Let $f, g : [a, b] \to [0, \infty)$ be continuous and non-decreasing functions such that for each $x \in [a, b]$ we have

$$\int_a^x \sqrt{f(t)}\,\mathrm{d}t \leqslant \int_a^x \sqrt{g(t)}\,\mathrm{d}t$$

and $\displaystyle\int_a^b \sqrt{f(t)}\,\mathrm{d}t = \int_a^b \sqrt{g(t)}\,\mathrm{d}t$.

Prove that

$$\int_a^b \sqrt{1 + f(t)}\,\mathrm{d}t \geqslant \int_a^b \sqrt{1 + g(t)}\,\mathrm{d}t$$

Solution:

Let $F(x) = \displaystyle\int_a^x \sqrt{f(t)}\,\mathrm{d}t$ and $G(x) = \displaystyle\int_a^x \sqrt{g(t)}\,\mathrm{d}t$ be some functions.

It is satisfied that $f(x) = (F'(x))^2$ and $g(x) = (G'(x))^2$, so the inequality to prove is equivalent to

$$\int_a^b \sqrt{1 + (F'(x))^2}\,\mathrm{d}t \geqslant \int_a^b \sqrt{1 + (G'(x))^2}\,\mathrm{d}t$$

Then, it needs to be shown that the length of the graph of F between a and b is greater than the length of the graph of G.

Knowing that $F'(x) = \sqrt{f(x)} \geqslant 0$ and $F''(x) = \frac{f'(x)}{2\sqrt{f(x)}} \geqslant 0$ (because $f'(x) \geqslant 0$, since f is non-decreasing) then F is a convex function. The same applies for G.

Moreover, it is known that $F(a) = G(a) = 0$, also that $F(x) \leqslant G(x)$ for all $x \in [a, b]$, and also that $F(b) = G(b)$. Since both functions are convex, they intersect in $x = a$ and $x = b$, and G is over F, then the length of the graph of F is longer than the length of the graph of G. Hence the inequality is proven.