Let $C = \{4, 6, 8, 9, 10, \dots\}$ be the set of composite positive integers. For each $n \in C$, let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n$$

Solution:

The root criterion states that a series $\sum b_n$ is convergent if and only if $\lim\limits_{n\to\infty} \sqrt[n]{b_n} < 1$. Therefore, if $b_n = \left(\frac{a_n}{n}\right)^n$, the series converges if $\lim\limits_{n\to\infty} \frac{a_n}{n} < 1$.

It might me convenient to see some examples:

$$n = 4 \Longrightarrow a_4 = 4 \Longrightarrow 4! = 24 \Longrightarrow \frac{a_4}{4} = 1$$

$$n = 6 \Longrightarrow a_6 = 3 \Longrightarrow 3! = 6 \Longrightarrow \frac{a_6}{6} = \frac{1}{2}$$

$$n = 8 \Longrightarrow a_8 = 4 \Longrightarrow 4! = 24 \Longrightarrow \frac{a_8}{8} = \frac{1}{2}$$

$$n = 9 \Longrightarrow a_9 = 6 \Longrightarrow 6! = 720 \Longrightarrow \frac{a_9}{9} = \frac{2}{3}$$

$$n = 10 \Longrightarrow a_{10} = 5 \Longrightarrow 5! = 120 \Longrightarrow \frac{a_{10}}{10} = \frac{1}{2}$$

It seems that $\frac{a_n}{n} \leqslant \frac{2}{3}$ for $n > 4$. Next, we will show different forms of n:

Let $n = p_1 \cdot p_2$, being p_i prime numbers and $p_1 < p_2$, so $a_n = p_2$ because $p_2! = p_2 \cdot (p_2 - 1) \cdots p_1 \cdot (p_1 - 1) \cdots 1$. Therefore, $n \mid k$, and $\frac{a_n}{n} = \frac{1}{p_1} \leqslant \frac{1}{2}$, due to 2 is the least positive prime number.

If $n = p_1 \cdots p_m$, with $p_1 < \cdots < p_m$, then $a_n = p_m$ and $\frac{a_n}{n} = \frac{1}{p_1 \cdot p_2 \cdots p_{m-1}} \leqslant \frac{1}{2}$.

On the other hand, when $n = p^\alpha$, with $\alpha \geqslant 2$, we have that $a_n = \alpha p$ because $(\alpha p)! = \alpha p \cdots (\alpha - 1)p \cdots p \cdots 1$ is satisfied, and hence $n \mid (\alpha p)!$. Thus, the quotient $\frac{a_n}{n} = \frac{\alpha}{p^{\alpha-1}} \leqslant \frac{2}{3}$, ya que $n = 4 = 2^2$ is a special case.

In general terms, if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2}$, with $p_1^{\alpha_1} < p_2^{\alpha_2}$, we have that $a_n = \alpha_2 p_2$, and therefore $\frac{a_n}{n} \leqslant \frac{2}{3}$. This case can be generalized for m in a similar way as before.

Then, it is proved that $\frac{a_n}{n} \leqslant \frac{2}{3}$ for $n > 4$. Hence $\lim\limits_{n\to\infty} \frac{a_n}{n} \leqslant \lim\limits_{n\to\infty} \frac{2}{3} = \frac{2}{3} < 1$, so that the series of the problem statement converges.