Module:1

System of Linear Equations:

Gaussian elimination and Gauss Jordan methods - Elementary matrices- permutation matrix - inverse matrices - System of linear equations - - LU factorizations.

Foundations: **Matrix:** → Row ↓ Column Allocating data $\begin{bmatrix} \vdots & \vdots & \vdots \end{bmatrix}$

A system of mn numbers arranged in a rectangular form along with m-rows, n-columns denoted by [ ], ( )

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 8 & 9 \end{bmatrix}_{2\times 3}$$

Row matrix: A matrix having single row

i.e., $A = \begin{bmatrix} 2 & -2 & 5 & -8 & 7 & 9 \end{bmatrix}$ ...

Column matrix: A matrix having single column.

$$A = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

Square matrix: In mn arrangement if $m = n$.

$$A = \begin{bmatrix} 2 & 5 \\ 7 & 8 \end{bmatrix}_{2\times 2}$$

① **determinant of A:** [A should be square matrix].

then $|A|$ (or) $\det(A) = \begin{vmatrix} 2 & 5 \\ 7 & 8 \end{vmatrix}$

② **Trace of A:** Sum of principal diagonal elements.

Trace of A: $A\begin{pmatrix} 2 & 3 & 7 \\ 8 & 9 & 10 \\ 10 & 11 & 12 \end{pmatrix} = 2+9+12 = 23$ //

(iii) Singular matrix :-

A square matrix is said to be singular,

if its $\det(A) = 0$ (or) $|A| = 0$

otherwise, non-singular $\det(A) \neq 0$ (or) $|A| \neq 0$

$\Rightarrow$ non-singular.

* Existence of inverse of the matrix A is :

$|A| \neq 0$ [i.e, A is non-singular]

Diagonal matrix:

only principal diagonal members are present (may be few of them zero), rests are zero.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Unit matrix :

In, diagonal matrix all members in the diagonal are 1, rests are zero

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null matrix:

Any mn type matrix have all the entries are zero.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$$

Symmetric and Skew-symmetric matrix:

For square matrix $A = [a_{ij}]$ is said to be symmetric

$$a_{ij} = a_{ji} \quad \forall \; i \text{ and } j$$

(ie., $A = A^T$)

$$B = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \rightarrow B^T = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \Rightarrow B = B^T.$$

Symmetric

Skew-Symmetric:-

$$\text{If } \quad a_{ij} = -a_{ji} \quad \forall \; i \text{ and } j$$

so that principal diagonal elements are zero.

$$\Rightarrow \quad A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \qquad \Bigg| \qquad B = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$$

Symmetric matrix | Skew-symmetric

Triangular matrix:

A Square matrix all of whose elements below the leading diagonal are zero, is called Upper triangular matrix.

A Square matrix all of whose elements above the leading diagonal are zero, is called Lower triangular matrix.

Eg:

$$A = \begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \qquad \Bigg| \qquad A = \begin{bmatrix} a & 0 & 0 \\ h & b & 0 \\ g & f & c \end{bmatrix}$$

Upper triangular matrix | Lower Triangular matrix.

Systems of linear equations:-

m- linear equations with n- unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Here, x_i's are unknowns and a_{ij}'s are coefficients of the unknowns. b_i's are constants. (real or complex)

Solution?:

A sequence $(s_1, s_2, \ldots, s_n)$ is called the solution of the system if $s_i = x_i$ $(i = 1 \ldots n)$ satisfy the system of linear equations simultaneously.

Trivial Solution:

if $\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\}$ is called the trivial soln (zero soln)

$$\begin{cases} x + 2y = 0 \\ x - 5y = 0 \end{cases}$$

It is possible for Homogeneous system

Non-trivial soln

otherwise, non-trivial soln

Consider,

$$x + 2y = 0$$
$$x = -2y$$
$$\Rightarrow \frac{x}{-2} = \frac{y}{1} (= t)$$

$t \neq 0$

$$x = -2t$$
$$y = t$$

x & y (or $x+2y=0$) has many solutions

$x+2y=0$

if $t = 1$, $-2+2=0$

if $t = -2$, $4-4=0$

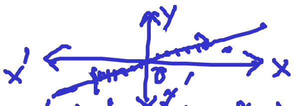
The system of two equations in two unknowns x and y is

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$ax + by = c$$
$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = c$$
$$A \quad X = B$$

In matrix form of the above said equations

$$\underbrace{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{B}$$



which is a straight line

$$y = \frac{c}{b} - \frac{a}{b}x.$$

$P = (x_0, y_0)$ is a solution of if and only if the point P lies on the line.

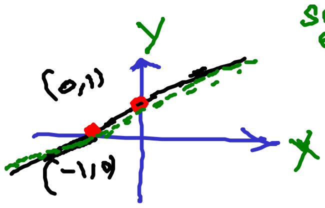
$\Rightarrow$ infinitely many solutions

Solution:

Case ①:

$$x - y = -1$$
$$2x - 2y = -2$$

same eqn

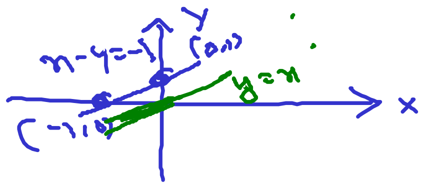


infinitely many solutions

① Straight itself they coincide

Case ②:

$$x - y = -1$$
$$x - y = 0$$

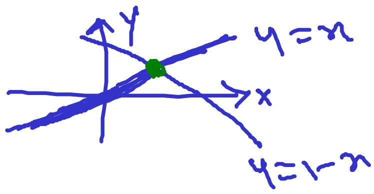


They are ∥el lines

No Solution:

Case ③

$$x + y = 1$$
$$x - y = 0$$



They cross at only one point

$\Rightarrow$ unique soln

$\Rightarrow$ The linear system may have either (i) No Solution

or (ii) Unique solution (one Solution)

(or) (iii) infinitely many solutions. //

Consider, $ax + by + cz = d$ where $(a, b, c) \neq (0, 0, 0)$

which is a plane in 3-space $\mathbb{R}^3$.

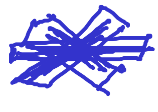
The solution set includes

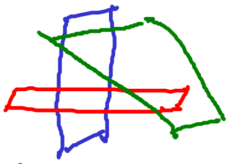
$\{(x,y,0) \mid ax+by=d\}$ in xy plane

$\{(x,0,z) \mid ax+cz=d\}$ in xz plane

$\{(0,y,z) \mid by+cz=d\}$ in yz-plane

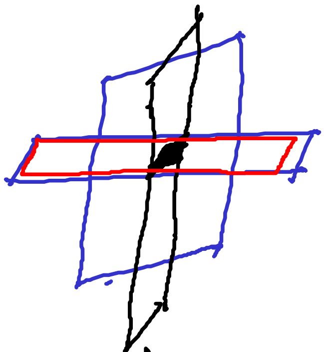
For three equations in three unknown:



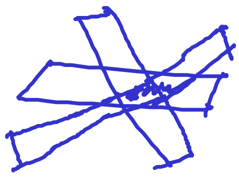


$\boxed{\text{No solution:}}$

Not having common intersection pts



only one soln



many pts intersect...

$\Rightarrow$ infinitely many solutions

Homogeneous & Non-Homogeneous

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$
$(*)$

(1) In $(*)$ if all b_i's $= 0$
$\Rightarrow$ Homogeneous System

(2) In $(*)$, some b_i's $\neq 0$
$\Rightarrow$ Non-Homogeneous system

Consistent and Inconsistent:

* The System of linear equations have at least one Soln is called Consistent

* The system if does not posses any soln is called inconsistent.

Suppose, $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

If $a_i = 0$ for $i = 1, 2, \ldots, n \implies 0 = b$

Thus, it has no solution if $b \neq 0$ (non-homogeneous)

(or) has infinitely many solutions (any n numbers x_i's are can be soln)

if $b = 0$ (homogeneous).

Simpler way,

Consider the System of equations [m- equations & n-unknowns]

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \rightarrow (1)$$

(1) will be written as in the Augmented form as follows:

$$\begin{array}{ccc} A & X & B \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \end{array}$$

$$\Rightarrow AX = B$$

Augmented form:-

$$\begin{array}{cc} A & B \\ \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \ddots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \end{array}$$

Elementary row operations:

✓[A]
✓[A|B]

1. The interchange of any two rows.
2. The multiplication of any row with a non-zero constant (number).
3. The addition of a constant multiple of the elements of any row corresponding to the elements of any other row.

Hint: [column operations also there, just change the rows as columns in the above rules]

Eg: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix} \longrightarrow$$

R_1: 1st Row $[1 \ \ 2 \ \ 3]$

R_2: 2nd Row $[5 \ \ 6 \ \ 7]$

R_3: 3rd Row $[-2 \ \ 1 \ \ 7]$

1st rule:

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 5 & 6 & 7 \\ 1 & 2 & 3 \\ -2 & 1 & 7 \end{bmatrix},$$

$R_1 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 7 \\ 5 & 6 & 7 \\ 1 & 2 & 3 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 7 \\ 5 & 6 & 7 \end{bmatrix}$$

2nd rule:

Constant multiple by a non-zero number

$5R_2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 25 & 30 & 35 \\ -2 & 1 & 7 \end{bmatrix},$$

$\frac{1}{2} R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -1 & 1/2 & 7/2 \end{bmatrix}$$

$-3R_1$

$$\begin{bmatrix} -3 & -6 & -9 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix}$$

3rd rule:

The addition of a constant multiple of the elements of any row corresponding to the elements of any other row.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix} \sim$$

In $R_1 + 5R_3$

$$A \sim \begin{bmatrix} \boxed{-9 \quad 7 \quad 38} \\ 5 \quad 6 \quad 7 \\ -2 \quad 1 \quad 7 \end{bmatrix}$$

$R_1 \to R_1 + 5R_3$

$R_1 : \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$5R_3 : \begin{bmatrix} -10 & 5 & 35 \end{bmatrix}$

$R_1 + 5R_3 : \begin{bmatrix} -9 & 7 & 38 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -12 & -11 & -7 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 11/2 & 7 & 17/2 \\ -2 & 1 & 7 \end{bmatrix}$$

$R_2 \to R_2 + \left(\frac{1}{2}\right) R_1$

Note:

(1) Elementary transformation [row operations] do not change either the order or rank of the matrix.

Rank of the matrix:-

$$A = \begin{bmatrix} \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \end{bmatrix}_{r \times r}$$

A matrix is said to be of rank r

(i) it has at least one non-zero minor of order r.

and (ii) every minor of order higher than r vanishes

The rank of a matrix A shall be denoted as $\rho(A)$.

Find the rank of the matrix

(1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

(2) $A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Solution:

1.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times 3} \approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}_{3\times 3}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

Here R_2 and R_3 are having same rows.

obviously, the 3rd order minor of A vanishes

Here, we are trying to consider 2nd order minors of A.

2nd order minor: $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$

$\Rightarrow \quad \rho(A) = 2.$

Note:

Suppose, In a problem $A \approx \begin{bmatrix} 3 & 6 & 7 \\ 5 & 6 & 7 \\ -1 & 0 & 0 \end{bmatrix}_{3\times 3}$

In some Cases:

$\det(A) = 0 \Rightarrow \text{rank}(A) = \rho(A) \neq 3$

So, the minor of order 2, here we are Considering

Choices:

$$\begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\ \begin{bmatrix} 3 & 6 \\ 5 & 6 \end{bmatrix} & \begin{bmatrix} 3 & 7 \\ 5 & 7 \end{bmatrix}, & \begin{bmatrix} 6 & 7 \\ 6 & 7 \end{bmatrix}, & \begin{bmatrix} 5 & 6 \\ -1 & 0 \end{bmatrix}, & \begin{bmatrix} 6 & 7 \\ 0 & 0 \end{bmatrix}, & \begin{bmatrix} 3 & 7 \\ -1 & 0 \end{bmatrix} \end{array}$$

$\det(B_3) = 0$ $\qquad$ $\det(B_5) = 0$

else, B_1, B_2, B_4 & B_6 have non-zero determinants.

$\Rightarrow \rho(A) = 2.$

(5)$\rightarrow$

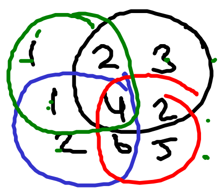
Hint:

$|A| = \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}_{3\times 3} = 0$ rank is not 3

$\Rightarrow$ may be $(r-1)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

&



$\begin{matrix} 1 & 3 \\ 1 & 2 \end{matrix}$ $\qquad$ $\begin{matrix} 1 & 2 \\ 2 & 5 \end{matrix}$

$\begin{matrix} 1 & 3 \\ 2 & 5 \end{matrix}$

② $A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}_{4\times 4} \Rightarrow A \sim \begin{bmatrix} 0 & 1 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix}_{4\times 4} \begin{matrix} \\ \to R_2 \\ \to R_3 \\ \\ \end{matrix}$

Here, R_2 and R_3 are same

$\Rightarrow |A| = 0$

$\Rightarrow \rho(A) \neq 4$.

So, Consider any minor of order 3,

B (say) $= \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \det(B) = -1(-3) + 3(1-0) = 3+3 = 6 \neq 0$

$\Rightarrow \text{rank}(A) = \rho(A) = 3$.

Definition: Two matrices (or system of linear equations) are said to be row equivalent if one can be transformed to the other by a finite sequence of elementary row operations.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times 3}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

In A

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

Here,
we performed two operations.

Here, A is row equivalent to B.

i.e., $A \sim B$

Note:- If two systems of linear equations are row equivalent, then they have same set of solutions.

Solve the System of linear equations:

$0x + 2y + 4z = 2$; $x + 2y + 2z = 3$; $3x + 4y + 6z = -1$

Solution:-

The given equations, will considered, in the Augmented form

$$\underbrace{\begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & 2 \\ 3 & 4 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}}_{B} \Rightarrow \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \left[\begin{array}{ccc|c} 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 6 & -1 \end{array}\right] \end{array}$$

Now, applying the elementary row operations:

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 6 & -1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \sim \left[\begin{array}{ccc|c} \boxed{1} & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 3 & 4 & 6 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & 0 & -10 \end{array}\right] \quad R_3 \to R_3 - 3R_1$$

$\underline{3 + (-3)1}$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -8 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\sim \begin{array}{c} \begin{array}{cccc} x & y & z & B \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \end{array} \quad \begin{array}{l} R_2 \to R_2/2 \\ R_3 \to R_3/4 \end{array}$$

Hint:

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & 0 & -10 \end{array}\right]$$

Forward elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

In the first row the non-zero entry is 1. This 1 is called first pivot. |||ly, In 2nd & 3rd row pivot is 1.

This form is called row-echelon form. (ref)

The associated system of equation is

$x + 2y + 2z = 3$
$0x + 1\cdot y + 2z = 1$
$0x + 0\cdot y + 1\cdot z = -2$

$\Rightarrow$ $\boxed{z = -2}$ and $y + 2z = 1$
$y = 1 - 2(-2)$
$\boxed{y = 5}$

and $x + 2y + 2z = 3$
$x + 2(5) + 2(-2) = 3$
$x + 6 = 3$
$\boxed{x = -3}$

The Solution of the system is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix}$.

The above said method is called Gaussian Elimination.

The process is called back substitution.

$$[A|B] \approx \begin{array}{c} \begin{array}{cccc} x & y & z & B \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad R_2 \to R_2 - 2R_3$$

$$\approx \left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad R_1 \to R_1 - 2R_3$$

$$7 - 2(5)$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

The corresponding system of equation is

$$x = -3$$
$$y = 5 \quad \Rightarrow \quad \text{Solution } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix}$$
$$z = -2$$

The whole process to obtain the RREF is called Gauss-Jordan elimination method.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \to \text{Reduced row-echelon form [RREF]}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \to \text{Row-echelon form [REF]}$$

Row Echelon Form (REF):

REF of an augmented matrix is of the following form;

1. The zero rows,if they exist,come last in the order of rows
2. The first non-zero entries in the non-zero rows are 1,called leading 1's
3. Below each leading 1 is a column of zeros.Thus, in any two consecutive non-zero rows,the leading 1 in the lower row appears farther to the right than the leading 1 in the upper row

Reduced row echelon form (RREF):

RREF of an augmented matrix is of the following form;

4.In REF,the above each leading one is a column of zeros

Find the REF and RREF:

(1) $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to$ RREF

(2) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \to$ RREF

(3) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right] \to$ RREF

(4) $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 6 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right] \to$ REF

Problem:- Solve the following system of linear equations by Gauss-Jordan elimination.

$$\left.\begin{array}{l} x_1 + 3x_2 - 2x_3 + 0x_4 = 3 \\ 2x_1 + 6x_2 - 2x_3 + 4x_4 = 18 \\ 0x_1 + x_2 + x_3 + 3x_4 = 10 \end{array}\right\} \quad (1)$$

Soln:- The augmented form of (1) is

REF
RREF

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 3 & -2 & 0 & 3 \\ 2 & 6 & -2 & 4 & 18 \\ 0 & 1 & 1 & 3 & 10 \end{array}\right]$$

$$\approx \left[\begin{array}{cccc|c} 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 2 & 4 & 12 \\ 0 & 1 & 1 & 3 & 10 \end{array}\right] \quad R_2 \to R_2 + (-2)R_1$$

$$\approx \left[\begin{array}{cccc|c} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\approx \left[\begin{array}{cccc|c} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right] \quad R_3 \to R_3/2 \quad \text{[REF form]}$$

$$\approx \left[\begin{array}{cccc|c} 1 & 3 & 0 & 4 & 15 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_3 \\ R_1 \to R_1 + 2R_3 \end{array}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$[A|B] \approx \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right] \quad R_1 \to R_1 - 3R_2 \quad \text{[RREF Form]}$$

From Augmented form to Equation form.

$$\left.\begin{array}{l} x_1 \qquad\qquad + x_4 = 3 \\ \quad x_2 \qquad + x_4 = 4 \\ \qquad\quad x_3 + 2x_4 = 6 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 3 - x_4 \\ x_2 = 4 - x_4 \\ x_3 = 6 - 2x_4 \end{array}$$

Let $x_4 = t \in \mathbb{R}$ [$\mathbb{R}$ - Set of real numbers]
t is arbitrary

The solution becomes $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3-t \\ 4-t \\ 6-2t \\ t \end{pmatrix}$

(or) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.

Based on the variable 't' the given System ① have infintely many solutions.

— x — x — x — x — x — x —

Basic variable: Among the variables in a system, the ones corresponding to the Columns leading 1's are called the basic variables. and the ones corresponding to the Columns without 1's, if they are any, are called the Free variables.

(x, y, z)

For example:

① $A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \rightarrow$ x, y, z are basic variables.

② $B = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \rightarrow$ $x, z \rightarrow$ basic Variables
$y \rightarrow$ free variable.

Solve the following system by Gauss-Jordan elimination

① $\left.\begin{array}{l} -2y - z = +1 \\ 4x - 10y + 3z = 5 \\ 3x - 3y = 6 \end{array}\right\}$

② $\begin{array}{l} x_1 + x_2 + x_3 = 3 \\ -3x_1 - 17x_2 + x_3 + 2x_4 = 1 \\ 4x_1 - 17x_2 + 8x_3 - 5x_4 = 1 \\ -5x_2 - 2x_3 + x_4 = 1 \end{array}$

Solution ①: Given in $AX = B$ form $\begin{bmatrix} 0 & -2 & -1 \\ 4 & -10 & 3 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \rightarrow$ (I)

Augmented form of (I):

$$[A|B] = \left[\begin{array}{ccc|c} 0 & -2 & -1 & 1 \\ 4 & -10 & 3 & 5 \\ 3 & -3 & 0 & 6 \end{array}\right]$$

In REF:

$$\approx \left[\begin{array}{ccc|c} 3 & -3 & 0 & 6 \\ 4 & -10 & 3 & 5 \\ 0 & -2 & -1 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 4 & -10 & 3 & 5 \\ 0 & -2 & -1 & 1 \end{array}\right] \quad R_1 \to \frac{R_1}{3}$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & -6 & 3 & -3 \\ 0 & 2 & 1 & -1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3/-1 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 2 & 1 & -1 \end{array}\right] \quad R_2 \to R_2/-6$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 2 & -2 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad R_3 \to \frac{R_3}{2} \longrightarrow \text{REF}$$

$$\approx \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad R_2 \to R_2 + \tfrac{1}{2}(R_3)$$

$$\begin{array}{cccc} x & y & z & B_i\text{'s} \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad R_1 \to R_1 + R_2$$

$\longrightarrow$ RREF

$\boxed{x = 2,\ y = 0,\ z = -1}$ unique solution

Consider the REF of some matrix A:

$$[A|B] = \underbrace{\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1 \end{array}\right]}_{[A|B]} \Rightarrow \rho(A) = 3 \quad \& \quad \rho(A|B) = 3$$

A

& The unknowns are x, y, z

So, $\rho(A) = \rho(A|B) = \#$ unknowns are same

$\Rightarrow$ The given matrix A is having unique (or) one soln

Consistent and Inconsistent: Consider the system AX=B

$\boxed{rank(A) = \rho(A)}$

\# of non-zero rows in matrix A of REF is called rank(A)

1. rank(A) = rank (A | B) and
   rank(A) = rank (A | B)=number of unknowns

   ======> unique solution [Consistent]

2. rank(A) = rank (A | B) and
   rank(A) = rank (A | B) < number of unknowns

   ======> infinitely many solutions

3. $rank(A) \neq rank(A|B)$

   ======> No solution

In AX = B

$[A \mid B]$ → $\rho(A)$ & $\rho(A|B)$

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ → unknown

Consistent	Inconsistent
At least one soln < unique soln, many solns.	No Solution

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 2 \\ 0 & 0 & 1 \end{bmatrix} \overset{REF}{\sim} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & -8 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 / -8 \end{matrix} \to \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\to R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank(A) = $\rho(A)$ = # of non-zero rows in REF

rank(A) = $\rho(A) = 2$

for example:

$$[A|B] \sim \left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right] \xrightarrow{REF} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & c \end{array}\right]$$

where $c \neq 0$

$\rho(A) = 2$ and $\rho(A|B) = 3$

$\Rightarrow \rho(A) \neq \rho(A|B) \Rightarrow$ No Solution

Consistent System: $\boxed{AX=B.}$

- one solution (unique solution) → The system has no free variable (only basic variables)
- Many solutions (infinitely many solutions) ↳ The system has atleast one free variable

Inconsistent System:

- No solution.

Consider $\underline{AX=B}$

$(A)_{m\times n}$ with if $\underline{m<n}$

$m \to$ # of equations

$n \to$ # of unknowns.

$\Rightarrow$ free variable always occured in $[A|B]$.

Consider: $\left.\begin{matrix} 2x+3y+z=8 \\ 3x+y+z=5 \end{matrix}\right\} \Rightarrow \begin{matrix} m=2 \\ n=3 \end{matrix}$ $\boxed{2<3}$

$$[A|B] \approx \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 3 & 1 & 1 & 5 \end{array}\right] \xrightarrow[\approx]{R_1/2} \left[\begin{array}{ccc|c} 1 & 3/2 & 1/2 & 4 \\ 3 & 1 & 1 & 5 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 3/2 & 1/2 & 4 \\ 0 & -7/2 & -7/2 & -7 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$$\begin{array}{c} \\ \approx \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z \to \underline{F.V} & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 3/2 & 1/2 & 4 \\ 0 & 1 & 1/7 & 2 \end{array}\right] \end{array} \quad R_2 \to R_2/(-7/2) \quad \underline{(REF)}$$

Let $\underline{z=t}$

$x = 4 - 3/2 - 1/2\, t$

$y = 2 - 1/7\, t$

$z = t$, $t \in \mathbb{R}$

$\underline{z=t}$ is called free variable

$x, y \to$ $\underline{\text{basic variables}}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5/2 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/2 \\ -1/7 \\ 1 \end{pmatrix}$$ ✓

This system has many $\underline{\text{solutions}}$

Problem:

For which values of "a" will the following system have

1. no solutions?
2. Exactly one solution?
3. Infinitely many solutions?

$$\begin{cases} x+2y-3z=4 \\ 3x-y+5z=2 \\ 4x+y+(a^2-14)z=a+2 \end{cases}$$

$$\left[\begin{matrix} * & * & * \\ * & * & * \\ * & * & * \end{matrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$[A|B] \approx \left[\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}\middle|\begin{matrix} b_1' \\ b_2' \\ b_3' \end{matrix}\right]$

$\rightarrow \rho(A) = \rho(A|B)$

Solution:-

$$[A|B] = \left[\begin{matrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ 4 & 1 & a^2-14 \end{matrix}\middle|\begin{matrix} 4 \\ 2 \\ a+2 \end{matrix}\right]$$

$$[A|B] \approx \left[\begin{matrix} 1 & 2 & -3 \\ 0 & -7 & 14 \\ 0 & -7 & a^2-2 \end{matrix}\middle|\begin{matrix} 4 \\ -10 \\ a-14 \end{matrix}\right] \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\approx \left[\begin{matrix} 1 & 2 & -3 \\ 0 & -7 & 14 \\ 0 & 0 & a^2-16 \end{matrix}\middle|\begin{matrix} 4 \\ -10 \\ a-4 \end{matrix}\right] \quad R_3 \to R_3 - R_2$$

$a-14+10$

Cases:-

if $a=4$, $\Rightarrow \rho(A) = 2$ & $\rho(A|B) = 2$ & # of unknowns $= 3$

$$\Rightarrow \left[\begin{matrix} 1 & 2 & -3 \\ 0 & -7 & 14 \\ 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} 4 \\ -10 \\ 0 \end{matrix}\right]$$

$\rho(A) = \rho(A|B) = 2 < 3$ # of unknowns

$\Rightarrow AX=B$ has many solutions.

For understanding:

$z = t$

$$\left[\begin{matrix} 1 & 2 & -3 \\ 0 & 1 & -2 \end{matrix}\middle|\begin{matrix} 4 \\ 10/7 \end{matrix}\right] \quad R_2 \to R_2/-7$$

: soln:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8/7 \\ 10/7 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$y - 2z = 10/7$

$y = 2z + 10/7$

$x = -2y + 3z + 4$

$x = -2\left[2t\right] - 2\left(\frac{10}{7}\right) + 3(t) + 4$

$x = -t + 8/7$

$4 - 20/7$

Case (2):

$$[A|B] \approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

Let $a = -4$: In R_3 $\left[\begin{array}{ccc|c} x & x & x & x \\ x & x & x & x \\ 0 & 0 & 0 & -8 \end{array}\right]$

$\Rightarrow$ $\rho(A) = 2$, $\rho(A|B) = 3$

$\Rightarrow \rho(A) \neq \rho(A|B)$

$\Rightarrow$ Not Consistent

$\Rightarrow$ No solution for $AX = B$.

Case (3):

$$[A|B] \approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

if $a \neq 4$ & $a = -4$ then

$$[A|B] \approx \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & c_1 & c_2 \end{array}\right] \Rightarrow c_1 \neq 0 \text{ \& } c_2 \neq 0$$

$\rho(A) = \rho(A|B) =$ Number of unknowns $= 3$

$\Rightarrow$ Consistent

$\Rightarrow$ unique solution for $AX = B$.

Inverse matrices

Definition For an $m \times n$ matrix A, an $n \times m$ matrix B is called a **left inverse** of A if $BA = I_n$, and an $n \times m$ matrix C is called a **right inverse** of A if $AC = I_m$.

Example From a direct calculation for two matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -3 \\ -1 & 5 \\ -2 & 7 \end{bmatrix},$$

we have $AB = I_2$, and $BA = \begin{bmatrix} -5 & 2 & -4 \\ 9 & -2 & 6 \\ 12 & -4 & 9 \end{bmatrix} \neq I_3$.

Thus, the matrix B is a right inverse but not a left inverse of A, while A is a left inverse but not a right inverse of B. Since $(AB)^T = B^T A^T$ and $I^T = I$, a matrix A has a right inverse if and only if A^T has a left inverse. □

$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$

$[A \to n\times n \text{ matrix}]$

if $(A)_{m\times n}$ matrix

$\Rightarrow AB = I = CA$

$m\times n$

i) $\Rightarrow B = A^{-1}$ [right

ii) & $C = A^{-1}$ [left]

$\Rightarrow B = C$

Inverse exist

Square matrix:

Lemma *If an $n \times n$ square matrix A has a left inverse B and a right inverse C, then B and C are equal, i.e., $B = C$.*

$BA = I = AC$

$\Rightarrow B = C$

Results: Let A be an invertible matrix and k any nonzero scalar. ~~Show that~~

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$;

(2) the matrix kA is invertible and $(kA)^{-1} = \frac{1}{k} A^{-1}$;

(3) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

A^{-1} is invertible

inverse exist

for A.

Theorem *The product of invertible matrices is also invertible, whose inverse is the product of the individual inverses in reverse order:*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

An $n \times n$ matrix is called an ***elementary matrix*** if it can be obtained from the $n \times n$ identity matrix I_n by performing a single elementary row operation.

EXAMPLE

Listed below are four elementary matrices and the operations that produce them.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$, $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2(-3)$ $\quad R_2 \leftrightarrow R_4$ $\quad R_1 \to R_1 + 3R_3$ $\quad R_1 \to R_1(1)$

$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}_{2\times2}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}_{4\times4}$, $E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$, $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

↑ Multiply the second row of I_2 by -3.

↑ Interchange the second and fourth rows of I_4.

↑ Add 3 times the third row of I_3 to the first row.

↑ Multiply the first row of I_3 by 1.

$\Rightarrow$ E_1, E_2, E_3 & E_4 are called elementary matrices.

Row Operations by Matrix Multiplication

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}_{3\times4}$$

Let $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 \to R_3 + 3R_1$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}_{3\times3}$$

$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$

which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$

which is precisely the same matrix that results when we add 3 times the first row of A to the third row.

$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 3R_1} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix} = EA$

$I \to E$ Row Operation on I That Produces E	$E \to I$ Row Operation on E That Reproduces I		
① Multiply row i by $c \neq 0$	Multiply row i by $1/c$	$R_i \to cR_i$	$R_i \to \frac{1}{c} R_i$
② Interchange rows i and j	Interchange rows i and j	$R_i \leftrightarrow R_j$	$R_i \leftrightarrow R_j$
③ Add c times row i to row j	Add $-c$ times row i to row j	$R_j \to R_j + cR_i$	$R_j \to R_j - cR_i$

Row Operations and Inverse Row Operations

① $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 7R_2} E = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \longrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $R_2 \to \frac{1}{7} R_2$

Multiply the second row by 7. | Multiply the second row by $\frac{1}{7}$.

③ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Interchange the first and second rows. | Interchange the first and second rows.

④ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 5R_2} E = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 5R_2} I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Add 5 times the second row to the first. | Add -5 times the second row to the first.

② $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_2 \to R_2 - 3R_1$

$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

2×2

$\downarrow$ $R_2 \to R_2 + 3R_1$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

$I \longrightarrow E$, $E \rightarrow$ invertible $\Rightarrow E^{-1}$ is also invertible matrix.

we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$\Rightarrow$ $E_k \cdots E_2 E_1 A = I_n$

$E_1, E_2, \ldots, E_k$ are invertible

$A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$

$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

We know that,

$E_k E_{k-1} \cdots E_2 \cdot E_1 A = I_n \Rightarrow E_k^{-1}(E_k E_{k-1} \cdots E_2 \cdot E_1 A) = E_k^{-1} \cdot I_n$

$\Rightarrow E_{k-1} E_{k-2} \cdots E_2 \cdot E_1 \cdot A = E_k^{-1} I_n$

$\Rightarrow E_{k-1}^{-1}(E_{k-1} E_{k-2} \cdots E_2 E_1 \cdot A) = E_{k-1}^{-1}(E_k^{-1} \cdot I_n)$

$\because E_k^{-1} \cdot E_k = I$

$E_{k-1}^{-1} E_{k-1} = I$

$\vdots$

$\Rightarrow E_{k-2} \cdot E_{k-3} \cdots E_2 E_1 A = E_{k-1}^{-1} E_k^{-1} I_n$

$\vdots$

$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{k-1}^{-1} E_k^{-1} I_n$

$$\boxed{A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{k-1}^{-1} E_k^{-1} I_n}$$

Inverse of A:

w.k.t, $A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{k-1}^{-1} E_k^{-1} I_n$

$A^{-1} = (E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{k-1}^{-1} E_k^{-1} I_n)^{-1}$

Reversal Law:

$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

$\boxed{(A^{-1})^{-1} = A}$

$\Rightarrow A^{-1} = (E_k^{-1})^{-1} (E_{k-1}^{-1})^{-1} \cdots (E_3^{-1})^{-1} (E_2^{-1})^{-1} (E_1^{-1})^{-1}$

$A^{-1} = E_k E_{k-1} \cdots E_3 E_2 E_1$

$A^{-1} = E_k \cdots E_2 E_1 I_n$

Conclusion:

(i) $A = I_n E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$

(ii) $A^{-1} = E_k E_{k-1} \cdots E_3 E_2 E_1 I_n$.

Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find elementary matrices E_1, E_2, and E_3 such that $E_3E_2E_1A = I_3$.

(b) Write A as a product of elementary matrices.

(c) Compute A^{-1} as a product of elementary matrices

Solution:

Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Elementary matrix.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2/4$

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2/4} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $R_2 \to R_2 - \frac{3}{4}R_3$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - \frac{3}{4}R_3$ $\approx \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $R_1 \to R_1 + 2R_3$

$E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 \to R_1 + 2R_3$ $\approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The elementary matrices are

$E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{bmatrix}$, $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now,

$$E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1/4 & -3/4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2+2 \\ 0 & 1 & 3/4-3/4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

② Write A as a product of elementary matrices:

Soln.

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$$

Here $k=3$

Conclusion:

(i) $A = I_n E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$

(ii) $A^{-1} = E_k E_{k-1} \cdots E_3 E_2 E_1 I_n$.

$$A = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\times} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

The elementary matrices are

$$E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/4 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I_3: $R_2 \to cR_2$ $\qquad$ I_3: $R_3 \to R_3 + 3R_1$ $\qquad$ I_3: $R_1 \leftrightarrow R_2$

① $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$; $\quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_1$

$R_2 \to \frac{1}{c} R_2$ $\qquad$ $R_3 \to R_3 - 3R_1$ $\qquad$ I_3: $R_1 \leftrightarrow R_2$

$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$; $\quad E_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

③ Compute A^{-1} as a product of elementary matrices

Solns

$$A^{-1} = E_3 E_2 E_1 I_n$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1/4 & -3/4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1/4 & -3/4 \\ 0 & 0 & 1 \end{bmatrix}$$

[X

Verify:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$|A| = 4$

adjoint$(A) \Rightarrow$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

Gauss-Jordan method to find the inverse of the given matrix with the elementary matrix Concepts.

method:

1. start with [A | I_n] is reduced into to the martrix form [I_n | B].

2.This matrix B is the inverse of the matrix A.

(I_n is the identity matrix of the size given matrix A.

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$

using G-J- Elimination

soln:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \quad R_2 \to R_2(-1)$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \quad R_3 \to R_3 + 2R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \quad R_2 \to R_2 - R_3 //$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -8 & 6 & -3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \quad R_1 \to R_1 - 3R_3 //$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

$$[A \mid I_3] \approx [I_3 \mid B]$$

$$B = A^{-1} = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$ find A^{-1} using G-J-E

$$\boxed{A^{-1} = \frac{1}{|A|} adj(A)}$$

Soln.:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$[A|I] \approx \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0} & & & -1 & 1 & 1 \end{array}\right] \quad R_3 \to R_3 + R_2$$

Since third row all members are zero

$\Rightarrow$ $\det(A) = 0$ $\Rightarrow$ Inverse does not exist

Definition A **permutation matrix** is a square matrix obtained from the identity matrix by permuting the rows.

Properties:

(1) A permutation matrix is the product of a finite number of elementary matrices each of which is corresponding to the "row-interchanging" elementary row operation.

(2) Any permutation matrix P is invertible and $P^{-1} = P^T$.

(3) The product of any two permutation matrices is a permutation matrix.

(4) The transpose of a permutation matrix is also a permutation matrix.

I_n by rowinterchanging

$P^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P^T$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\boxed{2!}$ $P = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P = \Bigg\{ \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3:\ R_1 \leftrightarrow R_2}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{R_1 \leftrightarrow R_3}, \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{R_2 \leftrightarrow R_3},$

$\boxed{3!}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg\}$

For an $n \times n$ Identity matrix the number of permutation matrices are $\underline{\underline{n!}}$

elementary row operations: $\boxed{\begin{array}{l} 1: cR_i \\ 2: R_i \to R_i \pm cR_j \\ 3: R_i \leftrightarrow R_j \end{array}}$

[the **elementary column operations** for a matrix by just replacing "row" by "column" in the definition of the elementary row operations.] $R \to C$

Show that if A is an $m \times n$ matrix and if E is an elementary matrix obtained by executing an elementary column operation on I_n, then AE is exactly the matrix that is obtained from A when the same column operation is executed on A.

E. Row operations: $E_k \cdots E_2 E_1 A = I_n$

E. Column operations: $A E_1 E_2 E_3 \cdots E_k = I_n.$

Theorem *Let A be an $n \times n$ matrix. The following are equivalent:*

(1) *A has a left inverse;*

(2) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\boxed{\mathbf{x} = \mathbf{0}}$;*

(3) *A is row-equivalent to I_n;*

(4) *A is a product of elementary matrices;*

(5) *A is invertible;*

(6) *A has a right inverse.*

$[A]_{n\times n}$

① if A has left inverse $BA = I_n$

② if A has right inverse $AC = I_n$

$\Rightarrow B = C$

③ A has inverse/ is invertible

④ $|A| \neq 0$ $\Rightarrow$ non-singular

⑤ A is a product of elementary matrices

⑥ $Ax = 0 \Rightarrow$ This system has only trivial solutions

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

⑦ A is row-equivalent to I_n.

Theorem *If A is an invertible $n \times n$ matrix, then for any column vector $\mathbf{b} = [b_1 \ \cdots \ b_n]^T$, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$. If A is not invertible, then the system has either no solution or infinitely many solutions according to whether or not the system is inconsistent.* □

$$b = [b_1 \ b_2 \ \cdots \ b_n]^T = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$Ax = b$$

$$\underset{n\times n}{\begin{bmatrix} A \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x \end{bmatrix}} = \underset{n\times 1}{\begin{bmatrix} b \end{bmatrix}}$$

One solution:

$$A^{-1}(Ax) = A^{-1}b$$

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b \Rightarrow x = A^{-1}b$$

a square matrix A is nonsingular if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$[|A| \neq 0]$

That is, a square matrix A is singular if and only if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, say $\mathbf{x}_0$.

$$\begin{array}{ccccc} A & & X & = & 0 \\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} & & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

, Here $|A| = 4-4 = 0$

$A \to$ singular

$$[A|0] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \qquad R_2 \to R_2 - 2R_1$$

$$\approx \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \rho(A|0) = 1 < \#$ of unknown (x_1, x_2) 2

$\Rightarrow$ This system have infinitely many solutions.

$$\Rightarrow \left[x_1 + 2x_2 = 0 \right.$$

$x_1 = -2x_2$,

Let $x_2 = t$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2t \\ t \end{pmatrix}, \quad t \in \mathbb{R}$$

Definition An $n \times n$ square matrix A is said to be **invertible** (or **nonsingular**) if there exists a square matrix B of the same size such that

$$AB = I = BA.$$

$B = A^{-1}$

Such a matrix B is called the **inverse** of A, and is denoted by A^{-1}. A matrix A is said to be **singular** if it is not invertible.

(1) singular $|A| = 0$

(2) Non-singular $|A| \neq 0 \longrightarrow$ Inverse exist

Definition Let $A = [a_{ij}]$ be an $m \times n$ matrix.

(1) A is called a **square matrix of order** n if $m = n$.

In the following, we assume that A is a square matrix of order n.

(2) The entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **diagonal entries** of A.

(3) A is called a **diagonal** matrix if all the entries except for the diagonal entries are zero.

(4) A is called an **upper (lower) triangular** matrix if all the entries below (above, respectively) the diagonal are zero.

The following matrices U and L are the general forms of the upper triangular and lower triangular matrices, respectively:

$\underline{\underline{U}}$

$a_{ij} = \begin{cases} 0, & i > j \\ c, & i < j \end{cases}$

L:

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

$a_{ij} = \begin{cases} c, & i > j \\ 0, & i < j \end{cases}$

Note that a matrix which is both upper and lower triangular must be a diagonal matrix, and the transpose of an upper (lower) triangular matrix is lower (upper, respectively) triangular matrix.

Solving Linear Systems by Factoring

If an $n \times n$ matrix A can be factored into a product of $n \times n$ matrices as

$$A = LU$$

where L is lower triangular and U is upper triangular, then the linear system $A\mathbf{x} = \mathbf{b}$ can be solved as follows:

From, Elementary Row operations → Row Swaping: is not allowed

Step 1. Rewrite the system $A\mathbf{x} = \mathbf{b}$ as $\quad LU\mathbf{x} = \mathbf{b} \quad (1)$

Step 2. Define a new $n \times 1$ matrix $\mathbf{y}$ by $\quad U\mathbf{x} = \mathbf{y} \quad (2)$

Step 3. Use 2 to rewrite 1 as $L\mathbf{y} = \mathbf{b}$ and solve this system for $\mathbf{y}$.

Step 4. Substitute $\mathbf{y}$ in 2 and solve for $\mathbf{x}$.

the problem of solving the single system $A\mathbf{x} = \mathbf{b}$ by the problem of solving the two systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$, the latter systems are easy to solve because the coefficient matrices are triangular.

An LU-Decomposition

Find an LU-decomposition of $\quad A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

Reduction to Row-Echelon Form	Elementary Matrix Corresponding to the Row Operation	Inverse of the Elementary Matrix
$\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$		
$\begin{bmatrix} 1 & 3 & 1 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$	$E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$E_1^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 4 & 9 & 2 \end{bmatrix}$	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix} \qquad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

is an LU-decomposition of A.

Find an L-U Decomposition of $A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

Solution:

Given matrix:

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}_{3\times 3}$$

Elementary matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Inverse of Elementary matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 3 & 1 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}; R_1 \to R_1/2$$

$$E_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 \to \frac{R_1}{2} \qquad E_1^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 \to 2R_1$$

$$\approx \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 4 & 9 & 2 \end{bmatrix} R_2 \to R_2 + 3R_1$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 \to R_2 + 3R_1 \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 \to R_2 - 3R_1$$

$$\approx \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} R_3 \to R_3 - 4R_1$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} R_3 \to R_3 - 4R_1 \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} R_3 \to R_3 + 4R_1$$

$$\approx \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} R_3 \to R_3 + 3R_2$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} R_3 \to R_3 + 3R_2 \qquad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} R_3 \to R_3 - 3R_2$$

$$A \approx \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} R_3 \to \frac{R_3}{7}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/7 \end{bmatrix} R_3 \to \frac{R_3}{7} \qquad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} R_3 \to 7R_3$$

This is an upper triangular matrix

$\Rightarrow$

Let $U = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Lower Triangular Matrix (L):

$$L = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix}$$

$$A = L \quad U$$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

[Verify!]

$LU = A$

Consider the system of linear equations

$$\underline{Ax = B}$$
$$(3\times 4)\ (4\times 1) = (3\times 1)$$

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = \mathbf{b}.$$

3×4 $\quad 4\times 1$ $\quad 3\times 1$

$$x = (x_1, x_2, x_3, x_4)^T$$

The elementary matrices for Gaussian elimination of A are easily found to be

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix},$$

so that

$$E_3E_2E_1A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix} = U.$$

Note that U is the matrix obtained from A after forward elimination, and $A = LU$ with

$$L = E_1^{-1}\, E_2^{-1}\, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix},$$

$$Ax = B$$

which is a lower triangular matrix with 1's on the diagonal. Now, the system

(Since $A = LU$)

$$L\mathbf{c} = \mathbf{b} : \begin{cases} c_1 & & & = & 1 \\ 2c_1 & + & c_2 & & = & -2 \\ -c_1 & - & 3c_2 & + & c_3 & = & 7 \end{cases}$$

$$LUx = B$$
$$Ux = Y$$
$$3\times 4\ \ 4\times 1 = \underline{3\times 1}$$
$$LY = B$$

$y = (y_1, y_2, y_3)^T$

resolves to $\mathbf{c} = (1, -4, -4)$ and the system

get Y:

$$U\mathbf{x} = \mathbf{c} : \begin{cases} 2x_1 & + & x_2 & + & x_3 & & & = & 1 \\ & - & x_2 & - & 2x_3 & + & x_4 & = & -4 \\ & & & - & 4x_3 & + & 4x_4 & = & -4 \end{cases}$$

Then $Ux = Y$

resolves to

$$X = (\quad)$$

$$\mathbf{x} = \begin{bmatrix} -1 + t \\ 2 + 3t \\ 1 - t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix},$$

for $t \in \mathbb{R}$. It is suggested that the readers find the solutions for various values of $\mathbf{b}$. $\square$

① Does every square matrix have L-U-Decomposition?

No. Some matrices may not have L-U-Decomposition since, if we applied Row interchanges $A \neq LU$

[R-EF] it is produced

$\Rightarrow PA = LU$

Try this: L-U-Decomposition

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

② Can a square matrix have more than one L-U-Decomposition?

Yes, have more than one L-U-Decomposition

$$A = \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -6 \\ 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + \left(\frac{2}{3}\right) R_1 \qquad E_1 = \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -2/3 & \boxed{1} \end{bmatrix}$$

$\downarrow$

U

$L = E_1^{-1}$

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ -2/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & -6 \\ 0 & 1 \end{bmatrix}}_{U}$$

Evaluate det(A) where $A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$

Evaluate, det(A), $A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$

$\det(A) = ?$