

Evaluate the product

$$\prod_{n=3}^{\infty} \frac{(n^3 + 3n)^2}{n^6 - 64}$$

**Solution:**

First, we can see that

$$\frac{(n^3 + 3n)^2}{n^6 - 64} = \frac{n^2}{n^2 - 4} \cdot \frac{(n^2 + 3)^2}{(n^2 - 2n + 4) \cdot (n^2 + 2n + 4)}$$

Let  $A_n = n^2 + 3$ , it follows that

$$A_{n+1} = (n+1)^2 + 3 = n^2 + 2n + 4$$

$$A_{n-1} = (n-1)^2 + 3 = n^2 - 2n + 4$$

$$\text{Let } P_k = \prod_{n=3}^k \frac{(n^3 + 3n)^2}{n^6 - 64} = \prod_{n=3}^k \left( \frac{n^2}{n^2 - 4} \cdot \frac{A_n^2}{A_{n-1} \cdot A_{n+1}} \right) = \prod_{n=3}^k \frac{n^2}{n^2 - 4} \cdot \prod_{n=3}^k \frac{A_n^2}{A_{n-1} \cdot A_{n+1}}.$$

On the one hand, it can be seen that

$$\begin{aligned} \prod_{n=3}^k \frac{n^2}{n^2 - 4} &= \prod_{n=3}^k \frac{n}{n-2} \cdot \prod_{n=3}^k \frac{n}{n+2} \\ &= \left( \frac{\cancel{3}}{1} \cdot \frac{\cancel{4}}{2} \cdot \frac{\cancel{5}}{\cancel{3}} \cdot \frac{\cancel{6}}{\cancel{4}} \cdots \frac{\cancel{k-2}}{\cancel{k-4}} \cdot \frac{k-1}{\cancel{k-3}} \cdot \frac{k}{\cancel{k-2}} \right) \cdot \prod_{n=3}^k \frac{n}{n+2} \\ &= \frac{(k-1) \cdot k}{1 \cdot 2} \cdot \left( \frac{\cancel{3}}{\cancel{5}} \cdot \frac{\cancel{4}}{\cancel{6}} \cdot \frac{\cancel{5}}{\cancel{7}} \cdot \frac{\cancel{6}}{\cancel{8}} \cdots \frac{\cancel{k-2}}{\cancel{k}} \cdot \frac{\cancel{k-1}}{k+1} \cdot \frac{k}{k+2} \right) \\ &= \frac{(k-1) \cdot k}{1 \cdot 2} \cdot \frac{3 \cdot 4}{(k+1) \cdot (k+2)} \\ &= 6 \cdot \frac{k^2 - k}{k^2 + 3k + 2} \end{aligned}$$

And on the other hand

$$\begin{aligned}
\prod_{n=3}^k \frac{A_n^2}{A_{n-1} \cdot A_{n+1}} &= \frac{A_3 \cdot \cancel{A_3}}{A_2 \cdot \cancel{A_4}} \cdot \frac{\cancel{A_4} \cdot \cancel{A_4}}{\cancel{A_3} \cdot \cancel{A_5}} \cdot \frac{\cancel{A_5} \cdot \cancel{A_5}}{\cancel{A_4} \cdot \cancel{A_6}} \cdots \frac{\cancel{A_{k-1}} \cdot \cancel{A_{k-1}}}{\cancel{A_{k-2}} \cdot \cancel{A_k}} \cdot \frac{\cancel{A_k} \cdot A_k}{\cancel{A_{k-1}} \cdot A_{k+1}} \\
&= \frac{A_3}{A_2} \cdot \frac{A_k}{A_{k+1}} = \frac{3^2 + 3}{3^2 - 2 \cdot 3 + 4} \cdot \frac{k^2 + 3}{k^2 + 2k + 4} \\
&= \frac{12}{7} \cdot \frac{k^2 + 3}{k^2 + 2k + 4}
\end{aligned}$$

$$\text{Then, } P_k = 6 \cdot \frac{k^2 - k}{k^2 + 3k + 2} \cdot \frac{12}{7} \cdot \frac{k^2 + 3}{k^2 + 2k + 4} = \frac{72}{7} \cdot \frac{k^2 - k}{k^2 + 3k + 2} \cdot \frac{k^2 + 3}{k^2 + 2k + 4}.$$

$$\text{And it follows } \lim_{k \rightarrow \infty} P_k = \lim_{k \rightarrow \infty} \left( \frac{72}{7} \cdot \frac{k^2 - k}{k^2 + 3k + 2} \cdot \frac{k^2 + 3}{k^2 + 2k + 4} \right) = \frac{72}{7}.$$

Finally:

$$\prod_{n=3}^{\infty} \frac{(n^3 + 3n)^2}{n^6 - 64} = \frac{72}{7}$$