Module 3

Row space, column space and null space

Let A be a $m \times n$ matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

We note that

$$\text{Row } 1 = r_1 = (a_{11}, a_{12}, a_{13}, \ldots, a_{1n})$$
$$\text{Row } 2 = r_2 = (a_{21}, a_{22}, a_{23}, \ldots, a_{2n})$$
$$\vdots$$
$$\text{Row } m = r_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$$

Since each r_i consists n co-ordinates, we may think each r_i is an element in $\mathbb{R}^n$.

$$\text{column } 1 = c_1 = (a_{11}, a_{21}, a_{31}, \ldots, a_{m1})$$
$$\text{column } 2 = c_2 = (a_{12}, a_{22}, a_{32}, \ldots, a_{m3})$$
$$\vdots$$
$$\text{column } n = c_n = (a_{1n}, a_{2n}, a_{3n}, \ldots, a_{mn})$$

Since each c_i consists m co-ordinates, we may think each c_i is an element in $\mathbb{R}^m$.

Row space of A is a subspace of $\mathbb{R}^n$ spanned by $\{r_1, r_2, \ldots, r_m\}$

Column space of A is a subspace of $\mathbb{R}^m$ spanned by $\{c_1, c_2, \ldots, c_n\}$.

We denote Row space by $R(A)$ and Column space by $C(A)$

Null space It is a subspace of the solution space of the homogeneous equation $AX = 0$. In symbols.

$$\text{Null space} = N(A) = \left\{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array} \right\}$$

Rank and Nullity Dimension of the column space is known as the rank of A. Dimension of the null space of A is known as nullity of A.

Remark:- For any matrix A

dimension of $R(A)$ = dimension of $C(A)$ = rank

Theorem:- Let A be a $m \times n$ matrix then

Rank + Nullity = n = no. of. columns in A.

(The above equation is known as Rank-Nullity Theorem)

Procedure to find a basis and dimension of row space, column space:-

1) Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

2) Find the reduced row echelon form U of A

3) The vectors in A corresponding to the columns containing leading 1's in U forms a basis for $C(A)$

4) The non zero rows of U forms a basis for $R(A)$

5) Find a basis and dimension of $N(U)$. Since $N(U) = N(A)$ conclude that basis and dimension of $N(A)$ is the basis and dimension of $N(U)$.

Pb:1 Find a basis and dimension of row space, column space and null space of

$$A = \begin{bmatrix} 1 & -3 & 2 & -3 & 9 \\ 2 & 0 & 1 & 3 & 3 \\ -2 & -4 & 1 & -9 & 7 \\ 1 & 3 & -1 & 6 & -6 \end{bmatrix}$$

Soln:-

$$\begin{bmatrix} 1 & -3 & 2 & -3 & 9 \\ 2 & 0 & 1 & 3 & 3 \\ -2 & -4 & 1 & -9 & 7 \\ 1 & 3 & -1 & 6 & -6 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & -3 & 9 \\ 0 & 6 & -3 & 9 & -15 \\ 0 & -10 & 5 & -15 & 25 \\ 0 & 6 & -3 & 9 & -15 \end{bmatrix} \begin{matrix} R_2 \to \frac{R_2}{3} \\ R_3 \to \frac{R_3}{-5} \\ R_4 \to \frac{R_4}{3} \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & -3 & 9 \\ 0 & 2 & -1 & 3 & -5 \\ 0 & 2 & -1 & 3 & -5 \\ 0 & 2 & -1 & 3 & -5 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & -3 & 9 \\ 0 & 2 & -1 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to \frac{R_2}{2}$$

$$\begin{bmatrix} 1 & -3 & 2 & -3 & 9 \\ 0 & 1 & -1/2 & 3/2 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 + 3R_2$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 3/2 & 3/2 \\ 0 & 1 & -1/2 & 3/2 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

In the reduced row echelon form leading 1's are in 1st and 2nd column and so 1st and 2nd column in A will be a basis of C(A)

Basis of $C(A) = \{(1, 2, -2, 1), (-3, 0, -4, 3)\}$

dimension of $C(A) = 2 =$ rank of A.

The non zero rows of U will be the basis of row space of A.

Basis of $R(A) = \{(1, 0, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}), (0, 1, \frac{-1}{2}, \frac{3}{2}, \frac{-5}{2})\}$

dimension of $R(A) = 2$.

Basis and dimension of $N(A)$

Since $N(U)=N(A)$, we find the basis and dimension of $N(U)$

$$U = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & \frac{-1}{2} & \frac{3}{2} & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N(U) = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \;\middle|\; U \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0 \right\}$$

$$= \left\{ (x_1, x_2, x_3, x_4, x_5) \;\middle|\; \begin{array}{l} x_1 + \frac{1}{2}x_3 + \frac{3}{2}x_4 + \frac{3}{2}x_5 = 0 \\ x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 - 5/2\, x_5 = 0 \end{array} \right\}$$

Since U is itself in reduced row echelon form, x_1, x_2 are basic variables and x_3, x_4, x_5 are free variables.

dimension of $N(U)$ = no. of free variables $= 3$

In order to find all the solutions of the system $UX=0$ we fix $x_3 = r$; $x_4 = s$; $x_5 = t$ then

$$x_1 = \frac{-1}{2}r - \frac{3}{2}s - \frac{3}{2}t \text{ and}$$

$$x_2 = \frac{1}{2}r - \frac{3}{2}s + 5/2\, t.$$

we write.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2}r - \frac{3}{2}s - \frac{3}{2}t \\ \frac{1}{2}r - \frac{3}{2}s + 5/2\, t \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{-3}{2} \\ \frac{-3}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{-3}{2} \\ 5/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus a basis of $N(U) = \left\{ \left(\frac{-1}{2}, \frac{1}{2}, 1, 0, 0\right), \left(\frac{-3}{2}, \frac{-3}{2}, 0, 1, 0\right), \left(\frac{-3}{2}, \frac{5}{2}, 0, 0, 1\right) \right\}$

$=$ Basis of $N(A)$.

Pb:2 Find a basis and dimension of the row space, column space and null space of

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 2 & 0 & 1 & -1 & 3 \\ 5 & -1 & 3 & 0 & 3 \\ 4 & -2 & 5 & 1 & 3 \\ 1 & 3 & -4 & -5 & 6 \end{bmatrix}$$

$$\begin{matrix} 1 & -5 & 3 \\ -6 & -6 & 6 \end{matrix}$$

Soln:-

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 2 & 0 & 1 & -1 & 3 \\ 5 & -1 & 3 & 0 & 3 \\ 4 & -2 & 5 & 1 & 3 \\ 1 & 3 & -4 & -5 & 6 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \\ R_4 \to R_4 - 4R_1 \\ R_5 \to R_5 - R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 3 \\ 0 & 4 & -7 & -5 & 3 \\ 0 & 2 & -3 & 3 & 3 \\ 0 & 4 & -6 & -6 & 6 \end{bmatrix} \begin{matrix} R_3 \to R_4 - 2R_2 \\ R_4 \to R_4 - R_2 \\ R_5 \to R_5 - 2R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 3 \\ 0 & 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to \frac{1}{2} R_2 \\ R_3 \to (-1) R_3 \end{matrix} \begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & -3/2 & -3/2 & 3/2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_2 + R_1$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 3/2 \\ 0 & 1 & -3/2 & -3/2 & 3/2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{1}{2} R_3 \\ R_2 \to R_2 + \frac{3}{2} R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 6 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The leading 1's in U are in 1st, 2nd and 3rd column, so the 1st, 2nd and 3rd column of A will be a basis of C(A)

Basis of $C(A) = \{(1,2,5,4,1), (-1,0,-1,-2,3), (2,1,3,5,-4)\}$

dimension of $C(A) = 3 = \text{rank}$.

The non zero rows of U forms a basis of R(A)

Basis of $R(A) = \{(1,0,0,0,0), (0,1,0,-3,6), (0,0,1,-1,3)\}$

dimension of $R(A) = 3$.

Pb:3 Let A be a 4×4 matrix with columns a_1, a_2, a_3, a_4. The reduced row echelon form of A is $U = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

If $a_1 = \begin{pmatrix} -3 \\ 5 \\ 2 \\ 1 \end{pmatrix}$ and $a_2 = \begin{pmatrix} 4 \\ -3 \\ 7 \\ -1 \end{pmatrix}$ then find a_3 and a_4.

Soln:-

In U the leading 1's are in 1st and 2nd column. So 1st and second column in A forms a basis of $C(A)$, then we have

Basis of $C(A) = \{a_1, a_2\} = \{(-3,5,2,1), (4,-3,7,-1)\}$

$$N(U) = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 + 2x_3 + x_4 = 0 \\ x_2 + x_3 + 4x_4 = 0 \end{array} \right\}$$

Here x_3, x_4 are free variables x_1, x_2 are basic variables.

Fix $x_3 = s$; $x_4 = t$ then $x_1 = -2s - t$; $x_2 = -s - 4t$.

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-t \\ -s-4t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ A basis of $N(A) = \{(-2,-1,1,0), (-1,-4,0,1)\}$.

Since $(-2, -1, 1, 0)$ is in the basis of $N(A)$ we have

$$A \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow (-2)a_1 + (-1)a_2 + a_3 = 0$$

$$\Rightarrow (-2)(-3, 5, 2, 1) + (-1)(4, -3, 7, -1) + a_3 = 0$$

$$\Rightarrow (6, -10, -4, -2) + (-4, 3, -7, 1) + a_3 = 0$$

$$\Rightarrow (2, -7, -11, -1) + a_3 = 0 \quad \therefore \boxed{a_3 = (-2, 7, 11, 1)}$$

Since $(-1, -4, 0, 1)$ is in the basis of $N(A)$, we have

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$-a_1 - 4a_2 + a_4 = 0 \Rightarrow a_4 = a_1 + 4a_2$$

$$= (-3, 5, 2, 1) + 4(4, -3, 7, -1)$$

$$\boxed{a_4 = (13, -7, 30, -3)}$$

Pb:4 Let A be a matrix of order 2021×2022. Suppose rank $A = 1011$ then what is nullity of A?

Ans:- We know that, rank A + nullity A = no. of. columns in A

$$1011 + \text{nullity } A = 2022$$

$$\boxed{\text{nullity } A = 1011}$$

Pb:5 Let A be a matrix of order 5×9. If nullity A is 6 then what is the dimension of $R(A)$?

Ans:- we know rank A + nullity $A = 9$

$\Rightarrow$ rank $A + 6 = 9 \Rightarrow$ rank $A = 3$.

we also know that dimension of $R(A)$ = rank A

$= 3$.

Pb:6 How many matrices of order 5×5 exists so that its rank and nullity are same?

Ans:- If rank = nullity then 2 rank $= 5$

this is not possible.

No such matrix exists.

Basis and dimension of null space of A :-

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 6 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad N(U) = \{ X \in \mathbb{R}^5 \mid UX = 0 \}$$

$$= \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \;\middle|\; \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 6 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \;\middle|\; \begin{array}{l} x_1 = 0 \\ x_2 - 3x_4 + 6x_5 = 0 \\ x_3 - x_4 + 3x_5 = 0 \end{array} \right\}$$

Here the free variables are x_4, x_5 and basic variables are x_1, x_2, x_3.

So the dimension of $N(U)$ is 2. To find all the solutions, we fix $x_4 = s$; $x_5 = t$ then $x_1 = 0$; $x_2 = 3s - 6t$; $x_3 = s - 3t$. So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3s - 6t \\ s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -6 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis of $N(U) = \{ (0, 3, 1, 1, 0), (0, -6, -3, 0, 1) \}$ = Basis of $N(A)$

dimension of $N(A) = 2$.