Constructing new linear transformations

Let V, W be two vector spaces. Suppose $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. In order to construct a linear transformation from V to W, we proceed by the following steps.

1) Choose n arbitrary elements $\{w_1, w_2, \ldots, w_n\}$ in W.

2) Define a map $T: V \to W$ such that $T(v_1) = w_1, T(v_2) = w_2, \ldots, T(v_n) = w_n$

3) Using (2) find image of any arbitrary element x in V

4) Conclude that T in (2) is a linear transformation.

Pb:1 Construct a linear transformation from $\mathbb{R}^3$ to $P_2(\mathbb{R})$

Ans:- We know $\alpha = \{e_1, e_2, e_3\}$ forms a basis of $\mathbb{R}^3$

consider the arbitrary collection of vectors $\beta = \{1, t^2, 3t+4t^2-1\}$

Define $T: \mathbb{R}^3 \to P_2(\mathbb{R})$ such that $T(e_1) = 1$

$$T(e_2) = t^2$$

$$T(e_3) = 3t + 4t^2 - 1$$

we know, $(x, y, z) = x e_1 + y e_2 + z e_3$

$$T(x,y,z) = x T(e_1) + y T(e_2) + z T(e_3)$$

$$= x \cdot 1 + y \cdot t^2 + z (3t + 4t^2 - 1)$$

$$T(x,y,z) = (x - z) \cdot 1 + 3z t + (y + 4z) t^2$$

This is a linear transformation.

Construct a linear transformation from $P_3(\mathbb{R})$ to $M_{2\times 2}(\mathbb{R})$

Soln:- A basis of $P_3(\mathbb{R})$ is $\{1, t, t^2, t^3\}$

A basis of $M_{2\times 2}(\mathbb{R})$ is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

Define $T: P_3(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ such that

$$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad T(t^2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T(t^3) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $T(a + bt + ct^2 + dt^3) = aT(1) + bT(t) + cT(t^2) + dT(t^3)$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Kernal and Image of a linear transformation

Let V and W be two vector spaces $T: V \to W$ be a linear transformation

Kernal of T is the set defined as

$$Ker(T) = \{ v \in V \mid T(v) = 0_W \}$$

Image of T is the set defined as

$$Im(T) = \{ T(v) \in W \mid v \in V \}$$

Pb: 1 Find the image and kernal of

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{by} \quad T(x, y) = (3x, 3x)$$

Soln:- $\ker(T) = \{ (x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0) \}$

$= \{ (x,y) \in \mathbb{R}^2 \mid (3x, 3x) = (0,0) \}$

$= \{ (x,y) \in \mathbb{R}^2 \mid x = 0 \}$ = Y axis.

$\mathrm{Im}(T) = \{ T(x,y) \in \mathbb{R}^2 \mid (x,y) \in \mathbb{R}^2 \}$

$= \{ (3x, 3x) \in \mathbb{R}^2 \mid (x,y) \in \mathbb{R}^2 \}$

= The line $y = x$.

Pb: 2 Let $T : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ defined by $T(p(t)) = \frac{d}{dt}(p(t))$

Soln:- $\ker(T) = \{ p(t) \in P_3(\mathbb{R}) \mid T(p(t)) = 0 \}$

$= \{ p(t) \in P_3(\mathbb{R}) \mid \frac{d}{dt} p(t) = 0 \}$

= All constant polynomials.

$\operatorname{Im}(T) = \{ T(p(t)) \in P_3(\mathbb{R}) \mid p(t) \in P_3(\mathbb{R}) \}$

= collection of all polynomials of degree atmost 2.

Theorem:- Let V and W be two vectors spaces $T: V \to W$ be a linear transformation then $\ker(T)$ is a subspace of V

$\operatorname{Im}(T)$ is a subspace of W.

Rank-Nullity Theorem:- Let V and W be two vector spaces. $T: V \to W$ be a linear transformation then

$$\dim(\ker(T)) + \dim(\operatorname{Im}(T)) = \dim V.$$

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (x+y,\ x-y)$

Find the dimension of $Ker(T)$ and $Im(T)$.

<u>Soln:-</u> $Ker(T) = \{(x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0)\}$

$= \{(x,y) \in \mathbb{R}^2 \mid x+y=0,\ x-y=0\}$

$= \{(0,0)\}$ $\quad \dim(Ker(T)) = 0.$

By Rank nullity Theorem

$\dim Ker(T) + \dim Im(T) = 2$

$\Rightarrow \dim Im(T) = 2.$

Let $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$ by $T(p(t)) = \frac{d}{dt} p(t)$

Find the $\ker(T)$ and $\operatorname{Im}(T)$.

Soln:- $\ker(T) = \{ p(t) \mid T(p(t)) = 0 \}$

$=$ all constant polynomials.

Thus $\dim \ker(T) = 1$. We know $\dim P_3(\mathbb{R}) = 4$.

By rank nullity theorem, $\dim \operatorname{Im}(T) = 3$.