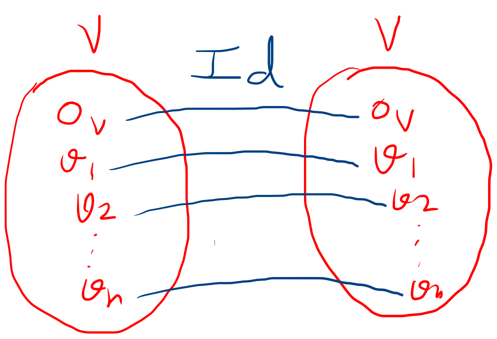
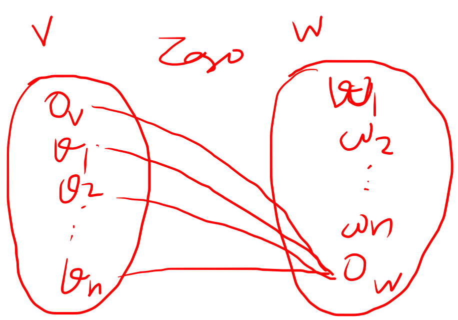
Identity linear transformation let V be a real vector space $I : V \to V$ defined by $I(v) = v$ is called as identity linear transformation.

Zero linear transformation let V and W be two real vector spaces $T : V \to W$ defined by $T(v) = 0_W$ is called a zero linear transformation





Example:- Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$. Construct a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ using A.

Soln:- $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(or) $T(x, y) = (3x - y,\ 2x + 5y)$

[Check, T is a linear transformation]

Remark:- Let A be a $m \times n$ matrix. Then the linear transformation defined using A is

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $T(X) = AX$.

Theorem:- Let $T: V \to W$ be a linear transformation of an n-dimensional vector space V into a vector space W. If $S = \{v_1, v_2, \ldots, v_n\}$ be a basis of V and

$$u = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n \quad \text{for some } k_1, k_2, \ldots k_n \in \mathbb{R}$$

then $T(u) = k_1 T(v_1) + k_2 T(v_2) + \cdots + k_n T(v_n)$.

Pb: 1 Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation for which we know that $T(1) = 1$; $T(t) = t^2$; $T(t^2) = t^3 + t$. Find

(a) $T(at^2 + bt + c)$ (b) $T(2t^2 - 5t + 3)$

Soln:- we know $\{1, t, t^2\}$ forms a basis of $P_2(\mathbb{R})$

Thus $at^2 + bt + c = a \cdot t^2 + b \cdot t + c \cdot t$

By the previous theorem, $T(at^2 + bt + c) = aT(t^2) + bT(t) + cT(1)$

$$= a(t^3 + t) + bt^2 + c \cdot 1$$

Thus $T(at^2 + bt + c) = at^3 + bt^2 + at + c.$

Hence, $T(2t^2 - 5t + 3) = 2t^3 - 5t^2 + 2t + 3.$

Pb: 2 let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation

Such that $T(1,0,0) = (3,2,1)$; $T(1,1,0) = (0,1,1)$; $T(1,1,1) = (0,0,2)$

find $T(x, y, z)$

soln:- Consider the collection $S = \{(1,0,0), (1,1,0), (1,1,1)\}$

If $k_1(1,0,0) + k_2(1,1,0) + k_3(1,1,1) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} k_1 + k_2 + k_3 &= 0; \\ k_2 + k_3 &= 0 \\ k_3 &= 0 \end{aligned}\right\} \Rightarrow k_1 = 0;\ k_2 = 0;\ k_3 = 0$$

Thus S is a linearly independent set and so it forms a basis.

Since $(x, y, z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1)$

Now, $T(x, y, z) = (x-y)\,T(1,0,0) + (y-z)\,T(1,1,0) + z\,T(1,1,1)$

$$= (x-y)(3,2,1) + (y-z)(0,1,1) + z(0,0,2)$$

$$= (3x-3y,\ 2x-2y+y-z,\ x-y+y-z+2z)$$

$$= (3x-3y,\ 2x-y-z,\ x+z)$$

$$\boxed{T(x, y, z) = (3x-3y,\ 2x-y-z,\ x+z)}$$

Pb:3 Let $T: P_1(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by

$$T(t+1) = t^2 - 1 \; ; \quad T(t-1) = t^2 + t$$

find (a) $T(at+b)$ (b) $T(7t+3)$

soln:- since $k_1(t+1) + k_2(t-1) = 0 \Rightarrow (k_1+k_2)t + (k_1-k_2) = 0$

$$\Rightarrow \left.\begin{array}{l} k_1+k_2=0 \\ k_1-k_2=0 \end{array}\right\} \Rightarrow \begin{array}{l} k_1=0 \\ k_2=0. \end{array}$$

Hence $\{(t+1), t-1\}$ is a linearly independent set and so it forms a basis.

Now, $at+b = \left(\frac{a+b}{2}\right)(t+1) + \left(\frac{a-b}{2}\right)(t-1)$

$$T(at+b) = \frac{a+b}{2}\, T(t+1) + \frac{a-b}{2}\, T(t-1)$$

$$= \frac{a+b}{2}(t^2-1) + \frac{a-b}{2}(t^2+t)$$

$$= at^2 + \left(\frac{a-b}{2}\right)t - \left(\frac{a+b}{2}\right)$$

and so $T(7t+3) = 7t^2 + \left(\frac{7-3}{2}\right)t - \left(\frac{7+3}{2}\right)$

$$= 7t^2 + 2t - 5.$$