Invertible Linear transformation

Recall:- Let $f: A \to B$ be a function. f is said to be invertible

if (*) f is one to one (*) f is onto.

One-one function f is said to be one to one if $f(x) = f(y)$

then $x = y$.

In other words $x \neq y \Rightarrow f(x) \neq f(y)$

Onto function f is said to be onto if every element in the codomain

has a pre image. (In other words) Range (f) = Co-domain.

Definition Let V and W be two vector spaces $T: V \to W$ be a linear transformation. T is said to be invertible if T is one to one and onto.

Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (2x, 3y)$. Is T invertible?

Ans:- T is one-one $T(x_1,y_1) = T(x_2,y_2)$

$\Rightarrow (2x_1, 3y_1) = (2x_2, 3y_2)$

$\Rightarrow 2x_1 = 2x_2$ and $3y_1 = 3y_2$

$\Rightarrow x_1 = x_2$ and $y_1 = y_2$.

$\therefore (x_1, y_1) = (x_2, y_2)$

T is one to one.

T is onto:- Let $(z_1, z_2) \in \mathbb{R}^2$ (in codomain) then it is easy to see that $T\left(\frac{z_1}{2}, \frac{z_2}{3}\right) = (z_1, z_2)$

$\therefore$ every element in the co-domain has a pre image.

Thus T is invertible.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (0,y)$. Is T investible?

Ans:- consider the vectors $(1,2)$ and $(5,2)$ in $\mathbb{R}^2$ (domain)

By the definition of T, $T(1,2) = (0,2)$

$T(5,2) = (0,2)$

So, $T(1,2) = T(5,2)$ but $(1,2) \neq (5,2)$

$\Rightarrow$ T is not one to one. Hence T is not investible.

Theorem: 1 Let $T: V \to W$ be a linear transformation then

(a) $\text{Ker}(T) = \{0_V\}$ if and only if T is one to one

(b) $\text{Im}(T) = W$ if and only if T is onto.

Theorem 2:— Let V be a real vector space. $T: V \longrightarrow V$ be a linear transformation. Then the following statements are equivalent

(a) T is invertible

(b) $\text{Ker}\, T = \{0_V\}$

(c) $\text{Im}(T) = V$.

Example:- Let $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $D(p(t)) = \frac{dp(t)}{dt}$

Is D invertible?

Ans:- $\ker(D) = \{ p(t) \mid D(p(t)) = 0 \}$

$$= \{ p(t) \mid \frac{dp(t)}{dt} = 0 \} = \text{All constant polynomials.}$$

By the previous theorem, we conclude that

D is not invertible, because $\ker(D) \neq \{0\}$.

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x,y,0)$

Is T is invertible?

Ans:- $(0,0,3)$ does not have pre image.

so T is not onto. Hence T is not invertible.

Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x,y,z) = (x,y)$. Is T invertible?

Ans:- Consider $(1,1,2)$ and $(1,1,3)$. By the definition of T, we have

$$T(1,1,2) = (1,1) = T(1,1,3)$$

But $(1,1,2) \neq (1,1,3)$.

so, T is not one to one.

Isomorphic vector spaces:-

1, 2, 3, 4, 5

I, II, III, IV, V,

Definition:- A linear transformation $T : V \to W$ is called isomorphism if T is invertible.

We say that the vector spaces V, W are isomorphic if there exists an isomorphism between V and W.

Theorem:- The vector spaces V and W are isomorphic if and only if $\dim V = \dim W$.

Examples:- (i) We know $\dim \mathbb{R}^{n+1} = n+1$

and $\dim P_n(\mathbb{R}) = n+1$.

Hence $\mathbb{R}^{n+1}$ is isomorphic to $P_n(\mathbb{R})$.

(ii) We know $\dim \mathbb{R}^{mn} = mn$ and

$\dim M_{m \times n}(\mathbb{R}) = mn$

Hence $\mathbb{R}^{mn}$ is isomorphic to $M_{m\times n}(\mathbb{R})$