General observation: consider a system $AX=B$ with m equations and n variables. Then we observe the following.

(i) If $m<n$ (no. of equations < no. of variables) then the system is either consistent or inconsistent. Suppose it is consistent then it has infinite solutions.

Example:-

$$x+y+z+w=2$$
$$x+y+z+w=3$$

Inconsistent

$$x+y+z+w=2$$
$$2x+2y+2z+2w=4$$

Consistent, has infinite solution.

(ii) If $(m \geq n)$ (no. of equations $\geq$ no. of variables) then the system is either consistent or inconsistent. Suppose it is consistent then it has either unique or infinite solutions.

Example:-

$$x+y=2$$
$$x+y=3$$
$$x+y=4$$

Inconsistent

$$x+y=2$$
$$2x+2y=4$$
$$3x+3y=6$$

Consistent, has infinite solution

$$x+y=2$$
$$2x+3y=1$$
$$2x+2y=4$$

Consistent has unique solution.

<u>Augmented matrix</u> consider the system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The augmented matrix for the above system is defined as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

<u>Why augmented matrix</u>

Consider the system

$$x + y = 2 \rightarrow (1)$$
$$x - y = 0 \rightarrow (2)$$

to solve this, (1) + (2) $\Rightarrow$ $2x = 2$

$$x = 1$$

apply $x = 1$ in (2) $\Rightarrow y = 1$

The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

Row 2 → Row 2 − Row 1

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & -2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ -2 \end{matrix}} \end{pmatrix}$$

The system corresponds to the above matrix is

$$x + y = 2$$
$$-2y = -2 \Rightarrow y = 1$$

We learn the following techniques to solve the system

1) Gauss elimination method
2) Gauss Jordan elimination
3) LU decomposition.

<u>Why elimination is preferable</u>:-

Consider $2x + y = 1$

$x + y = 2$ we solve this by determinant method and elimination method.

$① \Rightarrow \quad 2x + y = 1$

$② \times 2 \Rightarrow \quad 2x + 2y = 4$ (3 multiplication)

Subtract $\quad -y = -3$ (1 subtraction)

$y = \frac{-3}{(-1)}$ (1 division)

$\boxed{y = 3}$

Apply $y = 3$ in $① \Rightarrow 2x + 3 = 1$

$\Rightarrow 2x = 1 - 3$ (1 subtract)

$2x = -2$

$x = -1$ (1 division)

7 arithematic operations were used to get x and y.

matrix form of the given system is

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\Delta x = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$ (3 operation)

$\Delta y = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$ (3 operations)

$\Delta = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1$ (3 operations)

$x = \frac{\Delta x}{\Delta} = \frac{-1}{1} = -1$ (1 operation)

$y = \frac{\Delta y}{\Delta} = \frac{3}{1} = 3$ (1 operation)

Totall 11 operations are required.

$\therefore$ complexity is lesser