Gauss Jordan elimination

Reduced row echelon form:-

If a matrix in row echelon form satisfies the following additional conditions, then it is in reduced row echelon form

1) The leading entry in each non zero row is 1.
2) Each leading 1 is the only nonzero entry in its column.

Gauss Jordan procedure:- Consider the system $AX=B$. To solve this system by Gauss Jordan method, we proceed by the following steps

1) Form the augmented matrix $[A \ B]$
2) Apply row elementary operations successively to $[A \ B]$ so that we can get reduced row echelon form of $[A \ B]$
3) Using reduced row echelon form and backward substitution find the solution to the original problem.

Pb:1 Solve the system

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= -2 \\ 2x_1 - x_2 + x_3 + x_4 &= 0 \\ 3x_1 + 2x_2 - x_3 - x_4 &= 1 \\ x_1 + x_2 + 3x_3 - 3x_4 &= -8 \end{aligned}$$

by Gauss Jordan method.

Moreover find the basic and free variables.

Ans:- The augmented matrix is

1st pivot

$$\begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 1 & 0 \\ 3 & 2 & -1 & -1 & 1 \\ 1 & 1 & 3 & -3 & -8 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -3 & -1 & 3 & 4 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 2 & -2 & -6 \end{bmatrix} \begin{matrix} R_2 \to \frac{R_2}{(-3)} \\ R_3 \to \frac{R_3}{(-1)} \\ R_4 \to \frac{R_4}{2} \end{matrix}$$

2nd pivot

$$\begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & \frac{1}{3} & -1 & -4/3 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 2/3 & 0 & -2/3 \\ 0 & 1 & 1/3 & -1 & -4/3 \\ 0 & 0 & 11/3 & -1 & -17/3 \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix} \quad R_3 \to \frac{R_3}{\left(\frac{11}{3}\right)}$$

3rd pivot

$$\begin{bmatrix} 1 & 0 & 2/3 & 0 & -2/3 \\ 0 & 1 & 1/3 & -1 & -4/3 \\ 0 & 0 & 1 & \frac{-3}{11} & \frac{-17}{11} \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{2}{3}R_3 \\ R_2 \to R_2 - \frac{1}{3}R_3 \\ R_4 \to R_4 - R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 2/11 & 4/11 \\ 0 & 1 & 0 & \frac{-10}{11} & \frac{-9}{11} \\ 0 & 0 & 1 & \frac{-3}{11} & \frac{-17}{11} \\ 0 & 0 & 0 & \frac{-8}{11} & \frac{-16}{11} \end{bmatrix} \quad R_4 \to \frac{R_4}{\left(\frac{-8}{11}\right)}$$

4th pivot

$$\begin{bmatrix} 1 & 0 & 0 & \frac{2}{11} & 4/11 \\ 0 & 1 & 0 & \frac{-10}{11} & \frac{-9}{11} \\ 0 & 0 & 1 & \frac{-3}{11} & \frac{-17}{11} \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{2}{11}R_4 \\ R_2 \to R_2 + \frac{10}{11}R_4 \\ R_3 \to R_3 + \frac{3}{11}R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This is the reduced row echelon form, the respective system

$$\boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= -1 \\ x_4 &= 2 \end{aligned}}$$

Pb:2 Using Gauss Jordan method solve the system

$$-x_1 + x_2 - x_3 + 3x_4 = 0$$
$$3x_1 + x_2 - x_3 - x_4 = 0$$
$$2x_1 - x_2 - 2x_3 - x_4 = 0$$

Soln:- Augmented matrix corresponds to this system is

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} \quad R_1 \to (-1)R_1 \quad \begin{bmatrix} 1 & -1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

1st pivot

$$\begin{bmatrix} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \quad R_2 \to \frac{R_2}{(4)} \quad \begin{bmatrix} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - R_1 \end{matrix}$$

2nd pivot

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{bmatrix} \quad R_3 \to \frac{R_3}{(-3)} \quad \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \quad R_2 \to R_2 + R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

This is the reduced row echelon form and the reduced system is

$$x_1 - x_4 = 0$$
$$x_2 + x_4 = 0$$
$$x_3 - x_4 = 0$$

clearly this system has infinite solution. To find all solution take $x_4 = t$ then $x_3 = t$; $x_2 = -t$; $x_1 = t$

Hence $x_1 = t$; $x_2 = -t$; $x_3 = t$; $x_4 = t$ for all real t.

The leading is corresponds to the variables x_1, x_2, x_3 and so x_1, x_2, x_3 are basic variables, x_4 is the free variable.

Elementary matrices:- A matrix E obtained from the identity matrix by executing only one row elementary operation is called elementary matrix.

Example:- (i) Consider the matrix $\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$. This matrix is elementary because if we perform the operation $R_2 \to \frac{3}{2} R_2$ on the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 \\ 0 & 3/2 \end{bmatrix}$.

(ii) The matrix $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is elementary. Perform the operation $R_1 \longrightarrow R_1 + 3R_2$ on $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yields the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Remark:- Let E be an elementary matrix of order $m \times m$ and A be $m \times n$ matrix then EA is exactly the matrix that is obtained when the same elementary row operation E is executed on A

Invertible matrices:- Let A be a $m \times n$ matrix. A is said to be left invertible if there exists a $n \times m$ matrix B such that

$$BA = I_n$$

where I_n is the identity matrix of order n.

A is said to be right invertible if there exists a $n \times m$ matrix C such that $AC = I_m$ where I_m is the identity matrix of order m.

Example:- Consider the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$ then this matrix has left inverse $B = \begin{bmatrix} \frac{1}{2} & 0 & \alpha \\ 0 & \frac{1}{2} & \beta \end{bmatrix}$ where α, β are any two real numbers

$$BA = I_2$$

Note:- 1) In the previous example left inverse of A is not unique.

2) Right inverse of A does not exist.

Remark:- Let A be a $m \times n$ matrix. Then

(i) A need not be left or right invertible.

(ii) if A is left or right invertible it need not be unique.

Invertibility of a square matrix Let A be a $n \times n$ matrix. If B is the right inverse of A and C is the left inverse of A then $B = C$.

Theorem:- Let A be a $n \times n$ matrix then the following are equivalent.

(1) A has left inverse.

(2) The system $AX = 0$ has only zero solution.

(3) The reduced row echelon form of A is I_n.

(4) A is the product of elementary matrices.

(5) A has right inverse

(6) A is invertible.