## Application of rank and nullity to interpolation

Suppose that for $n+1$ distinct experimental input values $x_0, x_1, \ldots, x_n$, we obtained $n+1$ output values (say)

$y_0 = f(x_0),\ y_1 = f(x_1),\ y_2 = f(x_2), \ldots, y_n = f(x_n)$

The output values are supposed to be related to the inputs by a certain function $f$.

Aim:- Construct a polynomial $p(x)$ of degree less than or equal to $n$ which interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$ (ie)

$$p(x_0) = y_0;\ p(x_1) = y_1;\ p(x_2) = y_2, \ldots, p(x_n) = y_n.$$

Assume that $x_0, x_1, \ldots, x_n$ are all distinct.

Suppose the required polynomial is $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

Then we observe the following.

We have $p(x_0) = y_0 = a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n \rightarrow (1)$

$$p(x_1) = y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n \rightarrow (2)$$

$$p(x_2) = y_2 = a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_n x_2^n \rightarrow (3)$$

$$\vdots$$

$$p(x_n) = y_n = a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n \rightarrow (n+1)$$

The above equations represents a system of linear equations with $(n+1)$ equations and $n+1$ unknowns. The matrix form of the above equations are.

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & & x_n^n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}}_{Y}$$

The Coefficient matrix $V$ has determinant

$$\det V = \prod_{0 \le i < j \le n} (x_j - x_i)$$

Since $x_i$'s are distinct, $\det V \neq 0$ and so rank $V = n+1$.

Thus the system $VX = Y$ has a unique solution

Hence, we conclude that if $x_0, x_1, \ldots, x_n$ are all distinct then there exists a unique polynomial

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$ exists which interpolates

$p(x_i) = y_i$.

Remark:- 1) If we have $m$ values of $y$ with $m < n+1$ at $m$ distinct points $x_1, x_2, \ldots, x_m$ then there are as many interpolating polynomials as the null space of $A$ since in this case $A$ is a $m \times (n+1)$ matrix with $m < n+1$.

2) If we have $m$ values of $y$ with $m > n+1$ at $m$ distinct points $x_1, x_2, \ldots, x_m$ then there need not be any interpolating polynomials since the system could be inconsistent.

**Pb: 1** Given four points $(0,3), (1,0), (-1,2), (3,6)$ in the plan $\mathbb{R}^2$ find a interpolating polynomial of degree 3

**Soln:-** Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Given $p(0) = 3$; $p(1) = 0$; $p(-1) = 2$; $p(3) = 6$

Thus $p(0) = \boxed{3 = a_0} \rightarrow (1)$

$p(1) = 0 = a_0 + a_1 + a_2 + a_3 \rightarrow (2)$

$p(-1) = 2 = a_0 - a_1 + a_2 - a_3 \rightarrow (3)$

$p(3) = 6 = a_0 + 3a_1 + 9a_2 + 27a_3 \rightarrow (4)$

(2) + (3) $\Rightarrow 2a_0 + 2a_2 = 2$

$\Rightarrow a_2 = 1 - a_0$ since $a_0 = 3$

$\boxed{a_2 = -2}$

apply $a_0 = 3$, $a_2 = -2$ in (2) we get $0 = 1 + a_1 + a_3$

$\Rightarrow \boxed{a_1 + a_3 = -1} \rightarrow (5)$

apply $a_0 = 3$; $a_2 = -2$ in (4), we get $6 = -15 + 3a_1 + 27a_3$

$\boxed{7 = a_1 + 9a_3} \rightarrow (6)$

on Solving (5) (6), we get $\boxed{a_1 = -2; \; a_3 = 1}$

$\therefore$ Required polynomial is

$$\boxed{p(x) = 3 - 2x - 2x^2 + x^3}$$

Exercise:- 1) Find a polynomial $p(x) = a + bx + cx^2 + dx^3$ that satisfies $p(0) = 1$; $p'(0) = 2$; $p(1) = 4$; $p'(1) = 4$.

2) Find equation of a circle that passes through three points $(2,-2)$, $(3,5)$ and $(-4,6)$ in the plane $\mathbb{R}^2$.