Pb:1 Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$ using Gauss Jordan elimination and write A is the product of elementary matrices. Write A^{-1} is the product of elementary matrix.

Ans:-

Augmented Matrix	Row operations	elementary matrix	Inverse of elementary matrix
1st pivot $\left[\begin{array}{ccc\|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$	$R_2 \to R_2 - 2R_1$	$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$	$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\left[\begin{array}{ccc\|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$	$R_3 \to R_3 - R_1$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E_2$	$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
$\left[\begin{array}{ccc\|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$	$R_2 \to (-1)R_2$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$	$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\left[\begin{array}{ccc\|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$ 2nd pivot	$R_1 \to R_1 - 2R_2$	$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_4$	$E_4^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{bmatrix} \qquad R_3 \to R_3 + 2R_2 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = E_5 \qquad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

3rd pivot

$$\begin{bmatrix} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{bmatrix} \qquad R_1 \to R_1 - R_3 \qquad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_6 \qquad E_6^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{bmatrix} \qquad R_2 \to R_2 - R_3 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = E_7 \qquad E_7^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 0 & -1 & +1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{bmatrix}$$

We stop the process. $A^{-1} = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1}$$

$$A^{-1} = E_7 E_6 E_5 E_4 E_3 E_2 E_1.$$

Permutation Matrix A permutation matrix is a square matrix obtained from the same size identity matrix by a permutation of rows.

Note:-

1) Permutation matrix is row equivalent to identity matrix.
2) Every permutation matrix is a product of elementary row-interchange matrices.
3) If P is a elementary permutation matrix then

$$P^T = P = P^{-1} \; ; \quad P^2 = I$$

4) A product of permutation matrix is a permutation matrix.
5) Inverse of a permutation matrix is a permutation matrix and $P^T = P^{-1}$
6) Left multiplication by a permutation matrix rearranges to the corresponding rows.
7) Right multiplication by a permutation matrix rearranges the corresponding columns.
8) Some power of permutation matrix is the identity.

Example:-

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ is the permutation matrix.

LU decomposition

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$E_3 \qquad E_2 \qquad E_1 \qquad A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = U \qquad = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$\boxed{A = LU}$$

Note:- 1) Let A be a $n \times n$ matrix. LU decomposition of A exists only if all major subminors matrices are invertible.

2) Let A be a $m \times n$ matrix. LU decomposition of A does not exists if we interchange two rows of A to get the upper triangular matrix.

Example:-

1) $\begin{bmatrix} \boxed{1} & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ the subminor $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not invertible so LU does not exist.

2) $A=\begin{pmatrix} 0 & 3 & 4 \\ 1 & 5 & 7 \\ 6 & 8 & 7 \end{pmatrix}$ does not have LU decomposition.