Tanta University

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MATHEMATICS

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PART I

Dedication

To

All those in my life who know that love is reflected in love

2024/2025 **2024/2025** **2024/2025**

2024/2025 2024/2025 2024/2025

2024/2025 2024/2025 2024/2025

Contents	*Pages*

2024/2025 2024/2025 2024/2025

CHAPTER (1)

PROGRESSIONS AND SERIES

A sequence is quantities or numbers that sequence according to a specific rule governing the composition of the sequence. The sequence takes one of the following forms:

(1) An arithmetic progression (AP), or arithmetic sequence

❖ ***Definition:***

An arithmetic progression (or sequence) is a sequence where the difference d between successive terms is constant. The general term of an arithmetic sequence can be written in terms of its first term a_1, common difference d, and index n as follows:

$$a_n = a_1 + (n-1)\,d.$$

In fact, any general term that is linear in n defines an arithmetic sequence.

An arithmetic series is the sum of the terms of an arithmetic sequence.

For example:

3, 5, 7, 9, … is an arithmetic sequence with base 2.

11, 9, 7, 5, … is an arithmetic sequence with base −2.

x, 3x, 5x, 7x… is an arithmetic sequence with base 2x.

and natural numbers, 1, 3, 3, 4, … is an arithmetic sequence with base 1.

Generally, any arithmetic progression may be written in the form:

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ \ldots,\ a+(n-1)\,d$$

where:

a = the first term

d = the base (common difference)

❖ <u>*General term in arithmetic progression (T_n)*</u>

From the general term of an arithmetic sequence, the general term may be reached as follows:

The first term $t_1 = a$

The second term $t_2 = a + d$

-1) d. Thus, n^{th} term in an arithmetic sequence is determined by the formula:

$$T_n = a + (n-1)\,d \ \ldots\ldots\ldots\ldots \ (1)$$

NOTE: Formula (1) contains four elements (or variables): T_n, a, *n, d. Given three of the four elements, we can solve for the fourth.*

Example (1)

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 100th term: 7,10,13,16,19, …

SOLUTION

Begin by finding the common difference

$d = 10 - 7 = 3,$

Note that the difference between any two successive terms is 3. The sequence is indeed an arithmetic progression where $a_1 = 7$ and $d = 3$.

$$a_n = a_1 + (n - 1)\, d$$

$$= 7 + (n - 1) \times 3$$

$$= 7 + 3n - 3 = 3n + 4$$

Therefore, we can write the general term $a_n = 3n + 4$. Take a minute to verify that this equation describes the given sequence. Use this equation to find the 100 th term:

$$a_{100} = 3\,(100) + 4 = 304$$

Answer:

$$\boxed{a_n = 3n + 4;\ a_{100} = 304}$$

Example (2)

Find an equation for the general term of the given arithmetic sequence and use it to calculate its 75^{th} term: 6,4,2,0, −2, …

SOLUTION

Begin by finding the common difference,

$d = 4 - 6 = -2$

Next find the formula for the general term, here $a_1 = 6$ and $d = -2$.

$a_n = a_1 + (n-1)\,d$

$= 6 + (n-1)(-2)$

$= 6 - 2n + 2$

$= 8 - 2n$

Therefore, $a_n = 8 - 2n$ and the 75^{th} term can be calculated as follows:

$a_{75} = 8 - 2\,(75)$

$= 8 - 150$

$= -142$

<u>*Answer:*</u>

$$\boxed{a_n = 8 - 2n;\ a_{100} = -142}$$

❖ <u>*Sum of n terms of an arithmetic sequence:*</u>

(A) Finding Sum of n terms of an arithmetic sequence, given the first term, a, number of terms, n, and the last term l:

S = a + (a + d) + (a + 2d) + (a + 3d) + … + (*l – 2d*) + (*l – d*) + *l* ------------- *(1)*

Writing this series in reverse we have,

S = *l* + (*l* – d) + (*l* – 2d) + … + (a + d) + a ------------ (2)

(1) + (2) give,

$$2S = (a + l) + (a + l) + (a + l) + \ldots + (a + l)$$

$$2S = n\,(a + l)$$

Because the number of parentheses = number of terms = n

$$S = \frac{n}{2}(a + l) = \frac{n\,(a+l)}{2}$$

$$\text{Or } S = \frac{n}{2}(a_1 + a_n) = \frac{n\,(a_1 + a_n)}{2} \qquad \ldots\ldots\ldots\ldots\ldots\ (2)$$

Use this formula to calculate the sum of the first 100 terms of the sequence defined by $a_n = 2n - 1$. Here $a_1 = 1$ and $a_{100} = 199$.

$$S_{100} = \frac{n\,(a_1 + a_n)}{2}$$

$$= \frac{100\,(1 + 199)}{2}$$

$$= 10{,}000$$

(B) Finding Sum of n terms of an arithmetic sequence, given the first term, a, base/common difference d, number of terms n:

$$S = a + (a + d) + (a + 2d) + \ldots + [a + (n - 2)\,d] + [a + (n - 1)\,d] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

If we start with the last term (or writing this series in reverse), we have the same sum as follows:

$$S = [a + (n - 1)\,d] + [a + (n - 2)\,d] + \ldots + [a + 2d)] + (a + d) + a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

(1) + (2) give,

$2S = n\,[2a + (n-1)\,d$

$$\boxed{S = \frac{n}{2}[2a + (n-1)\,d] \;\ldots\ldots\ldots\; (3)}$$

The previous formula (3) can be found in a different way as follows:

$S = \frac{n}{2}[2a + (n-1)\,d],$

$l = T_n = a + (n-1)\,d$

And by substituting for l in the formula: $l = T_n = a + (n-1)\,d$, we get,

$S = \frac{n}{2}\;[a + a + (n-1)\,d$

$$\boxed{S = \frac{n}{2}[2a + (n-1)\,d] \;\ldots\ldots\; (3)}$$

NOTE: Formula (3) contains four elements (or variables): S, a, n, d. *Given three of the four elements, we can solve for the fourth.*

The following examples illustrate how to apply the previous mathematical formulas:

Example (3)

In the following sequence: 2, 5, 8, 11, … up to 20 terms. Find:

(A) t_9 *(B)* t_{15} *(C)* t_{20} *(D) Sum of the sequence (S)*

SOLUTION:

a = 2, d = 3, n = 20

(A) Finding t_9:

$T_n = a + (n - 1)\,d$

$t_9 = a + 8d = 2 + 8 \times 3 = 26$

<u>(B) Finding t_{15}</u>

$T_n = a + (n - 1)\,d$

$t_{15} = a + 14d = 2 + 14 \times 3 = 44$

<u>(C) Finding t_{20}</u>

$T_n = a + (n - 1)\,d$

$t_{20} = a + 19d = 2 + 19 \times 3 = 59$

(D) Finding Sum of the sequence (S): It can be found in two ways.

1st method: $S = \frac{n}{2}(a + l)$

$S = \frac{20}{2}(2 + 59) = 10 \times 61 = 610$

2nd method: $S = \frac{n}{2}[2a + (n - 1)\,d]$

$S = \frac{20}{2}[(2 \times 2 + 19 \times 3)] = 10\,[4 + 57] = 10 \times 61 = 610$

Example (4)

If the sum of the first three terms of an arithmetic sequence is 80, and the number of terms of the sequence is 20. Find the sum of the sequence.

SOLUTION

$a + (a + d) + (a + 2d) = 15$

$\therefore\ 3a + 3d = 15$ divide both sides by 5, we have

$\therefore\ a + d = 5$ $a = 5 - d$

$t_2 = a + d = 5$ $\therefore\ t_3 = a + 2d = 5 + d$

Therefore, $(5 - d) \times 5\ (5 + d) = 80$ divide both sides by 5, we get,
$(5 - d)\ (5 + d) = 16$

$25 - d^2 = 16$ $\therefore\ d^2 = 9$ $d = \pm\ 3$

Therefore, d (base or common difference) = 3 (increasing sequence), or d = – 3 (decreasing sequence).

<u>When d = 3</u>

$a = 5 - d$ $\therefore\ a = 5 - 3 = 2 = t_1$ $t_2 = 5$ $t_3 = 8$

$t_1 + t_2 + t_3 = 2 + 5 + 8 = 15$

Therefore, sequence is an increasing arithmetic sequence, when d = 3

The final term can therefore be found as follows:

$\because\ T_n = a + (n - 1)\ d$

$\therefore\ t_{20} = a + 19\ d = 2 + 19 \times 3 = 59$

And the sum of the sequence given the first term, last term, and number of terms, is:

$S = \frac{n}{2}(a + l)$

$S = \frac{20}{2}(2 + 59) = 10 \times 61 = 610$

NOTE: Sum of the sequence can be found using the following formula directly:

$S = \frac{n}{2}[2a + (n - 1)\,d]$

$S = \frac{20}{2}[(2 \times 2 + 19 \times 3)] = 10\,[4 + 57] = 10 \times 61 = 610$

Example (5)

An arithmetic sequence whose first term is 15, its base is 5, and its sum is 260. Find its number of terms.

SOLUTION

$a = 15, \qquad d = 5, \qquad S = 260$

$\because\ S = \frac{n}{2}[2a + (n - 1)\,d]$

$\therefore 260 = \frac{n}{2}[30 + 5n - 5] = \frac{n}{2}[5n + 25]$

$\therefore 520 = 5n^2 + 25n$ divide both sides by 5

$\therefore 104 = n^2 + 5n$ $\qquad \therefore\ n^2 + 5n - 104 = 0$

$\therefore\ (n + 13)(n - 8) = 0 \rightarrow n = 8$ or $n = -13$ (rejected)

$\therefore$ n (number of terms) = 8

Example (6)

If the general term in an arithmetic sequence is given by the formula: $t_r = 1 + 3r$.

Find: (A) the sum of the last 50 terms in the sequence, (B) What number is the term 216? [$t_{r=?} = 256 \rightarrow r = 85$]

SOLUTION

(A) Finding the sum of the last 50 terms in the sequence

r = 51, n = 100

$t_{51} = 1 + 3 \times 51 = 154$

$t_{100} = 1 + 3 \times 100 = 301$

$S = \frac{50}{2}(154 + 301) = 25 \times 455 = 11375$

(B) Finding $t_{r=?} = 256$

$\because T_r = 1 + 3r$

$\therefore 256 = 1 + 3r$ $\therefore 3r = 255$ $\therefore r = 85$

- **Arithmetic Means:** The terms between given terms of an arithmetic sequence are called arithmetic means.

Example (7)

Find all terms in between $a_1 = -8$ and $a_7 = 10$ of an arithmetic sequence. In other words, find all arithmetic means between the 1^{st} and 7^{th} terms.

SOLUTION

Begin by finding the common difference d.

In this case, we are given the first and seventh term:

$a_n = a_1 + (n-1)\,d$ Use n = 7.

$a_7 = a_1 + (7-1)\,d$

$a_7 = a_1 + 6\,d$

Substitute

a_1 = − 8 and a_7 = 10 into the above equation and then solve for the common difference d.

$10 = -8 + 6d \longrightarrow 18 = 6d \longrightarrow 3 = d$

Next, use the first term $a_1 = -8$ and the common difference d = 3 to find an equation for the nth term of the sequence.

$a_n = -8 + (n-1) \cdot 3$

$= -8 + 3n - 3$

$= -11 + 3n$

With $a_n = 3n - 11$, where n is a positive integer, find the missing terms.

Answer: −5, −2, 1, 4, 7

$a_1 = 3(1) - 11 = -8$

$a_2 = 3(2) - 11 = -5$

$a_3 = 3(3) - 11 = -2$

$a_4 = 3(4) - 11 = 1$

arithmetic means

$a_5 = 3(5) - 11 = 4$

$a_6 = 3(6) - 11 = 7$

$a_7 = 3(7) - 11 = 10$

In some cases, the first term of an arithmetic sequence may not be given.

Example (8)

The first row of seating in an outdoor amphitheater contains 26 seats, the second row contains 28 seats, the third row contains 30 seats, and so on. If there are 18 rows, what is the total seating capacity of the theater?

SOLUTION

Begin by finding a formula that gives the number of seats in any row. Here the number of seats in each row forms sequence:

$$26, 28, 30, \ldots$$

Note that the difference between any two successive terms is 2. The sequence is an arithmetic progression where $a_1 = 26$ and $d = 2$.

$$\begin{aligned} a_n &= a_1 + (n-1)\, d \\ &= 26 + (n-1) \cdot 2 \\ &= 26 + 2n - 2 \\ &= 2n + 24 \end{aligned}$$

Therefore, the number of seats in each row is given by $a_n = 2n + 24$. To calculate the total seating capacity of the 18 rows we need to calculate the 1st and the 18th terms:

$$a_1 = 26$$

$a_{18} = 2\ (18) + 24 = 60$

Use this to calculate the partial sum as follows:

$S_n = n\ (a_1 + a_n)\ /\ 2$

$S_{18} = 18\ (a_1 + a_{18})\ /\ 2$

$= 18\ (26 + 60)\ /\ 2$

$= 9\ (86)$

$= 774$

Answer:

There are 774 seats total.

Important Notes on Arithmetic Progression:

- An AP is a list of numbers in which each term is obtained by adding a fixed number to the preceding number.
- a is represented as the first term, d is a common difference, a_n as the n^{th} term, and n as the number of terms.
- In general, AP can be represented as a, a + d, a + 2d, a + 3d, ..., a + (n−1) d
- The n^{th} term of an AP can be obtained by $a_n = a + (n - 1)\ d$

- The sum of an AP can be obtained by $s_n = n/2\ [2a + (n - 1)\ d]$
- The graph of an AP is a straight line with the slope as the common difference.
- The common difference doesn't need to be positive always. For example, in the progression, 16, 8, 0, −8, −16, ... the common difference is negative (d = 8 − 16 = 0 − 8 = −8 − 0 = −16 − (−8) =... = −8).
- An arithmetic sequence is a sequence where the difference d between successive terms is constant.
- The general term of an arithmetic sequence can be written in terms of its first term a_1, common difference d, and index n as follows: $a_n = a_1 + (n - 1)\ d$.
- An arithmetic series is the sum of the terms of an arithmetic sequence.
- The nth partial sum of an arithmetic sequence can be calculated using the first and last terms as follows: $S_n = n\ (a_1 + a_n) / 2$.

(2) Geometric progression

A geometric progression, also known as a geometric sequence, is a mathematical sequence of non-zero numbers where each term after the first is found by multiplying the previous one by a fixed number called the *common ratio*. For example, the sequence 2, 6, 18, 54, ... is a geometric progression with a common ratio of 3. Similarly, 10, 5, 2.5, 1.25, ... is a geometric sequence with a common ratio of 1/2.

Examples of a geometric sequence are powers r^k of a fixed non-zero number r, such as 2^k and 3^k. The general form of a geometric sequence is

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$$

where r is the common ratio and a is the initial value.

 The sum of a geometric progression's terms is called a *geometric* *series*.

❖ Geometric Progression, Series & Sums

- Introduction

A geometric sequence is a sequence such that any element after the first is obtained by multiplying the preceding element by a constant called the common ratio which is denoted by r. The common ratio (r) is obtained by dividing any term by the preceding term, i.e.,

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = \frac{a_n}{a_{n-1}}$$

Where	*R*	common ratio
	a_1	*first term*
	a_2	*second term*
	a_3	*third term*
	a_{n-1}	*the term before the nth term*
	a_n	*the nth term*

The geometric sequence is sometimes called the geometric progression or GP, for short.

For example, the sequence 1, 3, 9, 27, 81 is a geometric sequence. Note that after the first term, the next term is obtained by multiplying the preceding element by 3.

The geometric sequence has its sequence formation: $a_1, a_1\, r,\, a_1 r^2, \ldots, a_1 r^{n-1}$.

To find the *nth* term of a geometric sequence we use the formula:

$$a_n = a_1 r^{n-1}$$

Where	*R*	common ratio
	a_1	first term
	a_{n-1}	the term before the *n* th term
	N	number of terms

❖ Sum of Terms in a Geometric Progression

Finding the sum of terms in a geometric progression is easily obtained by applying the formulas:

nth partial sum of a geometric sequence

$$\boxed{S_n = \frac{a_1\,(1-r^n)}{1-r}\ , \qquad r \neq 1}$$

sum to infinity

$$\boxed{S_\infty = \sum_{r=1}^{\infty} a\,r^{n-1} = \frac{a_1}{1-r}\ , \qquad -1 < r < 1}$$

Where	S_n	sum of GP with n terms
	S_∞	sum of GP with infinitely many terms
	a_1	the first term

R	common ratio
N	number of terms

Examples:

Write down a specific term in a Geometric Progression

Example (1) Write down the 8th term in the Geometric Progression 1, 3, 9, ...

SOLUTION

$a_1 = 1$; $a_2 = 3$; $a_3 = 9$; $n = 8$

r = $\frac{a_2}{a_1} = \frac{3}{1} = 3$

$a_8 = a_1 r^{8-1}$

$= (1)\ (3)^7$

$a_8 = 2187$

Finding the number of terms in a Geometric Progression

Example (2) Find the number of terms in the geometric progression 6, 12, 24, ..., 1536

SOLUTION

$a_1 = 6$; $a_2 = 12$; $a_3 = 24$; $a_n = 1536$

$r = \frac{a_2}{a_1} = \frac{12}{6} = 2$

$1536 = (6)\,(2)^{n-1}$

$2^8 = (2)^{n-1}$

$8 = n - 1$

$9 = n$

Hence, 1536 is the 9th term.

<u>Finding the sum of a Geometric Series</u>

Example (3) Find the sum of each of the $-2, \frac{1}{2}, -\frac{1}{8}, \ldots, -\frac{1}{32768}$

SOLUTION

$a_1 = -2$; $a_2 = \frac{1}{2}$; $a_3 = -\frac{1}{8}$; $a_n = -\frac{1}{32768}$

$r = \frac{a_2}{a_1} = \frac{\frac{1}{2}}{-2} = -\frac{1}{4}$

$-\frac{1}{32768} = (-2)\,(-\frac{1}{4})^{n-1}$

$-\frac{1}{65536} = (-\frac{1}{4})^{n-1}$

$$(-\tfrac{1}{4})^8 = (-\tfrac{1}{4})^{n-1}$$

$$8 = n - 1 \longrightarrow n = 9$$

$$S_9 = \frac{-2[\,(\,1-(-\frac{1}{4})^9]}{1-(-\frac{1}{4})} = -2\,\frac{1-(-\frac{1}{4})^9}{1-(-\frac{1}{4})}$$

$$= \frac{1-\frac{1}{262144}}{\frac{5}{4}} = \frac{\frac{262145}{262144}}{\frac{5}{4}}$$

$$S_9 = \frac{52429}{65536}$$

<u>Finding the sum of a Geometric Series to Infinity</u>

Work out the sum $\sum_{r=1}^{\infty} (\frac{1}{3})^r$

SOLUTION

$$\sum_{r=1}^{\infty} (\tfrac{1}{3})^r = (\tfrac{1}{3})^1 + (\tfrac{1}{3})^2 + (\tfrac{1}{3})^3 + \dots + (\tfrac{1}{3})^n + \dots$$

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots + (\tfrac{1}{3})^n + \dots$$

$$r = \frac{a_2}{a_1} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$$

$$S_\infty = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Example (4) Find the sum of each of the 1, 1.08, 1.08^{2}, …, 1.08^{19}

SOLUTION

n = 20, r = 1.08

$$S = a \times \frac{r^n - 1}{r - 1} = 1 \times \frac{(1.08)^{20} - 1}{(1.08) - 1}$$

$= 12.5\ (4.656 - 1)$

$= 12.5\ (3.656)$

$= 45.7$

Example (5) Find the sum of the following geometric progressions:

(A) 24, 12, 6, …………….

(B) 1, $\frac{1}{4}$, $\frac{1}{16}$, ……………….

(C) $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ ……………..

SOLUTION

(A) a = 24, r = $\frac{1}{2}$

$$S_{\infty} = \frac{a}{1 - r}$$

$$= \frac{24}{1-\frac{1}{2}} = \frac{24}{\frac{1}{2}} = 24 \times 2 = 48$$

(B) $a = 1,\ r = \frac{1}{4}$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}}$$

$$= \frac{4}{3}$$

(C) $a = \frac{1}{10},\ r = \frac{1}{10}$

$$S_\infty = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{10} \times \frac{10}{9}$$

$$= \frac{1}{9}$$

Important Notes on Geometric Progression:

- In a geometric progression, each successive term is obtained by multiplying the common ratio to its preceding term.
- The formula for the nth term of a geometric progression whose first term is a and common ratio is r is: $a_n = a\, r^{n-1}$.
- The sum of n terms in GP whose first term is (a) and the common ratio is (r) can be calculated using the formula:

  $S_n = \frac{a\,(1-r^n)}{1-r}$
- The sum of infinite GP formula is given as: $S_n = \frac{a}{1-r}$ where $|r|<1$.

Differences Between Arithmetic Progression (A.P.) and Geometric Progression (G.P.)

The following table explains the differences between arithmetic and geometric progression:

Property	Arithmetic progression	Geometric progression
Definition	It is a sequence in which the	It is a sequence in which the ratio of every

Property	Arithmetic progression	Geometric progression
	difference between every two consecutive terms is constant.	two consecutive terms is constant.
Common Difference/Ratio	d	R
General form	a, a + d, a + 2d, a + 3d, ...	a, ar, ar^2, ar^3, ...
nth term formula	a + (n – 1) d	a r^{n-1}
Sum of n terms formula	n/2 [2a + (n – 1) d]	(a (r^n –1)) / (r – 1)
How the terms vary?	The consecutive terms vary linearly.	The consecutive terms vary exponentially.

❖ <u>Series of power of natural numbers</u>

The natural numbers include the positive integers (also known as non-negative integers) and a few examples include 1, 2, 3, 4, 5, 6, … ∞. In other words, natural numbers are a set of all the whole numbers excluding 0.

Sum of natural numbers series

Sum of 'n' natural numbers

- 1 + 2 + 3 + 4 + ……….. + n

$= \sum_{r=1}^{n} r$ = n (n + 1)/2 = $\frac{n\,(n+1)}{2}$

Sum of square of natural numbers series

Sum of square of 'n' natural numbers

- $1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots\ldots + n^2$

$= \sum_{r=1}^{n} r^2$ = [n (n + 1) (2n + 1)]/6 = $\frac{n\,(n+1)(2n+1)}{6}$

Sum of cube of natural numbers series

Sum of cube of 'n' natural numbers

- $1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots\ldots + n^3$

$= \sum_{r=1}^{n} r^3$ = $[n (n+1)/2]^2$ = $[\frac{n\,(n+1)}{2}]^2$

❖ Solved Examples on Sum of Series of power of natural numbers

Examples 1: Find the sum of the first 100 natural numbers?

Solution: We can practice the arithmetic progression formula to obtain the sum of the first 100 natural numbers. Where a = 1, n = 100, and d = 1.

Sum of n terms of an arithmetic progression is given by the formula = $S_n = n/2\,[2a + (n-1)\,d]$

$S = 100/2\,[2 \times 1 + (100 - 1)1]$

$S = 50\,[2+100-1]$

$S = 5050$

Or $\sum_{r=1}^{n} r = n\,(n + 1)/2 = \frac{n\,(n+1)}{2}$

$\sum_{r=1}^{100} r = n\,(n + 1)/2 = \frac{100\,(101)}{2} = 5050$

Hence, the sum of the first 100 natural numbers is 5050.

Examples 2: Determine the sum of the first 50 natural numbers?

Solution: Assign S to be the required sum.

Therefore, S = 1 + 2 + 3 + 4 + 5 + ……………….... + 50

By the formula for finding the sum of first n natural numbers:

[n (n+1)] / 2

Here n = 50.

Therefore,

[n (n +1)] / 2 = [50 (50 +1)] / 2

[50 × 51] / 2 = 1275

Hence, the sum of the first 50 natural numbers is 1275.

Examples 3: Obtain the sum of the first 10 natural numbers.

Solution: n = 10

Applying equation for the sum of natural numbers, we get:

The sum of n natural numbers = [n (n +1)] / 2

$S = [10\ (10 + 1)] / 2$

$S = 55$

Accordingly, the sum of the first 10 natural numbers is 55.

Example 4: What is the sum of squares of 100 natural numbers from 1 to 100?

Solution: $n = 100$

Applying equation for the sum of squares of n natural numbers / terms we get:

The sum of squares of n terms = $[n\ (n + 1)\ (2n + 1)]/6$

$S = [100\ (100 + 1)\ (2\ (100) + 1] / 6$

$= 100\ (101)\ (201) / 6$

$S = 338{,}350$

Accordingly, the sum of the first 10 natural numbers is 338,350

Example 5: What is the sum of cubes of 50 natural numbers from 1 to 50?

Solution: $n = 50$

Applying equation for the sum of cubes of n natural numbers / terms, we get:

The sum of cubes of n terms = $[n\ (n + 1)/2]^2$

$S = [50\ (50 + 1)/2]^2$

$= [50\ (51)/2]^2$

$= [1275]^2$

$= 1{,}625{,}625$

Accordingly, the sum of cube of 50 natural numbers is 1,625,625

Example 6: find $\sum_{r=5}^{20} r^2$

Solution: S (From 5 to 20) = S [(From 1 to 20) – (From 1 to 4)]

$\sum_{r=5}^{20} r^2 = \sum_{r=1}^{20} r^2 - \sum_{r=1}^{4} r^2$

= [20 (20 + 1) /2] – [4 (4 + 1) /2]

= 210 – 10

= 200

Another solution:

5 + 6 + 7 + … + 20 is arithmetic progression (AB), where a = 5, d = 1, n = 16, and l (last term) = 20.

S = n/2 [(a + l)]

= 16/2 (5 + 20)]

= 8 × 25

= 200

Questions and Problems on Chapter:1
(Progressions and series)

(1) Find the general term of the arithmetic progression

–3, – (1/2), 2… [Answer: $a_n = \frac{5}{2}n - \frac{11}{2}$]

(2) Which term of the AP 3, 8, 13, 18... is 78? [Answer: 78 is the 16^{th} term (t_{16} or a_{16})]

(3) Find the sum of the first 5 terms of the arithmetic progression whose first term is 3 and 5^{th} term is 11. [Answer: The required sum of the first 5 terms is 35.]

(4) Which of the following sequences are in AP?

(A) 1,2,3,6,12, ...

(B) −3,3, −3,3, ...

(C) 1,2,4,8,16, ...

(D) 2,8,14,20,26, ...

(5) Write the sixth term of the AP whose first term is −20 and the common difference is 4.

(6) Check whether the given sequence, 9, 3, 1, 1/3, 1/9…… is in geometrical progression.

(7) How many terms of the series 1 + 3 + 9 +…. sum to 121?

A. 5 B. 6 C. 4 D. 3 (answer: A. 5)

(8) Explain the difference between geometric progression and arithmetic progression?

(9) Can zero be a part of a geometric series? A: No. If the first term is zero, then geometric progression will not take place.

(10) Five terms are in A.P. with common difference ≠ 0. If the first, third and fourth terms are in G.P then?

A. The fifth term is always zero

B. The third term is zero

C. The first term is always zero

D. Middle term is zero

E. Middle term is always 2

A: The correct option is A.

(11) Explain the difference between geometric progression and arithmetic progression? [A: *A sequence refers to a set of numbers arranged in some specific order. An arithmetic sequence is one where the difference between two consecutive terms is constant. While a geometric sequence is one where the ratio between two consecutive terms is constant.]*

(12) What is the explicit formula for the geometric sequence 4,12, 36,108…?

[$a_n = 4 \times 3^{n-1}$.]

(13) If the fourth term of a geometric progression with common ratio equal to half the initial term is 32, what is the 15th term? [$4 \times 2^{14} = 4 \times 16384 = 65{,}536$]

(14) Which of the following is the explicit formula for the geometric progression:

5, 10, 20, 40

(A) 2.5^{n-1} (B) $5 \cdot 2^{n-1}$ (C) $5 \cdot 2^{n+1}$ (D) $5 \cdot 5^{n}$

(15) Find the sum of the first 10 terms of the following geometric progression:

3, 15, 75, 3 75, 1875, … [Answer: $S_{10} = \frac{(3)5^{10}-(3)}{4}$] OR [$3\ \frac{5^{10}-1}{5-1}$]

(16) What is the sum of the first 10 terms of a geometric progression with initial term 2 and common ratio 3? [Answer: 59048]

(17) A girl puts one grain of rice in the first square of an 8 by 8 chess board. In the subsequent square, she puts twice that of the previous square, and she continues until she fills all the squares. How many total grains does she need?

(18) Find the sum of the geometric progression

$\frac{2}{3}, -1, \frac{3}{2}, \ldots$ up to 7 terms

(A) $\frac{95}{464}$ (B) $\frac{563}{191}$ (C) $\frac{463}{96}$ (D) $\frac{465}{93}$

(19) Calculate the following

$$5 - \frac{5}{3} + \frac{5}{9} - \frac{5}{27} + \dots \quad \left[\text{Answer: } S = \frac{15}{4}\right]$$

(20) Find the sum of the geometric progression

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$$

(21) $\frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \dots = ?$

(A) $\frac{1}{7}$ (B) $\frac{1}{6}$ (C) $\frac{1}{5}$ (D) $\frac{1}{4}$

* $\frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \dots = ?$ $S_\infty = \frac{\frac{1}{8}}{1-\frac{1}{2}} = \frac{1}{4}$ → (D)

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CHAPTER: 2

Determinants and Matrices

INTRODUCTION

Matrices and determinants are used to perform various arithmetic operations involving an array of elements. <u>Matrices</u> are a rectangular array of elements that are represented in the form of rows and columns. And <u>determinant</u> *is* calculated for a matrix and it is a <u>*single numeric value*</u> that has been computed from this array of elements. The matrix is represented with an alphabet in upper case and is written as A, and the determinant is represented as |A|.

Matrices and determinants have differences in their properties. The multiplication of a constant K with a matrix multiplies every element of the matrix, and the multiplication of a constant K with a determinant multiplies with the elements of any particular row or columns. Let us learn more about the properties, and differences between matrices and determinants with the help of examples, FAQs.

What are Matrices and Determinants?

Matrices and ***determinants*** represent an array of elements, and we compute a single element value for the entire determinant. Matrices is a plural form of a matrix, which is a rectangular array or a table where numbers or elements are arranged in a number of rows and columns. Matrices can be added or subtracted if only they have the same number of rows and columns whereas they can be multiplied if only columns in first and rows in second are exactly the same.

Matrices and determinants have a close relationship in math. A matrix is an array of elements that is denoted by M, and the determinant is the single numeric value to represent this matrix and is denoted as |M|. Let us look at the definition of a matrix and a determinant.

FIRST: Determinant of Matrix

The determinant of a matrix is a function that maps every square matrix to a unique number (real number or complex number). If A is the set of all square matrices (of all orders) and B is the set of all numbers (both real and complex) then the determinant function f is f: $A \to B$ and is defined as f(x) = y, where 'y' is the determinant of matrix 'x'.

Let us learn the process of finding determinant of the matrix for matrices of orders 1x1, 2x2, 3x3, etc. along with a few examples. Also, let us focus on the properties of determinants.

Determinant of a 2x2 Matrix

Let A = $\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $\rightarrow$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

bc ad

The determinant of a 2x2 matrix A = $abcd$ is |A| = a d – b c. It is simply obtained by cross multiplying the elements starting from top left and then subtracting the *products*.

Minor of Element of a Matrix

The *minor* of an element (aij)) of a square matrix of any order is the determinant of the matrix that is obtained by removing the row (i^{th} row) and the column (j^{th} column) containing the element. We can understand this by an example.

Example: For a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

column The minor of 6 $is,$ $\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$ = 1 (8) – 2 (7) = 8 –14 = – 6

The minor of 9 is, $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$ = 1 (5) – 2 (4) = 5 – 8 = –3

Co-factor of Element of a Matrix

The cofactor of an element a_{ij} of a square matrix of any order is its minor multiplied by $(-1)^{i+j}$. i.e.,

- Co-factor of an element = $(-1)^{\text{row number + number}}$ (minor of the element)

We found the minors of elements 6 and 9 in the previous example. Let us calculate the cofactors of the same elements now.

Example: For the same matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

- Since 6 is in the 2^{nd} row and 3^{rd} column, co-factor of 6 = $(-1)^{2+3}$ (minor of 6) = $(-1)^5$ (-6) = 6

- Since 9 is in the 3^{rd} row and 3^{rd} column, co-factor of 9 = $(-1)^{3+3}$ (minor of 9) = $(-1)^6$ (-3) = -3

The co-factors of elements of any matrix are nothing but the minors are multiplied by the alternative + and – signs (beginning with + sign for the first element of the first row).

Determinant of 3×3 Matrix

In the previous section, we have seen that the determinant of matrix is the sum of products of elements of any row (or any column) and their corresponding cofactors. Thus, here are the steps to find the determinant of matrix (a 3×3 matrix or any other matrix).

- Step 1: Choose any row or column. We usually choose the first row to find the determinant.

- Step 2: Find the co-factors of each of the elements of the row/column that we have chosen in Step 1.
- Step 3: Multiply the elements of the row/column from Step 1 with the corresponding co-factors obtained from Step 2.
- Step 4: Add all the products from Step 3 which would give the determinant of the matrix.

Example: Use the above steps to compute the determinant of 3 x 3 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

. Solution:

Step 1: We choose the first row with elements 1, 2, and 3.

Step 2: Let us compute the cofactors of these elements:

Co-factor of 1 = $(-1)^{1+1}$ Minor of 1 = $(-1)^{2}$

= 5 (9) – 6 (8) = –3

Co-factor of 2 = $(-1)^{1+2}$ Minor of 2 = $(-1)^{3}$

= 4 (9) – 6 (7) = 1 (– 6) = – 6

Co-factor of 3 = $(-1)^{1+3}$ Minor of 3 = $(-1)^{4}$

= 4 (8) – 5 (7) = – 3

Step 3: Multiply the elements by their cofactors.

1(co – factor of 1) = 1 (–3) = –3

2 (co – factor of 2 = 2 (6) = 12

3 (co- factor of 3) = 3 (– 3) = – 9

Step 4: Add them to get the determinant.

det A = –3 + 12 – 9 = 0.

All these steps can be summarized in a single step as follows:

det A = 1 (co-factor of 1) + 2 (co-factor of 2) + 3 (co-factor of 3)

= = 1 [5(9) – 6(8)] – 2 [4(9) – 6(7)] + 3 [4(8) – 5(7)]

= 1 (–3) – 2 (-6) + 3 (–3)

= –3 + 12 – 9 = 0

Note: Here we have used a negative sign with 2 in the second step because we get a minus sign while finding the co-factor of 2.

co-factor $\rightarrow$ *Minor with appropriate sign*.

The *appropriate sign* $\rightarrow$

(-1) $^{\text{number of row + number of column containing the element}}$ = *(–1)* $^{R + C}$.

Therefore, appropriate signs for minors of 3 × 3 matrix are:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Determinants of Matrix Formulas

The process of finding the determinant of matrix that is explained in the previous section can be used to find the determinant of a matrix of any [order](). But there are some tricks to find the determinants of 1x1, 2 x 2, and 3 x 3 matrices. These tricks are very useful as we come across finding the determinants of matrices of these orders only most of the time while solving problems.

Determinant of 1 × 1 Matrix

1x1 matrix is a row with just 1 row and 1 column and hence it has only one element. The determinant of any 1×1 matrix is always equal to the element of the matrix. i.e.,

- If A = $[x]_{1\times 1}$, then |A| (or) det A = x

Determinant of 2 × 2 Matrix

As we discussed earlier, its determinant is obtained by subtracting the product of elements of the non-principal diagonal from the product of the elements of the principal diagonal. i.e.,

- If A = $\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then |A| (or) det A = a d – b c

Determinant of 3 × 3 Matrix (Shortcut)

The shortcut to find the determinant of 3 x 3 matrix is, just write the matrix twice and apply the following trick. Here is the shortcut (easiest way) to find the determinant of 3 x 3 matrix

A = $\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$ $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$ $\rightarrow$ $\begin{array}{cccccc} a & b & c & a & b & c \\ p & q & r & p & q & r \\ x & y & z & x & y & z \end{array}$

$|A|$ (or) det A = a q z + b r x + c p y – a r y – b p z - c q x

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \rightarrow \begin{matrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 \end{matrix}$$

48 72 105 45 84 96

$|A|$(or) det A = (45 + 84 + 96) – (48 + 72 + 105)

= 225 – 225 = 0

Properties of Determinant of Matrix

The properties of determinants are useful in finding the determinant of a matrix without actually using the process of finding it. These are helpful in evaluating the complex determinants. These include how the determinant changes with respect to *elementary row operations*.

Property 1

"The determinant of a matrix is equal to the determinant of its *transpose*."

$$A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \rightarrow \begin{matrix} a & p & x & a & p & x \\ b & q & y & b & q & y \\ c & r & z & c & r & z \end{matrix}$$

$|A|$(or) det A = (a q z + p y c + x b r) – (a y r + p b z + x q c)

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{vmatrix} \rightarrow \begin{matrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 5 & 6 & 7 & 5 & 6 & 7 \\ 9 & 10 & 11 & 9 & 10 & 11 \end{matrix}$$

$|A|$(or) det A = (66 + 126 + 150) – (70 + 110 + 162)

= 342 – 342 = 0

A $\rightarrow$ A^T

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{vmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{vmatrix} \rightarrow$$

$$\begin{matrix} 1 & 5 & 9 & 1 & 5 & 9 \\ 2 & 6 & 10 & 2 & 6 & 10 \\ 3 & 7 & 11 & 3 & 7 & 11 \end{matrix}$$

$|A|$(or) det A = (66 + 150 + 126) – (70 + 110+ 162)

= 342 – 342

= 0

Property 2

"If any two rows (or columns) of a determinant are interchanged, then the sign of the determinant changes."

Example:

$$A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \rightarrow \begin{matrix} a & b & c & a & b & c \\ p & q & r & p & q & r \\ x & y & z & x & y & z \end{matrix}$$

$|A|$(or) det A = (a q z + b r x + c p y) – (a r y + b p z + c q x)

$$A = \begin{bmatrix} x & y & z \\ p & q & r \\ a & b & c \end{bmatrix} \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \longrightarrow \begin{matrix} x & y & z & x & y & z \\ p & q & r & p & q & r \\ a & b & c & a & b & c \end{matrix}$$

$|A|$(or) det A = - [(x q c + y r a + z p b) – (x r b + y p c + z q a)]

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} \longrightarrow \begin{matrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 2 & 1 & 0 \end{matrix}$$

= 4 – 7 = – 3

$R_1 \longleftrightarrow R_3$

.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} \longrightarrow \begin{matrix} 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 1 & 2 & 3 \end{matrix}$$

4 0 0 6 1 0

$|A|$(or) det A = (6 + 1 + 0) – (4 + 0 + 0)

= 7 – 4 = 3

Here, the first and third rows of the left side determinant are interchanged. We easily verified this by finding both determinants.

Property 3

"If any two rows (or columns) of a determinant are identical, then the determinant is 0."

Example:

$$A = \begin{bmatrix} a & b & c \\ a & b & c \\ x & y & z \end{bmatrix} \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} \Rightarrow \begin{matrix} a & b & c & a & b & c \\ a & b & c & a & b & c \\ x & y & z & x & y & z \end{matrix}$$

$|A|$(or) det A = (a b z + b c x + c a y) – (c b x + a c y + b a z) = 0

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 0 \end{vmatrix} \Rightarrow \begin{matrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 2 & 1 & 0 \end{matrix}$$

3 0 12 0 12 3

$|A|$(or) det A = (0 + 12 + 3) – (3 + 0 +12)

= 15 – 15 = 0

Property 4

"If all elements of a row (or column) of a matrix of a determinant are zeros, then the value of the determinant is 0."

Example

$$A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \Rightarrow \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ p & q & r & p & q & r \\ x & y & z & x & y & z \end{matrix}$$

0 0 0 0 0 0

$|A|$(or) det A = (0 + 0 + 0) – (0 + 0 +0) = 0

Property 5

"If each element of a row (or column) of a determinant is multiplied by a scalar k, then the value of the resultant determinant is k times the value of the original determinant."

Example:

$$A = \begin{bmatrix} ka & kb & kc \\ p & q & r \\ x & y & z \end{bmatrix} \begin{vmatrix} ka & kb & kc \\ p & q & r \\ x & y & z \end{vmatrix} \rightarrow \begin{matrix} ka & kb & kc & ka & kb & kc \\ p & q & r & p & q & r \\ x & y & z & x & y & z \end{matrix}$$

$$= k \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$A = \begin{bmatrix} 5(1) & 5(2) & 5(3) \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{vmatrix} 5(1) & 5(2) & 5(3) \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} \rightarrow \begin{matrix} 5 & 10 & 15 & 5 & 10 & 15 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 2 & 1 & 0 \end{matrix}$$

5 0 30 0 20 0

$|A|$(or) det A = (0 + 20 + 0) – (5 + 0 +30)

= 20 – 35

= – 15

$$k \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \rightarrow 5 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

= 5 (– 3)

= – 15

Property 6

"If each element of a row (or column) is multiplied by a constant and the elements are added to the corresponding elements of another row (or column), then the determinant remains unchanged."

Example:

$$A = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\\2 & 1 & 0\end{bmatrix}\begin{vmatrix}1 & 2 & 3\\0 & 1 & 1\\2 & 1 & 0\end{vmatrix} \rightarrow \begin{bmatrix}1 & 2 & 3\\2 & 5 & 7\\2 & 1 & 0\end{bmatrix}\begin{vmatrix}1 & 2 & 3\\2 & 5 & 7\\2 & 1 & 0\end{vmatrix}$$

$2R_1 + R_2 \rightarrow R_2$

$$A = \begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\\2 & 1 & 0\end{bmatrix}\begin{vmatrix}1 & 2 & 3\\0 & 1 & 1\\2 & 1 & 0\end{vmatrix}$$

$|A|$(or) det A = – 3 (as above)

$$A = \begin{bmatrix}1 & 2 & 3\\2 & 5 & 7\\2 & 1 & 0\end{bmatrix}\begin{vmatrix}1 & 2 & 3\\2 & 5 & 7\\2 & 1 & 0\end{vmatrix} \rightarrow \begin{array}{cccccc}1 & 2 & 3 & 1 & 2 & 3\\2 & 5 & 7 & 2 & 5 & 7\\2 & 1 & 0 & 2 & 1 & 0\end{array}$$

7 0 30 0 28 6

$|A|$(or) det A = (0 + 28 + 6) – (7 + 0 +30)

= 34 – 3 = – 3

Important Notes on Determinant of Matrix:

- The determinant of an *identity / unit matrix* is always 1.

$$A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}$$

$|A|$(or) det A = 1

- The determinant of a *diagonal matrix* is always the product of elements of its principal diagonal.

A = $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix}$

$|A|$(or) det A = 3 × 5 = 15

- The determinant of a matrix can be either positive, negative, or zero. (see above)
- The determinant of matrix is used in *Cramer's rule* which is used to solve the system of equations. (as we will see later)
- Also, it is used to find the *inverse of a matrix*. If the determinant of a matrix is not equal to 0, then it is an invertible matrix as we can find its inverse. (as we will see later)
- *If A is a square matrix of order 3 × 3, then |kA| = k^3 |A|, for any scalar k.*

Example: If k = 5, A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, then [k A] = $\begin{bmatrix} 5 & 0 & 5 \\ 0 & -10 & 2 \\ 5 & 5 & 15 \end{bmatrix}$

$$|kA| = \begin{vmatrix} 5 & 0 & 5 \\ 0 & -5 & 10 \\ 5 & 5 & 5 \end{vmatrix} = \begin{matrix} 5 & 0 & 5 & 5 & 0 & 5 \\ 0 & -5 & 10 & 0 & -5 & 10 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{matrix}$$

250 0 -125 -125 0 0

= (-125 + 0 + 0) – (250 + 0 –125)

= –125 – 125

= – 250

$$|kA| = k^3 \ |A| = 5^3 \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= (125)\ \begin{matrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 0 & -1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

$$2 \quad 0 \quad -1 \qquad -1 \quad 0 \quad 0$$

$= 125\ (-1\ -1)$

$= 125\ (-2) = -\ 250$

- A square matrix A is called *singular* if $|A| = 0$ and *non-singular* if $|A| \neq 0$.

USE OF DETERMINANTS TO SOLVE SYSTEMS OF LINEAR EQUATIONS

What is Cramer's Rule?

Cramer's rule is one of the methods used to solve a system of equations. This rule involves *determinants*. i.e., the values of the variables in the system are found with the help of determinants. Let us consider a system of equations in n variables $x_1, x_2, x_3, \ldots, x_n$ written in the *matrix* form AX = B, where

- A = the coefficient matrix which is a *square matrix*
- X = the column matrix with variables
- B = the column matrix with the constants (which are on the right side of the equations)

Cramer's Rule Formula

Here is the Cramer's rule formula to solve the system AX = B (or) to find the values of the variables $x_1, x_2, x_3, \ldots, x_n$. To solve the *system of equations*:

- Find det |A| and represent it by D.
- Find the determinants $D_{x_1}, D_{x_2}, D_{x_3}, \ldots, D_{x_n}$, where D_{x_i} is the determinant of matrix A where the i^{th} column is replaced by the *column matrix* B.
- We divide each of these determinants by D to find the value of the corresponding variables. i.e., $x_1 = D_{x_1}/D$, $x_2 = D_{x_2}/D$, ..., $x_n = D_{x_n} / D$.

Note that the system of equations has a unique solution only when $D \neq 0$.

Cramer's Rule For 2 x 2

Using the above formula, let us see how to solve a system of 2 equations in 2 variables using Cramer's rule. Here are the steps to solve this system of 2x2 equations in two unknowns x and y using Cramer's rule.

- Step-1: Write this system in matrix form is AX = B.
- Step-2: Find D which is the determinant of A. Also, find the determinants D_x and D_Y where D_x = det (A) where the first column is replaced with B D_Y = det (A) where the second column is replaced with B
- Step-3: Find the values of the variables x and y by dividing each of D_x and D_Y by D respectively.

Consider a system of two equations in two variables x and y.

$a_1 x + b_1 y = c_1$ and

$a_2 x + b_2 y = c_2$

Let us apply the above steps to solve the above system.

Step-1: Write this system in matrix form is AX = B, where

- A = $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$= the coefficient matrix
- X = $\begin{bmatrix} x \\ y \end{bmatrix}$= the variable matrix
- B = $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$= the constant matrix

Step-2: Calculate the determinants D, D_x, and D_Y, where

- D = det (A) = $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
- D_x = det (A) where the first column is replaced with B = $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$
- D_Y = det (A) where the second column is replaced with B =

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Step-3: Find x and y (when D ≠ 0) using

- $x = D_x/D$
- $y = \frac{D_Y}{D}$

Example 1: Solve the following system of 2 x 2 equations: 3x + 2y = 7 and x - y = -1.

Solution:

- The given system can be written in the matrix form AX = B

where, A = $\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$, and B = $\begin{bmatrix} 7 \\ -1 \end{bmatrix}$

Now, we will find the determinants.

D = det(A) = $\begin{vmatrix} 7 & 2 \\ -1 & -1 \end{vmatrix}$ = 7(-1) – 2(-1) = - 7 + 2 = - 5

D_x = $\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}$ = 3(-1) – 2(1) = - 3 – 2 = - 5

D_Y = $\begin{vmatrix} 3 & 7 \\ 1 & -1 \end{vmatrix}$ = 3(-1) – 7(1) = -3 – 7 = - 10

Now, by Cramer's rule for 2 equations,

x = $\frac{D_X}{D} = \frac{-5}{-5}$ = 1

y = $\frac{D_Y}{D} = \frac{-10}{-5}$ = 2

Answer: The solution of the given system is, x = 1 and y = 2

Cramer's Rule For 3 x 3

We will just extend the same process of Cramer's rule for 2 equations for a 3x3 system of *equations* as well. Here are the steps to solve this system of 3x3 equations in three variables x, y, and z by applying Cramer's rule.

- Step-1: Write this system in matrix form is AX = B.

- Step-2: Find D which is the determinant of A. i.e., D = det (A). Also, find the determinants D_x, D_Y, and D_z where
  D_x = det (A) where the first column is replaced with B
  D_Y = det (A) where the second column is replaced with B
  D_z = det (A) where the third column is replaced with B
- Step-3: Find the values of the variables x, y, and z by dividing each of D_x, D_Y, and D_z by D respectively.

Consider a system of three equations in three variables x, y, and z.

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$ and

$a_3 x + b_3 y + c_3 z = d_3$

Let us apply the above steps to solve 3 x 3 equations.

Step-1: We will write the system in matrix form is AX = B, where

A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b & c_3 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and B = $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$,

Step-2: Compute the determinants D, D_x, D_Y, and D_z. where

D = det (A) = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b & c_3 \end{bmatrix}$

D_x = det (A) where the first column is replaced with B = $\begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b & c_3 \end{bmatrix}$

D_Y = det (A) where the second column is replaced with B

$$= \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}$$

D_Z = det (A) where the third column is replaced with B = $\begin{bmatrix} a_1 & b_1 & d \\ a_2 & b_2 & d \\ a_3 & b_3 & d_3 \end{bmatrix}$

Step-3: Find the values of the variables x, y, and z (when D ≠ 0) using

$$x = \frac{D_X}{D} \qquad y = \frac{D_Y}{D} \qquad z = \frac{D_Z}{D} =$$

Example 2: Solve the following system of 3 equations in 3 variables using Cramer's rule:

x + y + z = 2

2x + y + 3z = 7

x - 3y + z = 6

Solution:

The given system can be written in the matrix form AX = B where, A =

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & -3 & 1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and B = $\begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$

D = det (A) = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & -3 & 1 \end{bmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & -3 & 1 \end{vmatrix} \rightarrow \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 2 & 1 & 3 \\ 1 & -3 & 1 & 1 & -3 & 1 \end{matrix}$

-9 2 1 1 3 -6

D = (1 + 3 -6) – (-9 + 2 +1)

= -2 – (-6) = -2 + 6 = 4

D_X = det (A) where the first column is replaced with B =

$$\begin{bmatrix} 2 & 1 & 1 \\ 7 & 1 & 3 \\ 6 & -3 & 1 \end{bmatrix} \begin{vmatrix} 2 & 1 & 1 \\ 7 & 1 & 3 \\ 6 & -3 & 1 \end{vmatrix} \rightarrow \begin{matrix} 2 & 1 & 1 & 2 & 1 & 1 \\ 7 & 1 & 3 & 7 & 1 & 3 \\ 6 & -3 & 1 & 6 & -3 & 1 \end{matrix}$$

-18 7 6 2 18 -21

= (2 + 18 – 21) – (-18 +7 + 6)

= -1 – (-5)

= -1 + 5

= 4

D_Y = det (A) where the second column is replaced with B =

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 3 \\ 1 & 6 & 1 \end{bmatrix} \begin{vmatrix} 1 & 2 & 1 \\ 2 & 7 & 3 \\ 1 & 6 & 1 \end{vmatrix} \rightarrow \begin{matrix} 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & 7 & 3 & 2 & 7 & 3 \\ 1 & 6 & 1 & 1 & 6 & 1 \end{matrix}$$

D_Y = (7 + 6 + 12) – (18 + 4 + 7)

= 25 – 29 = = - 4

D_Z= det (A) where the third column is replaced with B =

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 7 \\ 1 & -3 & 6 \end{bmatrix} \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 7 \\ 1 & -3 & 6 \end{vmatrix} \rightarrow \begin{matrix} 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 7 & 2 & 1 & 7 \\ 1 & -3 & 6 & 1 & -3 & 6 \end{matrix}$$

-21 12 2 6 7 -12

D_z = (6 + 7 -12) - (-21 + 12 + 2)

$= 1 - (-7)$

$= 1 + 7 = 8$

$x = \frac{D_X}{D} = \frac{4}{4} = 1$

$y = \frac{D_Y}{D} = \frac{-4}{4} = -1$

$z = \frac{D_Z}{D} = \frac{8}{4} = 2$

Answer: The solution of the given system is, x = 1, y = - 1 and z =S

Important notes on a determinant of Matrix

*The determinant of a matrix is obtained by multiplying the elements

any of its rows or columns by the corresponding cofactors and adding all the products. The determinant of a square matrix A is denoted by |A| or det (A).

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*To find the determinant of a square matrix:

Select any row or column

Find the cofactors of all the elements of the row or column that you have selected

Multiply the elements of the row or column by their corresponding cofactors

Add the products from the last step. This sum gives the determinant.

-

*The determinant [product](#) of two square matrices is the product of the determinants of the individual [matrices](#). i.e., for any two square matrices A and B each of order n x n, det (AB) = det(A) × det(B).

-

*For any two square matrices A and B of the same order, det (AB) = det A det B. So det (A^2) = det (A A) = det A det A = $(\det A)^2$. Soو the [determinant](#) of the square of a matrix is the square of the determinant of the matrix.

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*The determinants are used to:

- find the inverse of a matrix.
- [solve the system of equations](#)
- in Cramer's rule.

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*Using one of the [properties of determinants](#), when any two rows or columns of a matrix are equal, its determinant is zero. Using this, the determinant of a matrix whose all elements are equal is equal to 0.

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*Using one of the properties of [determinants](), when all the elements of any row or column are zeros, its determinant is zero. From this, the determinant of a matrix whose all elements are zeros is equal to 0.

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*Here are some easiest ways/formulas to find the [determinant]() of matrix.

- The determinant of a 1 × 1 matrix: If A = $[x]_{1\times1}$, then |A| = x.
- The determinant of a 2 × 2 matrix: If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then |A| = ad - bc.
- The determinant of a 3 × 3 matrix: If A = $\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$ then |A| = a (q z – r y) - b (p z – r x) + c (p y – q x).

SECOND: MATERICES

Definition of Matrix: In mathematics, a matrix (pl.: matrices) is a rectangular array or table of numbers, symbols, or expressions, with elements or entries arranged in rows and columns, which is used to represent a mathematical object or property of such an object. For example, $\begin{bmatrix} 1 & 7 & -9 \\ 10 & 5 & 3 \end{bmatrix}$ is a matrix with two rows and three columns. This is often referred to as a "two-by-three matrix", a " matrix", or a matrix of dimension 2 ×3. Matrices are commonly related to linear algebra.

Square matrices, matrices with the same number of rows and columns, play a major role in matrix theory. The determinant of a square matrix is a number associated with the matrix, which is fundamental for the study of a square matrix; for example, a square matrix is invertible if and only if it has a nonzero determinant.

Matrices are used in most areas of mathematics and scientific fields, either directly, or through their use in geometry and numerical analysis.

Matrix theory is the branch of mathematics that focuses on the study of matrices. It was initially a sub-branch of linear algebra, but soon grew to include subjects related to graph theory, algebra, combinatorics and statistics.

- Types of Matrices

1- Null Matrix:

$$\phi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \ldots .$$

2- Row Matrix:

$$g = \begin{pmatrix} a_{11} & a_{12} & \ldots \end{pmatrix} \; 1 \times n$$

a. Column Matrix:

2024/2025
2024/2025

$$h = \begin{pmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \end{pmatrix} \; m \times 1$$

3- Rectangular Matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix}$$

4- Square Matrix: A matrix with n rows and n columns is called a Square Matrix of order n.

$A = [a_{ij}]$ m × n is a square matrix if m = n

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 4 & 2 & 0 \\ 5 & 0 & 5 \end{bmatrix}$$

Square Matrix

5-Triangular Matrix

i. Lower Triangular Matrix

$$D = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

ii. Upper Triangular Matrix

$$B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

6- Vertical Matrix:

$$Q = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$$

Vertical Matrix

7- Horizontal Matrix

$$B = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 4 & 2 & 0 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

Horizontal Matrix

8- Diagonal Matrix

$$U = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Diagonal Matrix

9- Symmetric Matrix: Symmetric matrices: A square matrix A of size n × n is considered to be <u>symmetric</u> if and only if $A^T = A$. For example,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix} \text{ is symmetric matrix because } (A = A^t)$$

$$A^t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{pmatrix}$$

10- Skew-symmetric matrix (Reverse Matrix): Skew-symmetric matrices–A square matrix A of size n × n is considered to be <u>skew-symmetric</u> if and only if $A^T = -A$.

A Skew-symmetric $\Longleftrightarrow$ $A^T = -A$

$$A = \begin{bmatrix} 0 & 1 & -7 \\ -1 & 0 & -3 \\ 7 & 3 & 0 \end{bmatrix} \qquad -A = \begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 3 \\ -7 & -3 & 0 \end{bmatrix} = A^T$$

11- Identity Matrix:

$$\mathtt{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Identity Matrix

$A\,\mathtt{I} = \mathtt{I}\,A = A$ i.e., $\mathtt{I}$ represents multiplicative identity.

12- Scalar Matrix:

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{\text{Scalar Matrix}}$$

$$\begin{bmatrix} \sqrt{5} & 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 \\ 0 & 0 & 0 & \sqrt{5} \end{bmatrix}$$

Scalar Matrix

13- Transpose matrix:

If A = $\begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 5 \end{bmatrix}$. Find A^t

2 × 3

$A^t = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 5 \end{bmatrix}$

3 × 2

14- Singular Matrix: Square matrix is said to be a singular matrix if its determinant equals zero.

15- Idempotent Matrix: Matrix A is Idempotent Matrix if $A = A^2$. The values of the matrix do not change if it is multiplied by itself.

16- Involutory Matrix:

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $I^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} ; \quad I^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

17- Orthogonal Matrix: Any square matrix A is orthogonal if its transpose is equal to its inverse. i.e., $A^T = A^{-1}$

Matrix A is Orthogonal Matrix if $A.A^T = I$. That is identity/unit matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ ……… (1)}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ …… (2)}$$

(1) = (2). Therefore, A is orthogonal Matrix.

Also, $A.A^T = I$

18- Regular Matrix: If square matrix contains non-zero determinant. This means, it is a Regular Matrix/non - singular.

19. Singleton Matrix: A matrix contains single element is known as Singleton Matrix. Here, number of columns and rows equals 1. $[7]_{1\times 1}$

Singleton Matrix

The numbers, symbols, or expressions in the matrix are called its *entries* or its *elements*. The horizontal and vertical lines of entries in a matrix are called *rows* and *columns*, respectively.

20. Invertible Matrix: Any square matrix A is called invertible matrix, if there exists another matrix B, such that, AB = BA = I_n, where I_n is an identity matrix with n × n.

SIZE of Matrix

The size of a matrix is defined by the number of rows and columns it contains. There is no limit to the number of rows and columns, that a matrix (in the usual sense) can have as long as they are positive integers. A matrix with m rows and n columns is called an $m \times n$ matrix, or m -by- n matrix, where m and n are called its *dimensions*. For example, the matrix A above is a $m \times n$ matrix. Matrices with a single row are called *row vectors*, and those with a single column are called *column vectors*. A matrix with the same number of rows and columns is called a *square matrix*. A matrix with an infinite number of rows or columns (or both) is called an *infinite matrix*. In some contexts, such as computer algebra programs, it is useful to consider a matrix with no rows or no columns, called an *empty matrix*.

Overview of a matrix size

Name	Size	Example		Notation
Row vector	1× n	$\begin{bmatrix} 1 & 5 & 3 \end{bmatrix}$	A matrix with one row, sometimes used to represent a vector	a_i
Column vector	n× 1	$\begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$	A matrix with one column, sometimes used to represent a vector	j_i
Square matrix	n× n	$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 1 & 0 \\ 4 & 3 & 2 \end{bmatrix}$	A matrix with the same number of rows and columns, sometimes used to..represent linear..transformation from a vector space to itself,	A

			such as reflection, rotation, or shearing.	

Notation

The specifics of symbolic matrix notation vary widely, with some prevailing trends. Matrices are commonly written in square brackets or parentheses, so that an matrix is represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

This may be abbreviated by writing only a single generic term, possibly along with indices, as in

$A = (a_{ij})$, $[a_{ij}]$ or (a_{ij}) $1 \leq i \leq m$, $1 \leq j \leq n$

Or $A = (a_{i,j})$ $1 \leq i, j \leq n$ in the case that n = m.

Matrices are usually symbolized using upper-case letters (such as A in the examples above), while the corresponding lower-case letters, with two subscript indices (e.g., a_{ij}, or $a_{i,j}$), represent the entries.

The entry in the *i*-th row and *j*-th column of a matrix A is sometimes referred to as the i, j or (i, j) entry of the matrix, and commonly denoted by $a_{i,j}$ or a_{ij} Alternative notations for that entry are A$[i\ , j]$ and .$A_{i,j}$ For example, the (1,3) entry of the following matrix A is 5 (also denoted a_{13}, $a_{1,3}$, A$[1\ ,3]$ or $A_{1,3}$):

$$A = \begin{bmatrix} 2 & 3 & 5 & 0 \\ 0 & 1 & 7 & 8 \\ 13 & 1 & -4 & 10 \end{bmatrix}$$

Sometimes, the entries of a matrix can be defined by a formula such as $a_{i,j} = f\ (i, j)$. For example, each of the entries of the following matrix A is determined by the formula $a_{ij} = (i - j)$

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

In this case, the matrix itself is sometimes defined by that formula, within square brackets or double parentheses. For example, the matrix above is defined as A = $[i - j]$ or A = ((i – j)) . If matrix size is m × n , the above-mentioned formula $f\ (i, j)$. is valid for any i = 1, …, m and any j = 1, …, m . This can be specified separately or indicated using m × n as a subscript. For instance, the matrix A above is 3 × 4 , and can be defined as A = $[i - j]$ (i = 1, 2, 3; j = 1, …, 4) or A = $[i - j]_{3 \times 4}$.

Some programming languages utilize doubly subscripted arrays (or arrays of arrays) to represent an *m*-by-*n* matrix. Some programming languages start the numbering of array indexes at zero, in which case the entries of an *m*-by-*n* matrix are indexed by $0 < i < m - 1$ and $0 < j < n - 1$. This article follows the more common convention in mathematical writing where enumeration starts from 1.

Transpose of Matrix

The transpose of a matrix is done when we replace the rows of a matrix to the columns and columns to the rows. *Interchanging of rows and columns is known as the transpose of matrices.* In the matrix given below, we have row elements as row-1: 2, –3, –4, and row-2: –, 7, –7. On transposing, we will get the elements in column-1: 2, –3, -4, and column-2: –1, 7, -7, we can check that in the image given below:

$$A_{2\times3} = \begin{array}{c} \begin{bmatrix} 2 & -3 & -4 \\ -1 & 7 & -7 \end{bmatrix} \end{array} \begin{array}{l} \text{ROW 1} \\ \text{ROW 2} \end{array}$$

$$A^{T}{}_{3\times2} = \begin{array}{cc} \text{Column 1} & \text{Column 2} \\ \multicolumn{2}{c}{\begin{bmatrix} 2 & -1 \\ -3 & 7 \\ -4 & -7 \end{bmatrix}} \end{array}$$

Properties of transposition in matrices

There are various properties associated with transposition. For matrices A and B, given as,

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$, A and B being of the same order.

- $(KA)^T = KA^T$, K is any scalar(real or complex).
- $(AB)^T = B^T A^T$, A and B being conformable for the product AB. (This is also called reversal law.)

  Apart from these operations, we have several other operations on matrices like finding its trace, minors and cofactors, adjoint, inverse, etc. Let us learn each of these in detail in this section.

Trace of a Matrix

The trace of any matrix A, Tr(A) is defined as the sum of its diagonal elements. Some properties of trace of matrices are,

- tr (AB) = tr (BA)

- $tr(A) = tr(A^T)$
- tr(cA) = c tr(A), for a scalar 'c'
- tr (A + B) = tr(A) + tr(B)

Minor of Matrix

Minor for a particular element in the matrices is defined as the determinant of the matrix that is obtained when the row and column of the matrix in which that particular element lies are deleted, and the minor of the element a_{ij} is denoted as M_{ij}. For example, for the given matrix, minor of a_{ij} of the matrix A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is:

$$M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

Similarly, we can find all the minors of the matrix and will get a minor M of the given matrix A as:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Cofactor of Matrix

Cofactor of an element in the matrix A is obtained when the minor M_{ij} of the matrix is multiplied with $(-1)^{i+j8}$. The cofactor of a matrix is denoted as C_{ij}. If the minor of a matrix is M_{ij}, then the cofactor of the matrix would be:

$$C_{ij} = (-1)^{i+j} + M_{ij}.$$

On finding all the cofactors of the matrix, we will get a cofactor matrix C of the given matrix A:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Note: Be extra cautious about the negative sign while calculating the cofactor of the matrix.

Adjoint of Matrices:

The adjoint of matrices is calculated by finding the transpose of the cofactors of the elements of the given matrices. To find the adjoint of a matrix, we have to calculate the cofactors of the elements of the matrix and then transpose the cofactor matrix to get the adjoint of the given matrix. The adjoint of matrix A is denoted by adj(A). Let us understand this with an example: We have a matrix A = $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 2 \\ 1 & -1 & -2 \end{bmatrix}$

Then the minor matrix M of the given matrix would be:

$$M = \begin{bmatrix} -8 & -2 & -5 \\ 5 & -7 & -1 \\ -17 & 4 & 10 \end{bmatrix}$$

We will get the cofactor matrix C of the given matrix A as:

$$C = \begin{bmatrix} -8 & 2 & -5 \\ -5 & -7 & 1 \\ -17 & -4 & 10 \end{bmatrix}$$

Then the transpose of the cofactor matrix will give the adjoint of the given matrix:

$$adj(A) = C^T = \begin{bmatrix} -8 & -5 & -17 \\ 2 & -7 & -4 \\ -5 & 1 & 10 \end{bmatrix}$$

Transpose of Matrix

The [transpose of a matrix] is done when we replace the rows of a matrix to the columns and columns to the rows. Interchanging of rows and columns is known as the transpose of matrices. In the matrix given below, we have row elements as row-1: 2, -3, -4, and row-2: -1, 7, -7. On transposing, we will get the elements in column-1: 2, -3, -4, and column-2: -1, 7, -7, we can check that in the image given below:

Transpose of 2 × 3 Matrix

$$A_{2\times 3} = \begin{array}{cc} \begin{bmatrix} 2 & -3 & -4 \\ -1 & 7 & -7 \end{bmatrix} & \begin{array}{l} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \end{array} \end{array}$$

$$A^T_{3\times 2} = \begin{array}{c} \begin{array}{cc} \text{Column 1} & \text{Column 2} \end{array} \\ \begin{bmatrix} 2 & -1 \\ -3 & 7 \\ -4 & -7 \end{bmatrix} \end{array}$$

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There are various properties associated with transposition. For matrices A and B, given as,

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$, A and B being of the same order.
- $(KA)^T = KA^T$, K is any scalar (real or complex).
- $(AB)^T = B^T A^T$, A and B being conformable for the product AB. (This is also called reversal law.)

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- tr (AB) = tr (BA)
- $tr(A) = tr(A^T)$

- $tr(cA) = c\, tr(A)$, for a scalar 'c'
- $tr(A + B) = tr(A) + tr(B)$

Inverse of Matrices

- The inverse of any matrix is denoted as the matrix raised to the power (-1), i.e. for any matrix "A", the inverse matrix is denoted as A^{-1}. The inverse of a square matrix, A is A^{-1} only when: $A \times A^{-1} = A^{-1} \times A = I$. There is a possibility that sometimes the inverse of a matrix does not exist if the determinant of the matrix is equal to zero ($|A| = 0$). The inverse of a matrix is shown by A^{-1}. Matrices inverse is calculated by using the following formula:

$$A^{-1} = (1/|A|)\,(\text{Adj } A)$$

Where

- $|A|$ is the determinant of the matrix A and $|A| \neq 0$.
- Adj A is the adjoint of the given matrix A.

The inverse of a 2 × 2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is calculated

by: $A^{-1} = \frac{1}{a_{11}\, a_{22} - a_{12}\, a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Let us find the inverse of the 3 × 3 matrix we have used in the previous section: $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 5 & 2 \\ 1 & -1 & -2 \end{bmatrix}$

Since adj(A) = $\begin{bmatrix} -8 & -5 & -17 \\ 2 & -7 & -4 \\ -5 & 1 & 10 \end{bmatrix}$

And on calculating the [determinant](), we will get |A| = –33

Therefore, $A^{-1} = (1/-33) \times \begin{bmatrix} -8 & -5 & -17 \\ 2 & -7 & -4 \\ -5 & 1 & 10 \end{bmatrix}$

Hence, $A^{-1} = \begin{bmatrix} .24 & .15 & .51 \\ -.06 & .21 & .12 \\ .15 & -.03 & -.39 \end{bmatrix}$

Basic operations

Several basic operations can be applied to matrices. Some, such as *transposition* and *submatrix* do not depend on the nature of the entries. Others, such as *matrix addition*, *scalar multiplication*, *matrix multiplication*, and *row operations* involve operations on matrix entries and therefore require that matrix entries are numbers or belong to a [field]() or a [ring]().

In this section, it is supposed that matrix entries belong to a fixed ring, which is typically a field of numbers.

Addition, scalar multiplication, subtraction and transposition

Addition

The *sum* A+B of two *m*-by-*n* matrices A and B is calculated entry wise:

$(A + B)_{i,j} = A_{i,j} + B_{i,j}$, where $1 \le i \le m$ and $1 \le j \le n$.

For example,

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -5 & 4 \\ -1 & -2 & -3 \end{bmatrix}$

$= \begin{bmatrix} 1+(-1) & 2+(-5) & -1+4 \\ 0+(-1) & 5+(-2) & 3+(-3) \end{bmatrix} = \begin{bmatrix} 0 & -3 & 3 \\ -1 & 3 & 0 \end{bmatrix}$

Scalar multiplication

The product c A of a number c (also called a scalar in this context) and a matrix A is computed by multiplying every entry of A by c:$(cA)_{i,j} = c \cdot A_{i,j}$.

This operation is called *scalar multiplication*, but its result is not named "scalar product" to avoid confusion, since "scalar product" is often used as a synonym for "inner product". For example:

$5 \,.\, \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 5\,.1 & 5\,.2 & 5\,.-1 \\ 5\,.0 & 5\,.5 & 5\,.3 \end{bmatrix} = \begin{bmatrix} 5 & 10 & -5 \\ 0 & 25 & 15 \end{bmatrix}$

Properties of scalar multiplication in matrices

The different properties of matrices for scalar multiplication of any scalars K and l, with matrices A and B are given as,

- K (A + B) = KA + KB

  Let A = $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$, B = $\begin{bmatrix} 5 & 4 \\ -1 & 6 \end{bmatrix}$, and k = 2. Find K (A + B) = KA + KB

  K (A + B) = 2 $\left(\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ -1 & 6 \end{bmatrix}\right) = 2 \,.\, \begin{bmatrix} 6 & 7 \\ 1 & 13 \end{bmatrix}$

$$= \begin{bmatrix} 12 & 14 \\ 2 & 26 \end{bmatrix} \quad ..... (1)$$

$$KA + KB = 2 \,.\, \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} + 2 \,.\, \begin{bmatrix} 5 & 4 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 14 \end{bmatrix} + \begin{bmatrix} 10 & 8 \\ -2 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 14 \\ 2 & 26 \end{bmatrix} \quad .... \ (2)$$

FROM (1) AND (2) $\longrightarrow$ K (A + B) = KA + KB

- (K + l) A = KA + l A
- (Kl)A = K (l A) = l(KA)
- (–K) A = – (KA) = K(–A)
- 1–A = A
- (–1) A = –A

Subtraction

The subtraction of two $m \times n$ matrices is defined by composing matrix addition with scalar multiplication by –1:

A – B = A + (–1) . B

Transposition

The *transpose* of an *m*-by-*n* matrix A is the *n*-by-*m* matrix A^T (also denoted A^{tr} or ${}^tA$) formed by turning rows into columns and vice versa:

$(A^T)_{i,j} = A_{j,i}$. For example:

$$\begin{bmatrix} -1 & -5 & 4 \\ -1 & -2 & -3 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 \\ -5 & -2 \\ 4 & -3 \end{bmatrix}$$

2×3 3×2

Familiar properties of numbers extend to these operations on matrices: for example, addition is commutative, that is, the matrix sum does not depend on the order of the summands: A + B = B + A. The transpose is compatible with addition and scalar multiplication, as expressed by $(CA)^T = C(A^T)$ and $(A + B)^T = A^T + B^T$. Finally, $(A^T)^T = A$.

Example: Let A = $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{bmatrix}$, B = $\begin{bmatrix} -1 & -5 & 4 \\ -1 & -2 & -3 \end{bmatrix}$, $G = \begin{bmatrix} 3 & 5 & -1 \\ 7 & 2 & -3 \end{bmatrix}$ and C = 4. Find I. A + B II. B + A III. $(C\ G)^T$ Iv. $C(G^T)$ v. $(A^T)^T$

SOLUTION

I. A + B = $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -5 & 4 \\ -1 & -2 & -3 \end{bmatrix}, = \begin{bmatrix} 0 & -3 & 3 \\ -1 & 3 & 0 \end{bmatrix}$ … (1)

II. B + A = $\begin{bmatrix} -1 & -5 & 4 \\ -1 & -2 & -3 \end{bmatrix}, + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 3 \\ -1 & 3 & 0 \end{bmatrix}$ … (2)

From (1) and (2), we find A + B = B + A

III. $(CG)^T = \begin{bmatrix} 4.3 & 4.5 & 4.-1 \\ 4.7 & 4.2 & 4.-3 \end{bmatrix}^T = \begin{bmatrix} 12 & 20 & -4 \\ 28 & 8 & -12 \end{bmatrix}^T = \begin{bmatrix} 12 & 28 \\ 20 & 8 \\ -4 & -12 \end{bmatrix}$

iv. $C(G^T) = 4 \cdot G^T = 4 \cdot \begin{bmatrix} 3 & 5 & -1 \\ 7 & 2 & -3 \end{bmatrix}^T = 4 \cdot \begin{bmatrix} 3 & 7 \\ 5 & 2 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 12 & 28 \\ 20 & 8 \\ -4 & -12 \end{bmatrix}$

From III. and IV, we find $(CG)^T = C(G^T)$.

v. $(A^T)^T = \left(\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{bmatrix}^T \right)^T = \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{bmatrix}$

<u>Matrix multiplication</u>

I. $a_{1\times m} \cdot b_{m\times 1} = c_{1\times 1}$

II. Also, $a_{m\times 1} \cdot b_{1\times n} = C_{m\times n}$

III. $a_{1\times m} \cdot B_{m\times n} = c_{1\times n}$

IV. $A_{m\times n} \times b_{n\times 1} = c_{m\times 1}$

V. $A_{m\times n} \times B_{n\times p} = C_{m\times p}$

Example: If

$A = \begin{pmatrix} 3 & 2 \\ 1 & -1 \\ 5 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}$,

$X = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$, $Y = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

Find: AB, BA, XB, BX, YX, XY, B^TA^T, A^TB, BA.

SOLUTION

<u>Finding AB:</u> $A_{3\times 2} \quad \times \quad B_{2\times 2} = C_{3\times 2}$

=

$$AB = \begin{pmatrix} 3 & 2 \\ 1 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 4 + 2 \times 3 & 3 \times 5 + 2 \times 1 \\ 1 \times 4 + (-1) \times 3 & 1 \times 5 + (-1) \times 1 \\ 5 \times 4 + 4 \times 3 & 5 \times 5 + 4 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 17 \\ 1 & 4 \\ 32 & 29 \end{pmatrix}$$

<u>Finding B A:</u> $B_{2 \times 2} \times A_{3 \times 2}$

$\neq$

Multiplying the two matrices is impossible because the number of columns in the first matrix is not equal to the number of rows in the second matrix.

<u>The conclusion is that multiplication of matrices is not commutative.</u>

<u>Finding A^T B:</u> $A^T_{2 \times 3} \times B_{2 \times 2}$ is not impossible.

Finding $B^T A^T$: $B^T_{2 \times 2} \times A^T_{2 \times 3} = C_{2 \times 3}$

$$B^T A^T = \begin{pmatrix} 4 & 3 \\ 5 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 5 \\ 2 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 18 & 1 & 32 \\ 17 & 4 & 29 \end{pmatrix}$$

Finding XY : $X_{2 \times 2} \times Y_{3 \times 3}$ is not impossible.

Finding Y X : $Y_{3 \times 3} \times X_{2 \times 2}$ is not impossible.

<u>Finding BX:</u> $B_{2 \times 2} \times X_{2 \times 2} = C_{2 \times 2}$

$$BX = \begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 23 & 29 \\ 9 & 8 \end{pmatrix}$$

Finding X B: $X_{2\times 2} \times B_{2\times 2} = C_{2\times 2}$

$$XB = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} \times \begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 11 \\ 27 & 20 \end{pmatrix}$$

Note: Multiplying X, B <u>of the same size</u> is commutative, but not necessarily X B equals B X.

It is also noted that the following rules apply for multiplication of matrices:

(1) (AB) C = A (BC)

(2) A (B + C) = AB + AC

(3) (B + C) A = BA + CA

(4) K (AB) = (KA) B

Where k as (scalar) number.

(5) A I = I A = A

Where I is *identity/unit* matrix *of appropriate size.*

(6) A ϕ = ϕ A = ϕ

Where ϕ is a zero *matrix of appropriate size.*

(6) If we get a zero matrix from multiplying two matrices, this does not necessarily mean that either of the two multiplied matrices is zero. For example,

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 & 4 \\ 2 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(8) If a matrix is a square, we get: $A^2 = A \times A$, $A^3 = A^2 \times A$

The Rank of the Matrix, r(A):

The rank of a matrix is defined as the largest number of rows or columns that the matrix contains and which are linearly independent. The rank of the matrix is symbolized by r(A).

Example: Find the rank of the matrix: A = $\begin{pmatrix} 2 & 10 \\ -1 & -5 \\ 3 & 15 \end{pmatrix}$

SOIUTION

As c_1, c_2 are linearly correlated, because $c_1 = \frac{1}{5} c_2$, then the matrix contains one linearly independent column. Thus, r (A) = 1

The same result can be reached (by looking at the rows). Because the matrix has three linearly rows because:

$R_1 = -2R_2 = \frac{2}{3} R_3$

So, the matrix has one linearly independent row. Thus, its rank = 1. *That is*: r (A) = 1.

Example: Find the rank of the matrix:

$$A = \begin{pmatrix} 2 & 5 \\ -1 & -4 \\ 3 & 12 \end{pmatrix}$$

SOLUTION

As R_2 , R_3 are linearly correlated, because $R_3 = -3R_2$, then the matrix contains two linearly independent rows. Thus, r (A) = 2

Problems: (1) Find the following order of matrices:

(1) $A = \begin{pmatrix} 0 & 0 \dots & 0 \\ 0 & 0 & 0 \\ . & & . \\ . & & . \\ . & & . \\ 0 & 0 \dots & 0 \end{pmatrix} \rightarrow$ r (A) = 0 [r (A) $\geq$ 1 if A $\neq \phi$]

(2) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \\ 1 & 3 & 2 \end{bmatrix} \longrightarrow$ r (A) = 2

The Inverse of a Square Matrix (A^{-1}): Matrix A has an inverse if and only if (1) it is a square matrix, and (2) it is non-singular, i.e. $|A| \neq 0$.

In algebra, the inverse or reciprocal of the number a is a^{-1} (or $\frac{1}{a}$), the product of the number and its inverse is always equal to 1, or a . a^{-1}= 1. A similar relationship between a matrix and its inverse may be expressed.

Let A = a square matrix

A^{-1} = the inverse of *A*

Then A^{-1} . A = A . A^{-1} = *I*.

There are many methods in finding the inverse of matrix, such as:

- *The adjoint Matrix* method
- *The reduction method or the elementary operations method* (Gauss Way / Method)

Using *the Adjoint Matrix* in Finding the Inverse of Matrix

$$A^{-1} = \frac{adj.A}{|A|} = \frac{1}{|A|}\text{ (adj. A)}$$

Example: Find the inverse of each matrix *A, if the inverse exists, by adjoint Matri*x method. Aso, check your answer by the relationship

A . $A^{-1} = I$.

(1) A = $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$

(2) A = $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

SOLUTION

(1) $\because$ $A^{-1} = \frac{adj.matrix}{det.A}$

$\because$ Adj. Matrix = (Co-efficient matrix)^T

$= \begin{pmatrix} |-1| & -|2| \\ -|3| & |1| \end{pmatrix}^T = \begin{pmatrix} -1 & -3 \\ -2 & 1 \end{pmatrix}$

12-, $\because$ Det. A = $\begin{vmatrix} 1 & ...3 \\ 2 & -1 \end{vmatrix}$ = $\{1 \times (-1)\} - \{3 \times 2\} = -7$

13-$A^{-1} = \frac{1}{-7} \begin{pmatrix} -1 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1/7 & 3/7 \\ 2/7 & -1/7 \end{pmatrix}$

CHECK: A $A^{-1} = A^{-1}$ A = I_2 = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1/7 & 3/7 \\ 2/7 & -1/7 \end{pmatrix} = \begin{pmatrix} 1/7 & 3/7 \\ 2/7 & -1/7 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$(2)\quad \begin{pmatrix} \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \end{pmatrix}^{T}$$

$$(2)\quad A^{-1} = \frac{Adj.A}{Det.A} = \frac{\begin{pmatrix} 4 & 0 & -4 \\ -2 & 1 & 3 \\ -2 & -1 & 5 \end{pmatrix}}{4} =$$

$$\begin{pmatrix} 4/4 & 0 & -4/4 \\ -2/4 & 1/4 & 3/4 \\ -2/4 & -1/4 & 5/4 \end{pmatrix}$$

CHECK: $A\,A^{-1} = A^{-1}\,A = I_3$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4/4 & 0 & -4/4 \\ -2/4 & 1/4 & 3/4 \\ -2/4 & -1/4 & 5/4 \end{pmatrix}$$

$$= \begin{pmatrix} 4/4 & 0 & -4/4 \\ -2/4 & 1/4 & 3/4 \\ -2/4 & -1/4 & 5/4 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use of elementary operations in Finding the inverse

Operations That produce Row-Equivalent Matrices.

An augmented matrix is transformed into a row-equivalent matrix by performing any of the following row operations (In other words, either of the following transformations can be performed on matrix without change of its size or order):

(A) Two rows are interchanged $(R_i \leftrightarrow R_j)$

(2) A row is multiplied by a nonzero constant (k $R_i \rightarrow R_i$)

(3) A constant multiple of one row is added to another row (k R_i + R_j $\rightarrow$ R_j)

Note: The arrow $\longrightarrow$ means "replaces"

The following examples illustrate this:

Example: Find the inverse of the following matrices using the reduction method:

(1) A = $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$, (2) A = $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

SOLUTION

(1) First: Write [A | I_2] (Augmented matrix)

Second: We perform elementary operations on the rows of the augmented matrix to transform A into I_2 and to transform I_2 into A^{-1}, as follows:

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{-2R1+R2} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -7 & -2 & 1 \end{array}\right) \xrightarrow{-1/7\ R2}$$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2/7 & -1/7 \end{array}\right) \xrightarrow{-3R2+R1}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1/7 & 3/7 \\ 0 & 1 & 2/7 & -1/7 \end{array}\right)$$

(It is the same result we got using the Adjoint Matrix)

(2) First: Write [A | I_3] (Augmented matrix)

Second: We perform elementary operations on the rows of the augmented matrix to transform A into I_3 and to transform I_3 into A^{-1}, as follows:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R1/2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-R1+R2\ and-R1+R3} \left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 5/2 & -3/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 1/2 & -1/2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{2}{5}R2} \left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -3/2 & -1/5 & 2/5 & 0 \\ 0 & 1/2 & 1/2 & -1/2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R2+R1\ and-\frac{1}{2}R2+R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 4/5 & 3/5 & -1/5 & 0 \\ 0 & 1 & -3/5 & -1/5 & 2/5 & 0 \\ 0 & 0 & 4/5 & -2/5 & -1/5 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{5}{4}R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 4/5 & 3/5 & -1/5 & 0 \\ 0 & 1 & -3/5 & -1/5 & 2/5 & 0 \\ 0 & 0 & 1 & -1/2 & -1/4 & 5/4 \end{array}\right)$$

$$\xrightarrow{-\frac{4}{5}R3+R1 \ and -\frac{3}{5}R3+R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1/2 & 1/4 & 3/4 \\ 0 & 0 & 1 & -1/2 & -1/4 & 5/4 \end{array}\right)$$

So, $A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1/2 & 1/4 & 3/4 \\ -1/2 & -1/4 & 5/4 \end{pmatrix}$

(It is the same result we got using the Adjoint Matrix)

Solving systems of linear equations using matrices:

Any linear equation may be written as follows:

$a_1x_1 + a_2 x_2 = \text{c}$ where a_1, a_2, and c are constants.

Generally, the equation may be written in the form:

$a_1x_1 + a_2 x_2 + \ldots + a_nx_n = \text{c}$ where a_1, a_2, c *are constants. This equation is called linear equation in the variables (or unknowns):*

$x_1 + x_2 + \ldots + x_n = c$

For examples, the following equations are linear equations:

$x = 6$, $-2x + y + z = 6$, $x_1 - \frac{1}{5}x_2 + 5x_3 + x_4 = 9$

$$2x + 3y = 4$$

NOTE: The following equations are not linear equations:

$$2xy + 3z = 5 \qquad 2x^2 + 3y = 4$$

$$\sqrt{x_1} + 2x_2 - x_3 = 7 \qquad \frac{1}{x} - 2y + 3z = 10$$

❖ *Methods of Solving Linear Equations*

There several ways to solve instantaneous linear equations, they are:

- Elimination Method
- Substitution Method
- Graphing Method
- Determinants Method
- Matrices Method

Here, we will focus on the Matrices Method to solve instantaneous linear equations.

❖ *Solving Linear Equations in Two Variables (Unknowns)*

Example (1): Solve the following linear equations:

$x + y = 3$

$2x - y = 0$

By using:

(1) *Inverse of matrix* (2) *Reduced row echelon form (rref)*

SOLUTION

(1) *Using Inverse of matrix*

$\because$ X = A^{-1} B OR $\begin{pmatrix} x_1 \\ x_2 \\ : \\ . \\ x_n \end{pmatrix} = (A_{nn})^{-1} \begin{pmatrix} b_1 \\ b_2 \\ : \\ . \\ b_n \end{pmatrix}_{n\times 1}$

$(A_{nn})^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1}$ → Coefficient matrix

Where, $X = \begin{pmatrix} x_1 \\ x_2 \\ : \\ . \\ x_n \end{pmatrix}_{n\times 1} = \begin{pmatrix} x \\ y \end{pmatrix}$ → Variable matrix/ vector

$B = \begin{pmatrix} b_1 \\ b_2 \\ : \\ . \\ b_n \end{pmatrix}_{n\times 1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ → Constant matrix / vector

$A \times X = b$ → $x = \frac{b}{A} = \frac{1}{A} . b = A^{-1} . b \quad \therefore X = A^{-1} b$

$\because X = A^{-1} b$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}^{-1} \times \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/7 & 3/7 \\ 2/7 & -1/7 \end{pmatrix} \times \begin{pmatrix} 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore x = 1,\ y = 2$$

NOTE: The system of linear equation given in *Example (1)* *<u>consistent and independent.</u> Thus, the system has <u>Exactly one solution</u> or <u>Unique solution.</u>*

<u>NOTE:</u> *<u>A graph</u> of the system's equations <u>gives two lines intersecting at one point.</u>*

(2) *Using reduced row echelon form (rref)*

First: Write [A | I_3] (Augmented matrix) as follows:

$$\left(\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -1 & 0 \end{array}\right)$$ Second: We perform elementary operations on the rows of the augmented matrix to transform A into I_2 (or *reduced row echelon form (rref)* as follows:

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

The matrix is now in the Echelon form (note that the last line is a true statement with no variables, i.e., *the system is <u>consistent and dependent</u>*). The formula corresponds to the linear system:

x + 3y = 3 → x = – 3y + 3 → x = – 3 t + 3, where (t ∈ R).

<u>Therefore, the system has an infinite number of solutions (or infinitely many solutions)</u>

Example (2): Solve the following linear equations:

x + y = 3

$2x + 2y = 7$

By using:

(1) *Inverse of matrix (2) Reduced row echelon form (rref)*

SOLUTION

Because the coefficient matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ is not reversible (i.e., it has no inverse; $|A| = 0$). So, it cannot be solved using inverse of matrix method.

(2) *Using <u>reduced row echelon form</u> (rref)*

First: Write [A | I_3] (Augmented matrix) as follows:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 7 \end{array}\right)$$

Second: We perform elementary operations on the rows of the augmented matrix to transform A into I_2 (or *reduced row echelon form (rref)* as follows:

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 7 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 0 & 1 \end{array}\right)$$

<u>The matrix is now in the Echelon form (note that the last line is incorrect statement with no variables, i.e., the system is inconsistent.</u> Therefore, the system has <u>no solution.</u>

NOTE: A graph of the system's equations gives <u>two parallel lines.</u>

Therefore, there is no any point on this line <u>satisfies</u> the system.

From examples 1- 3, we conclude that the system with two equations in two unknowns has three possibilities:

- System has only one solution, if it is consistent and independent.
- System has an infinite number of solutions, if it is consistent and dependent.
- System has no solution, if it is inconsistent.
- The possibilities of solutions (graphically, algebraically and in matrices) are illustrated by the following statement:

Solutions:	Exactly one solution	Infinitely many solutions	No solution
	Consistent and Independent	*Consistent and Dependent*	*Inconsistent*
Graphing:	Intersecting Lines	Collinear Lines	Parallel Lines
Algebraic:	Unique value for variables	Produces true statement (with no variables)	Produces a false statemen (with no variables)
Matrix:	$\left(\begin{array}{cc\|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right)$	$\left(\begin{array}{cc\|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right)$ (Last row is a true statement)	$\left(\begin{array}{cc\|c} 1 & m & n \\ 0 & 0 & p \end{array}\right)$ (Last row is a false statement)

m, n $\in R$ and p $\neq 0$

Problems on Chapter 2: (Determinants and Matrices)

(1) If the determinant of matrix $\begin{bmatrix} 2x & 9 \\ 2 & x \end{bmatrix}$ is 0, then find the possible value(s) of x. [Answer: x = 3 or x = -3.]

(2) Find the value of the determinant of 4×4 matrix

$$\begin{bmatrix} 10 & 0 & 4 & -6 \\ 2 & 5 & 0 & 3 \\ 2 & 0 & 8 & -12 \\ 2 & 1 & -2 & 3 \end{bmatrix}$$ [Answer: - 216.]

(3) If A = $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix}$, then prove that $|4A| = 4^3\,|A|$.

(– 320 = – 320)

(4) Using Cramer's Rule, solve – if possible- the following problems:

(a) (1) X + 2y = 6   (3) x + y = 8   (5) X + 2y = 4

4x + y = -7   2x – y = 10   2x + y = 1

(2) 5x + y = -3   (4) 3x – y = 5   (6) x + 4y = 11

x + 4y = 26   x + y = 19   2x – y = 4

(7) 3x+y+z= 4   (9)2x+y+2z = 3   (10) x–3y+3z = –2

2x+ 2y + 3z =1   x– 2y –3z = 1   2x + y – 2z = 3

3x –y – 2z =7   3x + 2y + 4z =1   3x – y + z = 2

(8) 3x + 2y – z = 1

2x +2y + z = 13

x + y + 2z = 11

(5) Let u = (8, 1); v = (4, 5); $v^T = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$; $w = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$x = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$; $y = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$; and $z = (4, 8, 9)$

Perform the indicated operations in each of the following expressions.

(1) u + v	(6) w	(11) 2v	(16) v. v^T
(2) v – u	(7) X + y	(12) 3w	(17) u.w
(3) u – v	(8) Y – x	(13) 6z	(18) v.w
(4) v^T – w	(9) 5u	(14) 4x	(19) z.w
(5) w– v^T	(10) $\frac{2}{3}v$	(15) $\frac{1}{5}y$	(20) z.y

(6) Let $A = \begin{pmatrix} 3 & 9 \\ 2 & 4 \end{pmatrix}$; $B = \begin{pmatrix} 5 & 6 \\ 1 & 7 \end{pmatrix}$; $C = \begin{pmatrix} 1 & 2 \\ 3 & 9 \\ 8 & 4 \end{pmatrix}$;

$D = \begin{pmatrix} 5 & 1 \\ 2 & 7 \\ 3 & 6 \end{pmatrix}$; $E = \begin{pmatrix} 4 & 6 & 3 \\ 1 & 5 & 8 \end{pmatrix}$; $F = \begin{pmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{pmatrix}$

Perform the indicated operations in each of the following expressions.

(1) A + B	(8) $\frac{1}{3}E$	(15) E. C	(22) D. F
(2) C + D	(9) A. B	(16) F. D	(23) C. F
(3) C – D	(10) B. A	(17) C. A	(24) D. E
(4) A – B	(11) A. E	(18) D. B	(25) 7(C-B)
(5) 3a	(12) B. F	(19) F. C	(26) $\frac{1}{2}(D$ –A)
(6) 2B	(13) A. F	(20) E. D	

(7) $\frac{1}{2}$ C (14) B. E (21) C. E

(7) Let A = $\begin{pmatrix} 1 & -6 & 2 \\ -4 & 2 & 1 \end{pmatrix}$ B = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ C= $\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix}$

D = $\begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix}$ E = $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 6 & 1 \end{pmatrix}$ F= $\begin{pmatrix} 6 & 2 \end{pmatrix}$

G = $\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$ H = $\begin{pmatrix} 1 & 6 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ J = (4)

(a) State the size of each matrix

(b) Which matrices are square? [B, D, E, H, and J]

(c) Which matrices are upper triangular? Lower triangular? [H – D]

(d) Which are row vectors? [F, J]

(e) Which are column vectors? [G, J]

(8) Let

$$A = (A_{ij}) = \begin{pmatrix} 7 & 3 & 4 & 6 \\ 6 & 2 & 3 & -2 \\ 5 & 4 & 1 & 0 \\ 8 & 0 & 2 & 0 \end{pmatrix}$$

1. What is the order of A? [4×4]
2. Find the following entries?

(a) a_{21} (b) a_{42} (c) a_{32} (d) a_{34} (e) a_{44} (f) a_{55}

(9) Construct the matrix

(a) A = (A_{ij}) = if A is 2 × 4 and A_{ij} = – i + 2 j

(b) The matrix 2 × 4 matrix C = ((i + j) ^{2}). Then compute A^TC.

(6) Solve the matrix equations

(a) $\begin{pmatrix} 3_x & 2_y - 1 \\ z & 5_w \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 7 & 15 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 3 \\ x & 7 \\ 3y & 2z \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 6 & 7 \\ 2 & 7 \end{pmatrix}$

© $\begin{pmatrix} 4 & 2 & 1 \\ 3x & y & 3z \\ 0 & w & 7 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 \\ 6 & 7 & 9 \\ 0 & 9 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 2x & 7 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} y & 7 \\ 7 & y \end{pmatrix}$

(10) Find all values of x for which

$$\begin{pmatrix} x^2 + 2000x & \sqrt{x^2} \\ x^2 & In(e^x) \end{pmatrix} = \begin{pmatrix} 2001 & -x \\ 2001 - 2000x & x \end{pmatrix}$$

(11) If I = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; A = $\begin{pmatrix} 3 & -2 \\ -4 & 1 \\ 5 & 6 \end{pmatrix}$; B = $\begin{pmatrix} 3 & 1 & 4 & 2 \\ 1 & 7 & 3 & 6 \\ 1 & 4 & 2 & 2 \end{pmatrix}$

Compute $I^5 A^T B$

(12) Compute the required matrix, if it exists, given that

A = $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ B = $\begin{pmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ C = $\begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{pmatrix}$

0 = $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(a) A^2 (b) A^TA (c) B^4 (d) $A (B^T)^2 C$ (e) $(AIC)^T$
(f) $A^T(2C^T)$ (g) $(BA^T)^T$ (h) $(3A)^T$ (i) $(2I)^2 - 2 I^2$ (j) $(A^TC^TB)^0$ (k) A(I-0) (l) I^T0 (m) (AC) (AC)T (n) B^2-3B+2 I

(13) Express the matrix equations

x $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ − y $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$ = 3 $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ → [$x = -\frac{18}{39}$ $y = \frac{72}{39}$]

As a system of linear equations and solve.

(14) Perform the indicated operations of each the following:

(a) $\begin{pmatrix} 2 & 0 & 3 \\ -1 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 1 \\ 0 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$

(c) $\left(\begin{pmatrix} -2 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix} + 2 \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \end{pmatrix} \right) \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \left[\begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{pmatrix} \right]$

(e) $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 9 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

(15) If A = $\begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & -1 \\ 0 & 4 & 3 \end{pmatrix}$, B = $\begin{pmatrix} 0 & -2 & 3 \\ -2 & 4 & -2 \\ 3 & 1 & -1 \end{pmatrix}$,

and AB = (C_{ij}) .

Find:

(a) $C_{31} = 1$ (b) the size and number of entries of C. [$3\times 3, 9$]

(16) Solve each of the following systems of equations:

(a) Using the inverse of a square matrix,

(b) Using determinants, and

(17) If,

$[A] = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$; $[B] = \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix}$; C = 4

Prove that:

(a) $(AB)^T = B^T A^T$ (b) $A A^T = I$

(c) $(A + B)^T = A^T + B^T$ (d) $(B^2)^T = (B^T)^2$

(e) $(CA)^T = (A^T)$

(18) Find the entries of the matrix A that satisfy the equation:

$$2A - \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \end{pmatrix} = 3\begin{pmatrix} 3 & 0 & 5 \\ 2 & 1 & 4 \end{pmatrix}$$

(19) If, $A = \begin{pmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & 1 & 1 \end{pmatrix}$ Prove that $A^3 = A^{-1}$

(20) Find the inverse of the following matrix using coefficient matrix method, and reduction method.

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 3 & 3 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

 (17) If,

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$. Compute: (I) $B = I - A\,(A^T A)^{-1} A$, +9

(II) Prove: $B = B^2$

(21) (1) Using reduced row echelon form (rref), solve

(1) $x + 2y = 6$
$4x + y = -7$

(3) $x + y = 8$
$2x - y = 10$

(2) $5x + y = -3$
$x + 4y = 26$

(4) $3x - y = 5$
$x + y = 19$

(5) $x + 2y = 4$
$2x + y = 1$

(7) $3x + y + z = 4$
$2x + 2y + 3z = 1$

(6) $x + 4y = 11$

$2x - y = 4$

$3x - y - 2z = 7$

(8) $3x + 2y - z = 1$

$2x + 2y + z = 13$

$x + y + 2z = 11$

(9) $2x + y + 2z = 3$

$x - 2y - 3z = 1$

$3x + 2y + 4z = 1$

(10) $x - 3y + 3z = -2$

$2x + y - 2z = 3$

$3x - y + z = 2$

(22) Solve the following equations, using reduction method:

(a) $2x_1 + 3x_2 = 7$

$4x_1 + x_2 = 9$

(b) $2x - 4y + 3z = 1$

$x - 2y + 4z = 3$

$3x - y + 5z = 2$

(C) $2x + 2y = 7$

$-4x - 5y = 40$

(d) $5x + 2y = 1$

$7x + 4y = 8$

(e) $2x - 3y + z - 2 = 0$

$x + 5y - 4z + 5 = 0$

$4x + y - 3z + 4 = 0$

(f) $2x - 4y + 3z = 1$

$x - 2y + 4z = 3$

$3x - y + 5z = 2$

(g) $2x + 3y + 6z = 1$

$x + y - z = 0$

$5x + 2y + z = 2$

(h) $2x - y + z = 2$

$3x + 2y + 2z = -2$

$x - y + z = 1$

(i) $x + 2y + 3z = 5$

$2x - y - z = 1$

$x + 3y + 4z = 6$

(j) $2x + 3y = 7$

$6x + 9y = 21$

(k) $2x - 3y = 7$

$6x + 9y = 20$

(l) $2x - 5y + 2z = 0$

$x + 4y - 3z = 0$

(m) $3x + 2y - z = 0$

$2x - y + 3z = 0$

$x + y - z = 0$

(n) $3x + 2y - z = 0$

$2x - y + 3z = 0$

$x + y - z = 0$

(O) $3x + 2y + 2z = 0$

$5x + 2y + 3z = 0$

(p) $3x - y + 2z = 0$

$2x + 3y - 5z = 0$

$x + y + z = 0$

(q) $2x - y + 3z = 0$

$x + 2y - 5z = 0$

$3x + y - 2z = 0$

(23) Using reduced row echelon form (rref), solve

(b) 1. $x + 2y = 6$, $4x + y = -7$ 3. $x + y = 8$, $2x - y = 10$ 5. $X + 2y = 4$, $2x + y = 1$

2. $5x + y = -3$, $x + 4y = 26$ 4. $3x - y = 5$, $x + y = 19$ 6. $x + 4y = 11$, $2x - y = 4$

7. $3x+y+z= 4$ 9. $2x+y+2z = 3$ 10. $x - 3y+3z = -2$

$2x+ 2y + 3z =1$ $x- 2y -3z = 1$ $2x + y - 2z = 3$

$3x -y - 2z =7$ $3x + 2y + 4z =1$ $3x - y + z = 2$

8. $3x + 2y - z = 1$

$2x +2y + z = 13$

$x + y + 2z = 11$

(24) In the following problems, determine whether the statement is true or false.

(A) There exist two 1×1 matrices A and B such that AB ≠ BA

(B) There exist two 2×2 matrices A and B such that AB ≠ BA

(C) There exist two 2×2 matrices A and B such that AB is the 2×2 zero matrix.

(D) There exist two 1×1 matrices A and B such that AB is the 1×1 zero matrix.

(25) A square matrix is a diagonal matrix if all elements not on the principal diagonal are zero. So, a 2×2 diagonal matrix has the form

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

Where a and d are real numbers. Discuss the validity of each of the following statements. If the statement is always true, explain why. If not, give examples.

(A) If A and B are 2×2 diagonal matrices, then A + B is a 2×2 diagonal matrix.

(B) If A and B are 2×2 diagonal matrices, then A + B = B + A

(C) If A and B are 2×2 diagonal matrices, then A B is a 2×2 diagonal matrix.

(D) If A and B are 2×2 diagonal matrices, then A B = BA

(26) A square matrix is an upper triangular matrix if all elements below the principal diagonal are zero. So, a 2×2 upper triangular matrix has the form

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

Where a and d are real numbers. Discuss the validity of each of the following statements. If the statement is always true, explain why. If not, give examples.

(A) If A and B are 2×2 upper triangular matrices, then A + B is a 2×2 upper triangular matrix.

(B) If A and B are 2×2 upper triangular matrices, then A + B = B + A

(C) If A and B are 2×2 upper triangular matrices, then A B is a 2×2 upper triangular matrix.

(D) If A and B are 2×2 upper triangular matrices, then A B = BA

(27) *In the following problems, the matrix equation is not solved correctly. Explain the mistake and find the correct solution. Assume that the indicated inverses exist.*

(A) $AX = B$, $X = \frac{B}{A}$

(B) $XA = B$, $X = \frac{B}{A}$

(C) $XA = B$, $X = A^{-1} B$

(D) $A X = B$, $X = B A^{-1}$

(E) $A X = BA$, $X = A^{-1}BA$, $X = B$

(F) $XA = AB$, $X = AB A^{-1}$, $X = B$

(28) In the following problems, explain why the system cannot be solved by matrix inverse methods. Discuss methods that could be used and then solve the system.

(A) $-2x_1 + 4x_2 = -5$

$6x_1 - 12x_2 = 15$

(B) $x_1 - 3x_2 - 2x_3 = -1$

$-2x_1 + 6x_2 + 4x_3 = 3$

(C) $x_1 - 2x_2 + 3x_3 = 1$

$2x_1 - 3x_2 - 2x_3 = 3$

$x_1 - x_2 - 5x_3 = 2$

(29) For $n \times n$ matrices A and B, and $n \times 1$ columns matrices C, D, and X, solve each matrix equation in the following problems for X Assume that all necessary inverses exist.

(A) AX + BX = C

(B) AX – X = C

(C) AX + C = BX + D

(30) Discuss the number of solutions for a system of n equations in n variables if the coefficient

(A) has an inverse.

(B) Does not have an inverse.

(31) Discuss the number of solutions for a system corresponding to the reduced form shown below if

(A) $m \neq 0$

(B) $m = 0$ and $n \neq 0$

(C) $m = 0$ and $n = 0$

2024/2025
2024/2025

CALCULUS

FIRST: DEFFERENTIATION

SECOND: INTEGRATION

2024/2025 2024/2025 2024/2025

CHAPTER:3

DIFERENTIATION AND INTEGRATION

FIRST: DIFFERENTIATION

There is no doubt that most natural and economic phenomena are constantly changing. Speed changes with time, the price of the commodity changes according to availability, the profit earned by a manufacturer to produce a particular commodity change according to sales of the commodity, and other examples.

<u>The differentiation is a science that examines changes in different phenomena (variables)</u> i.e., measuring the change in a dependent variable as a result of a change occurred in an independent variable. If there is a relationship between the dependent variable y and the independent variable x on the general form y = f (x), and y = f (x) is a continuous function over the period in which the independent variable is defined. The change in y is Δ and we express this as follows: if y = f (x), then y + Δy = x + Δx. This means that if the independent variable x increases by Δx, the y variable increases by Δy. We can determine the value of Δy from the previous equation as follows:

Δy = f (x + Δx) – y = f (x + Δx) – f (x)

By dividing the two parties by Δx we get:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} \text{ -------------------------------- } (1)$$

= The ratio between the change in y (Δy) to the change in x (Δx), i.e., the average change in y.

Extracting limit to both sides as Δx → 0, i.e.

$$\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to} \frac{\mathrm{f(x+\Delta x)-f(x)}}{\Delta x} = \frac{dy}{dx} \text{ .... } (2)$$

The symbol $\frac{dy}{dx}$ (the first derivative or the first differential factor of the function) is sometimes written $y^{/}$ or $f^{/}(x)$ or $\frac{d}{dx}[f(x)]$.

It should be noted here that $\frac{dy}{dx}$ is not a ratio but the limit of an expression is reached by a specific mathematical process called the differentiation process. $\frac{dy}{dx}$ is also called the *"instantaneous rate of change of the function y*". It should also be noted that the previous steps that have brought us to a limit may be summarized in what is called the four–step process for finding the derivative of a *function f:*

Step 1 Find $f(x + \Delta x)$.

Step 2 Find $f(x + \Delta x) - f(x)$.

Step 3 Find $\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Step 4 Find $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

The following examples show how the four–step process is used to find both the *average change* in y $\left(\frac{\Delta y}{\Delta x}\right)$ and the *instantaneous rate of change in y* (i.e., the first derivative, $\frac{dy}{dx}$)

Example (1) Use the four-step process[1] to show that:

If $f(x) = e^{cx}$, then $f'(x) = c\,e^{cx}$

Solution

$f(x) = e^{cx}$

[1] In the process of finding the derivative of e^{ch}, we used (without proof) the fact that $\lim_{h \to 0} \frac{e^{ch}-1}{h} = c$

<u>Step 1</u>. $f(x + h) = e^c (x + h) = e^c x \cdot e^c h$

<u>Step 2</u>. $f(x + h) – f(x) = e^c (x + h) – ecx = ecx \cdot e^c h – e^c x$

$$= ecx\,(e^c h – 1)$$

<u>Step 3</u>. $\frac{f(x+h) - f(x)}{h} = \frac{e^{cx}\ (e^{ch}-1)}{h} = e^{cx} \left(\frac{e^{ch}-1}{h}\right)$

<u>Step 4</u>. $f'(x) = \lim_{h\to 0} e^{cx} \left(\frac{(e^{ch}-1}{h}\right) = e^{cx} \lim_{h\to 0} \frac{e^{ch}-1}{h}$

$$= c\, e^{cx}$$

Example (2) if $y = 3x^2$. Find:

1- Average change in y (i.e., $\frac{\Delta y}{\Delta x}$)

2- The instantaneous rate of change in y (i.e., the first derivative $\frac{dy}{dx}$)

Solution

$\because y = 3x^2$. Assuming that there has been a change in x of Δx, there is a change in y of Δy and the function after the change, becomes:

$y + \Delta y = 3(x + \Delta x)^2 = 3\,(x^2 + 2x\Delta x + (\Delta x)^2)$

$\therefore \Delta y = 3x^2 + 6x\Delta x + 3(\Delta x)^2 – y = 3x^2 + 6x\Delta x + 3(\Delta x)^2 – 3x^2 = 6x\Delta x + 3(\Delta x)^2$

$$\frac{\Delta y}{\Delta x} = \frac{6x\Delta x+3(\Delta x)^2}{\Delta x} = 6x + 3\,\Delta x$$

 2024/2025 2024/2025

1. Average change in y:

$\frac{\Delta y}{\Delta x}$ = 6x + 3Δx

For example, if x changed from x = 5 to x = 7 i.e., Δx = 7 – 5 = 2, then average change in y is:

$\frac{\Delta y}{\Delta x}$ = 6 × 5 + 3 × 2 = 36

2. The rate of change in y or the first derivative:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6x + 3\Delta x) = 6x$$

At x =2 (for e.g.) the *instantaneous rate of change,*

$\frac{dy}{dx}$ = 6 x 2 = 12

Example (3) if y = x^2 – 2x + 1. Find the rate of instantaneous change in y when x = 3.

Solution

y = x^2 – 2x + 1

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to} \frac{\mathrm{f(x+\Delta x)-f(x)}}{\Delta x} = \lim_{\Delta x \to} \frac{[(x+\Delta x)^2-2(x+\Delta x)+1]-[x^2-2x+1]}{\Delta x}$$

$$= \lim_{\Delta x \to} \frac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1]}{\Delta x} = \lim_{\Delta x \to} \frac{2x\Delta x+(\Delta x)^2-2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \Delta \to 0} 2x + \Delta x - 2$$

$\therefore \frac{dy}{dx}$ = (2x – 2)

At x = 4, the *instantaneous rate of change* is: $\frac{dy}{dx}$ = 4 × 2 – 2 = 6

Example (4) If the total cost function can be expressed by the following equation: C = 20 x^2 + 850 where x reflects the volume of production. Find the marginal cost when x = 5.

Solution

The marginal cost can be expressed as the limit of the ratio $\frac{\Delta y}{\Delta x}$ as Δx → 0, i.e.:

$$\frac{dc}{dx} = \lim_{\Delta x \to 0} \frac{\Delta c}{\Delta x} = \lim_{\Delta x \to} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to} \frac{[20(x+\Delta x)^2 + 825] - [20x^2 + 850]}{\Delta x}$$

$$= \lim_{\Delta x \to} \frac{[20(x^2 + 2x\Delta x + (\Delta x)^2 + 825] - [20x^2 + 850]}{\Delta x}$$

$$= \lim_{\Delta x \to} \frac{[40x^2 + 40x\Delta x + (\Delta x)^2 + 850] - [20x^2 + 850]}{\Delta x}$$

$$= \lim_{\Delta x \to} \frac{20x^2 + 40x\Delta x + 20(\Delta x)^2 + 850] - 20x^2 - 850]}{\Delta x} \lim_{\Delta x \to} \frac{40x\Delta x + 20(\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (40x + 20\Delta x)$$

$\therefore \frac{dc}{dx}$ = 40 x

That is, the marginal costs are 40x.When the production volume is 5 units, the marginal costs become:40 × 5 = 200

<u>Rules of Differentiation</u>

Instead of using the four–step process or general method to find (the first derivative) that require a lot of effort and time, there are rules or theories that greatly facilitate finding of differential derivatives for the function as they avoid the use of limits when finding derivatives. We will present those rules without proof, but we will focus on several examples that illustrate their use as follows:

<u>Rule (1): Constant Function Rule</u>

If y = *f(x)* = k is a constant function, then:

$\frac{d}{dx}(k) = \frac{dy}{dx}$ = $f'(x)$ = y' = 0. Therefore,

The derivative of any constant function is 0.

Note: When we write C' = 0 or $\frac{d}{dx}(C) = 0$, we mean that y' = $\frac{dy}{dx}$ = 0 when y = C.

<u>Rule (2): Power Rule:</u> If *y = f(x)* = x^n, where n is any real number, then $f'(x) = nx^{n-1}$

Also, $\frac{dy}{dx} = \frac{d}{dx}\,(\,x^n\,) = nx^{n-1}$ and $y^{/} = nx^{n-1}$

That is, the derivative of a constant power of x is the exponent times x raised to a power one less than the given power.

<u>Rule (3): Constant Factor Rule:</u> y = *k f(x)*

If *f* is a differentiable function and *k* is a constant, then *k f(x)* is differentiable and

$$\frac{d}{dx}\,[kf(x)] = \text{k}\,\frac{d}{dx}\,[f(x)] = \;k\,f^{/}(\text{x})$$

That is, the derivative of a constant times a function is the constant times the derivative of the function.

Example (5)

Differentiate the following functions:

(A) 1. y = 3 2. G(x) = $\sqrt{10}$ 3. s(t) = $(1{,}718{,}278)^{504.5}$

(B) y = x

(C) y = x^{-1}

(D) y = 6 x^5

(E) y = $x^{-1}x^5 - \frac{1}{5}x^3$

(F) y = $-\frac{8}{x^2} = -\frac{8}{\sqrt{x^3}}$

(G) $(3x^2)^5$

Solution

(A)

1. $\because$ y = 3 $\quad \therefore \frac{dy}{dx} = 0$

This is a self-evident matter because if we take any point on x, the corresponding value on y will not change which is 3. (Or because 3 is a constant function).

2. If g(x) = $\sqrt{10}$, then $g'(x) = 0$ *because g is a constant function. For example, the derivative of g when x = 4 is $g'(4) = 0$.*

3. If $s(t) = (1{,}718{,}278)^{504.5}$, then $\frac{ds}{dt} = 0$.

(B) y = x $\quad \therefore \frac{dy}{dx} = 1 \times x^0 = 1$

That is, the function differentiation for itself = 1

(C) (C) y = x^{-1} $\quad \therefore \frac{dy}{dx} = -1 \times x^{-2} = -x^{-2}$

(D) y = 6 x^5 $\quad \therefore \frac{dy}{dx} = 6 \times 5\, x^4 = 30\, x^4$

(E) $\because$ y = $-\frac{1}{5}x^3$ $\quad \therefore \frac{dy}{dx} = -\frac{1}{5} \times (3\, x^2) = -\frac{3}{5}x^2$

(F) $\because$ y = $-\frac{8}{\sqrt{x^3}} = \frac{-8}{x^{\frac{3}{2}}} = -8\, x^{\frac{-3}{2}}$

$\therefore \frac{dy}{dx} = -8 \times \frac{d}{dx}(x^{-\frac{3}{2}}) = -8 \times (-\frac{3}{2} x^{\left(-\frac{3}{2}-1\right)}) = 12\, x^{\frac{-5}{2}}$

$= \frac{12}{x^{\frac{5}{2}}} = \frac{12}{\sqrt{x^5}}$

(G) $\because$ y = $(3x^2)^5$ $\therefore \frac{dy}{dx}$ = $3^5 \times 10\, x^9 = 2430\, x^9$

<u>Rule (4): Sum or Difference Rule</u>

If *y = f(x)* = $f_1(x) \pm f_2(x)$, then

$\frac{d}{dx}[f_1(x) \pm f_2(x)] = \frac{d}{dx}[f_1(x)] \pm [\frac{d}{dx} f_2(x)]$. Or

$f'(x) = f_1'(x) \pm f_2'(x)$

Also, $y' = f_1' \pm f_2'$ $\qquad \frac{dy}{dx} = \frac{df_1}{dx} \pm \frac{df_2}{dx}$

That is, the derivative of the sum (difference) of two differentiable functions is the sum (difference) of their derivatives.

Note: This rule generalizes to the sum and difference of any given number of functions.

Example (6) Differentiating Sums and Differences of Functions

Differentiating the following functions:

(A) $y = 3x(x^4 + 3x^3 - 5x^2 + x + 1)$ (B) $y = (5x - 3)(2x^2 + x + 1)$

(C) $y = \sqrt{x}(\sqrt{x} + x - 1)$ (D) $y = (2x - 1)(3x + 1)(5x - 3)$

Solution

(A) $\because y = 3x\,(x^4 + 3x^3 - 5x^2 + x + 1) = 3x^5 + 9x^4 - 15x^3 + 3x^2 + 3x$

$\therefore \frac{dy}{dx} = 3\times 5x^4 + 9\times 4x^3 - 15\times 3x^2 + 3\times 2x + 3\times 1 = 15x^4 + 36x^3 - 45x^2 + 6x + 3$

(B) $\because$ $y = (5x-3)(2x^2+x+1) = 10x^3 + 5x^2 + 5x - 6x^2 - 3x - 3$

$= 10x^3 - x^2 + 2x - 3$

(We have converted the product of multiplying the two functions to the product of adding (or subtracting) the two functions so that we can use the sum (or difference) rule for several functions:

 $\therefore \frac{dy}{dx} = 30x^2 - 2x + 2$

(C) $\because$ $y = \sqrt{x}\,(\sqrt{x}+x-1) = x + x^{\frac{3}{2}} + x^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = 1 + \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} = 1 + \frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}}$

(d) $\because y = (2x-1)(3x+1)(5x-3)$

$\therefore \frac{dy}{dx} = 2\times(3x+1)(5x-3) + 3(2x-1)\times(5x-3) + 5(2x-1)(3x+1)$

<u>Rule (5): Product *Rule*</u>

If $f_1(x)$, $f_2(x)$ are differentiable functions, then the product $f_1\,f_2$ is differentiable, and

$\frac{d}{dx}[\, f_1\,(x) \times f_2\,(x)]$ = $f_1(x) \times f_2'(x) + f_2(x) \times f_1'(x)$

That is, the derivative of the product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the first.

$\frac{d}{dx}(product)$ = *(first (derivative of the second)) + (second (derivative of the first))*

The rule of the multiplication product can be generalized to include several functions.

Example (7) Find $\frac{dy}{dx}$ of the function:

y = $(5x-3)(2x^2+x+1)$

Solution

$\because y = (5x - 3)(2x^2 + x + 1)$

$\therefore \frac{dy}{dx} = (5x - 3)(4x + 1) + (2x^2 + x + 1)(5)$

$= 20x^2 - 7x - 3 + 10x^2 + 5x + 5 = 30x^2 - 2x + 2$

Which is the same result we got in the previous example(B)

Rule (6): The quotient Rule

If $f_1(x)$, $f_2(x)$ are two differentiable functions, and $f_2(x) \neq 0$, then the quotient f_1 / f_2 is also differentiable, and

$$\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] = \frac{f_2(x) \times f_1^{/}(x) - f_1(x) \times f_2^{/}(x)}{[f_2(x)]^2}$$

With the understanding about the denominator not being zero, we can write

$$\left(\frac{f_1}{f_2}\right)^{/} = \frac{f_2\, f_1^{/} - f_1\, f_2^{/}}{[f_2(x)]^2}$$

2024/2025 *That is, the derivative of the quotient of two functions is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.* 2024/2025 2024/2025

$$\frac{d}{dx}(quotient) =$$

$$\frac{(denominator)(derivative\ of\ numirator)-(numerator)\ (derivative\ of\ denominator)}{(denominator)^2}$$

Example (8) Find the derivative of the following functions:

(A) $y = \frac{x^5}{x^3}$ (B) $y = \frac{3x^2-1}{x+2}$

(C) $f(x) = \frac{1+\sqrt{x}}{1-\sqrt{x}}$ (d) $y = \frac{3}{2\sqrt{x}}$

Solution

(A) $\because y = \frac{x^5}{x^3} \quad \therefore \frac{dy}{dx} = \frac{x^3 \times 5\,x^4 - x^5\ \times 3\,x^2}{(x^3)^2} = \frac{5x^7 - 3\,x^7}{x^6} = \frac{2\,x^7}{x^6} = 2\,x$

Or in another way:

$\because y = \frac{x^5}{x^3} = x^2 \qquad \therefore\ y' = \frac{dy}{dx} = 2x$

(B) $\because y = \frac{3x^2-1}{x+2}$

$$\therefore \frac{dy}{dx} = \frac{(x+2)\times 6x-(3x^2-1)\times 1}{(x+2)^2} = \frac{6\,x^2+12x-3\,x^2+1}{(x+2)^2} = \frac{3\,x^2+12x+1}{(x+2)^2}$$

(C) $\because y = \frac{1+\sqrt{x}}{1-\sqrt{x}} = \frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}$

$$\therefore \frac{dy}{dx} = \frac{\left(1-x^{\frac{1}{2}}\right)\times\frac{1}{2}x^{-\frac{1}{2}}-\left(1+\frac{1}{2}x\right)\times-\frac{1}{2}x^{\frac{-1}{2}}}{(1-x^{\frac{1}{2}})^2} = \frac{\frac{1}{2}x^{\frac{-1}{2}}-\frac{1}{2}+\frac{1}{2}x^{\frac{-1}{2}}+\frac{1}{2}}{(1-x^{\frac{1}{2}})^2} = \frac{x^{\frac{-1}{2}}}{(1-x^{\frac{1}{2}})^2}$$

$$= \frac{1}{\sqrt{x}\,(1-\sqrt{x})^2}$$

(D) $\because y = \frac{3}{2\,\sqrt{x}} = \frac{3}{2}x^{\frac{-1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{3}{2}\times\left(-\frac{1}{2}\right)\left(x^{\frac{-3}{2}}\right) = -\frac{3}{4}\left(\frac{1}{x^{\frac{3}{2}}}\right) = -\frac{3}{4\,\sqrt{x^3}}$$

Example (9) assuming that f (x) = $\frac{1}{3}x^2$

Find:

(A) f (–3) (B) f (0) (C) $\frac{f(x+h)-f(x)}{h}$

Solution

(A) f (–3) = $\frac{1}{3}(-3)^2 = 3$

(B) f (0) = $\frac{1}{3}(0) = 0$

(C) $\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{3}(x+h)^2 - \frac{1}{3}x^2}{h} = \frac{\frac{1}{3}(x^2 + 2xh + h^2) - \frac{1}{3}x^2}{h}$

$= (\frac{2}{3}x + \frac{1}{3}h)$

<u>Rule (7): The Chain Rule</u> y = f (u), u = Φ (x)

If y is a differentiable function of u and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example (10) *If* $y = 7u^3$ *and* $u = 3x^5 + 1$, find $\frac{dy}{dx}$

Solution

By the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ = 21u^2 × 15x^4 = 315 x^4 u^2 = 315x^4 (3x^5 + 1)^2

Example (11) *If y = (3x^2+5x+1)^7*. Find $\frac{dy}{du}$

Solution

We assume that u = 3 x^2 + 5x +1. Thus y = u^7

$\therefore \frac{dy}{dx} = \frac{dy}{d\,u} \times \frac{du}{dx}$ = 7u^6 × (6x+5) = 7(3x^2 + 5x+1)^6(6x + 5)

= 7(6x + 5) (3x^2 + 5x +1)^6

Example (12) *If y =* $\sqrt[5]{2x^2 + x - 1}$, find $\frac{dy}{du}$

Solution

Rewriting, we have y = $(2\,x^2 + x - 1)^{\frac{1}{5}}$

By chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{5} u^{\frac{-4}{5}}$ ×(4x+1)

$= \frac{1}{5} \times \frac{1}{u^{\frac{4}{5}}}(4x+1) = \frac{1}{5}(4x+1) \times \frac{1}{\sqrt[5]{(2x^2+x-1)^4}}$

<u>Important result:</u> <u>(General Power Rule)</u>

If u(x) is differentiable function, n is any real number, and

y = f(x) = [u(x)]^n,

then

$$f'(x) = n\,[u(x)]^{n-1}\,u'(x)$$

Using simplified notation,

$$y' = nu^{n-1}u' \quad \text{or} \quad \frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \quad \text{where } u = u(x)$$

That is. If $y = (x^5 - 2x^3 + 9)^{10}$, then

$$\frac{dy}{dx} = \{Exponent\ (inside)^{exponent-1}\}\ (inside)'$$

$$= 10\,(x^5 - 2x^3 + 9)^9 \times (5x^4 - 6x^2)$$

As such, we can find the derivative of the function described in example (11) in a concise and quick manner as follows:

$$y = \sqrt[5]{2x^2 + x - 1} = (2x^2 + x - 1)^{\frac{1}{5}}$$

$$\frac{dy}{du} = \{Exponent\ (inside)^{exponent-1}\}\ (inside)'$$

$$= \frac{1}{5}\,(2x^2 + x - 1)^{\frac{-4}{5}}(4x + 1) = \frac{1}{5}\,(4x + 1)\,\frac{1}{5\,\sqrt[5]{(2x^2 + x - 1)^4}}$$

This is the same result we got earlier when using the chain rule.

Example (13)

Assuming that a ceramics factory has found that annual profits (estimated in millions of pounds) are given by the equation:

p(x) = 50 + $(\frac{3}{2}x^2 - 3x)^2$

x represents how much the factory spends (millions of pounds) on advertising. Find $\frac{dp}{dx}$ when x = 4

$$\frac{dp}{dx} = 0 + 2(\frac{3}{2}x^2 - 3x) \times (3x - 3)$$

And when x = 4 then:

$\frac{dy}{dx}$ = 2 $[(\frac{3}{2} \times (4)^2 - 3(4)] \times (3 \times 4 - 3) = 2 \times 12 \times 9 = 216$

So, when advertising expenses reach 4 million (x= 4), the profits made by the factory are increasing at a rate of 216 million per million spent on advertising.

Example (14) If $y = \frac{1}{x^3-5}$ *find* $\frac{dy}{dx}$

Solution

$\because y = \frac{1}{x^3-5} = (x^3 - 5)^{-1}$

$\therefore \frac{dy}{dx} = (-1)(x^3 - 5)^{-1-1} \frac{d}{dx}(x^3 - 5) = (-1)(x^3 - 5)^{-2}(3x^2)$

$$= -\frac{3x^2}{(x3-5)^2}$$

Example (15) If $y = \left(\frac{5x+7}{x^3+1}\right)^5$, find $\frac{dy}{dx}$

Solution: Since y is a power of a function, we first use the power rule:

$$\frac{dy}{dx} = 5\left(\frac{5x+7}{x^3+1}\right)^{5-1} \frac{d}{dx}\left(\frac{5x+7}{x^3+1}\right)$$

Now we use the quotient rule:

$$\frac{dy}{dx} = 5\left(\frac{5x+7}{x^3+1}\right)^4\left(\frac{(x^3+1)(5)-(5x+7)(3x^2)}{(x^3+1)^2}\right)$$

$$= \left(\frac{-5\,(10x^3-21x^2-5)\left(\frac{5x+7}{x^3+1}\right)^4}{(x^3+1)^2}\right)$$

Example (16) If $y = (x^2-5)^4(5x+1)^5$, find y'

Solution: Since y is a product, we first apply the product rule:

$$y' = (x^2-5)^4 \frac{d}{dx}(5x+1)^5 + (5x+1)^5 \frac{d}{dx}(x^2-5)^4$$

Now we use the power rule:

$$y' = (x^2-5)^4(5(5x+1)^4(5)) + (5x+1)^5(4(x^2-5)^3(2x))$$

$$= (x^2-5)^4(25(5x+1)^4) + (5x+1)^5(8x(x^2-5)^3$$

$$= 200x(x^2-5)^7(5x+1)^9.$$

*Example (17) If the demand equation for a manufacturer's product is

$$P = \frac{1000}{q+10}$$

Where p is in dollars, find the marginal-revenue function and evaluate it when q = 40.

Solution: $\because$ revenue = (price) (quantity); that is, r = pq

$\therefore$ The revenue function is

$$r = \left(\frac{1000}{q+10}\right) q = \frac{1000q}{q+10}$$

Thus, the marginal-revenue function is given by

$$\frac{dr}{dq} = \frac{(q+10)\,(1000q)' - (1000q)(q+10)'}{(q+10)^2}$$

$$= \frac{(q+10)\,(1000) - (1000q)(1)}{(q+10)^2} = \frac{10000}{(q+10)^2}$$

And

$$\frac{dr}{dq}\Big|_{q=40} = \frac{10000}{(40+10)^2} = \frac{10000}{2500} = 4$$

<u>*This means that selling one additional unit beyond 40 results in approximately \$4 more in revenue. In other words, selling the 41th unit results in approximately \$4 more in revenue.*</u>

*Example (18) Find the slope of the graph of $f(x) = (5x^3 - 3x + 7)(2x^5 + 10)$ when $x = 1$.

Solution: We find the slope by evaluating the derivative when $x = 1$. By using the product rule, we have

$f'(x) = (5x^3 - 3x + 7)(10x^4) + (2x^5 + 10)(15x^2 + 3)$
When $x = 1$, $f'(1) = 9(10) + 12(18) = 306$

The interpretations of the derivative:

(1) The tangent line (or tangent) to a curve at point *p* is the limiting position of secant lines PQ as Q approaches *p* along the curve. The slope of the tangent at *p* is called the slope of the curve at *p*.

If $y = f(x)$, the derivative of f at x is the function $f'(x)$ defined by the limit in the equation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Geometrically, the derivative gives the slope of the curve $y = f(x)$ at the point $(x, f(x))$. An equation of the tangent line at a particular point $(a, f(a))$ is obtained by evaluating $f'(a)$, which is the slope of the tangent line, and using the point-slope form of a line: $y - f(a) = f^{/}(a)(x - a)$. Any function that is differentiable at a point must also be continuous there.

(2) The derivative $\frac{dy}{dx}$ can also be interpreted as giving *the (instantaneous) rate of change of y with respect to x:*

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{change\ in\ y}{change\ in\ x}$$

In particular, if s = f(t) is a position function, where s is potion at time t, then

$$\frac{ds}{dt} = \textit{velocity at time t}$$

(3) *In economics, the term marginal is used to describe derivatives of specific types of functions*. This can be shown, as follows:

(A) If *c = f (q)* is a total -cost function (*c* is the total cost of *q* units of a product), then the rate of change

$$\frac{dc}{dq} \text{ is called marginal cost}$$

We interpret marginal cost as the approximate cost of one additional unit of output. (Average cost per unit, $\bar{c}$, is related to total cost c by $\bar{c} = \frac{c}{q}$, equivalently, $c = \bar{c}q$.)

(B) A total – revenue function $r = f(q)$ gives a manufacturer's revenue r for selling q units of product. (Revenue r and price p are related by $r = p\,q$.) The rate of change

$\frac{dr}{dq}$ is called marginal revenue

Which is interpreted as the approximate revenue obtained from selling one additional unit of output.

If r is the revenue that a manufacturer receives when the total output of m employees is sold, then the derivative $\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$ is called the marginal-revenue product and gives the approximate change in revenue that results when the manufacturer hires an extra employee.

(C) If C = f (I) is a consumption function, where I is national income and C is national consumption, then

$\frac{dC}{dI}$ <u>is marginal propensity to consume</u> and

$1 - \frac{dC}{dI}$ <u>is the marginal propensity to save</u>

For any function, <u>the relative rate of change of f (x) is</u>

$$\frac{f^{/}(x)}{f(x)}$$

Which compares the rate of change of f (x) with f (x) itself.

The percentage rate of change is

$$\frac{f^{/}(x)}{f\ (x)} \, .\ 100\%$$

<u>Marginal Average Cost, Revenue, and Profit</u>

Sometimes it is desirable to carry out marginal analysis relative to average cost (cost per unit), average revenue (revenue per unit), and average profit (profit per unit).

If x is the number of units of a product produced in some time interval, then

Cost per unit: average cost = $\bar{C}\ (x) = \frac{C\ (x)}{x}$

Managerial average cost = $\bar{C}^{\ /}(x) = \frac{d}{dx}\,\bar{C}\ (x)$

Revenue per unit: average revenue = $\bar{R}\ (x) = \frac{R(x)}{x}$

Managerial average revenue = $\bar{R}^{\ /}(x) = \frac{d}{dx}\,\bar{R}\ (x)$

Profit per unit: average profit = $\bar{P}\ (x) = \frac{P\ (x)}{x}$

Managerial average profit = $\bar{P}^{\ /}(x) = \frac{d}{dx}\,\bar{P}\ (x)$

Example (19):

The total cost and the total revenue (in dollars) for the production and sale of x hair dryers are given, respectively, by

$C(x) = 5x + 2340$ and $R(x) = 40x - 0.1x^2$

$0 \leq x \leq 400$

(A) Find the value of x where the graph of R(x) has a horizontal tangent line.

(B) Find the profit function P(x)

(C) Find the value of x where the graph of P(x) has a horizontal tangent line.

Solution

Total Cost $C(x) = 5x + 2340$

Total Revenue $R(x) = 40x - 0.1x^2$, $0 \leq x \leq 400$

(A) $R'(x) = 40 - .2x$

the graph of R(x) has a horizontal tangent line at the value(s) of x where $R'(x) = 0$, i.e.

$40 - .2x = o$

or $x = 200$

(B) $P(x) = R(x) - C(x) = 40x - 0.1x^2 - 5x - 2340$

$$= 35x - 0.1x^2 - 2340$$

(C) $P'(x) = 35 - .2x$ setting $P'(x) = 0$, we have

$$35 - .2x = 0 \text{ or } x = 175$$

Example (20): The price – demand equation and the cost function for the production of HDTVs are given, respectively, by

$$x = 9000 - 30p \qquad \text{and} \qquad C(x) = 15000 + 30x$$

Where x is the number of HDTVs that can be sold at a price of \$p per TV and $C(x)$ is the total cost (in dollars) of production x TVs.

(A) Express the price p as a function of the demand x, and find the domain of this function.

(B) Find the marginal cost.

(C) Find the revenue function and state its domain.

(D) Find the marginal revenue.

(E) Find $R'(3000)$ and $R'(6000)$ and interpret these quantities.

(F) Find the profit function in terms of x.

(G) Find the marginal profit.

(H) Find the $p^{/}(1500)$ and $p^{/}(4500)$ and interpret these quantities.

Solution

(A) $x = 9000 - 30p$ and $C(x) = 15000 + 30x$

$30p = 9000 - x$, $p = 300 - \frac{1}{30}x$, $0 \le x \le 9000$

(B) $C^{/}(x) = 30$

(C) $R(x) = x\left(300 - \frac{1}{30}x\right) = 300x - \frac{1}{30}x^2$, $0 \le x \le 9000$

(D) $R^{/}(x) = 300 - \frac{1}{30}x$

 (E) $R^{/}(x) = 300 - \frac{1}{30}(3000) = 100$, at a production level of 3000 sets,
revenue is increasing at the rate of 100 per set.

(F) $p(x) = R(x) - C(x) = 300x - \frac{1}{30}x^2 - (15000 + 30x)$

$$= -\frac{1}{30}x^2 + 270x - 15000$$

(G) $p^{/}(x) = -\frac{1}{15}x + 270$

(1) $p^{/}(1500) = -\frac{1}{15}(1500) + 270 = 170$, at a production level of 1500 sets, profit is increasing at the level of \$170 per set.

(2) $p^{/}(4500) = -\frac{1}{15}(4500) + 270 = -30$, at a production level of 4500 sets, profit is decreasing at the level of \$– 30 per set.

Example (21) A small machine shop manufactures drill bits used in the petroleum industry. The manager estimates that the total daily cost (in dollars) of producing x bits is

$$C(x) = 1000 + 25x - .1x^2$$

(A) Find $\bar{C}(x)$ and $\bar{C}'(x)$.

(B) Find $\bar{C}(10)$ and $\bar{C}'(10)$. Interpret these quantities.

(C) Use the results in part (B) to estimate the average cost per bit at a production level of 11 bits per day.

Solution

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{1000 + 25x - .1x^2}{x}$$

$$= \frac{1000}{x} + 25 - 0.1x \qquad \text{Average cost function}$$

$$\bar{C}'(x) = \frac{d}{dx}\bar{C}(x) = -\frac{1000}{x^2} - 0.1$$ Marginal average cost function

(B) $\bar{C}(10) = \frac{1000}{10} + 25 - 0.1(10) = \$124$

$\bar{C}'(10) = -\frac{1000}{10^2} - 0.1 = -\$10.10$

At a production level of 10 bits per day, the average cost of producing a bit is \$124. This cost is decreasing at the rate of \$10.10 per bit.

(D) If production is increased by 1 bit, then the average cost per bit will be decreased by approximately \$10.10. So, the average cost per bit at a production level of 11bits per day is approximately \$124 – \$10.10 = \$113.90.

Example (22)

The constant e and Continuous Compound Interest Formula

- The number e is defined as

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{s\to 0}(1+s)^{1/s} = 2.718\ 281\ 828\ 459\ \ldots$$

- If a principal P is invested at an annual rate r (expressed as a decimal) compounded continuously, then the amount A in the account at the end of t years is given by

$$A = P\,e^{rt}$$

Example (23): If \$1000 is invested at 10% compounded continuously, what amount will be in the account after 10 years? How much interest will be earned?

Solution

$A = Pe^{rt}$

$= 1000\, e^{(.10)\,(10)} = \$\ 2718.28$ (*10 % is equivalent to r = 0.10)*

The interest earned = \$ 2718.28 – \$ 1000 = \$ 1718.28

Rule 8: Derivatives of Exponential and logarithmic Functions
The derivative of e^x: $\frac{d}{dx} e^x = e^x$

That is, the derivative of the exponential function is the exponential.

 Example (24) Find $f^{/}(x)$ for

(A) $f(x) = 5e^x + 3x^e + 7x^3 + 1$ (B) $f(x) = x^3 - 6e^x + e^2$

Solution

(A) $f^{/}(x) = 5e^x + 3ex^{e-1} + 21x^2$

(B) $f^{/}(x) = 3x^2 - 6e^x$

Remember that e is a real number, so the power rule is used to find the derivative of x^e The derivative of the exponential function e^x, however, is e^x. Note that e^2 is a constant, so its derivative is 0.

The Derivative of ln x

Recall that the inverse of an exponential function is celled logarithmic function for $b > 0$ and $b \neq 1$. Therefore,

$$y = \log_b x \text{ is equivalent to } x = b^y.$$

Of all the possible bases for logarithmic function, the two most widely used are

$$\log x = \log_{10} x \qquad \text{Common logarithm (base 10)}$$

$$\ln x = \log_e x \qquad \text{Natural logarithm (base e)}$$

Therefore,

$$\frac{d}{dx} \ln x = \frac{1}{x} = \frac{1}{inside} (\text{inside})' \quad (x > 0)$$

Before finding the derivative of ln x, we use the following properties of logarithms:

If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\text{Log}_b b^x = x$
4. $b^{\text{Log}_b x} = x, \; x > 0$

5. $\text{Log}_b\, MN = \text{Log}_b\, M + \text{Log}_b\, N$

6. $\text{Log}_b\, \frac{M}{N} = \text{Log}_b\, M - \text{Log}_b\, N$

7. $\text{Log}_b\, M^p = p \log_b M$

8. $\text{Log}_b\, M = \text{Log}_b\, N$ if and only if $M = N$

Example (25) Find $f^{\prime}(x)$ for

(A) $f(x) = 5e^x + 3 \ln x$ (B) $f(x) = x^3 - \ln x^3$

Solution

(A) $f^{\prime}(x) = 5e^x + \frac{3}{x}$

(B) $f(x) = x^3 - 3 \ln x$ *(property 7)*

$$f^{\prime}(x) = 3x^2 - \frac{3}{x}$$

Other Logarithmic and Exponential Functions

Derivatives of Exponential and Logarithmic Functions

for $b > 0$ and $b \neq 1$

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx} b^x = b^x \ln b$$

for $b > 0$ and $b \neq 1$, and $x > 0$

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad \frac{d}{dx} \log_b x = \frac{1}{\ln b} \left(\frac{1}{x}\right)$$

Example (26) Find $f'(x)$ for

(A) $f(x) = 5^x - 7^x$

(B) $f(x) = \log_5 x^6$

Solution

(A) $f'(x) = 5^x \ln 5 - 7^x \ln 7$

(B) First, use a property of logarithms to rewrite $f(x)$.

$f(x) = \log_5 x^6$ Use $\log_b M^p = p \log_b M$

$f(x) = 6 \log_5 x$ Take the derivative of both

$f'(x) = \frac{6}{\ln 5} \left(\frac{1}{x}\right)$

Example (27) Find $y'(x)$ for

(A) $y = x^{10} + 10^x$ (B) $y = \log_2 x - 6 \log_2 x$

Solution

(A) $y' = 10x^9 + 10^x \ln 10$

(B) $y' = \left(\frac{1}{\ln 2} - \frac{6}{\ln 5}\right) \frac{1}{x}$

Example (28) Find $f^{/}(x)$ for

(A) y = $\log_e x^5$

**(B) y = $\frac{\log x}{x^5}$

(C) y = $\log_e (x^8 + 3x^7 + 2x + 1)^7$

(D) y = $x^5 \log_e (x^2 - 1)$

(E) y = $\log [(x-1)^3 (x+1)^5 (x+2)^7]$

(F) y = $\log \sqrt{\frac{1-3x}{1+3x}}$

 (G) y = $\log \sqrt{x^5 + x^3 + 1}$

Solution

(A) (y) = $\log_e x^5$

Using logarithm properties, we find

y = 5 log x $\therefore \frac{dy}{dx} = 5 \times \frac{1}{x} = \frac{5}{x}$

(B) y = $\frac{\log x}{x^5}$

$y^{/} = \left[\frac{1}{(x5)^2}\left(x^5 \times \frac{1}{x} - \log x \times (5x^4)\right)\right] = \left[\frac{1}{(x5)^2} x^4 - 5x^4 \log x\right] = \frac{x^4}{x^{10}}(1 - 5 \log x)$

$= \frac{1}{x^6}\ (1 - 5\log x)$

(C) y=$\log_e(x^8 + 3x^7 + 2x + 1)^7$= 7 $\log_e(x^8 + 3x^7 + 2x + 1)$

y/ = 7 × $\frac{1}{8x+3x^7+2x+1}$ × ($8x^7 + 21x^6 + 2$ = $\frac{7\ (\ 8x^7+21x^6+2\ )}{(x8+3x^7+2x+1)}$

(D) y = $x^5 \log_e (x^2 - 1)$

y/ = $x^5 \times \frac{1}{x^2-1} \times 2x + \log_e (x^2-1)\ (5x^4)$

$= \frac{2x^6}{x^2-1} + 5x^4\ log_e\ (\ x^2 - 1)$

(E) y = $\log\ [(x - 1)^3\ (x + 1\ )^5\ (\ x + 2\ )^7)]$

(Using logarithm properties)

∴ y = log $(x-1)^3$ + log $(x + 1)^5$ + log $(x + 2)^7$ = 3 log (x–1) + 5 log (x+1) + 7 log (x+2)

∴y/ = $3 \times \frac{1}{x-1} \times 1 + 5 \times \frac{1}{x+1} \times 1 + 7 \times \frac{1}{x+2} \times 1 = \frac{3}{x-1} + \frac{5}{x+1} + \frac{7}{x+2}$

(F) y = $\log \sqrt{\frac{1-x^3}{1+x^3}} = \log (\frac{1-x^3}{1+x^3})^{\frac{1}{2}} = \frac{1}{2}\log\left(\frac{1-x^3}{1+x^3}\right)$

$= \left[\frac{1}{2}[\log(\ 1 - x^3) - \log\ (1 + x^3)]\right]$

$\therefore \frac{dy}{dx} = \frac{1}{2}\ [\ \frac{-3x2}{1-x^3} - \frac{3x2}{1+x^3}] = -\frac{3}{2}x^2\ (\frac{1}{1-x^3} + \frac{1}{1+x^3})$

(G) y = log ($x^5\ \sqrt{x^3+1}$) = log x^5+ log ($x^3 + 1$)$^{\frac{1}{2}}$ = 5 log x + $\frac{1}{2}$ log (x^3 + (1)

$\therefore\ \frac{dy}{dx} = \frac{5}{x} + \frac{1}{2} \times \frac{3x^2}{x^3+1} = \frac{5}{x} + \frac{3}{2}\left(\frac{x^2}{x^3+1}\right)$

Example (29) Find $\frac{dy}{dx}$ for

(A) y = $a^{x^3+x^2+1}$ (B) y = $a^{x^{10}}$ (C) y = e^{3x^2-x-1}

(D) y = 5^u, u = $3x^3 + x^2 - 1$ (E) $y = 9^{x^3+2x^2-x+2}$ (F) y = $e^2 + x^e + 5^x$ + log x

Solution

(A) y = $a^{x^3+x^2+1}$

$\frac{dy}{dx} = a^{x^3+x^2+1} \times (3x^2 + 2x) \times \log_e a$

$= a^{x^3+x^2+1} \times (\log_e a)\ (3x^2 + 2x)$

(B) y = $a^{x^{10}}$

$\frac{dy}{dx} = a^{x^{10}} \times 10x^9 \times \log_e a = a^{x^{10}}\ (10x^9)\ (\log_e a\ )$

(C) y = e^{3x^2-x-1}

$\frac{d\ y}{dx} = e^{3x^2-x-1} \times (6x-1) = (6x-1)e^{3x^2-x-1}$

(D) $y = 5^u$, $u = 3x^3 + x^2 - 1$

$\frac{dy}{dx} = 5^u \times (9x^2 + 2x) \times \log_e 5$

$(E)\ \ y = 9^{x^3 + 2x^2 - x + 2}$

$\frac{dy}{dx} = 9^{x^3 + 2x^2 - x + 2}\,(\log 9) \times (3x^2 + 4x - 1)$

(F) $y = e^2 + x^e + 5^x + \log x$

$\frac{dy}{dx} = 0 + e\,x^{e-1} + 5^x\,(\log 5)\,(1) + \frac{1}{x}$

PROBLEMS

- ❖ find $f'(x)$ in the following problems:

(1) $f(x) = 5e^x + 3x + 1$

(2) $f(x) = - 2 \ln x + x^2 – 4$

(3) $f(x) = x^3 – 6 e^x$

(4) $f(x)$ = ex + x – ln x

(5) $f(x)$ = ln x3

(6) $f(x) = 5x – \ln x^5$

(7) $f(x) = \ln x^2 + 4ex$

(8) $f(x) = e^x + x^e$

(9) $f(x) = x x^e$

(10) $f(x) = -7e^x – 2x + 5$

(11) $f(x) = 6\ln x – x^3 + 2$

(12) $f(x) = 9e^x + 2x^2$

(13) $f(x) = \ln x + 2e^x – 3x^2$

(14) $f(x) = \ln x^8$

(15) $f(x) = 4 + \ln x^9$

(16) $f(x) = \ln x^{10} + 2\ln x$

(17) $f(x) = 3xe – 2ex$

(18) $f(x) = e ex$

- In the following problems, find the equation of the line tangent to the graph of fat the indicated value of x.

(19) $f(x) = 3 + \ln x$; $x = 1$

(20) $f(x) = 3ex$; $x = 0$

(21) $f(x) = \ln x3$; $x = e$

(22) $f(x) = 2 + ex$; $x = 1$

(23) $f(x) = 2\ln x$; $x = 1$

(24) $f(x) = ex + 1$; $x = 0$

(25) $f(x) = 1 + \ln x4$; $x = e$

(26) $f(x) = 5ex$; $x = 1$

- *in the following problems, find f $^{/}$(x) in two ways: (1) using the product or quotient rule and (2) simplifying first*

(27) *f(x)* = $x^4(x^3–1)$ (28) *f(x)* = $\frac{x^4+4}{x^4}$

- *In the following problems, find each indicated derivative and simplify.*

(29) *g(w) = (w–5)* $log_3 w$ (30) $\frac{d}{dx}$ *[(4$x^{1/2}$–1) (3$x^{1/3}$+ 2)]*

(31) $\frac{dy}{dx}$ *for y =* $\frac{10^x}{1+ x^4}$ (32) *y $^{/}$ for y =* $\frac{2\sqrt{x}}{x^2-3x+1}$

(33) *h $^{/}$(t) if h(t) =* $\frac{-0.05t^2}{2t+1}$ (34) $\frac{d}{dt}$ *[10^t log t]*

(35) $\frac{dy}{dx}$ *for y =* $\frac{x^2-3x+1}{\sqrt[4]{x}}$ (36) *y $^{/}$ for y =* $\frac{2\,x-1}{(x^3+2)\,(x^2-3)}$

(37) $\frac{dy}{du}$ *for y =* $\frac{u^2e^2}{1+\ln u}$

SECOND: INTEGRATION

Antiderivatives and indefinite Integrals

- operations in mathematics have reverses-addition and subtraction, multiplication and division, power and roots. We now know how to find the derivatives of many functions. The reverse operation, *antidifferentiation* (the reconstruction of a function from its derivative), will receive our attention in this section.

- A function *F* is an antiderivative of a function *f* if $F'(x) = f(x)$. Any two antiderivatives of *f* (*F* and *G*) differ at most by a constant, that is, F(x) = G(x) + k for some constant *k*. The most general antiderivative of *f* is called the indefinite integral of *f* and is denoted $\int f(x)dx$. Thus,

$$\int f(x)dx = F(x) + C$$

The symbol $\int$ is called an integral sign, *f(x)* is the integrand, and C is the constant of integration.

- Some elementary integration formulas are as follows:

$\int k\,dx = k\,x + C$ *k* a constant

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$ $n \neq -1$

$\int \frac{1}{x}\,dx = \ln x + C$ for $x > 0$

$$\int e^{x}\, dx = e^{x} + C$$

$$\int k\, f(x)\, dx = k \int f(x)dx + C \qquad k \text{ a constant}$$

$$\int [\, f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)dx$$

Another formula is the power rule for integration:

$$\int u^{n}\, du = \frac{u^{n+1}}{n+1} + C, \qquad \text{if } n \neq -1$$

Here u represents a differentiable function of x, and du is its differential. In applying the power rule to a given integral, it is important that the integral be written in a form that precisely matches the power rule. Other formulas are

$$\int e^{u}\, du = e^{u} + C$$

and $$\int \frac{1}{e^{u}}\, du = \ln |u| + C$$

If the rate of change of a function *f* is known – that is, if f' is known- then *f* is an antiderivative of f'. In addition, If we know that *f* satisfies an initial condition, then we can find the particular antiderivative. For example, if a marginal- cost function $\frac{dc}{dq}$ is given to us, then by integration, we can find the most general form of c. That form involves a constant of integration. However, if we are also given fixed costs (that is, costs involved when q = 0), then we can determine the value of the constant of integration and thus find the particular cost function

c. Similarly, if we are given a marginal- revenue function $\frac{dr}{dq}$, then by integration and by using the fact that r = 0 when q = 0, we can determine the particular revenue function r. Once r is known, the corresponding demand equation can be found by using the equation $p = \frac{r}{q}$.

DEFNITE INTEGRAL

Instead of evaluating definite integrals by using limits, we may be able to employ the Fundamental Theorem of Integral Calculus. Mathematically,

$$\int_a^b f(x)dx = F(x)\,\Big|_a^b = F(b) - F(a)$$

where F is any antiderivative of f.

Some properties of the definite integral are

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx \qquad \text{k a constant}$$

$$\int_a^b [\,f(x) \pm g(x)]\,dx = \int_a^b f(x)dx \pm \int_a^b G(x)dx$$

And

$$\int_a^c f(x)\,dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

EXAMPLE 1 Find the following integrals:

(A) $\int x^4 (x^2 + x + 1)\, dx$ (B) $\int (x^2 + 3)(x - 1)\, dx$

(C) $\int \frac{x^4-2}{x}\, dx$ (D) $\int (7x - 1)^2\, dx$

(E) $\int (3x - 5)^6\, dx$ (F) $\int \frac{5}{(3x+2)^3}\, dx$

(G) $\int \frac{5}{(3x+2)^1}\, dx$ (H) $\int e^{5x2+x+1}\, dx$

(I) $\int \frac{5}{\sqrt{3x-1}}\, dx$ (J) $\int \left(\frac{x^3}{3} - \frac{3}{x^3}\right) dx$

SOLUTION

(A) $\int x^4 (x^2 + x + 1)\, dx = \int (x^6 + x^5 + x^4)\, dx$

$= \int x^6\, dx + \int x^5\, dx + \int x^4\, dx$

$= \frac{1}{7} x^7 + \frac{1}{6} x^6 + \frac{1}{5} x^5 + k$

(B) $\int (x^2 + 3)(x - 1)\, dx$

$= \int (x^3 - x^2 + 3x - 3)\, dx$

$= \frac{1}{4} x^4 - \frac{1}{3} x^3 + \frac{3}{2} x^2 - 3x + k$

(C) $\int \frac{x^4-2}{x}\, dx = \int \left(x^3 - \frac{2}{x}\right) dx$

$= \frac{1}{4} x^4 - 2 \log x + k$

(D) $\int (7x-1)^2\,dx = \frac{1}{3}\ (7x-1)^3 \times \frac{1}{7}\ + k$

$$= \frac{1}{21}\ (7x-1)^3 + k$$

(E) $\int (3x-5)^6\,dx = \frac{1}{7}\ (3x-5)^7 \times \frac{1}{3}\ + k$

$$= \frac{1}{21}\ (3x-5)^7 + k$$

(F) $\int \frac{5}{(3x+2)^3}\,dx = 5\int (3X+2)^{-3}\,dx$

$$= 5 \times \frac{(3x+2)^{-3+1}}{-3+1} \times \frac{1}{3}\ + k$$

$$= -\frac{5}{6}\ (3x+2)^{-2} + k$$

(G) $\int \frac{5}{(3x+2)^1}\,dx = 5\int (3X+2)^{-1}\,dx$ (n = –1)

$\therefore 5\int (3X+2)^{-1}\,dx = \frac{\ln(3x+2)}{3}\ = \frac{5}{3}\ \ln(3x+2) + k$

(H) $\int e^{5x2+x+1}\,dx = e^{5x2+x+1} \times \frac{1}{10x+1} + k$

(I) $\int \frac{5}{\sqrt{3x-1}}\,dx = 5\int (3x-1)^{-1/2}\,dx$ (power rule)

$$= 5 \times \frac{1}{\frac{1}{2}} (3x-1)^{\frac{1}{2}} \times \frac{1}{3}\ + k$$

$$= \frac{10}{3}\ (3x-1)^{\frac{1}{2}} + k$$

$= \frac{10}{3} \sqrt{3x+1} + k$

(J) $\int \left(\frac{x^3}{3} - \frac{3}{x^3} \right) dx = \int \frac{x^3}{3}\, dx - \int \frac{3}{x^3}\, dx$

$= \frac{1}{3} \int x^3\, dx - 3 \int x^{-3}\, dx$

$= \frac{1}{3} \times \frac{1}{4} x^4 - 3 \int \frac{x^{-2}}{-2} + k$

$= \frac{1}{12} x^4 + \frac{3}{2} x^{-2} + k$

EXAMPLE 2 Find $\int_2^3 x^3\, dx$

 SOLUTION

$\int_2^3 x^3\, dx = \frac{1}{4} [\, x^4] = \frac{1}{4} [f(3) - f(2)]$

$= \frac{1}{4} [\, (3^4) - (2^4)]$

$= \frac{1}{4} [\, 81 - 16] = \frac{65}{4}$

EXAMPLE 3 Find $\int_0^3 \left(\frac{1}{x+5} + 4x \right) dx$

SOLUTION

3

|

$\int_0^3 \left(\frac{1}{x+5} + 4x\right) dx = \Big[\log(x+5) + 2x\Big]_0^3 = [\log 8 - \log 5] + 2\,[3^2 - 0]$

EXAMPLE 4 Find $\int_0^3 3x(x^2+3)^2\, dx$

SOLUTION

$\int_0^3 3x(x^2+3)^2\, dx = \int_0^3 (3x^5 + 18x^3 + 27x)\, dx$

$$= \left. \left(\frac{1}{2}x^6 + \frac{9}{2}x^4 + \frac{27}{2}x^2\right) \right|_0^3$$

$= [\frac{1}{2}(3)^6 + \frac{9}{2}(3)^4 + \frac{27}{2}(3)^2] - [0]$

$= 364.5 + 364.5 + 121.5 = 850.5$

EXAMPLE 5

If marginal-revenue function for a product,

$$\frac{dR}{dx} = R' = 1000 - .2\,x$$

and marginal - cost function for the same product,

$$\frac{dC}{dx} = .5\,x^2 + .02\,x \qquad \text{(where x size of production)}$$

Assume that fixed costs = 500

Find:

(A) Total revenue equation (B) Total cost equation

(C) Profit equation, then find profit at $x = 30$ units.

SOLUTION

(A) Total revenue equation

Total revenue (R)

$= \int R' \, dx = \int (1000 - .2\,x)\, dx = 1000\,x - .1x^2 + k$

So, Total revenue equation

$= 1000\,x - 0.1x^2$

(B) Total cost equation:

Since total cost (C)

$= \int C' \, dx$

Since k = fixed costs = 500

So, total costs

$= \int \frac{dC}{dx}\, dx = \int (.5\,x^2 + .02\,x)\, dx \quad \therefore \; C = \frac{.5}{3} x^3 + \frac{.02}{2} x^2 + 500$

Therefore, total costs equation

$= \frac{1}{6}x^3 + .01\,x^2 + 500$

(C) Profit equation: Since profit (P)

= Total revenue – total cost = R – C, then profit

$= (1000\,x - .1\,x^2) - (\frac{1}{6}x^3 + .01\,x^2 + 500)$

$= 1000\,x - .1\,x^2 - \frac{1}{6}x^3 - .01\,x^2 - 500$

$= -\frac{1}{6}x^3 - .11x^2 + 1000\,x - 500$

 Therefore, profit equation

$= -\frac{1}{6}x^3 - .11x^2 + 1000\,x - 500$

At $x = 30$

$P = = -\frac{1}{6}(30)^3 - .11(30)^2 + 1000\,(30) - 500$

$= -4500 - 99 + 30000 - 500 = 24901$

GENERAL PROBLEMS

<u>Answer what is required in each of the following</u>:

FIRST: CHOOSE THE CORRECT ANSWER IN EACH OF THE FOLLOWING:

(1) If $y = f(x) = (5x)^3$, then $y^{/}$

a. $3\,(5x)^2$ b. $5^3 \frac{d}{dx}(x)^3$ c. $125\, f^{/}(x^2)$ d. none of these

(2) Slope of the tangent line to the curve $y = \frac{3x^2 - 2}{x}$ when $x = 1$

a. 3 b. 4 c. 5 d. none of these

(3) The equation of the tangent line to the curve $y = \frac{3x^2 - 2}{x}$

a. $y = 3x - 4$ b. $y = 5x - 4$ c. $y = 4x - 5$ d. none of these

(4) If $f(x) = \frac{1}{e^x}$, then $y^{/}(x)$

a. $\frac{-e^x}{(e^x)^2}$ b. $(e^{-x})^{/}$ c. $-x\, e^{-x-1}$ d. none of these

(5) $\frac{dy}{dx}$ (derivative) can be interpreted as

a. slope of the tangent line to a curve b. velocity at time t

c. the (instantaneous) rate of change of y with respect to x

d. In economics, the term marginal means derivative

e. all of these

(6) $\frac{d}{dx}$ (x^6)

a. $\frac{d}{dx}$ (x^4) . $\frac{d}{dx}$ (x^2)

b. $\frac{d}{dx}$ (x^4. x^2)

c. $\frac{d}{dx}$ (x^4) (x^2) – $\frac{d}{dx}$ (x^2) (x^4)

d. none of these

SECOND: Determine whether the following statements are TRUE (T) or FALSE (F):

(7) If $f_1(x)$, $f_2(x)$ are differentiable functions, then the product f_1 f_2 is differentiable, and

$$\frac{d}{dx}[\,f_1\,(x) \times f_2\,(x)] \;= f_1\,(x) \times f_2'(x) - f_2(x) \times f_1'(x)$$

(8) If $f_1(x)$, $f_2(x)$ are differentiable functions, then the product f_1 f_2 is differentiable, and

$$\frac{d}{dx}[\,f_1\,(x) \times f_2\,(x)] \;= \;\frac{d}{dx}\,f_1\,(x) \times \frac{d}{dx}\,f_2\,(x)$$

(9) If $f_1\,(x)$, $f_2\,(x)$ are two differentiable functions, and $f_2\,(x) \neq 0$, then the quotient $f_1\,/\,f_2$ is also differentiable, and

$$\frac{d}{dx}\left[\frac{f_1\,(x)}{f_2\,(x)}\right] = \frac{f_2(x) \times f_1'(x) + f_1(x) \times f_2'(x)}{[f_2(x)]^2}$$

(10) $\int dx = x + C$

(11) $\int_a^b f(x)dx$ = F(x) $|_a^b$ = F(b) – F(a)

where F is any antiderivative of f.

Part II

EQUATIONS &FUNCTIONS

Solving Linear Equations in One Variable

A linear equation is an equation of a straight line, written in one variable. The only power of the variable is 1. Linear equations in one variable may take the form $ax+b=0$ and are solved using basic algebraic operations.

We begin by classifying linear equations in one variable as one of three types: identity, conditional, or inconsistent.

An identity equation is true for all values of the variable. Here is an example of an identity equation.
$3x=2x+x$

The solution set consists of all values that make the equation true. For this equation, the solution set is all real numbers because any real number substituted for x will make the equation true.

A conditional equation is true for only some values of the variable. For example, if we are going to solve the equation $5x+2=3x-6$, we have the following:

We collect all the terms containing X in one side ,and the constant terms in the other side ,then

$5x+2=3x-6 \Longrightarrow 5X\text{-}3x=-6\text{-}2 \Longrightarrow 2x=-8$ or $X=-4$

The solution set consists of one number: {−4}. It is the only solution and, therefore, we have solved a conditional equation.

An inconsistent equation results in a false statement. For example, if we need to solve 5x−15=5(x−4), we have the following:

5x−15=5x−20 .Subtract 5x from both sides.−15≠−20 False statement5x−15=5x−205x−15−5x=5x−20−5xSubtract 5x from

From each side of the equation ,we get a false relation between two sides ,where , $-15\neq-20$. There is no solution because this is an inconsistent equation.

Solving linear equations in one variable involves the fundamental properties of equality and basic algebraic operations. A brief review of those operations follows.

A GENERAL NOTE: LINEAR EQUATION IN ONE VARIABLE

A linear equation in one variable can be written in the form

$ax+b=0$

where ***a*** and ***b*** are real numbers, $a\neq0$

HOW TO: GIVEN A LINEAR EQUATION IN ONE VARIABLE, USE ALGEBRA TO SOLVE IT.

The following steps are used to manipulate an equation and isolate the unknown variable, so that the last line reads X= $\frac{-b}{a}$, if *x* is the unknown. There is no set order, as the steps used depend on what is given:

1. We may add, subtract, multiply, or divide an equation by a number or an expression as long as we do the same thing to both sides of the equal sign. Note that we cannot divide by zero.
2. Apply the distributive property as needed: a(b+c)=ab+ac.
3. Isolate the variable on one side of the equation.
4. When the variable is multiplied by a coefficient in the final stage, multiply both sides of the equation by the reciprocal of the coefficient.

Example 1: Solve the following equation: 2x+7=19.

Solution

This equation can be written in the form ax+b=0 by subtracting 19 from both sides. However, we may proceed to solve the equation in its original form by performing algebraic operations.

$2x+7=19.$

$2X+7-7=19-7$ or $2X=12$ Subtract 7 from both sides.

,

$X=6.$ Multiply both sides by $\frac{1}{2}$ *or devide by* 2.

The solution is $X=6$.

Solving an Equation Algebraically When the Variable Appears on Both Sides

Example 2: Solve the following equation: $4(x-3)+12=15-5(x+6)$.

Solution

Apply standard algebraic properties.

$4(x-3)+12=15-5(x+6)$

$4x-12+12=15-5x-30$

Apply the distributive property

$.4x=-15-5x$

Combine like terms.

$9x=-15$

Place x−terms on one side and simplify.

x=$\frac{-15}{9}=\frac{-5}{3}$

Try It 1

Solve the equation in one variable: −2(3x−1)+x=14−x.

Solution

Solving a Rational Equation

In this section, we look at rational equations that, after some manipulation, result in a linear equation. If an equation contains at least one rational expression, it is a considered a rational equation.

Recall that a rational number is the ratio of two numbers, such as $\frac{2}{3}\,or\,\frac{7}{2}$. A rational expression is the ratio, or quotient, of two polynomials. Here are three examples.

$$\frac{X+1}{X^2-4},\frac{1}{X-3},or\ \frac{4}{X^2+X-2}$$

Rational equations have a variable in the denominator in at least one of the terms.
Our goal is to perform algebraic operations so that the variables appear in the numerator. In fact, we will eliminate all denominators by multiplying both sides of the equation by the least common denominator (LCD).

Finding the LCD is identifying an expression that contains the highest power of all of the factors in all of the denominators. We do this because when the equation is multiplied by the LCD, the common factors in the LCD and in each denominator will equal one and will cancel out.

Example 3: Solving a Rational Equation

Solve the rational equation: $\frac{7}{2X} - \frac{5}{3X} = \frac{22}{3}$

Solution

We have three denominators; 2x,3x, and 3. The LCD must contain 2x,3x, and 3. An LCD of 6x contains all three denominators. In other words, each denominator can be divided evenly into the LCD. Next, multiply both sides of the equation by the LCD(6x)

.

(6x)[$\frac{7}{2X} - \frac{5}{3X}) = (6x)\left(\frac{22}{3}\right)$

Use the distributive property.

$(6x)(\frac{7}{2X})-(6x)(\frac{5}{3X})=(6X)(\frac{22}{3})$.

Cancel out the common factors.

$3(7)-2(5)=22(2x)$

Multiply remaining factors by each numerator.

$21-10=44X$ or $11=44X$, then

$X=\frac{44}{11}=4$

A common mistake made when solving rational equations involves finding the LCD when one of the denominators is a binomial—two terms added or subtracted—such as (x+1).

Always consider a binomial as an individual factor—the terms cannot be separated.

For example, suppose a problem has three terms and the denominators are x, x−1, and 3x−3. First, factor all denominators. We then have x, (x−1), and 3(x−1), as the denominators. (Note the parentheses placed around the second denominator) .Only the last two denominators have a common factor of (x−1). The x in the first denominator is separate from the x in the (x−1) denominators. An effective way to remember this is to write factored and binomial denominators in parentheses, and consider each parentheses as a separate unit or a separate factor. The LCD in this instance is found by multiplying together the x, one factor of (x−1), and the 3. Thus, the LCD is the following:

$$x(x-1)3=3x(x-1).$$

So, both sides of the equation would be multiplied by 3x(x−1). Leave the LCD in factored form, as this makes it easier to see how each denominator in the problem cancels out.

Another example is a problem with two denominators, such as x and x^2+2x.

Once the second denominator is factored as x^2+2x.=x(x+2), there is a common factor of *x* in both denominators and the LCD is x(x+2).

Sometimes we have a rational equation in the form of a proportion; that is, when one fraction equals another fraction and there are no other terms in the equation.

$$\frac{a}{b}=\frac{c}{d}$$

We can use another method of solving the equation without finding the LCD: cross-multiplication. We multiply terms by crossing over the equal sign.

$$\text{If } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a}{b} \times \frac{c}{d}.$$

Multiply a(d) and b(c), which results in ad=bc.

Any solution that makes a denominator in the original expression equal zero must be excluded from the possibilities.

A GENERAL NOTE: RATIONAL EQUATIONS

A rational equation contains at least one rational expression where the variable appears in at least one of the denominators.

HOW TO: GIVEN A RATIONAL EQUATION, SOLVE IT.

1. **Factor all denominators in the equation.**
2. **Find and exclude values that set each denominator equal to zero.**
3. **Find the LCD.**
4. **Multiply the whole equation by the LCD. If the LCD is correct, there will be no denominators left.**
5. **Solve the remaining equation.**
6. **Make sure to check solutions back in the original equations to avoid a solution producing zero in a denominator.**

Example 4: Solving a Rational Equation without Factoring

Solve the following rational equation:

$$\frac{2}{x}-\frac{3}{2}=\frac{7}{2x}$$

Solution

We have three denominators: x, 2, and 2x. No factoring is required. The product of the first two denominators is equal to the third denominator, so, the LCD is 2x. Only one value is excluded from a solution set, x=0. Next, multiply the whole equation (both sides of the equal sign) by 2x.

$2x[\frac{2}{x}-\frac{3}{2}]=2x[\frac{7}{2x}]$ **Distribute**

$2(2)-3x=7$ **Denominators cancel out**

$4-3x=7$

$-3x=3$ **,then** $x=-1$ **or** $\{-1\}$**.**

The proposed solution is x=−1, which is not an excluded value, so the solution set contains one number, x=−1, or {−1}, written in set notation.

Try It

Solve the rational equation: $\frac{2}{3x}=\frac{1}{4}-\frac{1}{6x}$

Example 5: Solving a Rational Equation by Factoring the Denominator

Solve the following rational equation: $\frac{1}{x} = \frac{1}{10} - \frac{3}{4x}$.

Solution

First find the common denominator. The three denominators in factored form are

X,10=2.5 ,and 4x=2.2x. The smallest expression that is divisible by each one of the denominators is 20x. Only x=0 is an excluded value. Multiply the hole equation by 20x.

$$20x\left(\frac{1}{x}\right) = 20x\left(\frac{1}{10} - \frac{3}{4x}\right)$$

20= 2x-15

2x=20+15 =35 or $x = \frac{35}{2} = 17.5$

Try It

Solve the rational equation: $-\frac{5}{2x} + \frac{3}{4x} = -\frac{7}{4}$.

Example 6: Solving Rational Equations with a Binomial in the Denominator

Solve the following rational equation and state the excluded values:

$$\frac{3x-6}{x}=\frac{3x}{x-6}$$

Solution

1. **Therefore, the LCD is the product x(x−6). However, for this problem, we can cross-multiply. (x-6)(5x-6)=x(5x) ,then collect the elements containing x in the left side and constant terms in the right side ;we have**

$$3x^2-24x+36=3x^2 \;\Rightarrow -24x=-36 \Rightarrow x=\frac{36}{24}=1.5$$

To check solutions back in the original equations to avoid a solution producing zero in a denominator.

$$\frac{3(1.5)-6}{1.5}=\frac{3(1.5)}{1.5-6} \Rightarrow \frac{-1.5}{1.5}=-1\ and\ \frac{4.5}{-4.5}=-1\ , i.e.\ two\ sides\ are\ equal$$

Finding a Linear Equation

Perhaps the most familiar form of a linear equation is the slope-intercept form, written as y=mx+b, where m is slope and b is y-intercept. Let us begin with the slope.

The Slope of a Line:

The slope of a line refers to the ratio of the vertical change in *y* over the horizontal change in *x* between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

$m=\frac{y_2-y_1}{x_2-x_1}$,

If the slope is positive, the line slants to the right. If the slope is negative, the line slants to the left. As the slope increases, the line becomes steeper. Some examples are shown in Figure 2. The lines indicate the following slopes: m=−3 for the line y=-3x-2 , m2 for the line y=2x+1,and $m=\frac{1}{3}$, for the line $y=\frac{1}{3}\,x+2$.

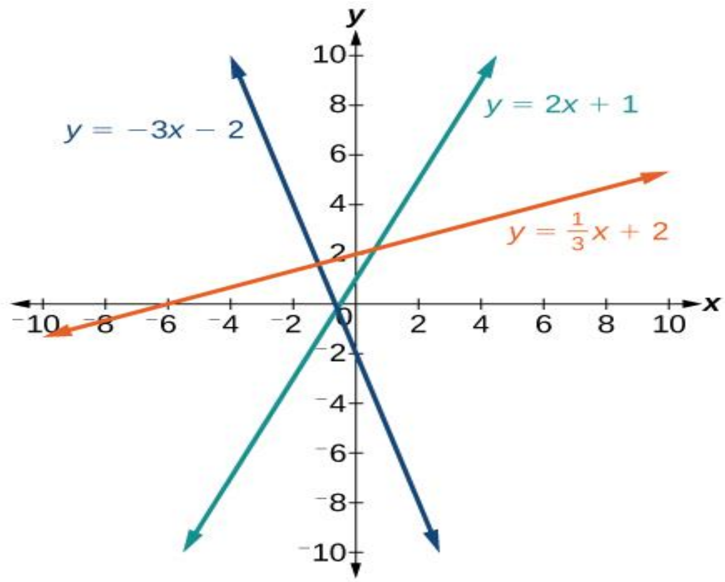

 Figure 2

A GENERAL NOTE: THE SLOPE OF A LINE

The slope of a line, *m*, represents the change in *y* over the change in *x*. Given two points, (x_1, y_1) and (x_2, y_2), the following formula determines the slope of a line containing these points:

Example 7: Finding the Slope of a Line Given Two Points

Find the slope of a line that passes through the points (2,−1) and (−5,3)..

Solution

We substitute the *y*-values and the *x*-values into the formula.

$$m=\frac{3+1}{-5+2}=\frac{4}{-3}=-1.33.$$

Try It 7

Find the slope of the line that passes through the points (−2,6) and (1,4).

Solution

Example 8: Identifying the Slope and *y*-intercept of a Line Given an Equation

Identify the slope and *y*-intercept, given the equation y=−34x−4.

Solution

As the line is in y=mx+b , i.e. in general slope = coefficient of x and y-intercept = the constant term, the given line has a slope of m=−34. The *y*-intercept is b=−4.

The *y*-intercept is the point at which the line crosses the *y*-axis.. We can always identify the *y*-intercept when the line is in slope-intercept form, as it will always equal *b.* Or, just substitute x=0 and solve for *y.*

The Point-Slope Formula

Given the slope and one point on a line, we can find the equation of the line using the point-slope formula.

Assume we know any two points $(x_1 , y_1), (x_2 , y_2)$ .If we choose unknown point in the same straight line (x,y). Take care that any straight line has the same slope what ever the two point you use ,then

$$m=\frac{y_2-y_1}{x_2-x_1}==\frac{y-y_1}{x-x_1}$$

$$m=\frac{y-y_1}{x-x_1} \text{ or } y-y_1=m(\mathrm{x}-x_1)$$

$$y=m(\mathrm{x}-x_1)+y_1$$

 This is an important formula, as it will be used in other areas of college algebra and often in calculus to find the equation of a tangent line. We need only one point and the slope of the line to use the formula. After substituting the slope and the coordinates of one point into the formula, we simplify it and write it in slope-intercept form.

Example 9: Finding the Equation of a Line Given the Slope and One Point

Write the equation of the line with slope m=−3,and passing through the point (4,8). Write the final equation in slope-intercept form.

Solution

Using the point-slope formula, substitute −3 for *m* and the point (4,8) for (x_1, y_1).

y−y1=m(x−x1) y−8=−3(x−4)

y−8=−3x+12

y=−3x+12+8 ,i.e. y=−3x+20 .

Try It 8

Given m=4, find the equation of the line in slope-intercept form passing through the point (2,5).

Solution

Example 10: Finding the Equation of a Line Passing Through Two Given Points

Find the equation of the line passing through the points (3,4) and (0,−3). Write the final equation in slope-intercept form.

Solution

First, we calculate the slope using the slope formula and two points.

$\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{y-y_{1}}{x-x_{1}}$ **(The left hand side is diffenetion of m) ,then**

$y-y_{1}=(\mathrm{x}-x_{1})[\frac{y_{2}-y_{1}}{x_{2}-x_{1}}]$ **or** $y=(\mathrm{x}-x_{1})[\frac{y_{2}-y_{1}}{x_{2}-x_{1}}]+y_{1}; i.e.$

$$y=(\mathrm{x}-3)[\frac{-3-4}{0-3}]+4$$

$$y=\frac{7}{3}x-3$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Standard Form of a Line

Another way that we can represent the equation of a line is in standard form. Standard form is given as

Ax+By=C$\Longrightarrow y=\frac{A}{B}x+\frac{C}{B}$ **,we devided each side by coffecient of y to transefere it to slope and y-intercept form.**

Example 11: Finding the Equation of a Line and Writing It in Standard Form

Find the equation of the line with m=−6 and passing through the point (14,−2)(41,−2). Write the equation in standard form.

<u>Solution</u>

We begin using the point-slope formula.

$y-(-2)=-6(x-14)$,$y+2=-6x+32$ $y-(-2)=-6(x-14)y+2=-6x+32$

From here, we multiply through by 2, as no fractions are permitted in standard form, and then move both variables to the left aside of the equal sign and move the constants to the right.

$2(y+2)=(-6x+32)22y+4=-12x+312x+2y=-12(y+2)=(-6x+32)22y+4=-12x+31$

$2x+2y=-1$

This equation is now written in standard form.

Try It 9

Find the equation of the line in standard form with slope $m=-13$ and passing through the point (1,13).

<u>Solution</u>

Vertical and Horizontal Lines

The equations of vertical and horizontal lines do not require any of the preceding formulas, although we can use the formulas to prove that the equations are correct. The equation of a vertical line is given as $x=c$,where *c* is a constant. The slope of a vertical line is undefined, and regardless of the *y*-value of any point on the line, the *x*-coordinate of the point will be *c*.

Suppose we need to find the equation of a line that contains the following set of points:(1,3),(1,5) ,and (3,5) . We can use the point-slope formula. First, we find the slope using any two points on the line.

m= $\frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{1-1} = \frac{2}{0} = \infty$ (undefined value)

The equation of a horizontal line is given as y=c, where *c* is a constant. The slope of a horizontal line is zero, and for any *x*-value of a point on the line, the *y*-coordinate will be *c*.

Suppose we want to find the equation of a line that contains the following set of points: (2,2),(0,2),, and (5,−2). We can use the point-slope formula. First, we find the slope using any two points on the line.

$$m=\frac{2-2}{0-2} = \frac{0}{-2} = 0$$

Use any point (say (5,-2)) in the formula, or use the *y*-intercept.

$$m=\frac{y+2}{x-5} \text{ or } y+2=0(x-5)=0 \text{ or } y=-2$$

The graph is a horizontal line through y=−2. Notice that all the *y*-coordinates are the same.

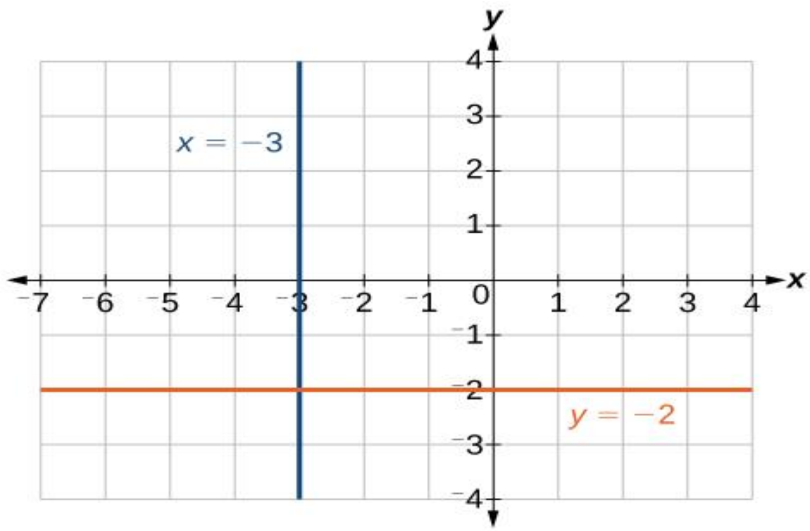

Figure 3. The line $x = -3$ is a vertical line. The line $y = -2$ is a horizontal line.

Determining Whether Graphs of Lines are Parallel or Perpendicular

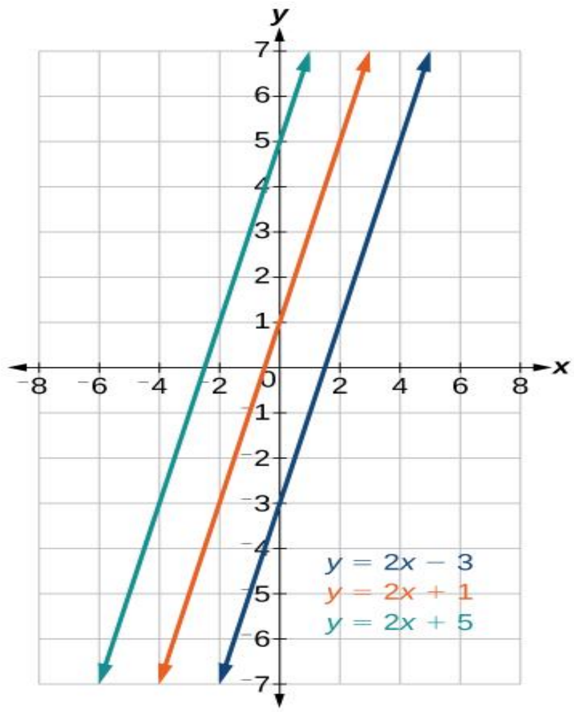

Figure 4. Parallel lines

Parallel lines have the same slope and different *y*-intercepts. Lines that are parallel to each other will never intersect. For example, Figure 4 shows the graphs of various lines with the same slope, m=2.

All the lines shown in the graph are parallel because they have the same slope and different *y*-intercepts.

Lines that are perpendicular intersect to form a 90^o angle. The slope of one line is the negative reciprocal of the other. We can show that two lines are

perpendicular if the product of the two slopes is $m_1 = 3$, $then\ \ m_2 \ = \frac{-1}{3}$.

For example, Figure 4 shows the graph of two perpendicular lines. One line has a slope of 3; the other line has a slope of $\frac{-1}{3}$.

Or $\quad m_1 * m_2 \ = -1$.

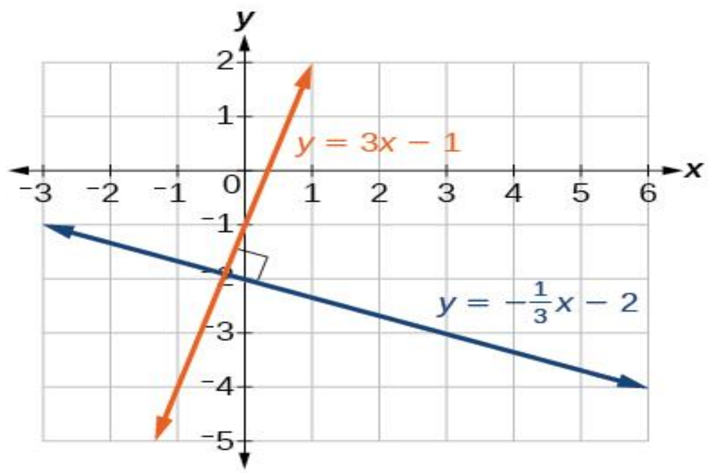

Figure 5. Perpendicular lines

Example 13: Graphing Two Equations, and Determining Whether the Lines are Parallel, Perpendicular, or Neither

Graph the equations of the given lines, and state whether they are parallel, perpendicular, or neither: 3y=−4x+3 and 3x−4y=8.

Solution

The first thing we want to do is rewrite the equations so that both equations are in slope-intercept form.

First equation:

$$3y=-4x+3 \quad \text{or} \quad y=-\frac{4}{3}x+1$$

Second equation:

$$3x-4y=8-4 \quad \text{or} \quad y=-\frac{3}{4}x-2$$

See the graph of both lines in Figure 6.

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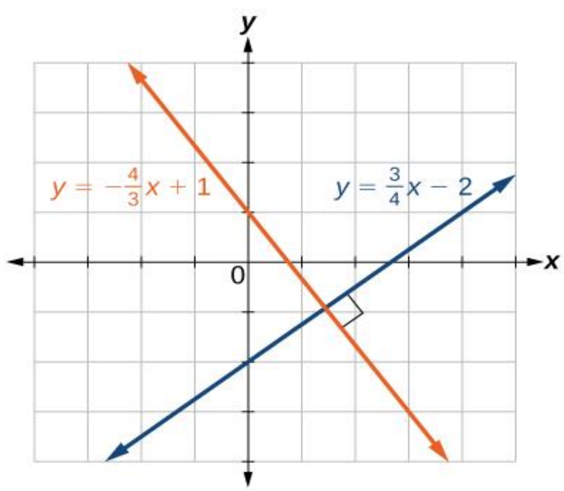


Figure 6

From the graph, we can see that the lines appear perpendicular, but we must compare the slopes.

$$m_1 = -\frac{4}{3} \; and \; m2 = \frac{3}{4} \quad , then \; m1 \cdot m2 = -1$$

The slopes are negative reciprocals of each other, confirming that the lines are perpendicular.

Try It 11

Graph the two lines and determine whether they are parallel, perpendicular, or neither: $2y - x = 10$ and $2y = x + 4$.

Solution

Writing the Equations of Lines Parallel or Perpendicular to a Given Line

As we have learned, determining whether two lines are parallel or perpendicular is a matter of finding the slopes. To write the equation of a line parallel or perpendicular to another line, we follow the same principles as we do for finding the equation of any line. After finding the slope, use the point-slope formula to write the equation of the new line.

HOW TO: GIVEN AN EQUATION FOR A LINE, WRITE THE EQUATION OF A LINE PARALLEL OR PERPENDICULAR TO IT.

1. Find the slope of the given line. The easiest way to do this is to write the equation in slope-intercept form.
2. Use the slope and the given point with the point-slope formula.
3. Simplify the line to slope-intercept form and compare the equation to the given line.

Example 14: Writing the Equation of a Line Parallel to a Given Line Passing Through a Given Point

Write the equation of line parallel to a $5x+3y=1$ and passing through the point $(3,5)$.

<u>Solution</u>

First, we will write the equation in slope-intercept form to find the slope.

$$5x+3y=1 \text{ or } y=-\frac{5}{3}x+\frac{1}{3}$$

The slope is $m=-\frac{5}{3}$. The *y*-intercept is 1, but that really does not enter our problem, as the only thing we need for two lines to be parallel is the same slope. The one exception is that if the *y*-intercepts are the same, then the two lines are the same line. The next step is to use this slope and the given point with the point-slope formula.

$$y-5=-\frac{5}{3}(x-3) \quad \text{or } y-5=-\frac{5}{3}x+5 \quad \text{i.e.}$$

$$y=-\frac{5}{3}x+10$$

The equation of the line is $\quad y=-\frac{5}{3}x+10$

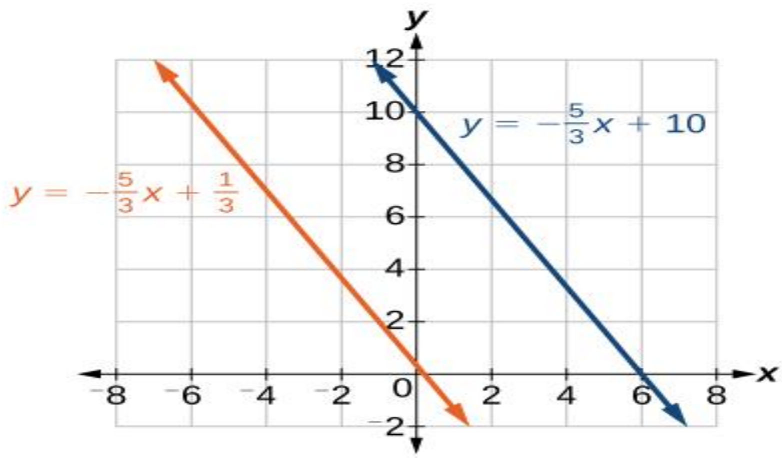


Figure 7

Example 15: Finding the Equation of a Line Perpendicular to a Given Line Passing Through a Given Point

Find the equation of the line perpendicular to 5x−3y+4=0(−4,1).

Solution

The first step is to write the equation in slope-intercept form.

$$5x-3y+4=0 \text{ or } 3y=5x+4 \text{ i.e.}$$

$$y=\frac{5}{3}x+\frac{4}{3}$$

We see that the slope is m= $\frac{5}{3}$. This means that the slope of the line perpendicular to the given line is the negative reciprocal, or $-\frac{3}{5}$. Next, we use the point-slope formula with this new slope and the given point.

$$y-1=-\frac{5}{3}(x+4) \text{ or } y=-\frac{5}{3}x-\frac{20}{3}+1 \quad i.e.$$

$$y=-\frac{5}{3}x-\frac{17}{3}$$

Setting up a Linear Equation to Solve a Real-World Application

To set up or model a linear equation to fit a real-world application, we must first determine the known quantities and define the unknown quantity as a variable. Then, we begin to interpret the words as mathematical expressions using mathematical symbols. Let us use the car rental example above. In this case, a known cost, such as \$0.10/mi, is multiplied by an unknown quantity, the number of miles driven.

Therefore, we can write 0.10x. This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it, according to the problem. As these amounts do not change, we call them fixed costs. Consider a car rental agency that charges $0.10/mi plus a daily fee of $50. We can use these quantities to model an equation that can be used to find the daily car rental cost .

$C=0.10x+50$

When dealing with real-world applications, there are certain expressions that we can translate directly into math.

HOW TO: GIVEN A REAL-WORLD PROBLEM, MODEL A LINEAR EQUATION TO FIT IT.

1. Identify known quantities.
2. Assign a variable to represent the unknown quantity.
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words as mathematical operations.
5. Solve the equation. Be sure the solution can be explained in words, including the units of measure.

Example 1: Modeling a Linear Equation to Solve an Unknown Number Problem

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution

Let x\displaystyle xx equal the first number. Then, as the second number exceeds the first by 17, we can write the second number as $x+17$. The sum of the two numbers is 31. We usually interpret the word *is* as an equal sign. $x+(x+17)=31$ or $2x+17=31$. Simplify and solve.$2x=14$ $x=7$ and $x+17=7+17=24$ then

then the two numbers are 7 and 24 .

Try It 1

Find a linear equation to solve for the following unknown quantities: One number is three more than twice another number. If the sum of the two numbers is 36\displaystyle 3636, find the numbers.

Solution

Example 2: Setting Up a Linear Equation to Solve a Real-World Application

There are two cell phone companies that offer different packages. Company A charges a monthly service fee of \$34 plus \$.05/min talk-time. Company B charges a monthly service fee of \$40 plus \$.04/min talk-time.

1. Write a linear equation that models the packages offered by both companies.
2. If the average number of minutes used each month is 1,160, which company offers the better plan?
3. If the average number of minutes used each month is 420, which company offers the better plan?
4. How many minutes of talk-time would yield equal monthly statements from both companies?

Solution

1. The model for Company *A* can be written as $A=0.05x+34$. This includes the variable cost of $0.05x$. plus the monthly service charge of \$34. Company *B*'s package charges a higher monthly fee of \$40, but a lower variable cost of $0.04x$. Company *B*'s model can be written as $B=0.04x+40$.

2. If the average number of minutes used each month is 1,160, we have the following:

$Company\ A = 0.05(1{,}160)$
$+34 = 58 + 34 = 92. Company\ B = 0.04(1{,}160) + 40 = 46.4 + 40 = 86.4.$

So, Company *B* offers the lower monthly cost of \$86.40 as compared with the \$92 monthly cost offered by Company *A* when the average number of minutes used each month is 1,160.

3. If the average number of minutes used each month is 420, we have the following:

$Company\ A = 0.05(420)$
$+34 = 21 + 34 = 55. Company\ B = 0.04(420) + 40 = 16.8 + 40 = 56.$

If the average number of minutes used each month is 420, then Company *A* offers a lower monthly cost of \$55 compared to Company *B*'s monthly cost of \$56.80.

4. To answer the question of how many talk-time minutes would yield the same bill from both companies, we should think about the problem in terms of (x,y): At what point are both the *x*-value and the *y*-value equal? We can find this point by setting the equations equal and solving for *x*.

0.05x+34=0.04x+400 or .01x=40-34=6

$$x = \frac{6}{.01} = 600 \text{ talk} - \text{time minutes}.$$

Check the *x*-value in each equation.

0.05(600)+34=640.04(600)+40=640.05(600)+34=640.04(600)+40=64

Therefore, a monthly average of 0600 talk-time minutes renders the plans equal.

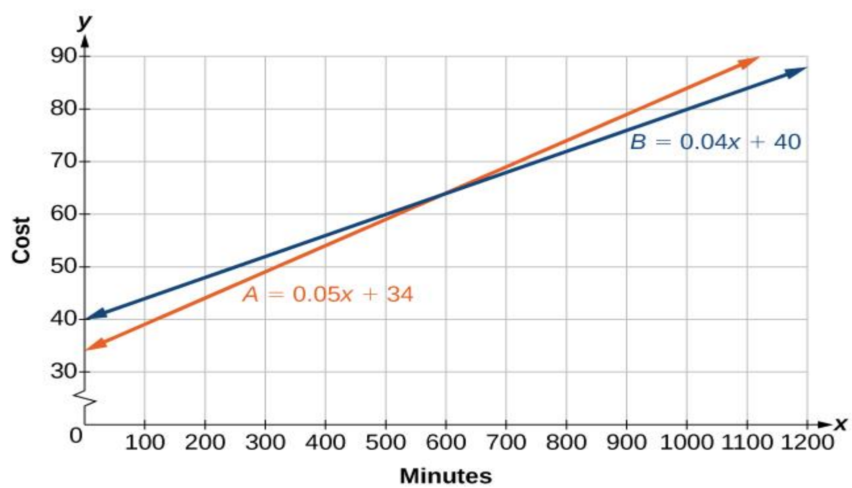


Figure 2

Using a Formula to Solve a Real-World Application

Many applications are solved using known formulas. The problem is stated, a formula is identified, the known quantities are substituted into the formula, the equation is solved for the unknown, and the problem's question is answered. Typically, these problems involve two equations representing two trips, two investments, two areas, and so on. Examples of formulae include the area of a rectangular region, A=LW; the perimeter of a rectangle, P=2(L+W. When there are two unknowns, we find a way to write one in terms of the other because we can solve only one variable at a time.

Example 3: Solving an Application Using a Formula

It takes a pearson 30 min to drive to work in the morning. He drives home using the same route, but it takes 10 min longer, and he averages 10 mi/h less than in the morning. How far does Andrew drive to work?

Solution

This is a distance problem, so we can use the formula d=rt, where distance equals rate multiplied by time. Note that when rate is given in mi/h, time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.

First, we identify the known and unknown quantities. pearson's morning drive to work takes 30 min, or $\frac{1}{2}$ h at rate r. His drive home takes 40 min, or $\frac{2}{3}$ h, and his speed averages 10 mi/h less than the morning drive. Both trips cover distance.

Write two equations, one for each trip.

d= $\frac{1}{2}r$ To work and d= $\frac{2}{3}$(r−10) .

As both equations equal the same distance, we set them equal and solve for *r*.

$\frac{1}{2}r$ = $\frac{2}{3}$(r−10) or $\frac{1}{6}r = \frac{20}{3}$ then

$$r = \frac{120}{3} = 40\ mph$$

We have solved for the rate of speed to work, 40 mph. Substituting 40 into the rate on the return trip yields 30 mi/h. Now we can answer the question. Substitute the rate back into either equation and solve for *d.*

d=40($\frac{1}{2}$) =20 or d =30($\frac{2}{4}$) $= 20$.

The distance between home and work is 20 mi.

Example 4: Solving a Perimeter Problem

The perimeter of a rectangular outdoor patio is 54 ft. The length is 3 ft greater than the width. What are the dimensions of the patio?

Solution

The perimeter formula is standard: $P=2(L+W)$. We have two unknown quantities, length and width. However, we can write the length in terms of the width as $L=W+3$. Substitute the perimeter value and the expression for length into the formula. It is often helpful to make a sketch and label the sides.

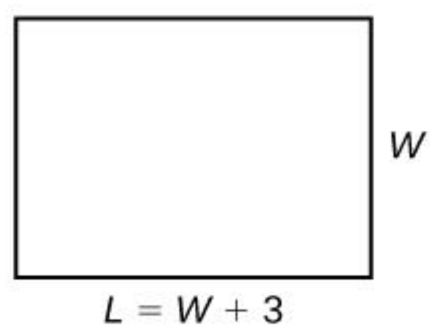


Figure 3

Now we can solve for the width and then calculate the length.

$$P=2(L+2W)=54$$

$$=2(W+3)+2W=54 \text{ or } 4W=54-6=48 \text{ ,then}$$

$$W=\frac{48}{4} = 12\ ft\ , and\ L = W + 3 = 12 + 3 = 15ft$$

The dimensions are L=15 ft and W=12 ft.

Try It 4

Find the dimensions of a rectangle given that the perimeter is 110 cm and the length is 1 cm more than twice the width.

Solution

Example 5: Solving an Area Problem

The perimeter of a tablet of graph paper is 48 in^2. The length is 6 in. more than the width. Find the area of the graph paper.

Solution

The standard formula for area is A=LW\displaystyle A=LWA=LW; however, we will solve the problem using the perimeter formula. The reason we use the perimeter formula is because we know enough information about the perimeter that the formula will allow us to solve for one of the unknowns. As both perimeter and area use length and width as dimensions, they are often used together to solve a problem such as this one.

We know that the length is 6 in. more than the width, so we can write length as L=W+6. Substitute the value of the perimeter and the expression for length into the perimeter formula and find the length.

$P=2L+2W=2(W+6)+2W=4W+12=48$ then

$4W=48-12=36;$

$W=\frac{36}{4}=9\ in. \quad and\ L=W+3=12in.$

Now, we find the area given the dimensions of L=15 in. and W=9 in.

Area=15(9) =135 in2

The area is 135 in^2.

Quadratic Equations

The computer monitor on the left in Figure 1 is a 23.6-inch model and the one on the right is a 27-inch model. Proportionally, the monitors appear very similar. If there is a limited amount of space and we desire the largest monitor possible, how do we decide which one to choose? In this section, we will learn how to solve problems such as this using four different methods.

Solving Quadratic Equations by Factoring

An equation containing a second-degree polynomial is called a quadratic equation. For example, equations such as $2x^2+3x-1$and $x^2-4=0$ are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

Often the easiest method of solving a quadratic equation is factoring. Factoring means finding expressions that can be multiplied together to give the expression on one side of the equation.

If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property, which states that if $a \cdot b=0$, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

Multiplying the factors expands the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication undoes the operation of factoring. For example, expand the factored expression (x−2)(x+3)\displaystyle \left(x - 2\right)\left(x+3\right)(x−2)(x+3) by multiplying the two factors together.

(x−2)(x+3)=x2+3x−2x−6=x2+x−6(x−2)(x+3)=x2+3x−2x−6=x2+x−6

The product is a quadratic expression. Set equal to zero, $x^2+x-6=0$ is a quadratic

equation. If we were to factor the equation, we would get back the factors we multiplied.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is 1 or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form, $ax^2+bx+c=0$, where *a*, *b*, and *c* are real numbers, and $a\neq 0$. The equation $x^2+x-6=0$ is in standard form.

2024/2025

We can use the zero-product property to solve quadratic equations in which we first have to factor out the greatest common factor (GCF), and for equations that have special factoring formulas as well, such as the difference of squares, both of which we will see later in this section.

A GENERAL NOTE: THE ZERO-PRODUCT PROPERTY AND QUADRATIC EQUATIONS

The zero-product property states

A quadratic equation is an equation containing a second-degree polynomial.e

Solving Quadratics with a Leading Coefficient of 1

In the quadratic equation $x^2+x-6=0$\, the leading coefficient, or the coefficient of x^2, is 1. We have one method of factoring quadratic equations in this form.

HOW TO: GIVEN A QUADRATIC EQUATION WITH THE LEADING COEFFICIENT OF 1, FACTOR IT.

1. Find two numbers whose product equals *c* and whose sum equals *b*.
2. Use those numbers to write two factors of the form $(x+k)$ or $(x-k)$, where *k* is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are 1 and −2, the factors are $(x+1)(x-2)$\displaystyle \left(x+1.
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

Example 1: Factoring and Solving a Quadratic with Leading Coefficient of 1

Factor and solve the equation: $x^2+x-6=0$..

Solution

To factor $x^2+x-6=0$, we look for two numbers whose product equals −6 and whose sum equals 1. Begin by looking at the possible factors of −6. The pair,3,-2 which give sum equal 1,so these are the two numberswill work. Then , write the factors,(x-2)(x+3)=0

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.

$(x-2)(x+3)=0$ if $(x-2)=0$;then $x=2$. Or if $(x+3)=0$; then $x=-3$

The two solutions are $x=2$ and $x=-3$.

We can see how the solutions relate to the graph in Figure 2. The solutions are the *x*-intercepts of $x^2+x-6=0$.

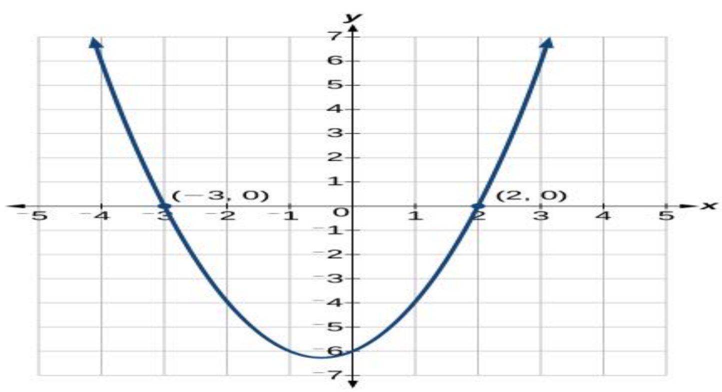


Figure 2

Try It 1

Factor and solve the quadratic equation: $x^2-5x-6=0$.

Solution

Example 2: Solve the Quadratic Equation by Factoring

Solve the quadratic equation by factoring: $x^2+8x+15=0$.

Solution

Find two numbers whose product equals 15 and whose sum equals 8. List the factors of 15.

$3\cdot5$. The numbers that add to 8 are 3 and 5. Then, write the factors, set each factor equal to zero, and solve.

$(x+3)(x+5)=0$,my be $(x+3)=0$;then $x=-3$. or $(x+5)=0$, then $x=-5$.

The solutions are $x=-3$ and $x=-5$.

Example 3: Using the Zero-Product Property to Solve a Quadratic Equation Written as the Difference of Squares

Solve the difference of squares equation using the zero-product property: $x^2-9=0$.

Solution

Recognizing that the equation represents the difference of squares, we can write the two factors by taking the square root of each term, using a minus sign as the operator in one factor and a plus sign as the operator in the other. Solve using the zero-factor property.

$x^2-9=0$ or $(x-3)(x+3)=0$ $(x-3)=0$ then $x=3$ or $(x+3)=0$ then $x=-3$.

Factoring and Solving a Quadratic Equation of Higher Order

When the leading coefficient is not 1, we factor a quadratic equation using the method called grouping, which requires four terms. With the equation in standard form, let's review the grouping procedures:

1. With the quadratic in standard form, $ax^2+bx+c=0$, multiply $a \cdot c$
2. Find two numbers whose product equals ac\displaystyle acac and whose sum equals b\displaystyle bb.
3. Rewrite the equation replacing the bx\displaystyle bxbx term with two terms using the numbers found in step 1 as coefficients of *x.*
4. Factor the first two terms and then factor the last two terms. The expressions in parentheses must be the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

Example 4: Solving a Quadratic Equation Using Grouping

Use grouping to factor and solve the quadratic equation: $4x^2+15x+9=0$.

<u>Solution</u>

First, multiply ac:4(9)=36. Then list the factors of 36 to get all the combination of two numbers gives us multiplication 36. The only pair of factors that sums to 15 is (3,12).

Rewrite the equation replacing the *b* term, 15x\d, with two terms using 3 and 12 as coefficients of *x*. Factor the first two terms, and then factor the last two terms.

4x2+3x+12x+9=0 x(4x+3)+3(4x+3)=0

(4x+3)(x+3)=0

$4x^2+3x+12x+9=$,i.e. x(4x+3)+3(4x+3)=0 or (4x+3)(x+3)=0

Solve using the zero-product property.

(4x+3)(x+3)=0 my be (4x+3)=0 ;the value of 4x=−3 or $x=\frac{-3}{4}$ and my be 4(x+3)=0 then value of x=−3.

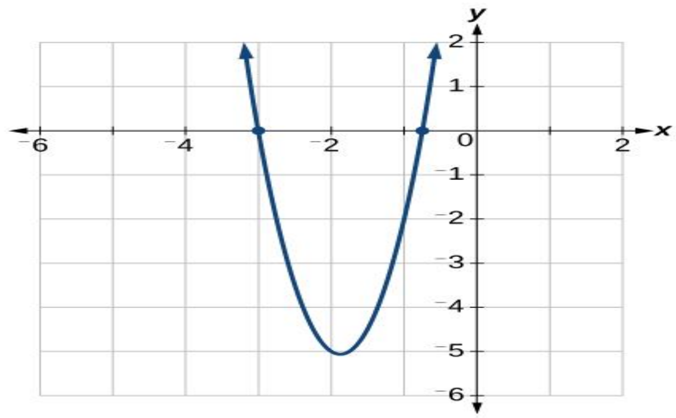


Figure 3

Example 5: Solving a Higher Degree Quadratic Equation by Factoring

Solve the equation by factoring: $-3x^3-5x^2-2x=0$ $-5x2-2x=0$.

<u>Solution</u>

This equation does not look like a quadratic, as the highest power is 3, not 2. Recall that the first thing we want to do when solving any equation is to factor out the GCF, if one exists. And it does here. We can factor out $-x$ from all of the terms and then proceed with grouping.

$-3x^3-5x^2-2x=0$ or $-x(3x^2+5x+2)=0$,One of our solutions $-x=0$ or $x=0$.

Use grouping on the expression in parentheses.$3x^2+5x+2=0$

$3x(x+1)+2(x+1)]=0$

(x+1)(3x+2)=0 ;then x+1=0 ;i.e. x=-1 and 3x+2=0 or x=$\frac{-2}{3}$.

Now, we use the zero-product property. Notice that we have three factors.

The solutions are x=0, x=$\frac{-2}{3}$, and x=−1.

Completing the Square

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a quadratic equation known as completing the square. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, *a*, must equal 1. If it does not, then divide the entire equation by *a*. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example $x^2+4x+1=0$, to illustrate each step.

1. Given a quadratic equation that cannot be factored, first add or subtract the constant term to the right sign of the equal sign.

   $x^2+4x=-1$.

2. Multiply the *b* term by $\frac{1}{2}$ and square it.

$\frac{1}{2}(4)=2^2=4$

3. Add $(\frac{1}{2}b)^2$ to both sides of the equal sign and simplify the right side.

We have

$x^2+4x+4=-1+4$; $x^2+4x+4=3$ i.e. $x^2+4x+4=3$

4. The left side of the equation can now be factored as a perfect square.

$x^2+4x+4=3$ then $(x+2)^2=(\sqrt{3})^2$

5. Use the square root property and solve.

$x+2=\sqrt{3}$ or $x=-2+\sqrt{3}$,and $-x-2=-\sqrt{3}$,then $x=-2+\sqrt{3}$

6. The solutions are $x=-2+\sqrt{3}$ and $x=-2-\sqrt{3}$.

Example 8: Solving a Quadratic by Completing the Square

Solve the quadratic equation by completing the square: $x^2-3x-5=0$.

Solution

First, move the constant term to the right side of the equal sign.

$x^2 - 3x = 5$

Then, take $\frac{1}{2}$ of the *b* term and square it.

$(\frac{-3}{2})^2) = \frac{9}{4}$.

Add the result to both sides of the equal sign.

$x^2 - 3x + (\frac{9}{4}) = 5 + (\frac{9}{4}) = \frac{29}{4}$. Factor the left side as a perfect square and simplify the right side.

$(x - \frac{3}{2})^2 = \sqrt{\frac{29}{4}}$ i.e. $(x - \frac{3}{2}) = \frac{\sqrt{29}}{2}$ or $x = \frac{3 \pm \sqrt{29}}{2}$

Using the Quadratic Formula

The fourth method of solving a quadratic equation is by using the quadratic formula, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by completing the square. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by −1 and obtain a positive a. Given $ax^2+bx+c=0$, $a\neq0$, we will complete the square as follows:

1. First, move the constant term to the right side of the equal sign:

$$ax^2+bx+c=0 .$$

2. Use the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

A GENERAL NOTE: THE QUADRATIC FORMULA

Written in standard form, $ax^2+bx+c=0$, any quadratic equation can be solved using the quadratic formula:

 where a, b, and c are real numbers and $a\neq0$.

HOW TO: GIVEN A QUADRATIC EQUATION, SOLVE IT USING THE QUADRATIC FORMULA

1. Make sure the equation is in standard form: ax^2+bx+c.
2. Make note of the values of the coefficients and constant term, a,.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

Example 9: Solve the Quadratic Equation Using the Quadratic Formula

Solve the quadratic equation: $x^2+5x+1=0$\.

<u>Solution</u>

Identify the coefficients: a=1,b=5,c=1. Then use the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-5\pm\sqrt{5^2-4.1.1}}{2.1}=\frac{-5\pm\sqrt{25-4}}{2}=\frac{-5\pm\sqrt{21}}{2}$$

$=-2.5\pm\frac{\sqrt{21}}{2}$**, then**

$x_1=-2.5+\frac{\sqrt{21}}{2}$ **, and** $x_2=-2.5-\frac{\sqrt{21}}{2}$

We have two roots (two solutions).

Example 10: Solving a Quadratic Equation with the Quadratic Formula

Use the quadratic formula to solve $x^2+x+2=0$.

<u>Solution</u>

First, we identify the coefficients: a=1,b=1, and c=2.

Substitute these values into the quadratic formula.

3. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4.1.2}}{2} = x = \frac{-1 \pm \sqrt{-7}}{2}$

4. Since the discriminant ($b^2 - 4ac$=-7) (negative value ,then there is no real solution for this quadratic equation.

Try It 8

Solve the quadratic equation using the quadratic formula: 9x2+3x−2=0\displaystyle 9{x}^{2}+3x - 2=09x2+3x−2=0.

Solution

The Discriminant

The quadratic formula not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the discriminant, or the expression under the radical, b^2−4ac. The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. The table below relates the value of the discriminant to the solutions of a quadratic equation.

Value of Discriminant	Results
$b^2-4ac=0$	One rational solution (double solution)
$b^2-4ac>0$	Two irrational solutions
$b^2-4ac<0$	Two complex solutions

Example 11: Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Use the discriminant to find the nature of the solutions to the following quadratic equations:

1. $x^2+4x+4=0$
2. $3x^2-5x-2=0$
3. $8x^2+14x+3=0$
4. $3x^2-10x+15=0$

Solution

Calculate the discriminant $b^2{-}4ac$ for each equation and state the expected type of solutions.

1. $x^2+4x+4=0$, $a=1, b=4$,and $c=4$,then $b^2-4ac=16-4.a.4=16-16=0$. This equation has one rational (two equal roots).
2. $3x^2-5x+-2=0$, $a=3, b=-5$,and $c=-2$,then $b^2-4ac= 25+24=49>0$. This equation has two irrational solutions .
3. $8x^2$^$+14x+3=0$. $a=8$,$b=14$, and $c=3$;then $b^2-4ac=196-4.8.3=196—96=100>0$.

This equation has two irrational solutions.

4. $3x^2+10x+15=0$. $a=3$,$b= 10$,and $c=15$,then $b^2-4ac = 100-180= -80<0$. $-4(3)(15)=-80$. There will be two complex solutions (non -real roots).

Example:

. A vacant lot is being converted into a community garden. The garden and the walkway around its perimeter have an area of 378 ft^2. Find the width of the walkway if the garden is 12 ft. wide by 15 ft. long.

Solution:

Let the width of the walkway =x, then the width and length of the walkway and garden are 12+x and 15+x respectively .The area =length*width=(12+x)(15+x)=378 ft^2 .

x^2 +27x +180=378 or x^2 +27x -198=0

b^2 - 4ac =$(27)^2$ +4(1)(198) =729+792= 1521>0 (we have two irrational solutions).

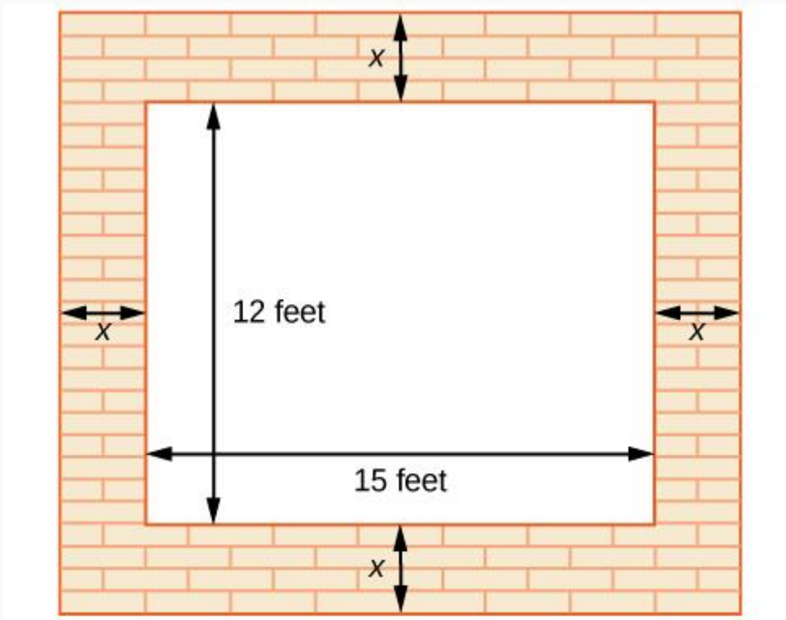
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-27\pm\sqrt{27^2-4.1.198}}{2.1}=\frac{-27\pm\sqrt{1521}}{2}=\frac{-27\pm39}{2}$$

$$x_1=\frac{-27+39}{2}=\frac{12}{2}=6\ ft\quad and\ x_2=\frac{-27-39}{2}=\frac{-66}{2}=-33\ refused$$

Then ,the width of walkway = 6 ft . the total length =15+6=21 ft ,and total width =12+6=18 ft .

Check : total area= W*L=21*18 +378 ft^2 = the given area.

2024/2025



Solving an Absolute Value Equation

Next, we will learn how to solve an absolute value equation. To solve an equation such as $|2x-6|=8$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value bars is 8\. This leads to two different equations we can solve independently.

$2x-6=8$ or $2x-6=-8$ or $2x=14$ and $2x=-2$ then $x=7$.

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

HOW TO: GIVEN AN ABSOLUTE VALUE EQUATION, SOLVE IT.

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c>0$, write and solve two equations: $ax+b=c$

   $ax+b=c$.

Example 8: Solving Absolute Value Equations

Solve the following absolute value equations:

a. $|6x+4|=8$ b. $|3x+4|=-9$ c. $|3x-5|-4=6$. d. $|-5x+10|=0$.

Solution

a. $|6x+4|=8$

Write two equations and solve each:

$6x+4=8$, $6x+4=-8$, then $6x=4$ and $6x=-12$ i.e. $x=\frac{2}{3}$ *and* $x=-2$.

b. $|3x+4|=-9$.

There is no solution as an absolute value cannot be negative.

c. |3x−5|−4=6.

Isolate the absolute value expression and then write two equations.

$$|3x-5|=10; then$$

The first equation is 3x-5=10 or 3x=15,i.e, x=5

The second equation –(3x-5) =10 ;then 3x=--10+5=-5 ,or x==$\frac{-5}{3}$.

There are two solutions x=5 ,and x ==$\frac{-5}{3}$.

d. |−5x+10|=0.

The equation is set equal to zero, so we must write only one equation.

−5x+10=0 ,or 5x=−10 ,then x=2.

There is one solution: x=2.

Other Types of Equations

Linear Inequalities and Absolute Value Inequalities

Introduction: Linear Inequalities and Absolute Value Inequalities

It is not easy to make the honor role at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.

Using Interval Notation

Indicating the solution to an inequality can be achieved in several ways.

We can use a number line as shown in Figure 2. The blue ray begins at $x=4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.

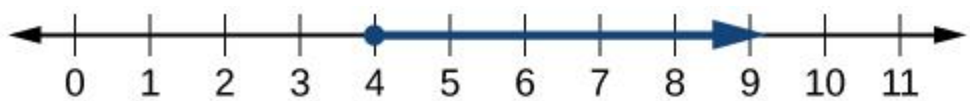


Figure 2

We can use set-builder notation: $\{x|x\geq 4\}$, which translates to "all real numbers *x* such that *x* is greater than or equal to 4." Notice that braces are used to indicate a set.

The third method is interval notation, in which solution sets are indicated with parentheses or brackets. The solutions to $x\geq 4$ are represented as $[4,\infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an interval, or a set of numbers in which a solution falls, are $[-2,6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1,0)$, all real numbers between, but not including -1 and 0; and $(-\infty,1]$, all real numbers less than and including 1. The table below outlines the possibilities.

 Example 1: Using Interval Notation to Express All Real Numbers Greater Than or Equal to *a*. Use interval notation to indicate all real numbers greater than or equal to -2.

Solution

Use a bracket on the left of -2 and parentheses after infinity: $[-2,\infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

Example 2: Using Interval Notation to Express All Real Numbers Less Than or Equal to *a* or Greater Than or Equal to *b*

Write the interval expressing all real numbers less than or equal to −1 or greater than or equal.

<u>Solution</u>

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1. So, this interval begins at −∞ and ends at −1, which is written as (−∞,−1].

The second interval must show all real numbers greater than or equal to 1, which is written as [1,∞). However, we want to combine these two sets.

.

(−∞,−1]∪[1,∞).

Linear Inequalities and Absolute Value Inequalities

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property

and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

A GENERAL NOTE: PROPERTIES OF INEQUALITIES

Addition Property If $a<b$, then $a+c<b+c$. Multiplication PropertyIf $a<b$ and $c>0$, then $ac<bc$.If $a<b$ and $c<0$, then $ac>bc$.Addition , then $a+c<b+c$ .Multiplication Property If $a<b$ and $c>0$, then $ac<bc$. If $a<b$ and $c<0$, then $ac>bc$.

These properties also apply to $a\leq b$, $a>b$, and $a\geq b$.

Example 3: Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following:

a. $x-15<4$ b. $6\geq x-1$ c. $x+7>9$

<u>Solution</u>

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

a. $x-15<4$; $x-15+15<4+15$ Add 15 to both sides.$x<19$.

b. $6 \geq x-1$;$6+1 \geq x-1+1$. Add 1 to both sides i.e, $7 \geq x$.

c. $x+7>9$l $x+7-7>9-7$,Subtract 7 from both sides getting .$x>2$.

Try It 3

Solve $3x-2<1$.

Solution

Example 4: Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

a. $3x<6$. b. $-2x-1 \geq 5$. c. $5-x>10$.

Solution

a. $3x<6$ multiply each side by $\frac{1}{3}$, *then* $\frac{1}{3}(3x)<(6)\frac{1}{3}$ *gives* $x<2$.

b. $-2x-1 \geq 5$ add 1to each side to get$-2x \geq 6$. Multiply by $\frac{-1}{2}$. We get $x \leq -3$. Reverse inequality.

c. $5-x>10$;subtract 5 from each side $-x>5$. Multiply by -1. We get $x<-5$,Reverse the inequality.$5-x>10$.

Try It 4

Solve 4x+7≥2x−3.

Solution

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

Example 5: Solving an Inequality Algebraically

Solve the inequality: 13−7x≥10x−4.

Solution

Solving this inequality is like solving an equation up until the last step. 13−7x≥10x−4 ;13−17x≥−4. Move variable terms to one side of the inequality.−17x≥−17 Isolate the variable term.x≤1.Or transfair -7x to left side and -4 to right side , we get

17x≤ 17.Dividing both sides by 17 .we get $x \leq 1$.

The solution set is given by the interval $(-\infty,1]$, or all real numbers less than and including 1.

Solve the inequality and write the answer using interval notation: $-x+4<12x+1$.

<u>Solution</u>

Example 6: Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: $-34x\geq -58+23x$.

<u>Solution</u>

We begin solving in the same way we do when solving an equation.

Adding -34x,and -58 to each side . We get $58\geq 34x++23x$ or $57x\leq 58$ or $x\leq \frac{58}{57}$.

The solution set is the interval $(-\infty , \frac{58}{57}]$.

Try It 6

Solve the inequality and write the answer in interval notation: $-56x\leq 34+83x$.

Solution

Linear Inequalities and Absolute Value Inequalities

Understanding Compound Inequalities

A compound inequality includes two inequalities in one statement. A statement such as $4<x\le 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

Example 7: Solving a Compound Inequality

Solve the compound inequality: $3\le 2x+2<6$..

Solution

The first method is to write two separate inequalities: $3\le 2x+2$ and $2x+2<6$. We solve them independently.

$3\le 2x+2 \text{and} 2x+2<61\le 2x2x<412\le xx<23\le 2x+2 \text{and} 2x+2<61\le 2x2x<412\le xx<2$

Then, we can rewrite the solution as a compound inequality, the same way the problem began. First simple inequality $3 \leq 2x + 2$ gives us $x \geq \frac{1}{2}$,and second simple inquality $x < 2$.

i.e. $\frac{1}{2} \leq x < 2$. In interval notation, the solution is written as $[\frac{1}{2}, 2)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.
3≤2x+2< 6. Isolate the variable term, and subtract 2 from all three parts.1≤2x<4 .Divide through all three parts by 2.3≤2x+2<61≤2x<4Isolate the variable term, and subtract 2 from all three parts 2, we get $\frac{1}{2}$≤x<2 .

We get the same solution: $[\frac{1}{2} . 2)$.

Try It 7

Solve the compound inequality 4<2x−8≤10.

Solution

Example 8: Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3+x>7x-2>5x-10$.

Solution

Lets try the first method. Write two inequalities:

$3+x>7x-2$ and $7x-2>5x-10$ $5>6x$. Then $6x<5$ or $x<\frac{5}{6}$ (first simple inquality).

$2x>-8$,then

$x>-4$. $(-4: \frac{5}{6})$.

Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line.

Figure 3

Try It 8

Solve the compound inequality: $3y<4-5y<5+3y$.

Solution

Linear Inequalities and Absolute Value Inequalities

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at (−x,0) has an absolute value of x. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form
$|A|<B$, $|A|\le B$, $|A|>B$, or $|A|\ge B$,

Where *A*, and sometimes *B*, represents an algebraic expression dependent on a variable *x.* Solving the inequality means finding the set of all x. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solve absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

A GENERAL NOTE: ABSOLUTE VALUE INEQUALITIES

For an algebraic expression *X,* and k>0, an absolute value inequality is an inequality of the form

Example 9: Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

<u>Solution</u>

We need to find the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in Figure 4, to represent the condition to be satisfied.

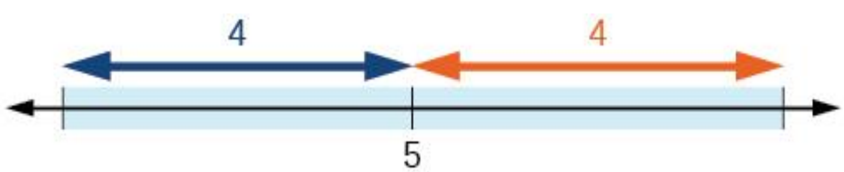


Figure 4

The distance from x to 5 can be represented using an absolute value symbol, $|x-5|$. Write the values of x that satisfy the condition as an absolute value inequality.

|x−5|≤4 - 5|.

We need to write two inequalities as there are always two solutions to an absolute value equation.
x−5≤4 and x−5≥−4 . Then x≤9 or x≥1 .

If the solution set is x≤9/, then the solution set is an interval including all real numbers between and including 1 and 9.

So |x−5|≤4 is equivalent to [1,9]\d in interval notation.

Try It 9

Describe all *x*-values within a distance of 3 from the number 2.

Solution

Example 10: Solving Absolute Value Inequality

Solve |x−1|≤3.

Solution

|x−1|≤3 i.e. −3≤x−1≤3, add 1 to each side to get −2≤x≤4, then interval solution is [−2,4] .

Example 11: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the equation $y=-\frac{1}{2}|4x-5|+3$, determine the *x*-values for which the *y*-values are negative.

<u>Solution</u>

We are trying to determine where $y<0$, which is when $-\frac{1}{2}|4x-5|+3<0$. We begin by isolating the absolute value.

$-\frac{1}{2}||4x-5|<-3$. Multiply both sides by -2, and reverse the inequality. $|4x-5|>6$.

Next, we solve for the equality $|4x - 5|=6$.

$4x-5=6$ or $4x-5=-6$,then add 5 to each side we get $4x=11$,and $4x=-1$,we divide each side by 4; we get $x=\frac{11}{4}$ $and\ x=-\frac{1}{4}$. Then interval solution of x $(-\frac{1}{4}: \frac{11}{4})$

Now, we can examine the graph to observe where the *y*-values are negative. We observe where the branches are below the *x*-axis. Notice that it is not important exactly what the graph looks like, if we know that it crosses the horizontal axis at $x=-\frac{1}{4}$, and $x=\frac{11}{4}$, and that the graph opens downward.

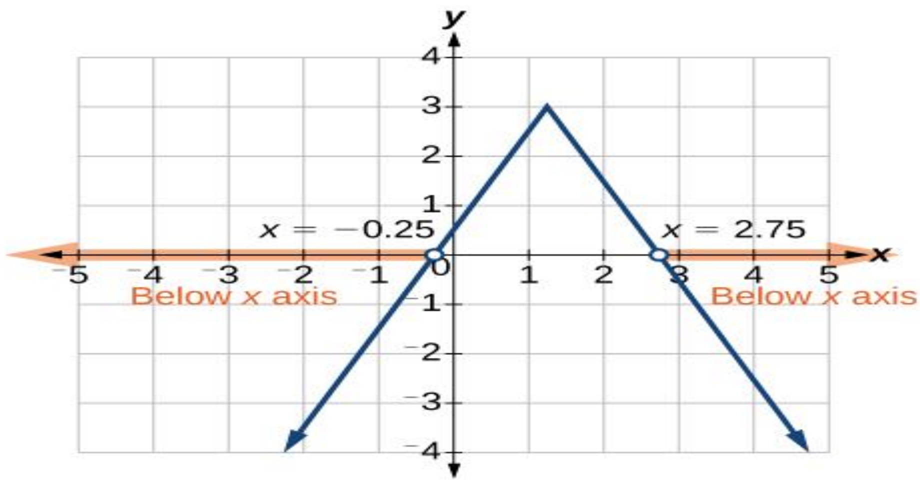


Figure 5

2024/2025 **Try It 10**

Solve −2|k−4|≤−6\displaystyle -2|k - 4|\le -6−2|k−4|≤−6.

<u>Solution</u>

<u>Introduction to Functions and Function Notation</u>

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two

quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determine whether a function is one-to-one

Some functions have a given output value that corresponds to two or more input values. For example, in the following stock chart the stock price was \$1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of \$1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in.

Letter grade	Grade point average
A	4.0
B	3.0

Letter grade	Grade point average
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let’s look again at the two simple functions sketched in (a)and (b) of Figure 10.

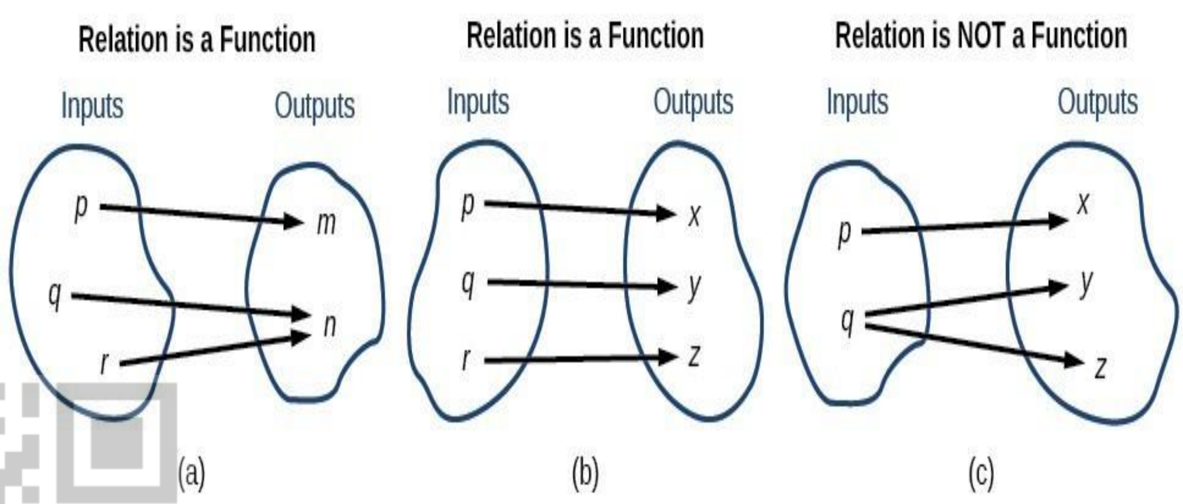


Figure 10

The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

A GENERAL NOTE: ONE-TO-ONE FUNCTION

A one-to-one function is a function in which each output value corresponds to exactly one input value.

Example 13: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A=\pi r^2$, so for any input, r, there is only one output, A. The area is a radius r.

Try It 7

1. **Is a balance a function of the bank account number?**

2. Is a bank account number a function of the balance?

3. Is a balance a one-to-one function of the bank account number?

Solution

Use the vertical line test to identify functions

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value y, and we sa is a function of x, or y=f(x) when the function is named f. The graph of the function is the set of all points (x,y) in the plane that satisfies the equation y=f(x). If the function is defined for only a few input values, then the graph of the function is only a few points, where the *x*-coordinate of each point is an input value and the *y*-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 11 tell us that f(0)=2 and f(6)=1. However, the set of all points satisfying y=f(x) is a curve. The curve shown includes (0,2) and (6,1) because the curve passes through those points.

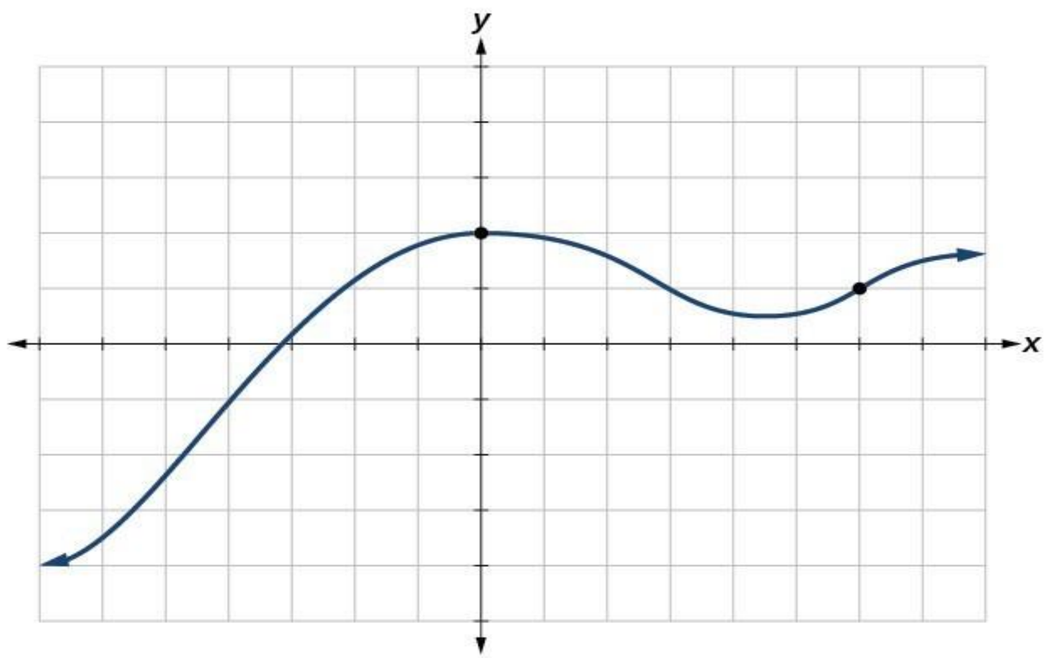


Figure 11

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does *not* define a function because a function has only one output value for each input value.

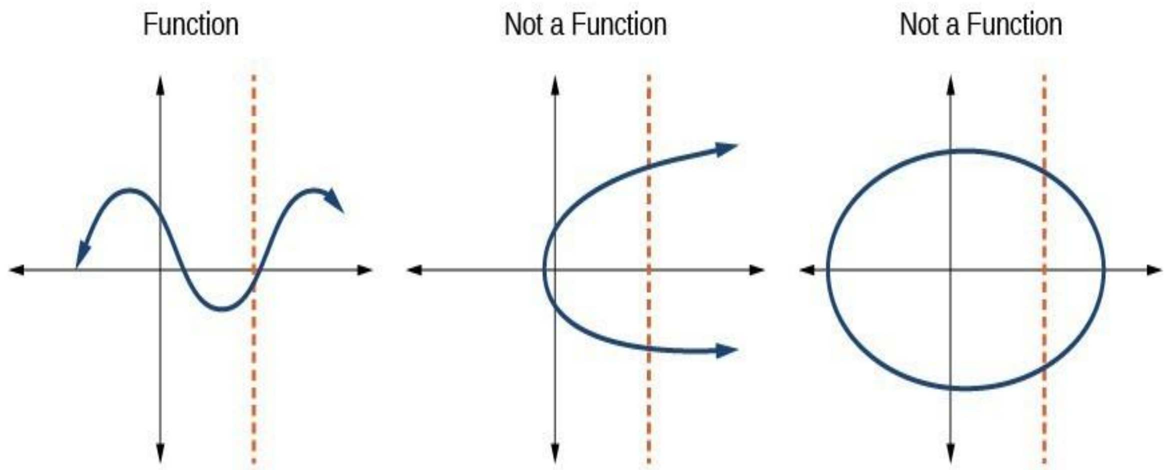


Figure 12

How To: Given a graph, use the vertical line test to determine if the graph represents a function.

1. **Inspect the graph to see if any vertical line drawn would intersect the curve more than once.**
2. **If there is any such line, determine that the graph does not represent a function.**

Example 14: Applying the Vertical Line Test

Which of the graphs represent(s) a function y=f(x)?

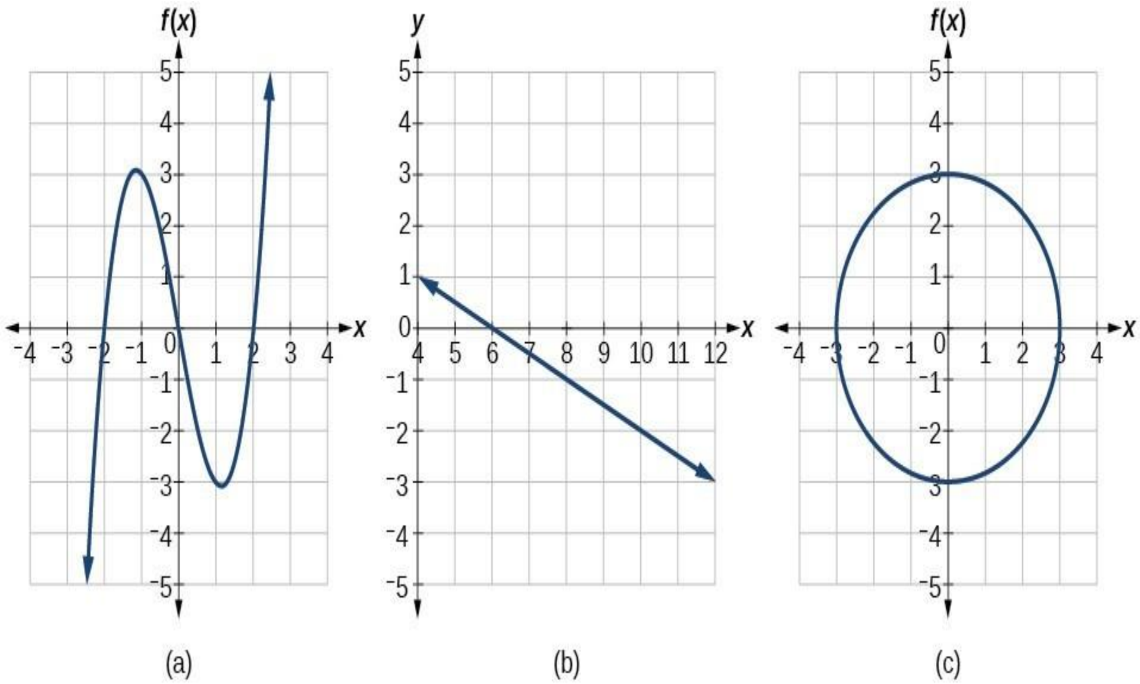


Figure 13

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 13. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most *x*-values, a vertical line would intersect the graph at more than one point.

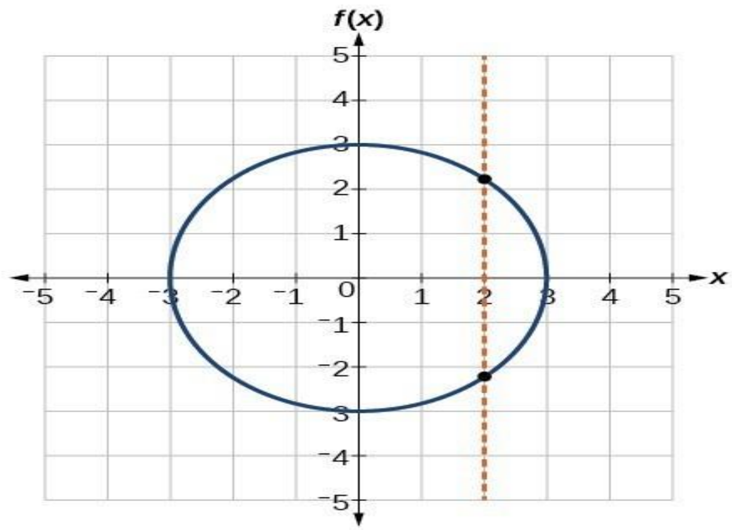


Figure 14

Try It 8

Does the graph in Figure 15 represent a function?

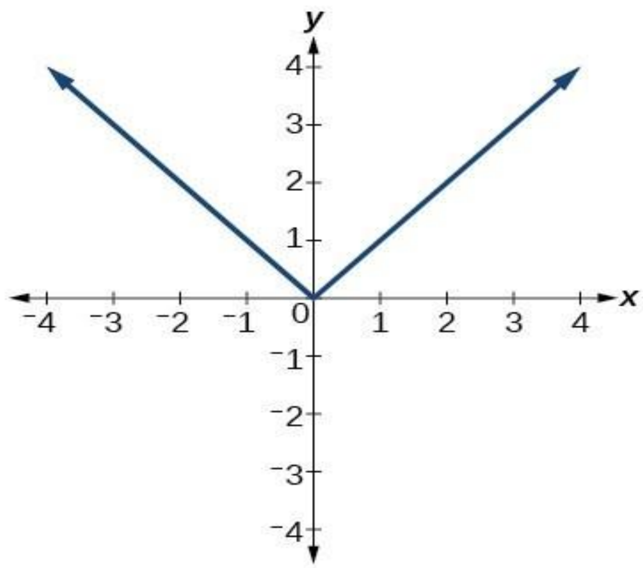


Figure 15

Solution

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the horizontal line test. Draw horizontal lines through the graph. If any horizontal line intersects

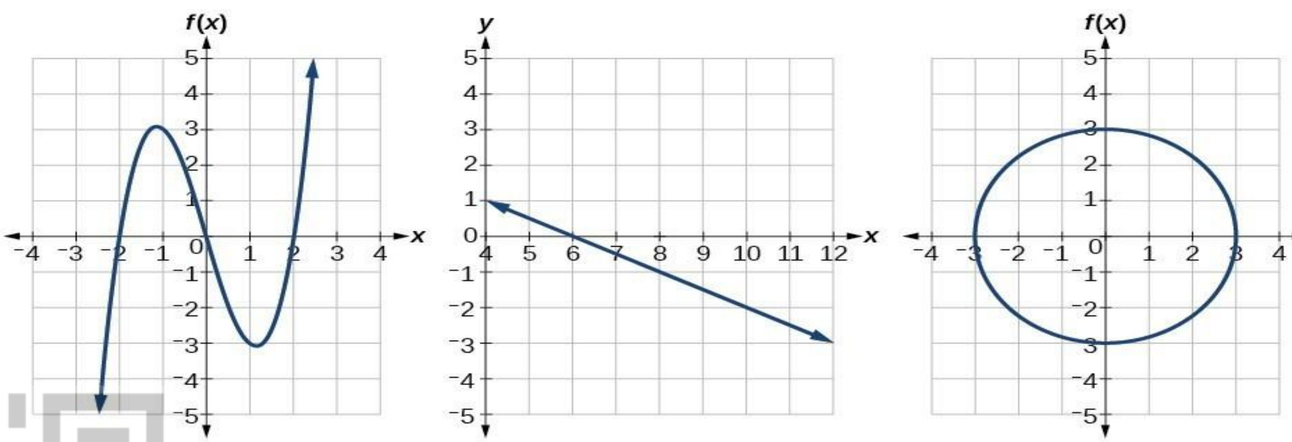
the graph more than once, then the graph does not represent a one-to-one function.

How To: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.

1. **Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.**
2. **If there is any such line, determine that the function is not one-to-one.**

Example 15: Applying the Horizontal Line Test

Consider the functions (a), and (b)shown in the graphs in Figure 16.



(a) (b) (c)

Figure 16

Are either of the functions one-to-one?

Solution

The function in (a) is not one-to-one. The horizontal line shown in Figure 17 intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

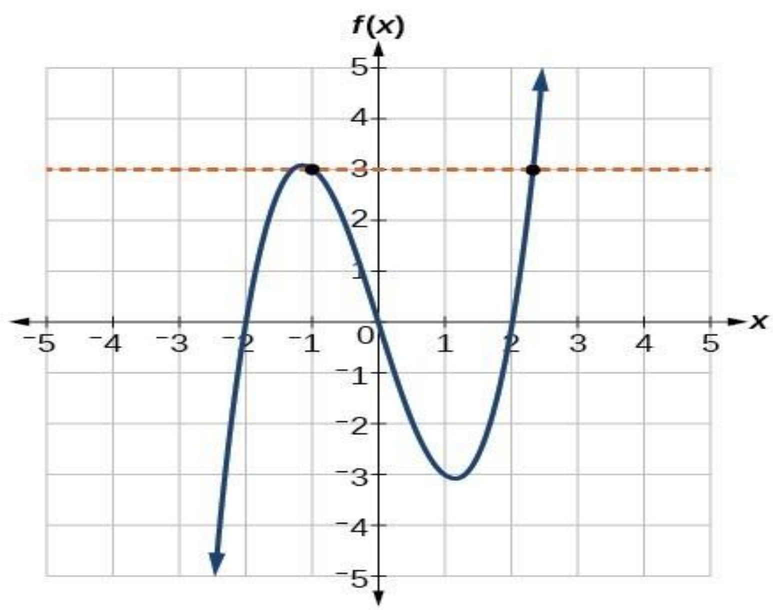


Figure 17

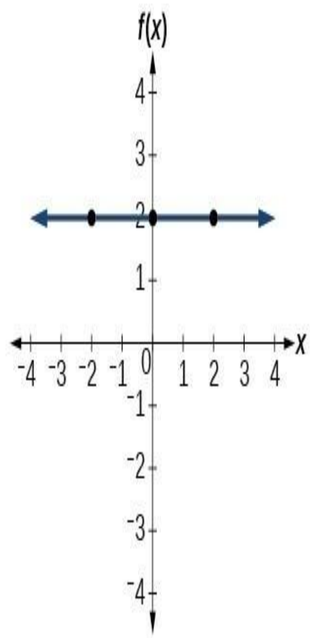
The function in (b) is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Graph the functions listed in the library of functions

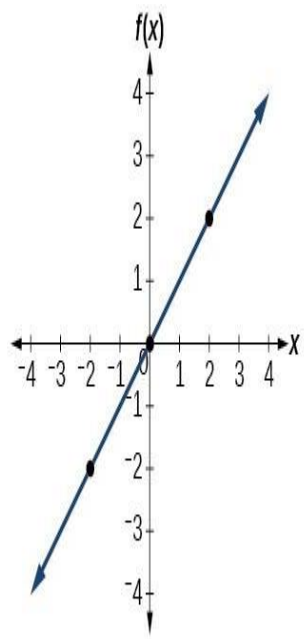
Identifying Basic Toolkit Functions

In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “toolkit functions,” which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and y=f(x) as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown below.

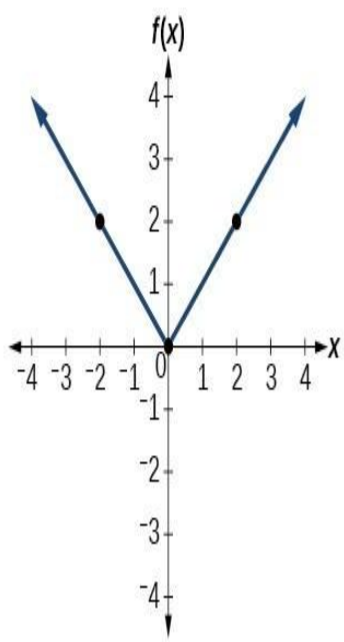
<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Constant</td><td>f(x)=c =cf(x)=c, where c is a constant</td><td>

<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>2</td></tr>
<tr><td>0</td><td>2</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
</table>

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Identity</td><td>f(x)=x</td><td>

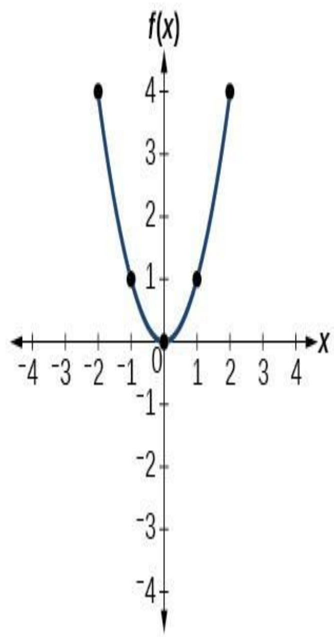
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>−2</td><td>−2</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>2</td><td>2</td></tr>
</table></td></tr>
</table>

Toolkit Functions

Name	**Function**	**Graph**
Absolute value	**$f(x)=\lvert x\rvert$**	

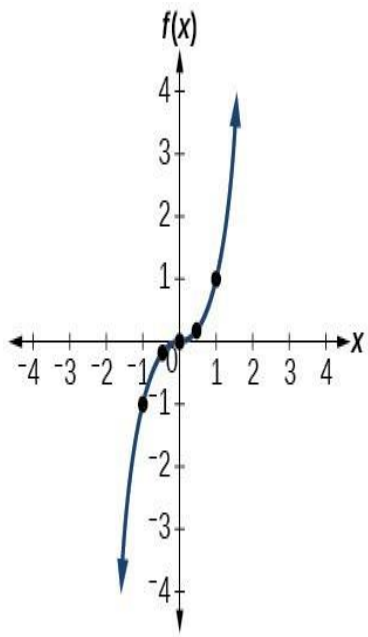
x	$f(x)$
−2	2
0	0
2	2

Toolkit Functions

Name	Function	Graph
Quadratic	$f(x)=x^2$	

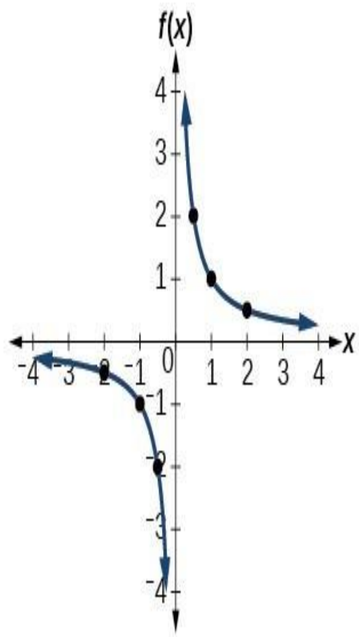
x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4

Toolkit Functions

Name	Function	Graph
Cubic	$f(x)=x^3$	

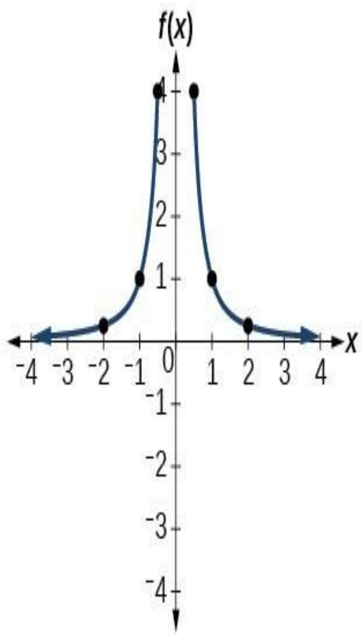
x	$f(x)$
−1	−1
−0.5	−0.125
0	0
0.5	0.125
1	1

Toolkit Functions

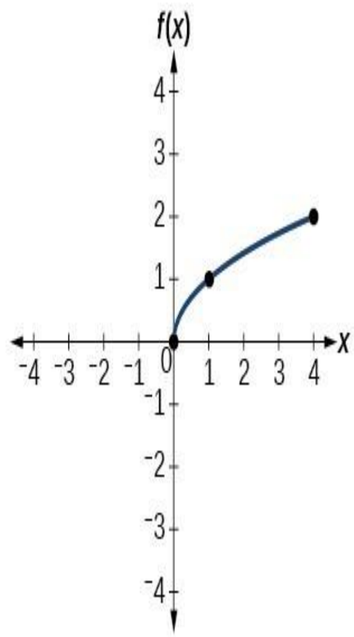
Name	Function	Graph
Reciprocal	$f(x)=\frac{1}{x}$	

x	$f(x)$
−2	−0.5
−1	−1
−0.5	−2
0.5	2
1	1
2	0.5

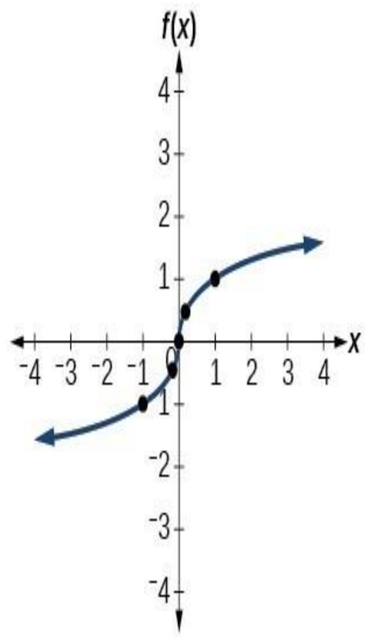
Toolkit Functions

Name	Function	Graph
Reciprocal squared	$f(x)=\frac{1}{x^2}$	

x	$f(x)$
−2	0.25
−1	1
−0.5	4
0.5	4
1	1
2	0.25

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Square root</td><td>$f(x)=\sqrt{x}$</td><td>
<table>
<tr><th>x</th><th>f(x)</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>1</td></tr>
<tr><td>4</td><td>2</td></tr>
</table>
</td></tr>
</table>

Toolkit Functions

Name	**Function**	**Graph**
Cube root	$f(x)=\sqrt[3]{x}$	

x	$f(x)$
−1	−1
−0.125	−0.5
0	0
0.125	0.5
1	1

Introduction to Rates of Change and Behaviors of Graphs

Gasoline costs have experienced some wild fluctuations over the last several decades. The table below lists the average cost, in dollars, of a gallon of gasoline for the years 2015–2022. The cost of gasoline can be considered as a function of year.

2024/2025 2024/2025 2024/2025

Y	5	6	7	8	9	0	1	2
C(y)	2.31	2.62	2.84	3.3	2.41	2.84	3.58	3.68

If we were interested only in how the gasoline prices changed between 2015 and 2022, we could compute that the cost per gallon had increased from $2.31 to $3.68, an increase of $1.37. While this is interesting, it might be more useful to look at how much the price changed *per year*. In this section, we will investigate changes such as these.

Use a graph to determine where a function is increasing, decreasing, or constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is

C(y)	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

positive, and the average rate of change of a decreasing function is negative. Figure 3 shows examples of increasing and decreasing intervals on a function.

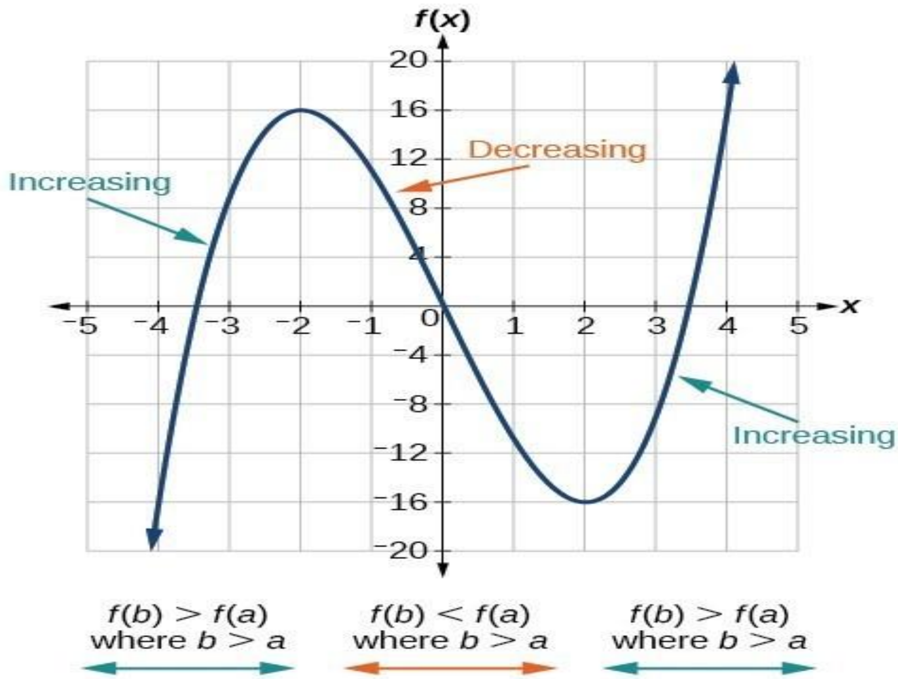


Figure 3. The function $f(x)=x^3-12x$ is increasing on $(-\infty,-2)\cup(2,\infty)$ and is decreasing on $(-2,2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a local maximum. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a local minimum. The plural form is “local minima.” Together, local maxima and minima are called local extrema, or local extreme values, of the function. (The singular form is “extremum.”) Often, the term *local* is replaced by the term *relative*. In this text, we will use the term *local*.

y	5	6	7	8	9	0	1	2
C(y)	2.31	2.62	2.84	3.3	2.41	2.84	3.58	3.68

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of *local* extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function’s entire domain.

For the function in Figure 4, the local maximum is 16, and it occurs at $x=2$ and $x=-2$. The local minimum is -16 and it occurs at $x=2$.

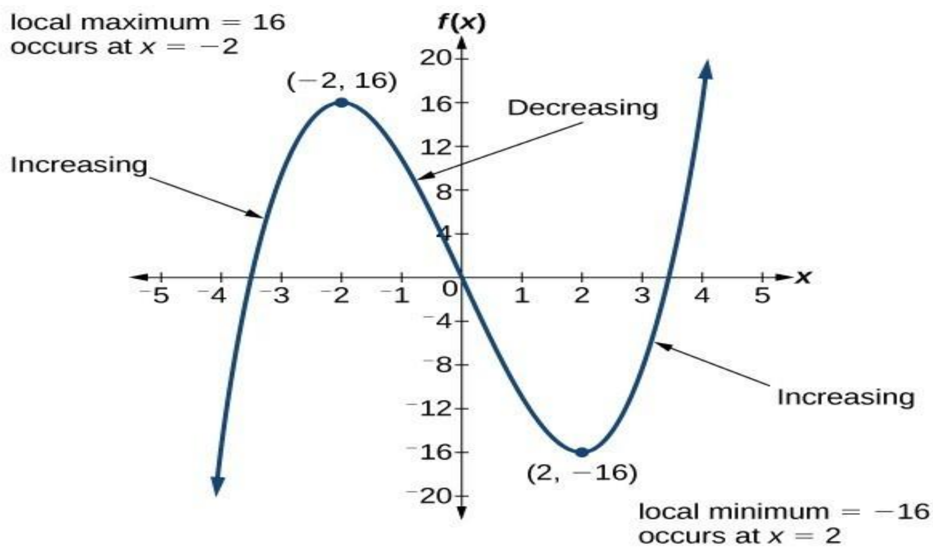


Figure 4

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 5 illustrates these ideas for a local maximum.

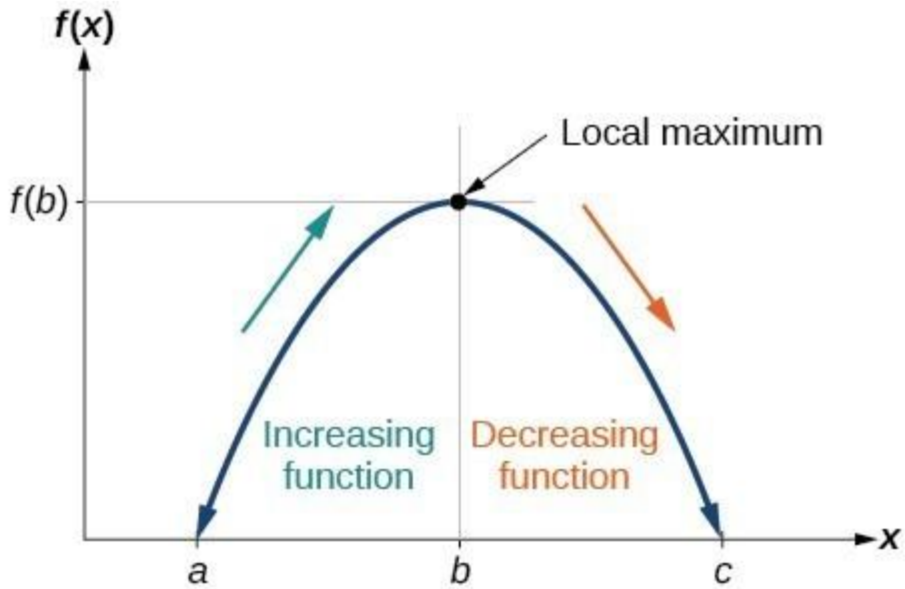


Figure 5. Definition of a local maximum.

These observations lead us to a formal definition of local extrema.

A General Note: Local Minima and Local Maxima

A function f is an increasing function on an open interval if f(b)>f(a) for any two input values a and b in the given interval where b>a .

A function f is a decreasing function on an open interval if f(b)<f(a) , for any two input values a and b in the given interval where b>a.

A function f has a local maximum at x=b if there exists an interval (a,c) such that, for any x in the interval (a,c), $f(x) \leq f(b)$. Likewise, f has a local minimum at x=b if there exists an interval (a,c) with $a<b<c$ such that, for any x in the interval (a,c).

Example 7: Finding Increasing and Decreasing Intervals on a Graph

Given the function f(t) in the graph below, identify the intervals on which the function appears to be increasing.

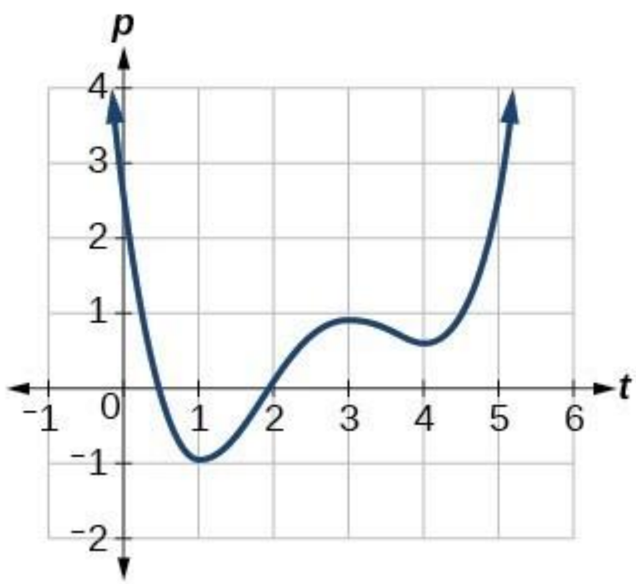


Figure 6

Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t=1$ to $t=3$ and from t=4\displaystyle t=4t=4 on.

In interval notation, we would say the function appears to be increasing on the interval $(1,3)$ and the interval $(4,\infty)$.

Analysis of the Solution

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t=1$, $t=3$, and $t=4$. These points are the local extrema (two minima and a maximum).

Example 8: Finding Local Extrema from a Graph

Graph the function $f(x)=2x+x3$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution

Using technology, we find that the graph of the function looks like that in Figure 7. It appears there is a low point, or local minimum, between x=2\displaystyle x=2x=2 and x=3\, and a mirror-image high point, or local maximum, somewhere between x=−3, x=-3 and x=−2.

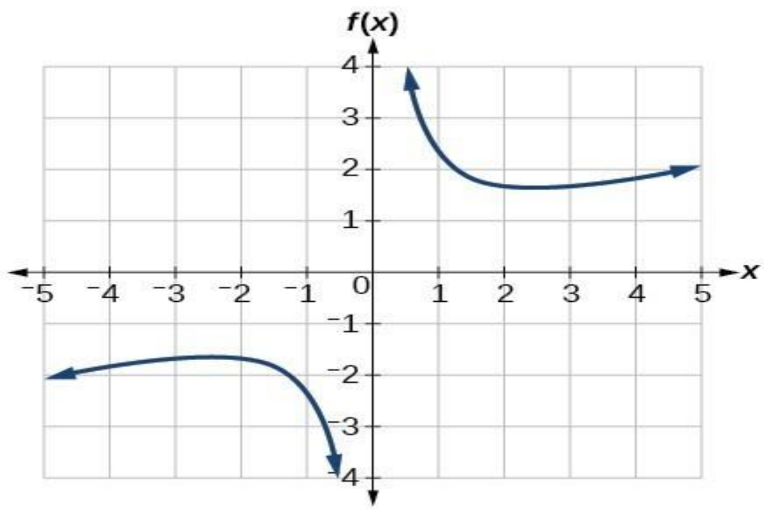
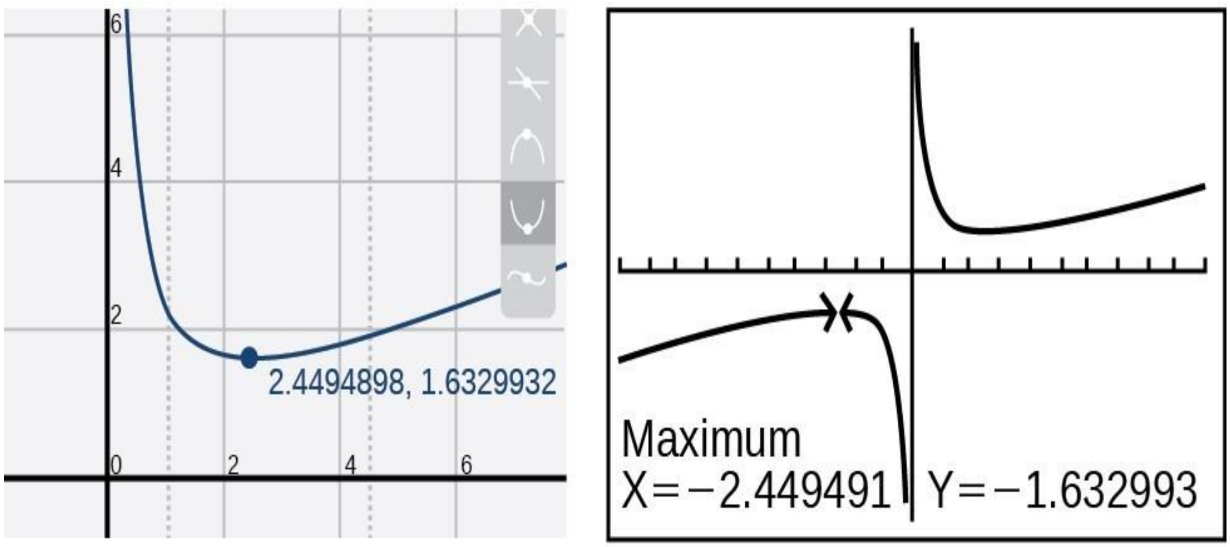


Figure 7

Analysis of the Solution

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure 7 provides screen images from two different technologies, showing the estimate for the local maximum and minimum.



(a) (b)

Figure 8

Based on these estimates, the function is increasing on the interval (−∞,−2.449)(−∞,−2.449)
and (2.449,∞)(2.449,∞) . Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at ±6, but determining this requires calculus.)

Try It 4

Graph the function $f(x)=x^3-6x^2-15x+20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

<u>Solution</u>

Example 9: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in Figure 9, find all local maxima and minima.

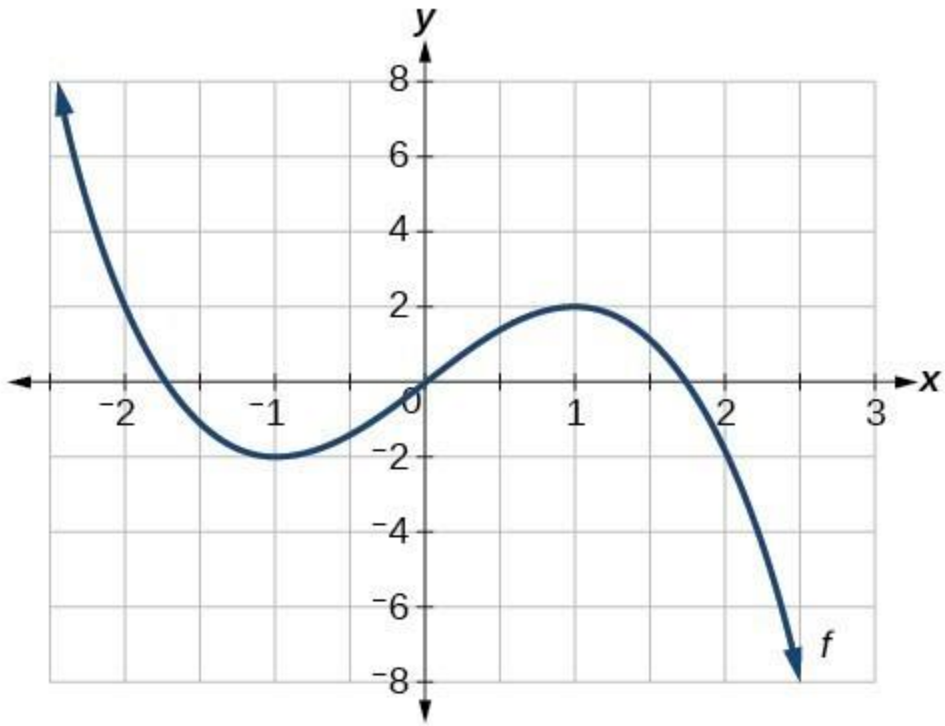


Figure 9

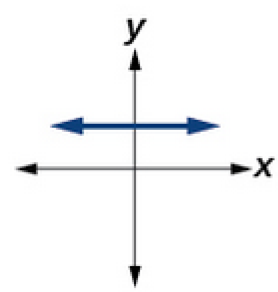
Solution

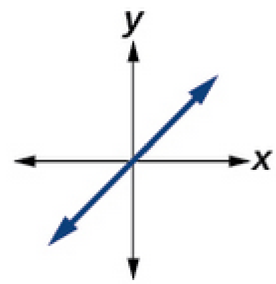
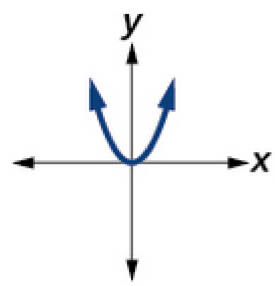
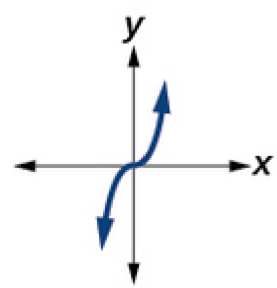
Observe the graph of f. The graph attains a local maximum at x because it is the highest point in an open interval around . The local maximum is the y\ -coordinate at x=1, which is 2.

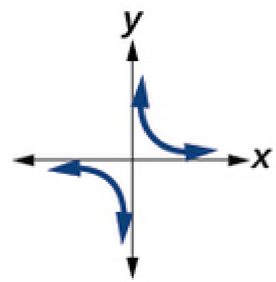
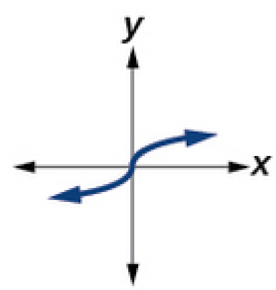
The graph attains a local minimum at $x=-1$ because it is the lowest point in an open interval around $x=-1$. The local minimum is the *y*-coordinate at $x=-1$, which is -2.

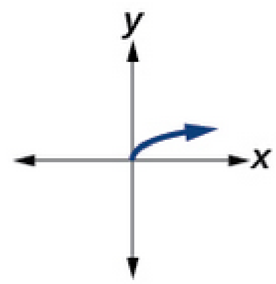
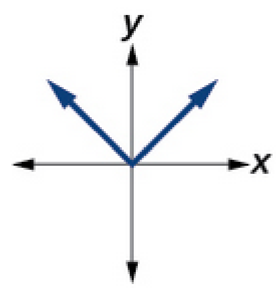
Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in the table below.

Function	Increasing/Decreasing	Example
Constant Function $f(x)=c$	Neither increasing nor decreasing	

Function	Increasing/Decreasing	Example
Identity Function $f(x)=x$	Increasing	
Quadratic Function $f(x)=x^2$	Increasing on $(0,\infty)$ Decreasing on $(-\infty,0)$ Minimum at x=0\displaystyle x=0x=0	
Cubic Function $f(x)=x^3$	Increasing	

Function	Increasing/Decreasing	Example
Reciprocal $f(x)=\frac{1}{x}$	Decreasing $(-\infty,0)\cup(0,\infty)$	
Reciprocal Squared $f(x)=\frac{1}{x^2}$	Increasing on $(-\infty,0)$ Decreasing on $(0,\infty)$	
Cube Root $f(x)=\sqrt[3]{x}$	Increasing	

Function	Increasing/Decreasing	Example
Square Root $f(x)=\sqrt{x}$	Increasing on (0,∞)	
Absolute Value f(x)=|x|	Increasing on (0,∞) Decreasing on (−∞,0)	

Use a graph to locate the absolute maximum and absolute minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-\displaystyle y\text{-}y- coordinates (output) at the highest and lowest points are called the absolute maximum and absolute minimum, respectively.

To locate absolute maxima and minima from a graph, we need to obs erve the graph to determine where the graph attains it highest and lowest points on the domain of the function. See Figure10.

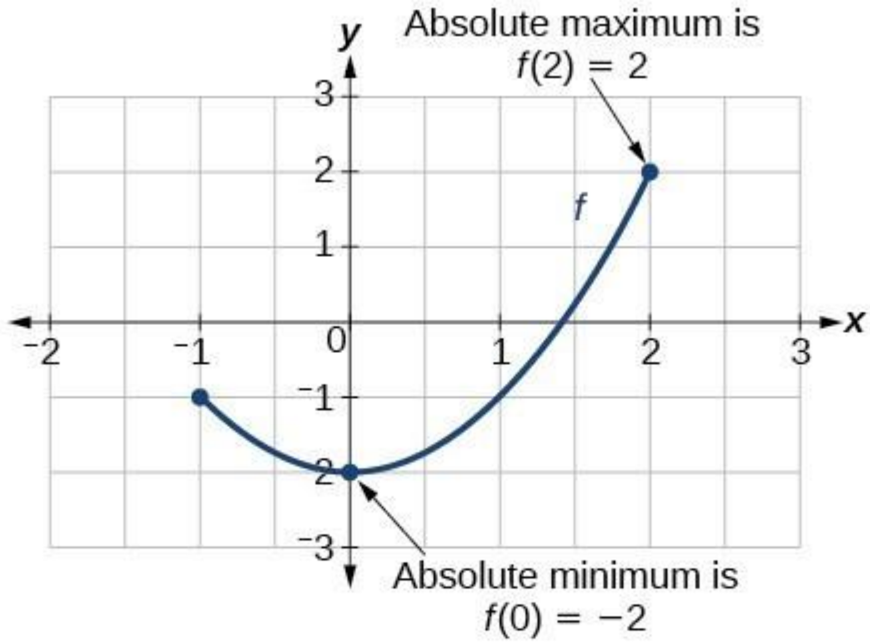


Figure 10

Not every function has an absolute maximum or minimum value. The toolkit function $f(x)=x^3$ is one such function.

A General Note: Absolute Maxima and Minima

The absolute maximum of f at x=c is f(c) where $f(c) \geq f(x)$ for all x.

The absolute minimum of f at x=d is f(d) where f(d)≤f(x) for all xin the domain of f.

Example 10: Finding Absolute Maxima and Minima from a Graph

For the function f\displaystyle ff shown in Figure 11, find all absolute maxima and minima.

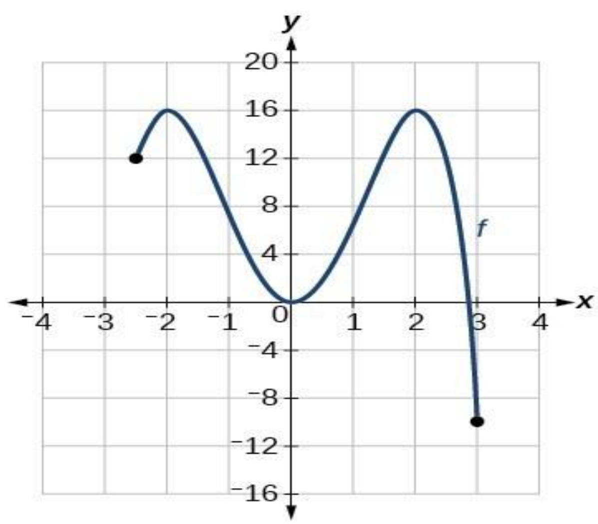


Figure 11

Solution

Observe the graph of f. The graph attains an absolute maximum in two locations, x=2 ,x=2, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the *y*-coordinate at $x=-2$ and $x=2$, which is 16.

The graph attains an absolute minimum at $x=3$, because it is the lowest point on the domain of the function’s graph. The absolute minimum is the *y*-coordinate at $x=3$, which is -10.

 2024/2025 2024/2025

BIS section(First Level)

Mathematical Assignment :

Student name :---

ID:---------------------------------------

<u>Solve these problems :</u>

a) Using Cramer's Rule, solve – if possible- the following problems:

$3x+y+z= 4 \qquad 2x+y+2z = 3 \qquad x-3y+3z = -2$

b) Express the matrix equations

$$x\begin{pmatrix}3\\2\end{pmatrix} - y\begin{pmatrix}-4\\7\end{pmatrix} = 3\begin{pmatrix}2\\4\end{pmatrix}$$

As a system of linear equations and solve