Let A be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A$$

(b) Express X in terms of A.

Solution:

Supposing that there existw a matrix $Y \neq X$ such that $Y + AY + YA^2 = A$. Since X also satisfies the equation, then

$$X + AX + XA^2 - (Y + AY + YA^2) = 0 \iff (X - Y) + A(X - Y) + (X - Y)A^2 = 0$$

Let $Z = X - Y$. It followa that $Z \neq 0$ due to $X \neq Y$. Hence

$$\begin{aligned} Z + AZ + ZA^2 = 0 \Longrightarrow A^2(Z + AZ + ZA^2)A &= 0 \\ \iff A^2ZA + A^3ZA + A^2ZA^3 &= 0 \\ \iff A^2ZA &= 0 \end{aligned}$$

$$\begin{aligned} Z + AZ + ZA^2 = 0 \Longrightarrow A(Z + AZ + ZA^2)A &= 0 \\ \iff AZA + A^2ZA + AZA^3 &= 0 \\ \iff AZA &= 0 \end{aligned}$$

$$\begin{aligned} Z + AZ + ZA^2 = 0 \Longrightarrow (Z + AZ + ZA^2)A &= 0 \\ \iff ZA + AZA + ZA^3 &= 0 \\ \iff ZA = 0 \Longrightarrow ZA^2 &= 0 \end{aligned}$$

$$\begin{aligned} Z + AZ + ZA^2 = 0 \Longrightarrow A^2(Z + AZ + ZA^2) &= 0 \\ \iff A^2Z + A^3Z + A^2ZA^2 &= 0 \\ \iff A^2Z &= 0 \end{aligned}$$

$$\begin{aligned} Z + AZ + ZA^2 = 0 \Longrightarrow A(Z + AZ + ZA^2) &= 0 \\ \iff AZ + A^2Z + AZA^2 &= 0 \\ \iff AZ &= 0 \end{aligned}$$

Therefore, we have concluded that $AZ = 0$ and $ZA^2 = 0$. Then

$$Z + AZ + ZA^2 = 0 \iff Z = 0$$

Contradiction. So $X = Y$ and thus if X satisfies the equation, it is unique.

The same process applies to find X in terms of A:

$$\begin{aligned} X + AX + XA^2 = A \Longrightarrow A^2(X + AX + XA^2)A = A^4 = 0 \\ \Longleftrightarrow A^2XA + A^3XA + A^2XA^3 = 0 \\ \Longleftrightarrow A^2XA = 0 \end{aligned}$$

$$\begin{aligned} X + AX + XA^2 = A \Longrightarrow A(X + AX + XA^2)A = A^3 = 0 \\ \Longleftrightarrow AXA + A^2XA + AXA^3 = 0 \\ \Longleftrightarrow AXA = 0 \end{aligned}$$

$$\begin{aligned} X + AX + XA^2 = A \Longrightarrow (X + AX + XA^2)A = A^2 \\ \Longleftrightarrow XA + AXA + XA^3 = A^2 \\ \Longleftrightarrow XA = A^2 \Longrightarrow XA^2 = A^3 = 0 \end{aligned}$$

$$\begin{aligned} X + AX + XA^2 = A \Longrightarrow A^2(X + AX + XA^2) = A^3 = 0 \\ \Longleftrightarrow A^2X + A^3X + A^2XA^2 = 0 \\ \Longleftrightarrow A^2X = 0 \end{aligned}$$

$$\begin{aligned} X + AX + XA^2 = A \Longrightarrow A(X + AX + XA^2) = A^2 \\ \Longleftrightarrow AX + A^2X + AXA^2 = A^2 \\ \Longleftrightarrow AX = A^2 \end{aligned}$$

And we have $AX = A^2$ y $XA^2 = 0$, so

$$X + AX + XA^2 = A \Longrightarrow X + A^2 = A \Longrightarrow X = A - A^2$$

From another point of view, we can define $X = aI + bA + cA^2$, for some $a, b, c \in \mathbb{R}$. And then

$$\begin{aligned} X + AX + XA^2 &= aI + bA + cA^2 + aA + bA^2 + cA^3 + aA^2 + bA^3 + cA^4 = \\ &= aI + bA + cA^2 + aA + bA^2 + aA^2 = \\ &= aI + (a+b)A + (a+b+c)A^2 = \\ &= A \iff \begin{cases} a & = 0 \\ a + b & = 1 \\ a + b + c & = 0 \end{cases} \iff \begin{cases} a = 0 \\ b = 1 \\ c = -1 \end{cases} \end{aligned}$$

And finally, we get $X = A - A^2$.