Module-3 21/02/22

Consider a matrix of size $\underline{m \times n}$

- Row Space
- Column Space
- Null space

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

w.k.t:

The row vectors of A:

$$r_1 = [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}]$$
$$r_2 = [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}]$$
$$\vdots$$
$$r_m = [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}]$$

Row vectors in $\mathbb{R}^n$

The column vectors of A:

$$c_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{bmatrix}, c_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, c_n = \begin{bmatrix} a_{mn} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in $\mathbb{R}^m$ formed from the columns of A

$(A)_{m \times n}$:

Row vector $\in \mathbb{R}^n$ | Column vector $\in \mathbb{R}^m$

Definition:

If A is $m \times n$ matrix, then

- the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the row space of A.
- the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the column space of A.
- The solution space of the homogeneous system of equations $\underline{AX = 0}$, which is a subspace of $\mathbb{R}^n$, is called Null space of A.

A system of linear equation $Ax=b$ is consistent if and only If b is in the Column Space of A

$$A = \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & & c_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where $c_1 c_2 c_3 \cdots c_n$ are column vectors of A.

$$Ax = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Since $Ax=b \Rightarrow c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = b$

So, b can be expressed as a linear Combination of the column vectors of A, equivalently, if and only if b is in the column space of A]

Eg: A vector 'b' in the Column Space of A:

Let us take $Ax=b$ be the linear System

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}}_{b}$$

show that b is in the Columnspace of A, and express b as a linear Combination of the column vectors of A.

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & 2 & -3 \end{array}\right]$$

$R_2 \to R_1 + R_2$

$R_3 \to R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 7 & 6 & -1 \end{array}\right]$$

$R_3 \to R_3 - \left(\frac{7}{5}\right) R_2$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 0 & \frac{37}{5} & \frac{51}{5} \end{array}\right]$$

By Gaussian-Elimination:

$$\frac{37}{5} x_3 = \frac{51}{5} \Rightarrow x_3 = \frac{51}{37}$$

$$5x_2 - x_3 = -8$$

$$5x_2 = x_3 - 8 = \frac{51}{37} - 8$$

$$5x_2 = -245/37$$

$$\boxed{x_2 = -49/37}$$

$$-x_1 + 3x_2 + 2x_3 = 1$$

$$x_1 = 3x_2 + 2x_3 - 1$$

$$x_1 = 3\left(\frac{-49}{37}\right) + 2\left(\frac{51}{37}\right) - 1$$

$$x_1 = \frac{-45}{37} - 1 = \frac{-45-37}{37}$$

Soln:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -82/37 \\ -49/37 \\ 51/37 \end{pmatrix}$$

$$\boxed{x_1 = \frac{-82}{37}}$$

$$x_1 \begin{array}{c} c_1 \\ \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \end{array} + x_2 \begin{array}{c} c_2 \\ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \end{array} + x_3 \begin{array}{c} c_3 \\ \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \end{array} = \begin{pmatrix} 1 \\ -9 \\ -3 \end{pmatrix}$$

$$\left(\frac{-82}{37}\right) \begin{array}{c} c_1 \\ \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \end{array} + \left(\frac{-49}{37}\right) \begin{array}{c} c_2 \\ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \end{array} + \left(\frac{51}{37}\right) \begin{array}{c} c_3 \\ \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \end{array} = \begin{pmatrix} 1 \\ -9 \\ \underline{\underline{-3}} \end{pmatrix}$$

$$-2\left(\frac{82}{37}\right) - \frac{49}{37} + 2\left(\frac{51}{37}\right)$$

$$= \frac{-164 - 49 + 102}{37}$$

$$= \frac{-111}{37} = -3$$

(Iıly the other two values of b can verify!)

Suggested: RREF:

Find the row space, column space and null space of the matrix A, where

$Ax = 0$

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

4×5

Soln:

Consider:

$$[A|\vec{0}] = \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$[A|\vec{0}] \approx \left[\begin{array}{ccccc|c} 1 & 1 & -1/2 & 0 & 1/2 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \qquad R_1 \to \frac{R_1}{2}$$

$$\approx \left[\begin{array}{ccccc|c} 1 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/2 & -3 & 3/2 & 0 \\ 0 & 0 & -3/2 & 0 & -3/2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\approx \left[\begin{array}{ccccc|c} 1 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & -3/2 & 0 & -3/2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \qquad R_2 \to \frac{2R_2}{3}$$

$$\approx \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + \frac{R_2}{2} \\ R_3 \to R_3 + \frac{3}{2} R_2 \end{array}$$

$$\approx \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{array}\right] \quad R_4 \to R_4 - R_2$$

$$\approx \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 / -3$$

$$\approx \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 + 2R_3 \end{array}$$

$[A|\vec{0}]$

$U \to$

$$\approx \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 2R_3 \\ R_4 \to R_4 - 3R_3 \end{array}$$

Basis for:

① Row Space of A : $R(A)$ = { In rref form, the non-zero rows (with pivotal elements) }

$$= \{ r_1, r_2, r_3 \}$$

$$R(A) = \{ (1,1,0,0,1), (0,0,1,0,1), (0,0,0,1,0) \}$$

$$\dim(R(A)) = 3 \quad \Big| \therefore \text{rank}(A)$$

Column Space (A):

$$[A|\bar{0}] = \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$c_1 \quad c_3 \quad c_4$ (columns of $[A|\bar{0}]$); $c_1 \quad c_3 \quad c_4$ (columns of RREF(A))

Column Space of A: $CS(A)$ = { The corresponding columns of $[A|\bar{0}]$ related to rref(A|0) having pivotal elements in the columnwise }

= { In rref(A|0) → c_1, c_3, c_4 }

$\Rightarrow CS(A) = \{$ In $[A|0]$ (from given matrix) of $\underline{c_1, c_3, c_4}$ $\}$ — c_1, c_3, c_4

$$CS(A) = \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\dim(CS(A)) = 3.$$

Null space of A: NS(A):

$$[A|\bar{0}] = \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \text{RREF}[A|\bar{0}] = \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$a \quad b \quad c \quad d \quad e$

b → free variable; e → free variable

Let us take $b = s$ & $e = t$

$$\bar{X} = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\bar{X} = s \cdot n_s + t \cdot n_t$$

from:

3rd eqn:

$d = 0$

2nd eqn:

$c + e = 0$

$c = -e$

$c = -t$

1st eqn: $a + b + e = 0$

$a = -b - e$

$a = -s - t$

Basis for null space $NS(A)$:

$$NS(A) = \left\{ \overset{u_1}{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}, \overset{u_2}{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}} \right\}, \quad s, t \in \mathbb{R}$$

$$\dim(NS(A)) = 2.$$

Find a basis for the row Space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$

Soln:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$R_3 \to R_3 + R_2$

$$A \approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{ref}$$

to get rref.

$$A \approx \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

Basis for row Space : $\text{rows}(A) = \{(1,0,7), (0,1,-2)\}$

$\dim(\text{rows}(A)) = 2$ Rank(2)

Basis for column space : $\text{Colms}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right\}$

$\dim(\text{colms}(A)) = 2.$

$\to \dim(r(A)) = \dim(c(A))$

Consider the matrix
$$U = \begin{array}{c} \begin{matrix} a & b & c & d & e & \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Find the basis for Null Space of U.!

Soln:

The given matrix itself in the form of rref(U)

Let $d = s$, $e = t$

1st eqn.
$a + 2d + 2e = 0$
$a = -2(d+e)$

3rd:
$c + 4d - e = 0$
$c = e - 4d$

4th:
$b - d + 3e = 0$
$b = d - 3e$

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} -2s - 2t \\ s - 3t \\ t - 4s \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$= s\, n_s + t\, n_t \quad , s, t \in \mathbb{R}$

Basis for Nullspace (A) : $\mathcal{N}(A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dim(\mathcal{N}(A)) = 2.$

(

Find the Basis for 1) row space of A : $\mathcal{R}(A)$

2) Column space of A: $\mathcal{C}(A)$

3) Null space of A : $\mathcal{N}(A)$

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}_{4\times 5} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

Soln:

$$[A|\bar{0}] = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 5 & 0 \\ -2 & -5 & 1 & -1 & -8 & 0 \\ 0 & -3 & 3 & 4 & 1 & 0 \\ 3 & 6 & 0 & -7 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{rref}([A|0])$$

① $\mathcal{R}(A)$: $\{ \underbrace{(1,0,2,0,1)}_{u_1}, \underbrace{(0,1,-1,0,1)}_{u_2}, \underbrace{(0,0,0,1,1)}_{u_3} \}$

$\dim(\mathcal{R}(A)) = 3.$

② $\mathcal{C}(A)$: $\{ (1,-2,0,3), (2,-5,-3,6), (2,-1,4,-7) \}$

$\dim(\mathcal{C}(A)) = 3.$

Null space: The solution space of $A\vec{x}=\vec{0}$, $x=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}$

$$\begin{array}{ccccc|c} & & \uparrow s & & \rightarrow t & \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ccccc} a & b & c & d & e \end{array}$$

1st eqn:
$a+2c+e=0$

Let $c=s$ & $e=t$

3rd eqn:
$d+e=0$
$d=-e=-t$

2nd eqn:
$b-c+e=0$
$b=c-e$
$=s-t$

1st eqn:
$a+2c+e=0$
$a=-2c-e$
$=-2s-t$

$$\vec{x}=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}-2s-t\\s-t\\s\\-t\\t\end{pmatrix}$$

$$=s\begin{pmatrix}-2\\1\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}-1\\-1\\0\\-1\\1\end{pmatrix}$$

$\vec{x}=s\cdot n_s + t\cdot n_t$

$$\text{Basis for } NS(A)=\left\{\overset{u_1}{\begin{pmatrix}-2\\1\\1\\0\\0\end{pmatrix}},\ \overset{u_2}{\begin{pmatrix}-1\\-1\\0\\-1\\1\end{pmatrix}}\right\}$$

$\dim(NS(A))=2.$

Rank theorem:- For any $m\times n$ matrix A,

- $\dim(R(A))+\dim(N(A))$
$=\text{rank}(A)+\text{nullity of } A = n$ [# of unknowns]

- $\dim(C(A))+\dim(N(A^T))$
$=\text{rank}(A)+\text{nullity of } A^T = m$ [# of equations].

Consider:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

plm: Find the rank and nullity of A.

Soln.

The rref(A):

$$[A|\bar{0}] \sim \begin{array}{c} \begin{array}{cccccc} \quad & \quad & c & x & e & \quad \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

4×5

The first three non-zero rows containing leading 1's in A, form a basis for $R(A)$:

$$\Rightarrow \dim(R(A)) = \text{rank}(A) = 3$$

$$\dim(CS(A)) = \text{rank}(A) = 3.$$

The Rank theorem:

$$\dim(R(A)) + \text{nullity of } A = n$$

$$\Rightarrow \text{nullity of } A = n - \dim(R(A))$$

$$= 5 - 3$$

$$\text{nullity of } A = 2$$

Result: Let A be $n \times n$ matrix. Then, A is invertible if and only if $\text{rank}(A) = n$.

① Find the nullity and rank of the following matrices

① $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix}$ ② $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}$

Soln.

①

$$A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix}_{3\times 4} \sim \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \to r_1 \\ \to r_2 \\ \to r_3 \end{array}$$

$\to n = 4.$

Basis for RS(A):

$$RS(A) = \{(1,0,-2,0), (0,1,1,0), (0,0,0,1)\}$$

$\dim RS(A) = 3$

By rank theorem:

$$\dim(R(A)) + \text{nullity of } A = n$$

$$3 + \text{nullity of } A = 4$$

$$\text{nullity of } A = 4-3 = 1$$

②

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}_{3\times 4} \longrightarrow \sim \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \to r_1 \\ \to r_2 \\ \\ \end{array}$$

$\left.\begin{array}{l} c = s \\ d = t \end{array}\right\}$

Basis for Row space of A: RS(A):

$$RS(A) = \{(1,0,3,-2), (0,1,-1,2)\}$$

$\dim RS(A) = 2.$

By rank theorem:

$$\dim(R(A)) + \text{nullity of } A = n$$

$$2 + \text{nullity of } A = 4$$

$$\text{nullity of } A = 4-2 = 2$$

$$\dim(NS(A)) = \text{nullity of } A = 2.$$

(2) Find the basis for $R(A)$, $CS(A)$, $NS(A)$ where A matrices are,

① $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix}$ ② $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}$

Soln:

①. $$A_{ct} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 7 & 0 \\ 2 & 3 & -1 & 9 & 0 \\ -1 & -2 & 0 & -5 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{cccc} c_1 & c_2 & & c_4 \end{array} \end{array} \begin{array}{l} \to r_1 \\ \to r_2 \\ \to r_3 \end{array}$$

$c_1 \quad c_2 \quad c_4 \quad 3\times 4 \to n = 4.$

- Basis for $RS(A)$:

$RS(A) = \{(1,0,-2,0), (0,1,1,0), (0,0,0,1)\}$

$\dim RS(A) = 3$

- Basis for Column Space of A: $CS(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 7 \\ 9 \\ -5 \end{pmatrix} \right\}$

$\dim(CS(A)) = 3$

- Basis for $NS(A)$: The solution space of $Ax=0$.

$$A_{ct} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 7 & 0 \\ 2 & 3 & -1 & 9 & 0 \\ -1 & -2 & 0 & -5 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{cccc} c_1 & c_2 & & c_4 \end{array} \end{array}$$

$c_1 \quad c_2 \quad c_4$

Let $c = s$:

From 3rd eqn: $d = 0$

From 2nd eqn: $b + c = 0 \Rightarrow b = -c = -s$

From 1st eqn: $a - 2c = 0 \Rightarrow a = 2c = a = 2s$

$$NS(A) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2s \\ -s \\ s \\ 0 \end{pmatrix} = s \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}$$

$NS(A) = \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$, $\dim(NS(A)) = 1$.

② $A = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 2 & 1 & 5 & -2 & 0 \end{array}\right] \sim \begin{array}{ccccc} a & b & c & d & \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] & & & & \end{array} \begin{array}{l} \to r_1 \\ \to r_2 \\ \\ \end{array}$

3×4

$\left.\begin{array}{l} c = s \\ d = t \end{array}\right\}$

- Basis for Row space of A: RS(A):

$$RS(A) = \{(1,0,3,-2), (0,1,-1,2)\}$$

$$\dim RS(A) = 2.$$

- Basis for CS(A) = $\left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \right\}$, $\dim CS(A) = 2.$

- NS(A) =

$$\begin{array}{ccccc} a & b & c & d & \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] & & & & \end{array}$$

Let $c = s$, $d = t$

From 2nd eqn:

$$b - c + 2d = 0$$
$$b = c - 2d$$
$$b = s - 2t$$

From 1st eqn:

$$a + 3c - 2d = 0$$
$$a = 2d - 3c$$
$$a = 2t - 3s$$

$$NS(A) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2t - 3s \\ s - 2t \\ s \\ t \end{pmatrix}$$

$$NS(A) = s \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

$NS(A) = s\,\mathbf{s} + t\,\mathbf{t}$

$$NS(A) = \left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \dim NS(A) = 2$$

Remark:

① $RS(A) = CS(A^T)$

② $CS(A) = RS(A^T)$

③ The system $Ax = b$ has a solution if and only if $b \in CS(A) \subseteq \mathbb{R}^m$

④ Let U be a rref(A). Then

(i) $RS(A) = RS(U)$

(ii) $NS(A) = NS(U)$

④ For an $m \times n$ matrix, $\dim RS(A) = \dim CS(A)$

$\Rightarrow$ The fundamental theorem.

⑤ $\dim NS(A) = \dim NS(U)$

= Number of free variables in $Ux = 0$.

⑥ $\dim RS(A) = \dim RS(U)$

= The numbers of non-zero row vectors in U

= The maximal number of linearly independent row vectors of A.

= The number of basic variables in $Ux = 0$.

= The maximal number of linearly independent column vectors of A

$\Rightarrow \dim RS(A) = \dim CS(A)$ //

(7) For $m \times n$ matrix A,

$$\text{rank}(A) \leq \min\{m, n\}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}_{3\times 4} \longrightarrow \sim \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow r_1 \\ \rightarrow r_2 \\ \\ \end{array}$$

$\because \text{rank}(A) \leq \min\{m,n\} \Rightarrow \text{rank}(A) \leq \min\{3,4\}$

$$\Downarrow$$

$$\underline{2 < 3} \quad ✓$$

(8) $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

Bases for Subspaces:

We will finding the bases for two subspaces $A+B$ and $A \cap B$ where A, B are subspaces of $n$space $\underline{\mathbb{R}^n}$.

Let $\alpha = \{a_1, a_2, \ldots, a_k\}$ be the bases for
& $\beta = \{b_1, b_2, \ldots, b_l\}$ A and B.

$(Q)_{n\times(k+l)}$ whose columns are those basis vectors.

i.e., $Q = [\underbrace{a_1 \; a_2 \; a_3 \; \cdots \; a_k}_{A} \; \underbrace{b_1 \; b_2 \; \cdots \; b_l}_{B}]_{n\times(k+l)}$

Let A and B be two subspaces of $\mathbb{R}^n$, and Q be the matrix defined above

1. $CS(Q) = A+B$, so that a basis for the column space $CS(Q)$ is a basis for $A+B$.

2. $NS(Q)$ can be defined with $A \cap B$ so that $\dim(A \cap B) = \dim NS(Q)$.

In general, $\dim(V+W) \neq \dim V + \dim W$

→ Theorem:

For any subspaces V and W of the n-space $\mathbb{R}^n$

$$\dim(V+W) + \dim(V \cap W) = \dim V + \dim W$$

Find a basis for a Subspace:

Let V and W be the subspaces of $\mathbb{R}^5$ with bases

V: $\{u_1, u_2, u_3\} = \{(1,3,-2,2,3), (1,4,-3,4,2), (1,3,0,2,3)\}$

W: $\{w_1, w_2, w_3\} = \{(2,3,-1,-2,9), (1,5,-6,6,1), (2,4,4,2,8)\}$

respectively.

Find the bases for $V+W$ and $V \cap W$.

Soln.

$$Q = \begin{bmatrix} V_1 & V_2 & V_3 & W_1 & W_2 & W_3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 3 & 5 & 4 \\ -2 & -3 & 0 & -1 & -6 & 4 \\ 2 & 4 & 2 & -2 & 6 & 6 \\ 3 & 2 & 3 & 9 & 1 & 8 \end{bmatrix}$$

Gauss-Jordan elimination form [rref(Q)]

$$\to U = \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 3 & 5 & 4 \\ -2 & -3 & 0 & -1 & -6 & 4 \\ 2 & 4 & 2 & -2 & 6 & 6 \\ 3 & 2 & 3 & 4 & 1 & 8 \end{bmatrix} \xrightarrow{\text{rref}(Q)} U = \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccccc} \uparrow & \uparrow & \uparrow & \downarrow & \downarrow & \uparrow \\ & & & \boxed{\text{free}} & & \end{array}$$

Basis for V+W: $CS(Q) = V+W$

The Columns $\{v_1, v_2, v_3, w_2\}$ in Q forms a basis for V+W.

$$\text{Basis for } V+W = \{(1,3,-2,2,3), (1,4,-3,4,2), (1,3,0,2,3), (2,4,4,2,8)\}$$

$\dim(V+W) = 4$

Basis for V∩W: $N(Q) = V \cap W$

w.r.t the solution of Homogeneous system $Ux = 0$

$$U = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccccc} & & & \uparrow & \uparrow & \end{array}$$

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \Rightarrow$$

Let $x_4 = s$ & $x_5 = t$

4th eqn: $x_6 = 0$,

3rd eqn: $x_3 - x_5 = 0$

$x_3 = x_5 = t$

2nd eqn: $x_2 - 3x_4 + 2x_5 = 0$

$x_2 = 3x_4 - 2x_5$

$x_2 = 3s - 2t$

1st eqn:

$x_1 + 5x_4 = 0.$

$x_1 = -5x_4$

$x_1 = -5s$

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -5s \\ 3s-t \\ t \\ s \\ t \\ 0 \end{pmatrix} = s\begin{pmatrix} -5 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$N(Q) = s \cdot n_s + t\, n_t \qquad , s,t \in \mathbb{R}$

Basis for $V \cap W = \{(-5,3,0,1,0,0),(0,-1,1,0,1,0)\}$

Now: $\dim(V \cap W) = 2.$

Find a basis for a subspace:

Let V and W be the subspaces of $\mathbb{R}^5$. with bases

$\{u_1, u_2, u_3\} = \{(1,3,-2,2,3), (1,4,-3,4,2), (1,3,0,2,3)\}$

$\{w_1, w_2, w_3\} = \{(2,3,-1,-2,9), (1,5,-6,6,1), (2,4,4,2,8)\}$ respectively.

verify $\dim(V+W) + \dim(V \cap W) = \dim V + \dim W$

For $\dim(V)$: $\quad V = \text{span}(v_1, v_2, v_3)$

$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \xrightarrow{\text{rref}(V)} V_u = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$RS(V) = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\dim(V) = 3.$

For $\dim(W)$:

$$W = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \Rightarrow W = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 5 & 4 \\ -1 & -6 & 4 \\ -2 & 6 & 2 \\ 9 & 1 & 8 \end{bmatrix} \xrightarrow{\text{rref}(W)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$RS(W) = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\dim W = 3$

By theorem:-

$$\dim(V+W) + \dim(V \cap W) = \dim V + \dim W$$

$$4 + 2 = 3 + 3$$

$$6 = 6$$

Yes, Satisfied

Sums and direct sums:

Suppose U and W are the subspaces of a vector space V.

$U+W$:

Define $U+W = \{ u+w : u \in U, w \in W \}$

Is $U+W$ a subspace of V?

Subspace:
(i) $0 \in U+W$
(ii) $ku + w \in U+W$

Proof:

Since U and W are subspaces.

$\Rightarrow 0 \in U, \; 0 \in W$

Hence, $0 + 0 = 0 \in U+W$

Suppose $v_1, v_2 \in U+W$, Then ~~there exist~~

$u_1, u_2 \in U$ & $w_1, w_2 \in W$

such that $v_1 = u_1 + w_1$ & $v_2 = u_2 + w_2$

since, U and W are the subspaces of V.

$\Rightarrow u_1 + u_2 \in U$ & $w_1 + w_2 \in W$

and for any scalar k,

$$k u_1 \in U$$
$$k w_1 \in W$$

Consider: Closure property (addition)

Addition

$$V_1 + V_2 = (u_1 + w_1) + (u_2 + w_2)$$
$$= (u_1 + u_2) + (w_1 + w_2)$$
$$v_1 + v_2 \in U + W$$

Closure property ~~(multiplication)~~

$$k v_1 = k(u_1 + w_1) = \underline{k u_1} + k w_1$$
$$k v_1 = U + W$$

$\Longrightarrow$ $U + W$ is also a Subspace of V.

Let V be the vector space of 2 by 2 matrices over $\mathbb{R}$.

Let $U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} ; a, b \in \mathbb{R} \right\}$

$W = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} ; a, c \in \mathbb{R} \right\}$

Describe $U + W$ & $U \cap W$

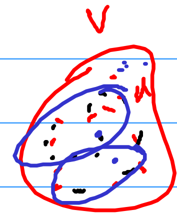
$$U + W = \{ u + w ; u \in U, w \in W \}$$

$$U + W = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} ; a, b, c \in \mathbb{R} \right\}$$

Let $U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} ; a, b \in \mathbb{R} \right\}$

$W = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} ; a, c \in \mathbb{R} \right\}$

$U \cap W = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, a \in \mathbb{R} \right\}$



Direct Sum:-

The vector space V is said to be direct sum of its subspaces U and W, denoted by $V = U \oplus W$

if every vector $v \in V$ can be written (as unique way) in one and only way as $v = u + w$ where $u \in U$, $w \in W$

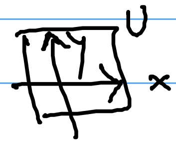
$\Rightarrow$ **Thm:** The vector space V is the direct sum of two subspaces U and W if and only if

(i) $V = U + W$

(ii) $U \cap W = \{0\}$

$\Rightarrow V = U \oplus W$

Eg: Consider the vector space $\mathbb{R}^3$:



Let U be the xy-plane

$U = \{ (a, b, 0) ; a, b \in \mathbb{R} \}$

Let W be the yz plane:

$W = \{ (0, b, c) ; b, c \in \mathbb{R} \}$

$U = \{(a,b,0) \; ; \; a,b \in \mathbb{R}\}$ → (xy plane)

$W = \{(0,b,c) \; ; \; b,c \in \mathbb{R}\}$ → (yz-plane)

U+W: $U+W = \{(a,b,c) \; ; \; a,b,c \in \mathbb{R}\}$

$\Rightarrow \mathbb{R}^3 = U+W$ (Sum).

Direct Sum?

Let $v = (3,5,7) \in \mathbb{R}^3$

Also,

(i) $(3,5,7) = \underbrace{(3,1,0)}_{\in U} + \underbrace{(0,4,7)}_{\in W}$

(ii) $(3,5,7) = \underbrace{(3,4,0)}_{\in U} + \underbrace{(0,1,7)}_{\in W}$

(iii) $(3,5,7) = \underbrace{(3,-4,0)}_{\in U} + \underbrace{(0,9,7)}_{\in W}$

Alternatively, $U \cap W = \{(0,b,0) \; ; \; b \in \mathbb{R}\} \Rightarrow U \cap W \neq \{(0,0,0)\}$

$\Rightarrow U+W = \mathbb{R}^3$ but not direct sum $V = U \oplus W$

Example for Direct Sum:

In $\mathbb{R}^3$: $U = \{(a,b,0) \; ; \; a,b \in \mathbb{R}\}$

$W = \{(0,0,c) \; ; \; c \in \mathbb{R}\}$

Example for Direct Sum:

In $\mathbb{R}^3$: $U = \{(a,b,0) ; a,b \in \mathbb{R}\}$

$W = \{(0,0,c) ; c \in \mathbb{R}\}$

Sum: $U+W = \{(a,b,c) ; a,b,c \in \mathbb{R}\} \Rightarrow \mathbb{R}^3 = U+W$

Direct Sum: $U \cap W = \{(0,0,0)\} \Rightarrow \mathbb{R}^3 = U \oplus W$

Alternatively,

$v = (a,b,c) \in \mathbb{R}^3$

$\Rightarrow (a,b,c) = (a,b,0) + (0,0,c)$

$\mathbb{R}^3 = U \oplus W$.

H.W: Let V be the supspace spanned by (in $\mathbb{R}^5$)

(1) $(v_1, v_2, v_3) = \{(1,3,-2,2,3), (1,4,-3,4,2), (2,3,-1,-2,10)\}$

Let W be the supspace spanned by (in $\mathbb{R}^5$)

$(w_1, w_2, w_3) = \{(1,3,0,2,1), (1,5,-6,6,3), (2,5,3,2,1)\}$

(1) Find a basis for $V+W$ & $\dim(V+W)$

(2) find a basis for $V \cap W$ & $\dim(V \cap W)$

(3) Verify: $\dim(V \cap W) = \dim V + \dim W - \dim(V+W)$

#2: Let $V = \{ (x,y,z,u) \in \mathbb{R}^4 ; \ y+z+u=0 \}$

$W = \{ (x,y,z,u) \in \mathbb{R}^4 ; \ x+y=0, \ z=2u \}$

be the two subspaces of $\mathbb{R}^4$. Find bases for $V+W$ & $V \cap W$.

Soln:

Given $V = \{ (x,y,z,u) \in \mathbb{R}^4 ; \ y+z+u=0 \}$

$y = -z-u$

Let $z = s$, $u = t$

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} x \\ -s-t \\ s \\ t \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Basis for } V = \left\{ \overset{b_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \overset{b_2}{\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}}, \overset{b_3}{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

(verify)

$\dim(V) = 3.$

$W = \{ (x,y,z,u) \in \mathbb{R}^4 : x+y=0, \ z=2u \}$

given $x+y=0$, $z=2u$.

$x=-y$ | $z=2u$.

Let $y=s$, $u=t \Rightarrow z=2t$

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 2u \\ u \end{pmatrix} = \begin{pmatrix} -s \\ s \\ 2t \\ t \end{pmatrix} = s \overset{\delta_1}{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}} + t \overset{\delta_2}{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}$$

$\to s, t \in \mathbb{R}.$

L.C: LI (or) L.D: b_4 b_5

$$c_1 \vec{\delta_1} + c_2 \vec{\delta_2} = \vec{0}$$

$$c_1(1,-1,0,0) + c_2(0,0,2,1) = (0,0,0,0)$$

$$(c_1,-c_1,0,0)+(0,0,2c_2,c_2)=(0,0,0,0)$$

$$\Rightarrow \quad (c_1,-c_1,2c_2,c_2)=(0,0,0,0)$$

$$\Rightarrow \quad c_1=0,\ c_2=0$$

$\therefore$ is linearly independent.

$\vec{s}_1, \vec{s}_2$ form a basis.

Basis for $W = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$, $\dim W = 2$

$\dim(V+W)$ & $\dim(V\cap W)$:

$$Q = \left[\underbrace{b_1\ b_2\ b_3}_{\text{basis for } V} \quad \underbrace{b_4 \quad b_5}_{\text{basis for } W} \;\middle|\; 0 \right]$$

$$= \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

$$\approx \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \end{array}\right] \quad R_3 \leftrightarrow R_4$$

$$\approx \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 \end{array}\right] \quad R_4 \to R_4 + R_2$$

$$\approx \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 3 & 0 \end{array}\right] \quad R_4 \to R_4 + R_3$$

free variable column

$$\approx \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{array}\right] \quad R_4 \to \frac{R_4}{-3}$$

rref(Q)

$$U \approx \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{array}\right] \quad R_1 \to R_1 - R_4$$

Basis for U $(RS(U))$ $\Big\}$ = Basis for $V+W$

$$= \{(1,0,0,0,3),\ (0,1,0,0,2),\ (0,0,1,0,1),\ (0,0,0,1,-3)\}$$

$\dim(V+W) = 4$

Basis for $N(Q)$ = Basis for $(V \cap W)$

$$U \approx \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{array}\right] \end{array}$$

$\therefore$ x_5 is free variable

$\Rightarrow$ Let $x_5 = s$

4th equal Last eqn

$x_4 - 3x_5 = 0$

$x_4 = 3x_5 \Rightarrow x_4 = 3s.$

3rd eqn

$x_3 + x_5 = 0 \Rightarrow x_3 = -x_5 \Rightarrow x_3 = -s$

2nd eqn

$x_2 + 2x_5 = 0 \Rightarrow x_2 = -2x_5 \Rightarrow x_2 = -2s$

1st eqn.

$x_1 + 3x_5 = 0$

$x_1 = -3x_5 \Rightarrow x_1 = -3s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3s \\ -2s \\ -s \\ 3s \\ s \end{pmatrix} = s \begin{pmatrix} -3 \\ -2 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}$$

Basis for $N(Q)$ = Basis for Q = Basis for $U \cap W$

$$= \left\{ s \begin{pmatrix} -3 \\ -2 \\ -1 \\ 3 \\ 1 \end{pmatrix}, s \in \mathbb{R} \right\}$$

$$\dim(U \cap W) = \underline{1}.$$

Verification:

$\dim(U+W) + \dim(U \cap W) = \dim U + \dim W$

$4 + 1 = 3 + 2$ //

verified //

Let V be the subspace of the V.S $\mathbb{R}^{100}$.

where $V = \{ (x_1, x_2, \ldots, x_{100}) \in \mathbb{R}^{100} ;$

$x_1 + x_2 + x_3 + \cdots + x_{50} = 0$

$x_{51} = 0, x_{52} = 0, \ldots, x_{100} = 0 \}$

Find basis and dimension of V.

Soln:

$x_1 = -(x_2 + x_3 + \cdots + x_{50})$

$x_{51} = x_{52} = \cdots = x_{100} = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{50} \\ x_{51} \\ x_{52} \\ \vdots \\ x_{100} \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 - x_4 \cdots - x_{50} \\ x_2 \\ x_3 \\ \vdots \\ x_{50} \\ x_{51} \\ x_{51} \\ \vdots \\ x_{51} \end{pmatrix}$$

$$= \underbrace{x_2 \begin{pmatrix} \\ \\ \end{pmatrix} + \cdots + x_{50} \begin{pmatrix} \\ \\ \end{pmatrix}}_{49 \text{ vectors}} + \underbrace{x_{51} \begin{pmatrix} \\ \\ \end{pmatrix}}_{\text{only one}}$$

$\Rightarrow \dim(V) = 50$ //

Let V and W be the subspaces of the vector space $P_3(\mathbb{R})$ spanned by

$$V(x) = \begin{cases} 3 - x + 4x^2 + x^3 & (u_1) \rightarrow v_1 = (3,-1,4,1) \\ 5 + 0x + 5x^2 + x^3 & (u_2) \quad v_2 = (5,0,5,1) \\ 5 - 5x + 10x^2 + 3x^3 & (v_3) \quad v_3 = (5,-5,10,3) \end{cases}$$

$$W(x) = \begin{cases} 9 - 3x + 3x^2 + 2x^3 & (w_1) \rightarrow w_1 = (9,-3,3,2) \\ 5 - x + 2x^2 + x^3 & (w_2) \rightarrow w_2 = (5,-1,2,1) \\ 6 + 0x + 4x^2 + 2x^3 & (w_3) \rightarrow w_3 = (6,0,4,2) \end{cases}$$

Find the bases and dimension of V, W, $V+W$, and $V \cap W$.

$\boxed{\text{Try!}}$

Find a basis for this / the set of all real-valued continuous functions $y = f(x)$ satisfying the diff^l eqn $(D^3 + 6D^2 + 11D + 6)y = 0$ is a vector space over $\mathbb{R}$.

Soln:

We can check $V = \{ f \mid f : \mathbb{R} \to \mathbb{R}, f\text{-cont} \}$ is V.S over $\mathbb{R}$.

under,

$$\left.\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x) \end{aligned}\right\}$$

Let $W = \{ f \in V \mid f \text{ is a solution of diff}^l \text{ eqn} \}$.

Find the soln for the given diff^l eqn.

Given $(D^3 + 6D^2 + 11D + 6)y = 0$.

$\Rightarrow m_1 = -1, m_2 = -2, m_3 = -3$

$y(x) = c_1 e^{-x} + c_2 e^{-2x} + c_3 \cdot e^{-3x}$

Solution Set $\{ \overset{f_1}{e^{-x}}, \overset{f_2}{e^{-2x}}, \overset{f_3}{e^{-3x}} \}$

$$c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} = \vec{0}$$

Wronskian:

$$W = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} & e^{-2x} & e^{-3x} \\ -e^{-x} & -2e^{-2x} & -3e^{-3x} \\ +e^{-x} & +4e^{-2x} & +9e^{-3x} \end{vmatrix}$$

$$= e^{-x}\left\{-18e^{-5x} + 12e^{-5x}\right\}$$
$$- e^{-2x}\left\{-9e^{-3x} + 3e^{-4x}\right\}$$
$$+ e^{-3x}\left\{-4e^{-3x} + 2e^{-3x}\right\}$$

$$W = -6e^{-6x} + 9e^{-5x} - 3e^{-6x} - 2e^{-6x}$$

$$W = -12e^{-6x} + 9e^{-5x} \neq 0$$

$\forall x \in \mathbb{R}$.

$\Rightarrow \{e^{-x}, e^{-2x}, e^{-3x}\} \rightarrow$ is Linearly Independent.

$S = \{f_1\ f_2\ f_3\}$ forms a basis for V.

Invertibility

$Ax = B$ $\quad (A)_{m\times n}$ $\quad B \in \mathbb{R}^m$

the following existence and uniqueness theorems for a solution of a system of linear equations $A\mathbf{x} = \mathbf{b}$ for an $m\times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^m$.

$(A)_{m\times n}(X)_{n\times 1} = B_{m\times 1}$

Theorem **(Existence)** *Let A be an $m\times n$ matrix. Then the following statements are equivalent.*

(1) *For each $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has* at least *one solution $\mathbf{x}$ in $\mathbb{R}^n$.*

(2) *The column vectors of A span $\mathbb{R}^m$, i.e., $\mathcal{C}(A) = \mathbb{R}^m$.*

(3) rank $A = m$, *and hence $m \leq n$.*

(4) *There exists an $n \times m$ right inverse B of A such that $AB = I_m$.*

$(A)_{m\times n}(A^{-1})_{(n\times m)} = I_m$

B

Theorem **(Uniqueness)** *Let A be an $m\times n$ matrix. Then the following statements are equivalent.*

(1) *For each $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has* at most *one solution $\mathbf{x}$ in $\mathbb{R}^n$.* unique

(2) *The column vectors of A are linearly independent.*

(3) $\dim \mathcal{C}(A) =$ rank $A = n$, *and hence $n \leq m$.*

(4) $\mathcal{R}(A) = \mathbb{R}^n$.

(5) $\mathcal{N}(A) = \{\mathbf{0}\}$.

(6) *There exists an $n \times m$ left inverse C of A such that $CA = I_n$.*

Problem 3.26 For each of the following matrices, discuss the number of possible solutions to the system of linear equations $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$:

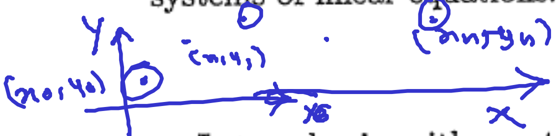
$$(1)\ A = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 2 & 3 \\ 3 & -7 \\ -6 & 1 \end{bmatrix},$$

$$(3)\ A = \begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix}, \qquad (4)\ A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 1 & 2 & -2 \end{bmatrix}.$$

$f(x) = \boxed{\cdots}$

Application: Interpolation

In many scientific experiments, a scientist wants to find the precise functional relationship between input data and output data. That is, in his experiment, he puts various input values into his experimental device and obtains output values corresponding to those input values. After his experiment, what he has is a table of inputs and outputs. The precise functional relationship might be very complicated, and sometimes it might be very hard or almost impossible to find the precise function. In this case, one thing he can do is to find a polynomial whose graph passes through each of the data points and comes very close to the function he wanted to find. That is, he is looking for a polynomial that approximates the precise function. Such a polynomial is called an **interpolating polynomial**. This problem is closely related to systems of linear equations.



$y = f(x)$

x	x_0	x_1	x_2	$\cdots$	x_n
y	y_0	y_1	y_2	$\cdots$	y_n

$(n+1)$ pts

Let us begin with a set of given data: Suppose that for $n+1$ distinct experimental input values $x_0, x_1, \ldots, x_n$, we obtained $n+1$ output values $y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n)$. The output values are supposed to be related to the inputs by a certain function f. We wish to construct a polynomial $p(x)$ of degree less than or equal to n which interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$: *i.e.*, $p(x_i) = y_i = f(x_i)$ for $i = 0, 1, \ldots, n$.

Note that if there is such a polynomial, it must be unique. Indeed, if $q(x)$ is another such polynomial, then $h(x) = p(x) - q(x)$ is also a polynomial of degree less than or equal to n vanishing at $n+1$ distinct points $x_0, x_1, \ldots, x_n$. Hence $h(x)$ must be the identically zero polynomial so that $p(x) = q(x)$ for all $x \in \mathbb{R}$.

In fact, the unique polynomial $p(x)$ can be found by solving a system of linear equations: If we write $p(x) = a_0 + a_1x + \cdots + a_nx^n$, then we are supposed to determine the coefficients a_i's. The set of equations

$p(x_0) = a_0 + a_1 x_0 + \cdots$

$p(x_1) =$

$p(x_2)$

$\vdots$

$p(x_n)$

$$p(x_i) = a_0 + a_1x_i + \cdots + a_nx_i^n = y_i = f(x_i),$$

for $i = 0, 1, \ldots, n$, constitutes a system of $n+1$ linear equations in $n+1$ unknowns a_i's:

$A \quad X = B$

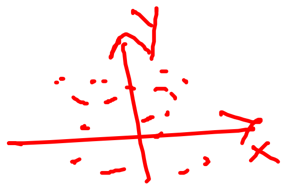
$$\underbrace{\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix}}_{(n+1)\times(n+1)} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} \begin{matrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_n) \end{matrix} \quad (n+1)$$

The coefficient matrix A is a square matrix of order $n+1$, known as **Vandermonde's matrix** (see Problem 2.10), whose determinant is

$$\det A = \prod_{0 \le i < j \le n} (x_j - x_i).$$

$0 \le i < j \le n$

Since the x_i's are all distinct, $\det A \ne 0$. It follows that A is nonsingular, and hence $A\mathbf{x} = \mathbf{b}$ always has a unique solution, which determines the unique polynomial $p(x)$ of degree $\le n$ passing through the given $n+1$ points (x_0, y_0), $(x_1, y_1), \cdots, (x_n, y_n)$ in the plane $\mathbb{R}^2$.



#Pb m: Given four points $(0, 3)$, $(1, 0)$, $(-1, 2)$, $(3, 6)$

$x_0 y_0 \quad x_1 y_1 \quad x_2 y_2 \quad x_3 y_3$

Given 4 points

$n+1 = 4$

$n = 3$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$

Soln: ~~In~~ the plane $\mathbb{R}^2$, let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be the polynomial passing through the given four points. Then, we have a system of equations

$$\begin{cases} a_0 & & & & & & & = 3 \\ a_0 & + & a_1 & + & a_2 & + & a_3 & = 0 \\ a_0 & - & a_1 & + & a_2 & - & a_3 & = 2 \\ a_0 & + & 3a_1 & + & 9a_2 & + & 27a_3 & = 6. \end{cases}$$

Solving this system, we find that $a_0 = 3$, $a_1 = -2$, $a_2 = -2$, $a_3 = 1$ is the unique solution, and the unique polynomial is $p(x) = 3 - 2x - 2x^2 + x^3$. □

(x, y) Consider $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

At $(0,3) \to 3 = a_0 + 0 + 0 + 0 \to a_0 = 3$

At $(1,0) \to 0 = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 \to a_0 + a_1 + a_2 + a_3 = 0$

At $(-1,2) \to 2 = a_0 + a_1(-1) + a_2(-1)^2 + a_3(-1)^3 \to a_0 - a_1 + a_2 - a_3 = 2$

At $(3,6) \to 6 = a_0 + a_1(3) + a_2(3)^2 + a_3(3)^3 \to a_0 + 3a_1 + 9a_2 + 27a_3 = 6$

$$\left.\begin{array}{l} a_0 = 3 \\ a_0 + a_1 + a_2 + a_3 = 0 \\ a_0 - a_1 + a_2 - a_3 = 2 \\ a_0 + 3a_1 + 9a_2 + 27a_3 = 6 \end{array}\right\} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 6 \end{bmatrix}$$

Since $a_0 = 3$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix}$$

REF | or | RREF (or) Solving System of eqns

we will get a_1, a_2, a_3.

$$[A|B] \cong \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ -1 & 1 & -1 & -1 \\ 3 & 9 & 27 & 3 \end{array}\right] \end{array} \to \cong \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 2 & 0 & -4 \\ 0 & 6 & 24 & 12 \end{array}\right] \begin{array}{l} R_2 \to R_1 + R_2 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 24 & 24 \end{array}\right] \; R_3 \to R_3 - 3R_2 \; \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_3 \to R_3 / 24$

Back Substitution

$a_3 = 1$

$2a_2 = -4$

$a_2 = -2$

$a_1 + a_2 + a_3 = -3$

$a_1 - 2 + 1 = -3$

$a_1 = -3 + 1$

$a_1 = -2$

$$x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \\ 1 \end{pmatrix}$$

$$= \quad p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$p(x) = 3 - 2x - 2x^2 + x^3$$

At $(0,3)$

At $(1,0)$

At $(-1,2)$

At $(3,6)$

find $p(2)$?

$p(2) = 3 - 2(2) - 2(2)^2 + (2)^3$

$p(2) = -1 - 8 + 8 = -1$

$p(2) = -1$

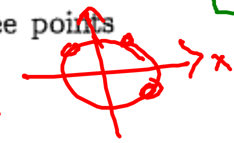
Problem 3.27 Let $f(x) = \sin x$. Then at $x = 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \pi$, the values of f are $y = 0, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 0$. Find the polynomial $p(x)$ of degree ≤ 4 that passes through these five points. (One may need to use a computer due to messy computation).

x	y
0	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$1/\sqrt{2}$
π	0

Problem 3.28 Find a polynomial $p(x) = a + bx + cx^2 + dx^3$ that satisfies $p(0) = 1$, $p'(0) = 2$, $p(1) = 4$, $p'(1) = 4$.

Problem 3.29 Find the equation of a circle that passes through the three points $(2, -2)$, $(3, 5)$, and $(-4, 6)$ in the plane $\mathbb{R}^2$.

$p(x) = a_0 + a_1 x + a_2 x^2$ | $n+1=3$, $n=2$



Pbm 3.27:

$n+1=5$

$n=4$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$(A)_{5\times 5}$ $\quad [A \mid B]$

Compute $a_0 \to a_4$

Pbm: 3.28:

$p(x) = a + bx + cx^2 + dx^3$

$p'(x) = 0 + b + 2cx + 3dx^2$

$p(1) = 4$ | $p(0) = 1$

$p'(1) = 4$ | $p'(0) = 2$

p $\quad$ p'

At $x = 0$: $\quad 1 = a$

At $x = 1$: $\quad 4 = a + b + c + d$

$\Rightarrow$ $a = 1,\ b = 2$

At $x = 0$

$2 = b + 0 + 0$

At $x = 1$

$4 = b + 2c + 3d$

$a + b + c + d = 4 \Rightarrow c + d = 1$

$b + 2c + 3d = 4 \Rightarrow 2c + 3d = 2$

$$
\begin{aligned}
- \quad 2c - 2d &= -2 \\
2c + 3d &= 2
\end{aligned}
$$

$$d = 0$$

$$c + d = +1$$

$$c = 1$$

$$\Rightarrow \quad p(x) = a + bx + cx^2 + dx^3$$

$$p(x) = 1 + 2x + x^2 + 0$$

Remark: **(1)** It is suggested that the readers think about the differences between this interpolation and the Taylor polynomial approximation to a differentiable function.

(2) Note again that the interpolating polynomial $p(x)$ of degree $\leq n$ is uniquely determined when we have the correct data, *i.e.*, when we are given precisely $n+1$ values of y at precisely $n+1$ distinct points $x_0, x_1, \ldots, x_n$.

However, if we are given fewer data, then the polynomial is under-determined: *i.e.*, if we have m values of y with $m < n+1$ at m distinct points $x_1, x_2, \ldots, x_m$, then there are as many interpolating polynomials as the null space of A since in this case A is an $m \times (n+1)$ matrix with $m < n+1$.

On the other hand, if we are given more than $n+1$ data, then the polynomial is over-determined: *i.e.*, if we have m values of y with $m > n+1$ at m distinct points $x_1, x_2, \ldots, x_m$, then there need not be any interpolating polynomial since the system could be inconsistent. In this case, the best we can do is to find a polynomial of degree $\leq n$ to which the data is closest. We will review this statement again in Section 5.8.

Theorem *For a square matrix A of order n, the following statements are equivalent.*

(1) *A is invertible.*

(2) $\det A \neq 0$.

(3) *A is row equivalent to I_n.*

(4) *A is a product of elementary matrices.*

(5) *Elimination can be completed: $PA = LDU$, with all $d_i \neq 0$.*

(6) *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^n$.*

(7) *$A\mathbf{x} = \mathbf{0}$ has only a trivial solution, i.e., $\mathcal{N}(A) = \{\mathbf{0}\}$.*

(8) *The columns of A are linearly independent.*

(9) *The columns of A span $\mathbb{R}^n$, i.e., $\mathcal{C}(A) = \mathbb{R}^n$.*

(10) *A has a left inverse.*

(11) $\operatorname{rank} A = n$.

(12) *The rows of A are linearly independent.*

(13) *The rows of A span $\mathbb{R}^n$*, i.e., $\mathcal{R}(A) = \mathbb{R}^n$.

(14) *A has a right inverse.*