Module - 6

Orthogonal matrix Let A be a square matrix. A is called as orthogonal if one of the following property holds

(i) Column vectors of A are orthonormal.

(ii) Row vectors of A are orthonormal.

(iii) $AA^T = I_n$ (I_n - identity matrix of order n)

(iv) $A^T A = I_n$

(v) $A^{-1} = A^T$

Example:- 1) Let $A = \begin{bmatrix} \frac{1}{\sqrt{10}} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$ then $A^T = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence A is an orthogonal matrix.

2) Consider $P = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for $0 < \theta < \pi/2$.

$P^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $P^T P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Here P is an orthogonal matrix.

Orthogonal linear transformation A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be orthogonal linear transformation if $\|T(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$

Example:—

$T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = \left(\frac{x - 3y}{\sqrt{10}}, \frac{3x + y}{\sqrt{10}} \right)$

<u>QR</u> <u>Factorization</u> If A is a $m \times n$ matrix of rank n then A can be factored into a product QR where Q is a $m \times n$ matrix with orthonormal column vectors and R is an upper triangular $n \times n$ matrix whose diagonal entries are always positive

[<u>Note!</u>— R must be invertible]

Procedure for QR factorization

Let $A = [v_1 \; v_2 \ldots, v_n]$ be a $m \times n$ matrix. Here we assume that $v_1, v_2, \ldots, v_n$ are all linearly independent

Suppose $Q = [q_1 \; q_2 \ldots q_n]$ then

$$q_1 = \frac{v_1}{\|v_1\|}$$

$$q_2 = \frac{v_2 - \langle v_2, q_1 \rangle q_1}{\| v_2 - \langle v_2, q_1 \rangle q_1 \|}$$

$$q_3 = \frac{v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2}{\| v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2 \|}$$

$$\vdots$$

$$q_n = \frac{v_n - \langle v_n, q_1 \rangle q_1 - \langle v_n, q_2 \rangle q_2 - \cdots - \langle v_n, q_{n-1} \rangle q_{n-1}}{\| v_n - \langle v_n, q_1 \rangle q_1 - \langle v_n, q_2 \rangle q_2 - \cdots \langle v_n, q_{n-1} \rangle q_{n-1} \|}$$

$$R = \begin{bmatrix} \langle v_1, q_1 \rangle & \langle v_2, q_1 \rangle & \langle v_3, q_1 \rangle & \cdots & \langle v_n, q_1 \rangle \\ 0 & \langle v_2, q_2 \rangle & \langle v_3, q_2 \rangle & \cdots & \langle v_n, q_2 \rangle \\ 0 & 0 & \langle v_3, q_3 \rangle & \cdots & \langle v_n, q_3 \rangle \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \langle v_n, q_n \rangle \end{bmatrix}$$

Pb:1 Find QR factorization of $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & -2 & 2 \end{bmatrix}$

Soln:- $v_1 = (1,-1,-1)$; $v_2 = (0,2,-2)$; $v_3 = (2,0,2)$

$\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = \sqrt{1^2+(-1)^2+(-1)^2} = \sqrt{3}$

$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$v_2 - \langle v_2, q_1\rangle q_1 = (0,2,-2) - \left\langle (0,2,-2)\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)\right\rangle \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$= (0,2,-2) - \left[0 + \frac{-2}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, -1/\sqrt{3}\right)$

$= (0,2,-2) - 0\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, -1/\sqrt{3}\right)$

$= (0,2,-2)$

$\|v_2 - \langle v_2, q_1\rangle q_1\| = \sqrt{0^2+2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$

$q_2 = \frac{v_2 - \langle v_2, q_1\rangle q_1}{\|v_2 - \langle v_2, q_1\rangle q_1\|} = \left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$v_3 - \langle v_3, q_1\rangle q_1 - \langle v_3, q_2\rangle q_2$

$= (2,0,2) - \left\langle (2,0,2),\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)\right\rangle \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$- \left\langle (2,0,2)\left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right\rangle \left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$= (2,0,2) - 0\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) - (-\sqrt{2})\left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$= (2,0,2) - 0 - (0,-1,1) = (2,1,1)$

$\|v_3 - \langle v_3, q_1\rangle q_1 - \langle v_3, q_2\rangle q_2\| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$

$\therefore q_3 = \left(\frac{2}{\sqrt{6}}, 1/\sqrt{6}, 1/\sqrt{6}\right)$

$$Q = [q_1 \quad q_2 \quad q_3] = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle v_1, q_1 \rangle & \langle v_2, q_1 \rangle & \langle v_3, q_1 \rangle \\ 0 & \langle v_2, q_2 \rangle & \langle v_3, q_2 \rangle \\ 0 & 0 & \langle v_3, q_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle (1,-1,-1), (\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}) \rangle & \langle (0,2,-2), (\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}) \rangle & \langle (2,0,2), (\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}) \rangle \\ 0 & \langle (0,2,-2), (0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) \rangle & \langle (2,0,2), (0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) \rangle \\ 0 & 0 & \langle (2,0,2), (\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}) \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} & 0 - \frac{2}{\sqrt{3}} + 2/\sqrt{3} & 2/\sqrt{3} + 0 - 2/\sqrt{3} \\ 0 & 0 + 2/\sqrt{2} + 2/\sqrt{2} & 0 + 0 - 2/\sqrt{2} \\ 0 & 0 & \frac{4}{\sqrt{6}} + 0 + \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

Problem:- Find QR factorization of $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

Soln:-

Let $v_1 = (1,1,1,1)$; $v_2 = (-1,4,4,-1)$; $v_3 = (4,-2,2,0)$

$\|v_1\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$

$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$v_2 - \langle v_2, q_1 \rangle q_1 = (-1,4,4,-1) - \langle (-1,4,4,-1), \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \rangle \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

$= (-1,4,4,-1) - \left[\frac{-1}{2} + 2 + 2 - \frac{1}{2}\right]\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

$= (-1,4,4,-1) - 3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

$= (-1,4,4,-1) - (3/2, 3/2, 3/2, 3/2)$

$= (-5/2, 5/2, 5/2, -5/2)$

$\|v_2 - \langle v_2, q_1 \rangle q_1\| = \sqrt{(-5/2)^2 + (5/2)^2 + (5/2)^2 + (-5/2)^2} = \sqrt{4 \cdot (25/4)} = 5$

$q_2 = \frac{v_2 - \langle v_2, q_1 \rangle q_1}{\|v_2 - \langle v_2, q_1 \rangle q_1\|} = \left(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$

$v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2$

$= (4,-2,2,0) - \langle (4,-2,2,0), \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \rangle \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

$- \langle (4,-2,2,0), \left(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right) \rangle \left(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}\right)$

$= (2,-2,2,-2)$

$\|v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2\| = \sqrt{2^2 + (-2)^2 + 2^2 + (-2)^2}$

$= \sqrt{4 \cdot 4} = 4$

$q_3 = \left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$

$$Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}$$

$$\langle v_1, q_1\rangle = \left\langle (1,1,1,1), \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)\right\rangle = 2$$

$$\langle v_2, q_1\rangle = \left\langle (-1,4,4,-1), \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)\right\rangle = 3$$

$$\langle v_3, q_1\rangle = \left\langle (4,-2,2,0), \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)\right\rangle = 2$$

$$\langle v_2, q_2\rangle = \left\langle (-1,4,4,-1), \left(\tfrac{-1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{-1}{2}\right)\right\rangle = 5$$

$$\langle v_3, q_2\rangle = \left\langle (4,-2,2,0), \left(\tfrac{-1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{-1}{2}\right)\right\rangle = -2$$

$$\langle v_3, q_3\rangle = \left\langle (4,-2,2,0), \left(\tfrac{1}{2}, \tfrac{-1}{2}, 1/2, -1/2\right)\right\rangle = 4$$

$$R = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$