Least square solution Consider a system $Ax=b$ where A is a $m\times n$ matrix and b is a $m\times 1$ matrix. Suppose that $Ax=b$ is inconsistent then we attempt to seek an $\hat{x}$ such that $\|A\hat{x}-b\|$ is as small as possible.

The closest such vector will be $\hat{x}$ such that $A\hat{x} = \text{Proj}_W(b)$ where W is the column space of A. In this case we say that $\hat{x}$ is the least square solution of $Ax=b$

Observation:- $b - \text{Proj}_W(b) \in C(A)^{\perp}$

$$b - \text{Proj}_W(b) \in N(A^T)$$

$$A^T(b - \text{Proj}_W(b)) = 0$$

$$A^T b = A^T \text{Proj}_W(b) = A^T A\hat{x}$$

$$\boxed{A^T A\hat{x} = A^T b}$$

Theorem:- Let A be a $m \times n$ matrix and let $b \in \mathbb{R}^m$ be any vector. Then a vector $\hat{x} \in \mathbb{R}^n$ is a least square solution of $Ax = b$ if and only if $\hat{x}$ is the solution of

$$\boxed{A^T A x = A^T b}$$

Pb: 1 Find all the least square solution of the system

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ 9 \end{bmatrix} = b$$

soln:-

Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 5 & 0 \end{bmatrix}$; $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 1 & 2 & 0 \end{bmatrix}$

$A^T A = \begin{bmatrix} 3 & 6 & 3 \\ 6 & 30 & 0 \\ 3 & 0 & 5 \end{bmatrix}$ $A^T b = \begin{bmatrix} 2 \\ 64 \\ -18 \end{bmatrix}$

We solve the system $A^T A x = A^T b$

$$\begin{bmatrix} 3 & 6 & 3 \\ 6 & 30 & 0 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 64 \\ -18 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 3 & 2 \\ 6 & 30 & 0 & 64 \\ 3 & 0 & 5 & -18 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \quad \begin{bmatrix} 3 & 6 & 3 & 2 \\ 0 & 18 & -6 & 60 \\ 0 & -6 & 2 & -20 \end{bmatrix} \quad \begin{array}{l} R_2 \to \frac{R_2}{6} \\ R_3 \to \frac{R_3}{2} \end{array}$$

$$\begin{bmatrix} 3 & 6 & 3 & 2 \\ 0 & 3 & -1 & 10 \\ 0 & -3 & 1 & -10 \end{bmatrix} \quad R_3 \to R_3 + R_2 \quad \begin{bmatrix} 3 & 6 & 3 & 2 \\ 0 & 3 & -1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Given system has infinite solutions

$$3x + 6y + 3z = 2$$
$$3y - z = 10$$

All the least square solutions are

$$x = \frac{-5t - 18}{3} \; ; \; y = \frac{10 + t}{3} \; ; \; z = t$$

Find all least square solution to the system $Ax = b$ where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Soln:-

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix}; \quad A^T b = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

we solve the system $A^T A x = A^T b$

$$\begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

By Gauss elimination,

$x = -\frac{8}{3} + 5t; \quad y = -5/3 + 3t; \quad z = t$