Orthogonal subspaces:- Let V be an inner product space, U and W be two subspaces of V. U and W are said to be orthogonal if

$\langle x, y \rangle = 0$ for all $x \in U$ and $y \in W$

Example:-

Consider $\mathbb{R}^2$ with usual inner product and subspaces

$$U = \{(x, \tfrac{1}{2}x) \mid x \in \mathbb{R}\} = \text{line } y = \tfrac{1}{2}x$$

$$W = \{(\tfrac{-1}{2}x, x) \mid x \in \mathbb{R}\} = \text{line } -2x = y$$

Then U and W are orthogonal.

Take $(x_0, \frac{1}{2}x_0) \in U$ and $(\frac{-1}{2}y_0, y_0) \in W$

$$\langle (x_0, \tfrac{1}{2}x_0), (\tfrac{-1}{2}y_0, y_0) \rangle = \frac{-1}{2}x_0y_0 + \frac{1}{2}x_0y_0$$
$$= 0$$

$\therefore$ U and W are orthogonal.

Note:- If U and W are orthogonal subspaces then we write it as $U \perp W$

Orthogonal Complement:-

Let U be a subspace of an inner product space V then the set

$$\{ y \in V \mid \langle x, y \rangle = 0 \text{ for all } x \in U \}$$

is called orthogonal complement of U.

we write it as $U^{\perp}$ (U perp)

Note:- $U^{\perp}$ is a subspace of V.

Example:- Consider the inner product space $\mathbb{R}^2$ with standard inner product. Suppose $U = \{(x, x) \mid x \in \mathbb{R}\}$ is a subspace. Find $U^\perp$

Soln:-

$$U^\perp = \{(z_1, z_2) \in \mathbb{R}^2 \mid \langle (z_1, z_2), (x, x) \rangle = 0 \text{ for all } (x, x) \in U\}$$

$$= \{(z_1, z_2) \in \mathbb{R}^2 \mid x z_1 + x z_2 = 0\}$$

$\therefore$ A suitable choice for $z_1 = x$ and $z_2 = -x$

Thus $U^\perp = \{(x, -x) \mid x \in \mathbb{R}\}$

Theorem:- Let V be an inner product space U be a subspace of V then

(a) $\dim U + \dim U^{\perp} = V$

(b) $(U^{\perp})^{\perp} = U$

(c) $V = U \oplus U^{\perp}$

Projections:- Let U and W be subspaces of an inner product space V. A linear transformation $T : V \to V$ is called a projection of V onto the subspace U along W if

(a) $V = U \oplus W$

(b) $T(x) = u$ for $x = u + w \in U \oplus W$.

Orthogonal Projections Let V be an inner product space and U be a subspace of V so that $V = U \oplus U^{\perp}$. The projection of V onto U along $U^{\perp}$ is called orthogonal projection of V onto U denoted as Proj_U.

Note:- For $x \in V$ the component vector $\mathrm{Proj}_U(x) \in U$ is called the orthogonal projection of x into U.

Theorem:- Let U be a subspace of an inner product space V and let $\{u_1, u_2, \ldots, u_m\}$ be an orthonormal basis of U. Then for any $x \in V$ the orthogonal projection $Proj_U(x)$ is given by

$$Proj_U(x) = \langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2 + \cdots + \langle x, u_m \rangle u_m$$

Theorem:- U be a subspace of an inner product space V. Let $x \in V$ then

$$\| x - Proj_U(x) \| \leq \| x - y \| \text{ for all } y \in U.$$

Pb. 1 Let W be the subspace of $\mathbb{R}^3$ with standard inner product. Suppose W is spanned by $v_1 = (1,1,2)$ and $v_2 = (1,1,-1)$ then
(a) Find $\text{Proj}_W (1,3,-2)$
(b) Find the shortest distance between $(1,3,-2)$ and W.

Soln:-

$\langle (1,1,2), (1,1,-1) \rangle = 1+1-2 = 0$

$\therefore \{v_1, v_2\}$ is an orthogonal set.

$\|v_1\| = \|(1,1,2)\| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$

$\|v_2\| = \|(1,1,-1)\| = \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$

Normalization of $v_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

Normalization of $v_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

An orthonormal basis to W is $\left\{\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)\right\}$

Take $b = (1,3,-2)$; $u_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$; $u_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$\text{Proj}_W(b) = \langle b, u_1 \rangle u_1 + \langle b, u_2 \rangle u_2$

$$= \left\langle (1,3,-2), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)\right\rangle \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$
$$+ \left\langle (1,3,-2), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)\right\rangle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$$= \left(\frac{1}{\sqrt{6}} + \frac{3}{\sqrt{6}} - \frac{4}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$$= 0 \cdot \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) + \frac{6}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$\text{Proj}_W(b) = (2, 2, -2)$

Shortest distance $= \|b - \text{Proj}_W(b)\|$

$$= \|(1,3,-2) - (2,2,-2)\| = \|(1,1,0)\|$$

$$= \sqrt{1^2+1^2+0^2} = \sqrt{2}.$$