2}\big(2\operatorname{tr}(ABAB) - 2\operatorname{tr}(A^2B^2)\big) = \\
&\quad = \frac{1}{2}(\operatorname{tr}(ABAB) + \operatorname{tr}(ABAB) - \operatorname{tr}(AABB) - \operatorname{tr}(AABB)) = \\
&\quad = \frac{1}{2}(\operatorname{tr}(ABAB) + \operatorname{tr}(A(BAB)) - \operatorname{tr}(A(ABB)) - \operatorname{tr}((AAB)B)) = \\
&\quad = \frac{1}{2}(\operatorname{tr}(ABAB) + \operatorname{tr}((BAB)A) - \operatorname{tr}((ABB)A) - \operatorname{tr}(B(AAB))) = \\
&\quad = \frac{1}{2}(\operatorname{tr}(ABAB) + \operatorname{tr}(BABA) - \operatorname{tr}(ABBA) - \operatorname{tr}(BAAB)) = \\
&\quad = \frac{1}{2}\operatorname{tr}(ABAB + BABA - ABBA - BAAB) = \\
&\quad = \frac{1}{2}\operatorname{tr}((AB - BA)^2)
\end{aligned}$$

Let $M = AB - BA + I$ and $N = M - I = AB - BA$.

From the condition $\operatorname{rank}(M) = \operatorname{rank}(AB - BA + I) = 1$, it is obtained that all eigenvalues of M are zero with multiplicity $n-1$, or there exists a unique eigenvalue that is non-zero (with multiplicity 1 and the rest are null eigenvalues with multiplicity $n-1$).

As it happens that

$$\begin{aligned}\operatorname{tr}(M) &= \operatorname{tr}(AB - BA + I) = \\ &= \operatorname{tr}(AB) - \operatorname{tr}(BA) + \operatorname{tr}(I) = \\ &= \operatorname{tr}(I) = n\end{aligned}$$

then M has a unique non-zero eigenvalue with value n, and the rest are null eigenvalues. Hence, the spectrum of the matrix M is $\sigma(M) = \{0, n\}$.

Since $N = M - I$, the spectrum of N is $\sigma(N) = \{-1, n-1\}$, and the spectrum of N^2 is $\sigma(N^2) = \{1, (n-1)^2\}$.

Notice that $\operatorname{tr}(ABAB) - \operatorname{tr}(A^2B^2) = \frac{1}{2}\operatorname{tr}((AB - BA)^2) = \frac{1}{2}\operatorname{tr}(N^2)$. Ant it is known that the trace of N^2 is the sum of its eigenvalues, namely:

$$\operatorname{tr}(N^2) = 1 \cdot (n-1) + (n-1)^2 = n(n-1)$$

An hence, it is proven that:

$$\operatorname{tr}(ABAB) - \operatorname{tr}(A^2B^2) = \frac{1}{2}\operatorname{tr}((AB - BA)^2) = \frac{1}{2}\operatorname{tr}(N^2) = \frac{1}{2}n(n-1)$$