Change of basis matrix (or) Transition matrix

Let α and β be two bases for a vector space V. Consider the identity transformation

$Id : V \longrightarrow V$ by $Id(v) = v$.

The matrix $[Id]_{\alpha}^{\beta}$ is called the transition matrix (or) change of basis matrix from α to β.

Pb: 1 Find the transition matrix $[Id]_{\alpha}^{\beta}$ where

$\alpha = \{(5,1), (1,2)\}$ and $\beta = \{(1,0), (0,1)\}$

Soln:- Define $Id : \mathbb{R}^2 \to \mathbb{R}^2$ by $Id(v) = v$ then

$$Id(5,1) = (5,1) = 5(1,0) + 1\cdot(0,1)$$

$$Id(1,2) = (1,2) = 1\cdot(1,0) + 2(0,1)$$

$$\therefore \quad [Id]_{\alpha}^{\beta} = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$$

Pb: 2 Find the transition matrix $[Id]_{\alpha}^{\beta}$ where

$\alpha = \{(1,1,1), (1,1,0), (1,0,0)\}$ and

$\beta = \{(2,0,3), (-1,4,1), (3,2,5)\}$.

Soln:- $Id : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $Id(v) = v$

$$T(1,1,1) = (1,1,1) = k_1(2,0,3) + k_2(-1,4,1) + k_3(3,2,5)$$

$$(1,1,1) = (2k_1 - k_2 + 3k_3,\ 4k_2 + 2k_3,\ 3k_1 + k_2 + 5k_3)$$

$$\Rightarrow \begin{aligned} 2k_1 - k_2 + 3k_3 &= 1 \\ 4k_2 + 2k_3 &= 1 \\ 3k_1 + k_2 + 5k_3 &= 1 \end{aligned} \qquad \begin{aligned} k_1 &= -2 \\ k_2 &= -1/2 \\ k_3 &= 3/2 \end{aligned}$$

$$T(1,1,1) = -2(2,0,3) + \left(\frac{-1}{2}\right)(-1,4,1) + \frac{3}{2}(3,2,5)$$

$$T(1,1,0) = (1,1,0) = k_1(2,0,3) + k_2(-1,4,1) + k_3(3,2,5)$$

$$\begin{aligned} 2k_1 - k_2 + 3k_3 &= 1 \\ 4k_2 + 2k_3 &= 1 \\ 3k_1 + k_2 + 5k_3 &= 0 \end{aligned} \qquad \begin{aligned} k_1 &= -13/3 \\ k_2 &= -7/6 \\ k_3 &= 17/6 \end{aligned}$$

$$T(1,1,0) = \left(\frac{-13}{3}\right)(2,0,3) + \left(\frac{-7}{6}\right)(-1,4,1) + 17/6\,(3,2,5)$$

$$T(1,0,0) = (1,0,0) = k_1(2,0,3) + k_2(-1,4,1) + k_3(3,2,5)$$

$$\begin{aligned} 2k_1 - k_2 + 3k_3 &= 1 \\ 4k_2 + 2k_3 &= 0 \\ 3k_1 + k_2 + 5k_3 &= 0 \end{aligned} \qquad \begin{aligned} k_1 &= -3 \\ k_2 &= -1 \\ k_3 &= 2 \end{aligned}$$

$$T(1,0,0) = (-3)(2,0,3) + (-1)(-1,4,1) + 2(3,2,5)$$

$$[Id]_{\alpha}^{\beta} = \begin{bmatrix} -2 & -1/2 & 3/2 \\ -13/3 & -7/6 & 17/6 \\ -3 & -1 & 2 \end{bmatrix}$$