Applied Linear Algebra MAT3004

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③ $T : R^3 \to R^3 \quad T(x, y, z) = (2x+y,\ y-z,\ 3x+y-z),$

$\forall\ (x, y, z) \in R^3$

<u>Ans</u>: $\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = (2,0,3) = 2\cdot(1,0,0) + 0\cdot(0,1,0) + 3\cdot(0,0,1)$

$T(0,1,0) = (1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (0,-1,-1) = 0(1,0,0) + -1(0,1,0) + -1(0,0,1)$

$$[T]_\alpha = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix} \quad (a)$$

(b) $[T]_\beta$? $\beta = \{(-1,1,2), (0,2,1), (1,1,4)\}$

$[T]_\beta = Q^{-1}[T]_\alpha Q, \quad Q = [Id]_\beta^\alpha \quad Q^{-1} = [Id]_\alpha^\beta$

$$Q = [Id]_\beta^\alpha = \cancel{[\ ]} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$ $\because$ α is standard basis.

$\cancel{T(1,0,0}\ Id(\cancel{-1,1,2})$

$Q^{-1} = [Id]_\alpha^\beta \quad -7/10 \quad 1/5 \quad 3/10$

$Id(1,0,0) = (1,0,0) = r_{11}(-1,1,2) + r_{12}(0,2,1) + r_{13}(1,1,4)$

$-r_{11} + r_{13} = 1, \quad r_{11} + 2r_{12} + r_{13} = 0, \quad 2r_{11} + r_{12} + 4r_{13} = 0$

$Id(0,1,0) = (0,1,0) = \frac{-1}{10}(-1,1,2) + \frac{3}{5}(0,2,1) + \frac{-1}{10}(1,1,4)$

$Id(0,0,1) = (0,0,1) = \frac{1}{5}(-1,1,2) + \frac{-1}{5}(0,2,1) + \frac{1}{5}(1,1,4)$

$$Q^{-1} = \begin{bmatrix} -7/10 & -1/10 & 1/5 \\ 1/5 & 3/5 & -1/5 \\ 3/10 & -1/10 & 1/5 \end{bmatrix}$$

$$[T]_\beta = Q^{-1}[T]_\alpha Q = \begin{bmatrix} -7/10 & -1/10 & 1/5 \\ 1/5 & 3/5 & -1/5 \\ 3/10 & -1/10 & 1/5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4/5 & -3/5 & -1/10 \\ -1/5 & 3/5 & -2/5 \\ 6/5 & 2/5 & -1/10 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -13/10 & -9/5 \\ 0 & 4/5 & -6/5 \\ -1 & 7/10 & 6/5 \end{bmatrix}$$

② ⓐ $T: M_{2\times 2} \to R^2$ $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{pmatrix} \max(a,b) \\ \min(a,b) \end{pmatrix}$

verify linear transform or not.

Ans: For property $T(x+y) = T(x) + T(y)$

Ex let $x = \begin{bmatrix} 100, & 0 \\ 1, & 2 \end{bmatrix}$, & $y = \begin{bmatrix} \cancel{200}\ 0, & 200 \\ 3, & 4 \end{bmatrix}$

$T(x) = \begin{pmatrix} \max(100,0) \\ \min(100,0) \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$

$T(y) = \begin{pmatrix} \max(0,200) \\ \min(0,200) \end{pmatrix} = \begin{pmatrix} 200 \\ 0 \end{pmatrix}$

$T(x+y) = T\left(\begin{bmatrix} 100, & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0, & 200 \\ 3 & 4 \end{bmatrix}\right) = T\left(\begin{bmatrix} 100, & 200 \\ 4 & 6 \end{bmatrix}\right)$

$= \begin{pmatrix} \max(100, 200) \\ \min(\cancel{100, 200}\ 100, 200) \end{pmatrix} = \begin{pmatrix} 200 \\ 100 \end{pmatrix}$

$T(x) + T(y) = \begin{pmatrix} 100 \\ 0 \end{pmatrix} + \begin{pmatrix} 200 \\ 0 \end{pmatrix} = \begin{pmatrix} 300 \\ 0 \end{pmatrix}$

$\therefore T(x+y) \neq T(x) + T(y) \Rightarrow T$ is not linear transformation

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ⓑ $T: P_2(R) \to R^3$, $T(a_0 + a_1 t + a_2 t^2) = (a_0 + 2a_1,\ a_2 - a_0,\ a_1 + 3a_2)$, $\forall (x, y) \in R^2$ or not.

(i) verify linear transformation

* $T(0) = 0$ $\quad T(0 + 0t + 0t^2) = (0, 0, 0)$ ✓

* $T(cx) = cT(x)$ ✓

$T(c(a_0 + a_1 t + a_2 t^2)) = T(ca_0 + ca_1 t + ca_2 t^2)$

$= (ca_0 + 2ca_1,\ ca_2 - ca_0,\ ca_1 + 3ca_2)$

$= c(a_0 + 2a_1,\ a_2 - a_0,\ a_1 + 3a_2) = \underline{cT(x)}$

ii) Find inverse

$T(a_0 + a_1 t + a_2 t^2) = (\underset{x}{\underset{\parallel}{a_0 + 2a_1}},\ \underset{y}{\underset{\parallel}{a_2 - a_0}},\ \underset{z}{\underset{\parallel}{a_1 + 3a_2}})$

$= (\ x,\ y,\ z\ )$

$\left[\begin{array}{ccc|c} 1 & 2 & 0 & x \\ -1 & 0 & 1 & y \\ 0 & 1 & 3 & z \end{array}\right] - \left[\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 0 & 1 & 3 & z \\ -1 & 0 & 1 & y \end{array}\right] \xrightarrow[\cancel{R_3 \to R_3/2}]{R_3 \to R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 0 & 1 & 3 & z \\ 0 & 2 & 1 & x+y \end{array}\right]$

$R_3 \to R_3 - 2R_2$
$R_3 \to R_3 / -5$
$\left[\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 0 & 1 & 3 & z \\ 0 & 0 & 1 & \frac{-x - y + 2z}{5} \end{array}\right]$

Rank (A) = Rank (A|B) = no of variable = 3

$\therefore$ invertible

also function is both one one and onto

⑤ Find QR decomposition of the matrix

$$\begin{array}{cccc} \begin{bmatrix} 2 & 1 & -1 & 4 \\ 1 & 2 & 2 & -3 \\ 2 & 1 & -1 & 0 \\ -3 & 0 & 0 & 1 \\ 4 & 1 & 1 & 2 \end{bmatrix} \\ v_1 \; v_2 \; v_3 \; v_3 \end{array}$$

$$\vec{u}_k = \vec{v}_k - \sum_{j=1}^{k-1} \text{Proj}_{\vec{u}_j}(\vec{v}_k)$$

$$\text{Proj}_{\vec{u}_j}(\vec{v}_k) = \frac{\langle \vec{u}_j, \vec{v}_k \rangle \vec{u}_j}{\| \vec{u}_j \|}$$

$$\vec{u}_1 = \vec{v}_1 = (2,1,2,-3,4)^T \quad \therefore \vec{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \left(\frac{\sqrt{34}}{17}, \frac{\sqrt{34}}{\cancel{34}}, \frac{\sqrt{34}}{17}, \frac{3\sqrt{34}}{34}, \frac{2\sqrt{34}}{17} \right)$$

$$\vec{u}_2 = \vec{v}_2 - \text{Proj}_{\vec{u}_1}(\vec{v}_2) = \vec{v}_2 - \frac{\langle \vec{u}_1, \vec{v}_2 \rangle \vec{u}_1}{\| \vec{u}_1 \|}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{\left\langle \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle}{\sqrt{\left\langle \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \right\rangle}} \right) \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7/17 \\ 29/17 \\ 7/17 \\ 15/17 \\ -3/17 \end{bmatrix}$$

$$\vec{e}_2 = \frac{\vec{u}_2}{|\vec{u}_2|} = \left(7/\sqrt{1173}, 29/\sqrt{1173}, 7/\sqrt{1173}, 5\sqrt{1173}/391, -\frac{\sqrt{1173}}{391} \right)$$

$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1}(\vec{v}_3) - \text{proj}_{\vec{u}_2}(\vec{v}_3)$$

$$= \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{\left\langle \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle}{\sqrt{\left[\begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \right]}} \right) \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} - \left(\frac{\left\langle \begin{bmatrix} 7/17 \\ 29/17 \\ 7/17 \\ 15/17 \\ -3/17 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle}{\sqrt{\langle u_2, u_2 \rangle}} \right) u_2$$

$u_3 = \left(-94/69,\ 64/69,\ -94/69,\ +8/23,\ 20/23\right)$

$\vec{e_3} = \vec{u_3}/\|\vec{u_3}\| = \left(-\frac{\sqrt{6486}}{136},\ \frac{16\sqrt{6486}}{3243},\ -\frac{\sqrt{6486}}{138},\ \frac{-2\sqrt{6486}}{1081},\ \frac{5\sqrt{6486}}{1081}\right)$

$\vec{u_4} = \vec{v_4} - \mathrm{Proj}_{\vec{u_1}}(\vec{v_4}) - \mathrm{Proj}_{\vec{u_2}}(\vec{v_4}) - \mathrm{Proj}_{\vec{u_3}}(\vec{v_4})$

$= \vec{v_4} - \frac{\langle u_1, v_4\rangle u_1}{\|u_1\|} - \frac{\langle u_2, v_4\rangle u_2}{\|u_2\|} - \frac{\langle u_3, v_4\rangle u_3}{\|u_3\|}$

$\vec{u_4} = \left(2,\ -42/47,\ -2,\ 98/47,\ 84/47\right)$

$\vec{e_4} = \vec{u_4}/\|\vec{u_4}\| = \left(\frac{\sqrt{141}}{24},\ \frac{-7\sqrt{141}}{376},\ -\frac{\sqrt{141}}{24},\ \frac{+49\sqrt{141}}{1128},\ \frac{7\sqrt{141}}{188}\right)$

The colums of the matrix Q are the orthonormalized vectors: Q

$$
\begin{array}{cccc}
\begin{bmatrix}
\frac{\sqrt{34}}{17} & \frac{7\sqrt{1173}}{1173} & -\frac{\sqrt{6486}}{138} & \frac{\sqrt{141}}{24} \\
\frac{\sqrt{34}}{34} & \frac{29\sqrt{1173}}{1173} & \frac{16\sqrt{6486}}{3243} & -\frac{7\sqrt{141}}{376} \\
\frac{\sqrt{34}}{17} & \frac{7\sqrt{1173}}{1173} & -\frac{\sqrt{6486}}{138} & -\frac{\sqrt{141}}{24} \\
-\frac{3\sqrt{34}}{34} & \frac{5\sqrt{1173}}{391} & -\frac{2\sqrt{6486}}{1081} & \frac{49\sqrt{141}}{1128} \\
\frac{2\sqrt{34}}{17} & -\frac{\sqrt{1173}}{391} & \frac{5\sqrt{6486}}{1081} & \frac{7\sqrt{141}}{188}
\end{bmatrix} \\
\vec{e_1} \quad \vec{e_2} \quad \vec{e_3} \quad \vec{e_4}
\end{array}
$$

$R = Q^T \cdot$ matrix

$$
\begin{bmatrix}
\frac{\sqrt{34}}{17} & \frac{\sqrt{34}}{34} & \frac{\sqrt{34}}{17} & -\frac{3\sqrt{34}}{34} & \frac{2\sqrt{34}}{17} \\
\frac{7\sqrt{1173}}{1173} & \frac{29\sqrt{1173}}{1173} & \frac{7\sqrt{1173}}{1173} & +\frac{5\sqrt{1173}}{391} & -\frac{\sqrt{1173}}{391} \\
-\frac{\sqrt{6486}}{138} & \frac{16\sqrt{6486}}{3243} & -\frac{\sqrt{6486}}{138} & -\frac{2\sqrt{6486}}{1081} & \frac{5\sqrt{6486}}{1081} \\
+\frac{\sqrt{141}}{24} & -\frac{7\sqrt{141}}{376} & -\frac{\sqrt{141}}{24} & +\frac{49\sqrt{141}}{1128} & \frac{7\sqrt{141}}{188}
\end{bmatrix}
\begin{bmatrix}
2 & 1 & -1 & 4 \\
1 & 2 & 2 & -3 \\
2 & 1 & -1 & 0 \\
-3 & 0 & 0 & 1 \\
4 & 1 & 1 & 2
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\sqrt{34} & \frac{5\sqrt{34}}{17} & \frac{\sqrt{34}}{17} & \frac{5\sqrt{34}}{17} \\
0 & \frac{\sqrt{1173}}{17} & \frac{41\sqrt{1173}}{1173} & -50/\sqrt{1173} \\
0 & 0 & \frac{2\sqrt{6486}}{69} & -118\sqrt{6486}/3243 \\
0 & 0 & 0 & 16\sqrt{141}/47
\end{bmatrix}
$$

Applied Linear Algebra MAT3004

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① $V = P_3(\mathbb{R})$, basis of W_1 $\alpha = \{1+t, 2t+t^2, 2+t^3\}$
and W_2 $\beta = \{t-2t^2, t+t^3\}$

ⓐ basis of $W_1 + W_2$ ⓑ basis of $W_1 \cap W_2$

Ans: ~~α~~ P_3 in form $a_0 + a_1 x + a_2 x^2$

$\therefore \alpha = \{(1,1,0,0), (0,2,1,0), (2,0,0,1)\}$

$\beta = \{(0,1,-2,0), (0,1,0,1)\}$

$\alpha = \{(1,1,0,0) \to v_1, (0,2,1,0) \to v_2, (2,0,0,1) \to v_3\}$

$\beta = \{(0,1,-2,0) \to x_1, (0,1,0,1) \to x_2\}$

Form $Q = [v_1, v_2, v_3, x_1, x_2]$

Find basis of $C(Q)$

$$= \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & -2 & 1 & 1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to \frac{R_2}{2}} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 \\ 0 & 0 & 1 & -5/2 & -1/2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 \\ 0 & 0 & 1 & -5/2 & -1/2 \\ 0 & 0 & 0 & 5/2 & 3/2 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4/(5/2)} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 \\ 0 & 0 & 1 & -5/2 & -1/2 \\ 0 & 0 & 0 & 1 & 3/5 \end{bmatrix}$$

Basis of $C(Q)$ = Basis of $W_1 + W_2$
$= \{(1,1,0,0), (0,2,1,0), (2,0,0,1), (0,1,-2,0)\}$

dim of $W_1 + W_2 = 3$

~~$N(Q) = N(U) = \{(a_0, a_1, a_2, a_3) \in \mathbb{R}$~~ ~~$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 \\ 0 & 0 & 1 & -5/2 & -1/2 \\ 0 & 0 & 0 & 1 & 3/5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0_V$~~

~~$\Rightarrow a_0 + 2a_2 = 0$~~

~~$a_1 - a_2 + 1/2$~~

Basis of $W_1 + W_2$ = $\{(1+t), (2t+t^2), (2+t^3), (t-2t^3)\}$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & -2 & 1 & 1 \\ 0 & 0 & 2 & -5 & -1 \\ 0 & 0 & 0 & 5 & 3 \end{bmatrix}$$

~~$x_5 = t$ $x_4 = \frac{-3t}{5}$~~

$x_5 = t$, $5x_4 + 3t = 0$, $x_4 = -\frac{3t}{5}$

$2x_3 - 5x_4 - x_5 = 0$, $2x_3 + 3t - t = 0$, $x_3 = -t$

$2x_3 - 2x_3 + x_4 + x_5 = 0$, $2x_3 + 2t - \frac{3t}{5} + t = 0$

$2x_2 + \frac{12t}{5} = 0$ $\quad x_2 = -\frac{6t}{5}$

$x_1 + 2x_3 = 0$, $x_1 - 2t = 0$, $x_1 = 2t$ //

Basis of nullspace $= \{2t, -\frac{6t}{5}, -t, -\frac{3t}{5}, t\}$

$= \{2, \frac{-6}{5}, -1, -3/5, 1\}$

Basis of $w_1 \cap w_2 = (t - 2t^2)(-\frac{3}{5}) + (t + t^3)$.

$= -\frac{3t}{5} + \frac{6t^2}{5} + t^2 + t^3 = t^3 + \frac{3t^2}{5} + \frac{6t^2}{5}$

$= t^3 + \frac{2t}{5} + \frac{6t^2}{5}$

Basis of $w_1 \cap w_2$ $= \{\frac{2t}{5}, \frac{6t^2}{5}, t^3\}$

$= t\{\frac{2}{5} + \frac{6}{5}t + t^2\}$

④ $(\mathbb{R}^3 \langle\rangle)$ $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = 2x_1y_1 + 3x_2y_2 + 5x_3y_3$

standard basis $\alpha = \{ \underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(0,0,1)} \}$

ⓐ i) matrix representation of inner product

$$A = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \langle v_1, v_3 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \langle v_2, v_3 \rangle \\ \langle v_3, v_1 \rangle & \langle v_3, v_2 \rangle & \langle v_3, v_3 \rangle \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} //$$

(ii) $v_1 = (1,3,2)$, $v_2 = (-2,2,2)$

angle b/w v_1 v_2

$\|v_1\| = \sqrt{14}$

$\|v_2\| = \sqrt{12}$

$$\theta = \cos^{-1}\left(\frac{\langle v_1, v_2 \rangle}{\|v_1\| \, \|v_2\|} \right)$$

$$= \cos^{-1}\left(\frac{\cancel{8} \cdot 4}{\cancel{2}\sqrt{7 \times 2 \times 3 \times \cancel{2}}} \right) = 78.6696^\circ //$$

$\text{norm}(v_1) = \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{12}} \right)$, $\text{norm}(v_2) = \left(\frac{-2}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right)$

4 (b) find orthonormal basis for

$\{ \overset{v_1}{1-t},\ \overset{v_2}{t+2t^2},\ \overset{v_3}{3t^2},\ \overset{v_4}{4t-t^3} \}$

~~$= \{ (1,-1,0,0), (0,1,2,0), (0,0,3,0), (4,0,0,-1) \}$~~

$\langle p(t), q(t) \rangle = \int_0^1 p(t)\, q(t)\, dt. \ \forall\ p(t)\ q(t) \in P_3$

$u_1 = v_1 \quad w_1 = \frac{u_1}{\|u_1\|}$

$\langle u_1, u_1 \rangle = \int_0^1 (1-t)(1-t)\, dt = \int_0^1 1+t^2-2t\, dt = \left[t + \frac{t^3}{3} - t^2 \right]_0^1$

$= 1 + \frac{1}{3} - 1 = 1/3, \quad \|u_1\| = \sqrt{\langle u_1, u_1 \rangle} = \sqrt{1/3}$

$w_1 = \frac{u_1}{\|u_1\|} = \frac{1-t}{\sqrt{1/3}} = \sqrt{3} - \sqrt{3}t$

$u_2 = v_2 - \langle v_2, w_1 \rangle w_1 = t + 2t^2 - \langle (t+2t^2), (\sqrt{3}-\sqrt{3}t) \rangle (\sqrt{3}-\sqrt{3}t)$

$\langle (t+2t^2), (\sqrt{3}-\sqrt{3}t) \rangle = \int_0^1 \sqrt{3}t + 2\sqrt{3}t^2 - 2\sqrt{3}t^3\, dt$

$= \left[\frac{\sqrt{3}t^2}{2} + \frac{\sqrt{3}t^3}{3} - \frac{2\sqrt{3}t^4}{4} \right]_0^1 = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{4} = \frac{1}{\sqrt{3}}$

$u_2 = t + 2t^2 - (1-t) = 2t^2 - 1 \quad w_2 = \frac{u_2}{\|u_2\|}$

$\langle u_2, u_2 \rangle = \int_0^1 4t^4 - 4t^2 + 1\, dt = \left[\frac{4}{5}t^5 - \frac{4}{3}t^3 + t \right]_0^1 = \frac{7}{15}$

$w_2 = \frac{2\sqrt{15}}{\sqrt{7}} t^2 - \sqrt{\frac{15}{7}}$

$u_3 = v_3 - \langle v_3, w_2 \rangle w_2 - \langle v_3, w_1 \rangle w_1$

$\langle v_3, w_2 \rangle = \int_0^1 (3t^2)\left(2\sqrt{\frac{15}{7}} t^2 - \sqrt{\frac{15}{7}} \right) dt = \int_0^1 6\sqrt{\frac{15}{7}} t^4 - 3\sqrt{\frac{15}{7}} t^2\, dt$

$= \left[\frac{6}{5} \sqrt{\frac{15}{7}} t^5 - \frac{3}{3} \sqrt{\frac{15}{7}} \right]_0^1 = \frac{1}{5} \sqrt{\frac{15}{7}}$

$\langle v_3, w_1 \rangle = \int_0^1 3t^2 (\sqrt{3} - \sqrt{3}t)\, dt = \int_0^1 -3\sqrt{3}t^3 + 3\sqrt{3}t^2\, dt$

$= \left[-\frac{3}{4}\sqrt{3}t^4 + \sqrt{3}t^3 \right]_0^1 = \frac{1}{4}\sqrt{3}$

$u_3 = 3t^2 - \frac{1}{5}\sqrt{\frac{15}{7}} \left(2\sqrt{\frac{15}{7}} t^2 - \sqrt{\frac{15}{7}} \right) - \frac{1}{4}\sqrt{3} (\sqrt{3} - \sqrt{3}t)$

$= 3t^2 - \frac{2}{5}t^2 + \frac{1}{5} - \frac{1}{4} + \frac{1}{4}t = \frac{13t^2}{5} + \frac{1}{4}t - \frac{1}{20}$

$\langle u_3, u_3 \rangle = \sqrt{\int_0^1 \left(\frac{13t^2}{5} + \frac{1}{4}t - \frac{1}{20} \right)\left(\frac{13}{5}t^2 + \frac{1}{4}t - \frac{1}{20} \right) dt} = \sqrt{\frac{9607}{6000}}$

$$w_3 = \frac{13}{5}\sqrt{\frac{6000}{9607}}\,t^2 + \frac{1}{4}\sqrt{\frac{6000}{9607}}\,t - \frac{1}{20}\frac{\sqrt{6000}}{\sqrt{9607}}$$

$$u_4 = v_4 - \langle v_4, w_1\rangle w_1 - \langle v_4, w_2\rangle w_2 - \langle v_4, w_3\rangle w_3$$

$$= (4t - t^3) - w_1 \int_0^1 (4t - t^3)(\sqrt{3}\,t \div t)\,dt - w_2 \int_0^1 (t \div t^3) -$$

$$(\sqrt{5}\cdot(2t^2 + 2t - 1)/3 - w_3 \int_0^1 (4t - t^3)\left(40t^2 - \frac{41}{3}t + 7\right)$$

$$= (4t - t^3) - \frac{37}{20}(1 - t^2) \times \frac{131}{108} + \frac{53}{540}(40t^2 - 41t + 7)$$

$$= (2160t - 540t^3 - 999 + 999t - 485130t^2 - 1310t +$$

$$6550 + 2120t^2 - 2173t + 371)/540$$

$$= (-540t^3 + 810t^2 - 342t + 27)/540$$

$$u_4 = (-20t^3 + 30t^2 - 12t + 1)/20$$

$$w_4 = \frac{u_4}{\|u_4\|} = -20t^3 + 3t^2 - 12t + 1 \Big/ \int_0^{1/2} (-20t^3 + 30t^2 - 12t)^2\,dt$$

$$= \sqrt{7}(-20t^3 + 30t^2 - 12t + 1)$$

hence orthogonal basis are

$$= \left\{ \sqrt{3}(1-t),\ 2\sqrt{\frac{15}{17}}\,t^2 - \sqrt{\frac{15}{17}},\ \cancel{\frac{13}{5}\sqrt{\frac{6000}{9607}}\,t^2}\right.$$

$$\left.\frac{1}{3}(4\cdot t^2 - 41t + 7),\ \sqrt{7}(-20t^3 + 30t^2 - 12t + 1)\right\}$$