Pb:1 Find all solutions of the system
$$5x + 3y + 4z = 4$$
$$y + 2z = 1$$
$$x - y + 2z = 0$$
$$2x + y + 6z = 2 \quad \text{by Gauss elimination.}$$

Soln:- The augmented matrix corresponds to this system,

$$\begin{bmatrix} \boxed{5} & 3 & 4 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & 1 & 6 & 2 \end{bmatrix}$$

1st pivot

$R_3 \to R_3 - \frac{1}{5}R_1$

$R_4 \to R_4 - \frac{2}{5}R_1$

$$\begin{array}{cccc} 1 & -1 & 2 & 0 \\ (-)\frac{1}{0} & \frac{3}{5} & \frac{4}{5} & \frac{4}{5} \\ \hline & -\frac{8}{5} & \frac{6}{5} & -\frac{4}{5} \end{array}$$

$$\begin{array}{cccc} 2 & 1 & 6 & 2 \\ 2 & \frac{6}{5} & \frac{8}{5} & \frac{8}{5} \\ \hline 0 & -\frac{1}{5} & \frac{22}{5} & \frac{2}{5} \end{array}$$

$$\Downarrow$$

$$\begin{bmatrix} 5 & 3 & 4 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & -\frac{8}{5} & \frac{6}{5} & -\frac{4}{5} \\ 0 & -\frac{1}{5} & \frac{22}{5} & \frac{2}{5} \end{bmatrix}$$

$R_3 \to 5R_3$

$R_4 \to 5R_4$

$$\sim \begin{bmatrix} 5 & 3 & 4 & 4 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & -8 & 6 & -4 \\ 0 & -1 & 22 & 2 \end{bmatrix}$$

2nd pivot

$R_3 \to R_3 + 8R_2$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 5 & 3 & 4 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \boxed{11} & 2 \\ 0 & 0 & 8 & 1 \end{bmatrix}$$

3rd pivot

$R_4 \to R_4 - \frac{8}{11}R_3$

$$\sim \begin{bmatrix} 5 & 3 & 4 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 11 & 2 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

The last matrix is the row echelon form of augmented matrix.

The system corresponds to echelon form is
$$5x + 3y + 4z = 4$$
$$y + 2z = 1$$
$$11z = 2$$
$$0 = -5.$$

Since $0 = -5$ has no sense the given system is inconsistent.

Pb:2 Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = b$

consistent for all possible b_1, b_2, b_3?

Soln:— The augmented matrix corresponds to this system is

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 + 3R_1 \end{array} \quad \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix} \quad R_3 \to 2R_3 - R_2$$

(1st pivot circled, 2nd pivot circled)

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & 2b_3 + 2b_1 - b_2 \end{bmatrix}$$

This is the echelon form of augmented matrix and the corresponding system is

$$x + 3y + 4z = b_1$$
$$14y + 10z = b_2 + 4b_1$$
$$0 = 2b_3 + 2b_1 - b_2.$$

Clearly the given system is consistent for all b_1, b_2, b_3

such that $\quad 2b_1 - b_2 + 2b_3 = 0$

In other words, the given system has solution for all b_1, b_2, b_3

which is in the plane $\quad 2x - y + 2z = 0$