Digital Assignment - I

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Applied Linear Algebra (MAT3004)

(1) LDU Factorization.

a) $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix}$

First find the Upper triangular matrix using row transformations.

$\Rightarrow A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$ E_1

$\approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 5 & 18 & 30 \end{bmatrix}$ $R_3 \to R_3 - 5R_1$ E_2

$\approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix}$ $R_3 \to R_3 - R_2$ E_3

$\approx \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ is the required upper triangular matrix

$\Rightarrow U = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

$\Rightarrow$ E_1^{-1} is $R_2 \to R_2 + 3R_1$

E_2^{-1} is $R_3 \to R_3 + 5R_1$

E_3^{-1} is $R_3 \to R_3 + R_2$

W.K.T. $E_n E_{n-1} \dots E_1 A = U$

$$\Rightarrow \quad A = \underbrace{E_1^{-1} E_2^{-1} \dots E_n^{-1}}_{L} \cdot U = LU$$

So, $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$, $E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$\Rightarrow L = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}$$

$A = LU$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}}_{U}.$$

Hence the given matrix is factorized into LDU..

b) $B = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

Finding the Upper triangular matrix using row operations.

$$B = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \qquad R_2 \to R_2 - \frac{b}{a} R_1 \quad E_1$$

$$\approx \begin{bmatrix} a & b \\ 0 & a - \frac{b^2}{a} \end{bmatrix}$$ is the required upper triangular matrix

$$\Rightarrow U = \begin{bmatrix} a & b \\ 0 & \frac{a^2-b^2}{a} \end{bmatrix}$$

$\Rightarrow E_1^{-1}$ is $R_2 \to R_2 + \frac{b}{a} R_1$

$$\Rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{bmatrix}$$

$$B = E_1^{-1} U$$

$$\Rightarrow L = E_1^{-1} = \begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}$$

$$B = L\ U$$

$$= \begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & \frac{a^2-b^2}{a} \end{bmatrix}$$

$$B = \underbrace{\begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a & b \\ 0 & \frac{a^2-b^2}{a} \end{bmatrix}}_{U}$$

Hence the given matrix is factorized into LDU.

<u>Relation between L and U:</u>

For any symmetric matrix of order 3x3,

1st row of L = a (1st column of U) and

1st column of L = a (1st row of U)

where $a \in \mathbb{R}$ and L and U are the lower triangular and upper triangular matrices obtained through LU decomposition. [a is the 1st element of the given matrix]

<u>NOTE</u>: Also, the Diagonal matrix (D) obtained through LDU decomposition is the Identity matrix of order 3 (I_3).

(2) $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

Applying row transformations to find elementary matrices

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 - cR_3$$

$$\approx \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 - bR_3$$

$$\approx \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 - aR_2$$

$$\approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is the reduced row Echelon form of A.

E_1 is $R_2 \to R_2 - cR_3 \Rightarrow E_1^{-1}$ is $R_2 \to R_2 + cR_3$

E_2 is $R_1 \to R_1 - bR_3 \Rightarrow E_2^{-1}$ is $R_1 \to R_1 + bR_3$

E_3 is $R_1 \to R_1 - aR_2 \Rightarrow E_3^{-1}$ is $R_1 \to R_1 + aR_2$

So, $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, $E_2^{-1} = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_3^{-1} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

WKT.

$$E_3 E_2 E_1 A = I_n$$

where I_n is reduced row echelon form of A, when A is invertible

$$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} I_n$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence A is represented as product of elementary matrices.

(3) Let V be a vector space.

To prove:

There exists unique element $y \in V$ such that $x + y = 0$. (where $x \in V$).

(3) Given.

V is a vector space,

$x \in V$ and $y \in V$.

T.P.

$\forall x \in V, \exists y \in V$ S.T $x+y=0$

Soln.

Since V is a vector space, for any $x \in V$, there always exists a vector $-x \in V$, called the negative of x, such that $x + (-x) = 0 \rightarrow (1)$

Now let's assume $x+y=0$ where $x, y \in V$. $\rightarrow (2)$.

On comparing (1) and (2), we can definitely say that $x+y=0$ only when $y=-x$. ($x \in V$, and $-x \in V$.)

So if $-x \in V$, then $y \in V$ and $x+y=0$.

$\Rightarrow x+y=0$, where $y=-x$ and $y \in V$

Hence Proved.

(4) Given:

Subset $W = \{(a_1, a_2 .. a_n) : a_1 + a_2 .. a_n = 1\}$.

T.P:

W is a subspace of $\mathbb{R}^n$ or not

Soln:

According to set W, Let $\alpha = (a_1, a_2, .. a_n)$

S.T $a_1 + a_2 + ... + a_n = 1$ and $a_1, a_2 .. a_n \in \mathbb{R}$

W.K.T:

i) The zero vector of $\mathbb{R}^n$ vector space is $(0_1, 0_2 ... 0_n)$.

$\Rightarrow 0_1 + 0_2 + 0_3 .. + 0_n = 0 \neq 1$

$\Rightarrow$ The zero vector $(0_1, 0_2 .. 0_n)$ doesnot belong to the set W because the sum of the coordinates is not 1.

Hence, we can conclude that W is not a subspace of the Vector Space $\mathbb{R}^n$.

(5) To check is following matrices are dependent / independent.

i) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A$

Applying row transformations to bring it to Reduced Row Echelon form,

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $R_1 \to R_1 - R_2$

(5) To check if the following matrices are dependent/independent.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Now, we know that

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\} \in M_{2\times 2}(\mathbb{R})$$

and lets take $c_1, c_2, c_3 \in \mathbb{R}$ which are from the scalar fields.

The linear combination of the given matrices would be,

$$c_1\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where, $\{a, b, c, d\} \in \mathbb{R}$.

To prove it is linearly independent, we assume $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\Rightarrow c_1\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 & c_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ c_2 & 0 \end{bmatrix} + \begin{bmatrix} c_3 & 0 \\ c_3 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_3 & c_1 + c_2 \\ c_2 + c_3 & c_1 + c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\Rightarrow c_1 + c_3 = 0 \Rightarrow c_1 = -c_3$

$c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \Rightarrow c_2 = c_3$

$c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$

WKT, $C_2 = C_3$ & $C_2 = -C_3$

$\Rightarrow C_2 = 0, \; C_3 = 0$

$\Rightarrow C_1 + C_3 = 0 \Rightarrow C_1 = 0.$

So we come to a conclusion that $C_1 = 0, C_2 = 0$ & $C_3 = 0$

only when $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence it is linearly independent.

(6) Give 3 different basis for $M_{2\times2}(\mathbb{R})$

i) Let $W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 2a+b-c=0 \right\}$ and, $a, b, c, d \in \mathbb{R}$

$\Rightarrow c = 2a+b.$

$\Rightarrow W_1 = \left\{ \begin{bmatrix} a & b \\ 2a+b & d \end{bmatrix} : c = 2a+b \right.$

This matrix can now be written as a linear combination of some matrices.

$$\Rightarrow \begin{bmatrix} a & b \\ 2a+b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = S$ and $S \subset M_{2\times2}(\mathbb{R})$

Hence we can say that span (S) exists for the given matrix set W_1.

Now to proove it is linearly independent, we assume

$$\begin{bmatrix} a & b \\ 2a+b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $a = 0$, $b = 0$, $2a + b = 0$ and $d = 0$

$\Rightarrow 2(0)+0=0$

$0=0$.

$\Rightarrow$ It is linearly independent.

Hence W_1, is a basis of $M_{2\times 2}(\mathbb{R})$.

ii) Let $W_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a-b=d \right\}$ where $a, b, c, d \in \mathbb{R}$.

Now this matrix can be written as a linear combination of matrices.

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & a-b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

where $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} = S$ and $S \subset M_{2\times 2}(\mathbb{R})$

Hence we say that span of (S) exists for the given matrix W_2.

Now to prove its linearly independent, we assume.

$$\begin{bmatrix} a & b \\ c & a-b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a=0,\ b=0,\ c=0,\ a-b=0$$

$a=b=0$.

$\Rightarrow$ It is lineary independent.

$\Rightarrow$ W_2 is linearly independent and $\text{span}(S) \subset M_{2\times 2}(\mathbb{R})$.

So W_2 is a basis of $M_{2\times 2}(\mathbb{R})$

iii) Let $W_3 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{array}{l} a+2b=c,\ a-2b=d \\ a,b,c,d \in \mathbb{R} \end{array} \right\}$

$$W_3 = \begin{bmatrix} a & b \\ a+2b & a-2b \end{bmatrix}$$

This can now be converted into linear combination of some matrices $\in M_{2\times2}(\mathbb{R})$.

$$\begin{bmatrix} a & b \\ a+2b & a-2b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}$$ where:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} \right\} = S$$ and $S \subset M_{2\times2}(\mathbb{R})$.

Hence we can say that span(S) exists for the given matrix set W_3.

To proove its linearly independent,

$$\begin{bmatrix} a & b \\ a+2b & a-2b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow a=0,\ b=0,$ $\quad a+2b=0$ $\quad a-2b=0$

$0+(0)=0$ $\quad 0-2(0)=0$

$0=0.$

$$\begin{bmatrix} a & b \\ a+2b & a-2b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ only when $a=0, b=0, c=0$ & $d=0$

Hence it is linearly independent.

So W_3 is a basis of $M_{2\times2}(\mathbb{R})$.

(7) Prove that null space (kernel) of a matrix $A_{m \times n}$ is a vector space.

Let A be an $m \times n$ real matrix, then the null-space $N(A)$ of A is defined by

$$N(A) = \{ x \in \mathbb{R}^n \mid Ax = O_m \}.$$

ie, the null space is the set of solutions to the homogeneous system $A\mathbf{x} = O_m$.

For this, we have to check 3 criterias.

i) The zero vector $O_n \in \mathbb{R}^n$ is in $N(A)$.

$\because$ $AO_n = O_m$, the vector $O_n \in \mathbb{R}^n$ is in $N(A)$.

ii) Take arbitrary vectors $x, y \in N(A)$. | If $x, y \in N(A)$, then $x+y \in N(A)$.

By defenition $Ax = O_m$ and $Ay = O_m$

$$\Rightarrow A(x+y) = Ax + Ay$$
$$= O_m + O_m$$
$$= O \qquad [\Rightarrow A(x+y) \in N(A)]$$

$\Rightarrow x+y \in N(A)$.

iii) If $x \in N(A)$ & $c \in \mathbb{R}$ then $cx \in N(A)$

Now take $x \in N(A)$ and a scalar $c \in \mathbb{R}$.

$\because$ $x \in N(A)$, $\Rightarrow Ax = O_m$.

Multiplying by scalar c on both sides,

$$cAx = cO_m$$

$\Rightarrow \quad cAx = O_m$

$Acx = O_m$

$A(cx) = O_m$

$\Rightarrow$ that $cx \in N(A)$.

So all the 3 conditions are met and concluded that the null space $N(A)$ is a subspace of $\mathbb{R}^n$.

(8) Find dimension of $M_{n\times n}(\mathbb{F})$

a) set of all diagonal matrices.

Let V be the vectorspace of the matrices $M_{n\times n}(\mathbb{F})$.

and let $W = \{ A \in V : A \text{ is a diagonal matrix} \}$.

$$\Rightarrow A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & a_{44} & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

is the generalised diagonal matrix $\in M_{n\times n}(\mathbb{F})$

where $\{ a_{11}, a_{22}, a_{33} \ldots a_{nn} \} \in \mathbb{F}$.

So A can be now written as linear combination as follows.

$$\Rightarrow A = a_{11}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \vdots \\ 0 & & & & \vdots \\ 0 & \cdots & & & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \cdots + a_{nn}\begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A = a_{11}E_{11} + a_{22}E_{22} + a_{33}E_{33} \cdots a_{nn}E_{nn}$$

where $\{ E_{11}, E_{22} \ldots E_{nn} \} \in M_{n\times n}(\mathbb{F})$ and E_{ij} is the matrix in which the only non zero element is in i^{th} row and j^{th} column, and the non-zero element is 1.

$\Rightarrow$ Basis, $\beta = \{ E_{11}, E_{22} \ldots E_{nn} \}$ is the basis for W.

$\Rightarrow \dim(W) = n$ because 'n' matrices are in β.

b) Set of all diagonal matrices whose traces are zero.

Let V be the vector space with square matrices $M_{n\times n}(\mathbb{F})$.

and $W = \{ A \in V : A$ is diagonal matrix with trace $= 0$.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$ is the generalised matrix $\in M_{n\times n}(\mathbb{F})$

and since its trace matrix, $\Rightarrow a_{11} + a_{22} \cdots + a_{nn} = 0$

$\Rightarrow a_{nn} = -a_{11} - a_{22} - a_{33} \cdots - a_{(n-1)(n-1)} \rightarrow (1)$

and $\{a_{11}, a_{22}, \cdots a_{nn}\} \in \mathbb{F}$

So A can be written in linear comb as.

$$A = a_{11}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \cdots a_{nn}\begin{bmatrix} 0 & & 0 \\ \vdots & 0 & 0 \\ 0 & \cdots 0 & 1 \end{bmatrix}$$

$$\Rightarrow A = a_{11}\begin{bmatrix} 1 & 0 & \cdots \\ 0 & & \vdots \\ \vdots & \cdots & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \cdots + (-a_{11} - a_{22} \cdots - a_{(n-1)(n-1)})\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & +1 \end{bmatrix}$$

$$A = a_{11}\begin{bmatrix} 1 & 0 & \cdots \\ \vdots & & \\ \vdots & & -1 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & -1 \end{bmatrix} + a_{33}\begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & & & \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\cdots + a_{(n-1)(n-1)}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ \vdots & +1 & 0 \\ 0 \cdots & 0 & -1 \end{bmatrix}$$

$= a_{11}E_{11} + a_{22}E_{22} \cdots a_{n-1(n-1)}E_{(n-1)(n-1)}$

on generalising the matrix, E_{ij} is the martix where the term $i = n, j = n$ is -1 and $i = j$ is 1.

and also, the set of matrices

$$E_{11}, E_{22} \cdots E_{(n-1)(n-1)} \in M_{n\times n}(\mathbb{F})$$

$\Rightarrow$ Basis, $\beta = \{E_{11}, E_{22} \cdots E_{(n-1)(n-1)}\}$ is the basis for W

and dimension of W is number of elements in β.

$\Rightarrow \dim(W) = n-1$

(c) Set of all symmetric matrices $(A^T = A)$.

Let V be the vector space with square matrices $M_{n\times n}(\mathbb{F})$.

and $W = \{A \in V : A$ is a symmetric matrix i.e. $(A^T = A)\}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ \vdots & & a_{32} & \vdots \\ & & & a_{n-1\,n} \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$ where $a_{12} = a_{21}$, $a_{13} = a_{31}$, $\cdots\cdots$, $a_{n(n-1)} = a_{(n-1)n}$.

and $\{a_{11}, a_{12} \cdots a_{nn}\} \in \mathbb{F}$

So A can be written as a linear combination as follows.

$$A = a_{11}\begin{bmatrix} 1 & 0 & 0 & \cdots \\ \vdots & & & \vdots \\ 0 & & & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \cdots \\ 0 & 0 & 0 \end{bmatrix} \cdots a_{nn}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cdots$$

$$\cdots + a_{12}\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & & \vdots \\ \vdots & & & \\ 0 & & & 0 \end{bmatrix} + a_{13}\begin{bmatrix} 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ & & & \ddots \end{bmatrix} + \cdots a_{(n-1)n}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & 0 \end{bmatrix}$$

$$A = \underbrace{\sum_{i=1}^{n} a_{ii} E_{ii}}_{n} + \sum_{\substack{i,j=1 \\ i\neq j,\ i<j}}^{n} a_{ij}(E_{ij} + E_{ji}).$$

where E_{ij} is the matrix which is symmetric.

Now finding the Basis, β

$$\beta = \left\{ \{E_{ii}\}_{i=1}^{n} \cup \{E_{ij} + E_{ji}\}_{\substack{i,j=1 \\ i<j}}^{n} \right\}$$

$\Rightarrow$ dimension is

$$\dim(W) = \underset{i=1,\, j=2 \text{ to } n}{n} + \underset{i=2,\, j=3 \text{ to } n}{(n-1)} + \underset{i=3,\, j=4 \text{ to } n}{(n-2)} \cdots + \underset{i=n-1,\, j=n}{1}$$

$$= n + (n-1) + (n-2) \cdots + 1$$ which is the sum of n natural numbers.

$$= \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

Hence the ~~basis~~ dimension for set of symmetric matrix is $\frac{n^2+n}{2}$.

(d) set of all skew-symmetric matrices ($A^T = -A$)

Let V be the V.S with square matrices $M_{n\times n}(\mathbb{F})$ and

$W = \{ A \in V : A$ is skew-symmetric matrix $(A^T = -A)\}$

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots \\ -a_{12} & 0 & a_{23} & \vdots \\ -a_{13} & -a_{23} & 0 & \\ \vdots & \cdots & & 0 \end{bmatrix}$$

$a_{ij} = -a_{ji} \quad i \neq j,$

$a_{ij} = a_{ji} \quad i = j$

This can be written as linear combination as,

$$A = a_{12}\begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & \ddots \; 0 \end{bmatrix} + a_{13}\begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & & \vdots \\ -1 & 0 & 0 & \cdots & \\ 0 & \cdots & & & 0 \end{bmatrix} \cdots a_{(n-1)n}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Now finding the basis B

$$B = \left\{ \left\{ E_{ij} - E_{ji} \right\}_{\substack{i,j=1 \\ i<j}}^{n} \right\}$$

$\Rightarrow$ The dimension will be,

$$\dim(W) = (n-1) + (n-2) \cdots + 2 + 1$$

$i=1, j=2$ to n $\quad$ $i=2, j=3$ to n $\quad$ $j=n-1$, $j=n-1$ to n

$$= 1 + 2 + 3 \cdots (n-2) + (n-1).$$

is the sum of $(n-1)$ natural numbers

$$= \frac{(n-1)(n-1+1)}{2}$$

$$\dim(W) = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$$

Hence the dimension for skew-symmetric matrix is $\frac{n^2-n}{2}$.

(9) Find basis for $M_{4\times4}(\mathbb{R})$

(a) Set of all diagonal matrices.

Let $W = \{A \in V : A \text{ is diagonal matrix}\}$

$$\Rightarrow A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \in M_{4\times4}(\mathbb{F}) \text{ and } \{a_{11}, a_{22}, a_{33} \,\&\, a_{44}\} \in \mathbb{R}.$$

$$\text{So, } A = a_{11}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + a_{33}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + a_{44}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \text{Basis, } \beta = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

(b) Set of all diagonal matrices with trace = 0.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \in M_{4\times4}(\mathbb{F}) \text{ and } \{a_{11}, a_{22}, a_{33}, a_{44}\} \in \mathbb{R}$$

also $a_{11} + a_{22} + a_{33} + a_{44} = 0$ $[\because \text{Trace}(A) = 0]$.

$$\Rightarrow a_{44} = -a_{11} - a_{22} - a_{33} = -(a_{11} + a_{22} + a_{33})$$

$$\Rightarrow A = a_{11}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + a_{33}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + a_{44}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= a_{11}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + a_{33}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (+a_{11} + a_{22} + a_{33})\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$A = a_{11}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix} + a_{22}\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix} + a_{33}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$$

$$\Rightarrow \text{Basis}, \beta = \left\{\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\right\}$$

(C) Set of real symmetric matrices. ($A^T = A$)

Let $W = \{ A \in V : A \text{ is symmetric matrix} \}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$ where $\{a_{11}, a_{12} \ldots a_{nn}\} \in \mathbb{F}$

and $a_{12} = a_{21}$, $a_{13} = a_{31}$,

i.e. $a_{ij} = a_{ji}$ (for $i \neq j$).

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & & \\ \vdots & & \ddots & \\ a_{1n} & & & a_{nn} \end{bmatrix}$$

Here $n = 4$

$\cancel{\Rightarrow A = a_{11}[}$

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$ where $\{a_{11}, \ldots a_{44}\} \in \mathbb{F}$

$$A = a_{11}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + a_{22}\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \ldots + a_{44}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix} +$$

$$+ a_{12}\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + a_{13}\begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix} \ldots + a_{34}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

$$\Rightarrow \text{Basis, } \beta = \left\{ \begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix} \right\}$$

$n = 4$

$$\Rightarrow \frac{n(n+1)}{2} = \frac{4(5)}{2} = 10 = \dim(W)$$

$\Rightarrow$ Number of matrices in basis $= 10$.

(d) Set of all skew-symmetric matrices ($A^T = -A$).

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}$$ is set of all skew-symmetric matrices of order 4×4.

i.e. $W = \{ A \in V : A$ is skew-symmetric matrix $\}$.

$$a_{ij} = -a_{ji}, \quad i \neq j$$
$$a_{ij} = a_{ji}, \quad i = j$$

$$A = a_{12} \begin{bmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} + a_{13} \begin{bmatrix} 0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0 \end{bmatrix} + a_{14} \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0 \end{bmatrix}$$

$$+ a_{23} \begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0 \end{bmatrix} + a_{24} \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{bmatrix}$$

$$+ a_{34} \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&+1\\0&0&-1&0 \end{bmatrix}$$

Hence, the basis,

$$\beta = \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \right\}$$

$n = 4$, $\Rightarrow \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 2 \times 3 = 6 = \dim(W)$.

$\Rightarrow$ Number of matrices is basis $= 6$.