太极图形课

第11讲 Fluid Simulation 02: The Grid-based Methods

taichi

Season Finale Alert

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  - Compiler Architect @ Taichi Graphics
  - THU → UIUC → Facebook (PyTorch) → Taichi

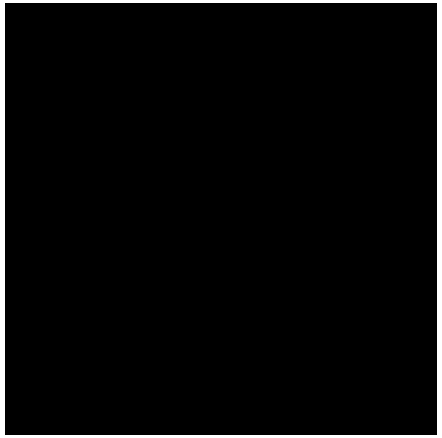
- Dec. 14th:
  - 手把手教你如何向Taichi仓库贡献代码，成为Taichi开发者

Where are we?

Procedural Animation

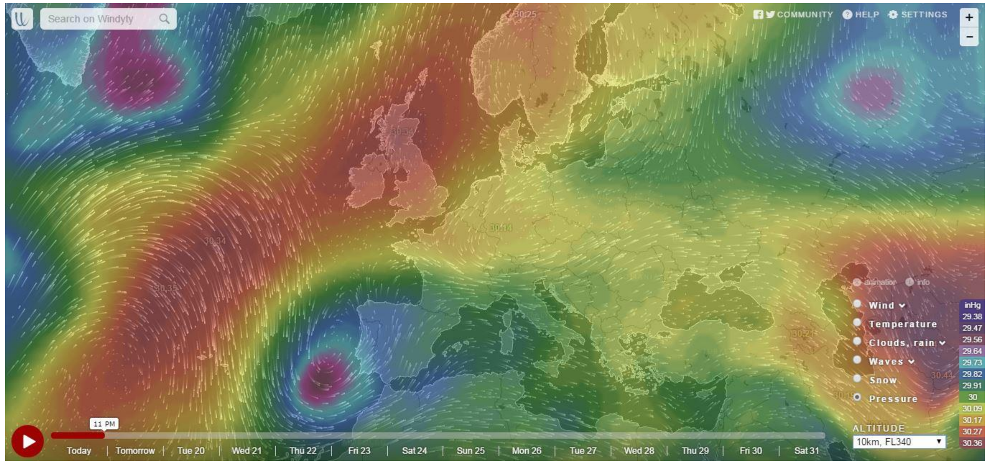
Rendering

Deformable Simulation

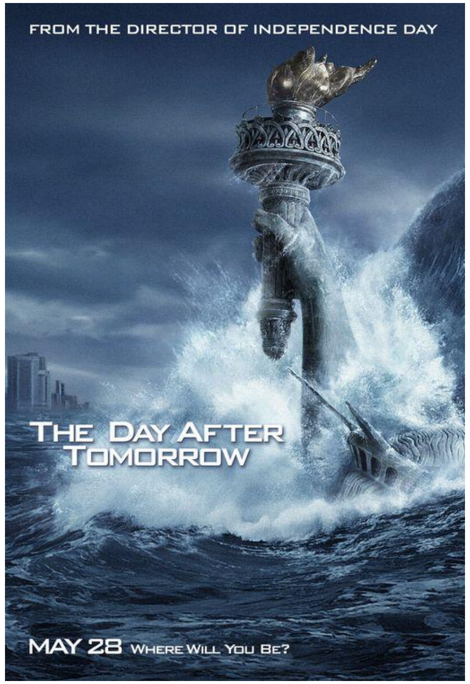


Fluid Simulation

Fluid simulation

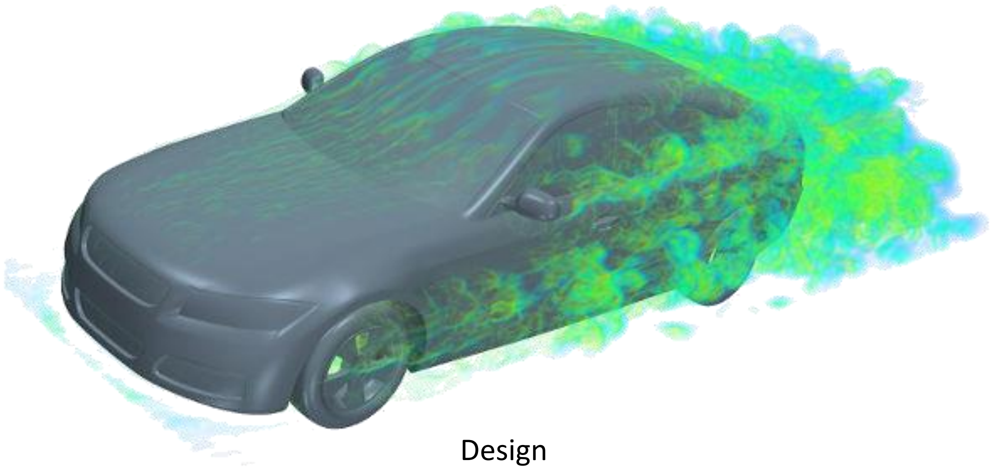


Forecast



VFX

Game

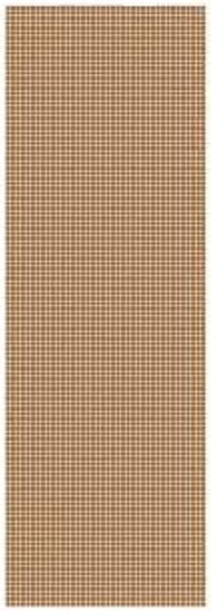


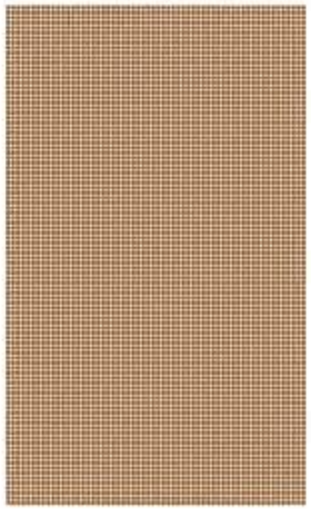
Design

Animation

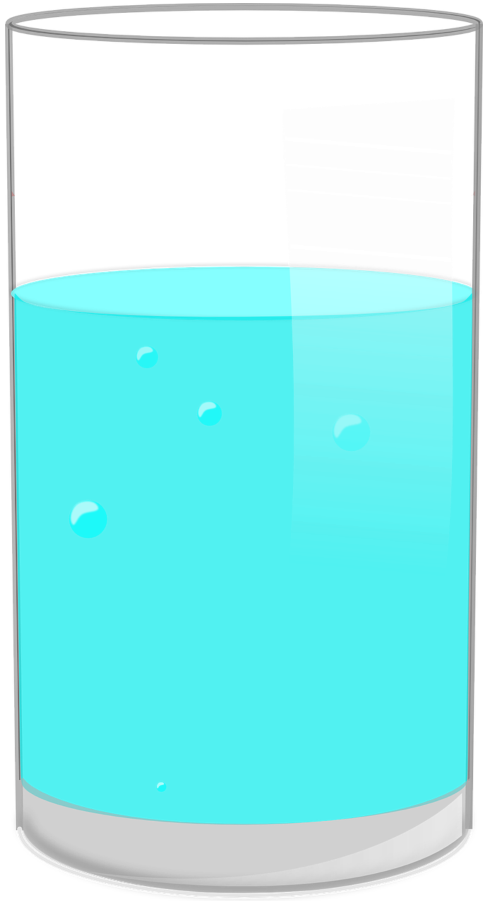
Recap

- Incompressible fluid dynamics
    - Incompressible Navier–Stokes equations
- Time discretization
    - Operator splitting
    - Integration with the weakly compressible assumption
- Spatial discretization
    - Smoothed particle hydrodynamics (SPH)
- Implementation details (WCSPH)
    - Simulation Pipeline
    - Boundary conditions
    - Neighbor search



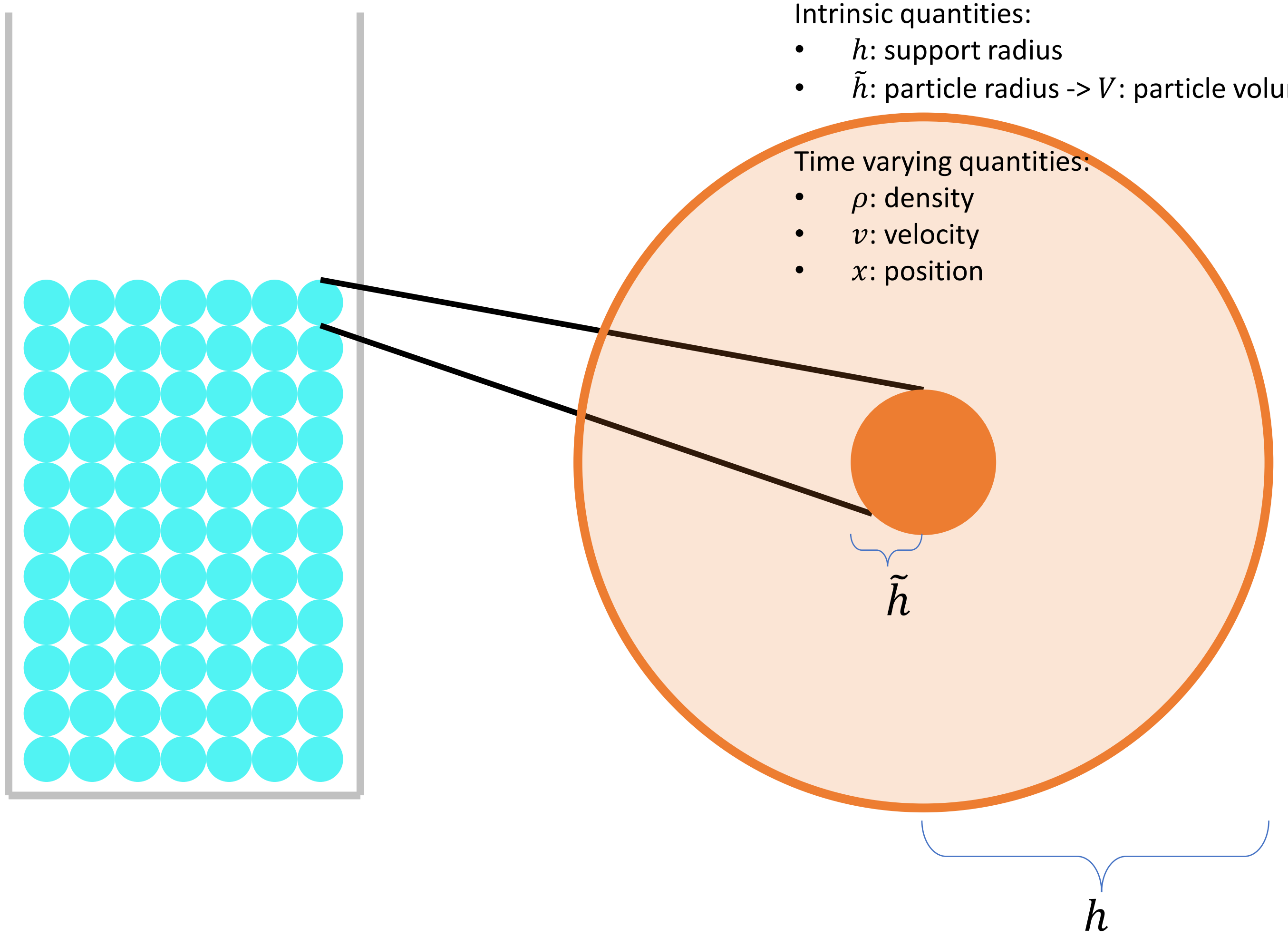


Lagrangian view

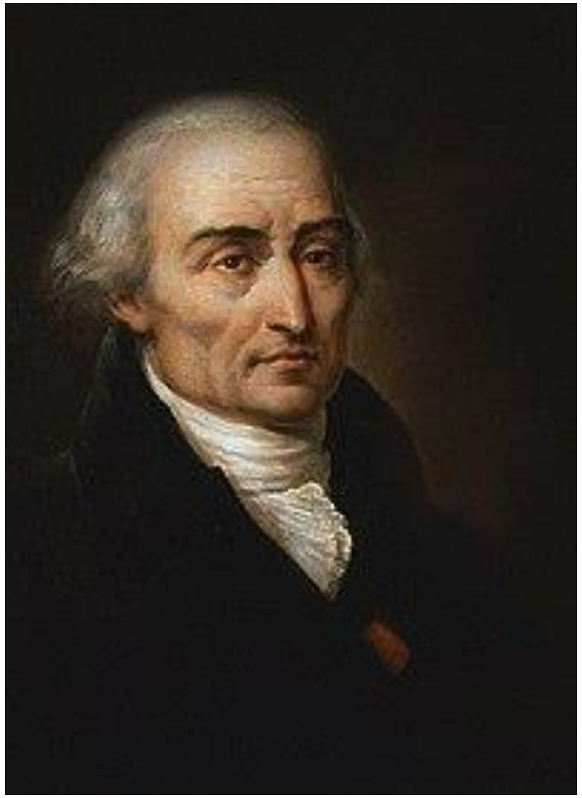


Intrinsic quantities:
- h: support radius
- $\tilde{h}$: particle radius -> V: particle volume

Time varying quantities:
- ρ: density
- v: velocity
- x: position

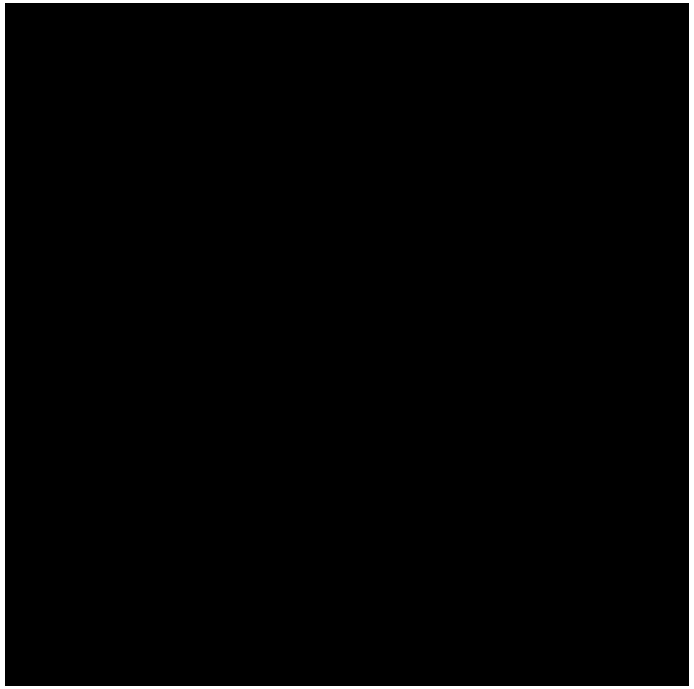


What will be covered today…



Code of the day

- Code:
  - https://github.com/taichi-dev/taichi/blob/master/python/taichi/examples/simulation/stable_fluid.py
- Code courtesy of 刘嘉枫 [@Hanke98]

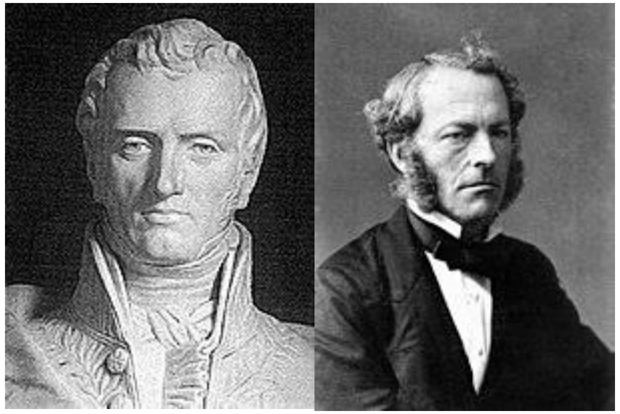


Outline today

- N-S equations and their time integration
  - Operator splitting
- From the Lagrangian view to the Eulerian view
  - Spatial derivatives using finite difference
  - MAC grid
- Advection
  - Material derivative
  - Quantity advection
- Projection
  - Poisson’s equation
  - Boundary conditions

N-S equations and the time integration

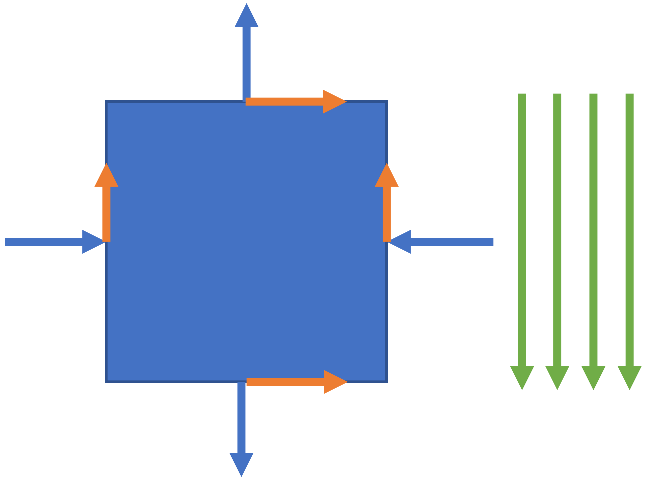
Incompressible Navier-Stokes equation



$$ma = f_{ext} + f_{pres} + f_{visc}$$

$$\rho \frac{Dv}{Dt} = \rho g - \nabla p + \mu \nabla^2 v$$

$$\nabla \cdot v = 0$$



Operator splitting: a toy example

- Let's integrate $\frac{dq}{dt} = 1 + 2$
- (We know that the answer is:$q^{n+1} = q^n + 3\Delta t$)

- Operator splitting:
  - $\tilde{q} = q^n + 1\Delta t$
  - $q^{n+1} = \tilde{q} + 2\Delta t$

Operator splitting: a general example

- Let's integrate $\frac{dq}{dt} = f(q) + g(q)$

- Operator splitting:
  - $\tilde{q} = q_n + \Delta t f(q^n)$
  - $q^{n+1} = \tilde{q} + \Delta t g(\tilde{q})$

Operator splitting: N-S equations

- Let's integrate $\frac{Dv}{Dt} = g - \frac{1}{\rho}\nabla p + \nu\nabla^2 v$

  $\nabla \cdot v = 0$

- Operator splitting:
  - Advection: $\frac{Dq}{Dt} = 0$, where q can be velocity, density, temperature etc.
  - Applying forces: $\frac{\partial v}{\partial t} = g + \nu\nabla^2 v$
  - Projection: $\frac{\partial v}{\partial t} = -\frac{1}{\rho}\nabla p \; s.t. \nabla \cdot v = 0$

One numerical time-stepping for N-S equations

- Given q^n, where q can be velocity, density, temperature etc.
  - Step 1 Advection:
    - $q^{n+1} = advect(v^n, \Delta t, q^n)$
    - $\tilde{v} = advect(v^n, \Delta t, v^n)$
  - Step 2 Applying forces:
    - $\tilde{\tilde{v}} = \tilde{v} + \Delta t(g + \nu\nabla^2\tilde{v})$
  - Step 3 Projection:
    - $v^{n+1} = project(\Delta t, \tilde{\tilde{v}})$
  - Return v^{n+1}, q^{n+1}

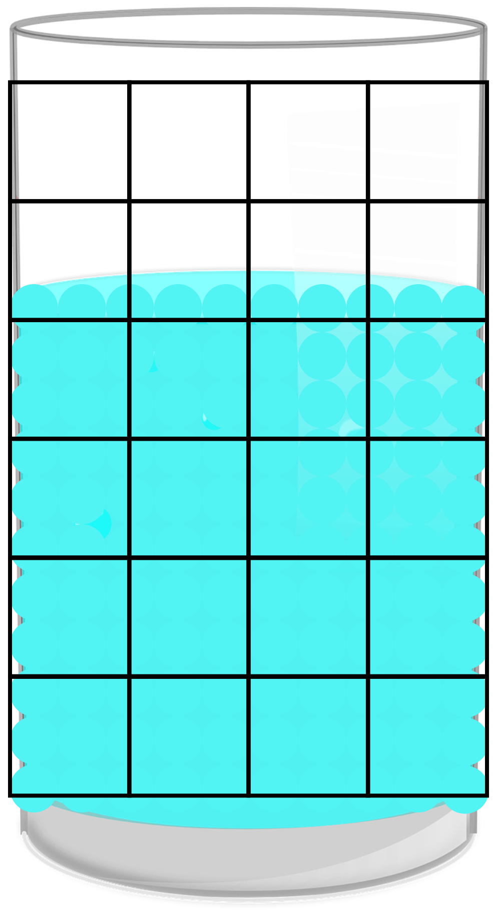
$$\frac{Dv}{Dt} = g - \frac{1}{\rho}\nabla p + \nu\nabla^2 v$$

$$\nabla \cdot v = 0$$

From the Lagrangian view to the Eulerian view

Lagrangian view v.s. Eulerian view

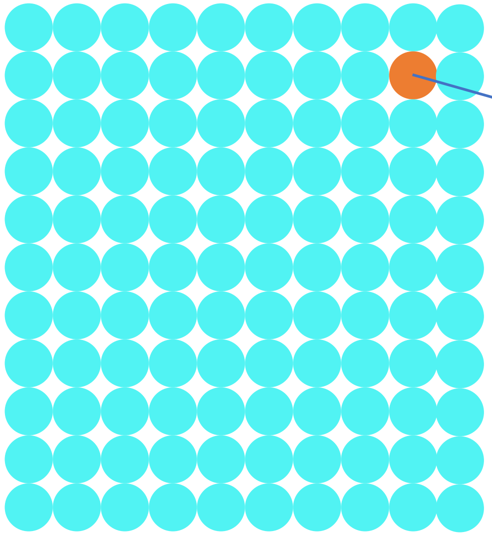
- Lagrangian view：



- Eulerian view：

Lagrangian view v.s. Eulerian view

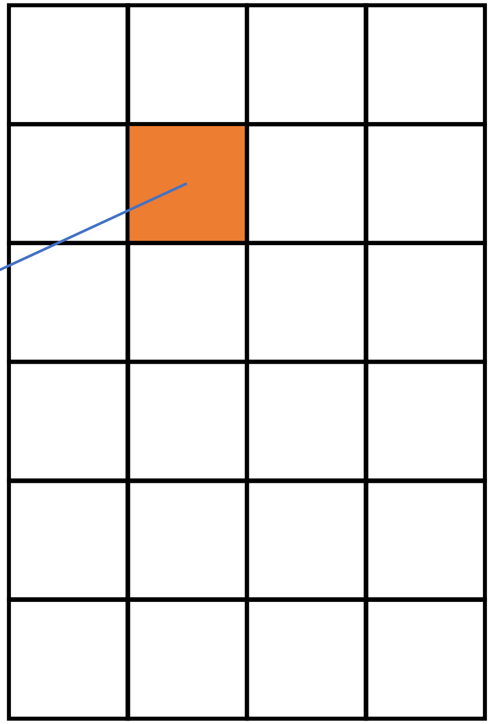
- Lagrangian view：



Position
Velocity
Density
Temperature
Volume
Support radius
…
etc.

Dynamic Markers

- Eulerian view：

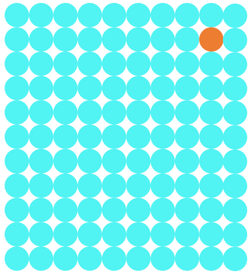


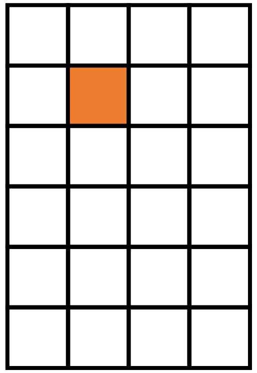
Grid index
Velocity
Density
Temperature
Grid size
…
etc.

Static Markers

Pros and cons

- Lagrangian view:
  - Pros:
    - Advection (Quantity preservation)
    - Boundary condition (Conformal discretization)
    - Coupling with solids
  - Cons:
    - Spatial derivative
    - High spatial discretization error
    - Neighbor search
    - Unbounded distortion
    - Explicit collision handling

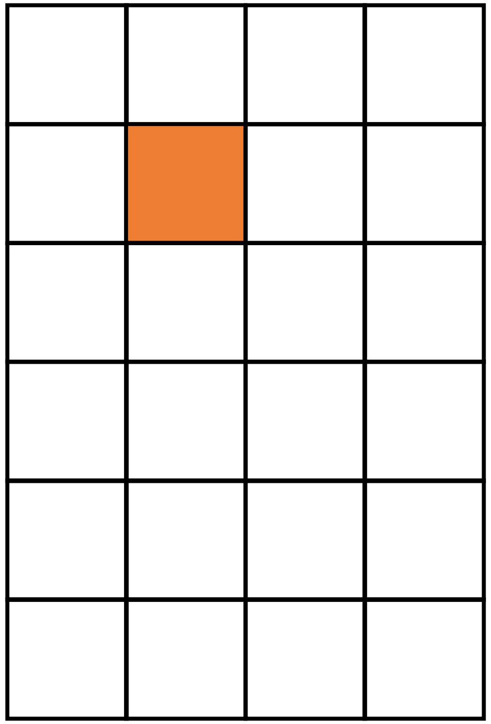




- Eulerian view:
  - Pros:
    - Spatial derivative for free (finite difference)
    - Low spatial discretization error
    - Fixed topology (good for neighbor search)
    - Bounded distortion
    - Collision free
  - Cons:
    - Advection
    - Boundaries
    - Coupling with solids

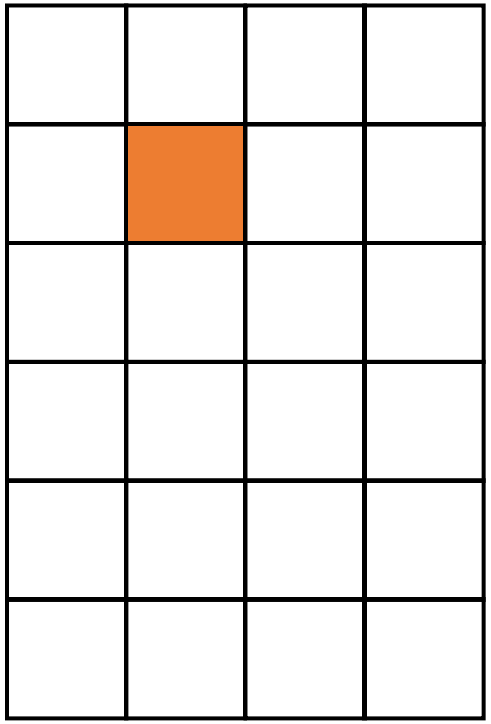
Spatial derivatives under the Eulerian viewpoint

- Spatial derivative in a grid:
  - $\nabla q_{i,j,k} = \begin{bmatrix} \partial q_{i,j,k}/\partial x \\ \partial q_{i,j,k}/\partial y \\ \partial q_{i,j,k}/\partial z \end{bmatrix}$
  - The dimensions can be ***decoupled*** when computing the spatial derivatives due to the ***structural grid***.



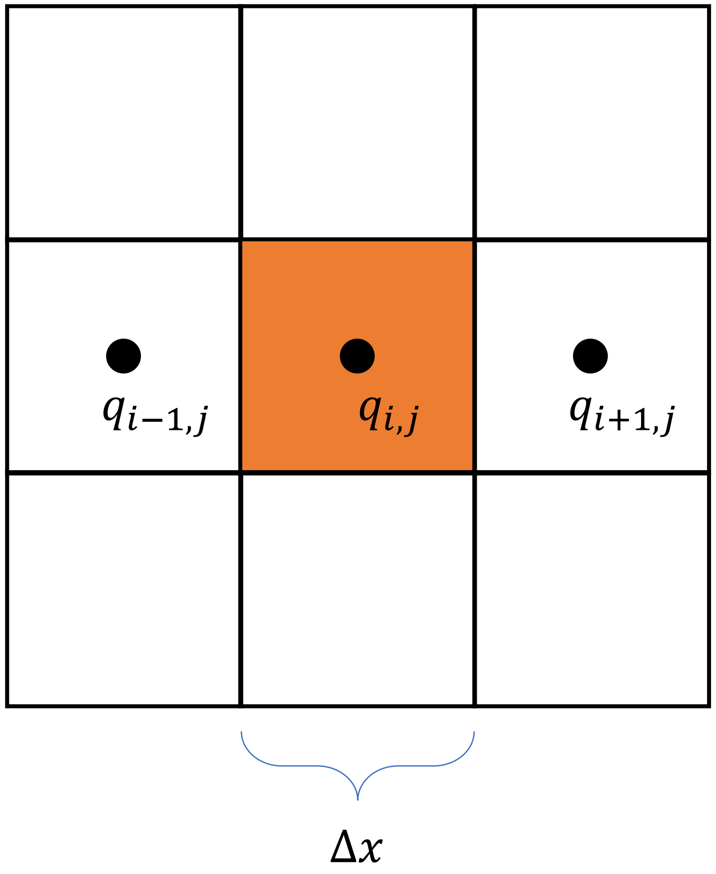
Spatial derivatives under the Eulerian viewpoint

- Spatial derivative in a grid:
  - $\nabla q_{i,j,k} = \begin{bmatrix} \partial q_{i,j,k}/\partial x \\ \partial q_{i,j,k}/\partial y \\ \partial q_{i,j,k}/\partial z \end{bmatrix}$
  - The dimensions can be ***decoupled*** when computing the spatial derivatives due to the ***structural grid***.



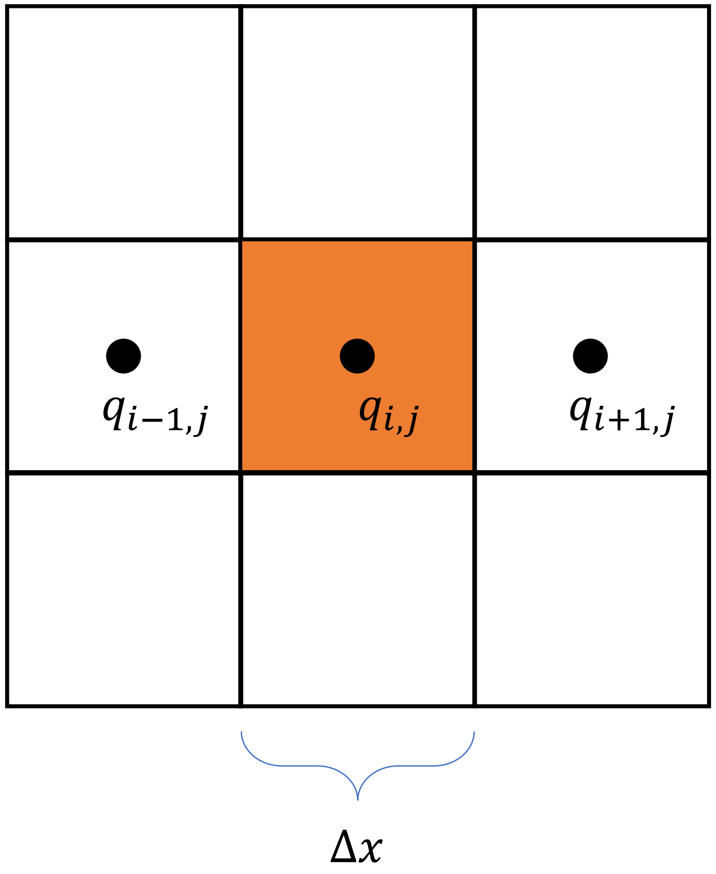
How to compute $\partial q/\partial x$?

- Finite difference, two options:
  - Forward difference:
    - $\left(\frac{\partial q}{\partial x}\right)_i \approx \frac{q_{i+1}-q_i}{\Delta x}$
    - Accurate to $\mathcal{O}(\Delta x)$
    - Biased
  - Central difference:
    - $\left(\frac{\partial q}{\partial x}\right)_i \approx \frac{q_{i+1}-q_{i-1}}{2\Delta x}$
    - Accurate to $\mathcal{O}(\Delta x^2)$
    - Unbiased



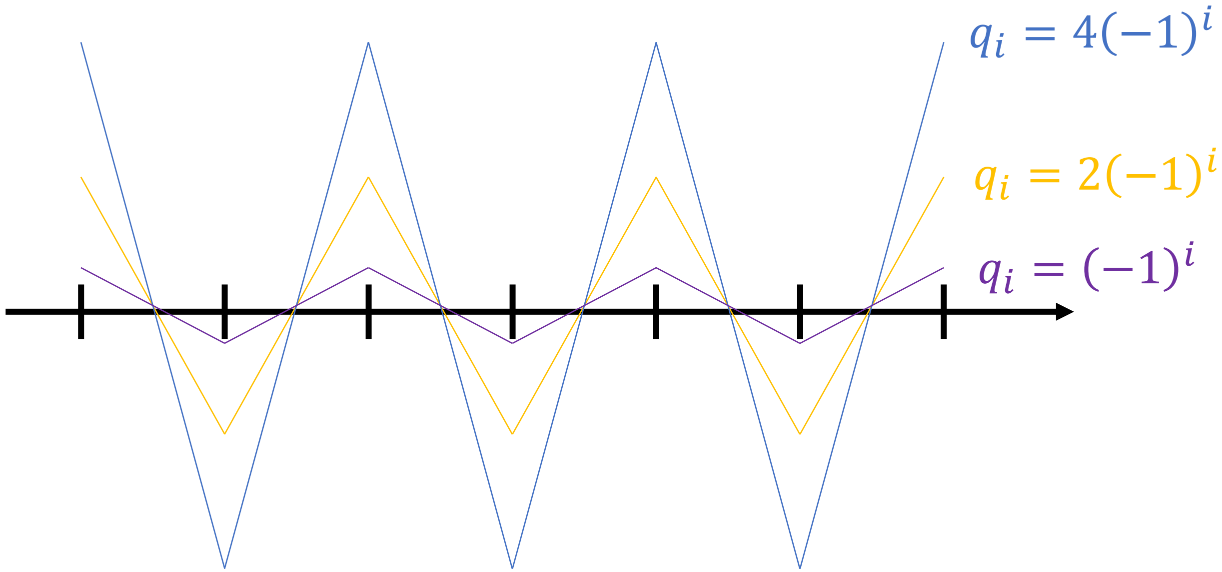
How to compute $\partial q / \partial x$?

- Finite difference, two options:
  - Forward difference:
    - $\left(\frac{\partial q}{\partial x}\right)_i \approx \frac{q_{i+1} - q_i}{\Delta x}$
    - Accurate to $\mathcal{O}(\Delta x)$
    - Biased
  - Central difference:
    - $\left(\frac{\partial q}{\partial x}\right)_i \approx \frac{q_{i+1} - q_{i-1}}{2\Delta x}$
    - Accurate to $\mathcal{O}(\Delta x^2)$
    - Unbiased



The problem of central difference $\left(\frac{\partial q}{\partial x}\right)_i \approx \frac{q_{i+1} - q_{i-1}}{2\Delta x}$:

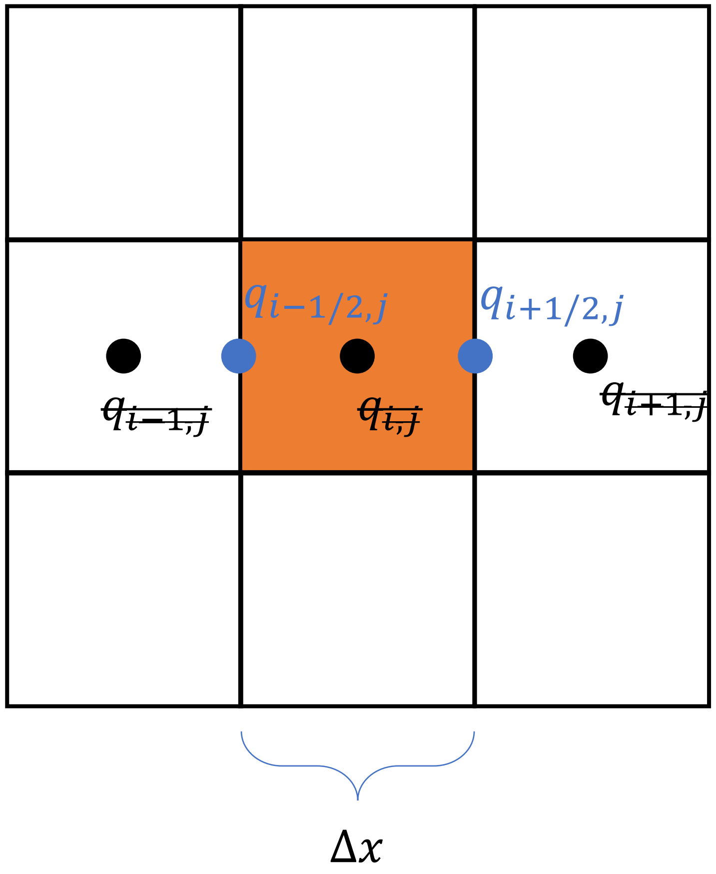
- Non-constant functions are able to register a zero spatial derivative:



Solution: central difference with a "staggered" grid

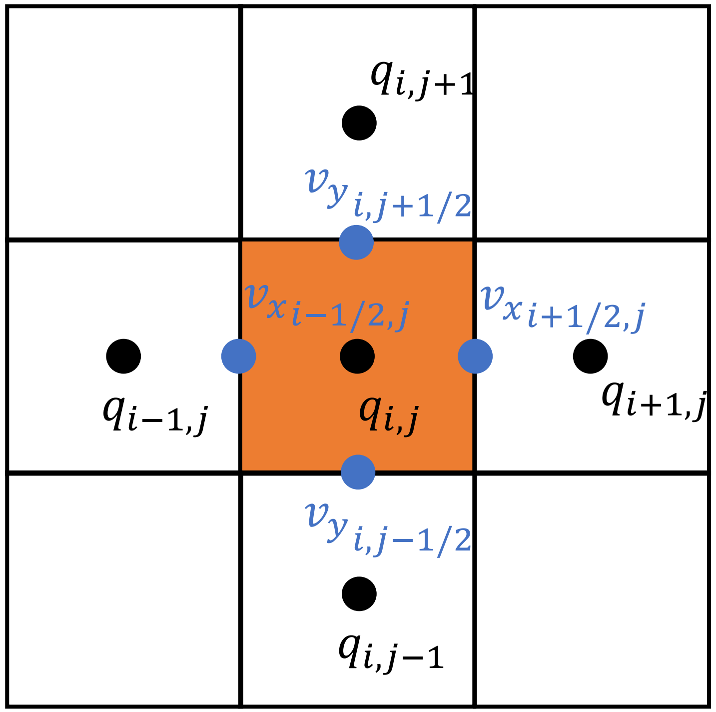
- $\left(\frac{\partial q}{\partial x}\right)_i \approx \frac{q_{i+1/2} - q_{i-1/2}}{\Delta x}$
- Also accurate to $\mathcal{O}(\Delta x^2)$
- Unbiased

- Usually we store the ***velocity*** using the staggered fashion.
- … and store the other (scalar) quantities in the grid centers:
  - e.g. temperature / density / pressure



Staggered grid for fluid simulation

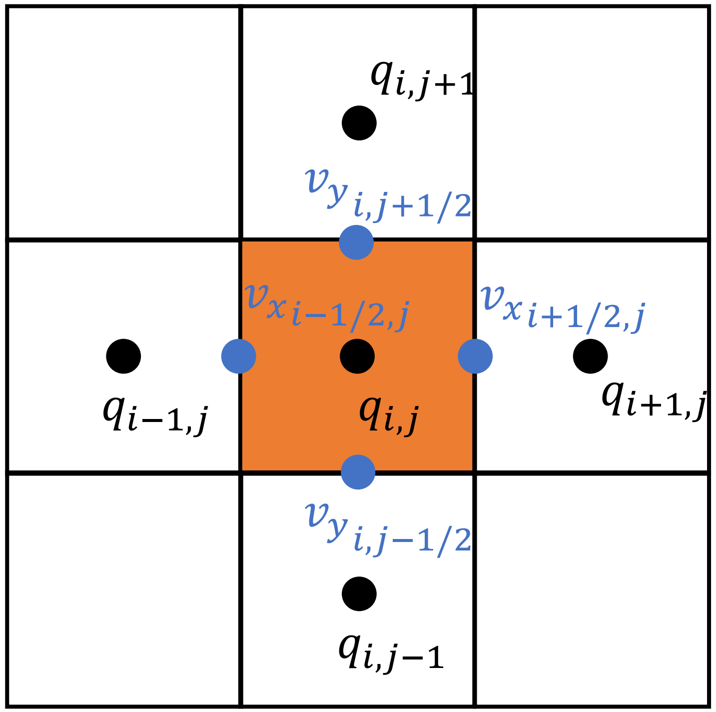
- Compositing a velocity vector in a staggered grid:
  - $v_{i,j} = \left[\frac{v_{x\,i-1/2,j} + v_{x\,i+1/2,j}}{2}, \quad \frac{v_{y\,i,j-1/2} + v_{y\,i,j+1/2}}{2}\right]$
  - $v_{i+1/2,j} = \left[v_{x_{i+1/2,j}}, \quad \frac{v_{y\,i,j-1/2} + v_{y\,i,j+1/2} + v_{y\,i+1,j-1/2} + v_{y\,i+1,j+1/2}}{4}\right]$
  - $v_{i,j+1/2} = \left[\frac{v_{x\,i-1/2,j} + v_{x\,i+1/2,j} + v_{x\,i-1/2,j+1} + v_{x\,i+1/2,j+1}}{4}, \quad v_{y\,i,j+1/2}\right]$



Note: The staggered grid is first introduced to computational fluid dynamics by Harlow and Welch [1965]. It was called the Marker-and-Cell (MAC) method. Sometimes the staggered grid is also called the MAC grid.

Quiz:

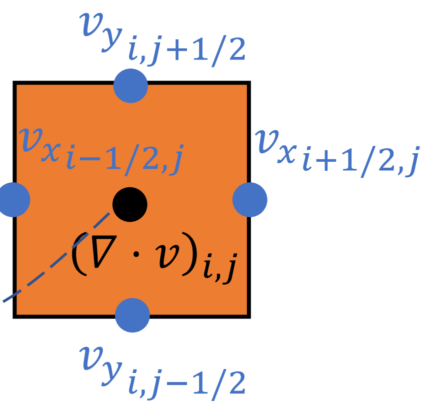
- For a $row \times col$ grid ($row = 3, col = 3$):
  - How many temperature T values do we need to store? $row \times col = 9$
  - How many horizontal velocity v_x values do we need to store? $row \times (col + 1) = 12$
  - How many vertical velocity v_y values do we need to store? $(row + 1) \times col = 12$



Staggered grid: another viewpoint (optional)

- Stokes Theorem (exterior calculus):

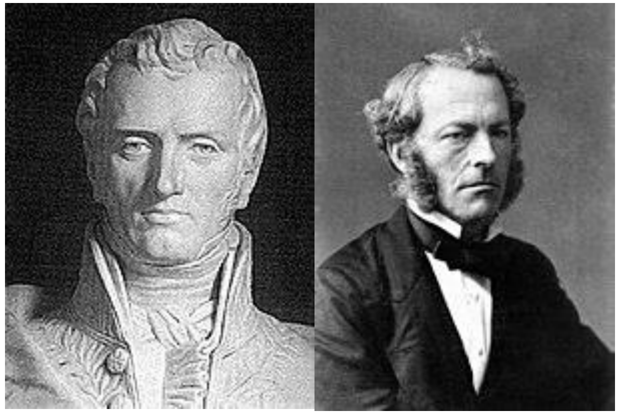
$$\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega$$



Further Readings:
Discrete Differential Geometry: An Applied Introduction [Crane 2019][Course][Video]
《简明微积分》-- 龚昇

Advection

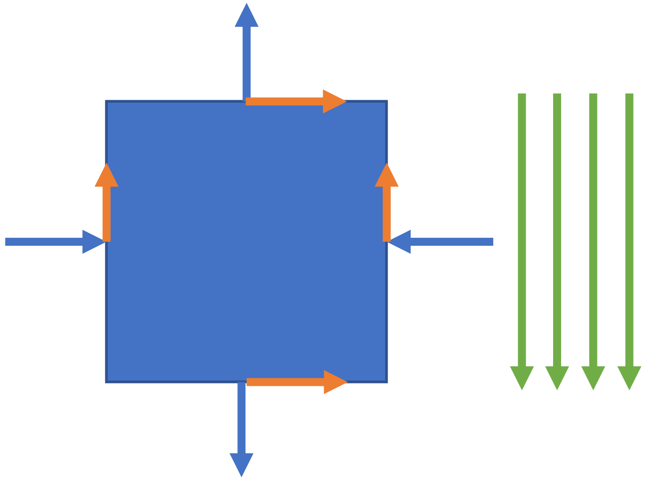
Revisit incompressible Navier-Stokes equation



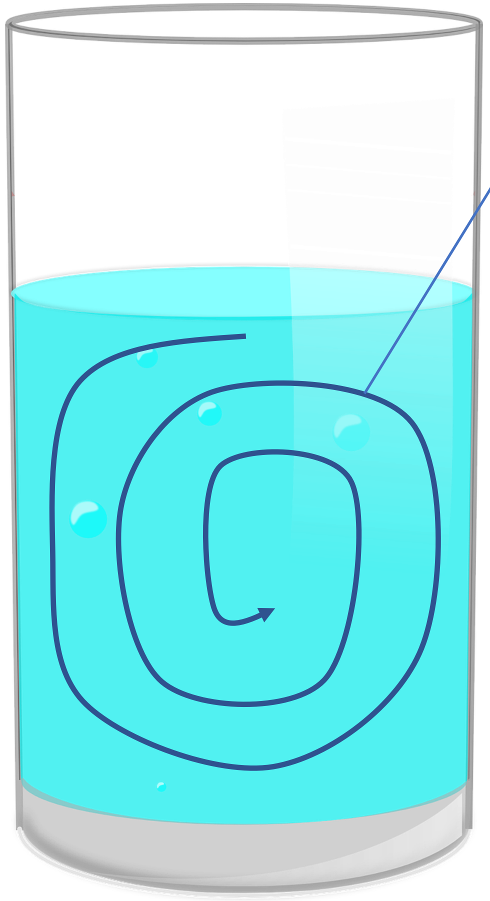
$$ma = f_{ext} + f_{pres} + f_{visc}$$

$$\rho \frac{Dv}{Dt} = \rho g - \nabla p + \mu \nabla^2 v$$

$$\nabla \cdot v = 0$$



Material derivative $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v \cdot \nabla f$

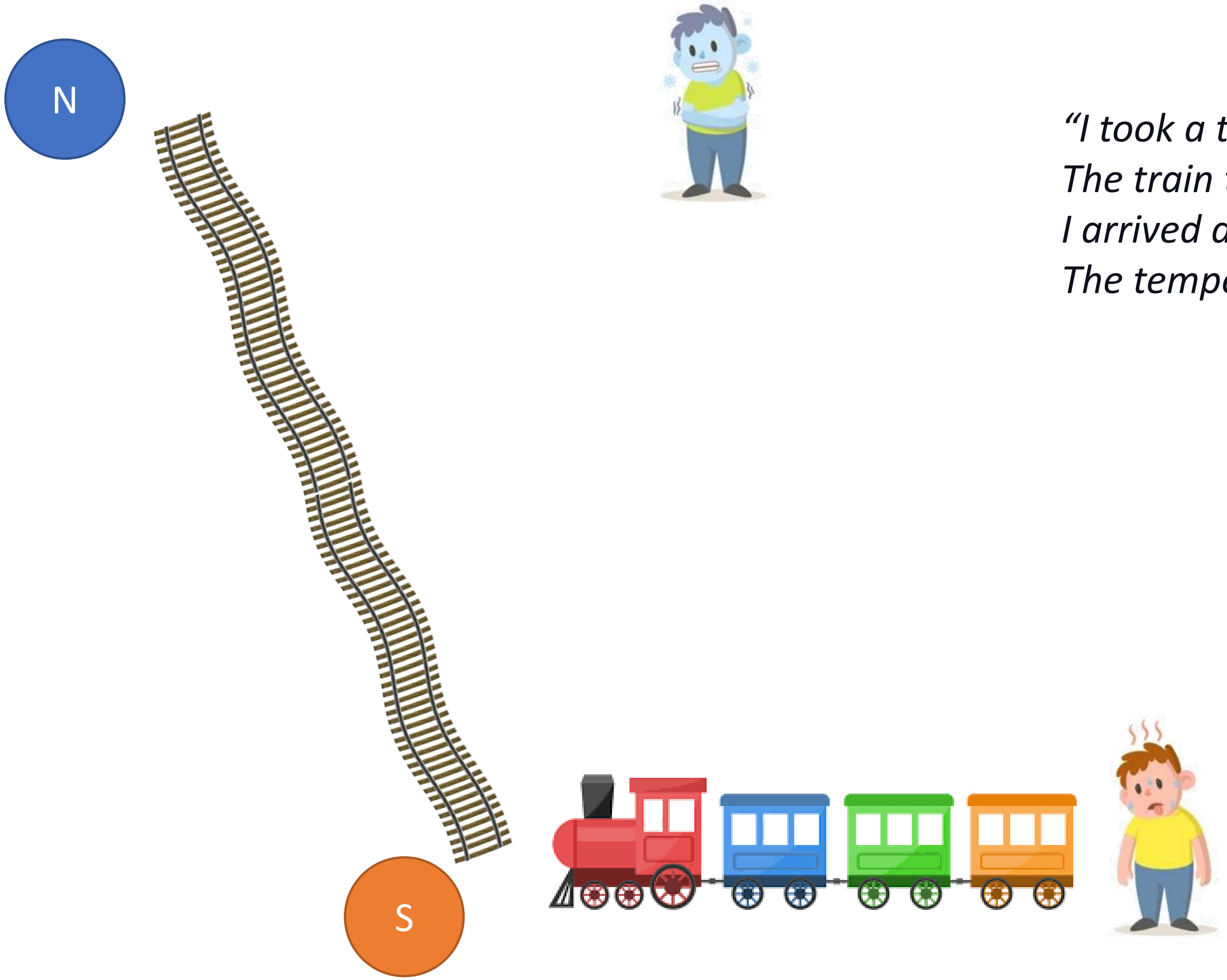


- Some functions we want to evaluate depends on both ***space*** and ***time*** coordinates:
  - $f = f(x, t)$
- We have the **total derivative w.r.t. *time*** of f expanded using chain rule:
  - $\frac{d}{\mathrm{dt}} f(x,t) = \frac{\partial f}{\partial t} + \frac{dx}{dt} : \frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} + v \cdot \nabla f$

- Other names for $\frac{Df}{Dt}$:

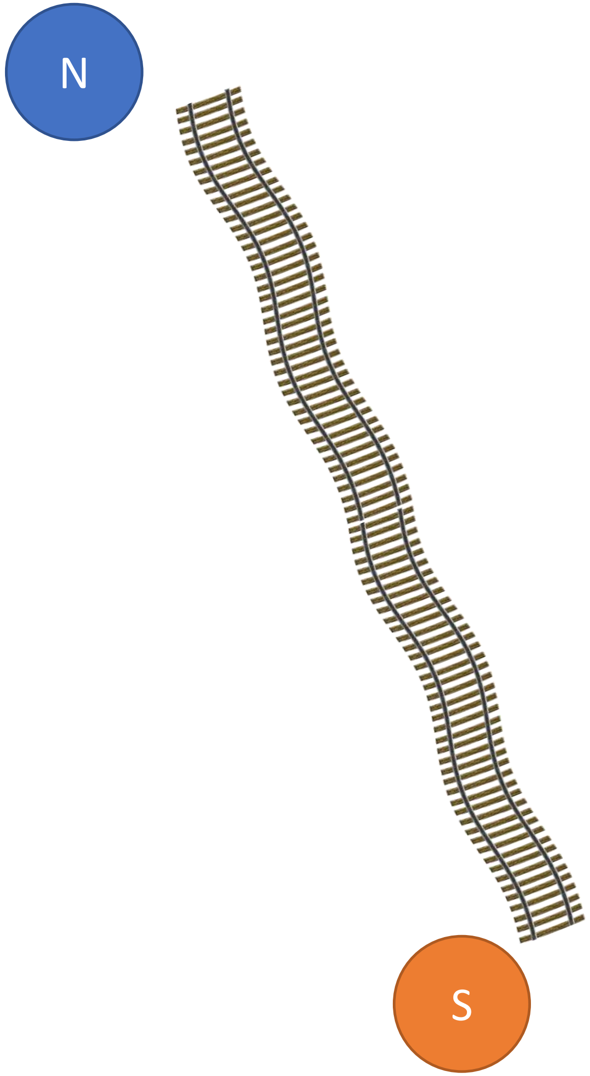
  advective derivative
  convective derivative
  derivative following the motion
  hydrodynamic derivative
  Lagrangian derivative
  particle derivative
  substantial derivative
  substantive derivative
  Stokes derivative

Material derivative explained using a train



“I took a train from the ***south*** *to the* ***north****.*
The train took off at ***1:00 PM****. It was* ***35 °C*** *outside.*
I arrived at ***7:00 PM****, and it was so freaking cold there.*
The temperature dropped to only ***5 °C****!”*

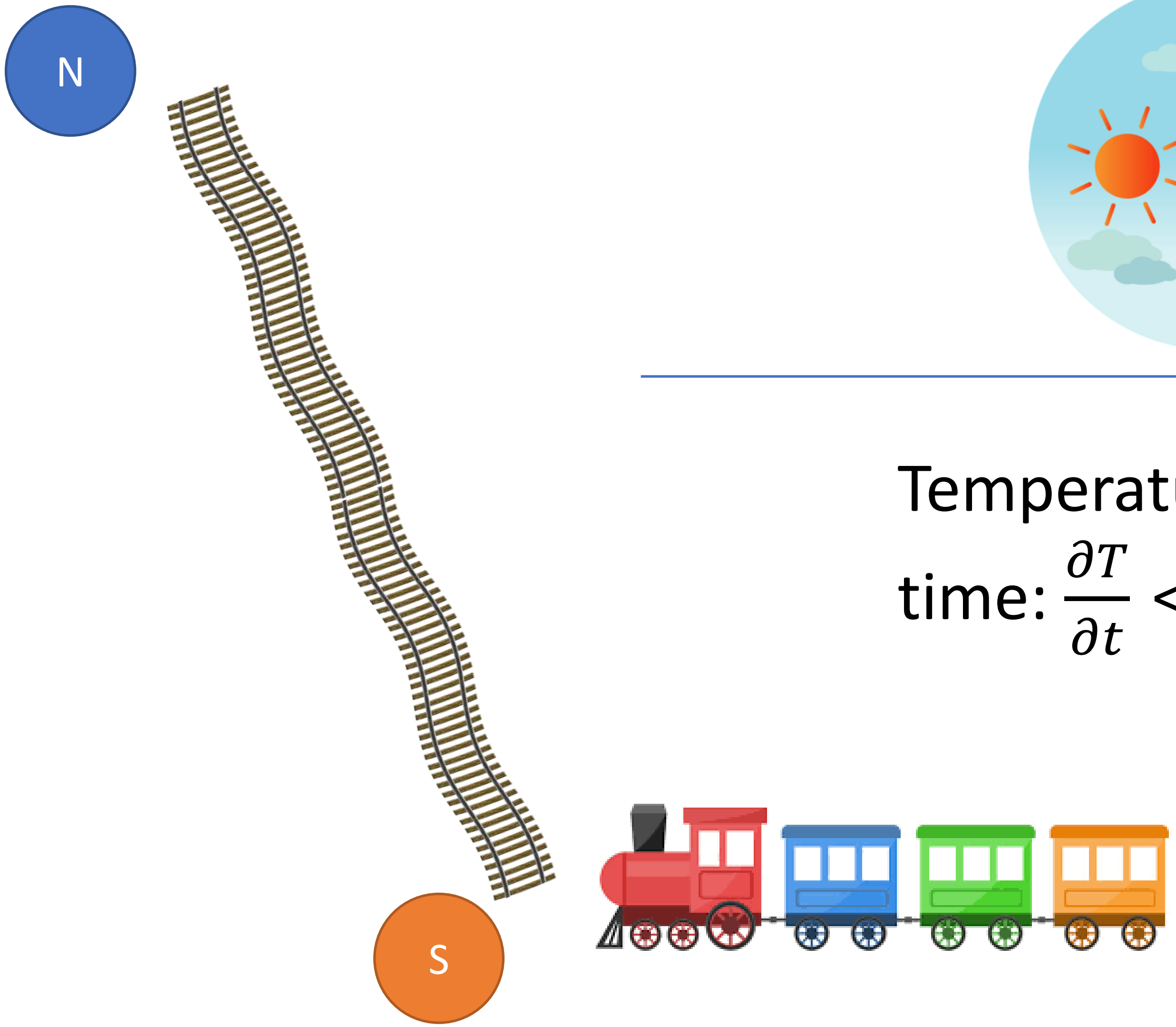
Material derivative explained using a train

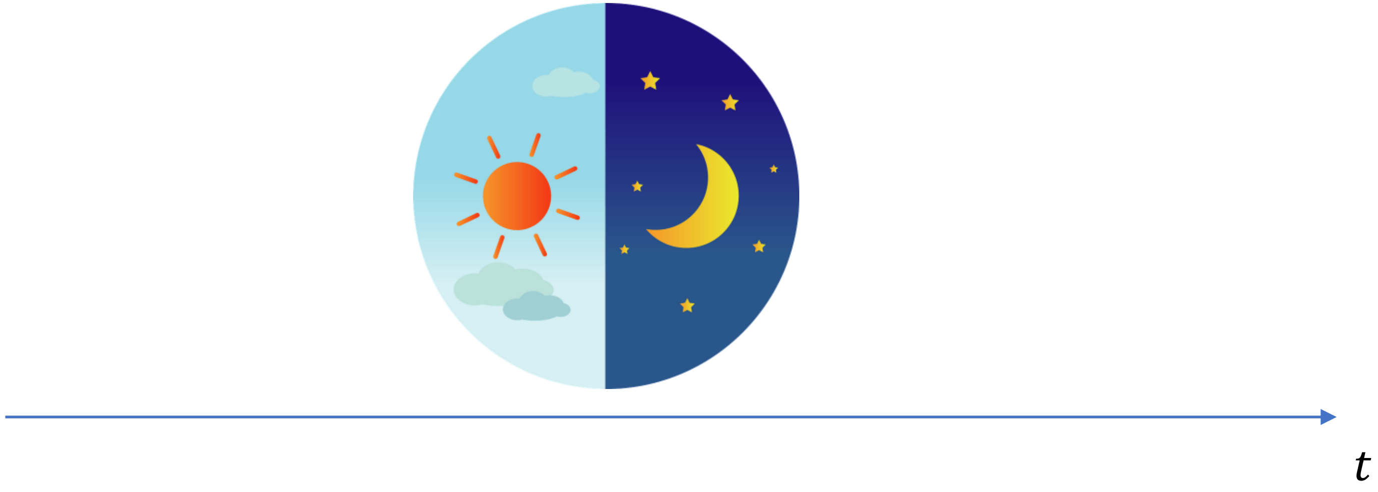


"I took a train from the ***south*** *to the* ***north****.*
The train took off at ***1:00 PM****. It was* ***35 °C*** *outside.*
I arrived at ***7:00 PM****, and it was so freaking cold there.*
The temperature dropped to only ***5 °C****!"*

Why was it so cold when I arrived?

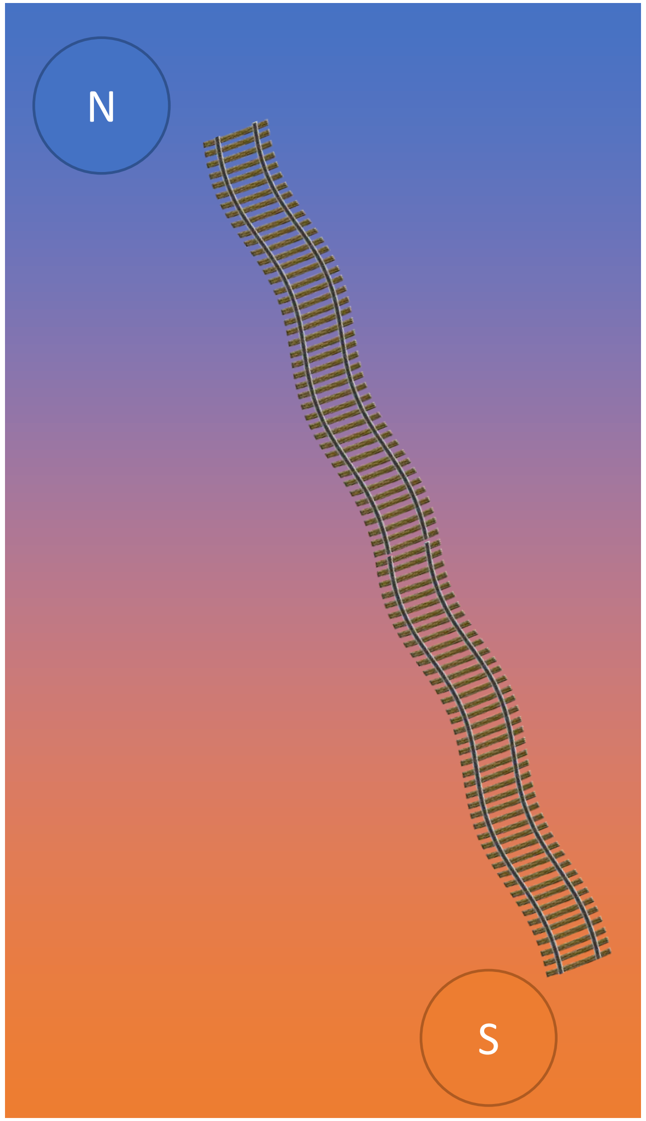
When we freeze the ***space*** coordinate

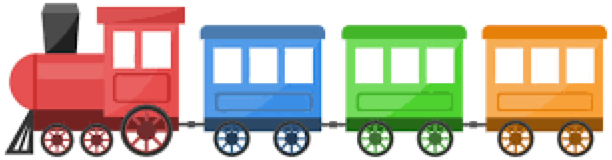




Temperature drops with time: $\frac{\partial T}{\partial t} < 0$

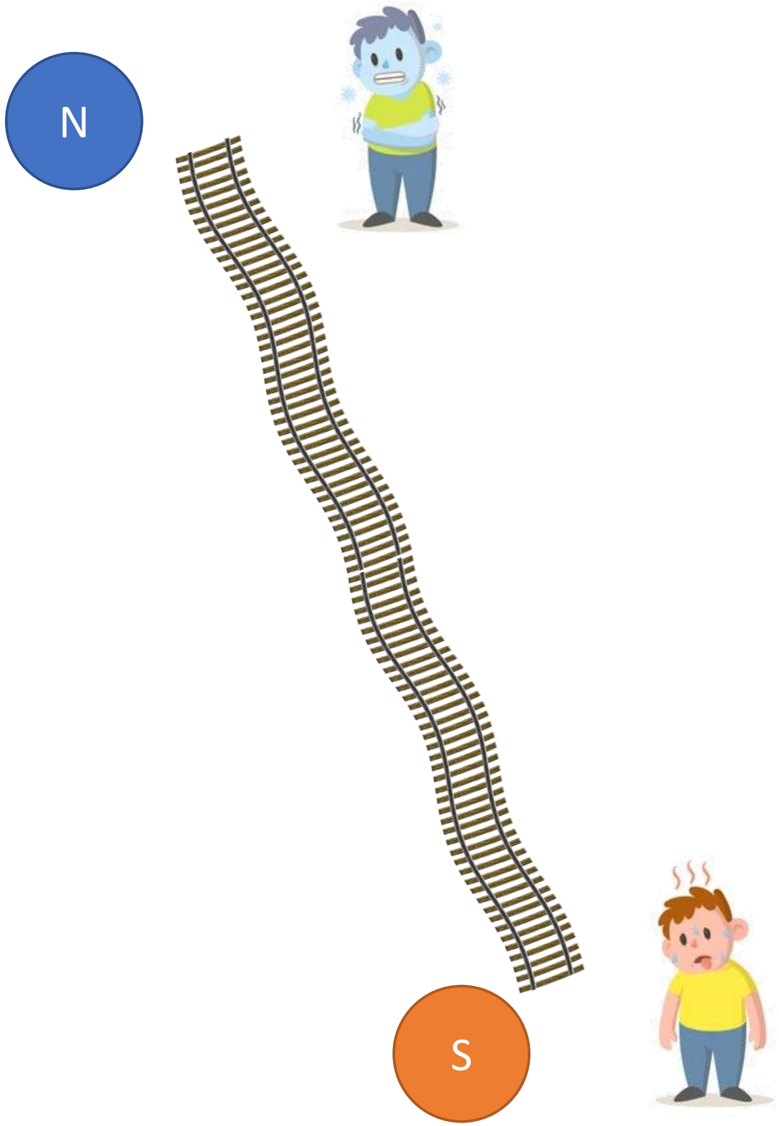
When we freeze the *time* coordinate





Temperature drops along my moving trajectory: $v \cdot \nabla T < 0$

Material derivative explained using a train



*“I took a train from the **south** to the **north**.*
*The train took off at **1:00 PM**. It was **35 °C** outside.*
*I arrived at **7:00 PM**, and it was so freaking cold there.*
*The temperature dropped to only **5 °C**!”*

It was getting colder because:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v \cdot \nabla T < 0$$

Material derivative of vectors

- $\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + v \cdot \nabla q$
- If $\boldsymbol{q}$ is a vector: $\boldsymbol{q} = \left[q_x, q_y, q_z\right]^T$
  - $\frac{D\boldsymbol{q}}{Dt} = \frac{\partial \boldsymbol{q}}{\partial t} + v \cdot \nabla \boldsymbol{q} = \frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{v}\colon \begin{bmatrix} \begin{bmatrix} \frac{\partial q_x}{\partial x} \\ \frac{\partial q_y}{\partial x} \\ \frac{\partial q_z}{\partial x} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial q_x}{\partial y} \\ \frac{\partial q_y}{\partial y} \\ \frac{\partial q_z}{\partial y} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial q_x}{\partial z} \\ \frac{\partial q_y}{\partial z} \\ \frac{\partial q_z}{\partial z} \end{bmatrix} \end{bmatrix} = \frac{\partial \boldsymbol{q}}{\partial t} + v_x \begin{bmatrix} \frac{\partial q_x}{\partial x} \\ \frac{\partial q_y}{\partial x} \\ \frac{\partial q_z}{\partial x} \end{bmatrix} + v_y \begin{bmatrix} \frac{\partial q_x}{\partial y} \\ \frac{\partial q_y}{\partial y} \\ \frac{\partial q_z}{\partial y} \end{bmatrix} + v_z \begin{bmatrix} \frac{\partial q_x}{\partial z} \\ \frac{\partial q_y}{\partial z} \\ \frac{\partial q_z}{\partial z} \end{bmatrix} = \frac{\partial \boldsymbol{q}}{\partial t} + \begin{bmatrix} v_x \frac{\partial q_x}{\partial x} + v_y \frac{\partial q_x}{\partial y} + v_z \frac{\partial q_x}{\partial z} \\ v_x \frac{\partial q_y}{\partial x} + v_y \frac{\partial q_y}{\partial y} + v_z \frac{\partial q_y}{\partial z} \\ v_x \frac{\partial q_z}{\partial x} + v_y \frac{\partial q_z}{\partial y} + v_z \frac{\partial q_z}{\partial z} \end{bmatrix}$

Material derivative of vectors (cont'd)

- If $\boldsymbol{q}$ is a vector: $\boldsymbol{q} = \left[q_x, q_y, q_z\right]^T$

  - $\frac{D\boldsymbol{q}}{Dt} = \frac{\partial \boldsymbol{q}}{\partial t} + v \cdot \nabla \boldsymbol{q} = \begin{bmatrix} \frac{\partial q_x}{\partial t} + v \cdot \nabla q_x \\ \frac{\partial q_y}{\partial t} + v \cdot \nabla q_y \\ \frac{\partial q_z}{\partial t} + v \cdot \nabla q_z \end{bmatrix}$

Material derivative of velocity

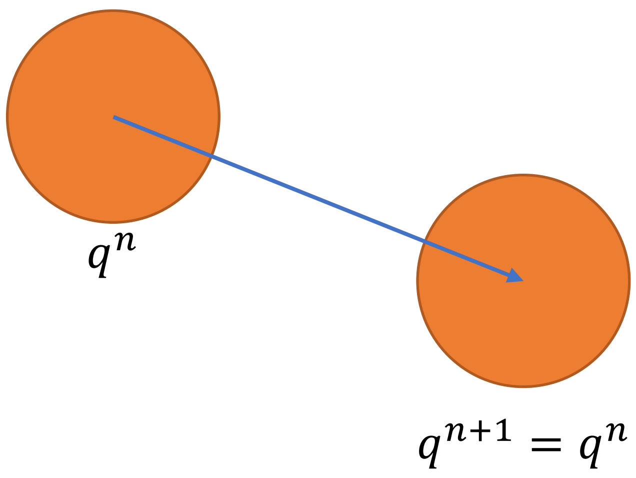
- The *“self-advection”*
  - $\frac{D\boldsymbol{v}}{Dt} = \frac{\partial \boldsymbol{v}}{\partial t} + v \cdot \nabla \boldsymbol{v} = \begin{bmatrix} \frac{\partial v_x}{\partial t} + v \cdot \nabla v_x \\ \frac{\partial v_y}{\partial t} + v \cdot \nabla v_y \\ \frac{\partial v_z}{\partial t} + v \cdot \nabla v_z \end{bmatrix}$
  - … is nothing but the material derivative of the velocity itself.

Advection: $\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + v \cdot \nabla q = 0$

- "Quantities flow with the velocity field"

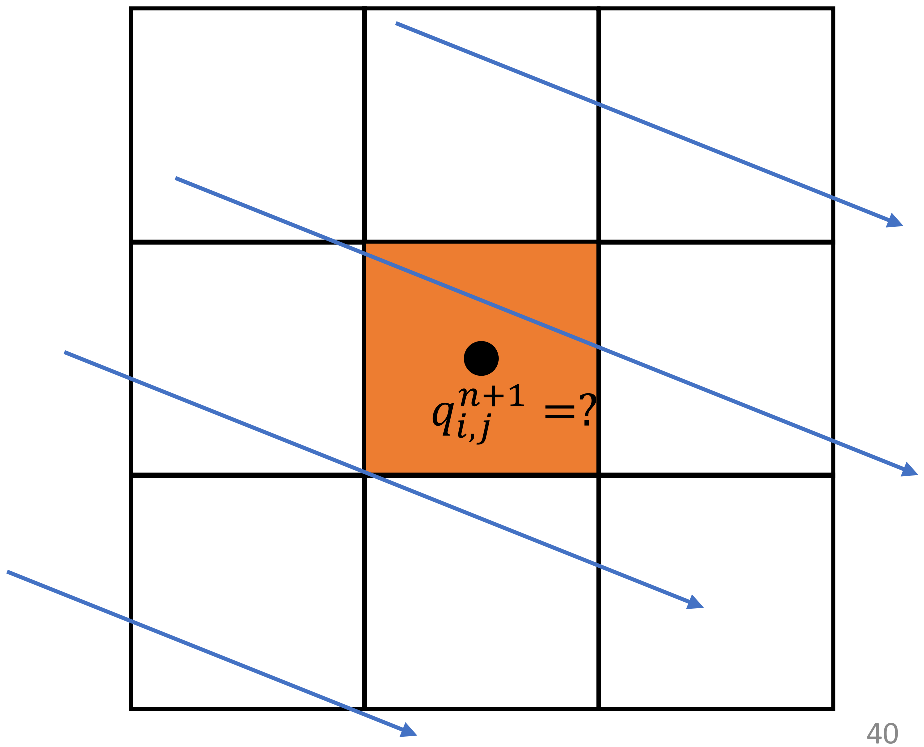
Lagrangian view

$q^{n+1} = q^n$ if $\frac{Dq}{Dt} = 0$



Eulerian view

$q_{i,j}^{n+1} = ?$

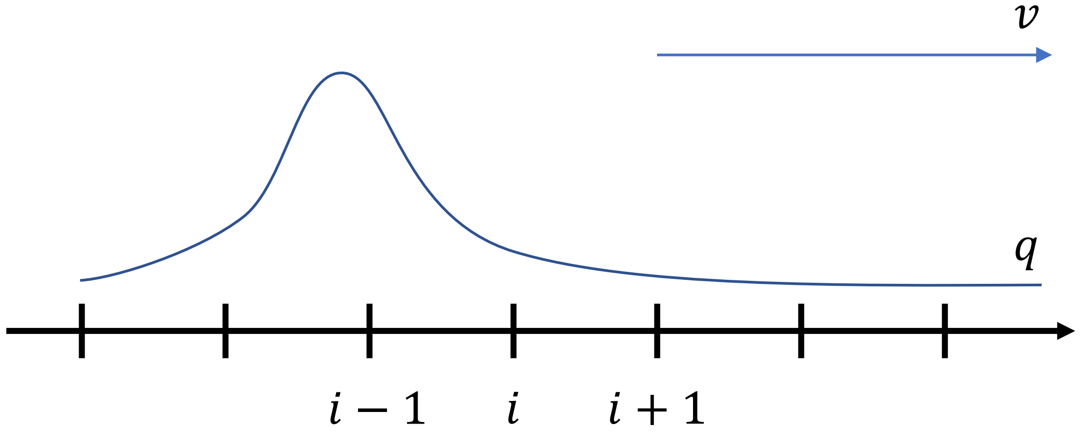


Attempt 1: Finite difference

- $\frac{\partial q}{\partial t} + v \cdot \nabla q = 0$
- $\Rightarrow \frac{q_i^{n+1} - q_i^n}{\Delta t} + v^n \cdot \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x} = 0$
- $\Rightarrow \ q_i^{n+1} = q_i^n - \Delta t v^n \cdot \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}$

Eulerian view

$q_i^{n+1} =?$



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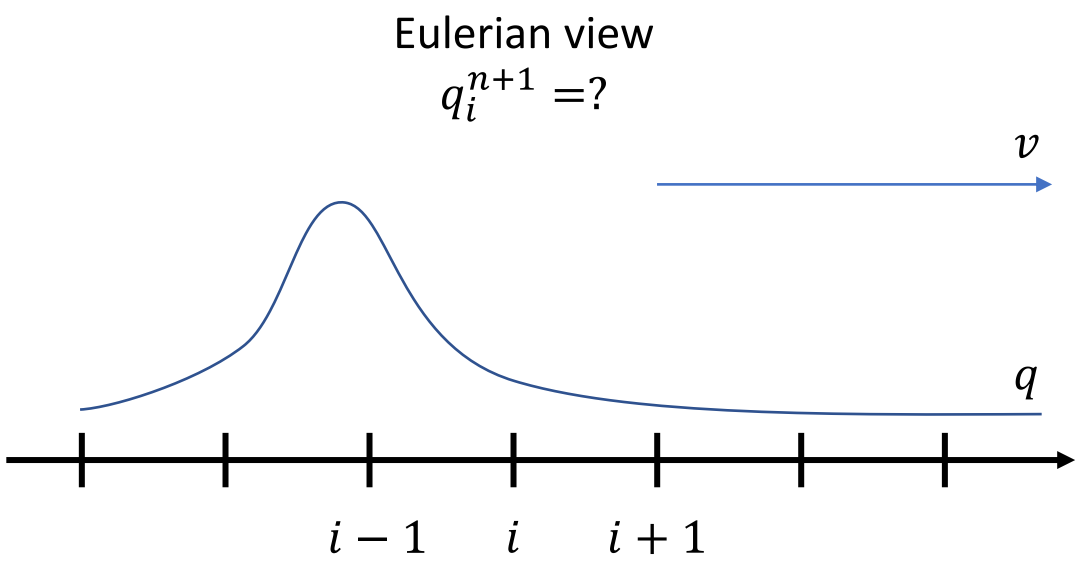


This advection scheme is ***unconditionally unstable***!

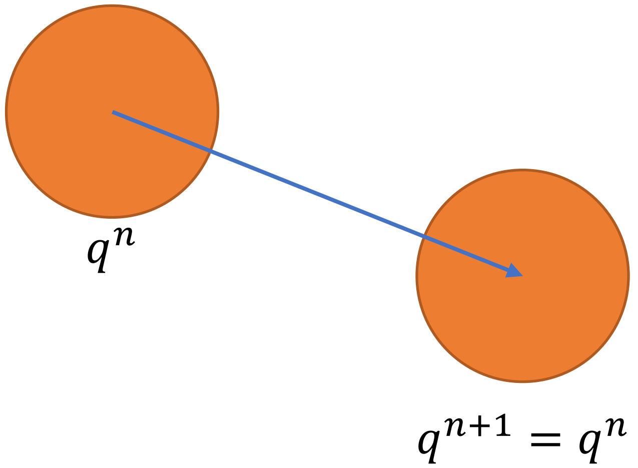
Attempt 2: “semi-Lagrangian”

- It is extremely simple to handle advection in the Lagrangian view…
  - shall we reuse this idea in our Eulerian grid too?
  - The answer is “yes”

  - $q^{n+1} \qquad\qquad = q^n$

Lagrangian view

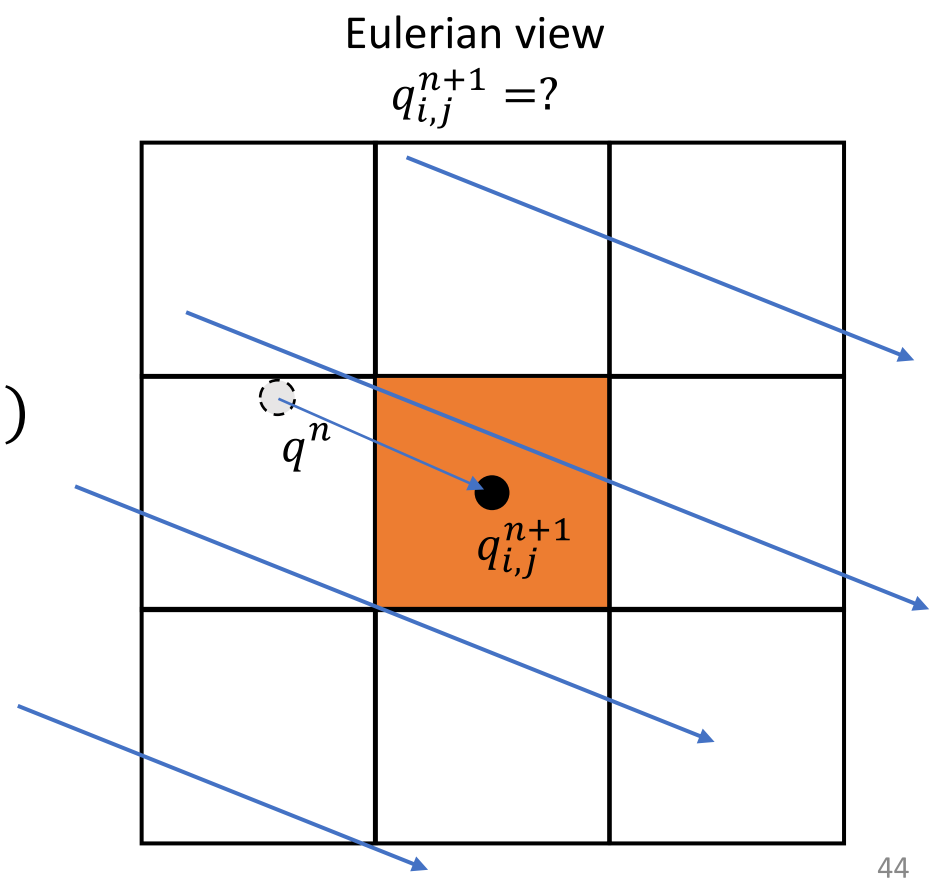
$q^{n+1} = q^n$ if $\frac{Dq}{Dt} = 0$



Attempt 2: “semi-Lagrangian”

- It is extremely simple to handle advection in the Lagrangian view…
  - shall we reuse this idea in our Eulerian grid too?
  - The answer is “yes”

- $q^{n+1}(x^{n+1}) = q^n(x^n) = q^n(x^{n+1} - \Delta t v^n)$

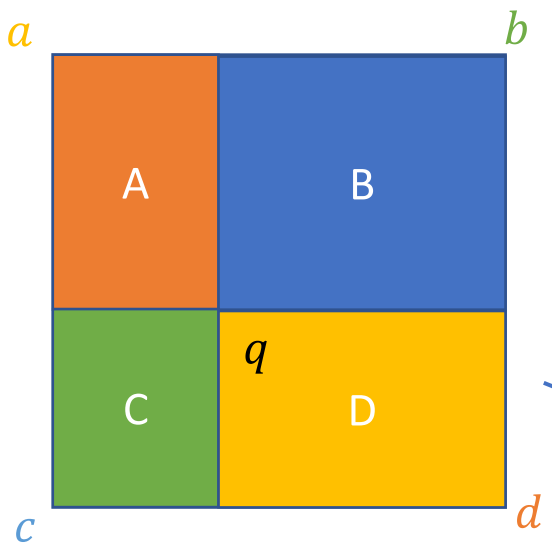


Attempt 2: “semi-Lagrangian”

- How do we get the value for $q^n(x^{n+1} - \Delta t v^n)$?
  - Interpolation!
  - $q^{n+1}(x^{n+1}) = \text{interpolate}(q^n, x^{n+1} - \Delta t v^n)$

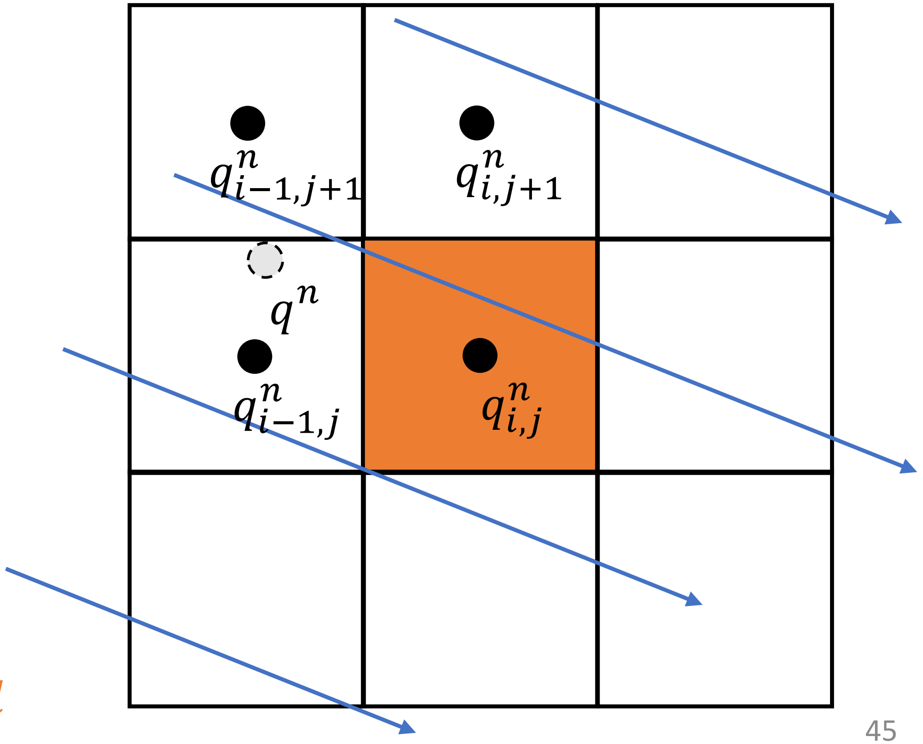
Bi-linear interpolation in 2D:

$$\begin{aligned} q &= lerp(a, b, c, d) \\ &= lerp\big(lerp(a, b), lerp(c, d)\big) \\ &= \frac{D * a + C * b + B * c + A * d}{A + B + C + D} \end{aligned}$$



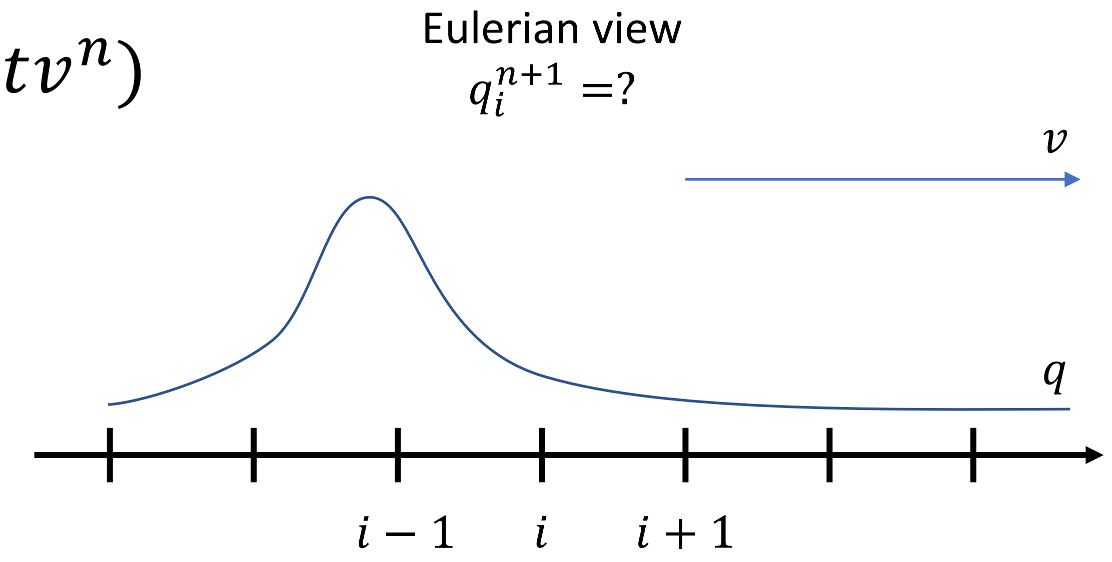
Eulerian view

$q^{n+1}_{i,j} = q^n$



Attempt 2: “semi-Lagrangian”

- $\frac{\partial q}{\partial t} + v \cdot \nabla q = 0$
- $\Rightarrow \; q_i^{n+1} = \text{interpolate}(q^n, x^{n+1} - \Delta t v^n)$



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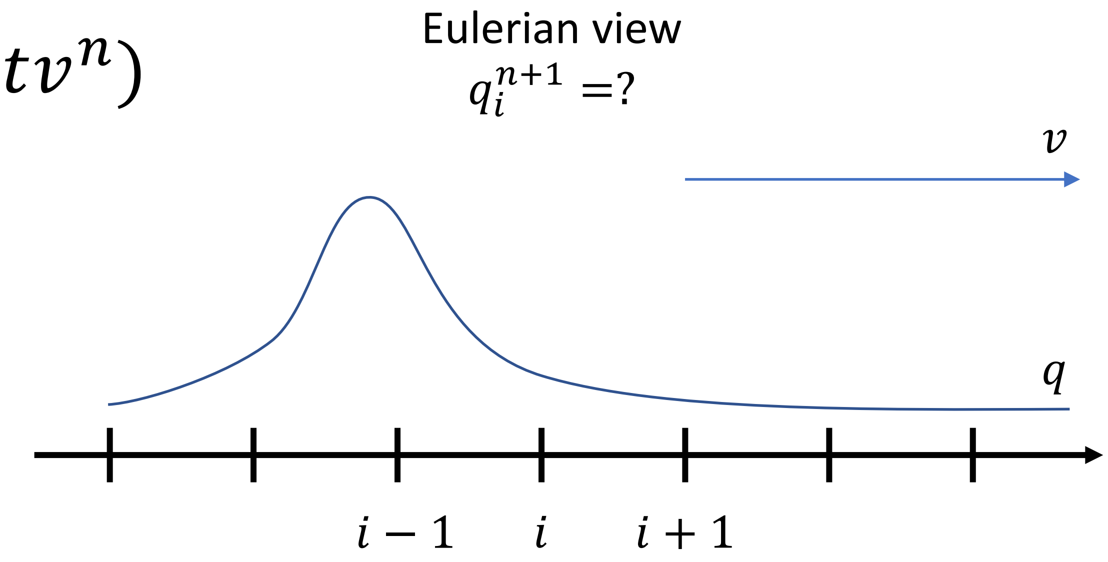
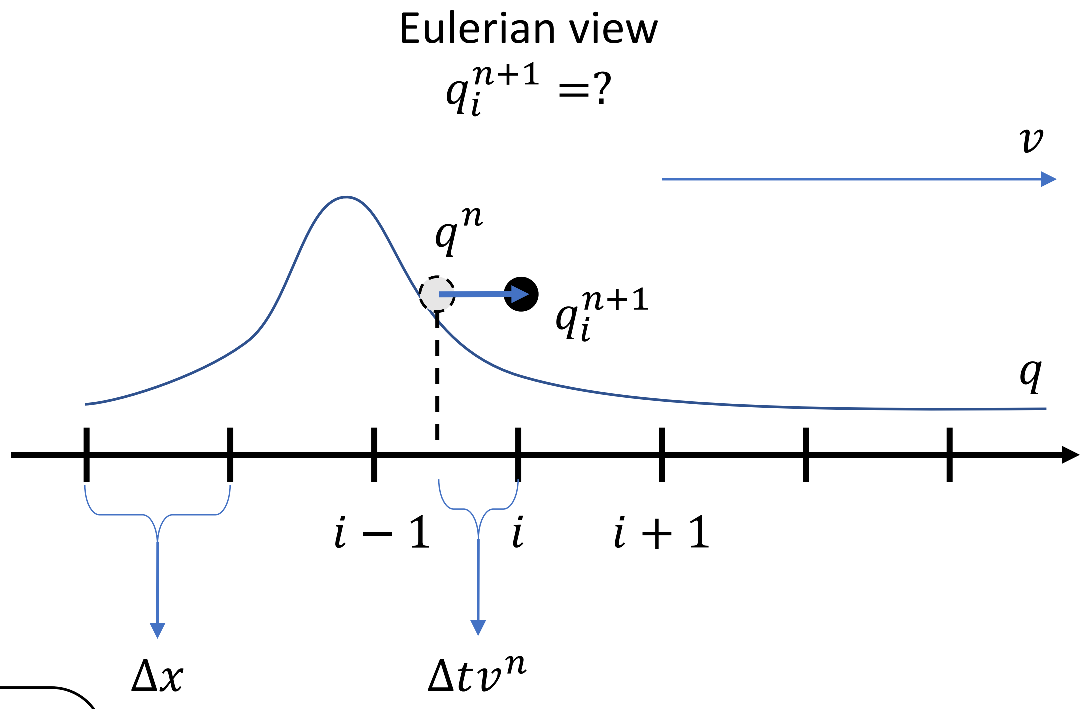


This advection scheme is ***unconditionally stable***!

Semi-Lagrangian advection: what is that?

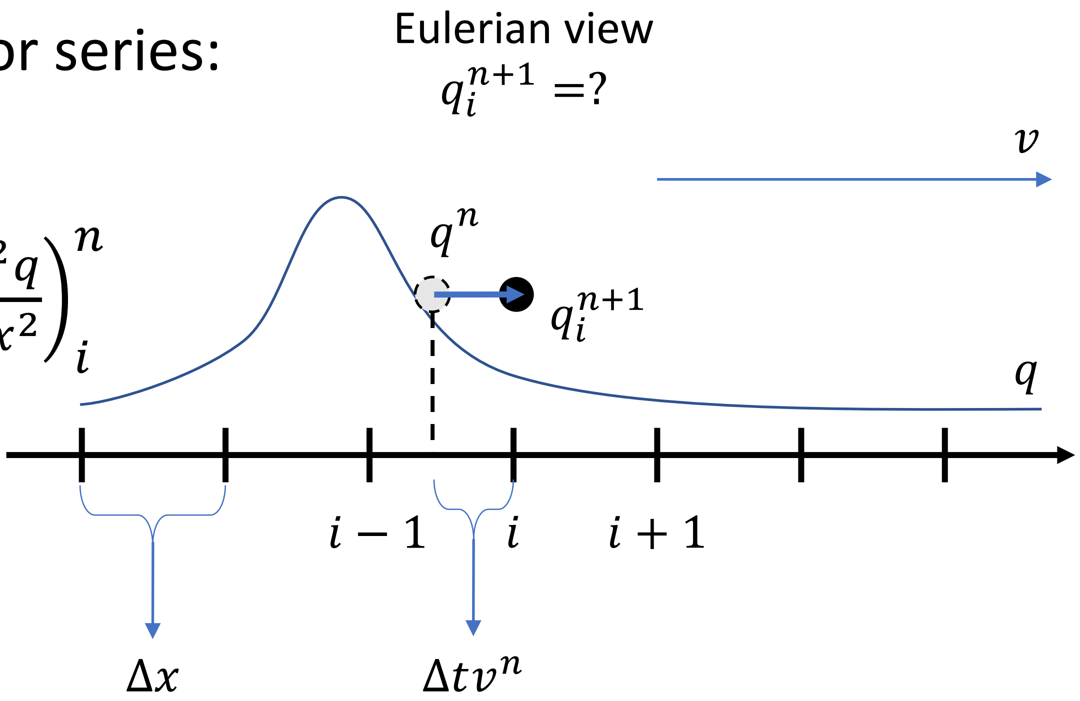
- What we want (in 1D):
  - $\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + v\frac{\partial q}{\partial x} = 0$
- Assuming $v^n \Delta t < \Delta x$:
  - $q_i^{n+1} = \frac{\Delta t v^n}{\Delta x} q_{i-1}^n + \left(1 - \frac{\Delta t v^n}{\Delta x}\right) q_i^n$
  - $\Rightarrow q_i^{n+1} = q_i^n - \Delta t v^n \frac{q_i^n - q_{i-1}^n}{\Delta x}$
  - $\Rightarrow \frac{q_i^{n+1} - q_i^n}{\Delta t} + v^n \frac{q_i^n - q_{i-1}^n}{\Delta x} = 0$



The semi-Lagrangian scheme is essentially a forward Euler scheme with a "velocity-aware" one-sided finite difference

Semi-Lagrangian advection: what do we lose?

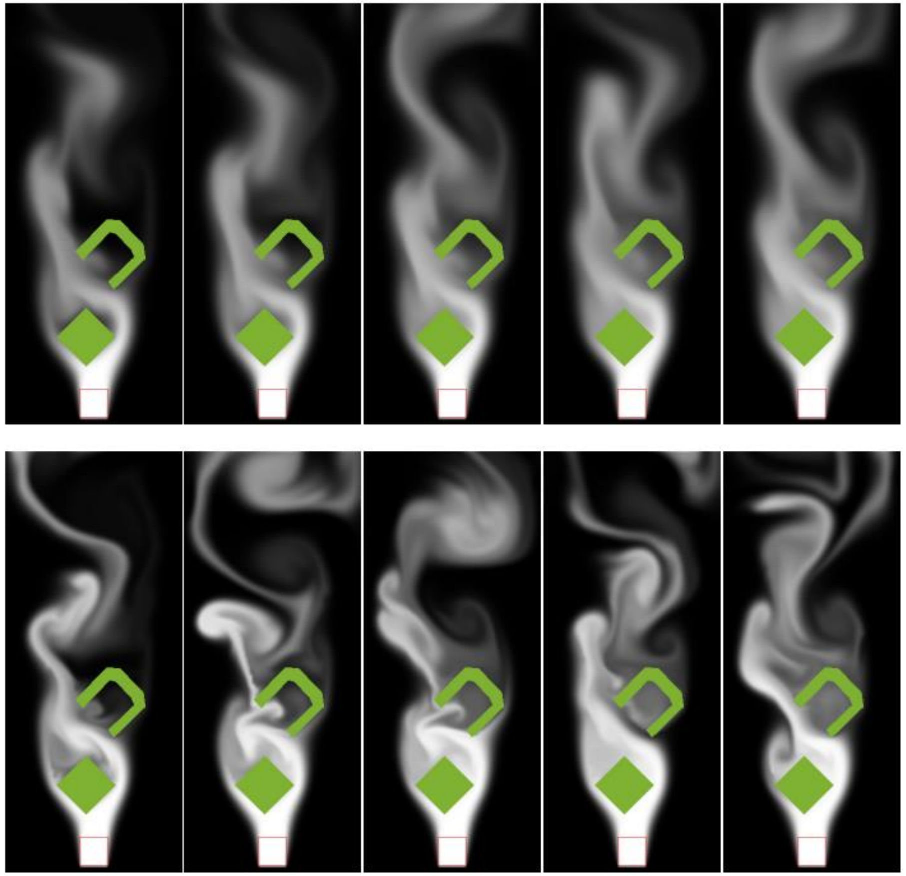
- $q_i^{n+1} = q_i^n - \Delta t v^n \frac{q_i^n - q_{i-1}^n}{\Delta x}$
- We can also expand q_{i-1}^n at q_i^n using the Taylor series:
  - $q_{i-1}^n \approx q_i^n - \left(\frac{\partial q}{\partial x}\right)_i^n \Delta x + \left(\frac{\partial^2 q}{\partial x^2}\right)_i^n \frac{\Delta x^2}{2}$
  - $\Rightarrow q_i^{n+1} \approx q_i^n - \Delta t v^n \left(\frac{\partial q}{\partial x}\right)_i^n + \frac{\Delta t v^n \Delta x}{2}\left(\frac{\partial^2 q}{\partial x^2}\right)_i^n$
  - $\Rightarrow \frac{q_i^{n+1} - q_i^n}{\Delta t} + v^n \left(\frac{\partial q}{\partial x}\right)_i^n \approx \frac{v^n \Delta x}{2}\left(\frac{\partial^2 q}{\partial x^2}\right)_i^n$
  - $\Rightarrow \frac{Dq}{Dt} \approx \frac{v^n \Delta x}{2}\frac{\partial^2 q}{\partial x^2} \neq 0$



The semi-Lagrangian advection scheme introduces “numerical dissipation/viscosity”

A better advection scheme with less dissipation?

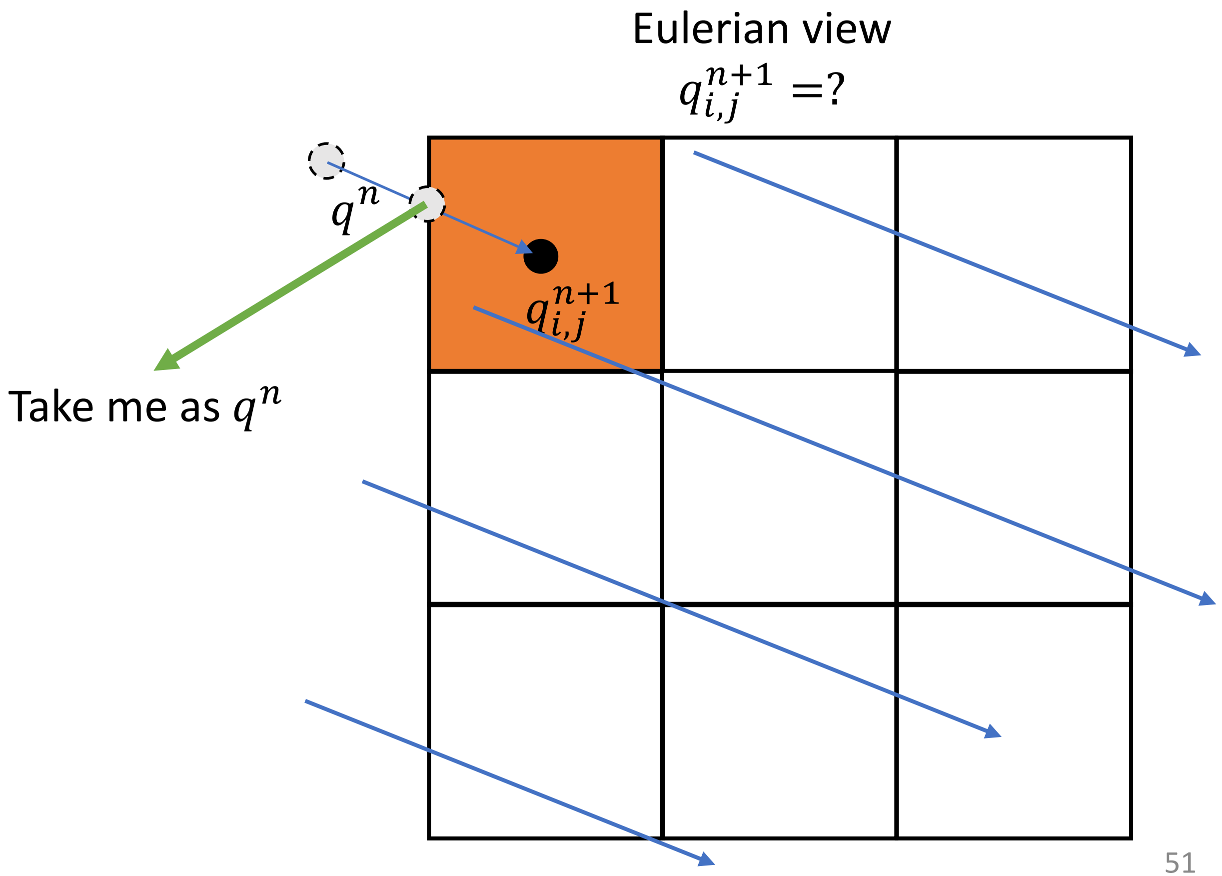
- Sharper interpolation
  - Cubic Hermite spline interpolation [Link]
- Better error correction schemes
  - MacCormack method [Link]
  - Back and Forth Error Compensation and Correction (BFECC) [Kim et al. 2005][Paper]



BFECC [Kim et al. 2005]

What if the backtracked “particle” is out-of-boundary?

- Simplest solution:
  - Take the value on the boundary



One numerical time-stepping for N-S equations

- Given q^n, where q can be velocity, density, temperature etc.
  - Step 1 Advection:
    - $q^{n+1} = advect(v^n, \Delta t, q^n)$
    - $\tilde{v} = advect(v^n, \Delta t, v^n)$
  - Step 2 Applying forces:
    - $\tilde{\tilde{v}} = \tilde{v} + \Delta t(g + \nu \nabla^2 \tilde{v})$
  - Step 3 Projection:
    - $v^{n+1} = project(\Delta t, \tilde{\tilde{v}})$
  - Return v^{n+1}, q^{n+1}

One numerical time-stepping for N-S equations

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Sometimes we can drop the viscous force term $\nu \nabla^2 \tilde{v}$ for water/smoke/"inviscid fluid" simulations. Sometimes we can even drop the gravitational force term g for smoke simulations.

One numerical time-stepping for N-S equations

- Given q^n, where q can be velocity, density, temperature etc.
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  - Step 3 Projection:
    - $v^{n+1} = project(\Delta t, \tilde{\tilde{v}})$
  - Return v^{n+1}, q^{n+1}

$$\frac{Dv}{Dt} = g - \frac{1}{\rho}\nabla p + \nu \nabla^2 v$$

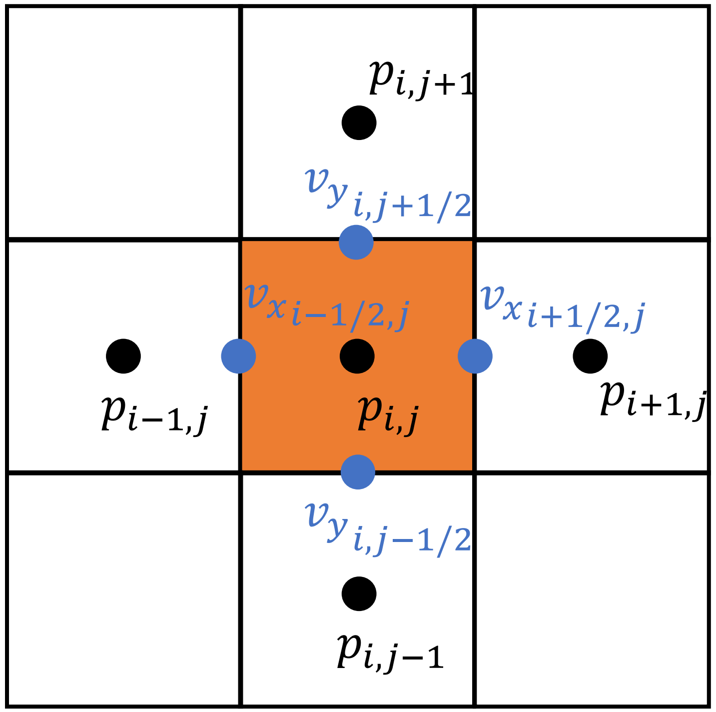
$$\nabla \cdot v = 0$$

Projection

The projection step

- We want to solve: $\dfrac{\partial v}{\partial t} = -\dfrac{1}{\rho}\nabla p$
  $\nabla \cdot v = 0$

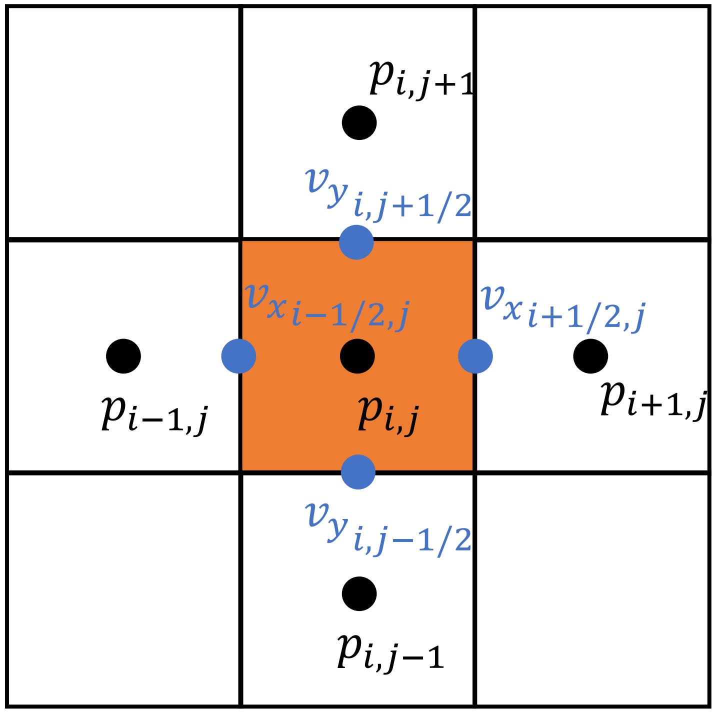
- Well, the finite difference comes to the rescue again:
  - $\dfrac{{v_x}_{i-1/2,j}^{n+1} - {v_x}_{i-1/2,j}^{n}}{\Delta t} = -\dfrac{1}{\rho}\dfrac{p_{i,j} - p_{i-1,j}}{\Delta x}$
  - s.t. $\underbrace{\dfrac{{v_x}_{i+1/2,j}^{n+1} - {v_x}_{i-1/2,j}^{n+1}}{\Delta x}}_{\frac{\partial v_x}{\partial x}} + \underbrace{\dfrac{{v_y}_{i,j+1/2}^{n+1} - {v_y}_{i,j-1/2}^{n+1}}{\Delta x}}_{\frac{\partial v_y}{\partial y}} = 0$



Finite difference

- $\frac{{v_x}_{i-1/2,j}^{n+1} - {v_x}_{i-1/2,j}^{n}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i,j} - p_{i-1,j}}{\Delta x}$ 1
- $\frac{{v_x}_{i+1/2,j}^{n+1} - {v_x}_{i+1/2,j}^{n}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i+1,j} - p_{i,j}}{\Delta x}$ 2
- $\frac{{v_y}_{i,j-1/2}^{n+1} - {v_y}_{i,j-1/2}^{n}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i,j} - p_{i,j-1}}{\Delta x}$ 3
- $\frac{{v_y}_{i,j+1/2}^{n+1} - {v_y}_{i,j+1/2}^{n}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i,j+1} - p_{i,j}}{\Delta x}$ 4

- $\frac{{v_x}_{i+1/2,j}^{n+1} - {v_x}_{i-1/2,j}^{n+1}}{\Delta x} + \frac{{v_y}_{i,j+1/2}^{n+1} - {v_y}_{i,j-1/2}^{n+1}}{\Delta x} = 0$



Finite difference

(1) $\dfrac{{v_x}_{i-1/2,j}^{n+1} - {v_x}_{i-1/2,j}^{n}}{\Delta t} = -\dfrac{1}{\rho}\dfrac{p_{i,j} - p_{i-1,j}}{\Delta x}$

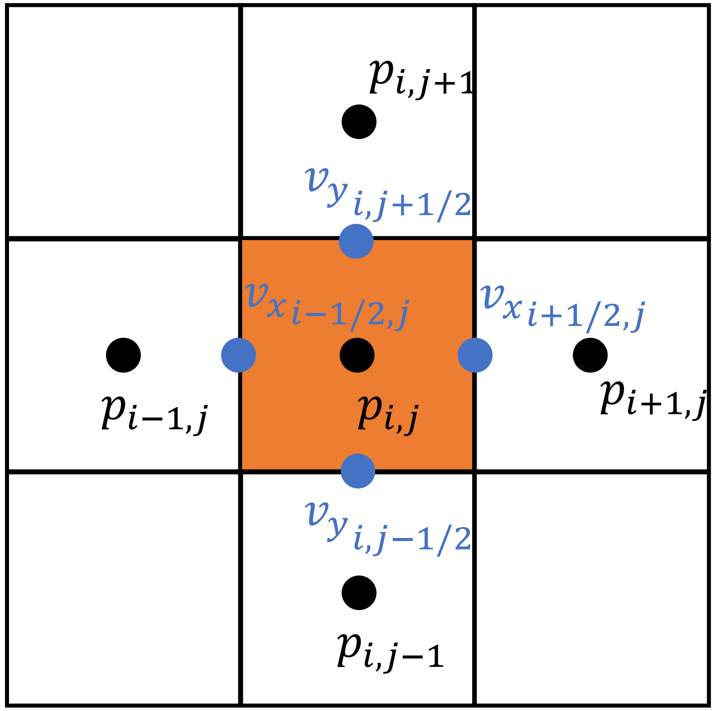
(2) $\dfrac{{v_x}_{i+1/2,j}^{n+1} - {v_x}_{i+1/2,j}^{n}}{\Delta t} = -\dfrac{1}{\rho}\dfrac{p_{i+1,j} - p_{i,j}}{\Delta x}$

(3) $\dfrac{{v_y}_{i,j-1/2}^{n+1} - {v_y}_{i,j-1/2}^{n}}{\Delta t} = -\dfrac{1}{\rho}\dfrac{p_{i,j} - p_{i,j-1}}{\Delta x}$

(4) $\dfrac{{v_y}_{i,j+1/2}^{n+1} - {v_y}_{i,j+1/2}^{n}}{\Delta t} = -\dfrac{1}{\rho}\dfrac{p_{i,j+1} - p_{i,j}}{\Delta x}$

(2) − (1) + (4) − (3) :

$$\frac{{v_x}_{i+1/2,j}^{n+1} - {v_x}_{i-1/2,j}^{n+1} + {v_y}_{i,j+1/2}^{n+1} - {v_y}_{i,j-1/2}^{n+1}}{\Delta t}$$
$$-\frac{{v_x}_{i+1/2,j}^{n} - {v_x}_{i-1/2,j}^{n} + {v_y}_{i,j-1/2}^{n} - {v_y}_{i,j-1/2}^{n}}{\Delta t}$$
$$= \frac{1}{\rho}\frac{4p_{i,j} - p_{i+1,j} - p_{i-1,j} - p_{i,j+1} - p_{i,j-1}}{\Delta x}$$



- $\dfrac{{v_x}_{i+1/2,j}^{n+1} - {v_x}_{i-1/2,j}^{n+1}}{\Delta x} + \dfrac{{v_y}_{i,j+1/2}^{n+1} - {v_y}_{i,j-1/2}^{n+1}}{\Delta x} = 0$

Finite difference

$$① \quad \frac{{v_x}^{n+1}_{i-1/2,j} - {v_x}^{n}_{i-1/2,j}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i,j} - p_{i-1,j}}{\Delta x}$$

$$② \quad \frac{{v_x}^{n+1}_{i+1/2,j} - {v_x}^{n}_{i+1/2,j}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i+1,j} - p_{i,j}}{\Delta x}$$

$$③ \quad \frac{{v_y}^{n+1}_{i,j-1/2} - {v_y}^{n}_{i,j-1/2}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i,j} - p_{i,j-1}}{\Delta x}$$

$$④ \quad \frac{{v_y}^{n+1}_{i,j+1/2} - {v_y}^{n}_{i,j+1/2}}{\Delta t} = -\frac{1}{\rho}\frac{p_{i,j+1} - p_{i,j}}{\Delta x}$$

$$(② - ① + ④ - ③)\,\frac{\Delta t}{\Delta x}:$$

$$\frac{{v_x}^{n+1}_{i+1/2,j} - {v_x}^{n+1}_{i-1/2,j} + {v_y}^{n+1}_{i,j+1/2} - {v_y}^{n+1}_{i,j-1/2}}{\Delta x}$$

$$-\frac{{v_x}^{n}_{i+1/2,j} - {v_x}^{n}_{i-1/2,j} + {v_y}^{n}_{i,j-1/2} - {v_y}^{n}_{i,j-1/2}}{\Delta x}$$

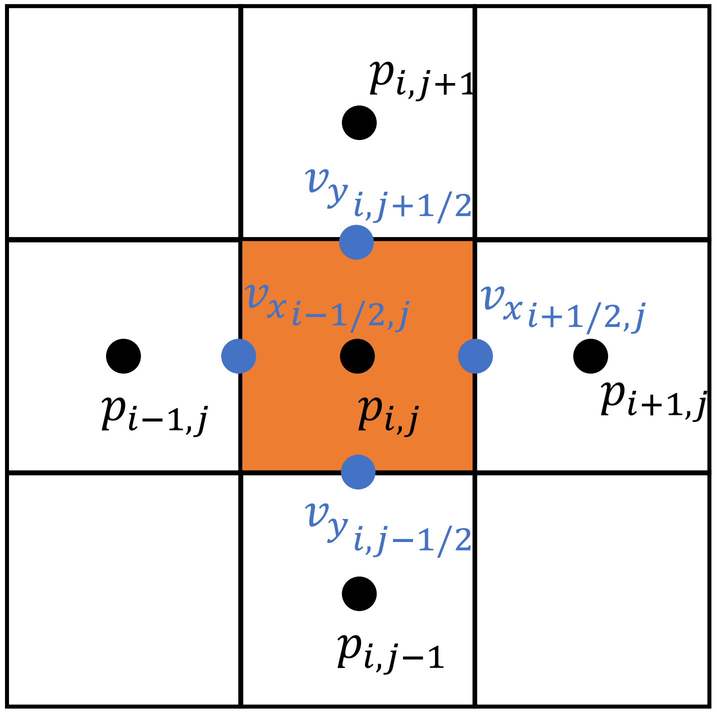
$$= \frac{\Delta t}{\rho}\frac{4p_{i,j} - p_{i+1,j} - p_{i-1,j} - p_{i,j+1} - p_{i,j-1}}{\Delta x^2}$$

$$\nabla \cdot v^{n+1}$$

$$-\nabla \cdot v^{n}$$

$$-\frac{\Delta t}{\rho}\nabla \cdot \nabla p$$

$$-\frac{\Delta t}{\rho}\nabla \cdot \nabla p = -\nabla \cdot v^{n}$$



- $\frac{{v_x}^{n+1}_{i+1/2,j} - {v_x}^{n+1}_{i-1/2,j}}{\Delta x} + \frac{{v_y}^{n+1}_{i,j+1/2} - {v_y}^{n+1}_{i,j-1/2}}{\Delta x} = 0$

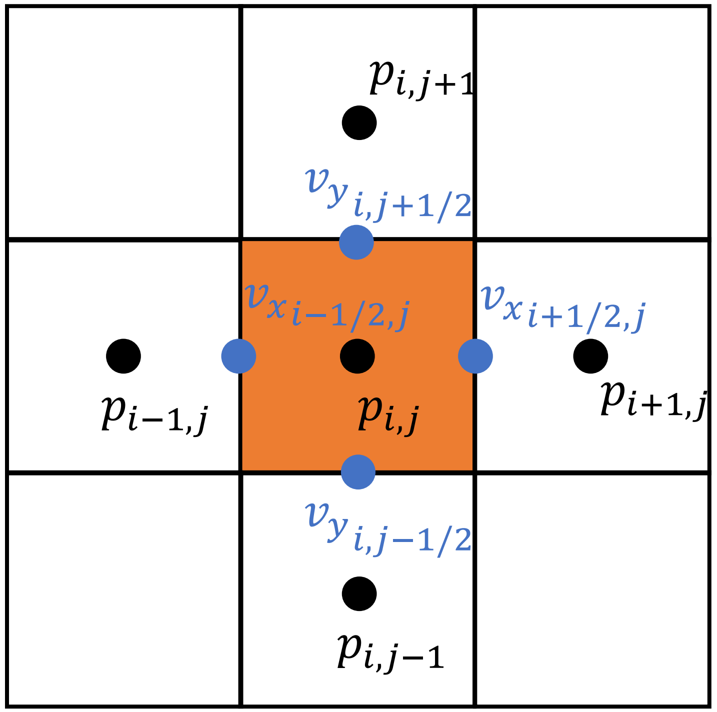
The projection step (A Poisson problem)

- $-\frac{\Delta t}{\rho}\nabla \cdot \nabla p = -\nabla \cdot v^n$

- Or:
  - $\frac{\Delta t}{\rho}\frac{4p_{i,j}-p_{i+1,j}-p_{i-1,j}-p_{i,j+1}-p_{i,j-1}}{\Delta x^2} = -\frac{{v_x}_{i+1/2,j}^n-{v_x}_{i-1/2,j}^n+{v_y}_{i,j-1/2}^n-{v_y}_{i,j-1/2}^n}{\Delta x}$

Another way to achieve the Poisson problem

- What we want (the pressure equation):
  - $\frac{\partial v}{\partial t} = -\frac{1}{\rho}\nabla p$ s.t. $\nabla \cdot v = 0$
- Discretize the pressure equation in time:
  - $v^{n+1} - v^n = -\frac{\Delta t}{\rho}\nabla p$ s.t. $\nabla \cdot v^{n+1} = 0$
- Apply divergence operator ($\nabla \cdot$) on both sides:
  - $-\nabla \cdot v^n = -\frac{\Delta t}{\rho}\nabla \cdot \nabla p$



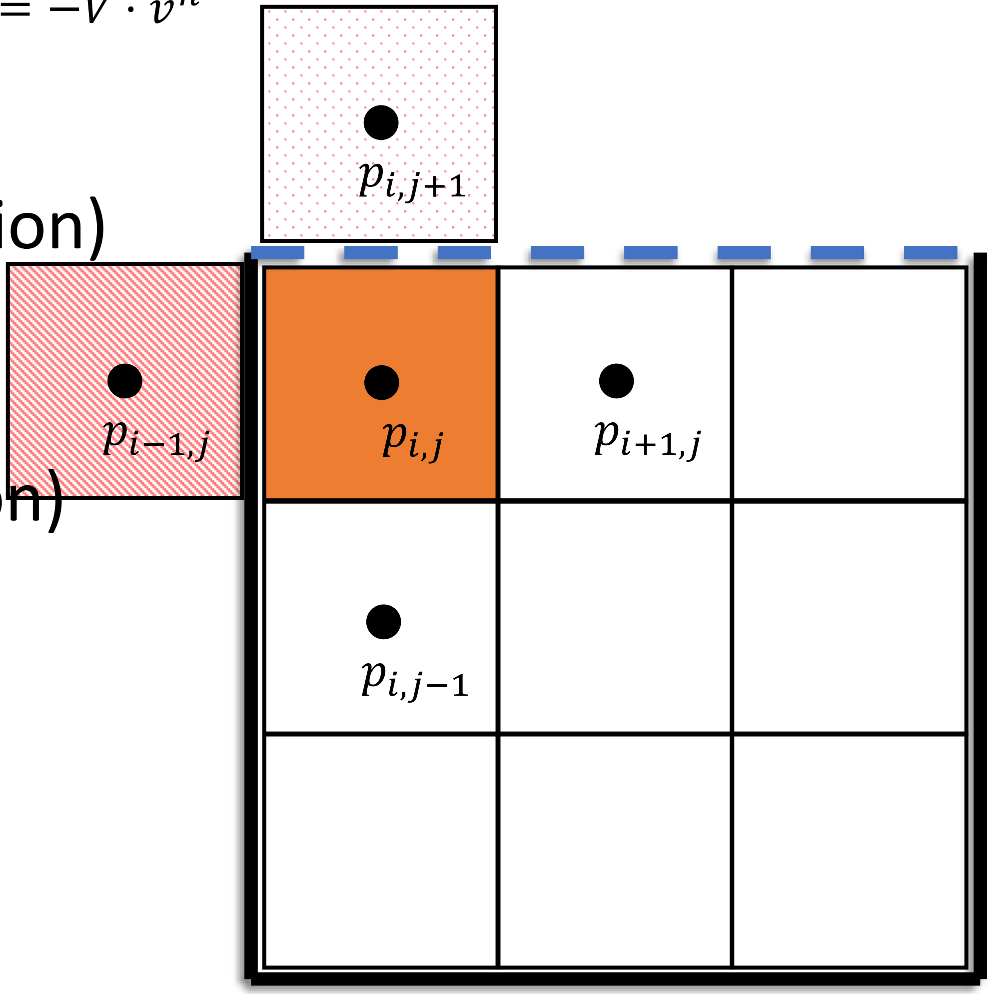
Pressure solve $-\frac{\Delta t}{\rho}\nabla \cdot \nabla p = -\nabla \cdot v^n$

- For every grid, there is one unknown $p_{i,j}$
- For every grid, there will be one equation:
  - $\frac{\Delta t}{\rho}\frac{4p_{i,j}-p_{i+1,j}-p_{i-1,j}-p_{i,j+1}-p_{i,j-1}}{\Delta x^2} = -\frac{{v_x}^n_{i+1/2,j}-{v_x}^n_{i-1/2,j}+{v_y}^n_{i,j-1/2}-{v_y}^n_{i,j-1/2}}{\Delta x}$

- This requires us only a linear solve $Ap = -d$
- Once we have all the pressure values…
  - We can solve for the velocity update:
    - $v^{n+1}_{x_{i-1/2,j}} = v^n_{x_{i-1/2,j}} - \frac{\Delta t}{\rho}\frac{p_{i,j}-p_{i-1,j}}{\Delta x}$

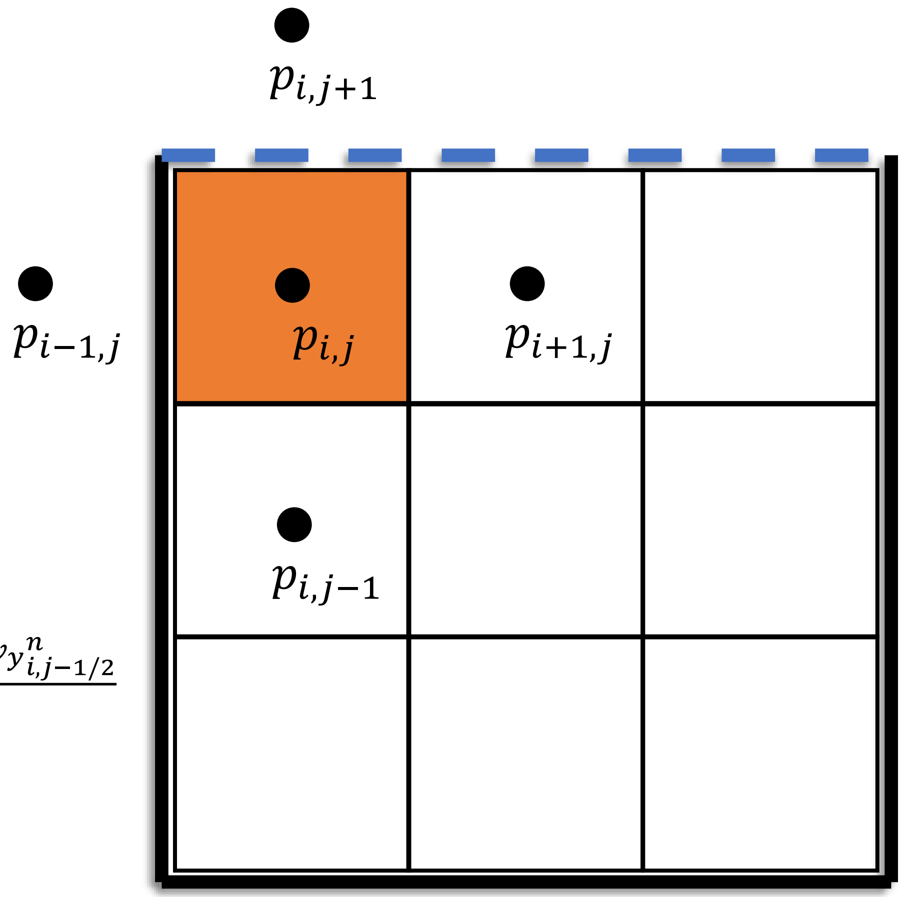
Boundary conditions $-\frac{\Delta t}{\rho}\nabla \cdot \nabla p = -\nabla \cdot v^n$

- Free surface (Dirichlet boundary condition)
  - $p = 0$ for void grids
- Solid wall (Neumann boundary condition)
  - $v^{n+1} \cdot n = v^{solid} \cdot n$
    - or $v_x^{n+1} = v_x^{solid}$, $v_y^{n+1} = v_y^{solid}$
  - For solid grids:
    - $v_x^{solid} = v_{x_{i-1/2,j}}^{n+1} = v_{x_{i-1/2,j}}^{n} - \frac{\Delta t}{\rho}\frac{p_{i,j} - p_{i-1,j}}{\Delta x}$
    - $\Rightarrow p_{i-1,j} = p_{i,j} - \frac{\rho \Delta x}{\Delta t}\left(v_{x_{i-1/2,j}}^{n} - v_x^{solid}\right)$



The Poisson's equation with boundaries

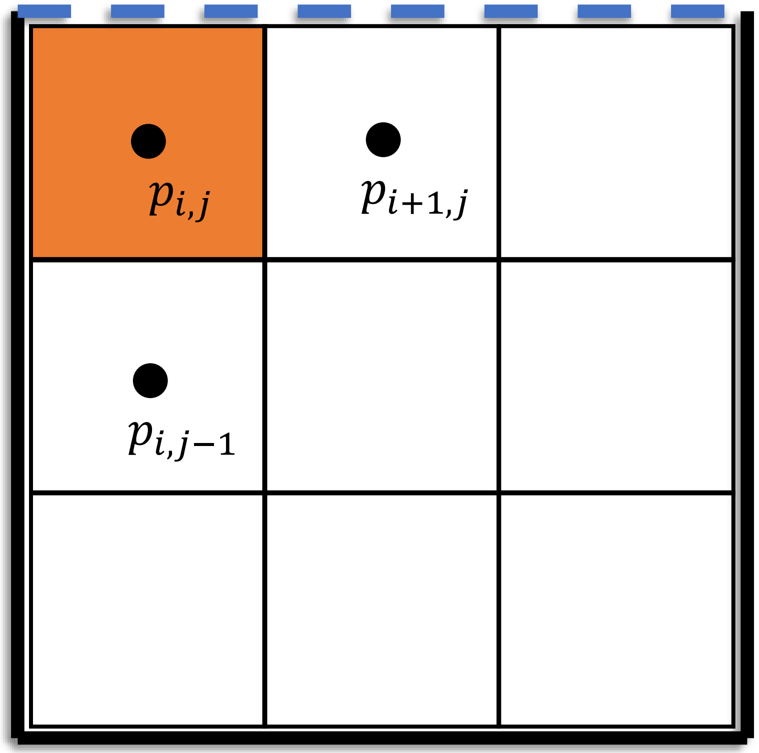
- Dirichlet boundary:
  - $p_{i,j+1} = 0$
- Neumann boundary:
  - $p_{i-1,j} = p_{i,j} - \frac{\rho \Delta x}{\Delta t}\left(v_{x_{i-1/2,j}}^{n} - v_x^{solid}\right)$
- The original Poisson's equation:
  - $\frac{\Delta t}{\rho}\frac{4p_{i,j}-p_{i+1,j}-p_{i-1,j}-p_{i,j+1}-p_{i,j-1}}{\Delta x^2} = -\frac{{v_x}_{i+1/2,j}^{n}-{v_x}_{i-1/2,j}^{n}+{v_y}_{i,j-1/2}^{n}-{v_y}_{i,j-1/2}^{n}}{\Delta x}$
- The Poisson's equation with boundaries:
  - $\frac{\Delta t}{\rho}\frac{4p_{i,j}-p_{i+1,j}-\left(p_{i,j}-\frac{\rho\Delta x}{\Delta t}\left(v_{x_{i-1/2,j}}^{n}-v_x^{solid}\right)\right)-0-p_{i,j-1}}{\Delta x^2} = -\frac{{v_x}_{i+1/2,j}^{n}-{v_x}_{i-1/2,j}^{n}+{v_y}_{i,j-1/2}^{n}-{v_y}_{i,j-1/2}^{n}}{\Delta x}$
  - Or equivalently:
  - $\frac{\Delta t}{\rho}\frac{3p_{i,j}-p_{i+1,j}-p_{i,j-1}}{\Delta x^2} = -\frac{{v_x}_{i+1/2,j}^{n}-v_x^{solid}+{v_y}_{i,j-1/2}^{n}-{v_y}_{i,j-1/2}^{n}}{\Delta x}$



The Poisson's equation with boundaries

- $Ap = -d$

$$
\mathbf{A} = \frac{\Delta t}{\rho \Delta x^2}
\begin{array}{c}
\left[\begin{array}{ccccccccc}
+3 & -1 & & -1 & & & & & \\
-1 & +4 & -1 & & -1 & & & & \\
 & -1 & +3 & & & -1 & & & \\
-1 & & & +3 & -1 & & -1 & & \\
 & -1 & & -1 & +4 & -1 & & -1 & \\
 & & -1 & & -1 & +3 & & & -1 \\
 & & & -1 & & & +2 & -1 & \\
 & & & & -1 & & -1 & +3 & -1 \\
 & & & & & -1 & & -1 & +2
\end{array}\right]
\end{array}
\begin{array}{l}
\\
\text{Dirichlet} \\
\\
\\
\text{Interior} \\
\\
\text{Neumann} \\
\\
\\
\end{array}
$$



Linear solvers $Ax = b$ [Lecture 09]

- Direct solvers:
  - Inversion: $x = \mathrm{A}^{-1}b$
  - Factorization: $\mathrm{A} = \begin{cases} \mathrm{LU} & , if\ \mathrm{A}\ is\ a\ square\ matrix \\ \mathrm{LDL^T} & , if\ \mathrm{A} = \mathrm{A^T} \\ \mathrm{LL^T} & , if\ \mathrm{A} = \mathrm{A^T}\ and\ \mathrm{A} \succ 0 \end{cases}$
- Iterative solvers:
  - Stationary iterative linear solvers: Jacobi / Gauss-Seidel / SOR / Multigrid
  - Krylov subspace methods: Conjugate Gradient (CG) / biCG / CR / MinRes / GMRes

Put things together

- Given q^n, where q can be velocity, density, temperature etc.
  - Step 1 Advection:
    - $q^{n+1} = advect(v^n, \Delta t, q^n)$
    - $\tilde{v} = advect(v^n, \Delta t, v^n)$
  - Step 2 Applying forces:
    - $\tilde{\tilde{v}} = \tilde{v} + \Delta t(g + \nu \nabla^2 \tilde{v})$
  - Step 3 Projection:
    - $v^{n+1} = project(\Delta t, \tilde{\tilde{v}})$
  - Return v^{n+1}, q^{n+1}

$$\frac{Dv}{Dt} = g - \frac{1}{\rho}\nabla p + \nu \nabla^2 v$$

$$\nabla \cdot v = 0$$

Remark

Remark

- N-S equations and their time integration
  - Operator splitting
- From the Lagrangian view to the Eulerian view
  - Spatial derivatives using finite difference
  - MAC grid
- Advection
  - Material derivative
  - Quantity advection
- Projection
  - Poisson's equation
  - Boundary conditions

$$\frac{Dv}{Dt} = g - \frac{1}{\rho}\nabla p + \nu\nabla^2 v \qquad \nabla \cdot v = 0$$

Operator splitting:

Advection: $\frac{Dq}{Dt} = 0$

forcing: $\frac{\partial v}{\partial t} = g + \nu\nabla^2 v$

Projection: $\frac{\partial v}{\partial t} = -\frac{1}{\rho}\nabla p \; s.t. \nabla \cdot v = 0$

Remark

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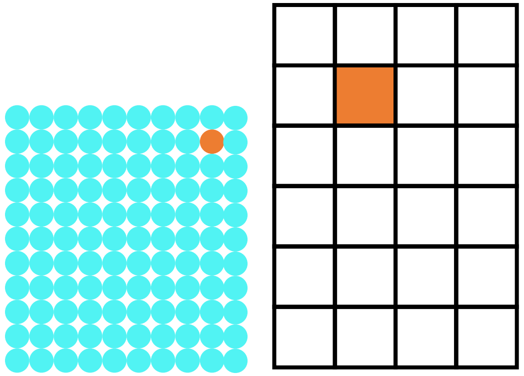

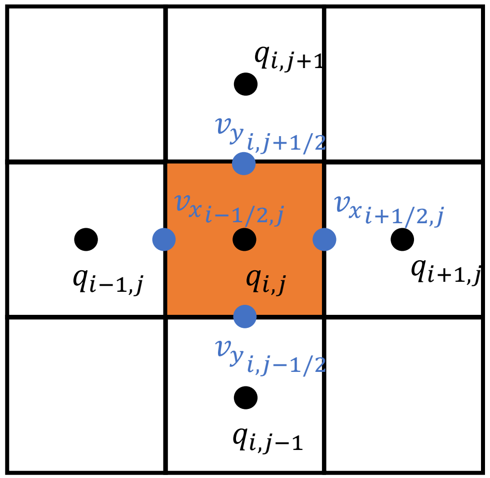

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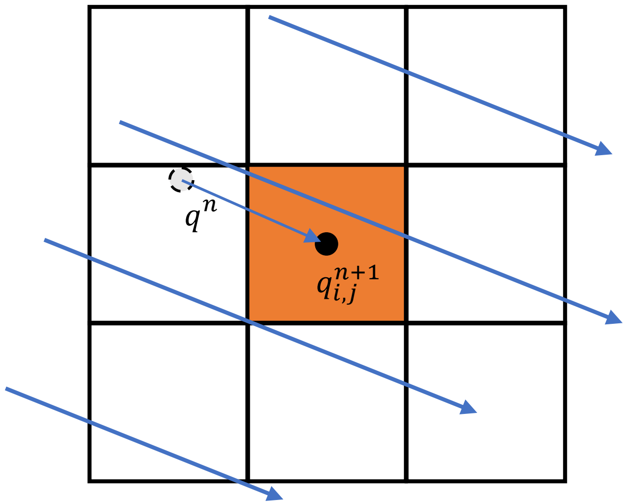

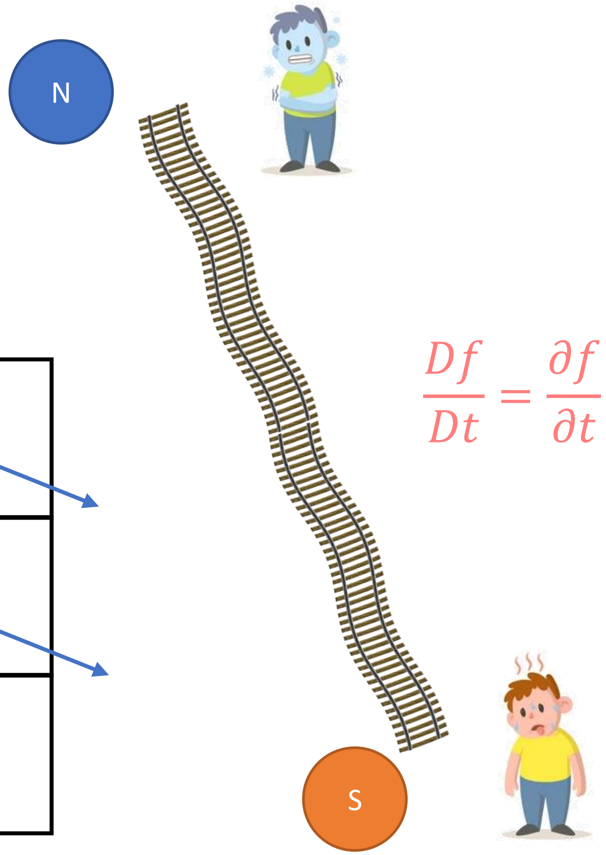

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v \cdot \nabla f$$

Remark

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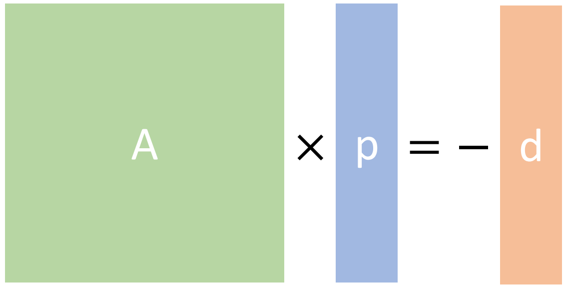
$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\nabla p$$

$$\nabla \cdot v = 0$$

$$-\nabla \cdot v^n = -\frac{\Delta t}{\rho}\nabla \cdot \nabla p$$



Remark

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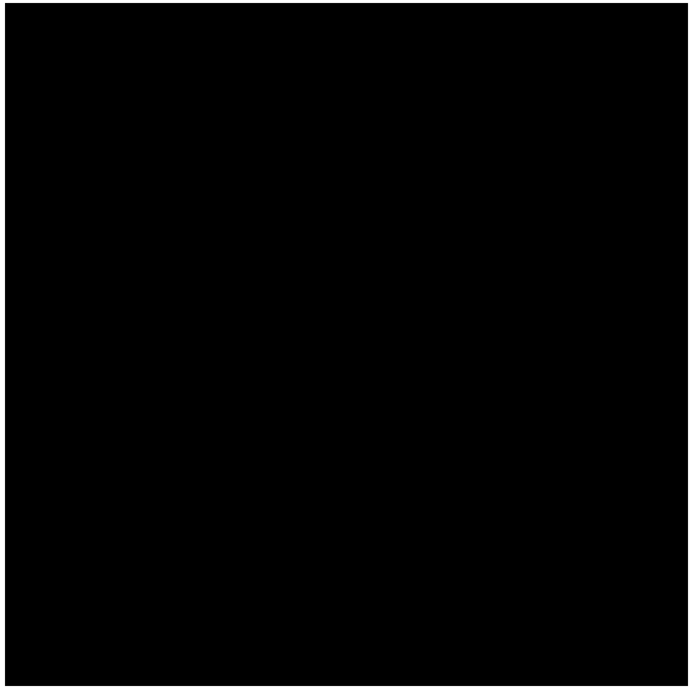
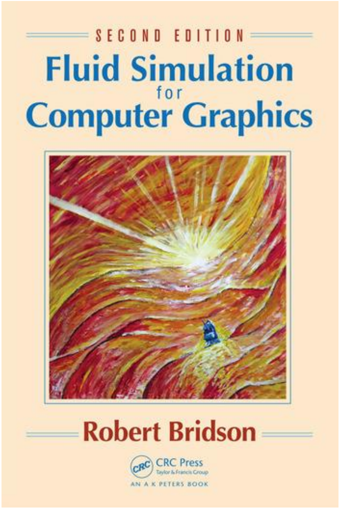
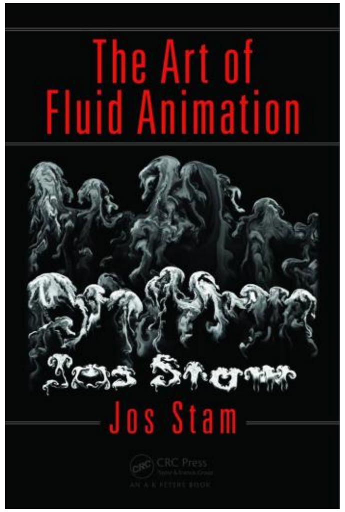


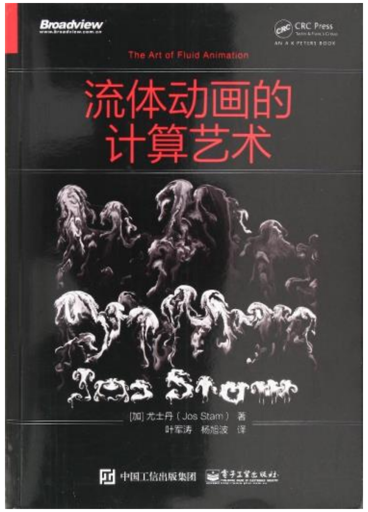
Further readings

- *Stable Fluids* [Stam 1999][Paper][Slides]
- *Fluid Simulation,* Chapter 1 ~ 6 [Bridson and Müller 2007][Course Notes]

- Books
  - *Fluid Simulation for Computer Graphics* [Bridson 2015]
  - 科普向：
    - *The Art of Fluid Animation* [Stam 2015]
    - 《流体动画的计算艺术》 -- 叶军涛、杨旭波译







Homework

Homework Today

- Check Taichi examples:
  - https://github.com/taichi-dev/taichi/blob/master/python/taichi/examples/simulation/stable_fluid.py

- Try:
  - free surface (Dirichlet boundary condition)
  - buoyancy/vorticity confinement force for smoke [Chapter 5]
  - sharper interpolation schemes [Link]
  - MacCormack method [Link] / BFECC [Paper]
  - Conjugate gradient linear solvers

Candidate projects for your final

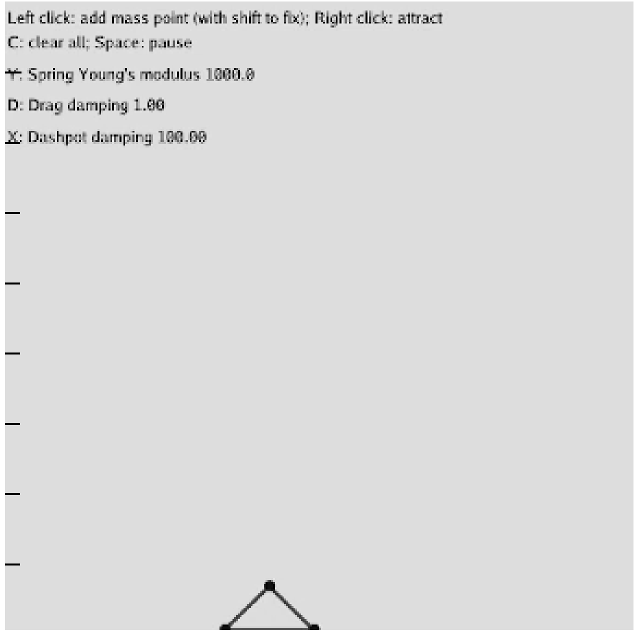
- Candidate topics:
  - Good performance! [Section 4.3]
    - Multigrid-preconditioned CG / Modified-incomplete-Cholesky-preconditioned CG
  - Complex boundaries [Section 4.5]
  - Eulerian water [Chapter 6]
  - Advection-reflection method [Paper]
  - Render your Eulerian fluid with your own renderer

- Both 2D and 3D projects are great!
  - As long as your pictures look great ☺

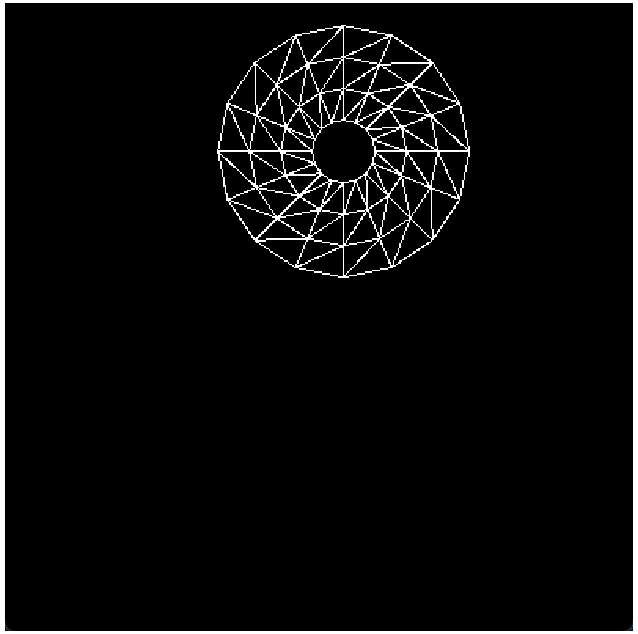
Final project

- 死线：2022年1月3日
- 要求：
  - 使用大作业模板
  - 需要有设计文档，如果有参照代码也需要标明
- 题材：
  - 任何使用Taichi完成的内容（图形学更佳）
  - 可以参考每节图形课后给出的大作业选题灵感 [参考第07,09,10,11讲]
  - 鼓励实现任意图形学论文/图形学课程内容
  - 可以在小作业的基础上完成大作业 (Homework Promotion!)
- 形式：
  - 使用 GitHub/Gitee提交并邀请tgc01@taichi.graphics加入你的代码仓
  - 允许三人以下合作，记得管理多人合作的git commits
- 奖励：
  - 太极图形课第一季结业证书一份+神秘Taichi小礼物一份

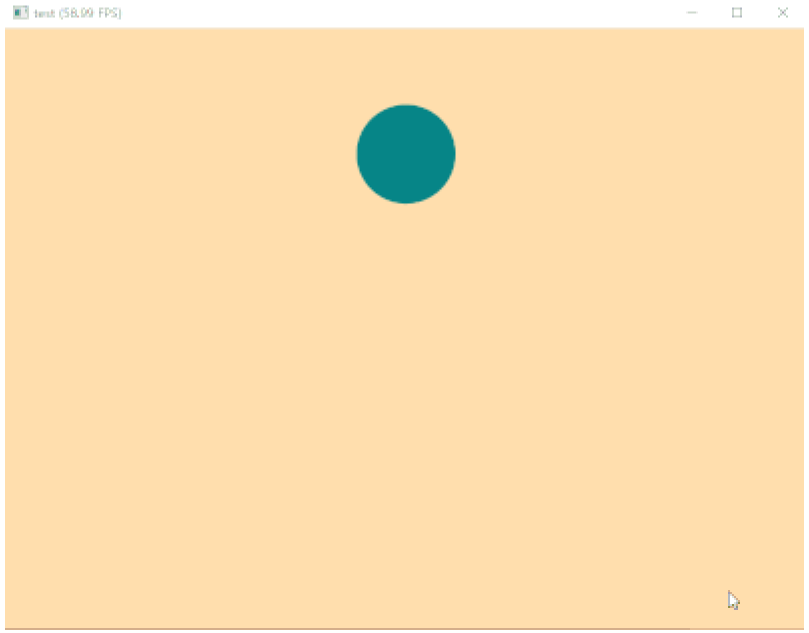
Excellent homework assignments



[@Vineyo]

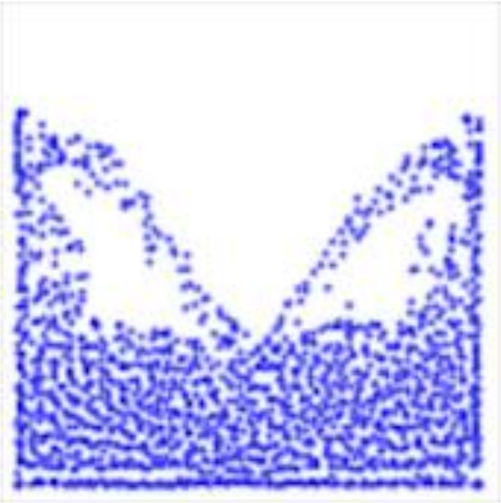


[@cflw]

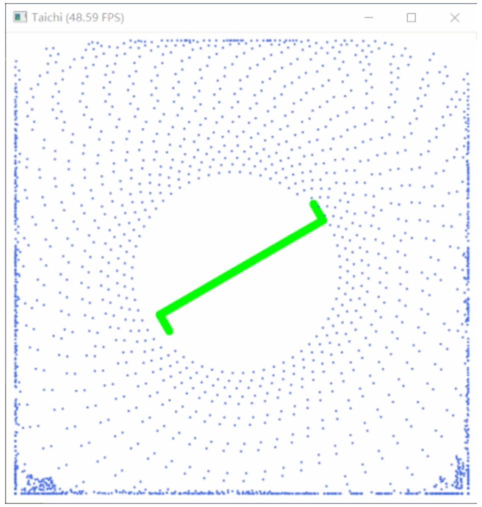


[@niushuqing123]

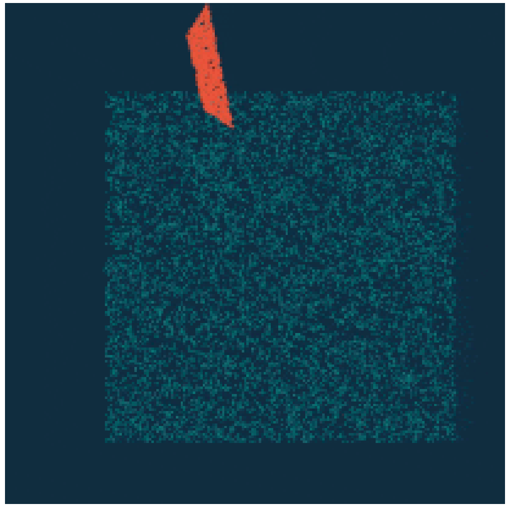
Excellent homework assignments



[@kphmd]



[@Pierce-qiang]



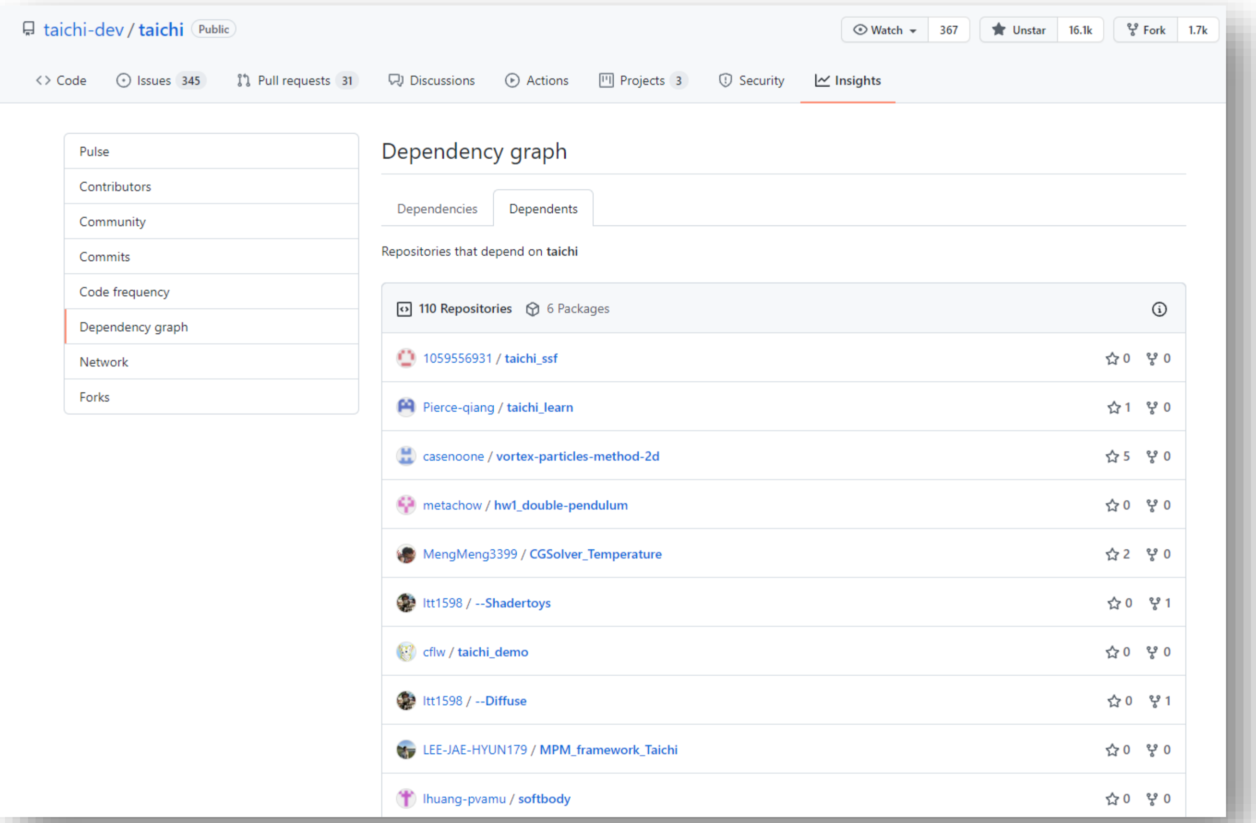
[@runck]

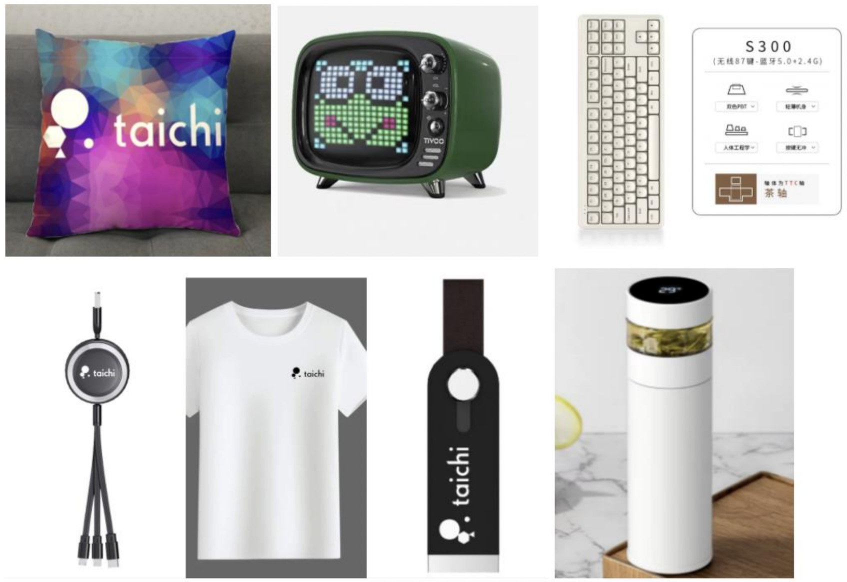
Excellent homework assignments

[@moxunbai]

Gifts for the gifted

- Use Template for your homework
- Next check in the next week! Dec. 14, 2021





Questions?

本次答疑：12/09 🡐作业分享也在这里

下次直播：12/14 🡐小作业抽奖以及第一季大结局

直播回放：Bilibili 搜索「太极图形」

主页&课件：https://github.com/taichiCourse01

主页&课件(backup)：https://docs.taichi.graphics/tgc01