Linearly dependent and independent set

Definition Let V be a real vector space and $\{v_1, v_2, \dots, v_n\}$ be a collection of vectors in V. We say $\{v_1, v_2, \dots, v_n\}$ are linearly independent if the vector equation $x_1 v_1 + x_2 v_2 + \dots + x_n v_n = 0_V$ has only zero solution (which means $x_1 = 0; x_2 = 0; \dots x_n = 0$)

We say $\{v_1, v_2, \dots, v_n\}$ are linearly dependent if the vector equation $x_1 v_1 + x_2 v_2 + \dots + x_n v_n = 0_V$ has a non zero solution.

Note:- 1) Consider the collection $\{v_1, v_2\}$ in a real vector space V if $v_1 = k v_2$ for some scalar $k \neq 0$ then $\{v_1, v_2\}$ is a linearly dependent set.

2) Suppose $L = \{v_1, v_2, \dots, v_k\}$ is a linearly independent set of V and $S = \{w_1, w_2, \dots, w_s\}$ is a spanning subset of V Then

$$\boxed{L \subseteq S}$$

3) Any subset of a linearly independent set is linearly independent but super set need not be.

4) Any linearly independent set does not contain zero vector

Pb: 1 Is the collection of vectors $\{ v_1 = (1,0,1,2),\ v_2 = (0,1,1,2),\ v_3 = (1,1,1,3) \}$ in $\mathbb{R}^4$ linearly independent?

Ans:- we form the vector equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = (0,0,0,0)$$

$$x_1 (1,0,1,2) + x_2 (0,1,1,2) + x_3 (1,1,1,3) = (0,0,0,0)$$

we get

$$x_1 + x_3 = 0 \rightarrow (1)$$

$$x_2 + x_3 = 0 \rightarrow (2)$$

$$x_1 + x_2 + x_3 = 0 \rightarrow (3)$$

$$2x_1 + 2x_2 + 3x_3 = 0 \rightarrow (4)$$

From (1), (2), we have $x_1 = -x_3$; $x_2 = -x_3$. Apply this in (3)

we get $\boxed{x_3 = 0}$ thus $\boxed{x_1 = 0;\ x_2 = 0}$

Hence $\{v_1, v_2, v_3\}$ are linearly independent.

Pb: 2 Is the set $S = \{t^2+1,\ t-1;\ 2t+2\}$ linearly independent in $P_2(\mathbb{R})$?

Ans:- Take $p_1 = t^2+1$; $p_2 = t-1$; $p_3 = 2t+2$

Form the vector equation $x_1 p_1 + x_2 p_2 + x_3 p_3 = 0$

$\Rightarrow x_1(t^2+1) + x_2(t-1) + x_3(2t+2) = 0$

$\Rightarrow x_1 t^2 + (x_2 + 2x_3)t + (x_1 - x_2 + 2x_3) = 0$

we have $x_1 = 0; \rightarrow (1)$

$x_2 + 2x_3 = 0 \rightarrow (2)$

$x_1 - x_2 + 2x_3 = 0 \rightarrow (3)$. Since $x_1 = 0$ we have $-x_2 + 2x_3 = 0 \rightarrow (4)$

$(2) + (4) \Rightarrow 4x_3 = 0 \Rightarrow x_3 = 0$. From (2); $x_2 = 0$.

Hence $\{t^2+1;\ t-1;\ 2t+2\}$ is a linear independent set.

Pb: 3 Is the set $\{p_1 = t^2+t+2;\ p_2 = 2t^2+t;\ p_3 = 3t^2+2t+2\}$ linearly independent in $P_2(\mathbb{R})$?

Ans:- From the equation $x_1p_1 + x_2p_2 + x_3p_3 = 0$

$$\Rightarrow x_1(t^2+t+2) + x_2(2t^2+t) + x_3(3t^2+2t+2) = 0$$

$$\Rightarrow (x_1+2x_2+3x_3)t^2 + (x_1+x_2+2x_3)t + (2x_1+2x_3) = 0$$

We get

$$x_1 + 2x_2 + 3x_3 = 0 \rightarrow (1)$$

$$x_1 + x_2 + 2x_3 = 0 \rightarrow (2)$$

$$2x_1 \qquad + 2x_3 = 0 \rightarrow (3)$$

We solve by Gauss elimination

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -4 & -4 & 0 \end{bmatrix} \quad R_3 \to R_3 - 4R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

. The reduced system is

$$x_1 + 2x_2 + 3x_3 = 0$$
$$-x_2 - x_3 = 0$$

This system has infinite solution. A non zero solution is

$$\boxed{x_1 = -1;\ x_2 = -1;\ x_3 = 1}$$

Thus $\{p_1, p_2, p_3\}$ is linearly dependent.

Pb. 4 Determine if $S = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ is linearly dependent or independent?

Soln:-

$$x_1 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} + x_3 \begin{bmatrix} -3 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + 2x_2 - 3x_3 & 2x_1 - x_2 + x_3 \\ x_1 + x_2 & 2x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we get
$$x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0$$
$$x_1 + x_2 = 0$$
$$2x_2 + x_3 = 0$$

. We solve by Gauss elimination

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \quad \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -5 & 7 & 0 \\ 0 & -1 & +3 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & -5 & 7 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 5R_2 \\ R_4 \to R_4 + 2R_2 \end{matrix} \quad \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \quad R_4 \to 8R_4 + 7R_3$$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced system is
$$x_1 + 2x_2 - 3x_3 = 0$$
$$-x_2 + 3x_3 = 0$$
$$-8x_3 = 0$$

Hence $x_1 = 0;\ x_2 = 0;\ x_3 = 0$

Hence S is a linearly independent set.