Module-2

Recall:- consider the system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n$$

Matrix form of the above system is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Vector form representation is

$$\underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{v_1} x_1 + \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{v_2} x_2 + \underbrace{\begin{bmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{bmatrix}}_{v_3} x_3 + \cdots + \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{v_n} x_n = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$

then

$$v_1 x_1 + v_2 x_2 + \cdots + v_n x_n = b$$

Example:- Consider the system

$$2x_1 + x_2 + x_3 + x_4 = 1$$
$$4x_1 + x_2 + 0x_3 + x_4 = -2$$
$$-2x_1 + 2x_2 + x_3 + x_4 = 7$$

Then the vector form of this equation is

$$\underbrace{\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}}_{v_1} x_1 + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{v_2} x_2 + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{v_3} x_3 + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{v_4} x_4 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = b$$

$v_1 = [2 \quad 4 \quad -2]^T \rightarrow$ is an element in 3d space.

$v_2 = [1 \quad 1 \quad 2]^T \rightarrow$ " "

$v_3 = [1 \quad 0 \quad 1]^T \rightarrow$ "

Recall:- A point in 3D can be represented as follows.

(x_0, y_0, z_0), $x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$, $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$

<u>Recall:-</u> Let (x_0, y_0, z_0), (x_1, y_1, z_1) and (x_2, y_2, z_2) be three points in 3D

(i) <u>Addition of vectors</u> $(x_0, y_0, z_0) + (x_1, y_1, z_1) = (x_0 + x_1, y_0 + y_1, z_0 + z_1)$

(ii) <u>Scalar multiplication</u> $k(x_0, y_0, z_0) = (kx_0, ky_0, kz_0)$

<u>Vector space</u>:- Let V be a non empty set (collection of well defined objects). V is said to be a real vector space if it satisfies the following properties

(a) $x+y \in V$ for all $x, y \in V$

(b) $(x+y)+z = x+(y+z)$ for all $x, y, z \in V$

(c) There exists an element $0_V \in V$ such that $x + 0_V = x$ for all $x \in V$

(d) For every $x \in V$ there exists $-x \in V$ such that $x + (-x) = 0_V$

(e) $x+y = y+x$ for all $x, y \in V$

(e1) For $k \in \mathbb{R}$, $x \in X$, $kx \in V$

(f) $k(x+y) = kx + ky$ for all $x, y \in V$, $k \in \mathbb{R}$

(g) $(k+l)x = kx + lx$ for all $k, l \in \mathbb{R}$, $x \in V$

(h) $(kl)x = k(lx)$ for all $k, l \in \mathbb{R}$, $x \in V$

(i) $1 \cdot x = x$ for all $x \in V$.

where $\mathbb{R}$ is the set of real numbers.

<u>Notation</u> $\mathbb{R}$ — set of real numbers

$\mathbb{R}^2$ — 2 dimensional space $= \{(x,y) \mid x,y \in \mathbb{R}\} = \mathbb{R}\times\mathbb{R}$

$\mathbb{R}^3$ — 3 dimensional space $= \{(x,y,z) \mid x,y,z \in \mathbb{R}\} = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$

$\vdots$

$\mathbb{R}^n$ — n dimensional space $= \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots x_n \in \mathbb{R}\}$

$$= \underbrace{\mathbb{R}\times\mathbb{R}\times \cdots \times\mathbb{R}}_{n \text{ times.}}$$

<u>Example:-</u> Consider $\mathbb{R}^2 = \{(x,y) \mid x\in\mathbb{R}, y\in\mathbb{R}\}$. Define addition on $\mathbb{R}^2$ by

$$(x_1, y_1) + (x_2, y_2) = (x_1+x_2, y_1+y_2)$$

scalar multiplication by

$$k(x,y) = (kx, ky)$$ then $\mathbb{R}^2$ is a real vector space.

<u>Proof:-</u> (a) Let $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$ then

$$(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$$

clearly $(x_1+x_2, y_1+y_2) \in \mathbb{R}^2$

(b) (x_1,y_1), (x_2,y_2) and (x_3,y_3) in $\mathbb{R}^2$

$$[(x_1,y_1)+(x_2,y_2)] + (x_3,y_3) = (x_1+x_2, y_1+y_2)+(x_3,y_3) = (x_1+x_2+x_3, y_1+y_2+y_3)$$

$$(x_1,y_1) + [(x_2,y_2)+(x_3,y_3)] = (x_1,y_1) + [(x_2+x_3, y_2+y_3)] = (x_1+x_2+x_3, y_1+y_2+y_3)$$

$$[(x_1,y_1)+(x_2,y_2)] + (x_3,y_3) = (x_1,y_1) + [(x_2,y_2)+(x_3,y_3)]$$

(c) The zero vector $(0,0)$ satisfies the property

$$(x,y)+(0,0) = (x,y) \quad \text{for all } (x,y)\in\mathbb{R}^2$$

(d) For every $(x,y)\in\mathbb{R}^2$ the vector $(-x,-y)\in\mathbb{R}^2$ such that

$$(x,y)+(-x,-y) = (x-x,\ y-y) = (0,0)$$

(e) For (x_1,y_1), $(x_2,y_2)\in\mathbb{R}^2$ we have

$$(x_1,y_1)+(x_2,y_2) = (x_1+x_2,\ y_1+y_2)$$

$$(x_2,y_2)+(x_1,y_1) = (x_2+x_1,\ y_2+y_1)$$

Hence $(x_1,y_1)+(x_2,y_2) = (x_2,y_2)+(x_1,y_1)$

(f) For $(x,y)\in\mathbb{R}^2$ and $k\in\mathbb{R}$ we have $k(x,y)=(kx,ky)$

clearly $(kx,ky)\in\mathbb{R}^2$

(g) Let $k\in\mathbb{R}$, (x_1,y_1) and (x_2,y_2) are in $\mathbb{R}^2$ then

$$k[(x_1,y_1)+(x_2,y_2)] = k[(x_1+x_2,\ y_1+y_2)] = (k(x_1+x_2),\ k(y_1+y_2))$$

$$= (kx_1+kx_2,\ ky_1+ky_2)$$

$$= k(x_1,y_1)+k(x_2,y_2)$$

(h) Let $k,l\in\mathbb{R}$, $(x,y)\in\mathbb{R}^2$ then

$$(k+l)(x,y) = ((k+l)x,\ (k+l)y) = (kx+lx,\ ky+ly)$$

$$= k(x,y)+l(x,y)$$

(i) $1\cdot(x,y) = (1\cdot x,\ 1\cdot y) = (x,y)$

Hence $\mathbb{R}^2$ is a real vector space.

Example 2:- Consider the set $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$. We define addition on $\mathbb{R}^3$ by $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

scalar multiplication by $k(x, y, z) = (kx, ky, kz)$

Then $\mathbb{R}^3$ is a real vector space with respect to the addition and scalar multiplication defined above.

Example 3:- Consider the set

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, x_3, \ldots, x_n \in \mathbb{R}\}$$

Define addition on $\mathbb{R}^n$ by

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

scalar multiplication on $\mathbb{R}^n$ by

$$k(x_1, x_2, \ldots, x_n) = (kx_1, kx_2, \ldots, kx_n)$$

Then $\mathbb{R}^n$ is a real vector space with respect to the addition and scalar multiplication defined above.

Note:- The addition and scalar multiplication defined on the example 3 is called as standard addition and standard scalar multiplication.

Remark:- In order to prove a non empty set is not a vector space, it is enough to give an example which does not obey a axiom of the vector space.

Example of a non vector space. Consider $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

Define addition on $\mathbb{R}^3$ by $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1, z_1)$

scalar multiplication by $k(x, y, z) = (kx, ky, kz)$. Is $\mathbb{R}^3$ a real vector space defined on the above addition and scalar multiplication.

Soln:-

Consider the vectors $(1, 1, 1)$, $(2, 3, 4)$. We observe the following $(1, 1, 1) + (2, 3, 4) = (3, 1, 1)$ and

$$(2, 3, 4) + (1, 1, 1) = (3, 3, 4)$$

Hence the sum is not commutative. Hence $\mathbb{R}^3$ is not a real vector space.

Consider $\mathbb{R}^2 = \{(x,y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}$. we define addition by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and scalar multiplication by

$$k(x,y) = (k^2 x, k^2 y)$$

Is $\mathbb{R}^2$ a real vector space with respect to the addition and scalar multiplication defined above.?

<u>Ans:-</u> consider the vector $x = (2, 2)$ and the scalars $k = -1$; $l = 1$

we observe the following

$$(k+l)x = (-1+1)(2,2) = 0 \cdot (2,2) = (0 \cdot 2, 0 \cdot 2) = 0$$

$$kx + lx = (-1)(2,2) + 1 \cdot (2,2)$$

$$= ((-1)^2 \cdot 2, (-1)^2 2) + (1 \cdot 2, 1 \cdot 2)$$

$$= (2,2) + (2,2) = (4,4).$$

clearly $(k+l)x \neq kx + lx$.

Hence $\mathbb{R}^2$ is not a real vector space with respect to the above addition and scalar multiplication.