Polynomial vector space consider the set

$$P_n(\mathbb{R}) = \{ a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 \mid a_n, a_{n-1}, a_{n-2}, \ldots a_0 \in \mathbb{R} \}$$

For $m, r \leq n$ and $m > r$, we define addition by

$$(a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0) + (b_r x^r + b_{r-1} x^{r-1} + \cdots + b_1 x + b_0)$$
$$= a_m x^m + a_{m-1} x^{m-1} + \cdots + (a_r + b_r) x^r + (a_{r-1} + b_{r-1}) x^{r-1} + (a_1 + b_1) x + (a_0 + b_0)$$

For $m \leq n$, we define the scalar multiplication by

$$k(a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0) = (k a_m) x^m + (k a_{m-1}) x^{m-1} + \cdots + (k a_1) x + (k a_0)$$

Then $P_n(\mathbb{R})$ is a real vector space with respect to the addition and scalar multiplication defined above.

Example 1) $P_1(\mathbb{R}) = \{ a_1 x + a_0 \mid a_1, a_0 \in \mathbb{R} \}$
= set of all polynomials of degree atmost 1.

Zero vector is zero polynomial

2) $P_2(\mathbb{R}) = \{ a_2 x^2 + a_1 x + a_0 \mid a_2, a_1, a_0 \in \mathbb{R} \}$
= set of all polynomials of degree atmost 2

3) $P_3(\mathbb{R}) = \{ a_3 x^3 + a_2 x^2 + a_1 x + a_0 \mid a_3, a_2, a_1, a_0 \in \mathbb{R} \}$.
= set of all polynomials of degree atmost 3

Note:- The addition and scalar multiplication defined above is known as standard addition and standard scalar multiplication on $P_n(\mathbb{R})$.

Matrix vector Spaces: We denote a $m \times n$ matrix by $[a_{ij}]_{m\times n}$

Consider the set

$$M_{m\times n}(\mathbb{R}) = \left\{ [a_{ij}]_{m\times n} \;\middle|\; a_{ij} \in \mathbb{R} \right\}$$

We define addition on $M_{m\times n}(\mathbb{R})$ by

$$[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$$

We define scalar multiplication on $M_{m\times n}(\mathbb{R})$ by

$$k[a_{ij}]_{m\times n} = [k a_{ij}]_{m\times n}$$

Then $M_{m\times n}(\mathbb{R})$ is a real vector space with respect to the addition and scalar multiplication defined above.

Example:- 1) $M_{2\times 3}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} : a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23} \in \mathbb{R} \right\}$

2) $M_{3\times 3}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{32} & a_{33} \end{bmatrix} : \begin{array}{l} a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, \\ a_{31}, a_{32}, a_{33} \in \mathbb{R} \end{array} \right\}$

Note:- The addition and scalar multiplication defined above are known as standard addition and standard scalar multiplication on $M_{m\times n}(\mathbb{R})$.

Subspaces:– Let V be a real vector space. A non empty subset W of V is said to be a subspace of V if it possess the following properties

(a) $0_V \in W$

(b) $x+y \in W$ for all $x, y \in W$

(c) $kx \in W$ for all $x \in W$, $k \in \mathbb{R}$

Example: 1 Consider the real vector space $\mathbb{R}^2$ with respect to standard addition and standard scalar multiplication. Does the set

$$W = \{(x,y) \in \mathbb{R}^2 \mid 2x - 5y = 0\}$$ forms a subspace of $\mathbb{R}^2$?

Ans:– 1) Clearly $(0,0) \in W$ because $2 \cdot 0 - 5 \cdot 0 = 0$

2) $(x_1, y_1), (x_2, y_2) \in W$ then $2x_1 - 5y_1 = 0$ and
$2x_2 - 5y_2 = 0$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

We note that $2(x_1 + x_2) - 5(y_1 + y_2) = (2x_1 - 5y_1) + (2x_2 - 5y_2)$
$= 0 + 0 = 0$

Thus $(x_1 + x_2, y_1 + y_2) \in W$.

3) $k(x_1, y_1) = (kx_1, ky_1)$

Now, $2(kx_1) - 5(ky_1) = k(2x_1 - 5y_1) = k \cdot 0 = 0$

Thus W is a vector subspace of $\mathbb{R}^2$.

Pb:2 Consider the real vector space $\mathbb{R}^3$ with standard addition and standard scalar multiplication. Let

$$W_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x - 2y + 5z = 0\} \rightarrow \text{plane passing through origin}$$

$$W_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x \le 0;\ y \le 0\} \rightarrow$$

$$W_3 = \{(x,y,z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2\}$$

Is W_1, W_2, W_3 are vector subspaces of $\mathbb{R}^3$?

Ans:- For W_1 1) clearly $(0,0,0) \in \mathbb{R}^3$ because $0 - 2\cdot 0 + 5\cdot 0 = 0$

2) Let $(x_1,y_1,z_1), (x_2,y_2,z_2) \in W_1$ then $x_1 - 2y_1 + 5z_1 = 0$ and

$$x_2 - 2y_2 + 5z_2 = 0$$

$(x_1,y_1,z_1) + (x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$. Now

$$(x_1+x_2) - 2(y_1+y_2) + 5(z_1+z_2) = (x_1 - 2y_1 + 5z_1) + (x_2 - 2y_2 + 5z_2)$$

$$= 0 + 0 = 0$$

Hence $(x_1+x_2, y_1+y_2, z_1+z_2) \in W_1$

3) Let $k \in \mathbb{R}$, $k(x_1,y_1,z_1) = (kx_1, ky_1, kz_1)$. Now

$$kx_1 - 2ky_1 + 5kz_1 = k(x_1 - 2y_1 + 5z_1) = k\cdot 0 = 0$$

Thus $k(x_1,y_1,z_1) \in W_1$

W_1 is a subspace.

<u>For W_2</u> consider the vector $x = (-1, -1, 0)$ in W_2 and scalar $k = -5$

$$kx = (-5)(-1, -1, 0) = (5, 5, 0) \notin W_2$$

Hence W_2 is not a subspace.

<u>For W_3</u> consider the vector $X = (1, 1, \sqrt{2})$; $Y = (1, 1, -\sqrt{2})$ in W_3

$$X + Y = (2, 2, 0) \notin W_3.$$

Hence W_3 is not a subspace.

<u>Remarks:-</u> 1) The only subspaces of $\mathbb{R}^2$ is the line passing through origin

2) The only subspaces of $\mathbb{R}^3$ is the line and plane passing through origin.

3) For any vector space V, $\{0_V\}$ and V are the trivial subspaces.

4) Consider the set $W_n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_n\}$

W_n forms a subspace of $\mathbb{R}^n$ only if $b_n = 0$

In the real vector space $P_2(\mathbb{R})$ with respect to standard addition and standard scalar multiplication, consider the sets

$$E_1 = \{a_0 + a_1 t + a_2 t^2 \mid a_0 = 0\}$$
$$E_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_0 + a_1 + a_2 = 3\}$$

Does E_1, E_2 form a vector subspace of $P_2(\mathbb{R})$?

Ans:- 1) clearly $0 \in E_1$ (zero polynomial)

2) Let $p(t) = a_0 + a_1 t + a_2 t^2$ then $a_0 = 0$

$q(t) = b_0 + b_1 t + b_2 t^2$ then $b_0 = 0$

$p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$ we also have $a_0 + b_0 = 0$

Thus $p(t) + q(t) \in E_1$

3) Let $k \in \mathbb{R}$, $kp(t) = k(a_0 + a_1 t + a_2 t^2) = (ka_0) + (ka_1)t + (ka_2)t^2$

since $a_0 = 0$, we have $ka_0 = 0$

Thus $kp(t) \in E_1$. Hence E_1 is a subspace.

E_2 is not a subspace, because zero polynomial is not in E_2.

Problem 3:- In the real vector space $M_{n\times n}(\mathbb{R})$, consider the set

$W_1 = \{A \in M_{n\times n}(\mathbb{R}) \mid A = A^T\}$ and (set of symmetric matrices)

$W_2 = \{A \in M_{n\times n}(\mathbb{R}) \mid A = -A^T\}$ (set of anti symmetric matrices)

Does W_1, W_2 forms a vector subspace of $M_{n\times n}(\mathbb{R})$?

Ans:- (1) Let $A, B \in W_1$ then $A = A^T$; $B = B^T$

$$(A+B)^T = A^T + B^T = A + B \quad \text{Thus } A+B \in W_1$$

(2) Let $k \in \mathbb{R}$, $(kA)^T = kA^T = kA$ thus $kA \in W_1$

(3) clearly zero matrix is a symmetric matrix, $0 \in W_1$

Hence W_1 is a subspace.

(check!) W_2 is a subspace.