Linear Combinations Let V be a real vector space. A vector y in V is a linear combination of vectors $u_1, u_2, \ldots, u_n$ in V if there exists scalars $k_1, k_2, \ldots, k_n$ in $\mathbb{R}$ such that $y = k_1u_1 + k_2u_2 + \cdots + k_nu_n$

Note:- In order to justify a vector y is the linear combination of $u_1, u_2, \cdots u_n$, we have to find a solution to the vector equation $y = k_1u_1 + k_2u_2 + \cdots + k_nu_n$.

Pb: 1 Does the vector $(6, 14, -8)$ in $\mathbb{R}^3$ can be written as a linear combination of $(1,2,3)$, $(2,3,7)$, $(3,5,6)$?

Ans:- Take $y = (6, 14, -8)$; $u_1 = (1,2,3)$; $u_2 = (2,3,7)$, $u_3 = (3,5,6)$

Form the vector equation $y = k_1 u_1 + k_2 u_2 + k_3 u_3$

$$(6, 14, -8) = k_1(1,2,3) + k_2(2,3,7) + k_3(3,5,6)$$

$$= (k_1 + 2k_2 + 3k_3,\ 2k_1 + 3k_2 + 5k_3,\ 3k_1 + 7k_2 + 6k_3)$$

$$\Rightarrow \quad k_1 + 2k_2 + 3k_3 = 6$$

$$2k_1 + 3k_2 + 5k_3 = 14$$

$$3k_1 + 7k_2 + 6k_3 = -8$$

we solve this by Gauss elimination.

1st pivot

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 3 & 5 & 14 \\ 3 & 7 & 6 & -8 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -1 & -1 & 2 \\ 0 & 1 & -3 & -26 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -4 & -24 \end{bmatrix}$$

reduced system is

$$\left.\begin{aligned} k_1 + 2k_2 + 3k_3 &= 6 \\ -k_2 - k_3 &= 2 \\ -4k_3 &= -24 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} k_1 &= 16 \\ k_2 &= -4 \\ k_3 &= 6 \end{aligned}$$

$$\therefore \quad y = 16u_1 - 4u_2 + 6u_3.$$

Pb: 2 Determine whether $p(t) = t^2+t+2$ is a linear combination of $p_1 = t^2+2t+1$; $p_2 = t^2+3$; $p_3 = t-1$

Ans:- Form the vector equation $p = k_1 p_1 + k_2 p_2 + k_3 p_3$.

we solve this equation

$$(t^2+t+2) = k_1(t^2+2t+1) + k_2(t^2+3) + k_3(t-1)$$
$$= t^2(k_1+k_2) + t(2k_1+k_3) + (k_1+3k_2-k_3)$$

Equation the coefficients of respective powers,

$$k_1+k_2 = 1$$
$$2k_1+k_3 = 1 \qquad \text{we solve this Gauss elimination.}$$
$$k_1+3k_2-k_3 = 2$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & -1 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{bmatrix} R_3 \to R_3 + R_2 \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced system is $k_1 + k_2 = 1$; $-2k_2 + k_3 = -1$. This system has infinite solution. To find a solution fix $k_2 = 1$ then

$$k_1 = 0 \; ; \quad k_3 = 1$$

A linear combination is $p = 0 \cdot p_1 + 1 \cdot p_2 + 1 \cdot p_3$.

Pb: 3 Does the polynomial $f = 3t^2 - 3t + 1$ a linear combination of $p_1 = t^2 - t$; $p_2 = t^2 - 2t + 1$; $p_3 = 1 - t^2$?

Ans:- Form the vector equation $f = k_1 p_1 + k_2 p_2 + k_3 p_3$

we solve this.

$$3t^2 - 3t + 1 = k_1(t^2 - t) + k_2(t^2 - 2t + 1) + k_3(1 - t^2)$$

$$= (k_1 + k_2 - k_3)\, t^2 + (-k_1 - 2k_2)\, t + (k_2 + k_3)$$

Equating the co-efficients, we get

$$k_1 + k_2 - k_3 = 3 \rightarrow (1)$$

$$-k_1 - 2k_2 = -3 \rightarrow (2)$$

$$k_2 + k_3 = 1 \rightarrow (3)$$

(3) $\Rightarrow$ $k_3 = 1 - k_2$ apply this in (1),

we get, $k_1 + k_2 - (1 - k_2) = 3$

$$\Rightarrow k_1 + 2k_2 = 4 \rightarrow (4)$$

From (2), (4), we have $4 = 3$ (this has no sense)

Thus the given system has no solution and so linear combination is not possible.