Kelvin Wakes

ES 123 Final Project

Alessio Pignatelli, Egemen Bostan

April 2021

Abstract

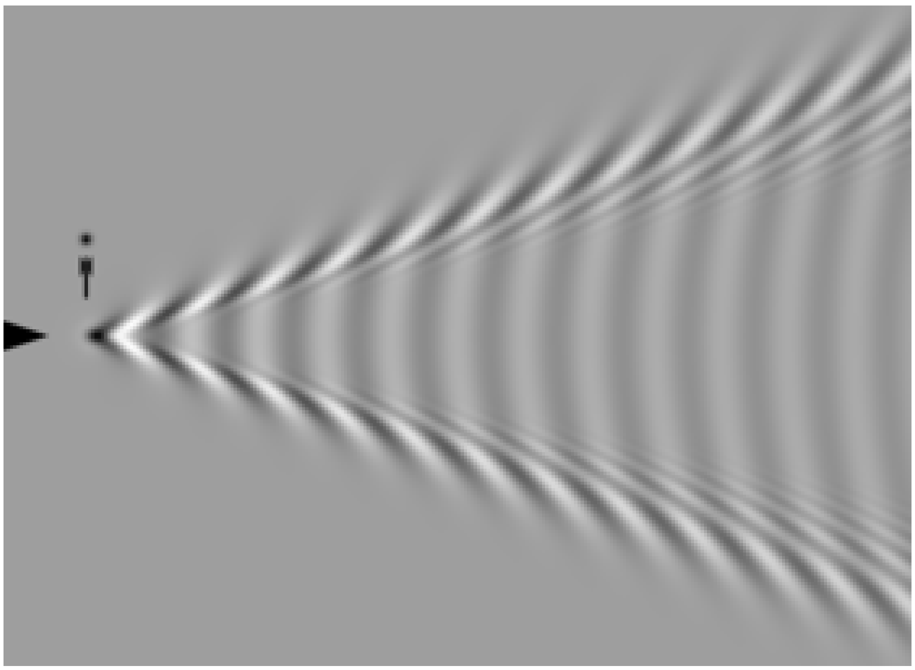


Figure 1: When a ship (or any other object) moves on the surface of water, it produces a V-shaped wake behind it. Image from [21].

In this project, we analyze the wakes that form as a ship (or any other object) travels in deep water. It is well-known that (under some assumptions) the wakes form a pattern that is shown in Figure 1, with the angle α known as the Kelvin's angle that takes a value of $\alpha \approx 19.47°$. We discuss the derivation of such angle, as well as the validity of the assumptions made in the first section of this report. We then proceed with simplified models and simulations of the phenomenon, evaluating for each their validity and limitations. Our attempt is to visualize how dispersion, the property of media where the velocity of waves depend on the wavelength, allows for the formation of pattern of interference, which constitute the Kelvin wake. Figure 3 shows our results for the 1D model, while figure 4 focuses on the 2D system and the difficulty in modelling dispersion using Fourier Transform. The third section of the report presents the results of the simple experiments we conducted at the pool. We recorded videos of wake formation both in a kiddie pool, which has depth = 30 cm, and in a regular pool of depth 1.5 m. Although we could make quantitative statement about the experiment, we can clearly see that the wake pattern does form, however we can see that the half-angle between the wedges is different in the two cases. This was to be expected: while the Kelvin Model assumes infinite depth, at lower depth the angle also depend on the depth of the water, because the wave frequencies do as well.

1 Kelvin Wake Derivation

This derivation is largely based on [The16]. We'll use in-text citations for other sources when needed.

1.1 Phase velocity and group velocity

As the ship moves and waves are generated from it, there are two relevant velocities that we need to consider: the phase velocity and the group velocity. We can think of the phase velocity as the velocity at which a specific peak (could be any specific point) propagates along the wave. Instead, by group velocity we refer to the velocity at which a set of waves within a frequency band travels as a group. [San15]
We will assume linear waves, which implies amplitude and slope of the waves are small, which is reasonable for calm water. Let us write down the equation for the elevation of the wave with respect to the zero level of the sea without waves. Letting $\eta(x,t)$ be such elevation as a function of position x and time t, we can write

$$\eta(x,t) = \epsilon \cos(kx - \omega t)$$

where ϵ is simply a multiplicative factor accounting for the amplitude of the wave, while the wavenumber k is defined as

$$k = \frac{2\pi}{\lambda}$$

where λ is the wavelength. ω is the frequency.
From the equation for $\eta(x,t)$ it is clear that the peaks of the wave move to the right with velocity $\frac{\omega}{k}$. Looking back at our definition of phase velocity, we can conclude that

$$v_{phase} = \frac{\omega}{k}$$

Now, let us consider a group of wave that move close together, so that we can determine the group velocity. For simplicity, let us look at only two waves. We can write

$$\eta(x,t) = \epsilon \cos(k_1 x - \omega_1 t) + \epsilon \cos(k_0 x - \omega_0 t)$$

for waves with wavenumbers k_1 and k_0 and frequencies ω_1 and ω_0 respectively. Recalling the trig identity for addition of cosines, we can rewrite the above as

$$\eta(x,t) = 2\epsilon \cos\left(\frac{k_1 + k_0}{2}x - \frac{\omega_1 + \omega_0}{2}t\right) \cos\left(\frac{k_1 - k_0}{2}x - \frac{\omega_1 - \omega_0}{2}t\right)$$

The first cosine term describes waves with wavenumber and frequency being the average of those of the original waves. The second cosine term instead gives waves with a wavenumber that is half the difference of the original two, this correspond to a large wavelength if the two original numbers are close as is the case for waves in the same group. Such wave is the group wave. We can consider the smaller waves as a group, represented by a single wave of large wavelength.
Given that the second cosine term describes the wave group, then the group velocity is simply given by

$$v_{group} = \frac{\omega_1 - \omega_0}{k_1 - k_0}$$

Regarding the frequency as a function of the wavenumber, and taking the limit of $k_1 \to k_0$ we have

$$v_{group} = \lim_{k_1 \to k_0} \frac{\omega(k_1) - \omega(k_0)}{k_1 - k_0} = \frac{d\omega}{dk}$$

For waves in infinitely deep water (We will verify the impact of these assumption in the "Experiment" section) we know [Per10] that

$$\omega = \sqrt{gk}$$

where g is simply the gravitational acceleration. The above implies that

$$v_{phase} = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}}$$

and

$$v_{group} = \frac{d}{dk}\left(\sqrt{gk}\right) = \frac{1}{2}\sqrt{\frac{g}{k}}$$

Hence we have

$$v_{group} = \frac{1}{2} v_{phase}$$

1.2 Kelvin Angle Derivation

Let us carefully look at the statement of the problem as it appears from Figure 1: we are looking for a pattern of waves that look fixed from the point of view of an observer on the boat. Now, that means that the boat needs to move at a velocity such that it keeps up with the movement of the peaks of the waves. If the boat is moving in the direction of propagation of the waves, then it must move at the same speed as the wave, so that the waves appear stationary, hence fixed, in the frame of the boat. If instead the boat is moving at an angle, then it has to move faster in order to keep up with the phase velocity of the wave. Looking at the left side of following diagram

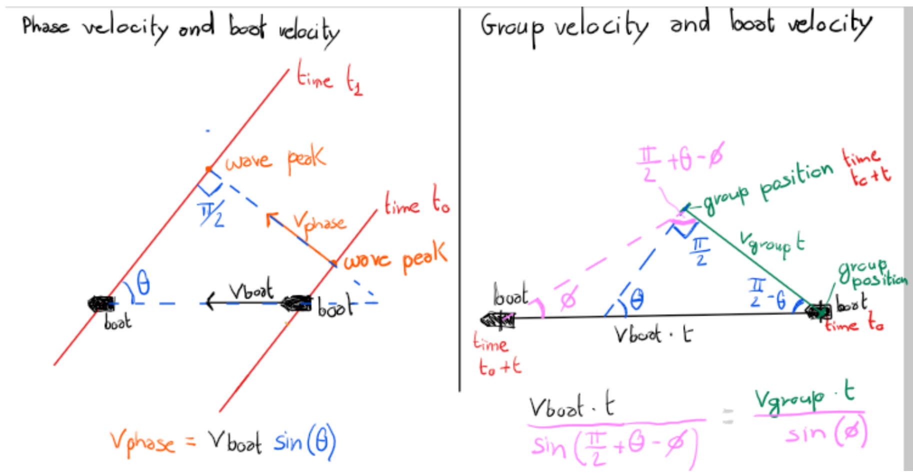


Figure 2: Relationship of the boat velocity with the phase velocity (left) and the group velocity (right)

we can trivially write down a relationship between the speed of the boat, the phase velocity of the wave and the angle between the direction of these two velocities, which we call θ. We have

$$\frac{v_{phase}}{v_{boat}} = \sin(\theta)$$

Now, let us consider the group of waves in the wave, they have been generated from the boat and have moved in a straight line from the point of generation. After some time t they have moved a distance $v_{group}t$ while the boat has moved a distance $v_{boat}t$. The right side of Figure 2 better explains the geometry.
In the diagram, we are labelling as ϕ the angle between the current position of the ship and the position of the wave, while we keep labelling as θ the angle between the direction of the boat velocity and the direction of velocity of the wave. The other angles are obtained just by addition identities for the angles of triangles. We can now use the Law of Sines, which implies

$$\frac{v_{group}t}{\sin(\phi)} = \frac{v_{boat}t}{\sin\left(\frac{\pi}{2}+\theta-\phi\right)}$$

Now, let us concisely write down a system of equations including the two relationships we have just derived from the geometry of the waves and the relationship we previously derived by the phase and group velocities. We have

$$\begin{cases} \frac{v_{group}t}{\sin(\phi)} = \frac{v_{boat}t}{\sin\left(\frac{\pi}{2}+\theta-\phi\right)} \\ \frac{v_{phase}}{v_{boat}} = \sin(\theta) \\ v_{group} = \frac{1}{2}v_{phase} \end{cases}$$

We can rearrange and simplify the first equation of the system as

$$\frac{v_{group}}{v_{boat}} = \frac{\sin(\phi)}{\sin\left(\frac{\pi}{2}+\theta-\phi\right)}$$

Now, plugging in the third equation in the LHS of the above yields

$$\frac{v_{phase}}{2v_{boat}} = \frac{\sin(\phi)}{\sin\left(\frac{\pi}{2}+\theta-\phi\right)}$$

Then, using the second equation we can write the above as

$$\frac{1}{2}\sin(\theta) = \frac{\sin(\phi)}{\sin\left(\frac{\pi}{2}+\theta-\phi\right)}$$

which we can rearrange as

$$\sin\left(\frac{\pi}{2}+\theta-\phi\right)\sin(\theta) = 2\sin(\phi)$$

Now, using trigonometry it can be shown via lengthy but fairly straight-forward calculations that the above is equivalent to

$$\frac{1}{2}\left(\sin(\phi)+\sin(2\theta-\phi)\right) = 2\sin(\phi)$$

which yields

$$\sin(\phi) = \frac{\sin(2\theta-\phi)}{3}$$

Looking back at our definition of ϕ as well as Figure 1, we know that

$$\alpha = max(\phi)$$

which also implies since $\alpha < 90°$

$$\sin(\alpha) = max(\sin(\phi)) = \frac{1}{3}$$

Hence we can conclude that

$$\alpha = \sin^{-1}\left(\frac{1}{3}\right) \approx 19.47°$$

2 Modeling

In this section, we will attempt to verify this solution through modeling and experimentation. As evident from the initial derivation, the models will center around the solutions to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

where c is the wave speed. We will build upon this model to reach the desired solutions.

2.1 1D

One dimensional space is the simplest starting point for modeling and will provide insights for higher dimensions.

2.1.1 Non-dispersive medium

In a non-dispersive medium, the wave velocity (c) is constant. This represents the simplest form of the wave equation. In this form, analytical solutions are available, yet a model also has to start here.

Because the surface gravity waves have no constant speed, instead of setting an arbitrary speed, we investigate acoustic waves, which obey the same wave equation. This will also help validate the findings since we are familiar with the uniform spread of acoustic waves. Let $u(x,t)$ be the pressure that will describe the acoustic wave.

We introduce a term $s(x,t)$ that expresses the acceleration undergone by $u(x,t)$ due to external excitations. Note that it can vary in space and time. Hence our equation becomes:

$$\partial_t^2 u(x,t) = c^2\ \partial_x^2 u(x,t) + s(x,t).$$

We utilize the 3-point finite difference method to calculate second derivatives of time and space as described in Igel's Computational Seismology. [Ige16]. An example for the time derivative is shown below. Note that n refers to the step and i refers to a specific position in x.

Integration scheme:

$$u_i^{n+1} \;=\; c_i^2 \mathrm{d}t^2 \left[\partial_x^2 u\right] + 2u_i^n - u_i^{n-1} + \mathrm{d}t^2 s_i^n.$$

Space derivatives:

$$\partial_x^2 u \;=\; \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\mathrm{d}x^2}.$$

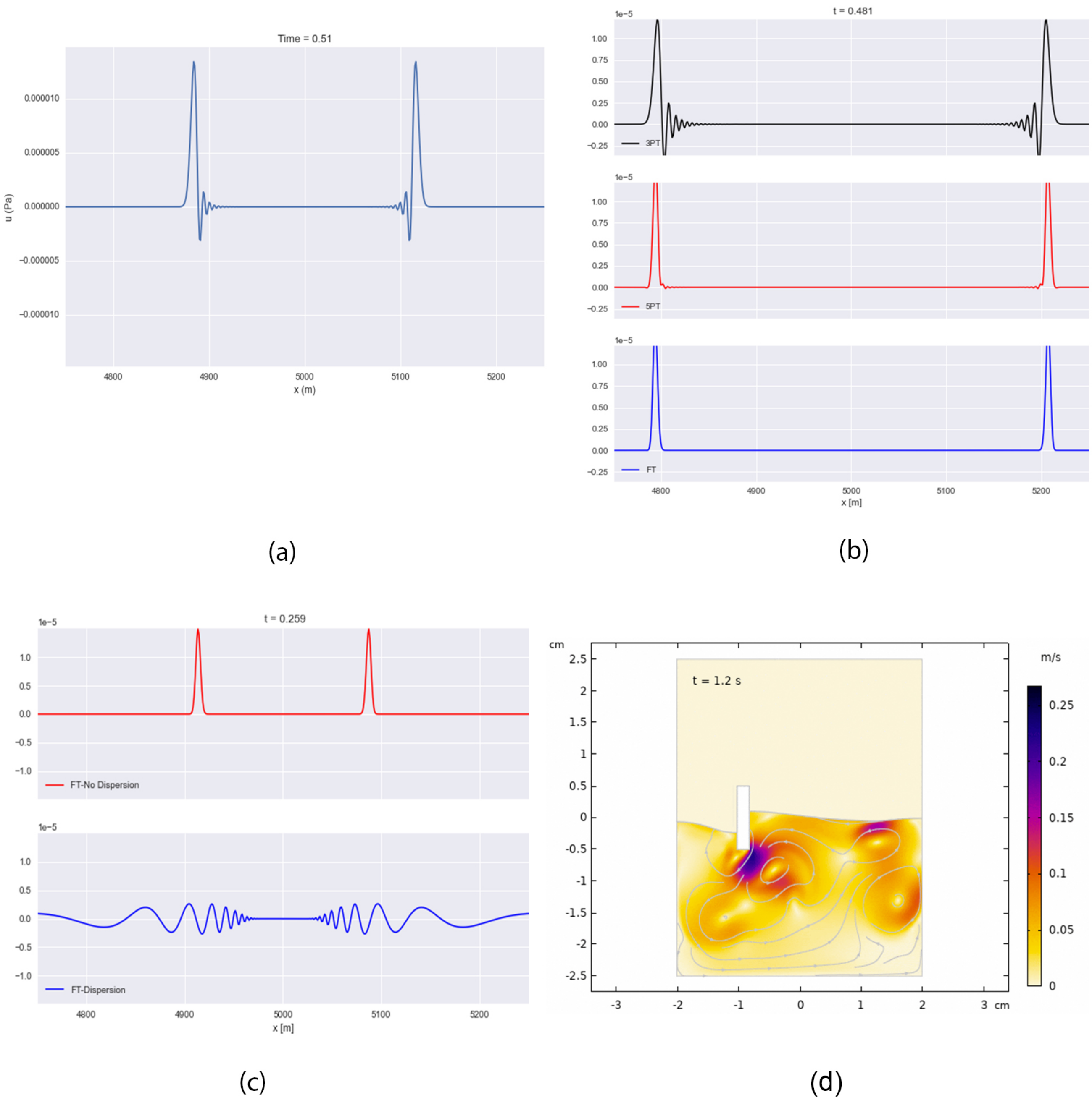


Figure 3: Various plots for 1 dimensional wave equation models.
(a) Two identical shapes travelling away in opposite direction from the source at $x = 5000m$. Note that initial shape was a Gaussian curve and that the smaller waves are a result of numerical dispersion.
(b) Three methods are compared. Note that the Fourier Transform method shows less dispersion than 5-point finite differences method, which is significantly better than the 3-point finite differences method.
(c) Dispersive medium solution is in blue. Since wavenumber is inversely proportional with wave speed squared, larger wavelength components spread faster.
(d) We could not run the CFD Simulation in large enough domains to observe the dispersion of propagated waves. Note that the rectangle is the 'source' that moves and provides the external excitation.

We set boundary conditions to $u(0,t) = 0$,$u(x_{max},t) = 0$, $u^{'}(0,t) = 0$,$u^{'}(x_{max},t) = 0$ and vary the source term $s(x,t)$ or initial conditions $u(x,0)$, $u^{'}(x,0)$ to obtain the expected wave equation solution in Figure 3 (a) - waves propagate in both directions with the same speed and half the amplitude of the initial shape.

Numerical integration methods of time extrapolation for the wave equation introduces erroneous dispersion - the approximations in the finite difference or any other iterative integration based method manifest as the wave 'spreading' - which is similar to dispersion of waves in a medium. To minimize the effects of this error, we can improve the numerical method's accuracy by using a five-point finite difference approximation [Ige16]:

$$\partial_x^2 u(x,t) = \frac{-u(x+2\mathrm{d}x,t) + 16u(x+\mathrm{d}x,t) - 30u(x,t) + 16p(x-\mathrm{d}x,t) - u(x-2\mathrm{d}x,t)}{12\mathrm{d}x^2}$$

Another method available to minimize the errors of numerical dispersion is solving the PDE using Fourier Transform, as discussed in ES123. F.T. enables the reduction of PDEs to easily solved ODEs in fourier domain, which can be transformed back into real domain. In this case, we transform the space variable x to obtain variable k , the wavenumber. We can use F.T. to only calculate the space derivatives or completely solve the PDE in wavenumber domain, and afterwards transform to space domain. Both methods are viable, but we prefer to only calculate the space-derivatives in Fourier domain here.

$$\partial_x^2 u(x,t) = \mathscr{F}^{-1}[(ik)^2 u(k,t)] = \mathscr{F}^{-1}[-k^2 u(k,t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -k^2 u(k,t) e^{ikx} dk$$

where F is the Fourier Transform and F^{-1} is the Inverse Fourier Transform.
We implement the three methods and compare their numerical dispersion in Figure 3 (b).

2.1.2 Dispersive medium (like deep water)

As shown in the derivation for Kelvin Wake angle, in deep water bodies, surface waves have a speed $c = \sqrt{\frac{g}{k}}$. This renders the previous 3 and 5 point finite difference methods invalid since c does not depend on medium, but the nature of the waves in the 1-d space. Since F.T. calculates wavenumbers, we can modify the F.T. solution to include the k dependent c term inside the space derivative calculation using Fourier transform, as in:

$$\begin{aligned} \partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t) &= \mathscr{F}^{-1}[c^2 (ik)^2 u(k,t)] \\ &= \mathscr{F}^{-1}[-gku(k,t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -gku(k,t) e^{ikx} dk \end{aligned}$$

Having solved the right-hand side in Fourier Domain, we inverse transform and use the same numerical time-extrapolation method to get the solution with water-wave dispersion in 1D, shown compared to the FT solution of nondispersive medium in Figure 3 (c). For the sake of consistency, this solution for water waves is modified with $g = 334^2$ to replicate the effect with a similar scale in acoustic waves ($c(k=1) = c_0$). Since acoustic ways have a constant speed value to compare against, we can visualize the differences.

2.1.3 Simulation - COMSOL

Early on in our preparations, we realized that a COMSOL simulation of the phenomena was not possible given the space and time-scale (and our lack of experience with COMSOL). It was not possible for us to shrink the scale or reduce the dimensions sufficiently to reduce computational load and still observe the Kelvin wake angle. However, in this section, we present a simple 1-D simulation that closely aligned with the numerical analysis above.

Based on the methods outlined in a COMSOL library application, we model the free water surface with a Laminar 2 Phase Flow interface under the CFD module [Fon18].The resulting study is in Figure 3 (d).

2.2 2D

2-D Modeling of water surface waves proved significantly more challenging. Here we present two attempts - the Fourier Transform method outlined in 1-D modeling and the implementation of an alternative integral describing the surface from National Institute of Standards and Technology's Digital Library of Mathematical Functions [Sta].

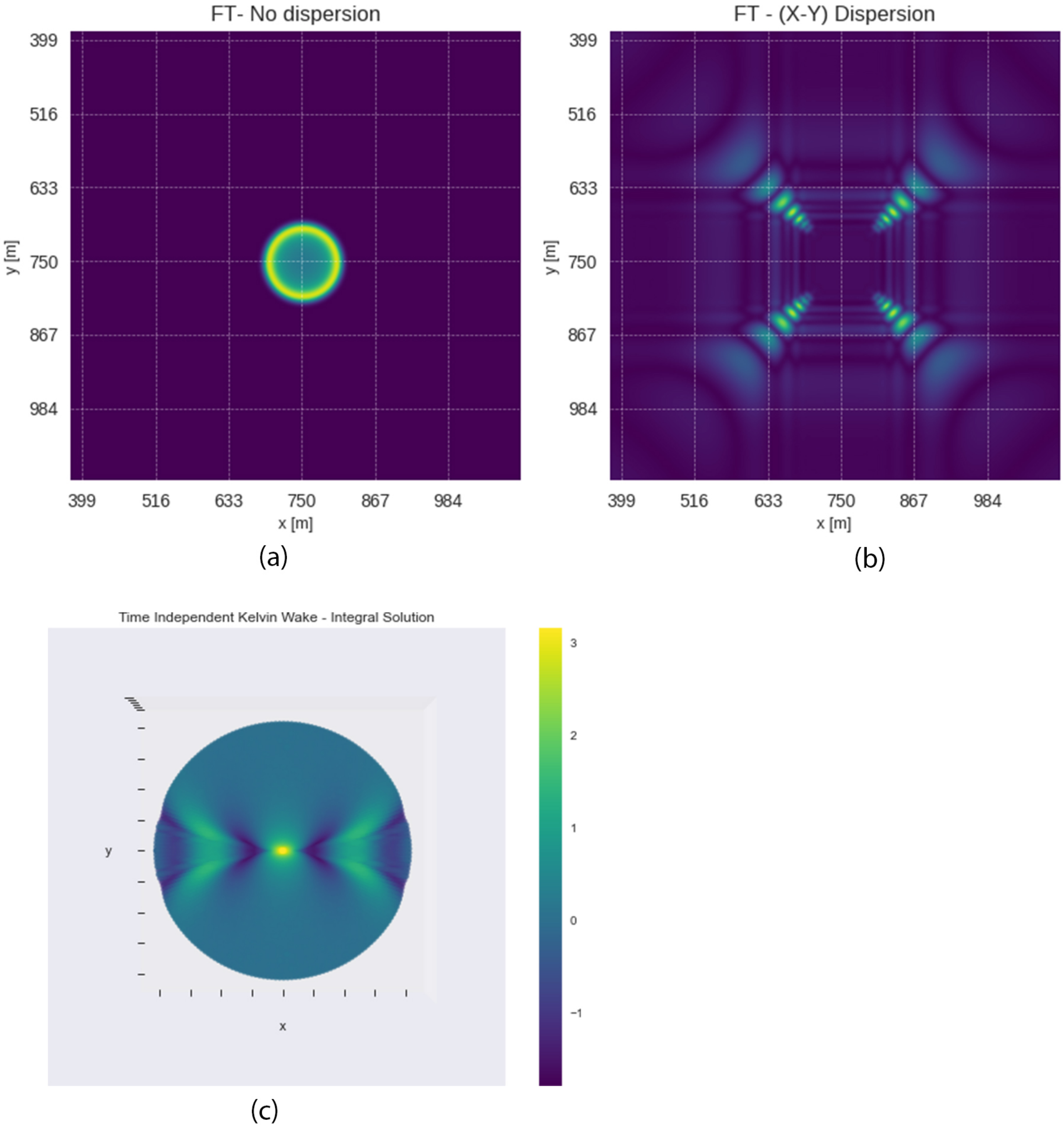


Figure 4: Various plots for 2 dimensional wave equation models.
(a - b) Comparison between no dispersion and dispersion schemes for an acoustic wave. Note the unrealistic dispersion pattern in the right plot. Since the source is radially symmetrical, we would expect radial symmetry in dispersion.
(c) 2-D Moving object reference frame solution for an object moving along x axis. Note that as r increases, there is more spatial distance between the same angular distance in a cylindrical coordinate grid, thus resolution decreases.

2.2.1 Fourier Transform Solutions

The wave equation in two dimensions is very similar:

$$\partial_t^2 u(x, y, t) = c^2\ (\partial_x^2 u(x, y, t) + \partial_y^2 u(x, y, t)) + s(x, t).$$

Thus, a natural first-step is to extend the previous 1D F.T. method to 2D. Noting the exponentially increasing computational load, we reduce the number of grid points in our model and change the source function, but beyond these changes, the model follows the same reasoning outlined in 1D. The results of this method are compared to a no dispersion F.T. scheme in Figure 4 (a) and (b).

However, this method is incorrect. When we implement F.T. with the speed term inside the transform, we treat the 2-d space as only consisting of waves along x and along y. However, the waves expand circularly- along any direction, there are waves with the same wavelength. This dispersion method 'misses' the presence of these waves, and disperses only along x and y. The result is the disappearance of the uniform circular wavefront and unrealistic dispersion in Figure 4 (b), compared to (a).

An interesting phenomenon also emerges at the diagonals of the region - where the method is most realistic since the wavefront travels equally in x and y directions. The shape looks similar to the solutions for the Kelvin Wake angle (the dark regions align with the caustic regions). However, the angle is not correct. We do not see a physical significance of this pattern, though it warrants a deeper investigation.

Solving the wave equation in 2D using cylindrical coordinates will solve this issue, as wavelength would only be a factor of r, creating an axi-symmetrical 2D system that can be reduced to a 1D system. However, in the earth reference frame, this would only capture the waves at the origin of the cylindrical coordinate system. Thus, to propagate all waves at all points, we would need to repeat the process for all grid points in the model. This immensely increases the computational load, making it unfeasible. Instead, we can focus on the ship reference frame, in which the waves are in steady-state.

2.2.2 Ship Reference Frame Solution

Since the patter nof waves do not vary over time in moving object/ship's reference frame, solving the 2-D wave equation becomes easier. Solving the wave equation in this reference frame using Fourier Tranforms yields the integral [Sta]:

$$z(\phi, \rho) = \int_{-\pi/2}^{\pi/2} \cos\left(\rho \frac{\cos(\theta + \phi)}{\cos^2\theta}\right) \mathrm{d}\theta$$

where $\rho = \frac{gr}{V^2}$, r, ϕ, z are the cylindrical coordinates, θ is a dummy variable for integration and V is the magnitude of ship velocity.

From this expression, Kelvin wake angle can be also derived by looking at the stationary points for the integrand. Alternatively, for a given velocity, this integral can be evaluated to obtain the water surface levels.

There are many numerical methods for solving this integral. We've attempted to come up with custom methods, which work but are significantly slower than the numerical integrator in the SciPy package. Thus, we choose to solve the integral for $0 \leq \phi \leq 2\pi$ and $r \leq r \leq r_0$, where r_0 is arbitrary. The solution to this integral can be seen in Fig 4 (c).

Note that since the integral is time-independent, the solution includes motion in both directions. We can employ a step function to remove one 'side' of the solution. Also note that as the PDE in question is linear (in the model), we would introduce new objects/ships to the same plane by adding together individual solutions. This is equivalent to solving the wave equation in cylindrical coordinates in different points of the plane, as suggested in the previous section.

2.2.3 End Goal: GPU assisted 2D F-T solutions

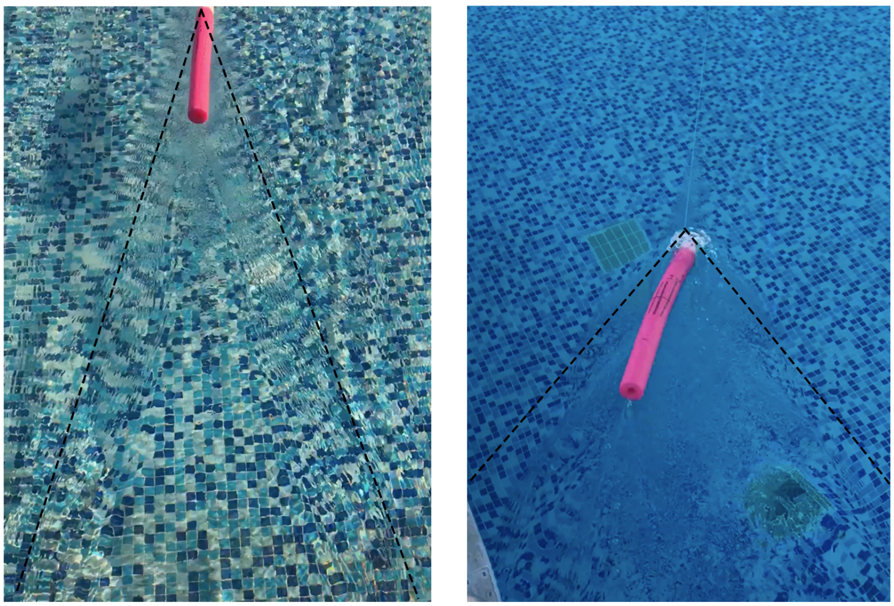
Graphical Processing Units(GPUs) of modern computers are very efficient in F.T. methods as they are a very common tool in generating graphics. Ricky Reusser on github uses F.T. in a 2-D plane to successfully time extrapolate water surface waves [Reu20]. Unlike our python based model, this model uses GPU resources to rapidly calculate Fourier transforms, which we believe are the key to the feasibility of this method. Beyond this similar method, Reusser also introduces a damping term to make sure the waves disappear over time - which is another realistic factor we have overlooked. It also enables the user to manipulate the speed of an object and the gravity constant to see the effects of speed and Froude number on the pattern of Kelvin wakes.

3 Experiment

3.1 Kelvin Angle and boat speed

In this section, we present simple experiments to observe the Kelvin wake. We pull a pool floatie across the surface of the pool with a string and record the water surface. Because it was not possible to place the video recorder vertically, we are not able to verify the wake angle, but we still observe the overall pattern.

An important assumption in the derivation has been that the water depth is large compared to the scale of surface displacements. To check the significance of this assumption, we also repeat the experiment in a kiddie pool. The wakes are presented in Figure 5.



(a) (b)

Figure 5: (a) shows the wake in a kiddie pool (depth = 30 cm) and (b) shows the wake in a regular pool (depth = 1.5 m). While perspectives are slightly different, the wake in (a) is narrower than in (b) and it is much less noticeable.

Upon experimentation, we see that the Kelvin wake is prominent and easily identifiable at large depths. However, as depth decreases, the assumptions of the calculations do not hold and the resulting pattern is less regular and has a narrower wake. That is because at lower depth, the frequency of waves also depends on the depth itself so that equation

$$\omega = \sqrt{gk}$$

does not hold anymore.

For more detailed documentation of the Kelvin wake, please refer to the video accompanying this write-up.

4 Conclusion and future development

With this project, we are able to verify the pattern that emerges out of the assumptions made to calculate the Kelvin wake and its angle. Experimental evidence suggests the validity of the shape, but a lack of angle measurement in our method limits us from making stronger conclusions.

In terms of modeling, the major insufficiency of our method has been the lack of a CFD component to replicate this phenomenon. While the reasons for this has been outlined, it remains a critical aspect to verify the validity of this 100-year old insight at a molecular level. What we have presented have been numerical solutions to an existing model for the wave, not a first-principles based recreation of the Kelvin wake, which we had hoped to do in COMSOL. This way, we would understand if the speed of waves is proportional to the square root of wavelength, as has been the central assumption of our (and Kelvin's) model.

What our models also lack is a damping term that would describe the dissipation of the waves over time - as it stands, our 'standing water' would never come to rest after an initial excitation, which is not realistic. However, as our 2D pattern is time-independent in ship reference frame, this is not a concern for Figure **??**. On the other hand, for a successful earth-frame model, we would need to account for a damping term (that could be a part of the source term $s(x, y, t)$).

Another aspect that we can hope to improve in the future is the correct 2D F.T. solution for the dispersive medium. While we have identified the problem, we could not find a computationally feasible solution. Reusser's work has achieved this and given more time, our strategy would be to reverse engineering this method and then build upon it.

Overall, we believe that this project has helped us develop an appreciation for the challenges in converting an analytical model into a numerical one. At many points, we've seen the implications of discretization and the non-trivial mathematical framework behind the conversion of analytical, continuous models into numerical, discrete ones.

References

[Per10] Joy Perkinson. *Gravity Waves: The Kelvin Wedge and Related Problems*. Dec. 2010. URL: http://web.mit.edu/joyp/Public/HarvardWork/AM201FinalPaper.pdf (visited on 04/30/2021).

[San15] Santhanam. *Group velocity and phase velocity*. Feb. 2015. URL: http://www.iiserpune.ac.in/~santh/course/qmr/gvel.pdf (visited on 04/30/2021).

[Ige16] Heiner Igel. *Computational Seismology: A Practical Introduction*. Oxford University Press, Nov. 10, 2016. 352 pp. ISBN: 978-0-19-871741-6.

[The16] TheOKstuff. *Finding the shape of a SHIP WAKE*. Mar. 24, 2016. URL: https://www.youtube.com/watch?v=-x1htub23tA&list=PLGejbB28fBlBJqEtIhxZarsuGOE-xycdt&index=2 (visited on 04/30/2021).

[Fon18] Ed Fontes. *Two Methods for Modeling Free Surfaces in COMSOL Multiphysics®*. COMSOL Multiphysics. May 15, 2018. URL: https://www.comsol.com/blogs/two-methods-for-modeling-free-surfaces-in-comsol-multiphysics/ (visited on 04/26/2021).

[Reu20] Ricky Reusser. *Dispersion in Water Surface Waves*. ObservableHQ. June 18, 2020. URL: https://observablehq.com/@rreusser/dispersion-in-water-surface-waves (visited on 04/26/2021).

[21] *Wave*. Feb. 17, 2021. URL: https://en.wikipedia.org/wiki/Wake#/media/File:Kelvin_Wake_Fr=2.png (visited on 04/30/2021).

[Sta] National Institute of Standards {and} Technology. *DLMF: 36.13 Kelvin's Ship-Wave Pattern*. DLMF NIST. URL: https://dlmf.nist.gov/36.13 (visited on 04/26/2021).