Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a continuous function such that $\lim\limits_{x \to \infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n \to \infty} \int_0^1 f(nx) \, \mathrm{d}x = L$$

Solution:

By doing a change of variable, we obtain the following:

$$\left.\begin{aligned} t &= nx \\ \mathrm{d}t &= n \, \mathrm{d}x \end{aligned}\right\} \Longrightarrow \int_0^1 f(nx) \, \mathrm{d}x = \frac{1}{n} \cdot \int_0^n f(t) \, \mathrm{d}t$$

And using L'Hôpital's Rule, the result is proven:

$$\lim_{n \to \infty} \int_0^1 f(nx) \, \mathrm{d}x = \lim_{n \to \infty} \frac{1}{n} \cdot \int_0^n f(t) \, \mathrm{d}t = \lim_{n \to \infty} \frac{\dfrac{\mathrm{d}}{\mathrm{d}n} \left(\displaystyle\int_0^n f(t) \, \mathrm{d}t \right)}{\dfrac{\mathrm{d}}{\mathrm{d}n}(n)} = \lim_{n \to \infty} f(n) = L$$