Application of row, column and null space to invertibility of a matrix :-

Theorem: 1 Let A be a $m\times n$ matrix. Then the following statements are equivalent

(i) For each $b\in\mathbb{R}^m$, $AX=b$ has atleast one solution

(ii) Rank of $A=m$ hence $m\leq n$

(iii) There exists a $n\times m$ matrix B such that $AB=I_m$
(Here B is the right inverse of A)

(iv) $C(A)=\mathbb{R}^m$.

Theorem 2:- Let A be a $m\times n$ matrix. Then the following statements are equivalent

(i) For each $b\in\mathbb{R}^m$, $AX=b$ has atmost one solution for X in $\mathbb{R}^n$

(ii) $\mathrm{Rank}(A)=n$ and hence $n\leq m$

(iii) $R(A)=\mathbb{R}^n$

(iv) $N(A)=\{0\}$

(v) There exists a $n\times m$ matrix such that $CA=I_n$
(left inverse exists)

(vi) The column vectors of A are linearly independent

Theorem 3:- The linear system $Ax=b$ has a Solution if and only if $\text{rank } A = \text{rank } [A \mid b]$ that is rank A is equal to rank of the corresponding augmented matrix.

Pb: 1 Let $A = \begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix}$

Discuss about the existance of left or right inverse of A ?

Soln:–

$$\begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 2R_1 \end{array} \begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 2 & 2 & 2 & -5 \\ 0 & -3 & -3 & 6 & -3 \end{bmatrix}$$

$$\begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 + 3R_2 \end{array} \begin{bmatrix} 1 & 0 & -5 & -6 & -2 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix} \begin{array}{l} R_3 \to \frac{R_3}{(-2)} \\ R_4 \to \frac{R_4}{(-6)} \end{array}$$

$$\begin{bmatrix} 1 & 0 & -5 & -6 & -2 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \to R_1 + 6R_3 \\ R_2 \to R_2 - 2R_3 \end{array} \begin{bmatrix} 1 & 0 & -5 & 0 & 10 \\ 0 & 1 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_1 \to R_1 - 10R_4 \\ R_2 \to R_2 + 5R_4 \\ R_3 \to R_3 - 2R_4 \end{array} \begin{bmatrix} 1 & 0 & -5 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

leading 1's are in 1st, 2nd, 4th and 5th column.

So rank A = 4. Since A is a matrix of order 4×5 and rank A = 4 by theorem 1, A has right inverse.

Pb: 2 let $A = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$

where b_1, b_2, b_3, b_4, b_5 are some real numbers. Discuss the number of possible solutions to $AX = B$.

Soln:-

$$\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \quad \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 1 & 1 & -4 & 5 \end{bmatrix}$$

$$\begin{matrix} R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 - R_2 \end{matrix} \quad \begin{bmatrix} 1 & 0 & -11 & 11 & 1 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & -2 & -2 & 4 \end{bmatrix} \quad R_3 \to \frac{R_3}{(-2)} \quad \begin{bmatrix} 1 & 0 & -11 & 11 & 1 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\begin{matrix} R_1 \to R_1 + 11R_3 \\ R_2 \to R_2 - 3R_3 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 0 & 22 & 23 \\ 0 & 1 & 0 & -5 & -5 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Rank $A = 3$

Hence $AX = b$ has atleast one solution. by theorem 1.

Note:- Let A be a $m \times n$ matrix

(i) If rank $A = m$, $(AA^T)^{-1}$ exists then $B = A^T(AA^T)^{-1}$ is the right inverse of A.

(ii) If rank $A = n$, $(A^TA)^{-1}$ exists, then $B = (A^TA)^{-1}A^T$ is the left inverse of A.

Theorem:- Let A be a $n \times n$ matrix. The following statements are equivalent:

(a) A is invertible

(b) $\det A \neq 0$

(c) A is row equivalent to I_n

(d) $AX = b$ has a solution for every $b \in \mathbb{R}^n$

(e) $N(A) = \{0\}$

(f) $C(A) = \mathbb{R}^n$

(g) $R(A) = \mathbb{R}^n$

(h) The columns of A are linearly independent.

(i) The rows of A are linearly independent.

(j) A has left inverse.

(k) A has right inverse.

(l) rank $A = n$.