Kernel, Image of a linear transformation using its matrix

1) Suppose A is a $m \times n$ matrix then we can think A is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ (with respect to standard basis).

2) Let V and W be vector spaces with bases α and β, $T: V \longrightarrow W$ be a linear transformation such that $[T]_\alpha^\beta = A$ then

(*) $\ker T$ and $\ker A$ are isomorphic.

(*) $\dim \ker(T) = \dim(\ker(A)) = \text{Nullity } A$

(*) $\operatorname{Im}(T)$ and Column space of A are isomorphic

(*) $\dim \operatorname{Im}(T) = \text{Rank } A = \dim(C(A))$

3) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation such that $[T]_\alpha^\beta = A$. If α, β are standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$ then

$$\ker(T) = N(A); \quad \operatorname{Im}(T) = C(A)$$

4) $\dim \ker(T) + \dim \operatorname{Im}(T) = \dim V.$

Pb:1 Let $f: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by

$$f(x,y,z,t) = (x-y+z+t,\ 2x-2y+3z+4t,\ 3x-3y+4z+5t)$$

Find the following

(a) $[f]_\alpha^\beta$ where α, β are standard basis on $\mathbb{R}^4$ and $\mathbb{R}^3$

(b) Find $\operatorname{Ker}(f)$, $\operatorname{Im}(f)$

(c) Is f invertible?

Soln:- (a) Standard basis of $\mathbb{R}^4$ is $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

Standard basis of $\mathbb{R}^3$ is $\{(1,0,0), (0,1,0), (0,0,1)\}$

$f(1,0,0,0) = (1,2,3) = 1\cdot(1,0,0) + 2\cdot(0,1,0) + 3(0,0,1)$

$f(0,1,0,0) = (-1,-2,-3) = (-1)(1,0,0) + (-2)(0,1,0) + (-3)(0,0,1)$

$f(0,0,1,0) = (1,3,4) = 1\cdot(1,0,0) + 3(0,1,0) + 4(0,0,1)$

$f(0,0,0,1) = (1,4,5) = 1\cdot(1,0,0) + 4\cdot(0,1,0) + 5(0,0,1)$

$$[f]_\alpha^\beta = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} = A$$

We calculate column and null space of A.

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \quad \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

. This is the reduced row echelon of A

The leading 1's are in 1st, 3rd column of U

$\therefore$ Basis of $C(A) = \{(1,2,3), (1,3,4)\}$

$C(A) = \{\alpha(1,2,3) + \beta(1,3,4) \mid \alpha, \beta \in \mathbb{R}\}$

$Im(f) = \{\alpha(1,2,3) + \beta(1,3,4) \mid \alpha, \beta \in \mathbb{R}\}$.

$U = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; we know $N(A) = N(U)$

$$N(U) = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; \begin{aligned} x_1 - x_2 - x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned} \right\}$$

Here x_2, x_4 are free variables,

Fix ; $x_2 = s$; $x_4 = t$ then

$x_1 = s + t$; $x_3 = -2t$

Now,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s+t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ A basis of $N(A) = \{ (1, 1, 0, 0), (1, 0, -2, 1) \}$

$$N(A) = \{ \alpha (1, 1, 0, 0) + \beta (1, 0, -2, 1) \mid \alpha, \beta \in \mathbb{R} \}$$

$$Ker(f) = \{ \alpha (1, 1, 0, 0) + \beta (1, 0, -2, 1) \mid \alpha, \beta \in \mathbb{R} \}$$

Since $Ker(f) \neq \{0\}$ f is not one to one

Hence f is not invertible.

Pb:2 Let $T: P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by $T(p(t)) = \int_1^t p'(t)\,dt$

Then

(i) Find $[T]_\alpha$ where α is the standard basis of $P_3(\mathbb{R})$

(ii) Find dimension of $\ker(T)$, $\mathrm{Im}(T)$

(iii) Is $[T]_\alpha$ invertible?

Soln:- Standard basis of $P_3(\mathbb{R})$ is $\{1, t, t^2, t^3\}$

$T(1) = \int_1^t 0.dt = 0 \quad = 0.1 + 0.t + 0.t^2 + 0.t^3$

$T(t) = \int_1^t 1.dt = [t]_1^t = t - 1 \quad = (-1)1 + 1.t + 0.t^2 + 0.t^3$

$T(t^2) = \int_1^t 2t\,dt = [t^2]_1^t = t^2 - 1 \quad = (-1)1 + 0.t + 1.t^2 + 0.t^3$

$T(t^3) = \int_1^t 3t^2\,dt = [t^3]_1^t = t^3 - 1 \quad = (-1).1 + 0.t + 0.t^2 + 1.t^3$

$$[T]_\alpha = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We find the dimension of column space and null space of $[T]_\alpha$

$$\begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 + R_2 \quad \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 + R_3$$

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 + R_4 \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus Rank of $[T]_\alpha = 3 = \dim(\mathrm{Im}(T))$

By Rank nullity theorem,

$$\text{Rank } [T]_\alpha + \text{Nullity } [T]_\alpha = 4$$

$$\therefore \text{Nullity } [T]_\alpha = 4 - 3 = 1$$

$$\dim[\mathrm{Ker}(T)] = 1$$

Since $\mathrm{Ker}(T) \neq \{0\}$ T is not invertible

Pb: 3 Let $T: P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = \begin{bmatrix} a_0 + a_2 & -a_0 + a_3 \\ a_1 - a_2 & a_1 - a_3 \end{bmatrix}$$

Then (i) Find $[T]_\alpha^\beta$ where α, β are the standard basis on $P_3(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$

(ii) Find a basis and dimension of $\ker(T)$ and $\operatorname{Im}(T)$

(iii) Is T invertible?

soln:- Standard basis of $P_3(\mathbb{R})$ is $\{1, t, t^2, t^3\}$

Standard basis of $M_{2\times 2}(\mathbb{R})$ is $\left\{ \underset{E_{11}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{E_{12}}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{E_{21}}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{E_{22}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$

$$T(1) = T(1\cdot 1 + 0\cdot t + 0\cdot t^2 + 0\cdot t^3) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = 1\cdot E_{11} + (-1) E_{12} + 0\cdot E_{21} + 0\cdot E_{22}$$

$$T(t) = T(0\cdot 1 + 1\cdot t + 0\cdot t^2 + 0\cdot t^3) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0\cdot E_{11} + 0\cdot E_{12} + 1\cdot E_{21} + 1\cdot E_{22}$$

$$T(t^2) = T(0\cdot 1 + 0\cdot t + 1\cdot t^2 + 0\cdot t^3) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = 1\cdot E_{11} + 0\cdot E_{12} + (-1) E_{21} + 0\cdot E_{22}$$

$$T(t^3) = T(0\cdot 1 + 0\cdot t + 0\cdot t^2 + 1\cdot t^3) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = 0\cdot E_{11} + 1\cdot E_{12} + 0\cdot E_{21} + (-1) E_{22}$$

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

We find the reduced row echelon of $[T]_\alpha^\beta$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad R_2 \to R_2 + R_1 \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad R_4 \to R_4 - R_2 \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 + R_3 \\ R_4 \to R_4 - R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad R_4 \to (-1) R_4 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank of $[T]_\alpha^\beta = 4 = \dim(\mathrm{Im}(T))$

We also know that $\dim(M_{2\times 2}(\mathbb{R})) = 4$

$\therefore$ $\mathrm{Im}(T) = M_{2\times 2}(\mathbb{R})$

By rank nullity theorem, $\dim \ker(T) = 0$

$\therefore$ $\ker(T) = \{0\}$

Since $\ker(T) = \{0\}$, T is invertible.