Matrix representation of an inner product space:-

Let V be an inner product space. $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. Suppose x and y be two vectors in V then

$$\langle x, y \rangle = [x]_\alpha^T A \, [y]_\alpha$$

where $[x]_\alpha$ is the co-ordinate matrix of x with respect to α

$[y]_\alpha$ is the co-ordinate matrix of y with respect to α

$A = [a_{ij}]$ where $a_{ij} = \langle v_i, v_j \rangle$

We call A is the matrix associated with the given inner product.

Pb:1 Consider $\mathbb{R}^3$ with basis $\{e_1, e_2, e_3\}$ and

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + 3 x_2 y_2 + 5 x_3 y_3$$

Find the matrix representation of the above inner product.

Soln:- We calculate the following.

$$\langle e_1, e_1 \rangle = \langle (1,0,0), (1,0,0) \rangle = 1.$$

$$\langle e_1, e_2 \rangle = \langle (1,0,0), (0,1,0) \rangle = 0$$

$$\langle e_1, e_3 \rangle = \langle (1,0,0), (0,0,1) \rangle = 0$$

$$\langle e_2, e_1 \rangle = \langle e_1, e_2 \rangle = 0$$

$$\langle e_2, e_2 \rangle = \langle (0,1,0), (0,1,0) \rangle = 3.$$

$$\langle e_2, e_3 \rangle = \langle (0,1,0), (0,0,1) \rangle = 0$$

$$\langle e_3, e_1 \rangle = \langle e_1, e_3 \rangle = 0$$

$$\langle e_3, e_2 \rangle = \langle e_2, e_3 \rangle = 0$$

$$\langle e_3, e_3 \rangle = \langle (0,0,1), (0,0,1) \rangle = 5$$

The required matrix

$$A = \begin{bmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle & \langle e_1, e_3 \rangle \\ \langle e_2, e_1 \rangle & \langle e_2, e_2 \rangle & \langle e_2, e_3 \rangle \\ \langle e_3, e_1 \rangle & \langle e_3, e_2 \rangle & \langle e_3, e_3 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Pb:2 Consider $P_2(\mathbb{R})$ with basis $\{1, t, t^2\}$ and

$$\langle f(t), g(t) \rangle = \int_0^1 f(t)\, g(t)\, dt$$

Find the matrix representation of the above inner product.

Soln:- $v_1 = 1$; $v_2 = t$; $v_3 = t^2$

$$\langle v_1, v_2 \rangle = \langle 1, t \rangle = \int_0^1 t\, dt = \left[\tfrac{t^2}{2}\right]_0^1 = \tfrac{1}{2}$$

$$\langle v_1, v_1 \rangle = \langle 1, 1 \rangle = \int_0^1 1\, dt = [t]_0^1 = 1$$

$$\langle v_1, v_3 \rangle = \langle 1, t^2 \rangle = \int_0^1 t^2 dt = \left[\tfrac{t^3}{3}\right]_0^1 = \tfrac{1}{3}$$

$$\langle v_2, v_1 \rangle = \langle v_1, v_2 \rangle = \tfrac{1}{2}$$

$$\langle v_2, v_2 \rangle = \langle t, t \rangle = \int_0^1 t^2 dt = \left[\tfrac{t^3}{3}\right]_0^1 = \tfrac{1}{3}$$

$$\langle v_2, v_3 \rangle = \langle t, t^2 \rangle = \int_0^1 t^3 dt = \left[\tfrac{t^4}{4}\right]_0^1 = \tfrac{1}{4}$$

$$\langle v_3, v_1 \rangle = \langle v_1, v_3 \rangle = \tfrac{1}{3}$$

$$\langle v_3, v_2 \rangle = \langle v_2, v_3 \rangle = \tfrac{1}{4}$$

$$\langle v_3, v_3 \rangle = \langle t^2, t^2 \rangle = \int_0^1 t^4 dt = \left[\tfrac{t^5}{5}\right]_0^1 = \tfrac{1}{5}$$

$$A = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \langle v_1, v_3 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \langle v_2, v_3 \rangle \\ \langle v_3, v_1 \rangle & \langle v_3, v_2 \rangle & \langle v_3, v_3 \rangle \end{bmatrix} = \begin{bmatrix} 1 & \tfrac{1}{2} & \tfrac{1}{3} \\ \tfrac{1}{2} & \tfrac{1}{3} & \tfrac{1}{4} \\ \tfrac{1}{3} & \tfrac{1}{4} & \tfrac{1}{5} \end{bmatrix}$$