Module - 5

Inner product space

Recall:- Consider $\vec{a} = x_1\vec{i} + x_2\vec{j}$ and $\vec{b} = y_1\vec{i} + y_2\vec{j}$

dot product $\vec{a}\cdot\vec{b} = (x_1\vec{i} + x_2\vec{j})(y_1\vec{i} + y_2\vec{j})$

$$= x_1 y_1 + x_2 y_2$$

In $\mathbb{R}^2$, $\vec{a} = (x_1, x_2)$ and $\vec{b} = (y_1, y_2)$

$$\vec{a}\cdot\vec{b} = x_1 \cdot y_1 + x_2 y_2$$

We known $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

If $\vec{c} = (z_1, z_2)$

(2) $(\vec{a}+\vec{b})\cdot\vec{c} = (x_1+y_1, x_2+y_2)\cdot(z_1, z_2) =$

$$= z_1(x_1+y_1) + z_2(x_2+y_2)$$

$$= \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$$

(3) $\vec{a}\cdot\vec{a} = x_1^2 + x_2^2 \geq 0$. Moreover if $\vec{a}\cdot\vec{a} = 0$

$\Rightarrow x_1^2 + x_2^2 = 0$

$\Rightarrow x_1 = 0,\ x_2 = 0$

Thus $\vec{a} = 0$

(4) If $\vec{a} = \vec{0}$ then $\vec{a}\cdot\vec{a} = 0$

Recall:- Let $\vec{a} = x_1 \hat{i} + x_2 \hat{j}$; $\vec{b} = y_1 \hat{i} + y_2 \hat{j}$

then length of $\vec{a} = \sqrt{x_1^2 + x_2^2}$ we denote it by $|\vec{a}|$

If $\vec{a}$ and $\vec{b}$ are perpendicular to each other then

$$\vec{a} \cdot \vec{b} = 0$$

Angle between $\vec{a}$ and $\vec{b}$ is $\theta = \cos^{-1}\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$

Inner product space definition

An inner product on a real vector space V is a function that associates a real number $\boxed{\langle x, y\rangle}$ to each pair of vectors x and y in V in such a way that the following rules are satisfied for all vectors x, y and z in V and all scalars k in $\mathbb{R}$

(i) $\langle x, y\rangle = \langle y, x\rangle$

(ii) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(iii) $\langle kx, y\rangle = k\langle x, y\rangle$

(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0_V$.

Inner product space A pair $(V, \langle , \rangle)$ of a vector space V and an inner product $\langle , \rangle$ is called a inner product space.

Pb:1 For $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$

define $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + 3 x_2 y_2 + 5 x_3 y_3$

Is $\langle , \rangle$ an inner product on $\mathbb{R}^3$.

Soln:-

(i) $\langle \bar{x}, \bar{y} \rangle = \underline{x_1 y_1 + 3 x_2 y_2 + 5 x_3 y_3} = y_1 x_1 + 3 y_2 x_2 + 5 y_3 x_3$

$= \langle \bar{y}, \bar{x} \rangle$

(ii) Let $\bar{z} = (z_1, z_2, z_3)$

$\langle \bar{x} + \bar{y}, \bar{z} \rangle = \langle (x_1, x_2, x_3) + (y_1, y_2, y_3), (z_1, z_2, z_3) \rangle$

$= \langle (x_1 + y_1, x_2 + y_2, x_3 + y_3), (z_1, z_2, z_3) \rangle$

$= (x_1 + y_1) z_1 + 3 (x_2 + y_2) z_2 + 5 (x_3 + y_3) z_3$

$= (x_1 z_1 + 3 x_2 z_2 + 5 x_3 z_3) + (y_1 z_1 + 3 y_2 z_2 + 5 y_3 z_3)$

$= \langle \bar{x}, \bar{z} \rangle + \langle \bar{y}, \bar{z} \rangle$

(iii) Let $k \in \mathbb{R}$, $k\bar{x} = (kx_1, kx_2, kx_3)$

$\langle k\bar{x}, \bar{y} \rangle = \langle (kx_1, kx_2, kx_3)(y_1, y_2, y_3) \rangle$

$= k x_1 y_1 + 3 k x_2 y_2 + 5 k x_3 y_3$

$= k (x_1 y_1 + 3 x_2 y_2 + 5 x_3 y_3) = k \langle \bar{x}, \bar{y} \rangle$

(iv) $\langle \bar{x}, \bar{x} \rangle = x_1^2 + 3 x_2^2 + 5 x_3^2 \geq 0$

If $\langle \bar{x}, \bar{x} \rangle = 0 \Rightarrow x_1^2 + 3 x_2^2 + 5 x_3^2 = 0 \Rightarrow$ $x_1 = 0$
$x_2 = 0$
$x_3 = 0$

$\therefore \bar{x} = (0, 0, 0)$

If $\bar{x} = (0, 0, 0)$ then $\langle \bar{x}, \bar{x} \rangle = 0$.

Thus $\langle , \rangle$ is an inner product on $\mathbb{R}^3$.

Pb:2 For $\overline{x} = (x_1, x_2, x_3)$ and $\overline{y} = (y_1, y_2, y_3)$ on $\mathbb{R}^3$

define $\langle \overline{x}, \overline{y} \rangle = x_1 y_3 - x_2 y_1 - x_3 y_2$

Does $\langle , \rangle$ an inner product on $\mathbb{R}^3$?

Soln:- let $\overline{x} = (0, 1, 0)$; $\overline{y} = (1, 0, 0)$

$\langle \overline{x}, \overline{y} \rangle = \langle (0, 1, 0), (1, 0, 0) \rangle = 0 \cdot 0 - 1 \cdot 1 - 0 \cdot 0 = -1$

$\langle \overline{y}, \overline{x} \rangle = \langle (1, 0, 0), (0, 1, 0) \rangle = 1 \cdot 0 - 0 \cdot 0 - 0 \cdot 1 = 0$

Thus $\langle \overline{x}, \overline{y} \rangle \neq \langle \overline{y}, \overline{x} \rangle$, Hence $\langle , \rangle$ is not a inner product.

Pb: 3 Let $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ on $\mathbb{R}^2$

Define $\langle \bar{x}, \bar{y} \rangle = x_1 y_2 + x_2 y_1$

Does $\langle , \rangle$ an inner product on $\mathbb{R}^2$?

Soln:- Let $\bar{x} = (1, -2)$

$$\langle \bar{x}, \bar{x} \rangle = \langle (1, -2), (1, -2) \rangle = 1(-2) + (-2)\cdot 1 = -4 < 0$$

Since $\langle \bar{x}, \bar{x} \rangle < 0$, $\langle , \rangle$ is not an inner product.

Remark:-

1) Let $\bar{x} = (x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\bar{y} = (y_1, y_2) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ on $\mathbb{R}^2$

then the standard inner product on $\mathbb{R}^2$ is defined as $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + x_2 y_2$.

2) Let $\bar{x} = (x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\bar{y} = (y_1, y_2, y_3) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

on $\mathbb{R}^3$ then the standard inner product on $\mathbb{R}^3$ is

$\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$

3) Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ and $\bar{y} = (y_1, y_2, \ldots y_n)$ then standard inner product on $\mathbb{R}^n$ is

$\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n$.

Pb: 4 Consider the vector space $P_n(\mathbb{R})$. For $p(x)$, $q(x)$ in $P_n(\mathbb{R})$ define $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx$

Does $\langle , \rangle$ an inner product on $P_n(\mathbb{R})$?

Soln:-

(i) $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx = \int_0^1 q(x)p(x)\,dx = \langle q(x), p(x)\rangle$

(ii) $\langle p(x)+q(x), r(x)\rangle = \int_0^1 [p(x)+q(x)]r(x)\,dx$

$= \int_0^1 p(x)\,r(x)\,dx + \int_0^1 q(x)\,r(x)\,dx$

$= \langle p(x), r(x)\rangle + \langle q(x), r(x)\rangle$

(iii) $\langle k\,p(x), q(x)\rangle = \int_0^1 k\,p(x)q(x)\,dx = k\int_0^1 p(x)\,q(x)\,dx$

$= k\langle p(x), q(x)\rangle$

(iv) $\langle p(x), p(x)\rangle = \int_0^1 p(x)\cdot p(x)\,dx = \int_0^1 [p(x)]^2\,dx \geq 0$

Suppose $\langle p(x), p(x)\rangle = 0 \Rightarrow \int_0^1 [p(x)]^2\,dx = 0$

$\Rightarrow p(x) = 0$

conversely if $p(x)=0$ then $\langle p(x), p(x)\rangle = 0$

Thus $\langle , \rangle$ is an inner product on $P_n(\mathbb{R})$.

<u>Properties</u> Let V be an inner product space with inner product $\langle , \rangle$ then

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, ky \rangle = k \langle x, y \rangle$

<u>Proof:-</u>

(a) $\langle x, y+z \rangle = \langle y+z, x \rangle$ [1st axiom of Inner product]

$= \langle y, x \rangle + \langle z, x \rangle$ [2nd axiom of Inner "]

$= \langle x, y \rangle + \langle x, z \rangle$ [1st axiom of Inner product]

(b) $\langle x, ky \rangle = \langle ky, x \rangle$ [1st axiom of inner product]

$= k \langle y, x \rangle$ [3rd axiom of "]

$= k \langle x, y \rangle$ [1st axiom "]