Practice Problems:

Basis of subspace:

1.

Find a basis and dimension of the subspace W of $\mathbf{R}^3$ where

(a) $W = \{(a,b,c) : a+b+c = 0\}$, (b) $W = \{(a,b,c) : (a=b=c)\}$

(a) Note that $W \neq \mathbf{R}^3$, because, for example, $(1,2,3) \notin W$. Thus, $\dim W < 3$. Note that $u_1 = (1,0,-1)$ and $u_2 = (0,1,-1)$ are two independent vectors in W. Thus, $\dim W = 2$, and so u_1 and u_2 form a basis of W.

(b) The vector $u = (1,1,1) \in W$. Any vector $w \in W$ has the form $w = (k,k,k)$. Hence, $w = ku$. Thus, u spans W and $\dim W = 1$.

2.

Let V be the vector space of 2×2 matrices over K. Let W be the subspace of symmetric matrices. Show that $\dim W = 3$, by finding a basis of W.

Recall that a matrix $A = [a_{ij}]$ is symmetric if $A^T = A$, or, equivalently, each $a_{ij} = a_{ji}$. Thus, $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ denotes an arbitrary 2×2 symmetric matrix. Setting (i) $a=1$, $b=0$, $d=0$; (ii) $a=0$, $b=1$, $d=0$; (iii) $a=0$, $b=0$, $d=1$, we obtain the respective matrices:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We claim that $S = \{E_1, E_2, E_3\}$ is a basis of W; that is, (a) S spans W and (b) S is linearly independent.

(a) The above matrix $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = aE_1 + bE_2 + dE_3$. Thus, S spans W.

(b) Suppose $xE_1 + yE_2 + zE_3 = 0$, where x, y, z are unknown scalars. That is, suppose

$$x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Setting corresponding entries equal to each other yields $x=0$, $y=0$, $z=0$. Thus, S is linearly independent. Therefore, S is a basis of W, as claimed.

Interpolation:

1. Find a polynomial $p(x)$ of degree 3, such that $p(0) = 1, p'(0) = 2, p(1) = 4, p'(1) = 4$.
2. Find the equation of circle passing through points (2,-2), (3,5) and (-4,6).

Ans: (1) $1 + t + t^2$ (2) $x^2 + 2xy + y^2 - 4y - 20 = 0$

Linear Transformation:

1. Which of the following map is linear transform:

- $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_1(x,y) = (|x|, 0)$.
- $T_2 : \mathbb{P}_2 \to \mathbb{R}^2$, $T_2(a_0 + a_1 t + a_2 t^2) = (a_0 + a_1, a_2)$
- $T_3 : \mathbb{P}_2 \to \mathbb{R}^3$, $T_3(a_0 + a_1 t + a_2 t^2) = (a_1 - a_2, a_0 + 1)$
- $T_4 : \mathbb{M}_{2\times 2} \to \mathbb{R}^3$, $T_4\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d, b+c, a)$.

- $T_5 : \mathbb{M}_{2\times 2} \to \mathbb{R}^3$, $T_5(A) = A^T$.
- $T_6 : \mathbb{P}_3 \to \mathbb{P}_2$, $T_6(p(t)) = \frac{d}{dt}p(t)$.

Ans:

- T_1 is not linear transform, as scalar multiplication property not satisfies. $(\alpha T_1)(x, y) \neq \alpha(T_1(x, y))$.
- T_2 is a L.T.
- T_3 not L.T. as $T_3(0, 0) \neq (0, 0)$.
- T_4 is L.T.
- T_5 and T_6 are also L.T.

2. Check the invertibility of the following linear transformations and also find the inverse transform if exist:

- $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_1(x, y) = (x + y, x - y)$
- $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$, $T_2(x, y, z) = (2x + 3y, z - 4y, x + z)$
- $T_3 : \mathbb{P}_2 \to \mathbb{R}^3$, $T_3(a_0 + a_1t + a_2t^2) = (a_0 - a_1, 0, a_1 - a_2)$.
- $T_4 : \mathbb{P}_2 \to \mathbb{R}^3$, $T_4(a_0 + a_1t + a_2t^2) = (0, a_2 - a_0, a_1 - a_3)$.
- $T_5 : \mathbb{P}_2 \to \mathbb{R}^3$, $T_5(p(t)) = T_3(p(t)) + T_4(p(t))$.

Ans:

- T_1 is invertible.
- T_2 is also invertible
- T_3 and T_4 both are not onto, so not invertible.
- T_5 is invertible.

3. **Find the matrix of transformation of the following linear transformations:**

- $T_1 : \mathbb{P}_2 \to \mathbb{R}^3$, $T_1(a_0 + a_1t + a_2t^2) = (a_0 - a_1, 0, a_1 - a_3)$. Find $[T]_\alpha^\beta$, α and β are the standard basis of $\mathbb{P}_2$ and $\mathbb{R}^3$ respectively.
- $T_2 : \mathbb{R}^3 \to \mathbb{R}^4$, $T_2(x, y, z) = (x + 2y, y + 3z, y - z, x + z)$. find $[T]_\alpha^\beta$ and $[T]_\alpha^\gamma$, where α is the standard basis of $\mathbb{R}^3$, β is the standard basis of $\mathbb{R}^4$ and $\gamma = \{(1, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0)\}$.
- $T_3 : \mathbb{R}^4 \to \mathbb{R}^4$, $T_3(x, y, z, w) = (x + 2y, y + 3z, w + y - z, x + z)$. Find $[T]_\alpha$ and hence find $[T]_\beta$ using similarity transform, where α is the standard basis of $\mathbb{R}^4$ and $\beta = \{(1, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0)\}$.
- Given $T_4 : \mathbb{R}^3 \to \mathbb{R}^3$, $T_4(x, y, z) = (2x, y + z, 3y)$ and $T_5 : \mathbb{R}^3 \to \mathbb{R}^3$, $T_5(x, y, z) = (-x, y - z, 4y)$. Also α are β are the bases of $\mathbb{R}^3$, where α is the standard basis and $\beta = \{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$. Then find (i) $[T_4]_\alpha^\beta$ (ii) $[T_5]_\alpha^\beta$ (iii) $[T_4 + T_5]_\alpha^\beta$ (iv) $[T_4^{-1}]_\alpha^\beta$ (v) $[T_5^{-1}]_\alpha^\beta$ (vi) $[(T_4 + T_5)^{-1}]_\alpha^\beta$.

Ans:

- $[T_1]_\alpha^\beta = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$
- $[T_2]_\alpha^\beta = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, $\quad [T_2]_\alpha^\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -1 & 1 \\ -2 & 2 & -1 \\ -1 & -3 & 1 \end{bmatrix}$.
- $[T_3]_\alpha = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $[Id]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $[T_3]_\beta = ([\mathrm{Id}]_\beta^\alpha)^{-1}[T_1]_\alpha^\beta[\mathrm{Id}]_\beta^\alpha$.

Gram-Schmidt ortho-normalization process:

1. $\alpha = \{(2,3,1,1),(1,0,2,5),(2,1,3,0),(1,1,1,1)\}$ is a basis of $\mathbb{R}^4$. Use the Gram-Schmidt ortho-normalization process to transform α into orth-onormal basis.

2. $\beta = \{2+t, 2t^2, 3-t^2\}$is a basis of $\mathbb{P}_2$. Use the G-S ortho-normalization process to transform β into ortho-normal basis.

ans: (1) $e_1 = (2\frac{\sqrt{15}}{15}, \frac{\sqrt{15}}{5}, \frac{\sqrt{15}}{15}, \frac{\sqrt{15}}{15})^T$, $e_2 = (-\frac{\sqrt{615}}{615}, -3\frac{\sqrt{615}}{205}, 7\frac{\sqrt{615}}{615}, 22\frac{\sqrt{615}}{615})^T$, $e_3 = (2\frac{\sqrt{246}}{123}, -5\frac{\sqrt{246}}{246}, 13\frac{\sqrt{246}}{246}, -\frac{\sqrt{246}}{41})^T$, $e_4 = (0,0,0,0)^T$.

Matrix representation of linear transform:

1. $(\mathbb{R}^3, \langle\rangle)$ is an inner product space with $\{(1,0,0),(1,1,1),(1,1,0)\}$ as basis of $\mathbb{R}^3$. Find the matrix representation of the inner product.

2. $(\mathbb{P}_2, \langle\rangle)$ is an inner product space with $\{1+t, t, 2t^2\}$ as basis of $\mathbb{R}^3$. Find the matrix representation of the inner product.

Ans: (1) $\langle x, y\rangle = x^T A y$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

QR-Decomposition/Factorization:

1. Find the QR factorization of the following matrices

(i) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 4 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

Ans: (i) $Q = \begin{bmatrix} 0.447 & 0.365 & -0.816 \\ 0 & 0.913 & 0.408 \\ 0.894 & -0.183 & 0.408 \end{bmatrix}$, $R = \begin{bmatrix} 2.24 & 3.58 & -1.34 \\ 0 & 1.1 & -0.183 \\ 0 & 0 & 0.408 \end{bmatrix}$,

(ii) $Q = \begin{bmatrix} 0.229 & -0.677 & -0.699 \\ 0.229 & 0.701 & -0.604 \\ 0.918 & 0.0484 & 0.254 \\ 0.229 & -0.218 & 0.286 \end{bmatrix}$, $R = \begin{bmatrix} 4.36 & 2.06 & 1.84 \\ 0 & 2.18 & -0.822 \\ 0 & 0 & 1.4 \end{bmatrix}$