Magnitude or length of a vector

Let V be an inner product space. The magnitude or length of a vector x denoted by $\|x\|$ is defined as $\|x\| = \sqrt{\langle x, x \rangle}$

The distance between two vectors x and y denoted by $d(x,y)$ is defined as

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y \rangle}$$

Angle between two vectors The real number θ in the interval $[0, \pi]$ that satisfies

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \, \|y\|}$$

is the angle between x and y.

Pb:1 Consider the inner product space $\mathbb{R}^3$ with standard inner product.

Find the length of $(3,5,-4)$ and

Find the distance between $(1,3,-2)$, $(0,1,0)$

Soln:- For $\bar{x}=(x_1,x_2,x_3)$ and $\bar{y}=(y_1,y_2,y_3)$

$$\langle \bar{x},\bar{y}\rangle = x_1y_1+x_2y_2+x_3y_3$$

(i) Take $\bar{x}=(3,5,-4)$

$$\text{length of } \bar{x} = \sqrt{\langle \bar{x},\bar{x}\rangle} = \sqrt{\langle (3,5,-4),(3,5,-4)\rangle}$$

$$= \sqrt{3\cdot 3+5\cdot 5+(-4)(-4)}$$

$$= \sqrt{9+25+16} = 5\sqrt{2}$$

(ii) Take $\bar{x}=(1,3,-2)$; $\bar{y}=(0,1,0)$

$$d(\bar{x},\bar{y}) = \|\bar{x}-\bar{y}\| = \|(1,3,-2)-(0,1,0)\|$$

$$= \|(1,2,-2)\| = \sqrt{\langle (1,2,-2),(1,2,-2)\rangle}$$

$$= \sqrt{1\cdot 1+2\cdot 2+(-2)\cdot(-2)} = \sqrt{1+4+4}$$

$$= 3.$$

Pb:2 Consider the inner product space $\mathbb{R}^3$ with inner product.

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + 3x_2y_2 + 5x_3y_3$$

Find the angle between $(1,2,3)$ and $(2,-1,3)$

Soln:-

take $\bar{x} = (1,2,3)$; $\bar{y} = (2,-1,3)$

$$\|\bar{x}\| = \sqrt{\langle \bar{x}, \bar{x} \rangle} = \sqrt{\langle (1,2,3), (1,2,3) \rangle} = \sqrt{1^2 + 3\cdot 2^2 + 5\cdot 3^2}$$

$$= \sqrt{1+12+45} = \sqrt{58}$$

$$\|y\| = \sqrt{\langle \bar{y}, \bar{y} \rangle} = \sqrt{\langle (2,-1,3), (2,-1,3) \rangle} = \sqrt{2^2 + 3\cdot(-1)^2 + 5\cdot 3^2}$$

$$= \sqrt{4+3+45} = \sqrt{52}$$

$$\langle \bar{x}, \bar{y} \rangle = \langle (1,2,3), (2,-1,3) \rangle = 1\cdot 2 + 3\cdot 2\cdot(-1) + 5\cdot 3\cdot 3$$

$$= 2 - 6 + 45 = 41$$

We know, $\cos\theta = \dfrac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\|\|\bar{y}\|} = \dfrac{41}{\sqrt{58}\sqrt{52}}$

$$\boxed{\theta = \cos^{-1}\left(\frac{41}{\sqrt{58}\sqrt{52}}\right)}$$

Pb: 2 Consider the inner product space $P_3(\mathbb{R})$ with inner product $\langle f(t), g(t) \rangle = \int_0^1 f(t) g(t)\, dt$

Find the angle between $2t^2$, $3t^3$

Soln:- Take $f(t) = 2t^2$; $g(t) = 3t^3$

$$\|f\| = \sqrt{\langle f(t), f(t) \rangle} = \sqrt{\langle 2t^2, 2t^2 \rangle} = \sqrt{\int_0^1 2t^2 . 2t^2 dt}$$

$$= 2\sqrt{\int_0^1 t^4 dt} = 2\sqrt{\left[\frac{t^5}{5}\right]_0^1}$$

$$= 2/\sqrt{5}$$

$$\|g\| = \sqrt{\langle g(t), g(t) \rangle} = \sqrt{\langle 3t^3, 3t^3 \rangle} = \sqrt{\int_0^1 3t^3 . 3t^3 dt}$$

$$= 3\sqrt{\int_0^1 t^6 dt} = 3\sqrt{\left[t^7/7\right]_0^1} = 3/\sqrt{7}$$

$$\langle f(t), g(t) \rangle = \int_0^1 2t^2 . 3t^3 dt = 6\int_0^1 t^5 dt = 6\left[\frac{t^6}{6}\right]_0^1 = 1$$

$$\cos\theta = \frac{\langle f(t), g(t) \rangle}{\|f\| \, \|g\|} = \frac{1}{(2/\sqrt{5})(3/\sqrt{7})}$$

$$= \frac{\sqrt{35}}{6}$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{35}}{6}\right)$$

Pb: 3 Let x and y be two vectors in an inner product space V. Suppose $\|x\|=1$; $\|y\|=1$; $\langle x, y\rangle = -1/2$ then find $\|x-y\|$.

Soln:-

$$\|x-y\| = \sqrt{\langle x-y, x-y\rangle} = \sqrt{\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle}$$

$$= \sqrt{\|x\|^2 - \langle x, y\rangle - \langle x, y\rangle + \|y\|^2}$$

$$= \sqrt{\|x\|^2 - 2\langle x, y\rangle + \|y\|^2}$$

Given $\|x\|=1$; $\|y\|=1$; $\langle x, y\rangle = -1/2$ then

$$\|x-y\| = \sqrt{1 - 2\cdot\left(\frac{-1}{2}\right) + 1} = \sqrt{1+1+1} = \sqrt{3}.$$

Pb:4 (Cauchy Schwartz inequality)

If x and y are two vectors in an inner product Space V then Show that $\langle x, y\rangle^2 \leq \langle x, x\rangle \langle y, y\rangle$

Soln:-

Recall:- $\langle x, 0_V\rangle = 0$ for any $x \in V$

$\langle 0_V, y\rangle = 0$ for any $y \in V$

Suppose either $x = 0_V$ or $y = 0_V$ then

$\langle x, y\rangle = 0$ and either $\|x\| = 0$ or $\|y\| = 0$

Hence, $\langle x, y\rangle^2 \leq \langle x, x\rangle \langle y, y\rangle$

Assume $x \neq 0_V$ and $y \neq 0_V$. For any $t \in \mathbb{R}$ we observe the following

$$0 \leq \langle tx + y, tx + y\rangle$$
$$= \langle tx, tx\rangle + \langle tx, y\rangle + \langle y, tx\rangle + \langle y, y\rangle$$
$$= t^2 \langle x, x\rangle + t\langle x, y\rangle + \langle tx, y\rangle + \|y\|^2$$
$$0 \leq t^2 \|x\|^2 + 2t\langle x, y\rangle + \|y\|^2 \rightarrow (1)$$

Equation (1) represents a quadratic equation interms of the variable 't'. From (1) it is clear that (1) has either no real root or repeated roots.

Hence we have

$$(2\langle x, y\rangle)^2 - 4 \cdot \|x\|^2 \|y\|^2 \leq 0$$
$$4\langle x, y\rangle^2 - 4\|x\|^2 \|y\|^2 \leq 0$$
$$4\langle x, y\rangle^2 \leq 4\|x\|^2 \|y\|^2$$
$$\Rightarrow \langle x, y\rangle^2 \leq \|x\|^2 \|y\|^2.$$

Hence proved

Pb: 5 Let V be an inner product space. If $x, y \in V$ then show that

(i) $\langle x, y \rangle \leq \|x\| \|y\|$

(ii) $\|x+y\| \leq \|x\| + \|y\|$

(iii) $\big| \|x\| - \|y\| \big| \leq \|x-y\|$

Soln:- We know that

(i) $\langle x, y \rangle^2 \leq \|x\|^2 \|y\|^2$

Take square root on both sides.

$\langle x, y \rangle \leq \|x\| \|y\|$

(ii) $\|x+y\| = \sqrt{\langle x+y, x+y \rangle}$

$= \sqrt{\|x\|^2 + 2\langle x, y \rangle + \|y\|^2} \rightarrow \circledast$

from (i), $\langle x, y \rangle \leq \|x\| \|y\|$ apply this in $\circledast$

we have

$\|x+y\| \leq \sqrt{\|x\|^2 + 2\|x\|\|y\| + \|y\|^2}$

$\leq \sqrt{(\|x\| + \|y\|)^2}$

$\therefore \|x+y\| \leq \|x\| + \|y\|.$

(iii) (try!)