Basis and dimension

Definition Let V be a real vector space and $\{u_1, u_2, \cdots, u_k\}$ be a collection of vectors in V. We say $\{u_1, u_2, \cdots, u_k\}$ forms a basis of V if

(a) $\{u_1, u_2, \cdots, u_k\}$ is a linearly independent set.

(b) $\{u_1, u_2, \cdots, u_k\}$ spans V.

Dimension Let V be a real vector space and S be a basis of V. The number of elements in S is known as dimension of V.

Standard basis

1) Consider the collection $\{(1,0),(0,1)\}$ in $\mathbb{R}^2$. Then we know that $\{(1,0),(0,1)\}$ spans $\mathbb{R}^2$ and $\{(1,0),(0,1)\}$ is a linearly independent set. Hence $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$. We call this as standard basis of $\mathbb{R}^2$. Dimension of $\mathbb{R}^2$ is 2.

2) The collection $\{e_1, e_2, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$ where $e_i = (0,0,\dots,0,\underset{i^{th}\text{ position}}{\boxed{1}},0,0,\dots,0)$. Dimension of $\mathbb{R}^n$ is n.

3) The collection $\{1, t, t^2\}$ spans $P_2(\mathbb{R})$ and it is also a linearly independent set. So $\{1, t, t^2\}$ forms a basis of $P_2(\mathbb{R})$. Dimension of $P_2(\mathbb{R})$ is 3

4) The collection $\{1, t, t^2, \dots, t^n\}$ forms a basis of $P_n(\mathbb{R})$. We call this as standard basis of $P_n(\mathbb{R})$. Dimension of $P_n(\mathbb{R})$ is $n+1$.

5) The collection $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ spans $M_{2\times2}(\mathbb{R})$ and it is a linearly independent set, so $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ forms a basis of $M_{2\times2}(\mathbb{R})$. Dimension of $M_{2\times2}(\mathbb{R})$ is 4

6) The collection $\{E_{ij} \in M_{m\times n}(\mathbb{R}) \mid E_{ij}$ is a $m\times n$ matrix whose $(i,j)^{th}$ entry is 1 and other entries are zero$\}$ forms a basis of $M_{m\times n}(\mathbb{R})$. We call this as standard basis of $M_{m\times n}(\mathbb{R})$. Dimension of $M_{m\times n}(\mathbb{R})$ is mn.

Example:- Consider the collection $\{(1,0),(0,1),(1,1)\}$ in $\mathbb{R}^2$. We note that $(1,0)+(0,1)-(1,1)=(0,0)$. Hence $\{(1,0),(0,1),(1,1)\}$ is not a basis. We also note the following

$$(a,b)=a(1,0)+b(0,1)+0\cdot(1,1)$$
$$(a,b)=(a-b)(1,0)+0\cdot(0,1)+b(1,1)$$
$$(a,b)=0\cdot(1,0)+(b-a)(0,1)+a(1,1)$$

The above says that every vector in $\mathbb{R}^2$ has more than one linear combination of $\{(1,0),(0,1),(1,1)\}$.

Theorem:- Let $\alpha=\{u_1,u_2,\ldots,u_k\}$ be a basis of a vector space V. Then each vector in V can be uniquely expressed as a linear combination of vectors in α.

Example:- The collections $E_1=\{(1,0),(0,1)\}$

$$E_2=\{(1,0),(1,1)\}$$
$$E_3=\{(4,5),(-2,-3)\}$$

clearly E_1, E_2, E_3 are all bases of $\mathbb{R}^2$. We note that every basis has same number of elements. (means dimension is unique)

Theorem:- If a basis for a vector space V consists of n vectors then so does every other basis.

Theorem:- Let V be a real vector space with dimension n.

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of n-vectors in V

(a) If S is linearly independent then it is a basis

(b) If S spans V then it is a basis of V.

Pb:- 1 Is the collection of vectors $S = \{v_1 = (0,0,1,1),\ v_2 = (-1,1,1,2),\ v_3 = (1,1,0,0),\ v_4 = (2,1,2,1)\}$ forms a basis of $\mathbb{R}^4$?

Ans:- We know that dimension of $\mathbb{R}^4$ is 4. Since S has 4 vectors, it is enough to check S is a linearly independent set.

We form the equation $x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = (0,0,0,0)$

$\Rightarrow x_1(0,0,1,1) + x_2(-1,1,1,2) + x_3(1,1,0,0) + x_4(2,1,2,1) = (0,0,0,0)$

$$\Rightarrow \begin{aligned} -x_2 + x_3 + 2x_4 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + \quad 2x_4 &= 0 \\ x_1 + 2x_2 \quad + x_4 &= 0 \end{aligned}$$

. We solve this Gauss elimination.

$$\begin{bmatrix} 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 2 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow[R_4 \to R_4 + R_2]{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 & 0 \end{bmatrix} \quad R_4 \to R_4 - 2R_3$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The reduced system
$$\begin{aligned} x_1 + 2x_2 \quad + x_4 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ x_3 + 2x_4 &= 0 \\ -x_4 &= 0 \end{aligned}$$

we get $x_1 = 0$; $x_2 = 0$; $x_3 = 0$; $x_4 = 0$.

Hence S is a linearly independent set and S forms a basis of $\mathbb{R}^4$.