Similarity transformation

Q:- Let V be a vector space α and β are bases of V. Consider $T: V \to V$ and the matrices $[T]_\alpha$ and $[T]_\beta$.

What is the relationship between $[T]_\alpha$ and $[T]_\beta$?

Ans:- Theorem:- Let $T: V \to V$ be a linear transformation. Consider α and β are two bases of V. Suppose $Q = [Id]_\beta^\alpha$ then

$$[T]_\beta = Q^{-1} [T]_\alpha Q$$

where $Q^{-1} = [Id]_\alpha^\beta$

Similar matrices:- For any square matrix A and B, A is said to be similar to B if there exists a non singular matrix Q such that

$$B = Q^{-1} A Q.$$

Pb:1 Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T(x,y,z) = (x+2y+z, \; -y, \; x+4z)$$

(i) Find $[T]_\alpha$ where α is the standard basis of $\mathbb{R}^3$

(ii) Find $[T]_\beta$ using similarity transformation

where $\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$.

Soln:- $\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$T(1,0,0) = (1,0,1) = 1(1,0,0) + 0\cdot(0,1,0) + 1\cdot(0,0,1)$$

$$T(0,1,0) = (2,-1,0) = 2(1,0,0) + (-1)(0,1,0) + 0\cdot(0,0,1)$$

$$T(0,0,1) = (1,0,4) = 1(1,0,0) + 0\cdot(0,1,0) + 4(0,0,1)$$

$$[T]_\alpha = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

We calculate the transition matrix $Q = [Id]_\beta^\alpha$

Define $Id: \mathbb{R}^3 \to \mathbb{R}^3$ by $Id(x,y,z) = (x,y,z)$

$$Id(1,0,0) = (1,0,0) = 1\cdot(1,0,0) + 0\cdot(0,1,0) + 0\cdot(0,0,1)$$

$$Id(1,1,0) = (1,1,0) = 1\cdot(1,0,0) + 1\cdot(0,1,0) + 0\cdot(0,0,1)$$

$$Id(1,1,1) = (1,1,1) = 1\cdot(1,0,0) + 1\cdot(0,1,0) + 1\cdot(0,0,1)$$

$$Q = [Id]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^{-1} = [Id]_\alpha^\beta$$

$$Id(1,0,0) = (1,0,0) = 1\cdot(1,0,0) + 0\cdot(1,1,0) + 0\cdot(1,1,1)$$

$$Id(0,1,0) = (0,1,0) = (-1)(1,0,0) + 1\cdot(1,1,0) + 0\cdot(1,1,1)$$

$$Id(0,0,1) = (0,0,1) = 0\cdot(1,0,0) + (-1)(1,1,0) + 1\cdot(1,1,1)$$

$$Q^{-1} = [Id]_\alpha^\beta = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

By the similarity theorem $[T]_\beta = Q^{-1}[T]_\alpha Q$.

$$[T]_\beta = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 5 \\ -1 & -2 & -6 \\ 1 & 1 & 5 \end{bmatrix}$$

Pb: 2 $D: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ such that $D(p(t)) = \frac{d\,p(t)}{dt}$

(i) Find $[D]_\alpha$ where α is the standard basis on $P_2(\mathbb{R})$

(ii) Find $[D]_\beta$ using similarity transformation where $\beta = \{1, 2t, 4t^2-2\}$

Soln:- $\alpha = \{1, t, t^2\}$

$D(1) = 0 = 0\cdot 1 + 0\cdot t + 0\cdot t^2$

$D(t) = 1 = 1\cdot 1 + 0\cdot t + 0\cdot t^2$

$D(t^2) = 2t = 0\cdot 1 + 2\cdot t + 0\cdot t^2$

$$[D]_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We calculate $Q = [Id]_\beta^\alpha$ and $Q^{-1} = [Id]_\alpha^\beta$

Define $Id: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $Id(p(t)) = p(t)$

$Id(1) = 1 = 1\cdot 1 + 0\cdot t + 0\cdot t^2$

$Id(2t) = 2t = 0\cdot 1 + 2\cdot t + 0\cdot t^2$

$Id(4t^2-2) = 4t^2-2 = (-2)\cdot 1 + 0\cdot t + 4t^2$

$$Q = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$Id(1) = 1 = 1\cdot 1 + 0\cdot 2t + 0\cdot(4t^2-2)$

$Id(t) = t = 0\cdot 1 + \frac{1}{2}(2t) + 0\cdot(4t^2-2)$

$Id(t^2) = t^2 = \frac{1}{2}\cdot 1 + 0\cdot 2t + \frac{1}{4}(4t^2-2)$

$$Q^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

By similarity theorem $[D]_\beta = Q^{-1}[D]_\alpha Q$

$$= \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Pb:3 Let $G: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $G(x,y,z) = (2y+z,\ x-4y,\ 3x)$

(i) Find $[G]_\alpha$ where α is the standard basis

(ii) Using similarity transformation find $[G]_\beta$ where $\beta = \{(1,0,0),\ (1,1,0),\ (1,1,1)\}$.