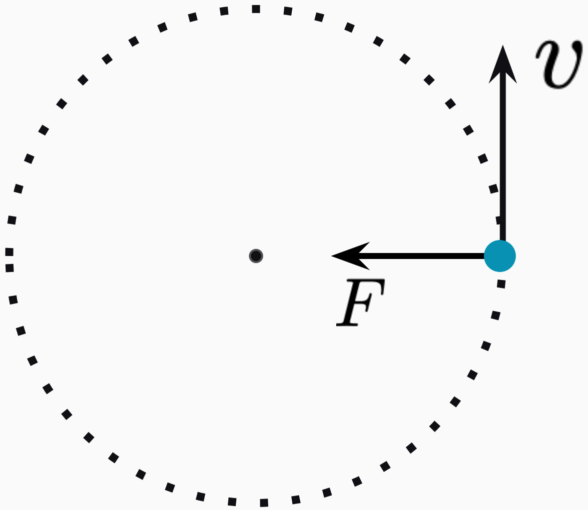
太极图形课

08讲 答疑

taichi

时间积分-天体匀速圆周运动

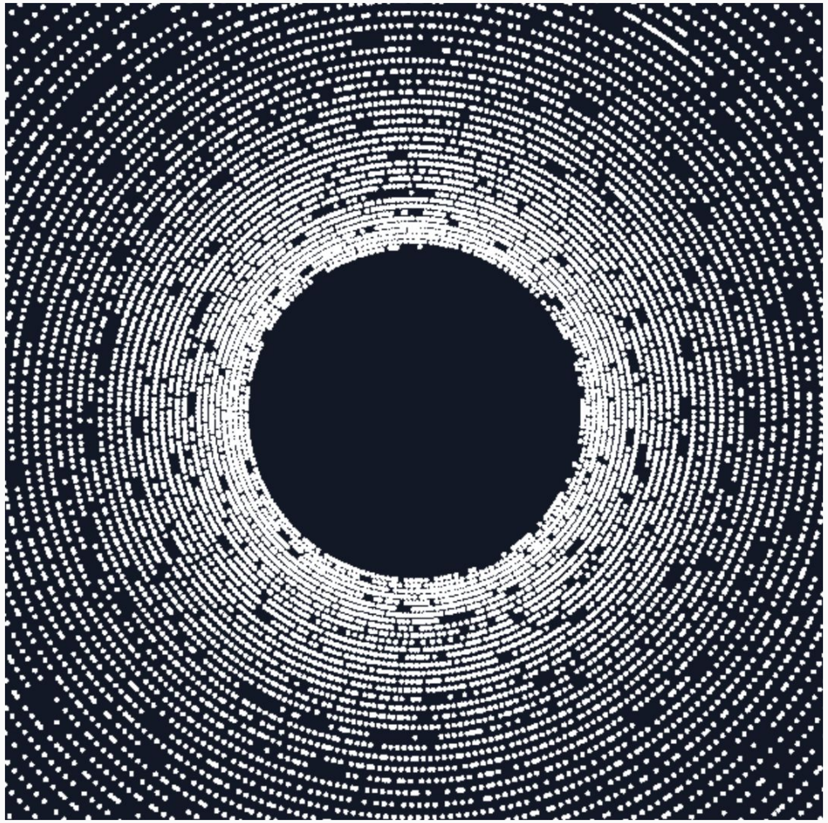
- 高中知识回顾: **万有引力(向心力)**

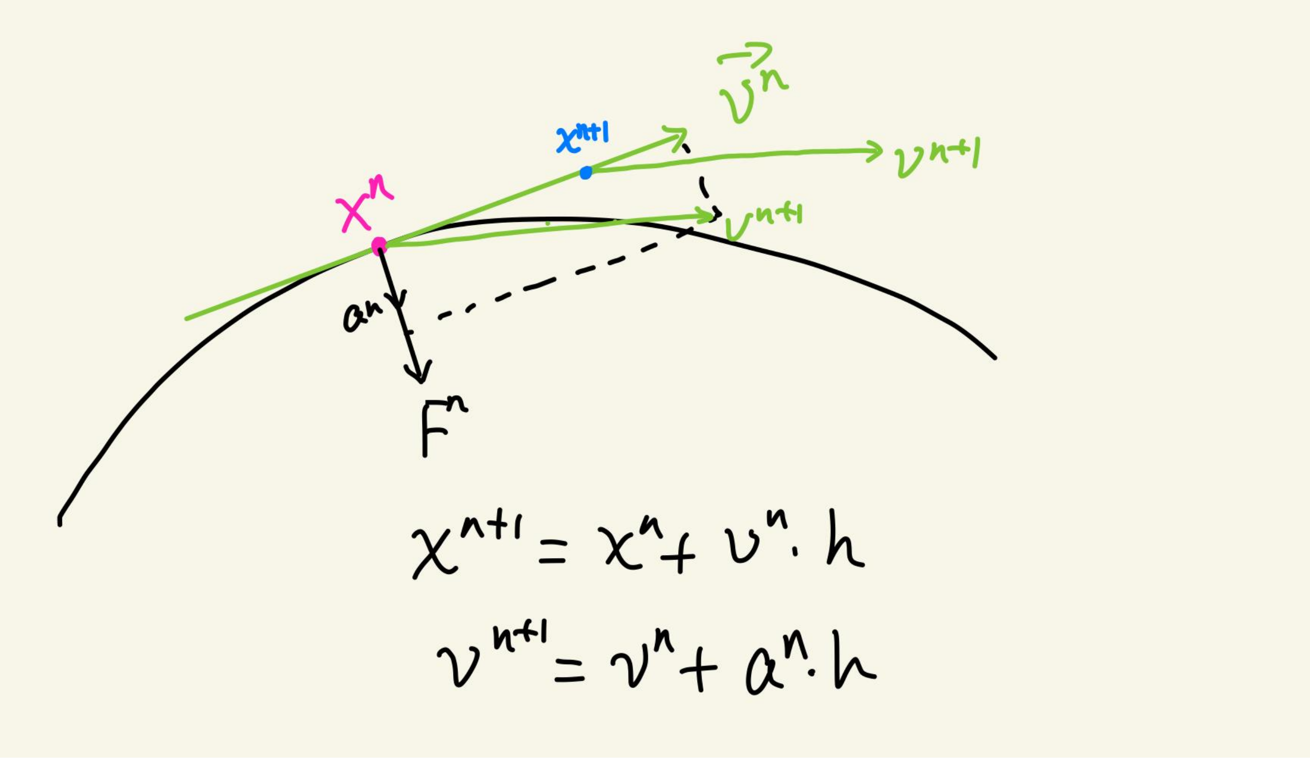


$$F = \frac{Gm_1m_2}{r^2}\hat{r} = m\frac{v^2}{r}\hat{r}$$

时间积分-天体匀速圆周运动

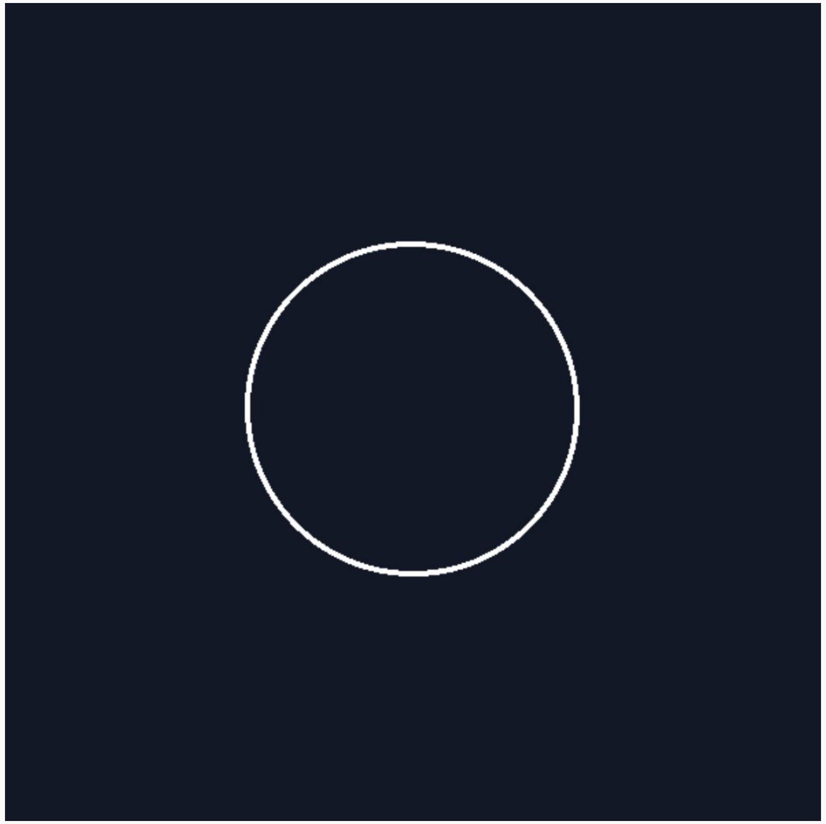
显式欧拉积分(Explicit Euler) :

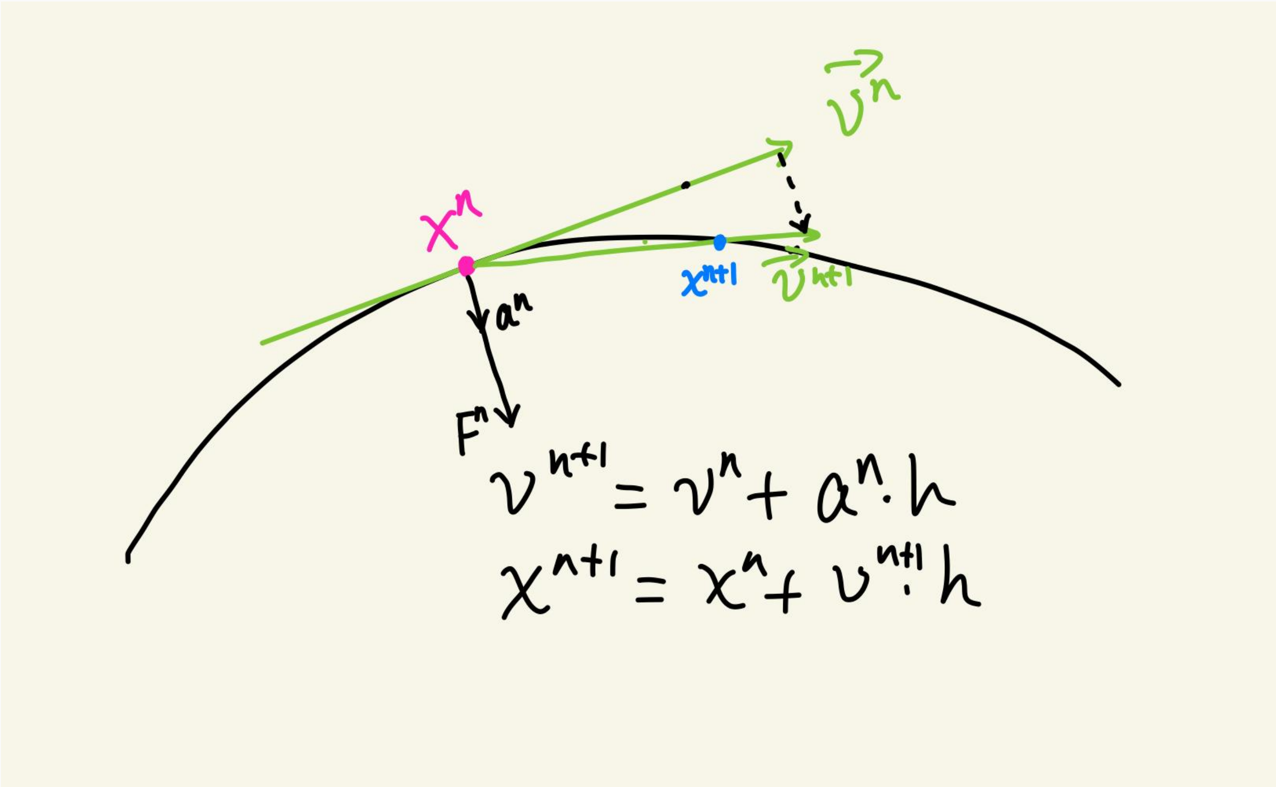




时间积分-天体匀速圆周运动

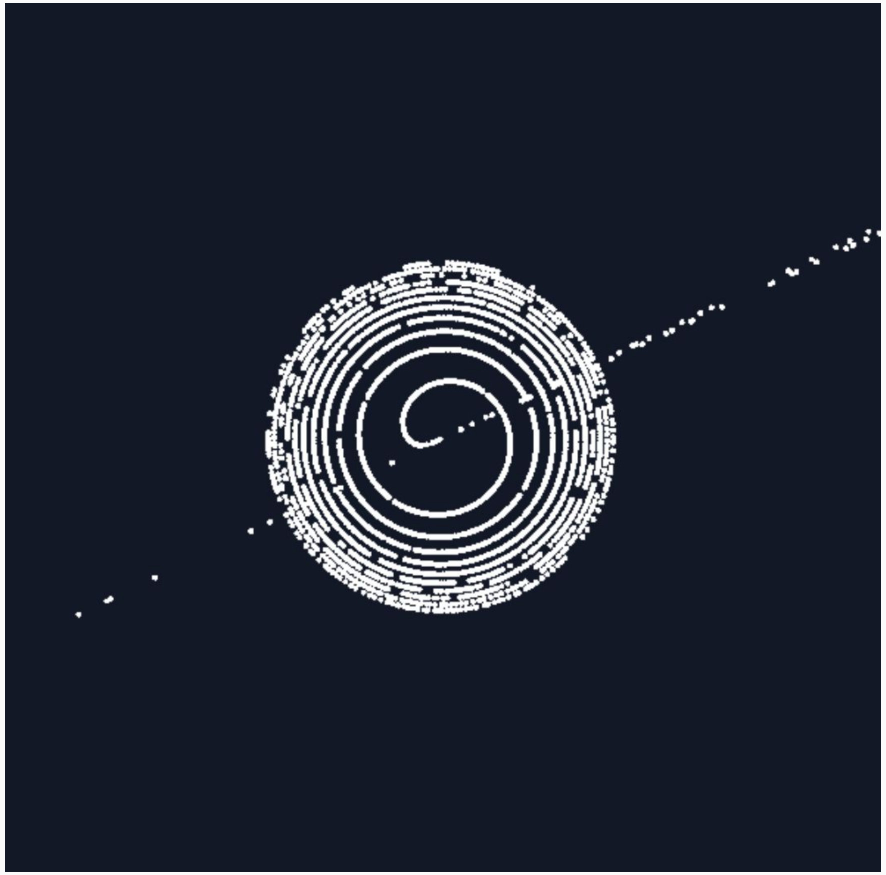
辛欧拉积分(Symplectic Euler)) :

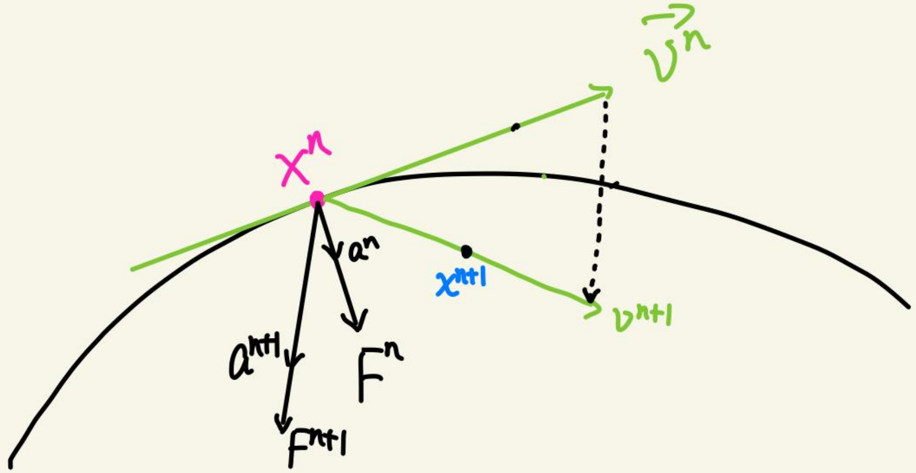




时间积分-天体匀速圆周运动

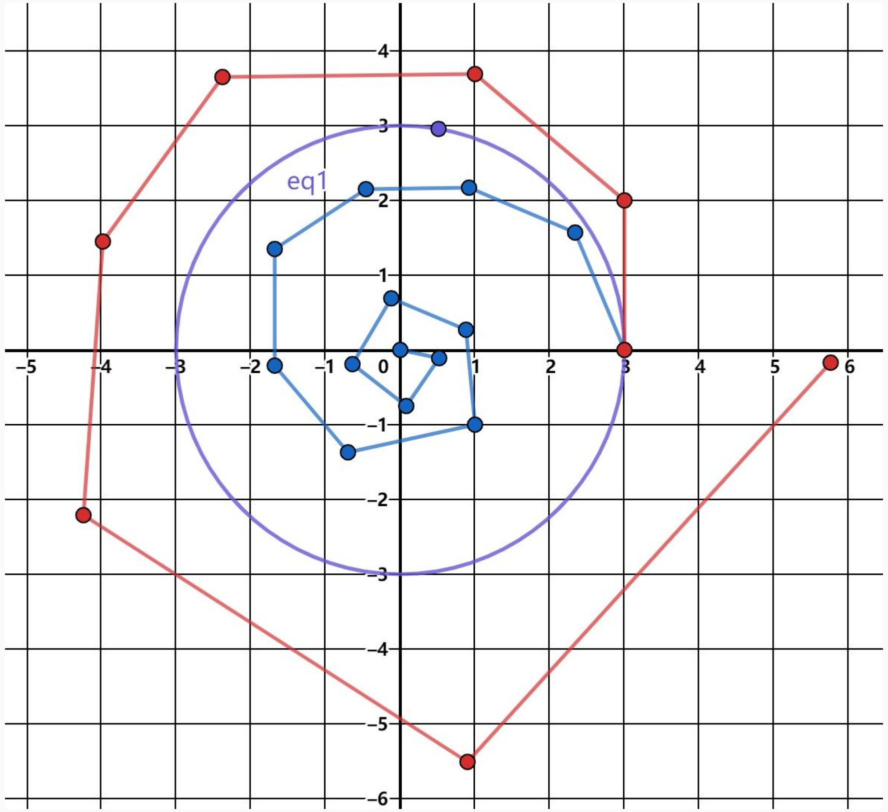
隐式欧拉积分(Implicit Euler) :





$$v^{n+1} = v^n + a^{n+1} \cdot h$$
$$x^{n+1} = x^n + v^{n+1} \cdot h$$

时间积分-天体匀速圆周运动




```python
@ti.kernel
def explicit_update(h: ti.f32):
    force[0] = -pos[0]
    pos[0] += vel[0] * h
    vel[0] += force[0]/mass[0] * h

@ti.kernel
def simplicit_update(h: ti.f32):
    force[0] = -pos[0]
    vel[0] += force[0]/mass[0] * h
    pos[0] += vel[0] * h

@ti.kernel
def implicit_update(h: ti.f32):
    I = ti.Matrix([[1.0, 0.0], [0.0, 1.0]])
    x = pos[0]
    t0 = x.norm()
    K = -I / (t0**3) + 3/(t0**5) * (x @ x.transpose())
    A = I - h**2/mass[0] * K
    force[0] = -pos[0]
    vel[0] = A.inverse() @ (vel[0] + h/mass[0] * force[0])
    pos[0] += vel[0] * h
```

https://zoo.taichi.graphics/playground/e1b3718bcd16f6d79846fea8a0ee1c20

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线性代数(2D)

变形梯度 (Deformation gradient):

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

矩阵的迹(trace) :

$$tr(F) = \sum_{i=1}^{2} F_{ii} = F_{11} + F_{22}$$

Forbenius norm:

$$\|F\|_F = \sqrt{tr(FF^T)} = \sqrt{\sum_{i=1}^{2}\sum_{j=1}^{2} F_{ij}^2} = \sqrt{F_{11}^2 + F_{12}^2 + F_{21}^2 + F_{22}^2}$$

Co-rotated FEM

Co-rotated linear elasticity

```python
@ti.kernel
def compute_total_energy():
    for i in range(N_triangles):
        Ds = compute_D(i)
        F = Ds @ elements_Dm_inv[i]
        # co-rotated linear elasticity
        R = compute_R_2D(F)
        Eye = ti.Matrix.cols([[1.0, 0.0], [0.0, 1.0]])
        element_energy_density = LameMu[None]*((F-R)@(F-R).transpose()).trace() + 0.5*LameLa[None]*(R.transpose()@F-Eye).trace()**2

        total_energy[None] += element_energy_density * elements_V0[i]

```

$$\Psi(F) = \mu \cdot \operatorname{tr}\left((F-R)(F-R)^\top\right) + \frac{\lambda}{2}\operatorname{tr}^2\left(R^\top F - I\right)$$

https://github.com/taichiCourse01/--Deformables/blob/master/fem-explicit.py#L160

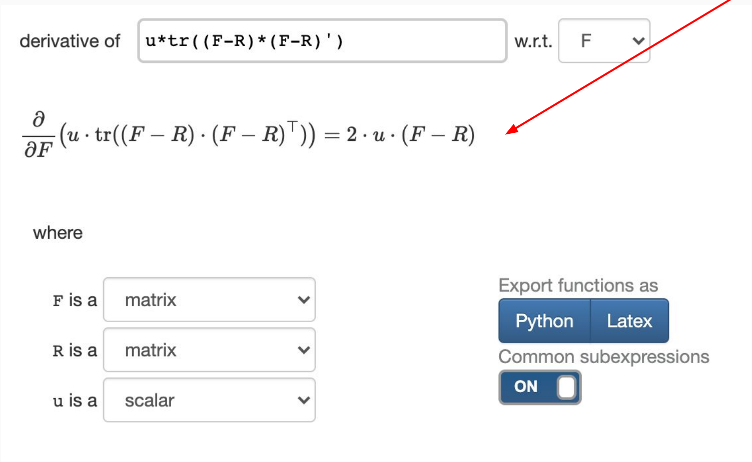
Co-rotated FEM

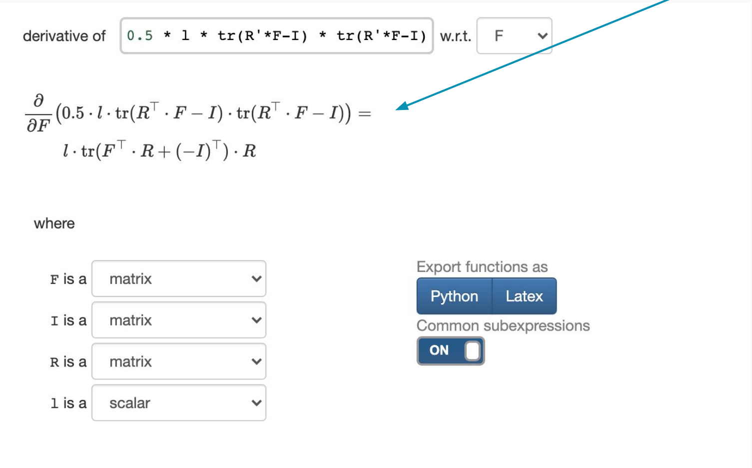
能量密度函数：

$$\Psi(F) = \mu \cdot \mathrm{tr}\left((F-R)(F-R)^\top\right) + \frac{\lambda}{2}\,\mathrm{tr}^2\left(R^\top F - I\right)$$

1-st Piola-Kirchhoff stress (PK1 stress):

$$P = \frac{\partial \Psi(F)}{\partial F} = \boxed{\frac{\partial\left(\mu \cdot \mathrm{tr}\left((F-R)(F-R)^\top\right)\right)}{\partial F}} + \boxed{\frac{\partial\left(\frac{\lambda}{2}\,\mathrm{tr}^2\left(R^\top F - I\right)\right)}{\partial F}}$$





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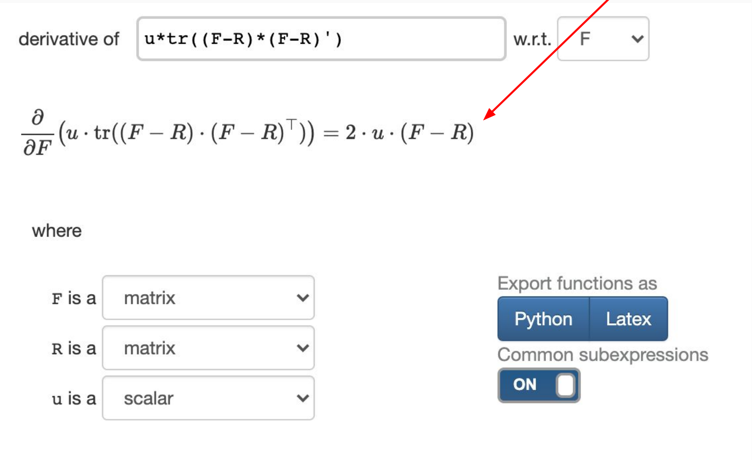
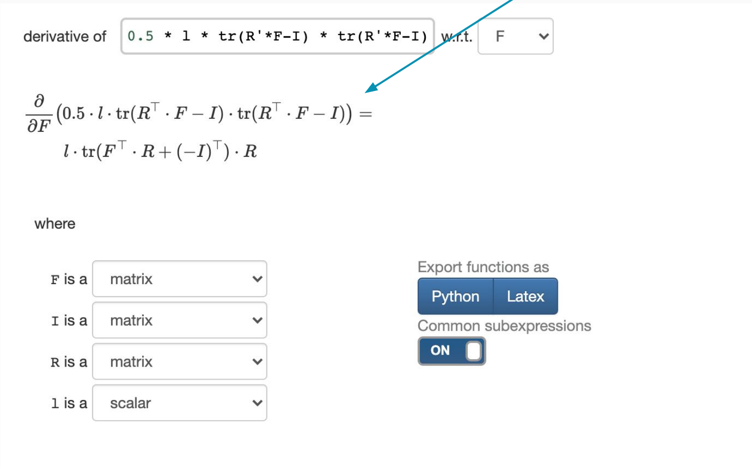
http://www.matrixcalculus.org/

Co-rotated FEM

PK1 stress tensor:

$$P = \frac{\partial \Psi(F)}{\partial F} = \boxed{\frac{\partial\left(\mu \cdot \operatorname{tr}\left((F-R)(F-R)^\top\right)\right)}{\partial F}} + \boxed{\frac{\partial\left(\frac{\lambda}{2}\operatorname{tr}^2\left(R^\top F - I\right)\right)}{\partial F}}$$





http://www.matrixcalculus.org/

$$P = \frac{\partial \Psi(F)}{\partial F} = 2\mu \cdot (F-R) + \lambda \operatorname{tr}(R^\top F - T)R$$

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Co-rotated FEM

PK1 stress tensor:

$$P = \frac{\partial \Psi(F)}{\partial F} = \boxed{2\mu \cdot (F - R)} + \boxed{\lambda \operatorname{tr}(R^\top F - T)R}$$

```python
    # gradient of elastic potential
    for i in range(N_triangles):
        Ds = compute_D(i)
        F = Ds@elements_Dm_inv[i]
        # co-rotated linear elasticity
        R = compute_R_2D(F)
        Eye = ti.Matrix.cols([[1.0, 0.0], [0.0, 1.0]])
        # first Piola-Kirchhoff tensor
        P = 2*LameMu[None]*(F-R) + LameLa[None]*((R.transpose())@F-Eye).trace()*R
```

https://github.com/taichiCourse01/--Deformables/blob/8706d393085b90aaf4197e32fdfb215d47d0fc1c/fem-explicit.py#L141

Co-rotated FEM

PK1 stress tensor:

$$P = \frac{\partial \Psi(F)}{\partial F} = \frac{\partial(\mu \cdot \mathrm{tr}\left((F-R)(F-R)^\top\right))}{\partial F} + \text{[redacted]}$$

令 $A = F - R$

$$\Psi_1(F) = \mu \cdot \mathrm{tr}\left(AA^\top\right) = \mu \cdot (A_{11}^2 + A_{12}^2 + A21^2 + A_{22}^2)$$

$$P_1 = \frac{\partial \Psi_1(F)}{\partial F} = \frac{\partial A}{\partial F} : \boxed{\frac{\partial \Psi_1(F)}{\partial A}}$$

$$\frac{\partial \Psi_1(F)}{\partial A} = \begin{bmatrix} \frac{\partial \Psi_1(F)}{\partial A_{11}} & \frac{\partial \Psi_1(F)}{\partial A_{12}} \\ \frac{\partial \Psi_1(F)}{\partial A_{21}} & \frac{\partial \Psi_1(F)}{\partial A_{22}} \end{bmatrix} = \mu \cdot \begin{bmatrix} 2A_{11} & 2A_{12} \\ 2A_{21} & 2A_{22} \end{bmatrix} = 2\mu \cdot A$$

Co-rotated FEM

PK1 stress tensor:

$$P_1 = \frac{\partial \Psi_1(F)}{\partial F} = \boxed{\frac{\partial A}{\partial F}} : \frac{\partial \Psi_1(F)}{\partial A}$$

$$A = F - R = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} F_{11} - R_{11} & F_{12} - R_{12} \\ F_{21} - R_{21} & F_{22} - R_{22} \end{bmatrix}$$

$$\frac{\partial A}{\partial F} = \begin{bmatrix} \frac{\partial A}{\partial F_{11}} & \frac{\partial A}{\partial F_{12}} \\ \frac{\partial A}{\partial F_{21}} & \frac{\partial A}{\partial F_{22}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \frac{\partial A_{11}}{\partial F_{11}} & \frac{\partial A_{12}}{\partial F_{11}} \\ \frac{\partial A_{21}}{\partial F_{11}} & \frac{\partial A_{22}}{\partial F_{11}} \end{bmatrix} & \begin{bmatrix} \frac{\partial A_{11}}{\partial F_{12}} & \frac{\partial A_{12}}{\partial F_{12}} \\ \frac{\partial A_{21}}{\partial F_{12}} & \frac{\partial A_{22}}{\partial F_{12}} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial A_{11}}{\partial F_{21}} & \frac{\partial_{12}}{\partial F_{21}} \\ \frac{\partial A_{21}}{\partial F_{21}} & \frac{\partial A_{22}}{\partial F_{21}} \end{bmatrix} & \begin{bmatrix} \frac{\partial A_{11}}{\partial F_{22}} & \frac{\partial A_{12}}{\partial F_{22}} \\ \frac{\partial A_{21}}{\partial F_{22}} & \frac{\partial A_{22}}{\partial F_{22}} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \in R^{(2\times 2)\times(2\times 2)}$$

Co-rotated FEM

PK1 stress tensor:

$$P_1 = \frac{\partial \Psi_1(F)}{\partial F} = \frac{\partial A}{\partial F} : \frac{\partial \Psi_1(F)}{\partial A} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right] : \mu \begin{bmatrix} 2A_{11} & 2A_{12} \\ 2A_{21} & 2A_{22} \end{bmatrix} = 2\mu \cdot A$$

更简单的方法:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad vec(A) = \begin{bmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \end{bmatrix}$$

$$\mathrm{vec}(P_1) = \mathrm{vec}(\frac{\partial A}{\partial F})\,\mathrm{vec}(\frac{\partial \Psi_1(F)}{\partial A}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot u \begin{bmatrix} 2A_{11} \\ 2A_{21} \\ 2A_{12} \\ 2A_{22} \end{bmatrix} = 2u \begin{bmatrix} A_{11} \\ A_{21} \\ A_{12} \\ A_{22} \end{bmatrix} = 2u \cdot \mathrm{vec}(A)$$

Co-rotated FEM

PK1 stress tensor:

$$P = \frac{\partial \Psi(F)}{\partial F} = \frac{\partial(\mu \cdot \operatorname{tr}\left((F-R)(F-R)^\top\right))}{\partial F} + \frac{\partial(\frac{\lambda}{2}\operatorname{tr}^2\left(R^\top F - I\right))}{\partial F}$$

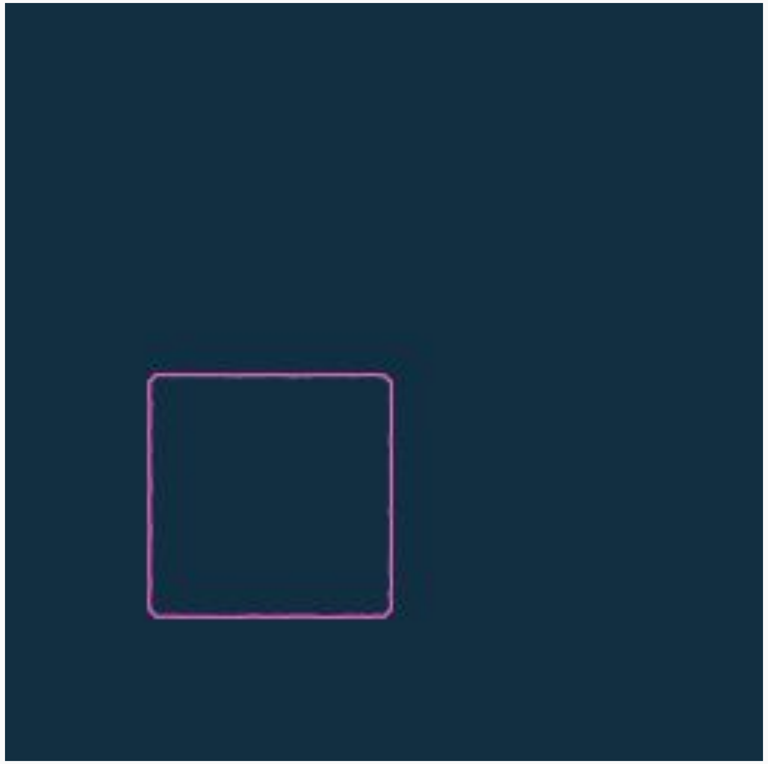
$$P_2 = \frac{\partial(\frac{\lambda}{2}\operatorname{tr}^2\left(R^\top F - I\right))}{\partial F} = \lambda \operatorname{tr}\left(R^\top F - I\right)) \frac{\partial \operatorname{tr}(R^T F - I)}{\partial F}$$

$$\frac{\partial \operatorname{tr}(R^T F - I)}{\partial F} = \frac{\partial(R^T F)}{\partial F} : \frac{\partial \operatorname{tr}(R^T F - I)}{\partial(R^T F - I)} = R$$

$$P_2 = \lambda \operatorname{tr}\left(R^\top F - I\right))R$$

$$P = \frac{\partial \Psi(F)}{\partial F} = P_1 + P_2 = 2\mu \cdot (F - R) + \lambda \operatorname{tr}(R^\top F - T)R$$

优秀作业展示：Marching Squares



作业链接：https://forum.taichi.graphics/t/1-mpm/1775

@wangfeng70117

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