LU decomposition consider a matrix A. To find LU decomposition we proceed by the following steps

(i) Apply row elementary operations to A successively untill we get the row echelon form of A, Call this row echelon form as matrix U

(ii) The entry l_{ij} $(i>j)$ in the matrix L is the multiplier co-efficient because of which the respective position becomes Zero in U and $\boxed{l_{ii} = 1 \text{ for all } i}$

Solving $AX=B$ by LU decomposition

1) Find LU decomposition of A

2) Take $Y = UX$

3) Solve the system $LY = B$ for Y.

4) solve the system $UX = Y$ for X

Pb:1 Solve the system $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}$ by LU factorization method.

Soln:-

Step 1

1st pivot

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix} \quad R_2 \to R_2 - (-2)R_1 \quad \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 8 & -17 \end{bmatrix} \quad R_3 \to R_3 - 4R_2$$

2nd pivot

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

This is the row echelon form. Call $U = \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

step 2 Take $UX = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

step 3:- Solve $LY = B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}$

$$\Rightarrow y_1 = 0 ; \; y_2 = 4 ; \; y_3 = 1$$

Step 4:- Solve $UX = Y \Rightarrow \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$

$$\Rightarrow \boxed{x = 2 ; \; y = 0 ; \; z = -1}$$

Solve the system $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix}$ by LU factorization method.

Soln:–

1st pivot

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - (-1)R_1 \end{matrix} \quad \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix} \quad R_3 \to R_3 - (-2)R_2 \quad \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

2nd pivot

This is the row echelon form. Take $U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

Take $UX = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Solve $LY = B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix} \Rightarrow$

$y_1 = 1$

$y_2 = -6$

$y_3 = -2$

Solve $UX = Y \Rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ -2 \end{bmatrix} \Rightarrow$

$$\boxed{\begin{matrix} x = 1/2 \\ y = 3 \\ z = -1 \end{matrix}}$$