Basis and dimension of Subspaces. Consider the real vector space $\mathbb{R}^n$ and

$$W = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array} \right\}$$

To obtain a basis and dimension of W we proceed by the following steps

1) Write the given system in matrix form $AX = 0$ and find the reduced row echelon form of A by Gauss Jordan elimination.
2) Find the number of free variables associated with the system and that number is the dimension of W.
3) If the solution of $AX = 0$ is zero then W has no basis and dimension of W is zero.
4) If the solution $x$ contains arbitrary constants then write $x$ as the linear combination of $x_1, x_2, \ldots x_p$ with $s_1, s_2, \ldots s_p$ as coefficients (means) $x = s_1 x_1 + s_2 x_2 + \cdots + s_p x_p$.
5) The set of vectors $\{x_1, x_2, \ldots, x_p\}$ is a basis for W.

**Pb:1** Find a basis and dimension of

$$W = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 = 0;\ x_2 - x_3 = 0;\ x_1 + x_3 = 0 \}$$

**soln:-**

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_2]{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced system is $x_1 + x_3 = 0$, $x_2 - x_3 = 0$. Here $x_3$ is the free variable so dimension of $W$ is one.

To find a basis fix $x_3 = t$ then $x_1 = -t$; $x_2 = t$. We note that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Basis of $W = \{ (-1, 1, 1) \}$.

Pb: 2 Find a basis and dimension of

$$W = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \;\middle|\; \begin{array}{l} x_1 - x_2 + 2x_3 + 3x_4 + 4x_5 = 0 \\ -x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0 \\ x_1 - x_2 + 3x_3 + 5x_4 + 6x_5 = 0 \\ 3x_1 - 4x_2 + x_3 + 2x_4 + 3x_5 = 0 \end{array} \right\}$$

Soln:-

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 4 & 0 \\ -1 & 2 & 3 & 4 & 5 & 0 \\ 1 & -1 & 3 & 5 & 6 & 0 \\ 3 & -4 & 1 & 2 & 3 & 0 \end{bmatrix} \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1 \end{array} \begin{bmatrix} 1 & -1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 5 & 7 & 9 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & -1 & -5 & -7 & -9 & 0 \end{bmatrix} \begin{array}{l} R_1 \to R_1 + R_2 \\ R_4 \to R_4 + R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 7 & 10 & 13 & 0 \\ 0 & 1 & 5 & 7 & 9 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \to R_1 - 7R_3 \\ R_2 \to R_2 - 5R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & -4 & -1 & 0 \\ 0 & 1 & 0 & -3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced system is

$$\begin{aligned} x_1 - 4x_4 - x_5 &= 0 \\ x_2 - 3x_4 - x_5 &= 0 \\ x_3 + 2x_4 + 2x_5 &= 0 \end{aligned}$$

. Here $x_4, x_5$ are free variables.

Hence dimension of $W$ is 2. Fix $x_4 = s$; $x_5 = t$ then.

$$x_1 = 4s + t; \quad x_2 = 3s + t; \quad x_3 = -2s - 2t.$$

Now

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4s+t \\ 3s+t \\ -2s-2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 4 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Basis of $W = \{ (4, 3, -2, 1, 0), (1, 1, -2, 0, 1) \}$.

Theorem:- Let V be a real vector space with dimension n. If W is a subspace of V with dimension m then $m \leq n$

Pb: 3 Consider the subspace

$$W = \{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_0 = a_1 ;\ a_2 = a_3 \} \text{ in } P_3(\mathbb{R})$$

Find a basis and dimension of W.

Soln:- Since $a_0 = a_1$; $a_2 = a_3$, we have.

$$a_0 + a_1 t + a_2 t^2 + a_3 t^3 = a_0 + a_0 t + a_2 t^2 + a_2 t^3$$
$$= a_0 (1+t) + a_2 (t^2 + t^3) \longrightarrow (*)$$

We claim $\{ (1+t), (t^2+t^3) \}$ is a basis for W.

Clearly $p_1 = 1+t$ and $p_2 = t^2 + t^3$ are in W.

From $(*)$ it is clear that $\{ (1+t), (t^2+t^3) \}$ spans W.

We prove $\{ (1+t), t^2+t^3 \}$ is a linearly independent set.

$$x_1 (1+t) + x_2 (t^2+t^3) = 0$$

$$\Rightarrow x_1 + x_2 t + x_2 t^2 + x_2 t^3 = 0$$

$\Rightarrow x_1 = 0$; $x_2 = 0$. Hence $\{ 1+t, t^2+t^3 \}$ is a linearly independent set.

Hence $\{ (1+t), t^2+t^3 \}$ forms a basis of W.

Note:- Some times dimension of a subspace can be calculated by finding arbitary entries which we can fill up freely in an arbitrary element of that subspace.

Pb: 4 Find a basis and dimension of

$$W = \{ A \in M_{3\times 3}(\mathbb{R}) \mid A = A^T \} = \text{set of } 3\times 3 \text{ symmetric matrices.}$$

Soln:-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ is symmetric only if $a_{12} = a_{21}$; $a_{13} = a_{31}$; $a_{23} = a_{32}$

Thus a symmetric matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$. In this matrix

$a_{11}, a_{22}, a_{33}, a_{12}; a_{13}; a_{23}$ has freedom to fix. So

dimension of $W$ is 6.

Try to prove $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$ forms a basis of $W$.