A four-digit number $YEAR$ is called *very good* if the system

$$\begin{cases} Yx + Ey + Az + Rw & = & Y \\ Rx + Yy + Ez + Aw & = & E \\ Ax + Ry + Yz + Ew & = & A \\ Ex + Ay + Rz + Yw & = & R \end{cases}$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good $YEAR$s in the 21st century (the 21st century starts in 2001 and ends in 2100).

Solution:

Let M, X and N be the following matrices:

$$M = \begin{pmatrix} Y & E & A & R \\ R & Y & E & A \\ A & R & Y & E \\ E & A & R & Y \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \qquad N = \begin{pmatrix} Y \\ E \\ A \\ R \end{pmatrix}$$

The system of linear equations $MX = N$ has a unique solution if and only if $\det(M) \neq 0$. The system can have more than one solution (indeed, infinite) if $\det(M) = 0$ and also $\operatorname{rank}(M) = \operatorname{rank}(M|N)$. It could also happen that the system does not have any solution when $\operatorname{rank}(M) < \operatorname{rank}(M|N)$. All of these comes from the Rouché-Capelli theorem.

The cases we are interested in are the ones that make $\det(M) = 0$ y $\operatorname{rank}(M) = \operatorname{rank}(M|N)$. We can notice that years in the 21st century satisfy $Y = 2$ always and $E = 0$ except for 2100. This case can be checked particularly:

$$\det(M) = \begin{vmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 15 \neq 0$$

Now we can discard year 2100, hence, we can fix $Y = 2$ and $E = 0$. Now we proceed to calculate the determinant of the coefficients matrix depending on A and R:

$$
\begin{aligned}
\det(M) &= \begin{vmatrix} 2 & 0 & A & R \\ R & 2 & 0 & A \\ A & R & 2 & 0 \\ 0 & A & R & 2 \end{vmatrix} = (2+A+R) \begin{vmatrix} 1 & 1 & 1 & 1 \\ R & 2 & 0 & A \\ A & R & 2 & 0 \\ 0 & A & R & 2 \end{vmatrix} \\
&= (2+A+R) \begin{vmatrix} 1 & 0 & 0 & 0 \\ R & 2-R & -R & A-R \\ A & R-A & 2-A & -A \\ 0 & A & R & 2 \end{vmatrix} \\
&= (2+A+R) \begin{vmatrix} 2-R & -R & A-R \\ R-A & 2-A & -A \\ A & R & 2 \end{vmatrix} \\
&= (2+A+R) \begin{vmatrix} 2+A-R & 0 & 2+A-R \\ R-A & 2-A & -A \\ A & R & 2 \end{vmatrix} \\
&= (2+A+R)(2+A-R) \begin{vmatrix} 1 & 0 & 1 \\ R-A & 2-A & -A \\ A & R & 2 \end{vmatrix} \\
&= (2+A+R)(2+A-R) \begin{vmatrix} 1 & 0 & 0 \\ R-A & 2-A & -R \\ A & R & 2-A \end{vmatrix} \\
&= (2+A+R)(2+A-R) \begin{vmatrix} 2-A & -R \\ R & 2-A \end{vmatrix} \\
&= (2+A+R)(2+A-R) \left((2-A)^2 + R^2\right)
\end{aligned}
$$

We can observe that $\det(M) = 0$ if some of the three factors of the expression is zero. The first factor $(2+A+R) \geqslant 2$ since $0 \leqslant A, R \leqslant 9$. The second factor is zero when $A = R - 2$, so $R \geqslant 2$. These cases are $2002, 2013, \ldots, 2079$. And the third factor equals zero when $A = 2$ and $R = 0$, that is, year 2020.

Having computed the values that make the determinant equal zero, we need to verify that $\operatorname{rank}(M) = \operatorname{rank}(M|N)$. The third case can be done particularly because it is unique:

$$
(M|N) \sim \left(\begin{array}{cccc|c} 2 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)
$$

Since $\operatorname{rank}(M) = \operatorname{rank}(M|N) = 2$, year 2020 is a *very good* year.

For the second case, we need to calculate the rank of the matrices depending on R:

$$
\begin{aligned}
(M|N) &\sim \left(\begin{array}{cccc|c} 2 & 0 & R-2 & R & 2 \\ R & 2 & 0 & R-2 & 0 \\ R-2 & R & 2 & 0 & R-2 \\ 0 & R-2 & R & 2 & R \end{array}\right) \\
&\sim \left(\begin{array}{cccc|c} R & R & R & R & R \\ R & 2 & 0 & R-2 & 0 \\ R-2 & R & 2 & 0 & R-2 \\ 0 & R-2 & R & 2 & R \end{array}\right) \\
&\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 2-R & -R & -2 & -R \\ 0 & 2 & 4-R & 2-R & 0 \\ 0 & R-2 & R & 2 & R \end{array}\right) \\
&\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 2-R & -R & -2 & -R \\ 0 & R & 4 & 4-R & R \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2-\frac{R}{2} & 1-\frac{R}{2} & 0 \\ 0 & R & 4 & 4-R & R \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2-\frac{R}{2} & 1-\frac{R}{2} & 0 \\ 0 & 0 & 4-2R+\frac{R^2}{2} & 4-2R+\frac{R^2}{2} & R \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)
\end{aligned}
$$

Since $4 - 2R + \dfrac{R^2}{2} \neq 0$ for integer values of R between 2 and 9, we have that $\operatorname{rank}(M) = \operatorname{rank}(M|N) = 3$.

To sum up, the list of *very good* years within the 21st century are: 2002, 2013, 2020, 2024, 2035, 2046, 2057, 2068 y 2079.