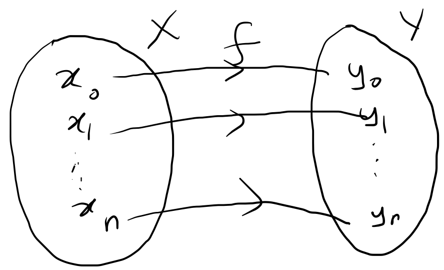
Module – 4

Linear transformation

Function :- Let X and Y be two non empty sets. A function from X to Y is an assignment of an element Y to each element of X. The set X is called the domain of the function and the set Y is called the co-domain of the function.



Consider the system

$$\left.\begin{aligned} 2x + y &= 0 \\ x - 2y &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Consider a function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Then finding a solution to $\begin{aligned} 2x + y &= 0 \\ x - 2y &= 0 \end{aligned}$ is sam as obtaining elements in $\mathbb{R}^2$ which are mapped to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

<u>Linear transformation</u> Let V and W be vector spaces. A linear transformation T of V into W is a function assigning a unique vector $T(u)$ in W to each u in V such that:

1) $T(u+v) = T(u) + T(v)$ for every u, v in V

2) $T(ku) = kT(u)$ for every $k \in \mathbb{R}$ and u in V

3) $T(0_V) = 0_W$

<u>Note:-</u> In the definition above, in (a) $+$ in $u+v$ on the left side of the equation refers to addition operation in V, whereas $+$ in $T(u) + T(v)$ refers to addition operation in W. Similarly, in (b) the scalar product ku is in V and scalar product $kT(u)$ in W.

<u>Example: 1</u> Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $T(x, y, z) = (x, y)$. Prove that T is a linear map.

<u>Soln:-</u> $T(0,0,0) = (0,0)$

Let (x_1, y_1, z_1), (x_2, y_2, z_2) are in $\mathbb{R}^3$. Now

$$T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$= (x_1 + x_2, y_1 + y_2) = (x_1, y_1) + (x_2, y_2)$$

$$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$$

Let $k \in \mathbb{R}$, $T(k(x_1, y_1, z_1)) = T(kx_1, ky_1, kz_1) = (kx_1, ky_1) = k(x_1, y_1)$

$$= k\,T(x_1, y_1, z_1)$$

Thus T is a linear map.

Example 2:- Consider a map $T : P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by

$$T(p(t)) = t\,p(t) + t^3$$

Is T a linear transformation?

Ans:- consider the zero polynomial $p(t) = 0$.

$$T(0) = t \cdot 0 + t^3 = t^3 \neq 0$$

So T does not map zero polynomial to zero polynomial.

Hence T is not a linear transformation.

Example 3:– Let $T : M_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

Is T a linear transformation?

Ans:– $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 + 0 = 0$

Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$. Now

$$T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right)$$

$$= a_1 + a_2 + d_1 + d_2 = (a_1 + d_1) + (a_2 + d_2)$$

$$= T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = ka + kd = k(a + d)$$

$$= k\,T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$\therefore$ T is a linear transformation.