# Vector spaces of linear transformation

Let $V$ and $W$ be two vector spaces

$L(V,W) = \{T \mid T : V \to W$ is a linear transformation$\}$

For $S, T \in L(V,W)$ define $S+T$ and $kS$ by

$$(S+T)(v) = S(v) + T(v)$$

$$(kS)(v) = k\,S(v)$$

Clearly $S+T$, $kS$ are linear transformation from $V$ to $W$. Hence $L(V,W)$ is a real vector space.

Note:- If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ then $L(\mathbb{R}^n, \mathbb{R}^m)$ is isomorphic to $M_{m \times n}(\mathbb{R})$.

**Theorem 1** Let $V$ and $W$ be vector spaces with bases $\alpha$ and $\beta$. Let $T, S : V \to W$ be a linear transformation. Then

(i) $[T+S]_\alpha^\beta = [T]_\alpha^\beta + [S]_\alpha^\beta$

(ii) $[kS]_\alpha^\beta = k[S]_\alpha^\beta$

If $S$ is invertible then $[S^{-1}]_\beta^\alpha = \left([S]_\alpha^\beta\right)^{-1}$

**Theorem 2:-** Let $V$, $W$ and $Z$ be vector spaces with bases $\alpha, \beta$ and $\gamma$ respectively. Suppose that $S : V \to W$ and $T : W \to Z$ are linear transformation then

$$[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma \, [S]_\alpha^\beta$$


$$(T \circ S)(v) = T(S(v))$$

Pb:1 $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ be linear transformations defined by

$$T_1(x, y, z) = (3x,\ y+z)$$

$$T_2(x, y, z) = (2x - 3z,\ y)$$

Find (a) $T_1 + T_2$ (b) $5T_1$ (c) $4T_1 - 5T_2$

(d) $T_1 \circ T_2$ and $T_2 \circ T_1$ if possible.

Soln:-

(a) $(T_1 + T_2)(x, y, z) = T_1(x, y, z) + T_2(x, y, z)$

$$= (3x, y+z) + (2x - 3z, y)$$

$$= (5x - 3z,\ 2y + z)$$

(b) $5T_1(x, y, z) = 5(3x, y+z) = (15x, 5y + 5z)$

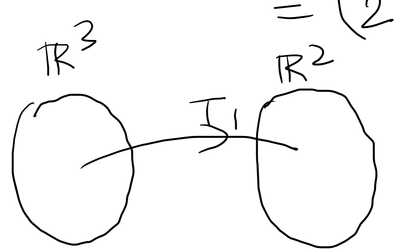
(c) $(4T_1 - 5T_2)(x, y, z) = (4T_1)(x, y, z) - (5T_2)(x, y, z)$

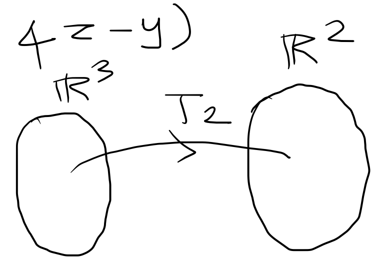
$$= 4(3x, y+z) - 5(2x - 3z, y)$$

$$= (12x, 4y + 4z) - (10x - 15z, 5y)$$

$$= (2x + 15z,\ 4z - y)$$

(d)




Since $\mathbb{R}^3$ and $\mathbb{R}^2$ are not related

$T_1 \circ T_2$, $T_2 \circ T_1$ is not possible.

Pb: 2 Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x,y,z) = (2x, 3y, 4z)$

$G: \mathbb{R}^3 \to \mathbb{R}^3$ by $G(x,y,z) = (5z, 4y+x, y)$

Find (i) $[T+G]_\alpha$ (ii) $[T-G]_\alpha$ (iii) $[5T]_\alpha$

(iv) $[T^{-1}]_\alpha$ (v) $[T \circ G]_\alpha$ (vi) $[G \circ T]_\alpha$

where $\alpha$ is the standard basis of $\mathbb{R}^3$.

Soln:- $\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = (2,0,0)$

$T(0,1,0) = (0,3,0)$

$T(0,0,1) = (0,0,4)$

$$[T]_\alpha = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$G(1,0,0) = (0,1,0)$

$G(0,1,0) = (0,4,1)$

$G(0,0,1) = (5,0,0)$

$$[G]_\alpha = \begin{bmatrix} 0 & 0 & 5 \\ 1 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) $$[T+G]_\alpha = [T]_\alpha + [G]_\alpha = \begin{bmatrix} 2 & 0 & 5 \\ 1 & 7 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

(b) $$[T-G]_\alpha = [T]_\alpha - [G]_\alpha = \begin{bmatrix} 2 & 0 & -5 \\ -1 & -1 & 0 \\ 0 & -1 & 4 \end{bmatrix}$$

(c) $$[5T]_\alpha = 5[T]_\alpha = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

(d) $[T^{-1}]_\alpha = [T]_\alpha^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

(e) $[T \circ G]_\alpha = [T]_\alpha [G]_\alpha = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 5 \\ 1 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 10 \\ 3 & 12 & 0 \\ 0 & 4 & 0 \end{bmatrix}$

(f) $[G \circ T]_\alpha = [G]_\alpha [T]_\alpha = \begin{bmatrix} 0 & 0 & 5 \\ 1 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 20 \\ 2 & 12 & 0 \\ 0 & 3 & 0 \end{bmatrix}$