Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

Solution:

For $n = 1$ the only possible matrix is $A_1 = \begin{pmatrix} 0 \end{pmatrix}$, whose rank is $\operatorname{rank}(A_1) = 0$. For $n = 2$, the matrices are like

$$A_2 = \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix}, m, n > 0 \Longrightarrow |A_2| = -mn \neq 0 \Longrightarrow \operatorname{rank}(A_2) = 2$$

For $n \geqslant 3$ it can be seen that the matrix is

$$A_n = \begin{pmatrix} 0^2 & 1^2 & \dots & (n-1)^2 \\ (-1)^2 & 0^2 & \dots & (n-2)^2 \\ \vdots & \vdots & \ddots & \vdots \\ (1-n)^2 & (2-n)^2 & \dots & 0^2 \end{pmatrix} = \left((i-j)^2\right)_{i,j=1}^{n} = \left(i^2 - 2ij + j^2\right)_{i,j=1}^{n} =$$

$$= \begin{pmatrix} 1^2 & 1^2 & \dots & 1^2 \\ 2^2 & 2^2 & \dots & 2^2 \\ \vdots & \vdots & \ddots & \vdots \\ n^2 & n^2 & \dots & n^2 \end{pmatrix} - 2 \cdot \begin{pmatrix} 1 \cdot 1 & 1 \cdot 2 & \dots & 1 \cdot n \\ 2 \cdot 1 & 2 \cdot 2 & \dots & 2 \cdot n \\ \vdots & \vdots & \ddots & \vdots \\ n \cdot 1 & n \cdot 2 & \dots & n \cdot n \end{pmatrix} + \begin{pmatrix} 1^2 & 2^2 & \dots & n^2 \\ 1^2 & 2^2 & \dots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^2 & 2^2 & \dots & n^2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1^2 & 1^2 & \dots & 1^2 \\ 4 & 4 & \dots & 4 \\ \vdots & \vdots & \ddots & \vdots \\ n^2 & n^2 & \dots & n^2 \end{pmatrix} - \begin{pmatrix} 2 & 4 & \dots & 2n \\ 4 & 8 & \dots & 4n \\ \vdots & \vdots & \ddots & \vdots \\ 2n & 4n & \dots & 2n^2 \end{pmatrix} + \begin{pmatrix} 1 & 4 & \dots & n^2 \\ 1 & 4 & \dots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 4 & \dots & n^2 \end{pmatrix}$$

Therefore, the smallest possible rank of A_n is

$$\operatorname{rank}(A_n) = \operatorname{rank}\left(\begin{pmatrix} 1^2 & 1^2 & \dots & 1^2 \\ 4 & 4 & \dots & 4 \\ \vdots & \vdots & \ddots & \vdots \\ n^2 & n^2 & \dots & n^2 \end{pmatrix} - \begin{pmatrix} 2 & 4 & \dots & 2n \\ 4 & 8 & \dots & 4n \\ \vdots & \vdots & \ddots & \vdots \\ 2n & 4n & \dots & 2n^2 \end{pmatrix} + \begin{pmatrix} 1 & 4 & \dots & n^2 \\ 1 & 4 & \dots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 4 & \dots & n^2 \end{pmatrix}\right) \leqslant$$

$$\leqslant \operatorname{rank}\begin{pmatrix} 1^2 & 1^2 & \dots & 1^2 \\ 4 & 4 & \dots & 4 \\ \vdots & \vdots & \ddots & \vdots \\ n^2 & n^2 & \dots & n^2 \end{pmatrix} + \operatorname{rank}\begin{pmatrix} -2 & -4 & \dots & -2n \\ -4 & -8 & \dots & -4n \\ \vdots & \vdots & \ddots & \vdots \\ -2n & -4n & \dots & -2n^2 \end{pmatrix} +$$

$$+ \operatorname{rank}\begin{pmatrix} 1 & 4 & \dots & n^2 \\ 1 & 4 & \dots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 4 & \dots & n^2 \end{pmatrix} =$$

$$= 1 + 1 + 1 = 3$$

Hence, the minimum rank of A_n for $n \geqslant 3$ is 3.