02/02/22

Vector space:

non empty set V together the operation $+$ and $\cdot$

A *vector space* is a set V equipped with two operations:

(i) *Addition*: adding any pair of vectors $\mathbf{v}, \mathbf{w} \in V$ gives another vector $\mathbf{v}+\mathbf{w} \in V$; Closure

(ii) *Scalar Multiplication*: multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\,\mathbf{v} \in V$;

subject to the following axioms, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$:

Abelian Group (i) + (a)-(d)

(a) *Commutativity of Addition*: $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.

(b) *Associativity of Addition*: $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.

(c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$.

(d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}=(-\mathbf{v})+\mathbf{v}$.

(e) *Distributivity*: $(c+d)\mathbf{v}=(c\,\mathbf{v})+(d\,\mathbf{v})$, and $c\,(\mathbf{v}+\mathbf{w})=(c\,\mathbf{v})+(c\,\mathbf{w})$.

(f) *Associativity of Scalar Multiplication*: $c\,(d\,\mathbf{v})=(c\,d)\,\mathbf{v}$.

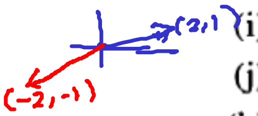
(g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\,\mathbf{v}=\mathbf{v}$.

Vector Space $\Leftarrow (V, +, \cdot)$

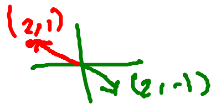
The following identities are elementary consequences of the vector space axioms:

$-1(2,1) = (-2,-1)$

(h) $0\,\mathbf{v}=\mathbf{0}$; $0 \in \mathbb{R}$ [Scalar]



(i) $(-1)\,\mathbf{v}=-\mathbf{v}$; $u = (2,-1)$, $-v = (-2,1)$



(j) $c\,\mathbf{0}=\mathbf{0}$.

(k) If $c\,\mathbf{v}=\mathbf{0}$, then either $c=0$ or $\mathbf{v}=\mathbf{0}$.

n-space

$\bar{0} = (0, \ldots, 0)$ n-tuples

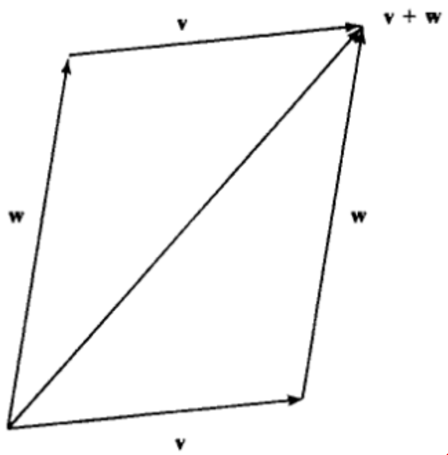
Vector addition and scalar multiplication are defined in the usual manner:

$$\mathbf{v}+\mathbf{w}=\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{pmatrix}, \qquad c\,\mathbf{v}=\begin{pmatrix} c\,v_1 \\ c\,v_2 \\ \vdots \\ c\,v_n \end{pmatrix}, \qquad \text{whenever} \quad \mathbf{v}=\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \ \mathbf{w}=\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}.$$

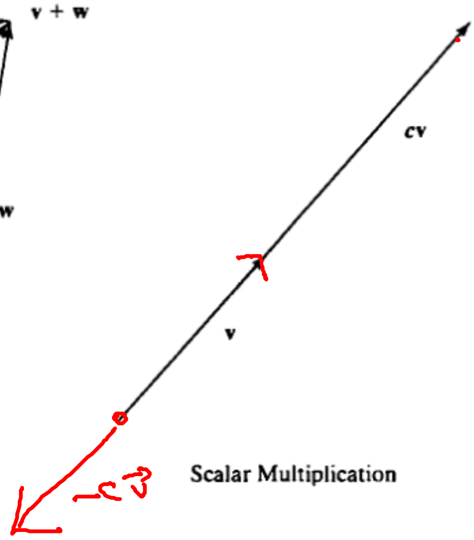
The zero vector is $\mathbf{0}=(0,\ldots,0)^T$. The fact that vectors in $\mathbb{R}^n$ satisfy all of the vector space axioms is an immediate consequence of the laws of vector addition and scalar multiplication.

Zero vector $\{0\} \rightarrow$ Vector Space

Vector space operations in $\mathbb{R}^n$.



Vector Addition



Scalar Multiplication

Consider the space

$$\mathcal{P}^{(n)} = \{\ p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\ \}$$

consisting of all real polynomials of degree $\leq n$. Addition of polynomials is defined in the usual manner; for example,

$$\underbrace{(x^2 - 3x)}_{p(x)} + \underbrace{(2x^2 - 5x + 4)}_{q(x)} = 3x^2 - 8x + 4. \quad \in \mathcal{P}_{(2)}(\mathbb{R})$$

Note that the sum $p(x) + q(x)$ of two polynomials of degree $\leq n$ also has degree $\leq n$. The zero element of $\mathcal{P}^{(n)}$ is the zero polynomial. We can multiply polynomials by scalars — real constants—in the usual fashion; for example if $p(x) = x^2 - 2x$, then $3\,p(x) = 3x^2 - 6x$.

$\mathcal{P}_n(\mathbb{R}) \rightarrow$ Vector space.

Warning: It is not true that the sum of two $\boxed{\text{polynomials of degree } n}$ also has degree n; for example

$$\underbrace{(x^2 + 1)}_{p(x)} + \underbrace{(-x^2 + x)}_{q(x)} = x + 1$$

has degree 1 even though the two summands have degree 2. This means that the set of polynomials of degree $= n$ is *not* a vector space.

① $\mathcal{P}_n(\mathbb{R})$ of degree n or less. $\rightarrow$ Vector Space.

② $V =$ Polynomial of degree n. $\rightarrow$ Not a vector Space

$\vec{a} + \vec{b} = \vec{c} \in V$

$V = \{\bar{a}, \bar{b}, \bar{c}\}$

Definition A *subspace* of a vector space V is a subset $W \subset V$ which is a vector space in its own right—under the same operations of vector addition and scalar multiplication and the same zero element.

→ V → non-empty set
→ $(V,+)$ → Abelian group
→ $(V,+,\cdot)$ → vector space

W is a subspace of V.

$\bar{a}, \bar{b} \in W$ ① Closure w.r.to addition $\bar{a}+\bar{b} \in W$

$c \in \mathbb{R}$ ② Closure w.r.to scalar multiplication → $c\bar{a} \in W$ with zero vector (additive identity)

In particular, a subspace W *must* contain the zero element of V. Proving that a given subset of a vector space forms a subspace is particularly easy: we only need check its *closure* under addition and scalar multiplication.

Proposition A nonempty subset $W \subset V$ of a vector space is a subspace if and only if (iff) (zero vector)

(a) for every $\mathbf{v}, \mathbf{w} \in W$, the sum $\mathbf{v} + \mathbf{w} \in W$, and

(b) for every $\mathbf{v} \in W$ and every $c \in \mathbb{R}$, the scalar product $c\,\mathbf{v} \in W$.

either

$\Longrightarrow$ or $cv + w \in W$

$v + cw \in W$

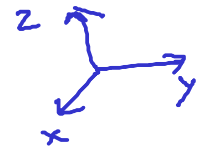
or $kv + lw \in W$

$k, l \in \mathbb{R}$

It will sometimes be convenient to combine the two closure conditions. to prove $W \subset V$ is a subspace it suffices to check that $c\,\mathbf{v} + d\,\mathbf{w} \in W$ for every $\mathbf{v}, \mathbf{w} \in W$ and $c, d \in \mathbb{R}$.

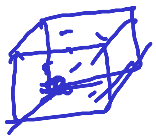
Let us list some examples of subspaces of the three-dimensional Euclidean space $\mathbb{R}^3$.

$\mathbb{R}^3 = \{(a,b,c) : a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R}\}$

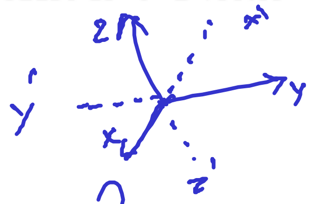


(a) The trivial subspace $W = \{\mathbf{0}\}$. Demonstrating closure is easy: since there is only one element $\mathbf{0}$ in W, we just need to check that $\mathbf{0} + \mathbf{0} = \mathbf{0} \in W$ and $c\,\mathbf{0} = \mathbf{0} \in W$ for any scalar c.

$a, b \in \mathbb{R} \Rightarrow a0 + b0 = 0 \in W$



(b) The entire space $W = \mathbb{R}^3$. Here closure is immediate because $\mathbb{R}^3$ is a vector space in its own right.



$\mathbb{R}^3 = \{(a,b,c)\ \forall a,b,c \in \mathbb{R}\}$

Identity: $\{(0,0,0)\}$

(c) The set of all vectors of the form $(x, y, 0)^T$, i.e., the $x\,y$ coordinate plane. To prove closure, we check that all sums

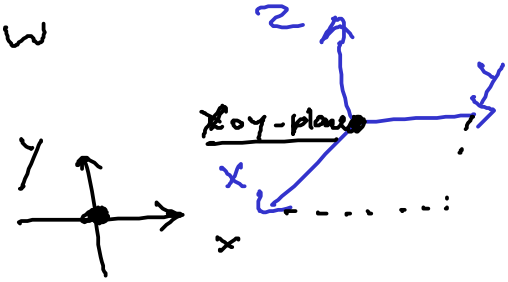
$$(x, y, 0)^T + (\widehat{x}, \widehat{y}, 0)^T = (x + \widehat{x}, y + \widehat{y}, 0)^T \in W$$

$x = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

and scalar multiples

$$c\,(x, y, 0)^T = (c\,x, c\,y, 0)^T \in W$$

of vectors in the $x\,y$–plane remain in the plane.



(d) The set of solutions $(x, y, z)^T$ to the homogeneous linear equation

$$3x + 2y - z = 0.$$

$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

Indeed, if $\mathbf{x} = (x, y, z)^T$ is a solution, then so is any scalar multiple $c\mathbf{x} = (c\,x, c\,y, c\,z)^T$ since

$3x + 2y - z = 0$

$$3(c\,x) + 2(c\,y) - (c\,z) = c\,(3x + 2y - z) = 0.$$

$cX = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix}$

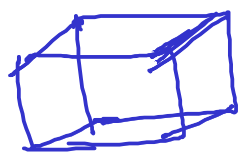
Moreover, if $\widehat{\mathbf{x}} = (\widehat{x}, \widehat{y}, \widehat{z})^T$ is a second solution, the sum

$$\mathbf{x} + \widehat{\mathbf{x}} = (x + \widehat{x}, y + \widehat{y}, z + \widehat{z})^T$$

is also a solution, since

$$3(x + \widehat{x}) + 2(y + \widehat{y}) - (z + \widehat{z}) = (3x + 2y - z) + (3\widehat{x} + 2\widehat{y} - \widehat{z}) = 0.$$

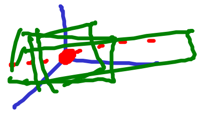
The solution space is, in fact, the two-dimensional plane passing through the origin with normal vector $(3, 2, -1)^T$.



$X = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

Subspaces of $\mathbb{R}^3$:

1. The zero space
2. A line passing through origin
3. A plane which is passing through the origin
4. $\mathbb{R}^3$ [whole $\mathbb{R}^3$]



(e) The set of all vectors lying in the plane spanned by the vectors $\mathbf{v}_1 = (2, -3, 0)^T$ and $\mathbf{v}_2 = (1, 0, 3)^T$. In other words, we consider all vectors of the form

$V = \mathbb{R}^3$

$W = \text{Span}\{v_1, v_2\}$

$$\mathbf{v} = a\,\mathbf{v}_1 + b\,\mathbf{v}_2 = a\begin{pmatrix}2\\-3\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\3\end{pmatrix} = \begin{pmatrix}2a+b\\-3a\\3b\end{pmatrix},$$

where $a, b \in \mathbb{R}$ are arbitrary scalars. If $\mathbf{v} = a\,\mathbf{v}_1 + b\,\mathbf{v}_2$ and $\mathbf{w} = \widehat{a}\,\mathbf{v}_1 + \widehat{b}\,\mathbf{v}_2$ are any two vectors in the span, so is

$$\begin{aligned} c\,\mathbf{v} + d\,\mathbf{w} &= c\,(a\,\mathbf{v}_1 + b\,\mathbf{v}_2) + d\,(\widehat{a}\,\mathbf{v}_1 + \widehat{b}\,\mathbf{v}_2) \\ &= (ac + \widehat{a}\,d)\mathbf{v}_1 + (bc + \widehat{b}\,d)\mathbf{v}_2 = \tilde{a}\,\mathbf{v}_1 + \tilde{b}\,\mathbf{v}_2, \end{aligned}$$

where $\tilde{a} = ac + \widehat{a}\,d$, $\tilde{b} = bc + \widehat{b}\,d$. This demonstrates that the span is a subspace of $\mathbb{R}^3$.

The following subsets of $\mathbb{R}^3$ are *not* subspaces.

$x = \begin{pmatrix}x\\y\\1\end{pmatrix}$, x, y

$\bar{0} = \begin{pmatrix}0\\0\\1\end{pmatrix} \notin P$

(a) The set P of all vectors of the form $(x, y, 1)^T$, i.e., the plane parallel to the xy coordinate plane passing through $(0, 0, 1)^T$. Indeed, $(0, 0, 0) \notin P$, which is the most basic requirement for a subspace. In fact, neither of the closure axioms hold for this subset.

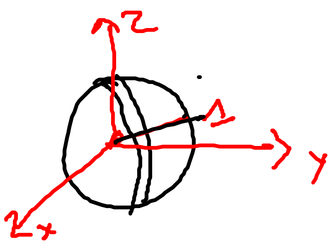
$\bar{a} = \begin{pmatrix}x\\y\\1\end{pmatrix}$

$\bar{b} = \begin{pmatrix}c\\d\\1\end{pmatrix}$

(b) The positive orthant $\mathcal{O}^+ = \{x \geq 0, y \geq 0, z \geq 0\}$. Although $\mathbf{0} \in \mathcal{O}^+$, and the sum of two vectors in $\mathcal{O}^+$ also belongs to $\mathcal{O}^+$, multiplying by negative scalars takes us outside the orthant, violating closure under scalar multiplication.

(c) The unit sphere $S_1 = \{\, x^2 + y^2 + z^2 = 1 \,\}$. Again, $\mathbf{0} \notin S_1$. More generally, curved surfaces, such as the paraboloid $P = \{\, z = x^2 + y^2 \,\}$, are not subspaces. Although $\mathbf{0} \in P$, most scalar multiples of elements of P do not belong to P. For example, $(1, 1, 2)^T \in P$, but $2(1, 1, 2)^T = (2, 2, 4)^T \notin P$.

$\bar{a} + \bar{b} = \begin{pmatrix}c+x\\d+y\\2\end{pmatrix}$



$c\bar{v} \in O^+$; $c \in \mathbb{R}$

if $c \geq 0 \in \mathbb{R} \Rightarrow c\bar{v} \in O^+$

if $c < 0 \in \mathbb{R} \Rightarrow -c\bar{v} \notin O^+$

Pbm: Show that W is a subspace of $V=\mathbb{R}^3$ where W is the xy plane which consists of those vectors whose third component is zero.

(or) $W = \left\{ (a,b,0) : a,b \in \mathbb{R} \right\}$

Proof:

Subspace:

① $\vec{0} \in W$

② $\alpha u + \beta v \in W$

① Since $\vec{0} = (0,0,0) \in W$

② For any vectors $u = (a,b,0) \in W$
$v = (a_1, b_1, 0) \in W$

Let $\alpha, \beta \in \mathbb{R}$

$$\alpha u + \beta v = \alpha(a,b,0) + \beta(a_1,b_1,0)$$
$$= (\alpha a, \alpha b, 0) + (\beta a_1, \beta b_1, 0)$$
$$= (\alpha a + \beta a_1, \alpha b + \beta b_1, 0+0)$$
$$= (\alpha a + \beta a_1, \alpha b + \beta b_1, \underline{0}) \in W$$

$\Rightarrow W = \left\{ (a,b,0) : a,b \in \mathbb{R} \right\}$ is a subspace of V where $\underline{V = \mathbb{R}^3}$

Verify W is a subspace or not in $V=\mathbb{R}^3$.

where W is consist of those vectors whose length does not exceed 1.

i.e., $W = \left\{ (a,b,c) : a^2+b^2+c^2 \leq 1 \right\}$

Proof:- $W = \{(a,b,c) : a^2+b^2+c^2 \le 1\}$ is not a subspace of $V = \mathbb{R}^3$.

① $\vec{0} \in W$
② $\alpha u + \beta v \in W$
$\alpha = 1, \beta = 1$

Suppose $u = (1,0,0) \in W$
$v = (0,1,0) \in W$

Now, $1.u + 1.v = (1,0,0) + (0,1,0)$
$= (1,1,0) \longrightarrow 1^2+1^2+0^2 \geq 1$
$= (1,1,0) \notin W$

Hence, W is not a subspace of $V = \mathbb{R}^3$.

Show that W is not a subspace of V where W consists of all matrices A for which $A^2 = A$..

i.e., $W = \{A = [a_{ij}] : A^2 = A\}$

Proof:- ① Suppose $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$, $\vec{0} * \vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$
$= \vec{0}$

② Unit matrix: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I^2 = I * I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\Rightarrow I \in W$

① $\vec{0} \in W$
② $\alpha \vec{u} \in W$
③ $u + \vec{v} \in W$

Let $\alpha = 2 \in \mathbb{R}$, $I \in W$

$\Rightarrow \alpha I = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Is $\alpha I \in W$? No,

Because, $(2I)^2 = 2I \cdot 2I$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I \neq 2I$$

$\Rightarrow$ W is not a subspace of V.

Theorem: The intersection of any number of subspaces of a vector space V is a subspace of V.

Proof: Let $\{W_i : i \in \mathbb{N}\}$ be a collection of subspaces of V. ~~space~~ V.

and let $W = \bigcap (W_i : i \in \mathbb{N})$

Since, each W_i is a subspace, $0 \in W_i$ for every $i \in \mathbb{N}$

$\Rightarrow 0 \in W$.

Suppose. $u, v \in W$, Then $u, v \in W_i$ for every $i \in \mathbb{N}$

Since each W_i is subspace.

$\Rightarrow \alpha u + \beta v \in W_i$ for each $i \in \mathbb{N}$

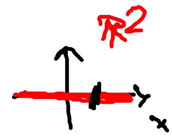
Hence, $\alpha u + \beta v \in W$

$\Rightarrow$ W is a subspace of V.space V.

(i.e.,) Any intersection subspaces are a subspace.

Is the union of two subspaces of a V.space V is again a subspace?

Proof: Let $V = \mathbb{R}^2$ and $W_1 = \{(a,0) : a \in \mathbb{R}\}$



$W_2 = \{(0,b) : b \in \mathbb{R}\}$

$\Rightarrow$ Thus, W_1 and W_2 are the subspaces of $\mathbb{R}^2$.

Let $u = (1,0)$, & $v = (0,1)$

Thus u and v both belonging to the union of $W_1 \cup W_2$

But, since $u, v \in W_1 \cup W_2$

$$\left.\begin{array}{l}\alpha = 1\\ \beta = 1\end{array}\right| \quad \alpha u + \beta v = u + v = (1,0) + (0,1) = (1,1) \notin W_1 \cup W_2$$

$\Rightarrow$ Hence, $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$.

there are only four fundamentally different types of subspaces $W \subset \mathbb{R}^3$ of three-dimensional Euclidean space:

(i) the entire three-dimensional space $W = \mathbb{R}^3$,

(ii) a plane passing through the origin,

(iii) a line passing through the origin,

(iv) a point—the trivial subspace $W = \{\mathbf{0}\}$.

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ belong to a vector space V. A sum of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \sum_{i=1}^{k} c_i\mathbf{v}_i,$$

where the coefficients $c_1, c_2, \ldots, c_k$ are any scalars, is known as a *linear combination* of the elements $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Their *span* is the subset $W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset V$ consisting of all possible linear combinations.

$V = \left\{ (u_1, u_2, \ldots, u_k) \right\}$ $u_i \in \mathbb{R}$

For instance, $3\mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3$, $8\mathbf{v}_1 - \frac{1}{3}\mathbf{v}_3 = 8\mathbf{v}_1 + 0\mathbf{v}_2 - \frac{1}{3}\mathbf{v}_3$, $\mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3$, and $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$ are four different linear combinations of the three vector space elements $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$. The key observation is that the span always forms a subspace.

$c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_k v_k$

$\sum_{i=1}^{k} c_i v_i$

The key observation is that the span always forms a subspace.

Proposition The span $W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of any finite collection of vector space elements forms a subspace of the underlying vector space.

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is called **linearly independent** when the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

If there are also nontrivial solutions, then S is called **linearly dependent.**

Examples of Linearly Dependent Sets

a. The set $S = \{(1, 2), (2, 4)\}$ in R^2 is linearly dependent because

$$-2(1, 2) + (2, 4) = (0, 0).$$

b. The set $S = \{(1, 0), (0, 1), (-2, 5)\}$ in R^2 is linearly dependent because

$$2(1, 0) - 5(0, 1) + (-2, 5) = (0, 0).$$

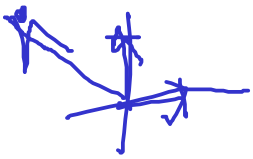
c. The set $S = \{(0, 0), (1, 2)\}$ in R^2 is linearly dependent because

$$1(0, 0) + 0(1, 2) = (0, 0).$$

$c_1(1,2) + c_2(2,4) = (0,0)$

$c_1 + 2c_2 = 0$

$2c_1 + 4c_2 = 0$



$(-2,5) = c_1(1,0) + c_2(0,1)$

$(-2,5) = -2(1,0) + 5(0,1)$

$V_3 = -2V_1 + 5V_2$

Testing for Linear Independence and Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent or linearly dependent, use the following steps.

1. From the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, write a system of linear equations in the variables $c_1, c_2, \ldots,$ and c_k.
2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$, then the set S is linearly independent. If the system also has nontrivial solutions, then S is linearly dependent.

LINEAR ALGEBRA APPLIED

Image morphing is the process by which one image is transformed into another by generating a sequence of synthetic intermediate images. Morphing has a wide variety of applications, including movie special effects, wound healing and cosmetic surgery results simulation, and age progression software. Morphing an image makes use of a process called warping, in which a piece of an image is distorted. The mathematics behind warping and morphing can include forming a linear combination of the linearly independent vectors that bound a triangular piece of an image, and performing an *affine transformation* to form new vectors and an image piece that is distorted.

A set of vectors $\{v_1, \dots v_k\}$ in a vector space V

is said to be linearly independent,

(if the vector equation, called the linear dependence of v_i's)

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

has only the trivial solution

$$c_1 = 0,\ c_2 = 0,\ c_3 = 0, \dots, c_k = 0$$

[All the scalars are zero]

otherwise, it is said to be linearly dependent.

For linearly dependent case:

$$c_1 x_1 + c_2 x_2 + \cdots + c_m x_m = \bar{0} \quad \longrightarrow (1)$$

has (non-trivial solution), for example if $c_m \neq 0$

$$(1) \Rightarrow \quad x_m = -\frac{c_1}{c_m} x_1 - \frac{c_2}{c_m} x_2 - \cdots - \frac{c_{m-1}}{c_m} x_{m-1}$$

$$\Rightarrow x_m = -[d_1 x_1 + d_2 x_2 + \cdots + d_{m-1} x_{m-1}]$$

Note:

a set of vectors is linearly dependent if and only if

atleast one of the vectors in the set can be written as the linear combination of the other vectors.

Pbm: Let $u = (1,-3,2)$, $v = (2,-1,1) \in \mathbb{R}^3$

Write $w = (1,7,-4)$ as a linear combination of u & v.

Soln:

$$w = \alpha u + \beta v$$

$$(1,7,-4) = \alpha(1,-3,2) + \beta(2,-1,1)$$

$$= (\alpha, -3\alpha, 2\alpha) + (2\beta, -\beta, \beta)$$

$$(1,7,-4) = (\alpha+2\beta,\ -3\alpha-\beta,\ 2\alpha+\beta)$$

$$\Rightarrow \left.\begin{aligned} \alpha+2\beta &= 1 \\ -3\alpha-\beta &= 7 \\ 2\alpha+\beta &= -4 \end{aligned}\right\} \Rightarrow \begin{array}{c} \qquad A \qquad\quad B \\ \begin{array}{cc} \alpha & \beta \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & 1 \\ -3 & -1 & 7 \\ 2 & 1 & -4 \end{array}\right] \end{array}$$

$$\begin{array}{c} \qquad A \qquad\quad B \\ \begin{array}{cc} \alpha & \beta \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & 1 \\ -3 & -1 & 7 \\ 2 & 1 & -4 \end{array}\right] \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 5 & 10 \\ 0 & -3 & -6 \end{array}\right] \quad \begin{aligned} &\to R_2 \to R_2 + 3R_1 \\ &R_3 \to R_3 - 2R_1 \end{aligned}$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \quad R_2 \to R_2/5 \ \& \ R_3 \to R_3/-3$$

Since $R_2 = R_3$

$\to R_3 \to R_3 - R_2$

$$\Rightarrow \begin{array}{c} \begin{array}{cc} \alpha & \beta \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$\rho(A) = \rho(A|B) = \#$ of unknowns

$\Rightarrow$ unique soln.

$$\Rightarrow \beta = 2, \quad \alpha + 2\beta = 1$$

$$\alpha + 2(2) = 1$$

$$\left.\begin{aligned} \Rightarrow \alpha &= -3 \\ \beta &= 2 \end{aligned}\right\}$$

(or)

$\alpha + 2\beta = 1$

$-3\alpha - \beta = 7$

$2\alpha + \beta = -4$

$$\left.\begin{aligned} -3\alpha - \beta &= 7 \\ 2\alpha + \beta &= -4 \end{aligned}\right\}$$

$-\alpha = 3$

$\alpha = -3$

$\alpha + 2\beta = 1$

$2\beta = +4$

$\beta = 2$

$$\Rightarrow (1,7,-4) = \alpha u + \beta v = -3(1,-3,2) + 2(2,-1,1)$$

Testing for Linear Independence

Determine whether the set of vectors in R^3 is linearly independent or linearly dependent.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

Soln:- Given $\{\overset{v_1}{(1,2,3)}, \overset{v_2}{(0,1,2)}, \overset{v_3}{(-2,0,1)}\}$

Linear Combination $c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$

$$c_1 (1,2,3) + c_2 (0,1,2) + c_3 (-2,0,1) = (0,0,0)$$

$$(c_1, 2c_1, 3c_1) + (0c_2, c_2, 2c_2) + (-2c_3, 0, c_3) = (0,0,0)$$

$\Rightarrow$

$$c_1 + 0c_2 - 2c_3 = 0$$
$$2c_1 + c_2 + 0c_3 = 0$$
$$3c_1 + 2c_2 + c_3 = 0$$

$$\Rightarrow \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right]$$

All the method
$\Delta = 0 \rightarrow$ Linearly dependent
$\Delta \neq 0 \rightarrow$ independent

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 7 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$
$R_3 \to R_3 / -1$

$$\approx \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_3 = 0$
$c_2 + 4c_3 = 0 \Rightarrow c_2 = 0$
$c_1 + (-2) c_3 = 0$
$\Rightarrow c_1 = 0$

$$\Rightarrow c_1 = 0, c_2 = 0, c_3 = 0$$

$\Rightarrow$ The set $S = \{v_1, v_2, v_3\}$

is linearly independent

Testing for Linear Independence

Determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

$$S = \{\underbrace{1 + x - 2x^2}_{\mathbf{v}_1},\ \underbrace{2 + 5x - x^2}_{\mathbf{v}_2},\ \underbrace{x + x^2}_{\mathbf{v}_3}\}$$

$\vec{0} \in P_2(\mathbb{R})$ $\quad$ $\vec{0} = 0x^2 + 0x + 0$

soln: $c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$

$$c_1(1+x-2x^2) + c_2(2+5x-x^2) + c_3(x+x^2) = 0x^2 + 0x + 0$$

$$x^2(-2c_1 - c_2 + c_3) + x(c_1 + 5c_2 + c_3) + 1(c_1 + 2c_2 + 0c_3) = 0x^2 + 0x + 0$$

$$\Rightarrow \begin{aligned} -2c_1 - c_2 + c_3 &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ c_1 + 2c_2 + 0c_3 &= 0 \end{aligned} \qquad [A|B] \Rightarrow \begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 1 & 5 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 1 & 5 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \qquad R_1 \to R_1/-2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 9/2 & 3/2 & 0 \\ 0 & 3/2 & 1/2 & 0 \end{array}\right] \qquad \begin{aligned} R_2 &\to R_2 - R_1 \\ R_3 &\to R_3 - R_1 \end{aligned}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \qquad \begin{aligned} R_2 &\to 2R_2 \\ R_3 &\to 2R_3 \end{aligned}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3 \to R_3 - \frac{1}{3}(R_2)$$

$\rho(A) = \rho(A|B) = 2 < \#$ of unknowns 3 (c_1, c_2, c_3)

$\Rightarrow$ This system has infinitely many solns

$\Rightarrow$ The set $S = \{v_1, v_2, v_3\}$ is linearly dependent

$$c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2)=0x^2+0x+0$$

$$\rightarrow \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 9 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Let $c_3 = t$

- $9c_2 + 3c_3 = 0$

$3c_2 + c_3 = 0$

$3c_2 = -c_3 = -t$

$c_2 = -\frac{1}{3}t$

- $c_1 + \frac{1}{2}c_2 - \frac{1}{2}c_3 = 0$

$c_1 + \left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)t - \frac{1}{2}\left(-\frac{1}{3}\right)t = 0$

$c_1 = \frac{1}{6}t - \frac{1}{6}t = 0$

$$\vec{x} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{3}t \\ t \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix} t \qquad \boxed{\forall t \in \mathbb{R}}$$

$\underline{t = (-3)}$

$$c_1(1+x-2x^2)+c_2\underbrace{(2+5x-x^2)}_{v_2}+c_3\underbrace{(x+x^2)}_{v_3}=0x^2+0x+0$$

$0(\quad) + 1.(2+5x-x^2) + (-3).(x+x^2)$

$1 \cdot v_2 = 3 \cdot v_3$

$2+5x-x^2 = 3x+3x^2$

$v_2 = 3(x+x^2)$

$$\boxed{v_2 = 3v_3}$$

Testing for Linear Independence

Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$S = \left\{ \overset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \overset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \overset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

$M_{m \times n}$

$M_{2\times 2}$ (or) $M_{2,2}$

Soln:-

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

$\vec{0} \in M_{2\times 2}$

$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

$$c_1 \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2c_1 & c_1 \\ 0c_1 & c_1 \end{pmatrix} + \begin{pmatrix} 3c_2 & 0c_2 \\ 2c_2 & 1c_2 \end{pmatrix} + \begin{pmatrix} 1c_3 & 0c_3 \\ 2c_3 & 0c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$2c_1 + 3c_2 + 1c_3 = 0$$
$$c_1 + 0c_2 + 0c_3 = 0 \rightarrow \boxed{c_1 = 0}$$
$$0c_1 + 2c_2 + 2c_3 = 0 \rightarrow \because c_1 = 0, c_2 = 0 \Rightarrow c_3 = 0$$
$$c_1 + 1.c_2 + 0c_3 = 0 \quad \because c_1 = 0 \rightarrow \underline{c_2 = 0}$$

$\Rightarrow \boxed{c_1 = c_2 = c_3 = 0} \Rightarrow$ $S = \{v_1, v_2, v_3\}$ is linearly independent

Testing for Linear Independence

Determine whether the set of vectors in $M_{4,1}$ is linearly independent or linearly dependent.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$\underline{M_{4,1}}$

$\underline{\text{Soln:}}$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \vec{0} \qquad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in M_{4,1}$$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right] \qquad R_3 \to R_3 - R_1$$

$R_4 \to R_4/2 \longrightarrow \underline{R_3 \leftrightarrow R_4}$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \qquad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \qquad R_3 \to R_3/-4$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$R_4 \to R_4 - R_3$

$R_4 \to R_4/-1$

$$\begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \end{array}\right] \end{array}$$

$\Rightarrow c_4 = 0,\ c_3 = 0,\ c_2 = 0 \Rightarrow c_1 = 0$

S is L. Independent

THEOREM A Property of Linearly Dependent Sets

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors in S.

THEOREM Corollary

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

REMARK

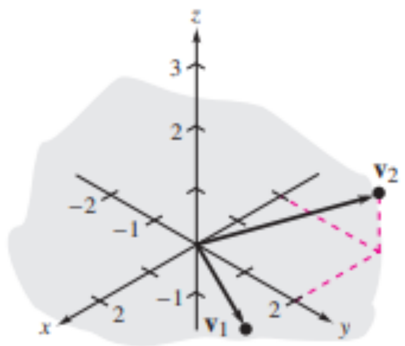
The zero vector is always a scalar multiple of another vector in a vector space.

Testing for Linear Dependence of Two Vectors

a. The set $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1, 2, 0), (-2, 2, 1)\}$ is linearly independent because $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of each other

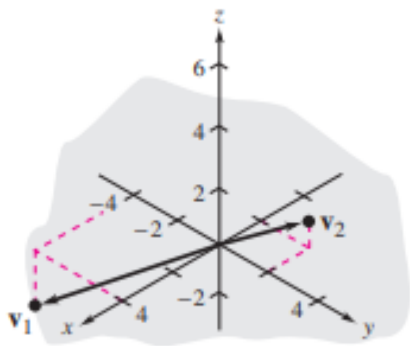
b. The set $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(4, -4, -2), (-2, 2, 1)\}$ is linearly dependent because $\mathbf{v}_1 = -2\mathbf{v}_2$

a.



$S = \{(1, 2, 0), (-2, 2, 1)\}$
The set S is linearly independent.

b.



$S = \{(4, -4, -2), (-2, 2, 1)\}$
The set S is linearly dependent.

$V_1 = -2V_2$

Span of a Set of Vectors

$S = \{v_1, v_2, v_3, \dots, v_n\}$

<u>Definition</u> The **span of a set** $\mathcal{S}$ **of vectors**, denoted span($\mathcal{S}$) is the set of all linear combinations of those vectors.

$V = \mathbb{R}, \mathbb{R}^2, \mathbb{R}^n, \dots, M_{m\times n}$

Spanning set

Definition. A subset S of a vector space V is called a **spanning set** for V if $\mathrm{Span}(S) = V$.

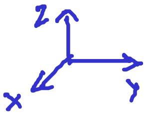
Examples.

- Vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ form a spanning set for $\mathbb{R}^3$ as

$$(x,y,z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

$\rightarrow V = \mathbb{R}^3$

$S = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$



$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(x,y,z) = ae_1 + be_2 + ce_3$

$\Rightarrow a = x,\ b = y,\ c = z$

- Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ form a spanning set for $\mathcal{M}_{2,2}(\mathbb{R})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

2×2

$\left[\begin{array}{cccc|c} c_1 & c_2 & c_3 & c_4 & \\ 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right]$

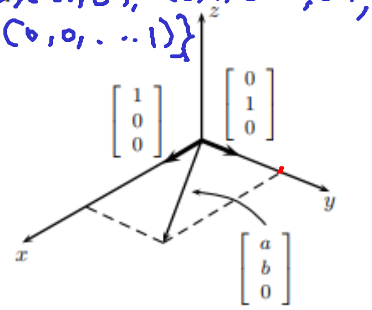
: Describe ~~the~~ the span of the set $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \boxed{0} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \boxed{0} \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

$a_1 \quad a_2$

$\vec{x} = a v_1 + b v_2$

$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\rightarrow \mathbb{R}^n = \{(1,0,\dots,0), (0,1,0,\dots,0), \dots (0,0,\dots,1)\}$



ANY vector with a zero third component can be written as a linear combination of these two vectors:

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is span set and it generates $\mathbb{R}^2 \subset \mathbb{R}^3$.

All the vectors with $x_3 = 0$ (or $z = 0$) are the xy plane in $\mathbb{R}^3$, so the span of this set is the xy plane. Geometrically we can see the same thing in the picture to the right.

Describe span $\left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right)$.

$\vec{x} = c_1(\vec{v_1}) + c_2(\vec{v_2})$

the span of this set is all vectors $\mathbf{v}$ of the form

$$\mathbf{v} = c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

which, because the two vectors are not scalar multiples of each other, we recognize as being a plane through the origin. It should be clear that all vectors created by such a linear combination will have a third component of zero, so the particular plane that is the span of the two vectors is the xy-plane. Algebraically we see that any vector $[a, b, 0]$ in the xy-plane can be created by

$$[A|B] = \begin{array}{cc} c_1 & c_2 \\ \end{array} \left[\begin{array}{cc|c} 1 & 3 & a \\ -2 & 1 & b \\ 0 & 0 & 0 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & 7 & 2a+b \end{array}\right] \quad R_2 \to R_2 + 2R_1$$

$$\approx \left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & 1 & \frac{2a+b}{7} \end{array}\right] \quad R_2 \to \frac{R_2}{7}$$

$$\approx \begin{array}{cc} c_1 & c_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & (a-3b)/7 \\ 0 & 1 & \frac{2a+b}{7} \end{array}\right] \quad R_1 \to R_1 - 3R_2$$

$$a - 3\left(\frac{2a+b}{7}\right) = \frac{7a - 6a - 3b}{7} = \frac{a-3b}{7}$$

$\rho(A) = \rho(A|B) = \#$ of unknown.

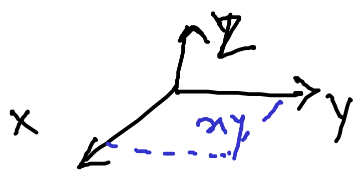
$\Rightarrow$ This system has unique solution

$$\Rightarrow \quad c_1 = \frac{a-3b}{7} \; ; \; c_2 = \frac{2a+b}{7}$$

$$\vec{v} = c_1 v_1 + c_2 v_2 = \left(\frac{a-3b}{7}\right)\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \left(\frac{2a+b}{7}\right)\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7a}{7} \\ \frac{7b}{7} \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

$$\left(\frac{a-3b}{7}\right)\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \left(\frac{2a+b}{7}\right)\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a-3b}{7} \\ \frac{-2a+6b}{7} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{6a+3b}{7} \\ \frac{2a+b}{7} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{7a}{7} \\ \frac{7b}{7} \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$



#3:

$$\text{Is } \mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ -4 \\ 1 \end{bmatrix} \text{ in the span of } \mathcal{S} = \left\{ \begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, & \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, & \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \end{array} \right\} ?$$

Soln.:

$\Rightarrow$ $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{v}$

"can we find scalars c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -4 \\ 1 \end{bmatrix} ?"$$

$Ax = B \longrightarrow [A|B] \longrightarrow \text{rref}([A|B])$

$$\left.\begin{aligned} c_1 + c_2 + 2c_3 &= 3 \\ 2c_1 - c_2 &= -2 \\ 3c_1 + c_2 - 3c_3 &= -4 \\ 4c_1 - c_2 + c_3 &= 1 \end{aligned}\right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & -1 & 0 & -2 \\ 3 & 1 & -3 & -4 \\ 4 & -1 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{cc} A & \quad\quad\quad B \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$\rho(A) = 3, \; \rho(A|B) = 4$

$\Rightarrow \rho(A) \neq \rho(A|B) \Rightarrow$ No Solution.

This tells us that the system above and to the left has no solution,

so there are no scalars c_1, c_2 and c_3 for which $\underline{\underline{c_1}} v_1 + \underline{\underline{c_2}} v_2 + \underline{\underline{c_3}} v_3 = \underline{\underline{v}}$

Thus $\mathbf{v}$ is not in the span of $\mathcal{S}$.

$S = \{v_1 \ldots v_k\} \longrightarrow c_1 v_1 + c_2 v_2 + \cdots + c_k \vec{v_k} \longrightarrow$ Subspace of V

$$\text{Is } \mathbf{v} = \begin{bmatrix} 19 \\ 10 \\ -1 \end{bmatrix} \text{ in span}(\mathcal{S}), \text{ where } \mathcal{S} = \left\{ \begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, & \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, & \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} \end{array} \right\} \subset \mathbb{R}^3 ?$$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{v}$?

Here we are trying to find scalars c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \\ -1 \end{bmatrix}.$$

$Ax = B$ → $\mathrm{rref}(A|B)$

$$\begin{aligned} 3c_1 - 5c_2 + c_3 &= 19 \\ -c_1 \quad + 7c_3 &= 10 \\ 2c_1 + c_2 - 4c_3 &= -1 \end{aligned}$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \\ c_1 \; c_2 \; c_3 \end{array} \Rightarrow \left.\begin{array}{l} c_3 = 2 \\ c_2 = -6 \\ c_1 = 4 \end{array}\right\}$$

$c_1 = 4$, $c_2 = -1$ and $c_3 = 2$, so $\mathbf{v}$ is in span(S).

X

~~Here we are trying to find scalars c_1, c_2 and c_3 such that~~

$12+5+2=19$

$-4+0+14=10$

$$\overset{4}{c_1}\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + \overset{-1}{c_2}\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + \overset{2}{c_3}\begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \\ -1 \end{bmatrix}$$

$8-1-8=-1$ ✓

Is $\mathbf{v} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$ in span$\left(\overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}\right)$?

$c_1v_1 + c_2v_2 + c_3v_3 = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \left.\begin{array}{l} c_1 = 4 \\ c_2 = 7 \\ c_3 = -1 \end{array}\right\}$$

$$4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

$\mathbf{v}$ is in span$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{w} = (4, -7, 3)$. Determine whether $\mathbf{w}$ belongs to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

$v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

$W = c_1v_1 + c_2v_2$

We have to check if there exist $r_1, r_2 \in \mathbb{R}$ such that $\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2$. This vector equation is equivalent to a system of linear equations:

$$\begin{cases} 4 = r_1 + 3r_2 \\ -7 = 2r_1 + r_2 \\ 3 = 0r_1 + r_2 \end{cases} \iff \begin{cases} r_1 = -5 \\ r_2 = 3 \end{cases}$$

Thus $\mathbf{w} = -5\mathbf{v}_1 + 3\mathbf{v}_2 \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

$\boxed{\begin{array}{l} \text{Subspace property:} \\ \forall u, v \in S \subset V, \; a, b \in \mathbb{R} \\ au + bv \in S \end{array}}$

Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

$c_1 v_1 + c_2 v_2 = V$

$(1,0)$

Alternative solution: First let us show that vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ belong to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

$$\mathbf{e}_1 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 1 \\ 5r_1 + 3r_2 = 0 \end{cases} \iff \begin{cases} r_1 = 3 \\ r_2 = -5 \end{cases}$$

$\begin{pmatrix}1\\0\end{pmatrix} = c_1\begin{pmatrix}2\\5\end{pmatrix} + c_2\begin{pmatrix}1\\3\end{pmatrix}$

$$\begin{pmatrix}0\\1\end{pmatrix} = \mathbf{e}_2 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 0 \\ 5r_1 + 3r_2 = 1 \end{cases} \iff \begin{cases} r_1 = -1 \\ r_2 = 2 \end{cases}$$

Thus $\mathbf{e}_1 = 3\mathbf{v}_1 - 5\mathbf{v}_2$ and $\mathbf{e}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$.
Then for any vector $\mathbf{w} = (a,b) \in \mathbb{R}^2$ we have

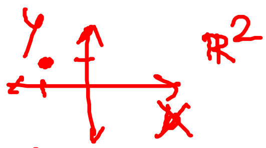
$$\boxed{\mathbf{w} = a\mathbf{e}_1 + b\mathbf{e}_2} = a(3\mathbf{v}_1 - 5\mathbf{v}_2) + b(-\mathbf{v}_1 + 2\mathbf{v}_2)$$
$$W = (3a - b)\mathbf{v}_1 + (-5a + 2b)\mathbf{v}_2.$$

Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Remarks on the alternative solution:
Notice that $\mathbb{R}^2$ is spanned by vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ since $(a,b) = a\mathbf{e}_1 + b\mathbf{e}_2$.
This is why we have checked that vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ belong to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Then

$$\mathbf{e}_1, \mathbf{e}_2 \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \implies \mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2) \subset \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$$
$$\implies \mathbb{R}^2 \subset \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \implies \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbb{R}^2.$$

In general, to show that $\mathrm{Span}(S_1) = \mathrm{Span}(S_2)$, it is enough to check that $S_1 \subset \mathrm{Span}(S_2)$ and $S_2 \subset \mathrm{Span}(S_1)$.



$(-2,-3) = -2(1,0) + (-3)(0,1)$

When?

$A = B$

① $A \subset B$

② $B \subset A$

from ① and ②

$\Rightarrow A = B$

Basis and Dimension of a Vector space V.

A set $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ of vectors in a vector space V is a **basis** for V if

(1) S spans V and → $L(S) = V$

(2) S is linearly independent.

$c_1 v_1 + c_2 v_2 = \vec{x}$

Is the set $S = \{(1, 1), (1, -1)\}$ a basis for R^2? → spans, LI

(1) Does S span R^2?

$c_1 v_1 + c_2 v_2 = \vec{x}$

$$\text{Solve } c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{array}{rcrcl} 1c_1 & + & 1c_2 & = & x \\ 1c_1 & - & 1c_2 & = & y \end{array}$$

$[A\,x\,|\,B]$:

$$\left(\begin{array}{rr|r} 1 & 1 & x \\ 1 & -1 & y \end{array}\right) \xrightarrow{r_2 - r_1 \to r_2} \left(\begin{array}{rr|c} 1 & 1 & x \\ 0 & -2 & y - x \end{array}\right)$$

This system is consistent for every x and y, therefore S spans R^2.

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$2c_1 = x + y$

$c_1 = \frac{x + y}{2}$

$c_2 = ?$

$\rho(A) = \rho(A|B) = 2$ = # no of un

$\rho(A) = \rho(A|B) = \#$ of unknowns $\Rightarrow 2$

(2) Is S linearly independent?

$$\text{Solve } c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{rcrcl} 1c_1 & + & 1c_2 & = & 0 \\ 1c_1 & - & 1c_2 & = & 0 \end{array}$$

$$\left(\begin{array}{rr|r} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{r_2 - r_1 \to r_2} \left(\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right)$$

The system has a unique solution $c_1 = c_2 = 0$ (trivial solution).

$c_1 = c_2 = 0$

Therefore S is linearly independent. Consequently, S is a basis for R^2.

$S \to L(S) = V = \mathbb{R}^2$, L.I $\Rightarrow$ S is a basis for $\mathbb{R}^2$.

$\dim(S) = 2$.

c_1 c_2 c_3

Is $S = \{(1,2,3),(0,1,2),(-1,0,1)\}$ a basis for R^3?

$\mathbb{R}^n \to$ V S

$\mathbb{R}^3 \to$ V.S

① $L(S) = V$
② L.I

Soln:

$c_1 \vec{v_1} + c_2 \vec{v_2} + c_3 \vec{v_3} = \vec{x}$

$c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1) = \vec{x}$

$\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$c_1 + 0c_2 - c_3 = a$
$2c_1 + c_2 + 0c_3 = b$
$3c_1 + 2c_2 + c_3 = c$

$\xrightarrow{[A|B]}$ $\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$ $\left[\begin{array}{ccc|c|c} 1 & 0 & -1 & a & 0 \\ 2 & 1 & 0 & b & 0 \\ 3 & 2 & 1 & c & 0 \end{array}\right]$

$[A|B] \approx \left[\begin{array}{ccc|c|c} 1 & 0 & -1 & a & 0 \\ 0 & 1 & 2 & b-2a & 0 \\ 0 & 2 & 4 & c-3a & 0 \end{array}\right]$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$\approx \left[\begin{array}{ccc|c|c} 1 & 0 & -1 & a & 0 \\ 0 & 1 & 2 & b-2a & 0 \\ 0 & 0 & 0 & c-5a & 0 \end{array}\right]$

$R_3 \to R_3 - 2R_1$

$c - 3a - 2(a)$

if

$\approx$ $\rho(A) = 2 \Rightarrow$

$\rho(A|B) = \begin{cases} 3 & \text{when } c \neq 5a \\ 2 & \text{when } c = 5a \end{cases}$

# of unknowns are 3 (c_1, c_2, c_3)

Let $c_3 = t$: $c_2 + 2c_3 = 0$

$c_2 = -2c_3 = -2t$

$c_1 - c_3 = 0$
$c_1 = c_3 = t$

$\vec{x_1} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t$; $t \in \mathbb{R}$

Here, $\vec{x} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t$, $t \in \mathbb{R}$, System have infinitely many solutions.

To say the vectors are L.I means, we need $c_1 = c_2 = c_3 = 0$

But here $\vec{x}$ have many solution.

$\Rightarrow S = \{v_1, v_2, v_3\}$ is linearly dependent

$\Rightarrow S = \{v_1, v_2, v_3\}$ is not a basis for vector space $\mathbb{R}^3$.

Is $S = \{(1,0),(0,1),(-2,5)\}$ a basis for R^2?

$\sigma_1 \quad \sigma_2 \quad c_3$

~~Not~~ form a lin... $c_3 \rightarrow$ free

$$\left[\begin{array}{ccc|c|c} 1 & 0 & -2 & x & 0 \\ 1 & 1 & 5 & y & 0 \end{array}\right]$$

#

$$S = \{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\vec{v_1}}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\vec{v_2}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\vec{v_3}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\vec{v_4}} \}$$

① $L(S) = V$ ✓

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \bar{x}$

② $c_1v_1 + c_2v_2 + \dots + c_4v_4 = \vec{0}$ → $c_1 = c_2 = c_3 = c_4 = 0$

is a basis for the vector space M_{22}.

$$\left[\begin{array}{cccc|c|c} 1 & 0 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & 0 & b & 0 \\ 0 & 0 & 1 & 0 & c & 0 \\ 0 & 0 & 0 & 1 & d & 0 \end{array}\right]$$

Yes, it is basis,

$\dim(S)$ for $M_{2\times 2} = 4$

$\dim(M_{m\times n}) = m \times n$

(1) $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} = \vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

is equivalent to:

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which is consistent for every $a, b, c,$ and d.
Therefore S spans M_{22}.

(2) $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} = \vec{0}$
is equivalent to:

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\bar{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

The system has only the trivial solution $\Rightarrow S$ is linearly independent.

Consequently, S is a basis for M_{22}.

$\dim(\mathbb{R}^4) = 4$

Standard basis →

Standard Basis:

$\mathbb{R}^4$: $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

① $V = \mathbb{R}^n$: $S = \{ e_1 = (1,0,0,\dots,0), e_2 = (0,1,\dots,0), \dots, e_n = (0,0,\dots,n) \}$

$\# S : \dim(S) = n$

$\dim(P_n) = n+1$

Is $S = \{\overset{v_1}{1}, \overset{v_2}{t}, \overset{v_3}{t^2}, \overset{v_4}{t^3}\}$ a basis for P_3?

(1) $c_1(1) + c_2(t) + c_3(t^2) + c_4(t^3)$
$= a + bt + ct^2 + dt^3$
has a solution for every $a, b, c,$ and d :
$c_1 = a, c_2 = b, c_3 = c, c_4 = d.$ → unique sol'n
Therefore S spans P_3.

(2) $c_1(1) + c_2(t) + c_3(t^2) + c_4(t^3) = \vec{0} = 0 + 0t + 0t^2 + 0t^3$
can only be solved by
→ $c_1 = c_2 = c_3 = c_4 = 0.$
Therefore S is linearly independent.

Consequently, S is a basis for P_3.

$\dim(P_3) = 3+1$
$|S| = 4$ //

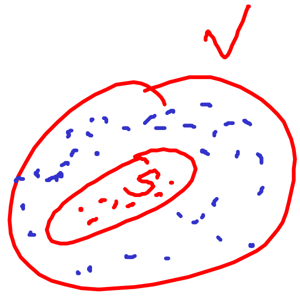
THEOREM

Let $S = \{\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}\}$ be a set of nonzero vectors in a vector space V. The following statements are equivalent:

(A) S is a basis for V,

(B) every vector in V can be expressed as a linear combination of the vectors in S in a **unique** way.

Basis



Proof (A)⇒(B)

- Every vector in V can be expressed as a linear combination of vectors in S because S spans V.
- Suppose $\vec{v}$ can be represented as a linear combination of vectors in S in **two ways**:

$$\vec{v} = c_1\vec{v_1} + \cdots + c_k\vec{v_k}$$
$$\vec{v} = d_1\vec{v_1} + \cdots + d_k\vec{v_k}$$

Subtract:

$$\vec{0} = (c_1 - d_1)\vec{v_1} + \cdots + (c_k - d_k)\vec{v_k}$$

Subtract:

$$\vec{0} = (c_1 - d_1)\vec{v_1} + \cdots + (c_k - d_k)\vec{v_k}$$

Since S is linearly independent, then

$$c_1 - d_1 = \cdots = c_k - d_k = 0$$

so that

$$c_1 = d_1$$
$$\vdots$$
$$c_k = d_k$$

The representation is **unique**.

Proof (B)⇒(A)

- (B)⇒ Every vector in V is in span S.
- **Zero vector** in V can be represented in a **unique** way as a linear combination of vectors in S:

$$\vec{0} = c_1\vec{v_1} + \cdots + c_k\vec{v_k}$$

This unique way must be: $c_1 = \cdots = c_k = 0$.
Therefore S is linearly independent.

Consequently, S is a basis for V.

Back to **EXAMPLE 2**:
$S = \{(1,1),(1,-1)\}$

Instead of showing that

- $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} = \overrightarrow{v}$ has a solution, and
- $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} = \overrightarrow{0}$ has a unique solution,

we can show

- $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} = \overrightarrow{v}$ has a unique solution.

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 1 & -1 & y \end{array}\right) \xrightarrow{r_2 - r_1 \to r_2} \left(\begin{array}{cc|c} 1 & 1 & x \\ 0 & -2 & y-x \end{array}\right)$$

Unique solution for every x and $y \Rightarrow S$ is a basis for R^2.

Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ be a set of nonzero vectors in a vector space V. Some subset of S is a basis for $W = \text{span } S$.

EXAMPLE Find a basis for
$\text{span}\{\underbrace{(1,2,3)}_{\overrightarrow{v_1}}, \underbrace{(-1,-2,-3)}_{\overrightarrow{v_2}}, \underbrace{(0,1,1)}_{\overrightarrow{v_3}}, \underbrace{(1,1,2)}_{\overrightarrow{v_4}}\}$.

Set $c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2} + c_3\overrightarrow{v_3} + c_4\overrightarrow{v_4} = \overrightarrow{0}$.

The corresponding system has augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 \\ 3 & -3 & 1 & 2 & 0 \end{array}\right)$$

which is equivalent
$(r_2 - 2r_1 \to r_2; r_3 - 3r_1 \to r_3; r_3 - r_2 \to r_3)$ to

$$\left(\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Can set c_2 and c_4 arbitrary. For example

- If $c_2 = 1, c_4 = 0$ then $\overrightarrow{v_2}$ can be expressed as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$.
- If $c_2 = 0, c_4 = 1$ then $\overrightarrow{v_4}$ can be expressed as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$.

Therefore, every vector in span S can be expressed as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$. Also note that $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$ are linearly independent. Consequently, they form a basis for span S.

Summarizing: The vectors corresponding to the columns with leading entries form a basis for W.

Different initial ordering of vectors, e.g., $\{\overrightarrow{v_2}, \overrightarrow{v_1}, \overrightarrow{v_3}, \overrightarrow{v_4}\}$ may change the basis obtained by the procedure above (in this case: $\overrightarrow{v_2}, \overrightarrow{v_3}$).

Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ span V and let $T = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots \overrightarrow{w_n}\}$ be a linearly independent set of vectors in V. Then $n \leq k$.

Let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_k}\}$ and $T = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots \overrightarrow{w_n}\}$ both be bases for V. Then $n = k$.

The dimension of a vector space V, denoted dim V, is the number of vectors in a basis for V.

$\dim(\{\overrightarrow{0}\}) = 0.$

- $\dim(R^n) = n$
- $\dim(P_n) = n + 1$
- $\dim(M_{mn}) = mn$

If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there exists a basis T for V, which contains S.

EXAMPLE

Find a basis for R^4 that contains the vectors $\overrightarrow{v_1} = (1,0,1,0)$ and $\overrightarrow{v_2} = (-1,1,-1,0)$.

Solution:

The natural basis for R^4:

$$\{\underbrace{(1,0,0,0)}_{\overrightarrow{e_1}}, \underbrace{(0,1,0,0)}_{\overrightarrow{e_2}}, \underbrace{(0,0,1,0)}_{\overrightarrow{e_3}}, \underbrace{(0,0,0,1)}_{\overrightarrow{e_4}}\}$$

Follow the procedure of **EXAMPLE** to determine a basis of span$\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}\}$.

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

has the reduced row echelon form:

$$\left(\begin{array}{cccccc|c} \boxed{1} & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

Answer: $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{e_1}, \overrightarrow{e_4}\}$.

Let V be an n-dimensional vector space, and let $S = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots \overrightarrow{v_n}\}$ be a set of n vectors in V.

(a) If S is linearly independent then it is a basis for V.

(b) If S spans V then it is a basis for V.