Pb:1 Using Gauss elimination solve the system

$$x+y+2z+3w=13$$
$$x-2y+z+w=8$$
$$3x+y+z-w=1$$

$3 \quad 1 \quad 1 \quad -1 \quad 1$

$3 \quad 3 \quad 6 \quad 9 \quad 39$

Soln:- Augmented matrix corresponds to this system is

1st pivot

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 13 \\ 1 & -2 & 1 & 1 & 8 \\ 3 & 1 & 1 & -1 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 0 & -2 & -5 & -10 & -38 \end{bmatrix} \begin{matrix} R_2 \to (-1)R_2 \\ R_3 \to (-1)R_3 \end{matrix}$$

3rd pivot

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 13 \\ 0 & 3 & 1 & 2 & 5 \\ 0 & 2 & 5 & 10 & 38 \end{bmatrix} R_3 \to 3R_3 - 2R_2 \begin{bmatrix} 1 & 1 & 2 & 3 & 13 \\ 0 & 3 & 1 & 2 & 5 \\ 0 & 0 & 13 & 26 & 104 \end{bmatrix} R_3 \to \frac{R_3}{13}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 13 \\ 0 & 3 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 & 8 \end{bmatrix}$$

This is the row echelon form. The system corresponds to row echelon form is

$$x+y+2z+3w=13 \to (1)$$
$$3y+z+2w=5 \to (2)$$
$$z+2w=8 \to (3)$$

This system has infinitely many solutions.

$(3) \Rightarrow \boxed{z=8-2w}$

$(2) \Rightarrow 3y = 5-z-2w$. put $z=8-2w$ then

$3y = 5-(8-2w)-2w = -3$

$\boxed{y=-1}$

$(1) \Rightarrow x = 13-y-2z-3w = 13+1-2(8-2w) -3w$

$= 14-16+4w-3w$

$\boxed{x = w-2}$

To find all the solutions of this system we fix $w=t$ then.

$x=t-2;\quad y=-1\quad;\ z=8-2t;\quad w=t$ for all real t.

Pb: 2 Determine all values of a for which the system

$$x + y - z = 2$$
$$x + 2y + z = 3$$
$$x + y + (a^2-5)z = a$$

has (i) No solution (ii) Unique solution (iii) Infinite solution.

Soln:- The augmented matrix is

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \quad \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-6 & a-2 \end{bmatrix}$$

1st pivot

This the row echelon form

The reduced system is

$$x + y - z = 2$$
$$y + 2z = 1$$
$$(a^2-6)z = (a-2)$$

(i) No solution If $a = \pm\sqrt{6}$ then this system has no solution.

(ii) Unique solution If $a = 2$ then this system has unique solution.

(iii) Infinite solution This system has no infinite solution for any any values of a.

Basic and free variables Consider the system $AX = B$. The variables corresponds to the leading non zero entry of each row in the row echelon form of augmented matrix $[A \; B]$ is known as the basic variables and other variables are known as free variables.

Example:-

1) Consider the system
$$x + 2y + 3z = 4$$
$$5x + 6y + 7z = 8$$
$$9x + 10y + 11z = 12$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 9R_3 \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \begin{array}{l} R_2 \to \frac{R_2}{4} \\ R_3 \to \frac{R_3}{8} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the row echelon form.

There are two leading non zero entries and they corresponds to variables x and y. Hence x, y are basic variables z is the free variable.