20BCE1025 Abhishek NN Applied Linear Algebra

MAT 3004 DA1

① $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$ A^{-1} by Gauss-Jordan elimination method.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 3 & -1 & -1 & 2 & 0 & 0 & 1 & 0 \\ -1 & 2 & 3 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 - 3R_1$

$R_4 \leftarrow R_4 + R_1$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -5 & -2 & 1 & 0 & 0 \\ 0 & -4 & -1 & -7 & -3 & 0 & 1 & 0 \\ 0 & 3 & 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow -R_2$

$R_1 \leftarrow R_1 - R_2$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 5 & 2 & -1 & 0 & 0 \\ 0 & -4 & -1 & -7 & -3 & 0 & 1 & 0 \\ 0 & 3 & 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 + 4R_2$

$R_4 \leftarrow R_4 - 3R_2$

1

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 5 & 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & 13 & 5 & -4 & 1 & 0 \\ 0 & 0 & 0 & -13 & -5 & 3 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3/3$

$R_4 \leftarrow R_4/-13$

$R_1 \leftarrow R_1 + R_3$

$R_2 \leftarrow R_2 - R_3$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3/13 & 8/39 & 1/3 & 7/39 \\ 0 & 1 & 0 & 2/3 & 1/3 & 1/3 & -1/3 & 0 \\ 0 & 0 & 1 & 13/3 & 5/3 & -4/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 5/13 & -3/13 & 0 & -1/13 \end{array}\right]$$

$R_2 \leftarrow R_2 - \frac{2}{3}R_4$

$R_3 \leftarrow R_3 - \frac{13}{3}R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3/13 & 8/39 & 1/3 & 7/39 \\ 0 & 1 & 0 & 0 & 1/13 & 19/39 & -1/3 & 2/39 \\ 0 & 0 & 1 & 0 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 5/13 & -3/13 & 0 & -1/13 \end{array}\right]$$

The right part is A^{-1}

solution to $AX = b$ $\quad X = [x_1, x_2, x_3, x_4]$ $\quad b = [1, 0, 1, -1]$

$AX = b \Rightarrow A^{-1}AX = A^{-1}b \Rightarrow IX = A^{-1}b \Rightarrow X = A^{-1}b$

$$X = \underset{4\times 4}{\begin{bmatrix} -3/13 & 8/39 & 1/3 & 7/39 \\ 1/13 & 19/39 & -1/3 & 2/39 \\ 0 & -1/3 & 1/3 & 1/3 \\ 5/13 & -3/13 & 0 & -1/13 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} -1/13 \\ -4/13 \\ 0 \\ 4/13 \end{bmatrix}$$

$\therefore$ $\underline{x_1 = -1/13, \; x_2 = -4/13, \; x_3 = 0, \; x_4 = 4/13}$

② (a) $x+2y-3z=4$, $3x-y+5z=2$, $4x+y+(a^2-14)z=a+2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right] \begin{array}{l} R_2 \leftarrow R_2-3R_1 \\ R_3 \leftarrow R_3-4R_1 \end{array} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -3 & a^2-2 & a-14 \end{bmatrix} R_2 \leftarrow R_2/-7$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & +10/7 \\ 0 & -3 & a^2-2 & a-14 \end{bmatrix} \begin{array}{l} R_3 \leftarrow R_3+3R_2 \\ \cancel{R_1 \leftarrow R_1-2R_2} \end{array} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-8 & a-\frac{68}{7} \end{bmatrix}$$

For no solution Rank(A) < Rank(A|B)

$\Rightarrow a^2-8=0$ and $a-\frac{68}{7} \neq 0 \Rightarrow a = \pm\sqrt{8}$ //

For unique solution Rank(A) = Rank(A|B) = # variables

$\Rightarrow a^2-8 \neq 0$ ~~and $a-68/7 \neq 0$~~ $\Rightarrow a = R - \{\pm\sqrt{8}\}$ //

For ∞ solution Rank(A) = Rank(A|B) ≠ (<) # variables

$\Rightarrow a^2-8=0$ and $a-68/7=0$ ∴ this is not possible

∴ no infinite case for this system of equation

(b) $x-y+z=1$, $x+3y+az=2$, $2x+ay+3z=3$

$$\left[\begin{array}{ccc|c} 1 & -1 & +1 & 1 \\ 1 & 3 & a & 2 \\ 2 & a & 3 & 3 \end{array}\right] \begin{array}{l} R_2 \leftarrow R_2-R_1 \\ R_3 \leftarrow R_3-2R_1 \end{array} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & a-1 & 1 \\ 0 & a+2 & 1 & 1 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2/4 \\ R_3 \rightarrow R_3 - R_2 \end{array}$$

$$\cancel{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & (a-1)/4 & 1/4 \\ 0 & a+2 & 1 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & (a-1)/4 & 1/4 \\ 0 & a-2 & 2-a & 0 \end{bmatrix}$$

⇒ For no solution Rank(A) < Rank(A|B) which is not possible because if a = 2 Rank(A) = Rank(B) = 2 or else it needs to be reduced further

⇒ For unique solution Rank(A) = Rank(A|B) = # variables

Not possible same reason as for no solution

⇒ For ∞ solution Rank(A) = Rank(A|B) < # variables

$\Rightarrow a-2=0,\ 2-a=0 \Rightarrow a=2$ //

③ $V = \{ f(x) ; a\cos(2x) + b\sin(2x) ; a, b \in R \}$

$f_1(x), f_2(x) \in V$, $f_1(x) = a_1\cos(2x) + b_1\sin(2x)$

$f_2 = a_2\cos(2x) + b_2\sin(2x)$

$f_1(x) + f_2(x) = (a_1+a_2)\cos(2x) + (b_1+b_2)\sin(2x)$

$\alpha f(x) = \alpha a\cos(2x) + \alpha b\sin(2x)$

Commutativity of Vector Addition to prove $f_2(x) + f_1(x)$ is equal to $f_1(x) + f_2(x)$

$f_2(x) + f_1(x) = (a_2+a_1)\cos(2x) + (b_2+b_1)\sin(2x)$

$= (a_1+a_2)\cos(2x) + (b_1+b_2)\sin(2x) = f_1(x) + f_2(x)$

($\because$ Real numbers are commutative)

Associativity of vector addition to prove

$f_1(x) + (f_2(x) + f_3(x)) = (f_1(x) + f_2(x)) + f_3(x)$

$= f_1(x) + ((a_2+a_3)\cos(2x) + (b_2+b_3)\sin(2x))$

$= (a_1 + (a_2+a_3))\cos(2x) + (b_1 + (b_2+b_3))\sin 2x$

$= (a_1+a_2+a_3)\cos(2x) + (b_1+b_2+b_3)\sin 2x$ LHS//

$= (a_1+a_2)\cos 2x + (b_1+b_2)\sin 2x + f_3(x)$

$= ((a_1+a_2)+a_3)\cos(2x) + ((b_1+b_2)+b_3)\sin(2x)$

$= (a_1+a_2+a_3)\cos 2x + (b_1+b_2+b_3) =$ RHS $=$ LHS//

Identity element of vector adition

$0_V = 0\cos(2x) + 0\sin(2x)$

$v + 0_V = (a+0)\cos(2x) + (b+0)\sin(2x) = v$

Inverse of vector addition

$-v = -a\cos(2x) - b\sin(2x)$

$v + -v = (a-a)\cos 2x + (b-b)\sin 2x = 0$

compatiblity of scalar multiplication with field multiplication ~~a(bv) = (ab)v~~ $p(qv) = (pq)v$

$p(qv) = p(q a\cos 2x + q b\sin(2x)) = pqa\cos 2x + pq\sin 2x$

$= (pq)(a\cos 2x + b\sin 2x) = (pq)v$

Identity element of scalar multiplication

$1 \cdot v = 1 \cdot a\cos(2x) + 1 \cdot b\sin 2x = v$//

Distributivity of scalar multiplication with respect to vector addition $u(f_1(x) + f_2(x)) = uf_1(x) + uf_2(x)$

$u((a_1+a_2)\cos(2x) + (b_1+b_2)\sin 2x) =$

$u(a_1\cos(2x) + b_1\sin 2x + a_2\cos 2x + b_2\sin 2x)$

$(ua_1\cos 2x + ub_1\sin 2x) + (ua_2\cos 2x + ub_2\sin 2x) = uf_1(x) + uf_2(x)$

Distributivity of scalar multiplication with respect to field addion ~~(a+b)~~ · ~~(u+v)~~ $(p+q)v = pv + qv$

$(p+q)v = (p+q)a\cos 2x + (p+q)b\sin 2x$

$= (pa\cos 2x + pb\sin 2x) + (qa\cos 2x + qb\sin 2x)$

$= pv + qv$

④ $V = \{ (1,3,-2,2,3),\ (1,+4,-3,4,2),\ (1,3,0,2,3) \}$

$W = \{ (2,3,-1,-2,9),\ (1,5,-6,6,1),\ (2,4,4,2,8) \}$

$\dim V = 3$, $\dim W = 3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 3 & 5 & 4 \\ -2 & -3 & 0 & -1 & -6 & 4 \\ 2 & 4 & 2 & -2 & 6 & 2 \\ 3 & 2 & 3 & 9 & 1 & 8 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - 2R_1 \\ R_5 \to R_5 - 3R_1 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & -3 & 2 & -2 \\ 0 & -1 & 2 & 3 & -4 & 8 \\ 0 & 2 & 0 & -6 & 4 & -2 \\ 0 & -1 & 0 & 3 & -2 & 2 \end{bmatrix}$$

$$\begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 2R_2 \\ R_5 \to R_5 + R_2 \end{array} \quad \begin{bmatrix} 1 & 0 & 1 & 5 & -1 & 4 \\ 0 & 1 & 0 & -3 & 2 & -2 \\ 0 & 0 & 2 & 0 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3/2 \\ R_4 \to R_4/2 \end{array} \quad \begin{bmatrix} 1 & 0 & 1 & 5 & -1 & 4 \\ 0 & 1 & 0 & -3 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 4 leading entry thus $\dim(W + \overset{V}{\cancel{W}}) = 4$ //
and two entries without leading element they are free variables $\therefore \dim N(Q) = \dim(W \cap \overset{V}{\cancel{W}}) = 2$ //

$\therefore \dim(W+V) + \dim(W \cap V) = \dim W + \dim V$

$4 + 2 = 3 + 3$ // $= 6$ //

⑤ ⓐ $A = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$, $R_2 \leftarrow -R_2$, $R_3 \leftarrow -R_3$ $\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 0 & +5 & +10 \\ 0 & 0 & +2 & +4 \end{bmatrix}$ $R_3 \leftarrow 5R_3 - 2R_2$, $R_2 \leftarrow R_2/5$

$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Row space Basis $= \{(1,2,1,5), (0,0,1,2)\} = \cancel{R(A)}$

dim $(R(A)) = 2$ //

Column space Basis $C(A) = \{(1,2,1), (1,-3,-1)\}$,

dim $(C(A)) = 2$ //

$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad x_1 + 2x_2 + x_3 + 5x_4 = 0$

$x_3 + 2x_4 = 0$

here x_1 and x_3 are basic variables, x_2 and x_4 are free variables, let $x_2 = p$, $x_4 = q$

$x_3 = -2q$, $x_1 = -2p - 3q$

$\therefore (x_1, x_2, x_3, x_4) = (-2p - 3q, p, -2q, q)$

$= (-2p, p, 0, 0) + (-3q, 0, -2q, q)$

$\therefore$ basis of Null space $= \{(-2,1,0,0), (-3,0,-2,1)\}$,

dim (~~basis~~ Null space) $= 2$ //

ⓑ $\begin{bmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{bmatrix}$ $R_3 \leftarrow R_3 - 2R_2$, $R_5 \leftarrow R_5 - 3R_2$ $\begin{bmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 2 & -2 & -4 & 7 \end{bmatrix}$ $R_3 \leftarrow R_3 + R_1$, $R_5 \leftarrow R_5 - 2R_1$

$\begin{bmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ $R_5 \leftarrow 2R_5 - 5R_3$, $R_4 \leftarrow R_4/-2$, $R_3 \leftarrow R_3/2$ $\begin{bmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Basis of Row space $= \{(1,1,-1,3,1), (0,1,-1,-2,1), (0,0,1,-1,-1/2), (0,0,0,0,1)\}$ dim $(R(A)) = 4$ //

Basis of $(C(A)) = \{(0,1,2,0,3), (1,1,1,0,+5), (-1,-1,-1,-2,-5), (1,1,3,1,10)\}$ dim $(C(A)) = 4$ //

$\begin{bmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad x_1 + x_2 - x_3 + 3x_4 + x_5 = 0$

$x_2 - x_3 - 2x_4 + x_5 = 0$

$x_3 - x_4 - 1/2 x_5 = 0$

$x_5 = 0$

~~The~~ Here x_4 is free variable let $x_4 = t$, $\Rightarrow x_3 = t$

$x_2 = 3t$, $x_1 = -5t$

$\therefore (x_1, x_2, x_3, x_4, x_5) = (-5t, 3t, t, t, 0) = t(-5,3,1,1,0)$

basis of null space $= \{(-5,3,1,1,0)\}$ // ~~dim (Null~~

dim (Null space) $= 1$ //