**Spanning set** Let $V$ be a real vector space. A collection of vectors $\{u_1, u_2, \dots, u_n\}$ in $V$ is said to span $V$ if every vector in $V$ is a linear combination of $\{u_1, u_2, \dots, u_n\}$.

**Remark:-**

1) The set $W = \{\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n \mid \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}\}$ is a subspace of $V$. We call such subspace $W$ is the subspace spanned by $\{u_1, u_2, \dots, u_n\}$

**Example 1** Does the collection $\{u_1 = (1,0), u_2 = (0,1)\}$ span $\mathbb{R}^2$?

**Ans:-** Let $(a,b) \in \mathbb{R}^2$. We note that $(a,b) = a(1,0) + b(0,1) = au_1 + bu_2$

Hence $\{(1,0), (0,1)\}$ span $\mathbb{R}^2$

**Example 2:-** Does the collection $\{u_1 = (1,0,0), u_2 = (0,1,0), u_3 = (0,0,1)\}$ span $\mathbb{R}^3$?

**Ans:-** Let $(a,b,c) \in \mathbb{R}^3$. We note that $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$$= au_1 + bu_2 + cu_3$$

Hence $\{u_1, u_2, u_3\}$ span $\mathbb{R}^3$.

**Remark:-** The collection $\{e_1, e_2, \dots, e_n\}$ span $\mathbb{R}^n$ where $e_i = (0, 0, \dots, 0, \underbrace{\boxed{1}}_{i\text{th position}}, 0, \dots)$

**Example 3:-** Does the collection $\{1, t, t^2\}$ span $P_2(\mathbb{R})$?

**Ans:-** Let $a_0 + a_1 t + a_2 t^2 \in P_2(\mathbb{R})$. We note that $a_0 + a_1 t + a_2 t^2 = a_0 \cdot 1 + a_1 \cdot t + a_2 \cdot t^2$

so $\{1, t, t^2\}$ span $P_2(\mathbb{R})$.

**Remark:-** The collection $\{1, t, t^2, \ldots, t^n\}$ span $P_n(\mathbb{R})$

**Example 4:-** Does the collection $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ span $M_{2\times 2}(\mathbb{R})$?

**Ans:-** Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$. We have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Hence $S$ span $M_{2\times 2}(\mathbb{R})$

**Remark:-** The set $S = \{ E_{ij} \in M_{m\times n}(\mathbb{R}) \mid E_{ij}$ is a $m \times n$ matrix whose $(i,j)^{th}$ position is 1 and other places are zero$\}$

span $M_{m\times n}(\mathbb{R})$.

Pb:1 Does the collection $\{u_1=(1,1,0,0);\ u_2=(1,2,-1,1);\ u_3=(0,0,1,1);\ u_4=(2,1,2,1)\}$ Span $\mathbb{R}^4$?

Soln:- Let $y=(a,b,c,d)\in\mathbb{R}^4$. Form the equation $y=\alpha_1u_1+\alpha_2u_2+\alpha_3u_3+\alpha_4u_4$

$$(a,b,c,d)=\alpha_1(1,1,0,0)+\alpha_2(1,2,-1,1)+\alpha_3(0,0,1,1)+\alpha_4(2,1,2,1)$$

$$\Rightarrow \begin{aligned} \alpha_1+\alpha_2+2\alpha_4 &= a\\ \alpha_1+2\alpha_2+\alpha_4 &= b\\ -\alpha_2+\alpha_3+2\alpha_4 &= c\\ \alpha_2+\alpha_3+\alpha_4 &= d \end{aligned}$$

we solve this by Gauss elimination.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a\\ 1 & 2 & 0 & 1 & b\\ 0 & -1 & 1 & 2 & c\\ 0 & 1 & 1 & 1 & d \end{array}\right] \xrightarrow{R_2\to R_2-R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a\\ 0 & 1 & 0 & -1 & b-a\\ 0 & -1 & 1 & 2 & c\\ 0 & 1 & 1 & 1 & d \end{array}\right] \begin{array}{l} R_3\to R_3+R_2\\ R_4\to R_4-R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a\\ 0 & 1 & 0 & -1 & b-a\\ 0 & 0 & 1 & 1 & c+b-a\\ 0 & 0 & 1 & -2 & d+a-b \end{array}\right] \xrightarrow{R_4\to R_4-R_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a\\ 0 & 1 & 0 & -1 & b-a\\ 0 & 0 & 1 & 1 & c+b-a\\ 0 & 0 & 0 & -3 & 2a-2b-c+d \end{array}\right]$$

The reduced System is
$$\begin{aligned} \alpha_1+\alpha_2+2\alpha_4 &= a\\ \alpha_2-\alpha_4 &= b-a\\ \alpha_3+\alpha_4 &= c+b-a\\ -3\alpha_4 &= 2a-2b-c+d \end{aligned}$$

$\alpha_1=4a-3b-c+d$; $\alpha_2=\dfrac{-5a+5b+c-d}{3}$; $\alpha_3=\dfrac{a+b+2c+d}{3}$; $\alpha_4=\dfrac{-2a+2b+c-d}{3}$

Thus

$$(a,b,c,d)=(4a-3b-c+d)u_1+\left(\frac{-5a+5b+c-d}{3}\right)u_2+\left(\frac{a+b+2c+d}{3}\right)u_3+\left(\frac{-2a+2b+c-d}{3}\right)u_4.$$

Pb: 2 Does the collection $S\{t^2+1,\ t^2+t,\ t+1\}$ span $P_2(\mathbb{R})$?

Ans:- let $a_0+a_1t+a_2t^2 \in P_2(\mathbb{R})$

$$a_0+a_1t+a_2t^2 = \alpha_1(t^2+1)+\alpha_2(t^2+t)+\alpha_3(t+1)$$

$$= (\alpha_1+\alpha_2)\,t^2+(\alpha_2+\alpha_3)\,t+(\alpha_1+\alpha_3)$$

$\Rightarrow$ $\alpha_1+\alpha_2 = a_0 \rightarrow (1)$

$\alpha_2+\alpha_3 = a_1 \rightarrow (2)$

$\alpha_1+\alpha_3 = a_2 \rightarrow (3)$

(2) $\Rightarrow \alpha_2 = a_1-\alpha_3$ apply this in (1), $(a_1+a_2)-2\alpha_3 = a_0$

(3) $\Rightarrow \alpha_1 = a_2-\alpha_3$

$$\Rightarrow \boxed{\alpha_3 = \frac{a_1+a_2-a_0}{2}}$$

$$\alpha_1 = a_2 - \frac{a_1+a_2-a_0}{2}$$

$$\boxed{\alpha_1 = \frac{a_0-a_1+a_2}{2}}$$

$$\alpha_2 = a_1 - \frac{a_1+a_2-a_0}{2}$$

$$\boxed{\alpha_2 = \frac{a_0+a_1-a_2}{2}}$$

Thus $a_0+a_1t+a_2t^2 = \left(\frac{a_0-a_1+a_2}{2}\right)(t^2+1)+\left(\frac{a_0+a_1-a_2}{2}\right)(t^2+t)$
$+\left(\frac{a_1+a_2-a_0}{2}\right)(t+1)$

$S$ span $P_2(\mathbb{R})$.