Orthogonal vectors Let V be an inner product space.

Two vectors x and y in V is said to be orthogonal if $\langle x, y\rangle = 0$.

Examples:- (i) Consider the inner product space $\mathbb{R}^2$ with standard inner product.

(*) $\langle (3,-1), (1,3)\rangle = 3\cdot 1 + (-1)\cdot 3 = 0$

Thus $\{(3,-1), (1,3)\}$ are orthogonal.

(*) $\langle (2,1), (1,2)\rangle = 2\cdot 1 + 1\cdot 2 = 4 \neq 0$

$\therefore \{(2,1), (1,2)\}$ are not orthogonal.

Consider the inner product Space $P_6(\mathbb{R})$ with inner product $\langle f(t), g(t) \rangle = \int_{-1}^{1} f(t)g(t)\,dt$. Then

(i) $\langle t, t^2 \rangle = \int_{-1}^{1} t \cdot t^2\,dt = \int_{-1}^{1} t^3\,dt = \left[\frac{t^4}{4}\right]_{-1}^{1} = 0$

$\therefore$ t and t^2 are orthogonal.

(ii) $\langle t^2, t^4 \rangle = \int_{-1}^{1} t^2 \cdot t^4 \cdot dt = \int_{-1}^{1} t^6\,dt = \left[t^7/7\right]_{-1}^{1}$

$= 2/7 \neq 0$

So t^2 and t^4 are not orthogonal.

Unit Vectors Let V be an inner product space. A vector x in V is said to be unit vector if $\|x\| = \sqrt{\langle x, x \rangle} = 1$

Example: 1 Consider the inner product space $\mathbb{R}^3$ with standard inner product. Then

$$\left\|\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\| = \sqrt{\left\langle\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\rangle}$$

$$= 1/3 + 1/3 + 1/3 = 1$$

So $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit vector

Example:- Consider $P_2(\mathbb{R})$ with inner product

$$\langle f(t), g(t)\rangle = \int_0^1 f(t)\, g(t)\, dt$$

Now, $\langle \sqrt{3}t, \sqrt{3}t\rangle = \int_0^1 \sqrt{3}\, t \cdot \sqrt{3}\, t\, dt = 3\int_0^1 t^2\, dt$

$$= 3\left[\frac{t^3}{3}\right]_0^1 = 1$$

thus $\sqrt{3}\,t$ is a unit vector in $P_2(\mathbb{R})$.

Normalization The process of obtaining a unit vector from a nonzero vector by multiplying the inverse of its length is called as normalization

Note:- If x is a given vector then the normalization of x is $\dfrac{x}{\|x\|}$.

Pb:1 Consider the inner product space $\mathbb{R}^3$ with standard inner product. Find the normalization of $(-2,3,4)$

Soln:-

$$\|(-2,3,4)\| = \sqrt{\langle(-2,3,4),(-2,3,4)\rangle}$$

$$= \sqrt{(-2)(-2)+3\cdot 3+4\cdot 4}$$

$$= \sqrt{4+9+16} = \sqrt{29}$$

Normalization of $(-2,3,4)$ is $\left(\frac{-2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)$

Pb:2 Consider the inner product space $P_3(\mathbb{R})$ with inner product $\langle f(t), g(t) \rangle = \int_0^1 f(t) g(t)\, dt$

Find the normalization of $t^2 - 3$

Soln:-

$$\|t^2-3\| = \sqrt{\langle t^2-3, t^2-3 \rangle} = \sqrt{\int_0^1 (t^2-3)^2\, dt} = 6/\sqrt{5}$$

Normalization of t^2-3 is $\dfrac{t^2-3}{\left(6/\sqrt{5}\right)} = \dfrac{\sqrt{5}\,(t^2-3)}{6}$.

Orthonormal set: A set of vectors $\{x_1, x_2, \ldots, x_k\}$ in an inner product space V is said to be orthonormal if $\langle x_i, x_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Note:- A set of vectors $\{x_1, x_2, \ldots, x_k\}$ of V is said to be orthonormal basis for V if

(i) $\{x_1, x_2, \ldots, x_k\}$ forms a basis of V

(ii) $\{x_1, x_2, \ldots x_k\}$ is an orthonormal set.

Example:– Consider the inner product space $\mathbb{R}^2$ with standard inner product. Which of the following is an orthonormal basis of $\mathbb{R}^2$

(i) $\{(1,0), (0,1)\}$ $\rightarrow$ orthonormal basis

(ii) $\{(1,0), (1,1)\}$ $\rightarrow$ not " "

(iii) $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right\}$

↳ orthonormal basis.

Soln:–

Projections:-

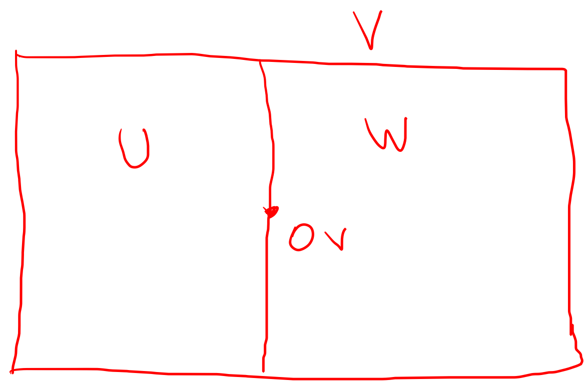
Direct Sum of subspaces:-

Let U and W be two subspaces of a vector space V. V is said to be direct sum of U and W if

(i) $V = U + W$

(ii) $U \cap W = \{0_V\}$

In this case we write $V = U \oplus W$



Example:- Consider $\mathbb{R}^2$ and subspaces

$$U = \{(x, 0) \mid x \in \mathbb{R}\}$$

$$W = \{(x, x) \mid x \in \mathbb{R}\}$$

For any $(x, y) \in \mathbb{R}^2$ we have

$$(x, y) = (x-y)(1, 0) + y(1, 1) = (x-y, 0) + (y, y)$$

So, $\mathbb{R}^2 = U + W$

and clearly $U \cap W = \{(0, 0)\}$

So, $\mathbb{R}^2 = U \oplus W$.