Matrices of linear transformations

Let V be a vector space with basis $\alpha = \{v_1, v_2, \dots, v_n\}$ and W be a vector space with basis $\beta = \{w_1, w_2, \dots, w_m\}$. Consider a linear transformation $T: V \to W$.

To find the matrix associated to T we proceed by the following steps.

(i) Find $T(v_1), T(v_2), \dots, T(v_n)$

(ii) If $T(v_1) = y_1$ then write $y_1 = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$

If $T(v_2) = y_2$ then write $y_2 = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$

$\vdots$

If $T(v_n) = y_n$ then write $y_n = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_n$

The matrix of T is

$$[T]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Note:- If $\alpha = \beta$ then $[T]_\alpha^\beta$ can be written as $[T]_\alpha$

Pb:1 Find the matrix representation of the following linear transformations with respect to the standard basis

(a) $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x,y,z) = (5x - 2y + 4z,\ x - 5y - 7z,\ y + 10z)$

(b) $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x,y,z) = (2y + z,\ x - 4y,\ 3x)$.

Soln:- standard basis of $\mathbb{R}^3$ is $\alpha = \{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}$

(a) $T(1,0,0) = (5, 1, 0) = 5(1,0,0) + 1(0,1,0) + 0\cdot(0,0,1)$

$T(0,1,0) = (-2, -5, 1) = (-2)(1,0,0) + (-5)(0,1,0) + 1(0,0,1)$

$T(0,0,1) = (4, -7, 10) = 4(1,0,0) + (-7)(0,1,0) + 10(0,0,1)$

$$[T]_\alpha = \begin{bmatrix} 5 & -2 & 4 \\ 1 & -5 & -7 \\ 0 & 1 & 10 \end{bmatrix}$$

(b) $T(1,0,0) = (0,1,3)$; $T(0,1,0) = (2,-4,0)$, $T(0,0,1) = (1,0,0)$

$$[T]_\alpha = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Pb:2 Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x,y,z) = (x+y,\ y-z)$

Find the matrix of T with respect to

$\alpha = \{(1,0,1), (0,1,1), (1,1,1)\}$ and

$\beta = \{(1,2), (-1,1)\}$

Soln:- $T(1,0,1) = (1+0,\ 0-1) = (1,-1)$

$T(0,1,1) = (0+1,\ 1-1) = (1,0)$

$T(1,1,1) = (2,0)$

Linear combination of $T(1,0,1)$ by β

$T(1,0,1) = (1,-1) = k_1(1,2) + k_2(-1,1)$

$(1,-1) = (k_1 - k_2,\ 2k_1 + k_2)$

$\Rightarrow \underbrace{k_1 - k_2 = 1 \ ;\ 2k_1 + k_2}_{\downarrow} = -1$

$3k_1 = 0 \Rightarrow \boxed{k_1 = 0}$

$\therefore \boxed{k_2 = -1}$

Thus, $T(1,0,1) = 0\cdot(1,2) + (-1)(-1,1)$

$$T(0,1,1) = (1,0) = k_1(1,2) + k_2(-1,1)$$

$$\Rightarrow \quad 1 = k_1 - k_2 \ ; \quad 0 = 2k_1 + k_2$$

$$3k_1 = 1 \Rightarrow \boxed{k_1 = 1/3}$$

$$\boxed{k_2 = \frac{-2}{3}}$$

$$T(0,1,1) = \frac{1}{3}(1,2) + \left(\frac{-2}{3}\right)(-1,1)$$

$$T(1,1,1) = (2,0) = \frac{2}{3}(1,2) + \left(\frac{-4}{3}\right)(-1,1) \quad \text{(Check!)}$$

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & \frac{2}{3} & -4/3 \end{bmatrix}$$