Basis and dimension of sum and intersection of subspaces:-

Exercise:- 1) Let V be a real vector space. W_1 and W_2 be subspaces of V

Define $W_1+W_2 := \{x+y \mid x \in W_1,\ y \in W_2\}$.

Prove that W_1+W_2 is a subspace of V

2) Define $W_1 \cap W_2 = \{z \mid z \in W_1 \text{ and } z \in W_2\}$. Prove that $W_1 \cap W_2$ is a subspace of V.

From the above problems, it is clear that W_1+W_2 and $W_1 \cap W_2$ are subspaces of V.

Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$.

A basis of W_1 is $\{v_1, v_2, \dots, v_k\}$ and a basis of W_2 is $\{w_1, w_2, \dots, w_l\}$. To find a basis of $W_1 + W_2$ and $W_1 \cap W_2$, we proceed by the following steps.

(i) Form the matrix $Q = [v_1\ v_2 \dots v_k\ w_1, w_2 \dots w_l]$

(ii) Find a basis and dimension of $C(Q)$ and conclude that

Basis of $C(Q)$ = Basis of $W_1 + W_2$ and

dimension of $C(Q)$ = dimension of $(W_1 + W_2)$

(iii) Find a basis and dimension of $N(Q)$.

(iv) Suppose a basis of $N(Q)$ is

$$x_1 = (a_{11}, a_{12}, a_{13}, \dots, a_{1k}, b_{11}, b_{12}, \dots, b_{1l})$$
$$x_2 = (a_{21}, a_{22}, a_{23}, \dots, a_{2k}, b_{21}, b_{22}, \dots b_{2l})$$
$$\vdots$$
$$x_s = (a_{s1}, a_{s2}, a_{s3}, \dots a_{sk}, b_{s1}, b_{s2}, \dots b_{sl})$$

then a basis of $W_1 \cap W_2$ is

$$y_1 = a_{11} v_1 + a_{12} v_2 + \dots + a_{1k} v_k$$
$$y_2 = a_{21} v_1 + a_{22} v_2 + \dots + a_{2k} v_k$$
$$\vdots$$
$$y_s = a_{s1} v_1 + a_{s2} v_2 + \dots + a_{sk} v_k$$

Pb:1 Let W_1 and W_2 be two subspaces of $\mathbb{R}^4$ with bases

$\alpha = \{(1,1,1,0), (1,1,0,1)\}$ and $\beta = \{(1,1,1,0), (0,1,0,1)\}$.

Find a basis and dimension of $W_1 + W_2$ and $W_1 \cap W_2$.

Soln:- Let $v_1 = (1,1,1,0)$; $v_2 = (1,1,0,1)$; $w_1 = (1,1,1,0)$, $w_2 = (0,1,0,1)$

Form $Q = [v_1 \quad v_2 \quad w_1 \quad w_2] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. We first find a basis of $C(Q)$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} R_4 \leftrightarrow R_2 \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_3 \to R_3 + R_2 \\ R_1 \to R_1 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - R_3 \\ R_4 \to R_4 - R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

The leading 1's are in 1st, 2nd and 4th column of U. The basis of $C(Q)$ is the 1st, 2nd and 4th column of Q.

Basis of $C(Q) = \{(1,1,1,0), (1,1,0,1), (1,1,1,0)\}$

$=$ Basis of $W_1 + W_2$

dimension of $W_1 + W_2 = 3$.

Basis and dimension of $W_1 \cap W_2$

$$N(Q) = N(U) = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0 \\ x_4 &= 0 \end{aligned} \right\}$$

Here the basic variables are x_1, x_2, x_4 and free variable is x_3

so dimension of $N(Q) = 1$ = dimension of $W_1 \cap W_2$.

To find all the solutions of $N(Q)$ fix $x_3 = t$ then $x_1 = -t$; $x_2 = 0$; $x_4 = 0$

We write $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. A basis of $N(Q) = \{(-1, 0, 1, 0)\}$

Let us take $a_{11} = -1$; $a_{12} = 0$; $b_{11} = 1$; $b_{12} = 0$

basis of $W_1 \cap W_2 = a_{11} v_1 + a_{12} v_2 = (-1)(1,1,1,0) + 0 \cdot (1,1,0,1)$

$$= (-1, -1, -1, 0)$$

A basis of $W_1 \cap W_2 = \{(-1, -1, -1, 0)\}$.

Pb: 2 Let $W_1 = \{(a,b,c,d) \in \mathbb{R}^4 \mid b-c-d=0\}$

$W_2 = \{(a,b,c,d) \in \mathbb{R}^4 \mid a-b=0,\ c=2d\}$

Find a basis and dimension of W_1, W_2, W_1+W_2 and $W_1 \cap W_2$.

Soln:- Basis of W_1

we have $b-c-d=0$. In this system the basic variable is b and free variable is a, c, d.

To find all Solution fix $a=r$; $c=s$; $d=t$ then $b=s+t$

we write
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} r \\ s+t \\ s \\ t \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ A basis of $W_1 = \{(1,0,0,0), (0,1,1,0); (0,1,0,1)\}$

dimension of $W_1 = 3$.

Basis of W_2

$W_2 = \{(a,b,c,d) \in \mathbb{R}^4 \mid a-b=0;\ c=2d\}$

In the system $\begin{matrix} a-b=0 \\ c-2d=0 \end{matrix}$ the basic variables are a, c and free variables are b, d. Fix $b=r$; $d=t$ then $a=r$; $c=2t$

we have
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} r \\ r \\ 2t \\ t \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Thus a basis of $W_2 = \{(1,1,0,0), (0,0,2,1)\}$

dimension of $W_2 = 2$.

Basis of $W_1 + W_2$ Take $v_1 = (1,0,0,0)$, $v_2 = (0,1,1,0)$; $v_3 = (0,1,0,1)$

$w_1 = (1,1,0,0)$; $w_2 = (0,0,2,1)$

Form $Q = [v_1 \quad v_2 \quad v_3 \quad w_1 \quad w_2]$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 + R_3]{R_2 \to R_2 + R_3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_4]{R_1 \to R_1 + R_4} \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \xrightarrow[R_4 \to (-1)R_4]{R_3 \to (-1)R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} = U$$

The leading 1's are in 1st, 2nd, 3rd and 4th column of U.

So 1st, 2nd, 3rd and 4th column of Q will be a basis of $C(Q)$

Basis of $C(Q) = \{(1,0,0,0), (0,1,1,0), (0,1,0,1), (1,1,0,0)\}$

$=$ Basis of $(W_1 + W_2)$

dimension of $W_1 + W_2 = 4$.

Basis of $W_1 \cap W_2$

$$N(Q) = N(U) = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \;\middle|\; \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 + 3x_5 = 0 \\ x_2 + 2x_5 = 0 \\ x_3 + x_5 = 0 \\ x_4 - 3x_5 = 0 \end{array} \right\}$$

In the given system x_1, x_2, x_3, x_4 are basic variables, x_5 is the free variable. To find all solutions, take $x_5 = t$ then

$x_1 = -3t$; $x_2 = -2t$; $x_3 = -t$; $x_4 = 3t$. We write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3t \\ -2t \\ -t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ -2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

A basis of $N(Q)$ is $\{(-3, -2, -1, 3, 1)\}$.

Take $a_{11} = -3$; $a_{12} = -2$; $a_{13} = -1$

A basis of $W_1 \cap W_2 = (-3)v_1 + (-2)v_2 + (-1)v_3$

$$= -3(1,0,0,0) + (-2)(0,1,1,0) + (-1)(0,1,0,1)$$

$$= (-3, \quad -2-1, \quad -2, \quad -1)$$

A basis of $W_1 \cap W_2 = \{(-3, -3, -2, -1)\}$.

dimension of $W_1 \cap W_2 = 1$.

Theorem:- Let V be a real vector space. Suppose W_1 and W_2 be two subspaces of V then

$$\dim(W_1+W_2) = \dim(W_1) + \dim W_2 - \dim(W_1 \cap W_2)$$

Problem:- Let W_1 and W_2 be two subspaces of $\mathbb{R}^5$ with bases

$$\alpha = \{(1,3,-2,2,3),\ (1,4,-3,4,2),\ (1,3,0,2,3)\}$$

$$\beta = \{(2,3,-1,-2,4),\ (1,5,-6,6,1),\ (2,4,4,2,8)\}$$

Prove that $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

Soln:- Dimension W_1+W_2

$$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 3 & 5 & 4 \\ -2 & -3 & 0 & -1 & -6 & 4 \\ 2 & 4 & 2 & -2 & 6 & 2 \\ 3 & 2 & 3 & 9 & 1 & 8 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - 2R_1 \\ R_5 \to R_5 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & -3 & 2 & -2 \\ 0 & -1 & 2 & 3 & -4 & 8 \\ 0 & 2 & 0 & -6 & 4 & -2 \\ 0 & -1 & 0 & 3 & -2 & 2 \end{bmatrix}$$

$$\begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 2R_2 \\ R_5 \to R_5 + R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 5 & -1 & 4 \\ 0 & 1 & 0 & -3 & 2 & -2 \\ 0 & 0 & 2 & 0 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to \frac{R_3}{2} \\ R_4 \to \frac{R_4}{2} \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 5 & -1 & 4 \\ 0 & 1 & 0 & -3 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 1 \\ 0 & 1 & 0 & -3 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_4 \\ R_2 \to R_2 + 2R_4 \\ R_3 \to R_3 - 3R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

There are 4 leading 1s in the reduced row echelon form.

Thus $\dim(W_1 + W_2) = 4$

$$N(U) = \left\{ (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \;\middle|\; \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \;\middle|\; \begin{matrix} x_1 + 5x_4 = 0 \\ x_2 - 3x_4 + 2x_5 = 0 \\ x_3 - x_5 = 0 \\ x_6 = 0 \end{matrix} \right\}$$

The free variables are x_4, x_5. Thus

dimension of $N(U) = 2 =$ dimension of $(W_1 \cap W_2)$

Since $\dim W_1 = 3$; $\dim W_2 = 3$, we have.

$$\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = 3 + 3 - 2$$
$$= 4 = \dim(W_1 + W_2)$$

Hence proved.