Practice Problems

Module- 1

1. Given matrix $A = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 2 & 2 & 3 & -1 \\ 4 & 2 & 0 & 1 \end{bmatrix}$, find
   i. Echelon form of A.
   ii. Reduced row echelon for of A.

   Ans: i $\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & -7 \end{bmatrix}$, ii. $\begin{bmatrix} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

2. Solve the following system of equation using Gauss-elimination method

$$x - y + 2z + w = 1$$
$$3x + y - w = 3$$
$$x + y + z = -1$$

   Ans: w=t free variable, solution is x=1-t/4, y=t/4, z=-t/2, w=t.

3. Given system of equation

$$x + y - z + w = 1$$
$$2x + y + z - w = 2$$
$$3x + 3y + cz + 4w = 4$$
$$2x + 2y - z + dw = e$$

   Find the values of parameters c,d and e, such the system of equations
   (i) Has infinitely many solutions
   (ii) Has unique solutions
   (iii) Has no solution.

   Ans: No solution: $c = -3, e \neq d + 1$ or $c \neq -3, d = 1$ and $e \neq 0$.

   Unique solution: $c \neq 3, d \neq 1$.

   Infinitely many solutions: $c = -3, e = d - 1$ or $c \neq 3, d = 1, e = 0$.

4. Use the Gauss-Jordon method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Ans: $\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$.

5. Find the LU decomposition of the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ and hence, us LU decomposition to find the solution of the following system of equations

$$x + 3y + 2z = 1$$
$$x + 2y + 2z = -1$$
$$2x + y + z = 1$$

Ans: L= $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/5 & 1 \end{bmatrix}$ $, U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ $, x = 1, y = 2, z = -3$,

Module 2:

1. Are the following sets vector spaces with the indicated operations? If not, why not?
   i) The set V of nonnegative real numbers; ordinary addition and scalar multiplication.
   ii) The set V of all polynomials of degree ≥ 3, together with 0; operations of Polynomials.
   iii) The set V of 2 × 2 matrices with equal column sums; operations of $M_{2\times2}$.
   iv) The set V of all ordered pairs (x, y) with the addition of $\mathbb{R}_2$, but using scalar multiplication $a(x, y) = (ax, -ay)$.
   v) The set V of all 2 × 2 matrices with the addition of $M_{2\times2}$ but scalar multiplication $*$ defined by $a * X = aX^T$.
   vi) The set V of complex numbers; usual addition and multiplication by a real number.

Ans: i. additive inverse does not exist. (Not a vector space)

ii) Not closed with respect to vector addition. (Not a vector space)

iii) Not closed with respect to vector addition. (Not a vector space)

iv) Multiplication with 1 property is not satisfied. (Not a vector space)

v) Multiplication with 1 property is not satisfied. (Not a vector space)

vi) Vector space.

2. Which of the following are subspaces of P_3 (Set of all polynomials of degree$\leq$ 3)? Support your answer
   a. $U = \{ f(x) \mid f(x) \in P_3, f(2) = 1\}$
   b. $U = \{xg(x) + (1-x)h(x) \mid g(x) \text{ and } h(x) \in P_2\}$

Ans: a. zero vector is not in U. Hence not a subspace.

b . Subspace. (will satisfy all property)

3. Which of the following are subspaces of $M_{2\times2}$(Set of all 2 × 2 matrices)? Support your answer.
   a. $U = \{A \mid A \in M_{2\times2}, A^2 = A\}$
   b. $U = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a + b = c + d; a, b, c, d \in \mathbb{R}\right\}$
   c. $U = \{A \mid A \in M_{2\times2}, BAC = CAB\}$, B and C fixed 2×2 matrices.

Ans: a. Not closed with respect to vector addition. (Not a subspace)

b . Subspace (all properties will be satisfied).

c . Subspace (all subspace properties will be satisfied).

4. Write each of the following vector as linear combination of $x+1,\ x^2\ +x$, and x^2+2,
   I. x^2+3x+2, II. $2x^2-3x+1$, III. x^2+1.

Ans: $a(x+1)+b(x^2\ +x)+c(x^2+2)$

I. $a=2, b=1, c=0.$
II. $a=-3, b=-0, c=2.$
III. $a=-1/3, b=1/3, c=2/3.$

5. Consider the vectors $p_1\ =\ 1+x+4x^2\ and\ p_2\ =\ 1+5x+x^2$ in P_2. Determine whether p_1 and p_1 lie in span $\{1+2x-x^2, 3+5x+2x^2\}$.

Ans: $p_1, p_2 \notin$ span $\{1+2x-x^2, 3+5x+2x^2\}$.

6. Find the value of a such that the following subsets are linearly independent in $\mathbb{R}^3$

a. {(1, −1, 0), (a, 1, 0), (0, 2, 3)}

b. {(2, a, 1), (1, 0, 1), (0, 1, 3)}

Ans: a. $a \neq -1$, b. . $a \neq -1/3$.

7. Find the basis and dimension of the following

a. $w=\{a(1+x)\ +b(x+x^2)$ | a and $b\ in\ \mathbb{R}\}$, subspace of P_2 (set of polynomials of degree atmost 2)

b. w={p(x) | p(x) = p(−x)}, subspace of P_2 (set of polynomials of degree atmost 2)

c. $w=\{A|A^T=-A\}$, subspace of $M_{2\times 2}$ (set of all 2×2 matrices).

d. $w=\left\{A|A\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}A\right\}$, subspace of $M_{2\times 2}$ (set of all 2×2 matrices).

Ans: a. basis={$(1+x), (x+x^2)$}, dim=2.

b . basis=$\{1, x^2\}$, dim=2.

c . basis= $\left\{\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right\}$, dim=2.

d . basis= $\left\{\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right\}$, dim=2.

Module-3

1. Find the rank, basis and dimension of the row space, column space and null space of the following matrix

$$A=\begin{bmatrix}1 & -2 & 2 & 1 & 3\\ 1 & 1 & -1 & 7 & 5\\ 2 & -4 & 4 & 2 & 6\\ -1 & 2 & -2 & 3 & 4\end{bmatrix}.$$

Ans: row reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -4.42 \\ 0 & 1 & -1 & 0 & -2.83 \\ 0 & 0 & 0 & 1 & 1.75 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Row space=<(1,-2,2,1,3),(1,1,-1,7,5),(2,-4,4,2,6),(-1,2,-2,3,4)>
  Basis={(1,0,0,-4.42),(0,1,-1,0,-2.83),(0,0,0,1,1.75)}, dim=3.
- Column space=span{(1,1,2,-1),(-2,1,-4,2),(2,-1,4,-2),(3,5,6,4)}
  Basis={(1,0,0,0),(0,1,0,0),(0,0,1,0)}, dim=3.
- Null space i.e. solution space of AX=0.
  Basis={(0,1,1,0,0),(4.82,2.83,0,1.75,1)}, dim=2=Nullity.