Gram-Schmidt orthonormalization process

Procedure:- Suppose $v_1, v_2, \dots, v_n$ be a basis of an inner product space V.

Step 1:- Take $u_1 = v_1$ calculate $w_1 = \frac{v_1}{\|v_1\|}$

Step 2:-

Calculate $u_2 = v_2 - \langle v_2, w_1 \rangle w_1$ and

$$w_2 = \frac{u_2}{\|u_2\|}$$

Step 3 Calculate $u_3 = v_3 - \langle v_3, w_1 \rangle w_1 - \langle v_3, w_2 \rangle w_2$ and

$$w_3 = \frac{u_3}{\|u_3\|}.$$

$\vdots$

Step n Calculate

$$u_n = v_n - \langle v_n, w_1 \rangle w_1 - \langle v_n, w_2 \rangle w_2 - \cdots - \langle v_n, w_{n-1} \rangle w_{n-1}$$

and $w_n = \frac{u_n}{\|u_n\|}$

conclude that $\{w_1, w_2, \dots, w_n\}$ forms an orthonormal basis to V.

Pb:1 Consider $P_2(\mathbb{R})$ with basis $\alpha = \{1, t, t^2\}$ and inner product $\langle f(t)\, g(t) \rangle = \int_{-1}^{1} f(t) g(t)\, dt$.

Find an orthonormal basis for $P_2(\mathbb{R})$.

Soln:- Take $v_1 = 1$; $v_2 = t$; $v_3 = t^2$

$u_1 = v_1 = 1$; $\|u_1\| = \sqrt{\langle u_1, u_1 \rangle} = \sqrt{\int_{-1}^{1} 1 \cdot 1 \cdot dt} = \sqrt{2}$

$\therefore w_1 = \frac{u_1}{\|u_1\|} = 1/\sqrt{2}$

$$u_2 = v_2 - \langle v_2, w_1 \rangle w_1 = t - \langle t, \tfrac{1}{\sqrt{2}} \rangle \tfrac{1}{\sqrt{2}} = t - \left[\int_{-1}^{1} \left(t \cdot \tfrac{1}{\sqrt{2}}\right) dt \right] \tfrac{1}{\sqrt{2}}$$

$$= t - (0) \tfrac{1}{\sqrt{2}} = t$$

$$\|u_2\| = \sqrt{\langle u_2, u_2 \rangle} = \sqrt{\int_{-1}^{1} t^2\, dt} = \sqrt{\left[t^3/3 \right]_{-1}^{1}} = \sqrt{2/3}$$

$$\therefore w_2 = \frac{u_2}{\|u_2\|} = \frac{t}{\left(\sqrt{2}/\sqrt{3}\right)} = \frac{\sqrt{3}}{\sqrt{2}}\, t.$$

$$u_3 = v_3 - \langle v_3, w_1 \rangle w_1 - \langle v_3, w_2 \rangle w_2$$

$$= t^2 - \langle t^2, \tfrac{1}{\sqrt{2}} \rangle \tfrac{1}{\sqrt{2}} - \langle t^2, \tfrac{\sqrt{3}}{\sqrt{2}} t \rangle \tfrac{\sqrt{3}}{\sqrt{2}}\, t$$

$$= t^2 - \left[\int_{-1}^{1} t^2 \cdot \tfrac{1}{\sqrt{2}}\, dt \right] \tfrac{1}{\sqrt{2}} - \left[\underbrace{\int_{-1}^{1} \left(t^2 \cdot \tfrac{\sqrt{3}}{\sqrt{2}}\, t\right) dt}_{\to 0} \right] \tfrac{\sqrt{3}}{\sqrt{2}}\, t$$

$$= t^2 - \frac{1}{2} \left[\frac{t^3}{3} \right]_{-1}^{1} = t^2 - \frac{1}{3}.$$

$$\|u_3\| = \sqrt{\langle u_3, u_3 \rangle} = \sqrt{\int_{-1}^{1} (t^2 - 1/3)^2\, dt} = \sqrt{8/45} =$$

$$w_3 = \frac{u_3}{\|u_3\|} = \frac{(t^2 - 1/3)}{\sqrt{8/45}} = \frac{\sqrt{45}}{\sqrt{8}} (t^2 - 1/3) =$$

$\therefore$ Orthonormal basis of $P_2(\mathbb{R})$ is

$$\left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}\, t, \frac{\sqrt{45}}{\sqrt{8}} (t^2 - 1/3) \right\}$$

Pb: 2 Find an orthonormal basis to W which has a basis $\{ v_1 = (1,1,1,1),\ v_2 = (1,2,0,1),\ v_3 = (2,2,4,0)\}$

Soln:-

$u_1 = v_1 = (1,1,1,1)$; $\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$

$\therefore w_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$u_2 = v_2 - \langle v_2, w_1\rangle w_1$

$= (1,2,0,1) - \langle (1,2,0,1), (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\rangle (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$

$= (1,2,0,1) - \left[\tfrac{1}{2} + 1 + 0 + \tfrac{1}{2}\right] (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$

$= (1,2,0,1) - 2(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$

$= (1,2,0,1) - (1,1,1,1) = (0, 1, -1, 0)$

$\|u_2\| = \sqrt{0^2+1^2+(-1)^2+0^2} = \sqrt{2}$

$w_2 = \frac{u_2}{\|u_2\|} = \left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

$u_3 = v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2$

$= (2,2,4,0) - \langle (2,2,4,0), (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\rangle (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$
$\quad - \langle (2,2,4,0), (0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0)\rangle (0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0)$

$= (2,2,4,0) - [1+1+2+0] (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$
$\quad - \left[0 + \tfrac{2}{\sqrt{2}} - \tfrac{4}{\sqrt{2}} + 0\right] (0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0)$

$= (2,2,4,0) - 4(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) + \tfrac{2}{\sqrt{2}}(0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0)$

$= (2,2,4,0) - (2,2,2,2) + (0,1,-1,0)$

$u_3 = (0, 1, 1, -2)$

$\|u_3\| = \sqrt{0^2+(+1)^2+1^2+(-2)^2} = \sqrt{6}$

$w_3 = \frac{u_3}{\|u_3\|} = \left(0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

Orthonormal basis to W is $\left\{ (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), (0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0), (0, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}}) \right\}$