University of Sadat City | جامعة مدينة السادات

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Artificial Intelligence* | كلية الحاسبات
والذكاء الاصطناعي

MATH 2

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2025-2024

Contants

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Chapter (1)
The indefinite integral

In this chapter we will begin with an overview of the problem of finding areas—we will discuss what the term "area" means, and we will outline two approaches to defining and calculating areas.Following this overview, we will discuss the fundamental theorem of Calculus, which is the theorem that relates the problems of finding tangent lines and areas, and we will discuss techniques for calculating areas. We will then use the ideas in this chapter to define the average value of a function, to continue our study of rectilinear motion, and to examine some consequences of the chain rule in integral calculus. We conclude the chapter by studying functions defined by integrals, with a focus on the natural logarithm function

(1-1) The concept of integration:

Assuming that we have the function $y = f(x)$ ***as in Figure (1-1) and we want to find the area*** $abcd$ ***and symbolize it*** F ***, we divide the line segment*** ab ***into*** n ***parts*** $a = x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b$

Then we draw the gradient shape $abcd$ ***. Therefore, the area of a figure is given by:***

$$F_n = y_\circ (x_1 - a) + y_1 (x_2 - x_1) + y_3 (x_3 - x_2) + \ldots. + y_{n-1} (b - x_{n-1})$$

Let:

$$(x_1 - a) = dx_\circ, (x_2 - x_1) = dx_1, (x_3 - x_2) = dx_2, \ldots., (b - x_{n-1}) = dx_{n-1}.$$

$$F_n = y_\circ dx_\circ + y_1 dx_1 + y_3 dx_2 + \ldots. + y_{n-1} dx_{n-1}.$$

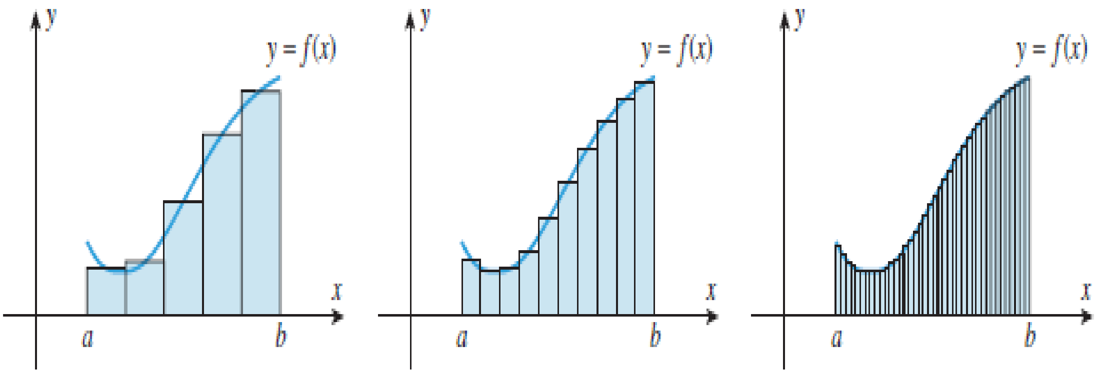


Figure (1-1)

We note that the area $abcd$ is $\lim\limits_{n\to\infty} F_n$. We will symbolize this limit with the symbol $\lim\limits_{n\to\infty} F_n = \int ydx$, where $\int$ it is the Latin letter in italics S , which is the first letter of the word summa, which means sum. Leibniz meant by the symbol $\int ydx$ that it is the sum of an infinite number of infinitesimal terms, and then he called this sum $\int ydx$ the integral.

Definition (1-1) :

The original function $f(x)$ on the given domain is the function $F(x)$ and whose derivative is equal to $f(x)$ the given domain, that is $F'(x) = f(x)$.:

For example: The original function of the function $3x^2$ is x^3 the function because $(x^3)' = 3x^2$, we have to

To note that the original function is not unique, as the function (x^3+1) is also an original function of the function x^3, as well as the function (x^3-5),......, etc.

Definition (1-1):

The general expression for all the original functions $f(x)$ is called the infinite integral of the function $f(x)$ and is denoted by: $\int f(x)dx$

i.e $\int f(x)dx = F(x)+c$ ***. For example:***

$$\int x^2 dx = \frac{1}{3}x^3 + c \Leftrightarrow \frac{d}{dx}\left[\frac{1}{3}x^3\right]$$

Properties of integration:

$$[1]\int f(x)dx = \int F'(x)dx = F(x)+c,$$

$$[2]\left[\int f(x)dx\right]' = f(x),$$

$$[3]d\int f(x)dx = f(x),$$

$$[4]\int\left[f(x)+g(x)-h(x)\right]dx =$$

$$= \int f(x)dx + \int g(x)dx - \int h(x)dx :$$

$$where\ f(x), g(x), h(x)\ are\ continuous\ on\ [a,b],$$

$$[5]\int af(x)dx = a\int f(x)dx = a\left[F(x)+c\right] =$$

$$= aF(x)+c_1 : a\ \ const.$$

Table of basic integrals:

$$[1]\int x^n dx=\frac{x^{n+1}}{n+1}+c,\ n\neq -1$$

$$[2]\left[\int f(x)dx\right]'=f(x),$$

$$[3]\int \frac{1}{x}dx=Ln\,x+c,$$

$$[4]\int e^x dx=e^x+c$$

$$[5]\int a^x dx=\frac{a^x}{Ln\,a}+c$$

$$[6]\int \sin ax\,dx=-\frac{1}{a}\cos ax+c,$$

$$[7]\int \cos ax\,dx=\frac{1}{a}\sin ax+c,$$

$$[8]\int \sec^2 ax\,dx=\frac{1}{a}\tan ax+c,$$

$$[9]\int co\sec^2 ax\,dx=-\frac{1}{a}co\,t\,ax+c,$$

$$[10]\int \sec ax\tan ax\,dx=\frac{1}{a}\sec ax+c$$

$$[11]\int co\sec ax\,co\,t\,ax\,dx=$$

$$=-\frac{1}{a}\mathrm{cosec}\,ax+c$$

$$[12]\int \frac{dx}{\sqrt{a^2-x^2}}=\begin{cases}\sin^{-1}\frac{x}{a}+c\\ -\cos^{-1}\frac{x}{a}\end{cases}$$

$$[13]\int \frac{dx}{x^2+a^2}=\begin{cases}\frac{1}{a}\tan^{-1}\frac{x}{a}+c\\ \frac{-1}{a}\cot^{-1}\frac{x}{a}+c\end{cases}$$

$$[14]\int \frac{dx}{x\sqrt{x^2-a^2}}=\begin{cases}\frac{1}{a}\sec^{-1}\frac{x}{a}+c\\ \frac{-1}{a}\mathrm{cosec}^{-1}\frac{x}{a}+c\end{cases},$$

$$[15]\int \sinh x\,dx=\cosh x+c,$$

$$[16]\int \cosh x\,dx=\sinh x+c$$

$$[17]\int \frac{dx}{\sqrt{a^2+x^2}}=\begin{cases}\sinh^{-1}\frac{x}{a}+c\\ Ln\left(x+\sqrt{a^2+x^2}\right)+c\end{cases}$$

$$[17]\int \frac{dx}{\sqrt{x^2-a^2}}=\begin{cases}\cosh^{-1}\frac{x}{a}+c\\ Ln\left(x+\sqrt{x^2-a^2}\right)+c,\ 0<a<|x|\end{cases}$$

$$[18]\int \frac{dx}{a^2-x^2}=\begin{cases}\frac{1}{a}\tanh^{-1}\frac{x}{a}+c\\ \frac{1}{2a}Ln\left(x+\frac{a+x}{a-x}\right)+c,\ 0<|x|<a\end{cases}$$

$$[19]\int \frac{dx}{x^2-a^2}=\frac{-1}{a}\coth^{-1}\frac{x}{a}+c$$

$$[20]\int \frac{dx}{x\sqrt{a^2-x^2}}=\begin{cases}-\frac{1}{a}\mathrm{sech}^{-1}\frac{x}{a}+c\\ -\frac{1}{a}Ln\left(x+\frac{\sqrt{a^2-x^2}}{x}\right)+c,\end{cases}$$

$$[21]\int \frac{dx}{x\sqrt{x^2+a^2}}=\begin{cases}\frac{-1}{a}\mathrm{cosec}^{-1}\frac{x}{a}+c\\ -\frac{1}{a}Ln\left(x+\frac{\sqrt{a^2+x^2}}{x}\right)+c,\end{cases}.$$

Theory (1-1):

Integration of the sum (difference) of two functions = sum (difference) of the integral of the two functions. That is

$$\int\left(f(x)\pm g(x)\right)dx=\int f(x)dx\pm\int g(x)dx$$

Theory (1-3):

$$(1)\int f(x)^n f'(x)dx=\frac{f(x)^{n+1}}{n+1}+c\,;n\neq-1.$$

$Proof\;:$

$put\;\; u=f(x)\Rightarrow du=f'(x)dx.$

$$\therefore\int f(x)^n f'(x)dx=u^n du=\frac{u^{n+1}}{n+1}+c=\frac{f(x)^{n+1}}{n+1}+c.$$

$$(2)\int(ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{a(n+1)}+c\,;n\neq-1.$$

$$(3)\int\frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+c.$$

$(Put\;\; u=f(x)\Rightarrow du=f'(x)dx)$

$$\therefore\int\frac{f'(x)}{\sqrt{f(x)}}dx=\int\frac{du}{\sqrt{u}}dx=2\sqrt{u}+c=2\sqrt{f(x)}+c.$$

Example (1-1):

$$(1)\int x^2dx=\frac{x^3}{3}+c\quad(2)\int x^3dx=\frac{x^4}{4}+c$$

$$(3)\int\frac{1}{x^6}dx=\int x^{-6}dx=\frac{x^{-5}}{-5}+c.\;(4)\int\sqrt{x}\,dx=\int x^{\frac{1}{2}}dx=\frac{2x^{\frac{3}{2}}}{3}+c=\frac{2\left(\sqrt{x}\right)^3}{3}+c.$$

Example (1-2): Calculate the following integrals:

$$(1)\int 4\cos x dx ,(2)\int\left(x^{2}+x\right)dx ,\ (3)\int\left(3x^{6}-2x^{2}+7x+1\right)dx ,$$

$$(4)\int\frac{\cos x}{\sin^{2}x}dx ,\ (5)\int\frac{x^{2}-2x^{4}}{x^{4}}dx ,\qquad (6)\int\frac{x^{2}}{x^{2}+1}dx .$$

Solution:

$$(1)\int 4\cos x dx = 4\int\cos x\, dx = 4\sin x + c$$

$$(2)\int\left(x^{2}+x\right)dx = \frac{x^{3}}{3}+\frac{x^{2}}{2}+c$$

$$(3)\int\left(3x^{6}-2x^{2}+7x+1\right)dx = \frac{3x^{7}}{7}-\frac{2x^{3}}{3}+\frac{7x^{2}}{2}+x+c.$$

$$(4)\int\frac{\cos x}{\sin^{2}x}dx = \int\frac{1}{\sin x}\frac{\cos x}{\sin x}dx = \int\csc x\,\cot x dx = -\csc x + c$$

$$(5)\int\frac{x^{2}-2x^{4}}{x^{4}}dx = \int\frac{1}{x^{2}}dx - \int 2dx = \int x^{-2}dx - \int 2dx =$$

$$= -x^{-1}-2x+c = -\frac{1}{x}-2x+c.$$

$$(6)\int\frac{x^{2}}{x^{2}+1}dx = \int\frac{x^{2}+1-1}{x^{2}+1}dx = \int\frac{x^{2}+1}{x^{2}+1}dx - \int\frac{1}{x^{2}+1}dx =$$

$$= \int dx - \int\frac{1}{x^{2}+1}dx = x-\tan^{-1}x + c.$$

Example (1-3): Calculate the following integrals

$$(1)\int x\left(1+x^{3}\right)dx ,\qquad (2)\int\left(x^{2}+2x\right)^{2}dx ,$$

$$(3)\int x^{\frac{1}{3}}(2-x)^{2}dx ,\qquad (4)\int\left(x^{2}+1\right)(2-x)dx .$$

Solution:

$$(1)\int x\left(1+x^{3}\right)dx=\int\left(x+x^{4}\right)dx=\frac{x^{2}}{2}+\frac{x^{5}}{5}+c$$

$$(2)\int\left(x^{2}+2x\right)^{2}dx=\int\left(x^{4}+4x^{3}+4x^{2}\right)dx=\frac{x^{5}}{5}+x^{4}+\frac{4x^{3}}{3}+c.$$

$$(3)\int x^{\frac{1}{3}}(2-x)^{2}dx=\int x^{\frac{1}{3}}\left(4-2x+x^{2}\right)dx=\int\left(4x^{\frac{1}{3}}-4x^{\frac{4}{3}}+x^{\frac{7}{3}}\right)dx=$$

$$=3x^{\frac{4}{3}}-\frac{12}{7}x^{\frac{7}{3}}+\frac{3}{10}x^{\frac{10}{3}}+c.$$

$$(4)\int\left(x^{2}+1\right)(2-x)dx=\int\left(2x^{2}-x^{3}-x+2\right)dx=\frac{2x^{2}}{3}-\frac{1}{4}x^{4}-\frac{x^{2}}{2}+2x+c.$$

Example (1-4): Calculate the following integrals:

$$(1)\int\left(\frac{1}{x}+\sec^{2}\pi x\right)dx,\quad (2)\int(x-8)^{23}dx,\ (3)\int\sin(x+9)dx$$

Solution:

$$(1)\int\left(\frac{1}{x}+\sec^{2}\pi x\right)dx=\int\frac{1}{x}dx+\int\sec^{2}\pi x\,dx=$$

$$=Ln\left|x\right|+\frac{1}{\pi}\tan\pi x+c.$$

$$(2)\int(x-8)^{23}dx=\frac{(x-8)^{24}}{24}+c.$$

$$(3)\int\sin(2x+9)dx=-\frac{1}{2}\cos(2x+9)+c.$$

***Example (1-5): Calculate the following integrals*:**

$$(1)\int\left(\frac{3x^{2}+1}{x^{3}+x}\right)dx\,,\ (2)\int\frac{\sec^{2}x}{\tan}dx\,,\ (3)\int\tan x dx\,,(4)\int\cot x dx\,,$$

$$(5)\int co\sec x dx\,,\ (6)\int\sec x dx\,,\quad (7)\int\cos 3x\ \cos 3x dx\,.$$

Solution:

$$(1)\int\left(\frac{3x^{2}+1}{x^{3}+x}\right)dx=\int\left(\frac{3x^{2}+1}{x^{3}+x}\right)dx=Ln\left|x^{3}+x\right|+c.$$

$$(2)\int\frac{\sec^{2}x}{\tan}dx=Ln\left|\tan x\right|+c.$$

$$(3)\int\tan x dx=\int\frac{\sin x}{\cos x}dx=-\int\frac{-\sin x}{\cos x}dx=$$

$$=-Ln\left|\cos x\right|+c.$$

$$(4)\int\cot x dx=\int\frac{\cos x}{\sin x}dx=Ln\left|\sin x\right|+c.$$

$$(5)\int\csc x dx=\int\csc x\left(\frac{\csc x+\cot x}{\csc x+\cot x}\right)dx=$$

$$=-\int\frac{-\csc^{2}x-\csc x\ \cot x}{co\sec x+\cot x}dx=Ln\left|\csc x+\cot x\right|+c.$$

$$(6)\int\sec x dx=\int\sec x\left(\frac{\sec x+\tan x}{\sec x+\tan x}\right)dx=$$

$$=\int\frac{\sec^{2}x+\sec x\ \tan x}{\sec x+\tan x}dx=Ln\left|\sec x+\tan x\right|+c.$$

$(7)\int \cos 5x \cos 3x dx$

$\because \cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$

$\therefore \int \cos 5x \cos 3x dx = \int\left(\frac{1}{2}\cos 8x + \frac{1}{2}\cos 2x\right)dx =$

$= \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + c.$

Example (1-6): Calculate the following integrals:

$(1)\int \frac{1}{2-2\sin x}dx$, $(2)\int \cos^2 x dx$, $(3)\int \tan^2 x dx$ $(4)\int \cot^2 x dx$,

$(5)\int \sin 5x \cos 7x dx$. $(6)\int 2\left(\tan x + \tan^3 x\right)dx$.

Solution:

$(1)\int \frac{1}{2-2\sin x}dx = \int \frac{1}{2(1-\sin x)}\frac{(1+\sin x)}{(1+\sin x)}dx =$

$= \int \frac{(1+\sin x)}{2\left(1-\sin^2 x\right)}dx = \frac{1}{2}\int \frac{(1+\sin x)}{\cos^2 x}dx =$

$= \frac{1}{2}\int\left[\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right]dx = \frac{1}{2}\int\left[\sec^2 x + \tan x \sec x\right]dx =$

$= \frac{1}{2}[\tan x + \sec x] + c.$

$(2)\int \cos^2 x dx = \int \frac{1}{2}[1+\cos 2x]dx = \frac{x}{2} + \frac{1}{2}\sin 2x + c.$

$(3)\int \tan^2 x dx = \int\left(\sec^2 x - 1\right)dx = \tan x - x + c.$

$$(4)\int \cot^2 x dx = \int \left(co\sec^2 x - 1\right)dx = -\cot x - x + c.$$

$$(5)\int \sin 7x \sin 5x dx = \frac{1}{2}\int (\cos 2x - \cos 12x)dx =$$

$$= \frac{1}{4}\sin 2x - \frac{1}{24}\sin 12x + c.$$

$$(6)\int 2\left(\tan x + \tan^3 x\right)dx = \int 2\tan x\left(1+\tan^2 x\right)dx =$$

$$= \int 2\tan x\left(1+\tan^2 x\right)dx = \int 2\tan x \sec^2 x dx = \tan^2 x + c.$$

Example (1-7): Calculate the following integrals:

$$(1)\int \frac{x^2+2x+2}{x^2+2}dx,\quad (2)\int \frac{dx}{xLnx}, x>0, (3)\int \sin^2 x \cos x\, dx.$$

$$(4)\int \sqrt{1+\tan^2 x}dx. (5)\int \sec^4 x dx. (6)\int \left(\frac{1}{x} + \sec^2 x\right)dx.$$

Solution:

$$(1)\int \frac{x^2+2x+2}{x^2+2}dx = \int \left(1 + \frac{2x}{x^2+2}\right)dx = x + Ln\left|x^2+2\right| + c.$$

$$(2)\int \frac{dx}{xLnx}, x>0,$$

$$\therefore \int \frac{dx}{xLnx} = \int \frac{\frac{1}{x}}{Lnx}dx == \int \frac{(Lnx)'}{Lnx}dx = Ln\left|Lnx\right| + c.$$

$$(3)\int \sin^2 x \cos x\, dx = \frac{1}{3}\sin^3 x + c.$$

$$(4)\int\sqrt{1+\tan^2 x}\,dx.$$

$$\because 1+\tan^2 x=\sec^2 x$$

$$\therefore \int\sqrt{1+\tan^2 x}\,dx=\int\sec x\,dx==Ln\left|\sec x+\tan x\right|+c.$$

$$(5)\int\sec^4 x\,dx=\int\sec^2 x\sec^2 x\,dx=\int\sec^2 x\left(1+\tan^2 x\right)dx=$$

$$=\int\sec^2 x+\int\sec^2 x\tan^2 x\,dx=\tan x+\frac{1}{3}\tan^3 x+c.$$

$$(6)\int\left(\frac{1}{x}+\sec^2 x\right)dx=Ln\,x+\tan x+c.$$

Theorem (1-4): ***If a function*** $f=f(x)$ ***is differentiable with respect to*** x ***,then:***

$$(i)\int e^x dx=e^x+c,\qquad (ii)\int f'(x)e^{f(x)}dx=e^{f(x)}+c,$$

$$(iii)\int a^x dx=\frac{a^x}{Ln\,a}+c\qquad (iv)\int f'(x)a^{f(x)}dx=\frac{a^{f(x)}}{\ln a}+c$$

Example (1-8): Calculate the following integrals:

$$(1)\int e^{4x+2}dx,\qquad (2)\int 7xe^{x^2}dx,(3)\int\frac{e^{\sin^2 x}}{e^{-\cos^2 x}}dx,$$

$$(4)\int\frac{e^x}{\left(3-e^x\right)^2}dx,\qquad (5)\int\frac{e^{2x}}{e^x+e^{2x}}dx,$$

Solution:

$$(1)\int e^{4x+2}dx = \frac{1}{4}e^{4x+2},$$

$$(2)\int 7xe^{x^2}dx = \frac{7}{2}\int 2xe^{x^2}dx = \frac{7}{2}e^{x^2}+c.$$

$$(3)\int \frac{e^{\sin^2 x}}{e^{-\cos^2 x}}dx = \int e^{\sin^2 x+-\cos^2 x}dx = \int e\,dx = ex+c.$$

$$(4)\int \frac{e^{x}}{\left(3-e^{x}\right)^{2}}dx = -\int \left(3-e^{x}\right)^{-2}\left(-e^{x}\right)dx = \left(3-e^{x}\right)^{-1}+c =$$

$$= \frac{1}{\left(3-e^{x}\right)}+c.$$

$$(5)\int \frac{e^{2x}}{e^{x}+e^{2x}}dx = \int \frac{e^{2x}}{e^{x}\left(1+e^{x}\right)}dx = \int \frac{e^{x}}{\left(1+e^{x}\right)}dx =$$

$$= Ln\left(1+e^{x}\right)+c.$$

Exercises (1-1)

1− **Calculate the following integrals:**

$$(1)\int\left(x^{3}\sqrt{x}+\sqrt[3]{x^{2}}\right)dx,\qquad (2)\int\frac{x^{5}+2x^{2}-1}{x^{4}}dx,$$

$$(3)\int\left(3\sin x-2\sec^{2}x\right)dx,(4)\int\left(\operatorname{cosec}^{2}x-\sec x\,\tan x\right)dx,$$

$$(5)\int\sec x\left(\sec x+\tan x\right)dx,\ (6)\int\frac{\sin x}{\cos^{2}x}dx.$$

2− **Calculate the following integrals:**

$$(1)\int(4x-3)^{9}\,dx,\quad (2)\int x^{3}\sqrt{5+x^{4}}\,dx,$$

$$(3)\int\sec 4x\,\tan 4x\,dx,\ (4)\int\frac{1}{1+16x^{2}}dx.$$

3-Calculate the following integrals:

$$(1)\int\frac{x}{x^{2}+1}dx,\qquad (2)\int\frac{x-1}{x^{2}-2x-7}dx$$

$$(3)\int\frac{x-4}{x^{2}-3x-4}dx,(4)\int\frac{x-1}{\sqrt{x}\left(1+\sqrt{x}\right)}dx,$$

$$(5)\int\frac{\sec^{2}x}{3+\tan x}dx,\ (6)\int\frac{\sin 2x}{2+\sin^{2}x}dx.$$

(1-2) Integration by substitution:

Sometimes we encounter functions that we want to integrate, but it is not easy to find a standard form for them from the table of basic integrals. In this case, we use some substitution for the independent variable x and write it in terms of another variable, for example $u, t, \theta, \ldots$ and thus we can convert the integral to any of the usual forms. This method is called the substitution integration method. .

First: algabric subsistution

Example (1-8): Calculate the following integrals.

$$(1)\int \left(x^2+1\right)^{50} 2x dx \quad (2)\int \frac{1}{1+3x^2} dx, (3)\int \sin^2 x \cos x dx.$$

Solution:

$$(1)\int \left(x^2+1\right)^{50} 2x dx.$$

$$put\ u = x^2+1 \Rightarrow du = 2x dx$$

$$\therefore \int \left(x^2+1\right)^{50} 2x dx = \int u^{50} du = \frac{u^{51}}{51}+c = \frac{\left(x^2+1\right)^{51}}{51}+c.$$

$$(2)\int \frac{1}{1+3x^2} dx,$$

$$put\ u = \sqrt{3}x \Rightarrow du = \sqrt{3}dx \Rightarrow \frac{1}{\sqrt{3}} du = dx$$

$$\therefore \int \frac{1}{1+3x^2} dx = \frac{1}{\sqrt{3}} \int \frac{1}{1+u^2} du =$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} u + c = \frac{1}{\sqrt{3}} \tan^{-1} \sqrt{3}x + c$$

$$(3)\int \sin^2 x\ \cos x dx\,.$$

$$put\ u=\sin x \Rightarrow du=\cos x dx \Rightarrow \frac{1}{\cos x}du=dx\,.$$

$$\int \sin^2 x\ \cos x dx=\int u^2 du=\frac{u^3}{3}+c=\frac{\sin^3 x}{3}+c.$$

Example (1-9): Calculate the following integrals

$$(1)\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx \qquad (2)\int x^4\sqrt[3]{2-5x^2}dx,\quad (3)\int \frac{e^x}{\sqrt{1-e^x}}dx\,.$$

Solution:

$$(1)\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx \quad .put\ u=\sqrt{x} \Rightarrow du=\frac{1}{2\sqrt{x}}dx \Rightarrow$$

$$2du=\frac{1}{\sqrt{x}}dx \Rightarrow 2udu=dx\,.$$

$$\therefore \int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx=\int \frac{e^u}{u}2udu=\int 2e^u du=2e^u+c=2e^{\sqrt{x}}+c.$$

$$(2)\int x^4\sqrt[3]{3-5x^5}dx$$

$$put\ u=3-5x^5 \Rightarrow du=-25x^4 dx \Rightarrow -\frac{1}{25}du=x^4 dx\,.$$

$$\therefore \int x^4\sqrt[3]{3-5x^5}dx=-\frac{1}{25}\int \sqrt[3]{u}du=-\frac{1}{25}\int u^{\frac{1}{3}}du=$$

$$=\left(-\frac{1}{25}\right)\left(\frac{3}{4}\right)u^{\frac{4}{3}}+c$$

$$=\left(-\frac{3}{100}\right)\left(3-5x^5\right)^{\frac{4}{3}}+c=\left(-\frac{3}{100}\right)\sqrt[3]{\left(3-5x^5\right)^4}+c.$$

$$(3) \because \int \frac{e^x}{\sqrt{1-e^x}} dx \text{ . } put\ u = e^x \Rightarrow du = e^x dx$$

$$\therefore \int \frac{e^x}{\sqrt{1-e^x}} dx = \int \frac{1}{\sqrt{1-u}} du = \int \sqrt{1-u}\, du = -\int -\sqrt{1-u}\, du =$$

$$= -2\sqrt{1-u} + c = -2\sqrt{1-e^x} + c..$$

Example (1-10): Calculate the following integrals

$(1) \int x^2 \sqrt{x-1} dx$ $\qquad$ $(2) \int \cos^3 x dx$,

$(3) \int \frac{1}{\sqrt{a^2+x^2}} dx$. $\qquad$ $(4) \int \frac{1}{\sqrt{2-x^2}} dx$.

$(5) \int 2^x \sin 2^x dx$. $\qquad$ $(6) \int x\, 3^x dx$

Solution:

$$(1) \int x^2 \sqrt{x-1} dx$$

$$put\ u = x-1, \Rightarrow x = u+1, du = dx$$

$$\therefore \int x^2 \sqrt{x-1} dx = \int (u+1)^2 \sqrt{u} du = \int \left(u^2 + 2u + 1\right) \sqrt{u} du =$$

$$= \int \left(u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}} \right) du = \frac{2u^{\frac{7}{2}}}{7} + \frac{4u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} + c =$$

$$= \frac{2}{7}(x-1)^{\frac{7}{2}} + \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + c =$$

$$= \frac{2}{7}\sqrt{(x-1)^7} + \frac{4}{5}\sqrt{(x-1)^5} + \frac{2}{3}\sqrt{(x-1)^3} + c.$$

$$(2)\, I=\int \cos^3 x\,dx\,.\quad Put\; u=\sin x \Rightarrow du=\cos x\,dx$$

$$\therefore I=\int \cos^3 x\,dx=\int \cos^2 x\,\cos x\,dx=\int\left(1-\underbrace{\sin^2 x}_{u^2}\right)\underbrace{\cos x\,dx}_{du}$$

$$=\int\left(1-u^2\right)du=u-\frac{u^3}{3}+c=\sin x-\frac{\sin^3 x}{3}+c.$$

$$(3)\, I=\int \frac{dx\,.}{\sqrt{x^2+a^2}}\quad Put\; u=\frac{x}{a}\Rightarrow a\,du=dx\,.$$

$$\therefore I=\int \frac{dx\,.}{\sqrt{x^2+a^2}}=\frac{1}{a}\int \frac{a\,du}{\sqrt{\left(\frac{x}{a}\right)^2+1}}=\sinh^{-1}\frac{x}{a}+c.$$

$$(5)\int 2^x \sin 2^x\,dx\,.$$

$$put\; u=2^x \Rightarrow du=2^x\,Ln2\,dx \Rightarrow dx=\frac{du}{2^x\,Ln2}.$$

$$\therefore I=\int 2^x \sin 2^x\,dx=\int u\sin u\,\frac{du}{uLn2}=-\frac{1}{Ln2}\cos u+c=$$

$$=-\frac{1}{Ln2}\cos 2^x+c.$$

$$(6)\int x\,3^{x^2}dx\,.\quad Put\; u=3^x \Rightarrow du=3^{x^2}2xLn3dx \Rightarrow dx=\frac{du}{3^{x^2}2xLn3}.$$

$$I=\int x\,3^{x^2}dx=\int xu\,\frac{du}{u\,2xLn3}=\int \frac{du}{2Ln3}=\frac{u}{2Ln3}+c=\frac{3^{x^2}}{2Ln3}+c.$$

Example (1-12): Calculate the following integrals

$$(1)\int \frac{dx}{(2+x)\sqrt{x+1}}\;,\quad (2)\int x\sqrt{x+1}\,dx\,,(3)\int x^3 \sin x^4\,dx\,,$$

$$(4)\int \operatorname{sech} x\,dx\,,\qquad (5)\int \frac{x}{x^4+25}dx\,.\quad (6)\int x^5\sqrt{x^3+1}\,dx\,.$$

Solution:

$$(1)\int \frac{dx}{(2+x)\sqrt{x+1}}\ ,$$

$$put\ u^2 = x+1, \Rightarrow u = \sqrt{x+1},\quad 2udu = dx$$

$$\therefore \int \frac{dx}{(2+x)\sqrt{x+1}} = \int \frac{2udu}{(1+u^2)u} = 2\int \frac{du}{1+u^2} =$$

$$= 2\tan^{-1}u + c = 2\tan^{-1}\sqrt{x+1} + c.$$

$$(2)\int x\sqrt{x+1}\,dx\ ,$$

$$put\ u = \sqrt{x+1},\ x = u^2 - 1\ and\quad 2udu = dx$$

$$\therefore \int x\sqrt{x+1}\,dx = 2\int (u^2-1)u^2du = 2\int (u^4 - u^2)du =$$

$$= 2\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + c = 2\left(\frac{\left(\sqrt{x+1}\right)^5}{5} - \frac{\left(\sqrt{x+1}\right)^3}{3}\right) + c =$$

$$= \left(\frac{2(x+1)^2\sqrt{x+1}}{5} - \frac{2(x+1)\sqrt{x+1}}{3}\right) + c.$$

$$(3)\int x^3\sin x^4\,dx\ ,\qquad put\ u = x^4, \Rightarrow du = 4x^3dx$$

$$\int x^3\sin x^4\,dx = \frac{1}{4}\int \sin u\,du = \frac{1}{4}\cos u + c = \frac{1}{4}\cos x^4 + c.$$

$$(4)\int \operatorname{sech} x\,dx = \int \frac{2}{e^x + e^{-x}}dx = \int \frac{2e^x}{e^{2x}+1}dx$$

$$put\ u = e^x, \Rightarrow du = e^x dx \Rightarrow$$

$$\therefore \int \operatorname{sech} x\,dx = \int \frac{2}{u^2+1}du = 2\tan^{-1}u + c = 2\tan^{-1}e^x + c.$$

$$(5)\int \frac{x}{x^4+25}dx,$$

$$put\, u = x^2, \Rightarrow du = 2xdx$$

$$\therefore \int \frac{x}{x^4+25}dx = \frac{1}{2}\int \frac{du}{u^2+5^2} = \frac{1}{2\times 5}\tan^{-1}\frac{u}{5}+c = \frac{1}{10}\tan^{-1}\frac{x^2}{5}+c$$

$$(6)\int x^5\sqrt{x^3+1}\,dx$$

$$put\, u^2 = x^3+1, \Rightarrow 2udu = 3x^2dx$$

$$\int x^5\sqrt{x^3+1}\,dx = \int \underbrace{x^3}_{u^2-1}\underbrace{\sqrt{x^3+1}}_{u}\;\underbrace{x^2dx}_{\frac{2}{3}udu} =$$

$$= \int (u^2-1)u\left(\frac{2}{3}u\right)du = \int \frac{2}{3}(u^4-u^2)du =$$

$$= \frac{2}{15}u^5 - \frac{2}{9}u^3 + c = \frac{2}{15}\left(\sqrt{x^3+1}\right)^5 - \frac{2}{9}\left(\sqrt{x^3+1}\right)^3 + c =$$

$$= \frac{2}{15}(x^3+1)^{\frac{5}{2}} - \frac{2}{9}(x^3+1)^{\frac{3}{2}} + c.$$

Example (1-13): Evaluate:

$$\int \frac{dx}{\sqrt{4x^2-9}}.$$

Solution:

$$I=\int\frac{dx}{\sqrt{4x^2-9}}.$$

$$Put: u=2x \Rightarrow du=2dx \Rightarrow dx=\frac{1}{2}du$$

$$\therefore I=\int\frac{dx}{\sqrt{4x^2-9}}=\frac{1}{2}\int\frac{du}{\sqrt{u^2-9}}=\frac{1}{2}\int\frac{du}{\sqrt{u^2-3^2}}=\frac{1}{2}\cosh^{-1}\frac{u}{3}+c.$$

$$\therefore I=\int\frac{dx}{\sqrt{4x^2-9}}=\frac{1}{2}\cosh^{-1}\frac{u}{3}+c=\frac{1}{2}\cosh^{-1}\frac{2x}{3}+c.$$

Example (1-13): Calculate the following integrals:

$$(1)\int e^{\sqrt{1-\sin x}}\sqrt{1+\sin x}\,dx \quad , \quad (2)\int x\cot x^2\,dx,$$

$$(3)\int\frac{\cos ec^2\sqrt{x}}{\sqrt{x}}dx. \qquad (4)\int e^{2x}\sec e^{2x}dx.$$

solution:

$$(1)\int e^{\sqrt{1-\sin x}}\sqrt{1+\sin x}\,dx$$

$$put\ u=\sqrt{1-\sin x}, \Rightarrow du=\frac{-\cos x}{2\sqrt{1-\sin x}}dx=$$

$$=\frac{-\cos x}{2\sqrt{1-\sin x}}\frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}dx=\frac{-\cos x\sqrt{1+\sin x}}{2\sqrt{1-\sin^2 x}}dx=$$

$$=\frac{-\sqrt{1+\sin x}}{2}dx \Rightarrow -2du=\sqrt{1+\sin x}\,dx$$

$$\therefore\int e^{\sqrt{1-\sin x}}\sqrt{1+\sin x}\,dx=-2\int e^u du=$$

$$=-2e^u+c=-2e^{\sqrt{1-\sin x}}+c.$$

$$(2)\int x\cot x^{2}\,dx,$$

$$put\ u=x^{2},\Rightarrow du=2xdx$$

$$\therefore I=\int x\cot x^{2}\,dx=\int x\cot x^{2}\,dx=\frac{1}{2}\int \cot u\,du=$$

$$=\frac{1}{2}Ln\left|\sin u\right|+c=\frac{1}{2}Ln\left|\sin x^{2}\right|+c.$$

$$(3)\int \frac{\cos ec^{2}\sqrt{x}}{\sqrt{x}}dx.\quad put\ u=\sqrt{x},\Rightarrow du=\frac{1}{2\sqrt{x}}dx$$

$$\therefore I=\int \frac{\cos ec^{2}\sqrt{x}}{\sqrt{x}}dx=2\int \cos ec^{2}u\,du=-2\cot u+c=-2\cot\sqrt{x}+c.$$

$$(4)\int e^{2x}\sec e^{2x}dx.\quad put\ u=e^{2x},\Rightarrow du=2e^{2x}dx$$

$$\therefore\int e^{2x}\sec e^{2x}dx=\frac{1}{2}\int \sec u du=$$

$$=\frac{1}{2}Ln\left|\sec u+\tan u\right|+c=\frac{1}{2}Ln\left|\sec e^{2x}+\tan e^{2x}\right|+c$$

(1-2-1) Second: Triangular substitutions:

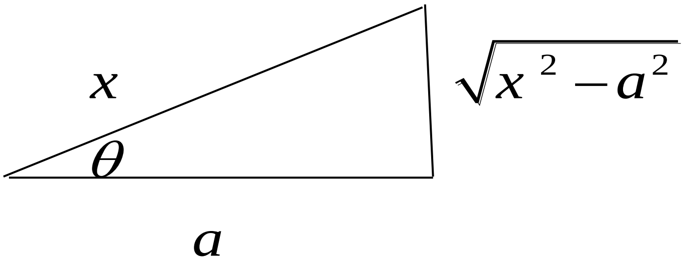
If the function to be integrated is in any of the following forms: $\sqrt{x^{2}-a^{2}},\sqrt{a^{2}-x^{2}},\sqrt{a^{2}+x^{2}}$
Trigonometric or hyperbolic substitutions are used to take one of the standard forms from the basic integrals table:

1 – If the function to be integrated is in the form:

$\sqrt{x^{2}-a^{2}}$ ***then we put:*** $x=a\sec\theta$ ***.Then:***

$$\sqrt{x^{2}-a^{2}}=\sqrt{a^{2}\sec^{2}\theta-a^{2}}=\sqrt{a^{2}\left(\sec^{2}\theta-1\right)}=\sqrt{a^{2}\tan^{2}\theta}=|a|\tan\theta,$$

$$dx=a\sec\theta\tan\theta d\theta,\theta\in\left[0,\frac{\pi}{2}\right[\cup\left[\frac{\pi}{2},\pi\right[.$$



Also, in this case we can put $x = a\cosh\theta$

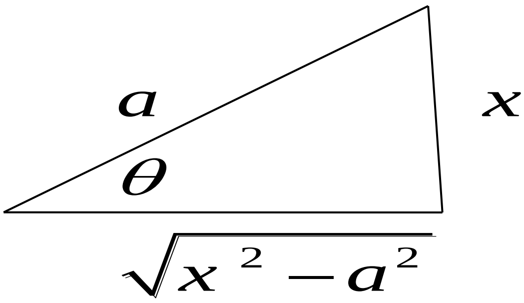
2-If the function to be integrated is in the form:

then we set $\sqrt{a^2-x^2}$ $x = a\sin\theta$

Then:

$$\sqrt{a^2-x^2} = \sqrt{a^2\left(1-\sin^2\theta\right)} = |a|\cos\theta,$$

$$dx = a\cos\theta d\theta; \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$



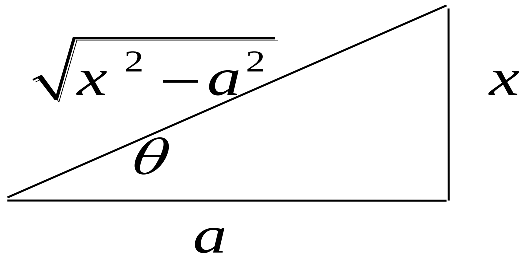
Also in this case we can put $x = a\tanh\theta$

3-If the function to be integrated is in the form:

$\sqrt{a^2+x^2}$ ***then we set*** $x = a\tan\theta$

Then:

$$\sqrt{a^2+x^2} = \sqrt{a^2\left(1+\tan^2\theta\right)} = |a|\sec\theta, dx = a\sec^2\theta d\theta; \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$



Also, in this case we can put $x = a\sinh\theta$

The previous three cases can be grouped in the following table:

The function to be integrated	***Trigonometric or hyperbolic substitution***
$\sqrt{x^2-a^2}$	$put \begin{cases} x = a\sec\theta \\ or \\ x = a\cosh\theta \end{cases}$
$\sqrt{a^2-x^2}$	$put \begin{cases} x = a\sin\theta \\ or \\ x = a\tanh\theta \end{cases}$
$\sqrt{a^2+x^2}$	$put \begin{cases} x = a\tan\theta \\ or \\ x = a\sinh\theta \end{cases}$

It is known to us that:

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \frac{1}{a}\sin^{-1}\frac{x}{a} + c, \qquad \int \frac{xdx}{\sqrt{x^2+a^2}} = \frac{1}{a}\sqrt{x^2+a^2} + c$$

While in some other cases, the integration may not be clear, so the appropriate substitution from the previous three cases is used, and the integration turns into one of the standard forms.

Rule (1): Prove that:

$$\boxed{\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.}$$

Proof: In order to find the integral $\int \sqrt{a^2 - x^2}\,dx$, we use the following substitution:

$$x = a\sin\theta \Rightarrow dx = a\cos\theta d\theta, \frac{x}{a} = \sin\theta \ and \ \theta = \sin^{-1}\frac{x}{a}.$$

$$\therefore \int \sqrt{a^2 - x^2}\,dx = \int \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta)d\theta =$$

$$= \int \sqrt{a^2(1 - \sin^2\theta)}\,(a\cos\theta)d\theta =$$

$$= \int \sqrt{a^2\cos^2\theta}\,(a\cos\theta)d\theta = \int a^2\cos^2\theta d\theta,$$

$$\because \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$\therefore \int \sqrt{a^2 - x^2}\,dx = \int a^2\cos^2\theta d\theta = \int \frac{a^2}{2}(\cos 2\theta + 1) =,$$

$$= \frac{a^2\sin 2\theta}{4} + \theta + c = \frac{a^2\sin\theta\cos\theta}{2} + \theta + c =$$

$$= \frac{a^2}{2}\left(\sin\theta\sqrt{1 - \sin^2\theta} + \theta\right) + c = \frac{a^2}{2}\left(\frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}} + \sin^{-1}\frac{x}{a}\right) + c.$$

$$\therefore \int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.$$

Example (1-14): Calculate the integral $\int \sqrt{4 - x^2}\,dx$:

Solution: Applying the previous rule we find:

$$\int \sqrt{4 - x^2}\,dx = \frac{x\sqrt{4 - x^2}}{2} + 2\sin^{-1}\frac{x}{2} + c.$$

Rule (2): Prove that:

$$\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+c$$

Proof:

$$x=a\sinh\theta \Rightarrow dx=a\cosh\theta d\theta, \frac{x}{a}=\sinh\theta$$

$$and\ \theta=\sinh^{-1}\frac{x}{a}.$$

$$\therefore \int\sqrt{a^2+x^2}\,dx=\int\sqrt{a^2+a^2\sinh^2\theta}\,(a\cosh\theta)d\theta=$$

$$=\int\sqrt{a^2(1+\sinh^2\theta)}\,(a\cosh\theta)d\theta=\int a^2\cosh^2\theta d\theta,$$

$$\because \cosh^2\theta=\frac{1}{2}(\cosh 2\theta+1)$$

$$\therefore \int\sqrt{a^2+x^2}\,dx=\int a^2\cosh^2\theta d\theta=\int\frac{a^2}{2}(\cosh 2\theta+1)=,$$

$$=\frac{a^2\sinh 2\theta}{4}+\theta+c=\frac{a^2\sinh\theta\cosh\theta}{2}+\theta+c=$$

$$\therefore \int\sqrt{a^2+x^2}\,dx=\frac{a^2}{2}\left(\sin\theta\sqrt{1+\sin^2\theta}+\theta\right)+c$$

$$\therefore \int\sqrt{a^2+x^2}\,dx=\frac{a^2}{2}\left(\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}+\sinh^{-1}\frac{x}{a}\right)+c.$$

$$\therefore \int\sqrt{a^2+x^2}\,dx=\left(\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}\right)+c$$

Example (1-15): Calculate the integral: $\int \sqrt{4+x^2}dx$

Solution:

Applying rule (2), we find

that $\int \sqrt{4+x^2}dx = \dfrac{x\sqrt{x^2+4}}{2} + 2\sinh^{-1}\dfrac{x}{2} + c$

Rule (3): Prove that:

$$\boxed{\int \sqrt{x^2-a^2}dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c}$$

To find the integral $\int \sqrt{x^2-a^2}dx$ ***, we use the following* substitution:**

$$x = a\cosh\theta \Rightarrow dx = a\sinh\theta d\theta, \frac{x}{a} = \cosh\theta$$

$$and\ \theta = \cosh^{-1}\frac{x}{a}.$$

$$\therefore \int \sqrt{x^2-a^2}dx = \int \sqrt{a^2\cosh^2\theta - a^2}\,(a\sinh\theta)d\theta =$$

$$= \int \sqrt{a^2\underbrace{(\cosh^2\theta - 1)}_{\sinh^2\theta}}\,(a\cosh\theta)d\theta = \int a^2\sinh^2\theta d\theta,$$

$$\left.\begin{aligned} &\because \cosh 2\theta = \cosh^2\theta + \sinh^2\theta \\ &\because \cosh^2\theta - \sinh^2\theta = 1 \end{aligned}\right\} \Rightarrow \frac{\cosh 2\theta - 1}{2} = \sinh^2\theta.$$

$$\therefore I = \int \sqrt{x^2-a^2}dx = \int a^2\sinh^2\theta d\theta = a^2\int\left(\frac{\cosh 2\theta - 1}{2}\right)d\theta =$$

$$I=\frac{a^2}{2}\left(\frac{\sinh 2\theta}{2}-\theta\right)+c=\frac{a^2}{2}\left(\sinh\theta\cosh\theta-\theta\right)+c=$$

$$=\frac{a^2}{2}\left(\cosh\theta\sqrt{\sinh^2\theta}-\theta\right)+c=\frac{a^2}{2}\left(\cosh\theta\sqrt{\cosh^2\theta-1}-\theta\right)+c=$$

$$=\frac{a^2}{2}\left(\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2-1}-\cosh^{-1}\frac{x}{a}\right)+c=$$

$$=\frac{a^2}{2}\left(\frac{x}{a}\sqrt{\left(\frac{x^2-a^2}{a^2}\right)}-\cosh^{-1}\frac{x}{a}\right)+c.$$

$$\therefore\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cos h^{-1}\frac{x}{a}+c.$$

Example (1-16): Calculate the f integral: $\int\sqrt{x^2-4}dx$

Solution:

Applying rule (2) we find that

$$\int\sqrt{x^2-4}dx=\frac{x\sqrt{x^2-4}}{2}-2coh\,\mathrm{h}^{-1}\frac{x}{2}+c$$

Example (1-17): Calculate the following integrals:

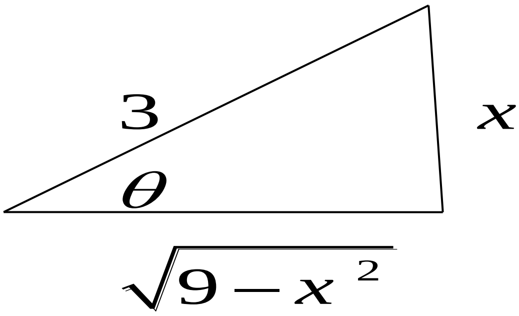
$$(i)\int\frac{dx}{x^2\sqrt{9-x^2}},\qquad (ii)\int\frac{\sqrt{x^2-9}}{x}dx\quad,\quad (iii)\int\frac{dx}{x\left(x^2+4\right)^{3/2}}.$$

Solution:

$$(i)\int\frac{dx}{x^2\sqrt{9-x^2}},$$

$$put\ \ x=3\sin\theta\Rightarrow dx=3\cos\theta d\theta,\sqrt{9-x^2}=3\cos\theta$$

$$x^2=9\sin^2\theta,$$



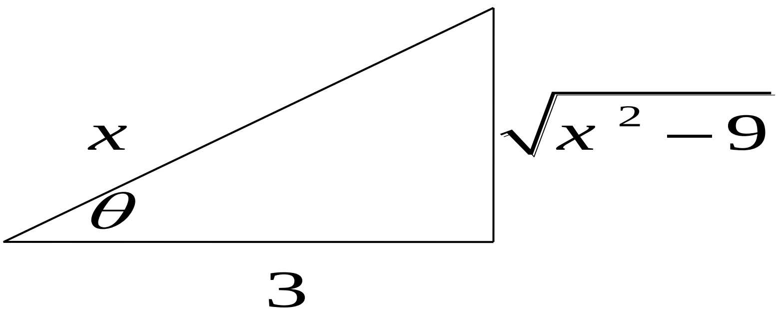
$$\therefore \int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta\, d\theta}{\left(9\sin^2\theta\right)\left(3\cos\theta\right)} =$$

$$= \int \frac{d\theta}{9\sin^2\theta} = \frac{1}{9}\int \sec^2\theta\, d\theta = \frac{-1}{9}\cot\theta + c =$$

$$= \frac{-1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + c$$

$(ii)\int \frac{\sqrt{x^2-9}}{x}dx$, *put* $x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\, d\theta,$

$\sqrt{9-x^2} = 3\tan\theta$ *and* $\theta = \sec^{-1}\dfrac{x}{3}.$



$$(ii)\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{3\tan\theta}{3\sec\theta}3\sec\theta\tan\theta\, d\theta =$$

$$= 3\int \tan^2\theta\, d\theta = 3\int \left(\sec^2\theta - 1\right)d\theta = 3\tan\theta - 3\theta + c =$$

$$= 3\sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + c.$$

$$(iii)\int \frac{dx}{x\left(x^{2}+4\right)^{3/2}}.$$

$$put\ \ x=2\tan\theta ,\Rightarrow dx=2\sec^{2}\theta\ d\theta .$$

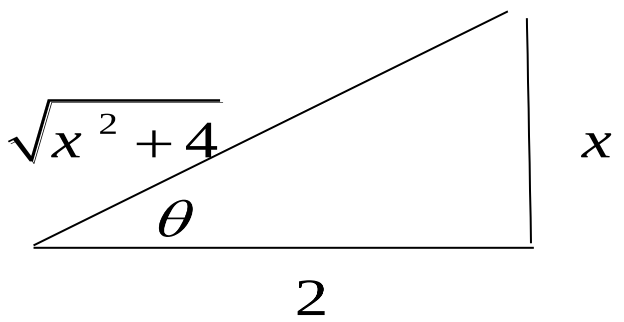
$$\therefore \int \frac{dx}{x\left(x^{2}+4\right)^{3/2}}=\int \frac{2\sec^{2}\theta\ d\theta}{2\tan\theta\underbrace{\left(\underbrace{4\tan^{2}\theta+4}_{4\sec^{2}\theta}\right)^{3/2}}_{8\sec^{3}\theta}}=$$

$$=\int \frac{\sec^{2}\theta\ d\theta}{8\tan\theta\sec^{3}\theta}=\int \frac{d\theta}{8\tan\theta\sec\theta}=\frac{1}{8}\int \frac{\cos^{2}\theta d\theta}{\sin\theta}=$$

$$=\frac{1}{8}\int \frac{1-\sin^{2}\theta d\theta}{\sin\theta}=\frac{1}{8}\int\left(\operatorname{cosc}\theta-\sin\theta\right)d\theta=$$

$$=\frac{1}{8}\left[-\left|Ln\left(\cot\theta+\operatorname{cosc}\theta\right)\right|+\cos\theta\right]+c=$$

$$=-\frac{1}{8}\left|Ln\left(\cot\theta+\operatorname{cosc}\theta\right)\right|+\frac{1}{8}\cos\theta+c.$$



من الشكل السابق نجد أن:

$$I=\int\frac{dx}{x\left(x^{2}+4\right)^{3/2}}=-\frac{1}{8}\left|Ln\left(\cot\theta+\csc\theta\right)\right|+\frac{1}{8}\cos\theta+c=$$

$$=-\frac{1}{8}\left|Ln\left(\frac{2}{x}+\frac{1}{x}\sqrt{x^{2}+4}\right)\right|+\frac{1}{8}\frac{2}{\sqrt{x^{2}+4}}+c=$$

$$=\frac{1}{8}\left|Ln\left(\frac{2\sqrt{x^{2}+4}}{x}\right)\right|+\frac{1}{4}\frac{1}{\sqrt{x^{2}+4}}+c.$$

***The integrals in the form:**

$$(i\ )I=\int\frac{dx}{ax^{2}+bx+c},(ii\ )I=\int\sqrt{ax^{2}+bx+c}\ dx,(iii\ )I=\int\frac{dx}{\sqrt{ax^{2}+bx+c}}.$$

In such cases, we complete the square for $ax^{2}+bx+c$

$$ax^{2}+bx+c=a\left(x^{2}+\frac{b}{a}x+\frac{c}{a}\right)=a\left(\left(x+\frac{b}{2a}\right)^{2}+\frac{c}{a}-\frac{b^{2}}{4a^{2}}\right)=$$

$$=a\left(\left(x+\frac{b}{2a}\right)^{2}+\underbrace{\frac{4ac-b^{2}}{4a^{2}}}_{k^{2}}\right)=a\left(\left(x+\frac{b}{2a}\right)^{2}\pm k^{2}\right).$$

$$put\ t=x+\frac{b}{2a}\Rightarrow dt=dx$$

$$\therefore I=\int\frac{dx}{ax^{2}+bx+c}=\frac{1}{a}\int\frac{dx}{t^{2}\pm k^{2}}$$

Thus, in both cases $(ii),(iii)$***, from the table of integrals, we can obtain the standard form corresponding to this integral.***

Example (1-18): Calculate the following integrals:

$$(i)\,I=\int\frac{dx}{2x^2+8x+20}\ ,\ (ii)\,I=\int\sqrt{x^2+x+1}\,dx\ ,(iii)\,I=\int\frac{dx}{x^2+2x+2}\ .$$

Solution:

$$(i)\,I=\int\frac{dx}{2x^2+8x+20}$$

$$\because\ 2x^2+8x+20=2\left(x^2+4x+10\right)=2\left((x+2)^2+6\right)$$

$$\therefore I=\int\frac{dx}{2x^2+8x+20}=\frac{1}{2}\int\frac{dx}{(x+2)^2+6}$$

$$put\ \ t=x+2\Rightarrow dt=dx$$

$$\therefore I=\int\frac{dx}{2x^2+8x+20}=\frac{1}{2}\int\frac{dx}{(x+2)^2+6}=$$

$$=\frac{1}{2}\int\frac{dt}{t^2+6}=\frac{1}{2}\frac{1}{\sqrt{6}}\tan^{-1}\frac{t}{\sqrt{6}}+c=\frac{1}{2\sqrt{6}}\tan^{-1}\frac{(x+2)}{\sqrt{6}}+c.$$

$$(ii)\because I=\int\sqrt{x^2+x+1}\,dx\ ,$$

$$\because x^2+x+1=\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)\Rightarrow I=\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$\because\int\sqrt{a^2+x^2}\,dx=\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+c$$

$$\therefore I=\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx=$$

$$=\left(x+\frac{1}{2}\right)\frac{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}{2}+\frac{3}{8}\sinh^{-1}\frac{\left(x+\frac{1}{2}\right)}{\sqrt{\frac{3}{4}}}+c.$$

$$(iii)\ I=\int\frac{dx}{\sqrt{x^2+2x+2}}$$

$$\because x^2+2x+2=\left((x+1)^2+1\right)$$

$$\therefore I=\int\frac{dx}{\sqrt{x^2+2x+2}}=\int\frac{dx}{\sqrt{(x+1)^2+1}}=\sinh^{-1}(x+1)+c.$$

Exercises (1-2)

1: Put true or false for each of the following statements:

$$(1)\int f(x)^{n+1} f'(x)\,dx = \frac{f(x)^{n+1}}{n+1} + c\,;\, n \neq -1. \qquad (\quad)$$

$$(2)\int a^x\,dx = \frac{a^x}{Ln\,a}. \qquad (\quad)$$

$$(3)\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a}\sin^{-1}\frac{x}{a} + c. \qquad (\quad)$$

$$(4)\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x) + c. \qquad (\quad)$$

$$(5)\int (x^2 + 2x)^2\,dx = \frac{x^5}{5} + x^4 + \frac{4x^3}{3} + c. \qquad (\quad)$$

$$(6)\int \frac{\cos x}{\sin^2 x}\,dx = -\csc^2 x + c. \qquad (\quad)$$

$$(7)\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + c \qquad (\quad)$$

$$(8)\int \cos^2 x\,dx = \frac{1}{2}(x - \sin x \cos x) + c. \qquad (\quad)$$

$$(9)\int \frac{3dx}{x\sqrt{16 - x^4}} = \frac{3}{4}\operatorname{sech}^{-1}\frac{x}{a} + c \qquad (\quad)$$

$$(10)\int \tan^2 x\,dx = \tan x + x + c \qquad (\quad)$$

2- Choose the correct answer

$(1)\int \cot x dx =$

$(A) Ln|\cos x|+c \quad (B) Ln|\sin x| (C) Ln|\sin x|+c \quad (D) Ln|\tan x|+c.$

$(2)\int x^3 \sin x^4 dx =$

$(A)\frac{1}{4}\cos x^4+c. (B)\frac{1}{2}\cos x^4+c. (C) 4\cos x^4+c. (D)\frac{1}{4}\cos x^4.$

$(3)\int \frac{dx}{xLnx} =, x>0,$

$(A) Ln|Ln2x| \ (B) Ln|Ln2x|+c \ (C) Ln|Lnx|. \ (D) Ln|Lnx|+c.$

$(4)\int \sin^2 x \cos x dx =$

$(A)\frac{\sin^2 x}{2}+c. (B)\frac{\sin^3 x}{3}+c. (C)\frac{1}{4}\tan^{-1} 4x+c. (D)\frac{\cos^3 x}{3}+c.$

$(5)\int \tan x dx =$

$(A)-Ln|\sin x|+c. (B) Ln|\cos x|+c. (C)-Ln|\cos x|+c. (D)-Ln|\sin x|+c.$

$(6)\int \sqrt{1+\tan^2 x}\, dx =$

$(A) Ln|\sec x+\tan x|+c. (B) Ln|\tan x|+c. (C)-Ln|\tan x|+c. (D)-Ln|\sin x|+c$

$(7)\int \frac{x^2+2x+2}{x^2+2} dx =$

$(A) x^2-Ln|x+2|+c. (B) Ln|x+2|+c. (C) x+Ln|x^2-2|+c \ (D) x+Ln|x^2+2|+c$

$(8)\int a^{f(x)} dx =$

$(A)\frac{a^{f(x)}}{f'(x)Ln a}+c \ (B)\frac{-a^{f(x)}}{f'(x)Ln a}. (C)\frac{a^{f(x)}}{Ln a} (D)\frac{a^{f(x)}}{f(x)Ln a}+c$

$(9)(1)\int x\, e^x dx =$

$(A) xe^x-e^x+c \ (B) xe^x+e^x+c. (C) xe^x+c \ (D)-xe^x+c$

$(10)\int \sec^2 x\sqrt{\tan x}\,dx =$

$(A)\frac{3}{2}\sqrt{\tan^3 x}+c.\ (B)\frac{2}{3}\sqrt{\tan^3 x}+c.(C)\frac{3}{2}\sqrt{\sec^3 x}+c.(D)\frac{2}{3}\sqrt{\sec^3 x}+c..$

$(11)\int e^{\cos x}dx =$

$(A)\sin x e^{\cos x}+c.\ (B)-\sin x e^{\cos x}+c.(C)\cos x e^{\cos x}+c.(D)-\cos x e^{\cos x}+c.$

$(12)\int \sqrt[3]{x^5-x^3}\,dx =$

$(A)\frac{8}{3}\sqrt[3]{x^2-1}+c.\ (B)\frac{8}{3}\sqrt[3]{x^5-x^3}+c.(C)\frac{3}{8}\sqrt[3]{x^2-1}+c.(D)\frac{3}{8}\sqrt[3]{x^5-x^3}+c$

$(13)\int 2\left(\tan x+\tan^3 x\right)dx =$

$(A)\tan^2 x+c.\ (B)\sec^2 x+c.(C)\sec^3 x+c.(D)\tan^3 x+c$

$(9)(1)\int x\ e^x dx =$

$(A)xe^x-e^x+c\ (B)xe^x+e^x+c.(C)xe^x+c\ (D)-xe^x+c$

$(10)\int e^{\cos x}dx =$

$(A)\sin x e^{\cos x}+c.\ (B)-\sin x e^{\cos x}+c.(C)\cos x e^{\cos x}+c.(D)-\cos x e^{\cos x}+c.$

$(11)\int \frac{Lnx^2}{Lnx}dx =$

$(A)Lnx+c.\ (B)\frac{x^2}{2}+c.(C)2x+c.(D)Otherwise.$

$(12)\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}dx =$

$(A)\frac{3}{2}\sqrt[3]{(\sin x-\cos x)^2}+c\ (B)\frac{3}{2}\sqrt[3]{(\sin x-\cos x)^3}+c$

$(C)\frac{3}{2}\sqrt[3]{(\sin x+\cos x)^2}+c\ .(D)Otherwise.+c$

•

$$(13)\int \frac{(2Lnx+3)^3}{x}dx =$$

$(A)\frac{1}{6}(2Lnx+3)^4+c$ $(B)\frac{1}{2}(2Lnx+3)^4+c$ $(C)\frac{1}{8}(2Lnx+3)^4+c$ $(D)Otherwise$

3-Calculate the following integrals using the corresponding substitution for each of them:

$$(1)\int 2x\left(x^2+2\right)^{24}dx\,;\; u=x^2+2,$$

$$(2)\int \cos^2 x\,\sin x dx\;\;;\; u=\cos x\,,$$

$$(3)\int \frac{1}{\sqrt{x}}\sin\frac{1}{\sqrt{x}}dx\;\;;\; u=\frac{1}{\sqrt{x}},$$

$$(4)\int \frac{3x}{\sqrt{4x^2+5}}dx\;\;;\; u=4x^2+5,$$

$$(5)\int x\sqrt{2x^2+1}dx\;\;;\; u=2x^2+1,$$

$$(6)\int \sqrt{\sin \pi x}\;\cos \pi x\,dx\;\;;\; u=\sin \pi x\,,$$

$$(7)\int (2x+7)\left(x^2+7x+3\right)^8 dx\;\;;\; u=x^2+7x+3,$$

$$(8)\int \cos x\left(\sin x+1\right)^8 dx\;\;;\; u=\sin x+1$$

$$(9)\int x\sec x^2\,dx\;\;;\; u=x^2,$$

$$(10)\int x^2\sqrt{x+1}dx\;\;;\; u=x+1,$$

4- Calculate the following integrals

$(1)\int \frac{3x}{\sqrt{9x^2+25}}dx$ $\quad$ $(2)\int \frac{e^x}{16-e^{2x}}dx$.

$(3)\int \frac{1}{5-4x^2}dx$ $\quad$ $(4)\int \frac{3x}{1+9x^2}dx$

$(5)\int x^2 \sin\left(x^3+1\right)dx$ $\quad$ $(6)\int \cos 3x \sqrt[5]{\sin 3x}\, dx$

$(7)\int \frac{1+e^x}{e^{2x}}dx$ $\quad$ $(8)\int \frac{e^{\cos x}}{\operatorname{cosec} x}dx$

$(9)\int e^{3x} \sin\left(e^{3x}+1\right)dx$ $\quad$ $(10)\int \frac{3x}{x\left(1+\sqrt{x}\right)}dx$

$(11)\int \frac{3x}{x\sqrt{16-x^4}}dx$ $\quad$ $(12)\int e^{-5x} \sin e^{-5x} dx$

$(13)\int e^{-3x} \cos e^{-3x} dx$

Chapter(2)
Methods of Integration

(2-1) Integrating by parts

Theory (2-1)

If two functions u, v ***are differentiable and each function*** u', v' ***is continuous, then:***

$$\boxed{\int uv'dx = uv - \int vu'dx \quad or \int udv = uv - vdu.}$$

Proof:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \Rightarrow (uv)' = u \bullet v' + u' \bullet v$$

$$\therefore uv = \int u \bullet v'dx + \int u' \bullet v\,dx$$

$$put\; dv = v'dx \quad , \quad du = u'dx$$

$$\therefore \int udv = uv - \int vdu$$

Example (2-1): Calculate the following integrals

$(1)\int xe^x dx,$ $\qquad (2)\int x^3e^x dx,$

$(3)\int x\cos x dx,$ $\qquad (4)\int Ln\, x dx.$

Solution:

$$(1)\int \underset{u}{x}\ \underset{dv}{e^x dx} = xe^x - \int e^x dx = xe^x - e^x + c$$

$$(2)\int \underset{u}{x^3}\,\underset{dv}{e^x dx} = x^3 e^x - 3\left[\int \underset{u}{x^2}\,\underset{dv}{e^x dx}\right] =$$

$$= x^3 e^x - 3\left[x^2 e^x - \int \underset{u}{2x}\,\underset{dv}{e^x dx}\right] =$$

$$= x^3 e^x - 3\left[x^2 e^x - 2\left(xe^x - \int \underset{dv}{e^x dx}\right)\right] =$$

$$= x^3 e^x - 3\left[x^2 e^x - 2\left(xe^x - e^x\right)\right] + c =$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c =$$

$$= e^x\left(x^3 - 3x^2 + 6x - 6\right) + c.$$

$$(3)\int \underset{u}{x}\,\underbrace{\cos x dx}_{dv} = x\sin x - \int \sin x dx = x\sin x + \cos x + c.$$

$$(4)\int \underbrace{\underset{u}{Ln\,x}\,\underset{dv}{dx}} = xLn\,x - \int x\left(\frac{1}{x}\right)dx = xLn\,x - x + c.$$

Example (2-2): Calculate the following integrals

$$(1)\int x\tan^{-1} x dx,\quad (2)\int x^n Ln\,x dx,\quad (3)\int e^x \sin x dx.$$

Solution:

$$(1)\int x\ \tan^{-1} x dx = \int \underbrace{\left(\tan^{-1} x\right)}_{u} \underbrace{x \quad dx}_{dv} =$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx =$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left(\int \frac{1+x^2-1}{1+x^2}dx\right) =$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left(\int 1 dx - \int \frac{1}{1+x^2}dx\right) =$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + c.$$

$$(2)\int \underbrace{x^n}_{dv} \underbrace{Ln\, x}_{u}\, dx = \frac{x^{n+1}}{n+1}Ln\, x - \int \frac{x^{n+1}}{n+1}.\frac{1}{x}dx =$$

$$= \frac{x^{n+1}}{n+1}Ln\, x - \frac{x^{n+1}}{(n+1)^2} + c.$$

$$(3) I = \int \underbrace{\sin x}_{u}\, \underbrace{e^x dx}_{dv} = e^x \sin x - \left[\int \underbrace{\cos x}_{u}\ \underbrace{e^x dx}_{dv}\right] =$$

$$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x dx\right] =$$

$$= e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x dx}_{I}.$$

$$2I = e^x \sin x - e^x \cos x + c \Rightarrow$$

$$I = \frac{1}{2}\left(e^x \sin x - e^x \cos x + c\right).$$

Example (2-3):Prove that:

$$I=\int e^{ax}\sin(bx)\,dx=\frac{e^{ax}}{a^2+b^2}\left[a\sin bx-b\cos bx\right]+c.$$

Solution:

$$(1)\ I=\int e^{ax}\sin(bx)\,dx=$$

$$=\int \underset{u}{\sin bx}\ \underset{dv}{e^{ax}dx}=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a}\left[\int \underbrace{\cos bx}_{u}\ \underbrace{e^{ax}dx}_{dv}\right]=$$

$$=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a}\left[\frac{1}{a}e^{ax}\cos bx+\frac{b}{a}\int e^{ax}\sin bx\,dx\right]=$$

$$=\frac{1}{a}e^{ax}\sin x-\frac{b}{a^2}e^{ax}\cos bx-\left(\frac{b}{a}\right)^2\underbrace{\int e^{ax}\sin x\,dx}_{I}.$$

$$\left[1+\left(\frac{b}{a}\right)^2\right]I=\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx+c\Rightarrow$$

$$\left(\frac{a^2+b^2}{a^2}\right)I=\left[\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx\right]+c=$$

$$I=\frac{a^2}{a^2+b^2}\left[\frac{1}{a}e^{ax}\sin bx-\frac{b}{a^2}e^{ax}\cos bx\right]+c$$

$$I=\frac{1}{a^2+b^2}\left[ae^{ax}\sin bx-be^{ax}\cos bx\right]+c=$$

$$I=\frac{e^{ax}}{a^2+b^2}\left[a\sin bx-b\cos bx\right]+c.$$

Example (2-4): Calculate the following integrals

$$(1)\int \sinh^{-1} x dx\ , \qquad (2)\int \sec^3 x dx\ ,$$

$$(3)\int \sin^{-1} x dx\ . \qquad (4)\int \frac{xe^x}{(x+1)^2} dx$$

Solution:

$$(1)\int \underbrace{\sinh^{-1} x}_{u} \underbrace{dx}_{dv} = x \sinh^{-1} x - \int \frac{x}{\sqrt{1+x^2}} dx =$$

$$= x \sinh^{-1} x - \frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}} dx = x \sinh^{-1} x - \frac{1}{2}\int \left(1+x^2\right)^{\frac{-1}{2}} 2x dx =$$

$$= x \sinh^{-1} x - \sqrt{1+x^2} + c$$

$$(2) I = \int \sec^3 x dx = \int \underbrace{\sec x}_{u} \underbrace{\sec^2 x dx}_{dv} =$$

$$= \sec x \tan x - \int \sec x \underbrace{\tan^2 x}_{\sec^2 x - 1} dx =$$

$$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx =$$

$$= \sec x \tan x - \int \left(\sec^3 x - \sec x\right) dx =$$

$$\sec x \tan x - \underbrace{\int \sec^3 x dx}_{I} + \int \sec x dx =$$

$$= \sec x \tan x - \underbrace{\int \sec^3 x dx}_{I} + \ln\left|\sec x + \tan x\right| + c$$

$$2\int \sec^3 x dx = \sec x \tan x + \ln\left|\sec x + \tan x\right| + c =$$

$$\int \sec^3 x dx = \frac{1}{2}\left(\sec x \tan x + \ln\left|\sec x + \tan x\right|\right) + c$$

Exercises (2-1)

Calculate the following integrals

$(1)\int x^{2} Logx dx$,

$(2)\int x^{3} e^{x} dx$,

$(3)\int \sin^{-1} x dx$.

$(4)\int e^{x} \cos x dx$

$(5)\int x \sin(2x+3) dx$.

$(6)\int e^{2x} \sin 5x dx$

$(7)\int x^{2} \sin x dx$.

$(8)\int x \sinh x dx$

$(11)\int x^{3} e^{-x} dx$.

$(10)\int x e^{-x} dx$

$(11)\int e^{ax} \cos bx dx$.

$(12)\int x^{n} \sin^{-1} x dx$

(2-3) Integration by successive reduction:

Integration by successive reduction is used to calculate integrals for functions that depend on constant quantities $m, n, \cdots$ ***, but it is usually used to deduce these relationships. Integration by division***

Example (2-5) Prove that:

$$1\ \ I_n = \int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}.$$

$$2\ \ \int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}\ \ , n \in \mathbb{N}, n > 2.$$

From there, find each of I_3, I_4 **.**

Solution:

$$1\ \ I_n = \int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx =$$

$$= \int \underbrace{\sin^{n-1} x}_{u}\, \underbrace{d\ \ -\cos x}_{dv}\ .$$

$$put\ u = \sin^{n-1} x \Rightarrow du = \ n-1\ \ \sin^{n-2} x\ \ \cos x,$$

$$dv = d\ \ -\cos x\ \ \Rightarrow v = -\cos x$$

$$\therefore I_n = \int \sin^{n-1} xd\ \ -\cos x\ \ = -\cos x \sin^{n-1} x +$$

$$\int\ n-1\ \ \sin^{n-2} x\ \ \cos^2 xdx =$$

$$= -\cos x \sin^{n-1} x +$$

$$+\ n-1\ \int\ \sin^{n-2} x\ \ 1 - \sin^2 x\ dx =$$

$$= -\cos x \sin^{n-1} x +$$

$$+(n-1)\left[\int \sin^{n-2} xdx - \int \sin^n xdx\right] =$$

$$= -\cos x \sin^{n-1} x +$$

$$+ \ n-1 \ \underbrace{\int \sin^{n-2} x dx}_{I_{n-2}} - \ n-1 \ \underbrace{\int \sin^{n} x dx}_{I_n} =$$

$$= -\cos x \sin^{n-1} x + \ n-1 \ I_{n-2} - \ n-1 \ I_n .$$

$$\therefore nI_n = -\cos x \sin^{n-1} x + \ n-1 \ I_{n-2}$$

$$\therefore I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2} .$$

$$\therefore I_n = \int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2} .$$

$$I_n = \int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2} .$$

$put \, n = 3 \Rightarrow$

$$I_3 = \int \sin^3 x \, dx = -\frac{1}{3} \cos x \sin^2 x + \frac{2}{3} I_1 .$$

$$\because I_1 = \int \sin x dx = -\cos x + c$$

$$I_3 = \int \sin^3 x \, dx = -\frac{1}{3} \cos x \sin^2 x - \frac{2}{3} \cos x + c .$$

$put \, n = 4 \Rightarrow$

$$I_4 = \int \sin^4 x \, dx = -\frac{1}{4} \cos x \sin^3 x + \frac{3}{4} I_2 .$$

$$\because I_2 = \int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2} I_0$$

$$I_0 = \int dx = x + c$$

$$I_4 = \int \sin^4 x\,dx =$$

$$= -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\left(-\frac{1}{2}\cos x \sin x + \frac{x}{2}\right) + c.$$

$$2 \quad \int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx =$$

$$= \int \underbrace{\cos^{n-1} x}_{u}\, \underbrace{d\ \sin x}_{dv} .$$

$$put\ u = \cos^{n-1} x \Rightarrow du = \ n-1 \quad \cos^{n-2} x \quad -\sin x \ ,$$

$$dv = d\ \sin x \ \Rightarrow v = \sin x$$

$$\therefore I_n = \int \underbrace{\cos^{n-1} x}_{u}\, \underbrace{d\ \sin x}_{dv} = \sin x \cos^{n-1} x$$

$$+\int (n-1)\ \cos^{n-2} x\ \sin^2 x dx =$$

$$= \sin x \cos^{n-1} x + +(n-1)\int \cos^{n-2} x(1-\cos^2 x)dx =$$

$$= \sin x \cos^{n-1} x +$$

$$+(n-1)\underbrace{\int \cos^{n-2} x dx}_{I_{n-2}} + (n-1)\underbrace{\int \cos^n x dx}_{I_n} =$$

$$= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore nI_n = \sin x \cos^{n-1} x + (n-1) I_{n-2}.$$

$$\therefore I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{(n-1)}{n} I_{n-2}.,n \in \mathbb{N}, n > 2$$

$$\therefore I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}., n \in \mathbb{N}, n > 2$$

$$\therefore I_3 = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3} I_1., I_4 = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4} I_2.,$$

Example (2-6) Prove that:

$$1\ I_n = \int \tan^n x\, dx = -\frac{1}{n-1}\tan^{n-1} x - I_{n-2}.$$

$$2\ \int \cot^n x\, dx = \frac{1}{n-1}\cot^{n-1} x + I_{n-2}\quad , n \in \mathbb{N}, n > 2.$$

From there, find each of $I_4 = \int \tan^4 x dx,\ I_5 = \int \cot^5 x dx,$.

Solution:

$$1\ I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \tan^2 x\, dx = . \qquad .$$

$$= \int \tan^{n-2} x\ \ \sec^2 x - 1\ \ dx =$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx =$$

$$= \int \tan^{n-2} x\, d\ \ \tan x\ \ - \underbrace{\int \tan^{n-2} x\, dx}_{I_{n-2}}\ =$$

$$= \frac{1}{n-1}\tan^{n-1} x - I_{n-2}.$$

$$\therefore I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$$

$$\because I_n = \int \tan^n x dx = \frac{1}{n-1}\tan^{n-1} x - I_{n-2} \Rightarrow$$

$$\therefore I_4 = \int \tan^n x dx = \frac{1}{3}\tan^3 x - I_2,$$

$$I_2 = \tan x - I_0,\; I_0 = \int dx = x + c.$$

$$\therefore I_4 = \int \tan^n x dx =$$

$$= \frac{1}{3}\tan^3 x - \quad \tan x - x \quad + c =$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + c$$

In the same way we can prove that:

$$2 \int \cot^n x\, dx = \frac{1}{n-1}\cot^{n-1} x + I_{n-2} \quad , n \in \mathbb{N}, n > 2.$$

$$I_4 = \int \tan^4 x dx,\; I_5 = \int \cot^5 x dx,$$

Example (2-7) Prove that:

$$1\; I_n = \int \sec^n x\, dx = -\frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}.$$

$$2 \int \cos ec^n x\, dx = \frac{1}{n-1}\cos ec^{n-2} x + \frac{n-2}{n-1} I_{n-2}$$

$$; n \in \mathbb{N}, n \geq 2.$$

Solution

$$1\; I_n = \int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx =$$

$$= \int \underbrace{\sec^{n-2} x}_{u} \underbrace{d \quad \tan x \quad}_{dv} = .$$

$$\therefore I_n = \sec^{n-2} x \tan x - \quad n-2 \quad \int \sec^{n-3} x \sec x \tan^2 x\, dx =$$

$$\therefore I_n = \sec^{n-2} x \tan x - \ n-2 \ \int \sec^{n-3} x \sec x \tan^2 x \, dx =$$

$$= \sec^{n-2} x \tan x - \ n-2 \ \int \sec^{n-2} x \ \sec^2 x - 1 \ dx =$$

$$= \sec^{n-2} x \tan x - \ n-2 \ \underbrace{\int \sec^{n} x \, dx}_{I_n} + \ n-2 \ \underbrace{\int \sec^{n-2} x \, dx}_{I_{n-2}}$$

$$\therefore \ n-1 \ I_n = \sec^{n-2} x \tan x + \ n-2 \ I_{n-2}$$

$$I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}.$$

$$\therefore I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$$

In the same way we can prove that:

$$2 \ \int \cos ec^n x \, dx = \frac{1}{n-1} \cos ec^{n-2} x + \frac{n-2}{n-1} I_{n-2}. \ ; n \in \mathbb{N}, n \geq 2.$$

(2-4) Integration by Partial Fractions

Before we begin to study integration using molecular fractions, we must give an overview of what has been previously studied regarding molecular fractions:

(2-4-1) Partial fractures

Sometimes it is necessary to express a specific fraction in the form of a number of simpler fractions, called partial fractions. This process is of special importance when performing some differentiation and integration operations, for

example: $$\frac{x+11}{x^2+2x-3} = \frac{3}{x-1} - \frac{2}{x+3}$$

The quantities $\frac{2}{x+3}, \frac{3}{x-1}$ are called partial fractions of the original fraction, and there are rules that can be followed to obtain the partial fractions of a particular fraction as long as the degree of the numerator is less than the degree of the denominator. However, if the degree of the numerator is equal to or greater than the degree of the denominator, the numerator must be divided by the denominator in ordinary ways to obtain a remainder that contains the degree of the numerator. Below the degree of status, the following rules apply to him:

1 – Every first-degree factor in the denominator in the form $a+bx$ has a corresponding partial fraction $\frac{A}{a+bx}$ as long as this factor is not repeated .

2- Every first-degree factor in the denominator is n repeated several times in the form

$ax+b^{\;n}$ corresponding to the sum of the partial fractions:

$$\frac{A_1}{ax+b}+\frac{A_2}{ax+b^{\;2}}+\frac{A_3}{ax+b^{\;3}}+\cdots+\frac{A_n}{ax+b^{\;n}}$$

3- Every factor of the second degree in the denominator in the form ax^2+bx+c that cannot be decomposed into two real radical factors of the first order is corresponding to a partial fractio $\frac{Ax+B}{ax^2+bx+c}$

4- Every second-order duplicate factor in the denominator cannot be decomposed into two real first-order factors of the form $ax^2 + bx + c\ ^2$ ***corresponding to it:***

$$\frac{A_1x + B_1}{ax^2 + bx + c\ ^2} + \frac{A_2x + B_2}{ax^2 + bx + c}$$

5-Every third-order factor in the denominator in the form $ax^3 + bx^2 + cx + d$ ***that cannot be decomposed into real first-order factors has a partial fraction*** :

$$\frac{Ax^2 + Bx + C}{ax^3 + bx^2 + cx + d}$$

And so on…

How to determine partial fraction constants:

First, you write an equation that equals the original fraction to the sum of its partial fractions, then multiply both sides of the equation by the denominator of the original fraction, so we obtain a new equation in which one side is the numerator of the original fraction and the other side contains all the imposed constants that can be determined by following one of the following two methods:

(***1) Since the equation is true for all values*** x ، ***, an appropriate value for the variable can be chosen to obtain a number of equations that can later be solved to obtain the values of these constants, as we will explain in the following examples.***

(2)Putting both sides of the equation in the form of a polynomial in powers x ،, then we obtain the values of the constants from equal power coefficients x ،,, starting with the coefficients of the highest power to facilitate calculations.

Example (2-8): Find: $\int \frac{x^3}{x^2-1}\,dx$

Solution:

In this example, it is clear that the degree of the numerator is greater than the degree of the denominator, so the numerator must be divided by the denominator using normal methods to obtain a remainder in which the degree of the numerator is less than the degree of the denominator, which is then analyzed into its partial fractions by following the previous rules

$$\cdot \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

To analyze a fraction $\frac{x}{x^2-1}$*, we find that the denominator can be decomposed into real, first-order, non-recurring factors, so we follow the first rule as follows:*

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \qquad (1)$$

Multiplying both sides by the denominator of the original fraction results in:

$$x = A(x+1) + B(x-1) \qquad (2)$$

$$x = Ax + A + Bx - B$$

$$\therefore x = (A+B)x + (A-B) \qquad (3)$$

To find the constants A, B we follow one of two methods:

(1) The method of substituting x appropriate values by putting $x=1$ in the equation $[2]$ we get:

$$1 = 2A \qquad \Rightarrow A = 1/2$$

Putting $x=-1$ in the equation $[2]$ we get:

$$-1 = -2B \qquad \Rightarrow B = 1/2$$

(2) The method of equating power coefficients x. From the equation $[3]$ by equating a coefficient x on both sides, it results that:

$$1 = A + B$$

Equating the absolute terms in the equation $[3]$ we get: $0 = A - B$

By solving these two equations, we get:

$$A = 1/2, \qquad B = 1/2$$

Substituting the values A, B into the equation $[1]$ we get:

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$$

$$\therefore I=\int \frac{x}{x^{2}-1}dx=\int x\,dx+\int \frac{1}{2(x-1)}dx+\int \frac{1}{2(x+1)}dx=$$

$$=\int x\,dx+\frac{1}{2}\int \frac{1}{x-1}dx+\frac{1}{2}\int \frac{1}{x+1}dx=$$

$$=\frac{x^{2}}{2}+\frac{1}{2}Ln|x-1|+\frac{1}{2}Ln|x+1|+c.$$

Example (2-9): Calculate the integral:

$$\int \frac{x^{2}+2}{(x-1)^{3}}dx$$

Solution:

In this example, the degree of the numerator is less than the degree of the denominator, and since the denominator contains a first-degree factor repeated three times, and by following the second rule,we get:

$$\frac{x^{2}+2}{(x-1)^{3}}=\frac{A}{(x-1)^{3}}+\frac{B}{(x-1)^{2}}+\frac{C}{x-1}$$

$$x^{2}+2=A+B(x-1)+C(x-1)^{2}$$

$$x=1: \qquad \Rightarrow A=3$$

$$x=0: \qquad \Rightarrow 2=A-B+C \Rightarrow B-C=1$$

$$x=2: \qquad \Rightarrow 6=3+B+C \Rightarrow B+C=3$$

We get:

$$A=3, \quad B=2, \quad C=1 \Rightarrow$$

$$\therefore \frac{x^{2}+2}{(x-1)^{3}}=\frac{3}{(x-1)^{3}}+\frac{2}{(x-1)^{2}}+\frac{1}{x-1}.$$

$$\therefore I = \int \frac{x^2+2}{(x-1)^3}dx = \int \frac{3}{(x-1)^3}dx + \int \frac{2}{(x-1)^2}dx + \int \frac{1}{x-1}dx =$$

$$= -\frac{3}{2}\frac{1}{(x-1)^2} - \frac{2}{x-1} + Ln|x-1| + c.$$

Example (2-10): Calculate the integral:

$$\int \frac{x^2+x+2}{(x+1)(x-1)^2}\, dx$$

Solution:

$$\because \frac{x^2+x+2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

$$x^2+x+2 = A(x-1)^2 + B(x+1) + C(x+1)(x-1)$$

$$x = 1: \quad \Rightarrow 4 = 2B \quad \Rightarrow B = 2$$

$$x = -1: \quad \Rightarrow 2 = 4A \quad \Rightarrow A = 1/2$$

$$x = 0: \quad \Rightarrow A + B - C \Rightarrow C = 1/2$$

$$\therefore \frac{x^2+x+2}{(x+1)(x-1)^2} = \frac{1}{2(x+1)} + \frac{2}{(x-1)^2} + \frac{1}{2(x-1)}.$$

$$\therefore I = \int \frac{x^2+x+2}{(x+1)(x-1)^2}dx = \int\left(\frac{1}{2(x+1)} + \frac{2}{(x-1)^2} + \frac{1}{2(x-1)}.\right)dx.$$

$$\therefore I = \int \frac{1}{2(x+1)}dx + \int \frac{2}{(x-1)^2}dx + \int \frac{1}{2(x-1)}dx =$$

$$\therefore I = \frac{1}{2}Ln|x+1| - \frac{2}{x-1} + \frac{1}{2}Ln|x-1| + c$$

Example (2-11): Calculate the integral:

$$\int \frac{x^2+2}{x^2-4}\,dx$$

Solution:

Since in the expression $\dfrac{x^2+2}{x^2-4}$ ***the degree of the numerator is equal to the degree of the denominator, therefore we perform the division process first:***

Example (2-18): Calculate the integral:

$$\int \frac{x^4+4x^3+11x^2+12x}{(x^2+2x+3)^2(x+1)}\,dx$$

Solution:

$$\frac{x^4+4x^3+11x^2+12x}{(x^2+2x+3)^2(x+1)}=\frac{Ax+B}{(x^2+2x+3)^2}+\frac{Cx+D}{x^2+2x+3}+\frac{E}{x+1}\Rightarrow$$

$$x^4+4x^3+11x^2+12x=$$

$$=(Ax+B)(x+1)+(Cx+D)(x^2+2x+3)(x+1)++E(x^2+2x+3)^2\Rightarrow$$

$$A=1,\ B=1,\ c=0, D=0, E=1$$

$$\therefore \frac{x^4+4x^3+11x^2+12x}{(x^2+2x+3)^2(x+1)}=\frac{x-1}{(x^2+2x+3)^2}+\frac{1}{x+1}\Rightarrow$$

$$\therefore \int\frac{x^4+4x^3+11x^2+12x}{(x^2+2x+3)^2(x+1)}dx=\int\frac{x+1}{(x^2+2x+3)^2}dx+\int\frac{1}{(x+1)}dx=$$

$$=\frac{1}{2}\int\frac{2x+2}{(x^2+2x+3)^2}dx+\int\frac{1}{(x+1)}dx=\frac{1}{2(x^2+2x+3)}+Ln(x+1)+c$$

Example (2-12): Calculate the integral:

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$$

Solution:

$$\because\ x^3+2x^2+x = x\left(x^2+2x+1\right) = x\left(x+1\right)^2$$

$$\therefore\ \frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{5x^2+20x+6}{x\left(x+1\right)^2} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{\left(x+1\right)^2} \Rightarrow$$

$$5x^2+20x+6 = A\left(x+1\right)^2 + Bx\left(x+1\right) + Cx$$

$$put\ x = 0,-1,1 \Rightarrow A=6,\ B=-1,\ c=9,$$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx = \int \frac{6}{x}\,dx - \int \frac{1}{x+1}\,dx + \int \frac{9}{\left(x+1\right)^2}\,dx =$$

$$= 6Ln\left|x\right| - Ln\left|x+1\right| - \frac{9}{x+1} + c = Ln\,\frac{x^6}{x+1} - \frac{9}{x+1} + c.$$

***** While performing integrations using partial fractions, we find some of the fractions in which the denominator is in the form ax^2+bx+c and cannot be analyzed. In such cases, the integration is found by completing the square as follows:***

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+c\right) = a\left(\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right).$$

Example (2-13): Calculate the integral: $\displaystyle\int \frac{dx}{x^2-4x+13}$

Solution:

$$\because\ x^2-4x+13 = \left(x-2\right)^2+9$$

$$\therefore\ \int \frac{dx}{x^2-4x+13} = \int \frac{dx}{\left(x-2\right)^2+9} = \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right)+c$$

Example (2-14): Calculate the integrals:

$$1\ \int \frac{dx}{4x^2+4x+2}, \qquad\qquad 2\ \int \frac{3x+5}{x^2+x+1}dx.$$

Solution:

$$1\ \int \frac{dx}{4x^2+4x+2},$$

$$\because\ 4x^2+4x+2=\frac{1}{4}\left(x^2+x+\frac{1}{2}\right)=\frac{1}{4}\left(\left(x+\frac{1}{2}\right)^2-\frac{1}{4}\right)$$

$$\therefore \int \frac{dx}{4x^2+4x+2}=\frac{1}{4}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}}.$$

$$put\ u=x+\frac{1}{2}\Rightarrow du=dx$$

$$\therefore \int \frac{dx}{4x^2+4x+2}=\frac{1}{4}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}}=\frac{1}{4}\int \frac{du}{u^2-\left(\frac{1}{2}\right)^2}=$$

$$=-\frac{2}{4}\tanh^{-1}2u=-\frac{1}{2}\tanh^{-1}2\left(x+\frac{1}{2}\right)+c.$$

$$(2)\int \frac{2x+1}{x^2-4x+13}dx.$$

$$\because\ x^2-4x+13=\left((x-2)^2+9\right)$$

$$\therefore \int \frac{2x+1}{x^2-4x+13}dx=\int \frac{2x+1}{(x-2)^2+9}dx$$

$$put\ u=x-2\Rightarrow du=dx, x=u+2$$

$$\therefore \int \frac{2x+1}{x^2-4x+13}dx = \int \frac{2x+1}{(x-2)^2+9}dx =$$

$$= \int \frac{2(u+2)+1}{u^2+3^2}du = \int \frac{2}{u^2+3^2}du + 5\int \frac{1}{u^2+3^2}du =$$

$$= Ln(u^2+9) + \frac{5}{3}\tan^{-1}\frac{u}{3} + c =$$

$$= Ln\left((x-2)^2+9\right) + \frac{5}{3}\tan^{-1}\frac{x-2}{3} + c =$$

$$= Ln(x^2-4x+13) + \frac{5}{3}\tan^{-1}\frac{x-2}{3} + c.$$

Example (2-15): Calculate the integrals:

$$(1)\ \int \frac{dx}{(e^x-1)^2}, \qquad (2)\ \int \frac{4e^x+6e^{-x}dx}{9e^x-4e^{-x}}.$$

Solution:

$$(1)\ \int \frac{dx}{(e^x-1)^2} = \int \frac{e^x dx}{e^x(e^x-1)^2}dx,. \quad put\, u = e^x \Rightarrow du = e^x dx.$$

$$\therefore \int \frac{dx}{(e^x-1)^2} = \int \frac{e^x dx}{e^x(e^x-1)^2}dx = \int \frac{du}{u(u-1)^2}dx$$

$$\because \frac{1}{u(u-1)^2} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{(u-1)^2} \Rightarrow$$

$$\therefore 1 = A(u-1)^2 + Bu(u-1) + Cu$$

$$put\, u = 1,0,1 \Rightarrow A = 1,\ B = -1,\ c = 1,$$

$$\therefore u(u-1)^2 = \frac{1}{u} - \frac{1}{u-1} + \frac{1}{(u-1)^2}$$

$$\therefore I=\int\frac{dx}{(e^{x}-1)^{2}}=\int\frac{du}{u(u-1)^{2}}=$$

$$=\int\frac{1}{u}du-\int\frac{1}{u-1}du+\int\frac{1}{(u-1)^{2}}du=$$

$$=Ln\,u-Ln(u-1)+\frac{1}{u-1}+c=Ln\,e^{x}-Ln(e^{x}-1)+\frac{1}{e^{x}-1}+c.$$

$$2\quad\int\frac{4e^{x}+6e^{-x}}{9e^{x}-4e^{-x}}dx=\int\frac{e^{x}(4e^{x}+6e^{-x})}{e^{x}(9e^{x}-4e^{-x})}dx=\int\frac{4e^{2x}+6}{9e^{2x}-4}dx.$$

$$put\,u=e^{x}\Rightarrow du=e^{x}dx$$

$$\therefore\int\frac{4e^{2x}+6}{9e^{2x}-4}dx=\int\left(\frac{4u^{2}+6}{9u^{2}-4}\right)\bullet\frac{du}{u}=\int\frac{4u^{2}+6}{u\left(9u^{2}-4\right)}du.$$

(2-5) Definite Integral

In this section we will establish two basic relationships between definite and indefinite Integrals that together constitute a result called the "Fundamental Theorem of Calculus ".One part of this theorem will relate the rectangle and antiderivative methods for calculating areas, and the second part will provide a powerful method for evaluating definite integrals using antiderivatives.

(2-5-1) The Fundamental theorem of calculus

As in earlier sections, let us begin by assuming that f is non negative and continuous on an interval [a, b], in which case the area A under the graph of f over the interval [a, b] is:

$$A=\int_{a}^{b}f(x)\,dx\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad(1)$$

represented by the definite integral.

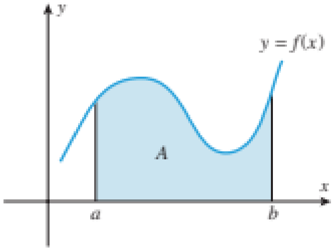


Figure (2-1)

represented by the definite integral

Recall that our discussion of the antiderivative method suggested that if A(x) is the area under the graph of f from a to b Figure (2 -1), then:

- •$A'\ x\ = f\ x$

-

•*A(a) = 0 The area under the curve from a to a is the area above the single point a, and hence is zero.*

•*A(b) = A The area under the curve from a to b is A.*

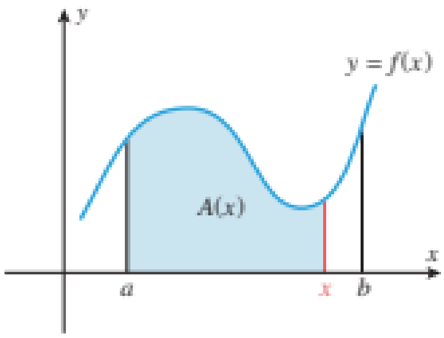


Figure (2- 2)

The formula •$A'\ x\ = f\ x$ *states that A(x) is an antiderivative of f (x), which implies that every other antiderivative of f (x) on [a, b] can be obtained by adding a constant A(x).Accordingly.*

Let $F(x)=A(x)+c$ ***be any antiderivative of f (x), and consider what happens when we subtract F (a) from F (b) :***

$$F(b)-F(a)=\left[A(b)+c\right]-\left[A(a)+c\right]=A(b)-A(a)=A-0=A.$$

Hence (1) can be expressed as:

$$A=\int_{a}^{b} f(x)\,dx=F(b)-F(a).$$

This equation states:

The definite integral can be evaluated by finding any antiderivative of the integrand and then subtracting the value of this antiderivative at the lower limit of integration from its value at the upper limit of integration.

Although our evidence for this result assumed that f is nonnegative on [a, b], this assump tion is notessential.

Theorem (2- 1): (The Fundamental Theorem of Calculus):

If f is continuous on [a, b], and F is any antiderivative of f on [a, b], then

$$\int_{a}^{b} f(x)\,dx=F(b)-F(a)\ \ldots\ldots\ldots\ldots\ldots\ (2)$$

Proof:

Let $x_1, x_2, x_3, \ldots, x_{n-1}$ ***be any points in [a, b] such that:*** $a<x_1, x_2, x_3, \ldots, x_{n-1}<b$

These values divide [a, b] into n subintervals

$$\left[a, x_1,\right], \left[x_1, x_2\right], \left[x_2, x_3\right], \ldots, \left[x_{n-1}, b\right] \ldots\ldots\ldots\ldots\ (3)$$

whose lengths, as usual, we denote by

$$\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n$$

By hypothesis, $F'(x) = f(x)$ ***for all x in [a, b], so*** $F'(x) = f(x)$ ***satisfies the hypotheses of the Mean-Value Theorem on each subinterval in (3). Hence, we can find points*** $x_1^*, x_2^*, x_3^*, \ldots, x_{n-1}^*$
in the respective subintervals in (3) such that:

$$F(x_1) - F(a) = F'(x_1^*)(x_1 - a) = f'(x_1^*)\Delta x_1,$$
$$F(x_2) - F(x_1) = F'(x_2^*)(x_2 - x_1) = f'(x_2^*)\Delta x_2$$
$$F(x_3) - F(x_2) = F'(x_3^*)(x_3 - x_2) = f'(x_3^*)\Delta x_3$$
.

.

$$F(b) - F(x_{n-1}) = F'(x_n^*)(b - x_{n-1}) = f'(x_n^*)\Delta x_n$$

Adding the preceding equations yields

$$F(b) - F(a) = \sum_{k=1}^{n} F(x_k^*)\Delta x_k \ldots\ldots\ldots\ldots\ldots (4)$$

Let us now increase n in such a way that max $\Delta x_k \to 0$***.***
Since f is assumed to be continuous, the right side of (4) approaches $\int_a^b f(x)\,dx$

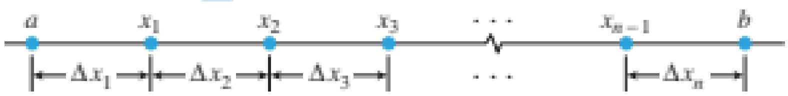


Figure (2- 3)

the left side of (4) is independent of n; that is, the left side of (4) remains constant as nincreases. Thus

$$F(b) - F(b) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$

It is standard to denote the difference F (b) − F (a)as:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a).$$

Example (2-17): Show that the area under the graph of the $y = 9 - x^2$ over the interval [0, 3] is 18 square units

Solution:

$$A = \int_0^3 9 - x^2 \; dx = \left(9x - \frac{1}{3}x^3\right)\Big|_0^3 = 27 - 9 - 0 = 18.$$

Example (2-18):Find the area under the curve y = cos x over the interval [0, π/2].

Solution:

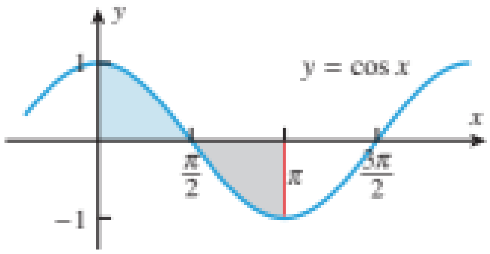


Figure (2- 4)

Since cos x ≥ 0 over the interval [0, π/2], the area A under the curve is

$$A = \int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} = \sin \pi/2 - \sin 0 = 1.$$

Example (2-19): Calculate

$$1 \; \int_4^9 x\sqrt{x} \; dx. \qquad 2 \; \int_0^{\pi/3} \sec^2 x dx. \qquad 3 \; \int_0^{\ln 3} e^{5x} dx.$$

Solution:

$$1 \; \int_4^9 x\sqrt{x} \; dx = \int_4^9 \left(x^{3/2}\right) dx == \left(\frac{2}{5}x^{5/2}\right)\Big|_4^9 =$$

$$= \left(\frac{2}{5}9^{5/2} - \frac{2}{5}4^{5/2}\right) = \frac{486}{5} - \frac{64}{5} = \frac{422}{5}.$$

2 $\displaystyle\int_0^{\pi/3} \sec^2 x dx = \left. \tan x \right|_0^{\pi/3} = \tan \pi/3 - \tan 0 = \sqrt{3}.$

3 $\displaystyle\int_0^{\ln 3} 5e^x dx = \left. 5e^x \right|_0^{\ln 3} = 5\left(e^{\ln 3} - e^0\right) = 5(3-1) = 10.$

Example (2-19): Find the total area between the curve $y = 1 - x^2$ and the x-axis over theinterval [0, 2].

***Solution:* area between the curve $y = 1 - x^2$ and the x-axis over the interval [0, 2] given as Figure (2-5)**

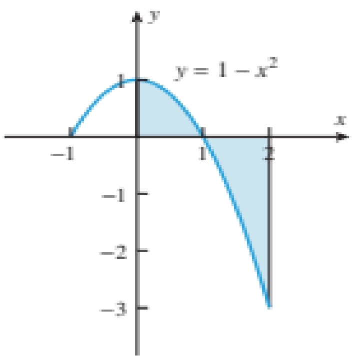


Figure (2-5)

$$Then: \ A = \int_0^2 \left(1 - x^2\right) dx = \left(x - \frac{x^3}{3}\right)\Bigg|_0^2 = 2 - \frac{8}{3} = -\frac{2}{3}.$$

***Definition(2-):* area formula If f and g are continuous functions on the interval $[a,b]$ and if f (x) ≥ g(x) for all x in $[a,b]$, then the area of the region bounded above by y = f (x), below by y = g(x), on the left by the line x = a, and on the right by the linex = b is:**

$$A = \int_a^b \left[f(x) - g(x)\right] dx.$$

Example (2-20): Find the area of the region enclosed by* $y^2 = x$ *and* $y = x - 2$*.

Solution:

To determine the appropriate boundaries of the region, we need to know where the curves $y^2 = x$ and $y = x - 2$ intersect.

$\because y^2 = x \quad y = x - 2 \Rightarrow y^2 = y + 2$

$\therefore y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0 \Rightarrow y = 2, y = -1.$

$at\ y = 2 \Rightarrow x = 4, at\ y = -1 \Rightarrow x = 1$

the curves $y^2 = x$ and $y = x - 2$ intersect at the points $(4,2)$, $(1,-1)$.

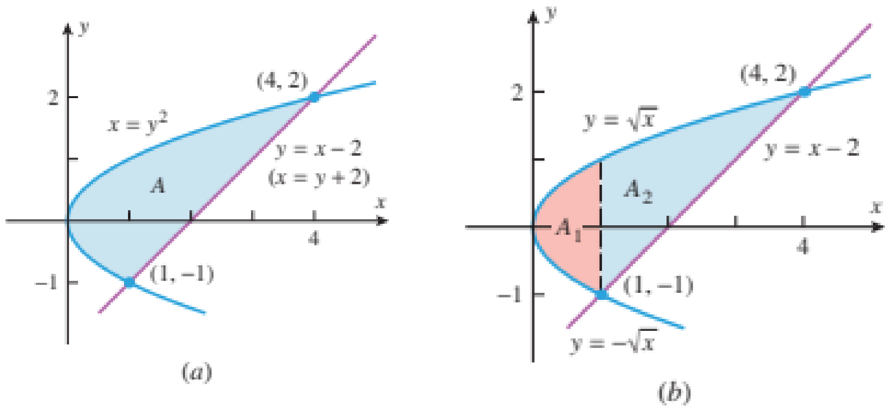


Figure (2-6)

$$y^2 = x \Rightarrow y = \pm\sqrt{x}$$

$$\therefore A_1 = \int_a^b \left[f(x) - g(x)\right] dx = \int_0^1 \left[\sqrt{x} - \left(-\sqrt{x}\right)\right] dx = 2\int_0^1 \sqrt{x}\, dx = \frac{4}{3}\left[x^{\frac{3}{2}}\right]_0^1 = \frac{4}{3}.$$

$$A_2 = \int_a^b \left[f(x) - g(x)\right] dx = \int_1^4 \left[\sqrt{x} - (x-2)\right] dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\right]_1^4 =$$

$$= \left[\left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right)\right] = \frac{19}{6}.$$

$$\therefore The\ area\ A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{27}{6} = \frac{9}{2}.$$

Example (2-21): Find the area of the region bounded above by $y = x + 6$, bounded below By $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution:

The region and a cross section are shown in Figure(2-). The cross section extends from $g(x) = x^2$ on the bottom to f (x) = x + 6 on the top. If the cross section is moved through the region, then its leftmost position will be x = 0 and and its rightmost position will be x = 2

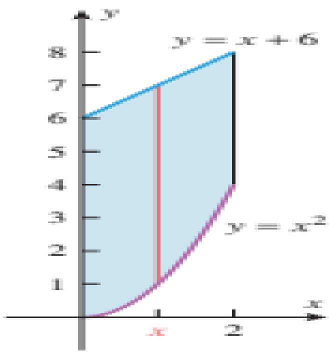


Figure (2-6)

$$\therefore A = \int_a^b [f(x) - g(x)]\, dx = \int_0^2 [x + 6 - x^2]\, dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 =$$

$$= \left[2 + 12 - \frac{8}{3}\right] - 0 = \frac{34}{3}.$$

Example (2-22): Evalute $\displaystyle\int_{-1/2}^{1/2} \frac{1}{\sqrt{1 - x^2}}\, dx.$

Solution:

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x \,\Big|_{-1/2}^{1/2} = \left(\sin^{-1}\left(\frac{1}{2}\right) - \sin\left({}^{-1}\frac{1}{2}\right)\right) = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{\pi}{6}.$$

Example (2-23): Evalute $\int_0^1 \tan^{-1} x\,dx$ **.**

Solution:Let

$$u = \tan^{-1} x,\ dv = dx \Rightarrow du = \frac{1}{1+x^2}, v = x.$$

$$\therefore \int_0^1 \tan^{-1} x\,dx = \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx =$$

$$= \left[x \tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx = \left[x \tan^{-1} x\right]_0^1 - \frac{1}{2}\left[Ln\ \ 1+x^2\ \ \right]_0^1 =$$

$$= \left[\frac{\pi}{4} - 0\right] - \frac{1}{2} Ln\,2.$$

Example (2-24): Evalute $\int_0^\pi x \sin 2x\,dx$ **.**

Solution:

$$Let: u = x,\ dv = \sin 2x \Rightarrow du = 1, v = -2\cos 2x$$

$$\therefore \int_0^\pi x \sin 2x\,dx = \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx =$$

$$= \left[-\frac{1}{2} x \cos 2x\right]_0^\pi + \frac{1}{2}\int_0^\pi \cos 2x dx = \left[-\frac{1}{2} x \cos 2x\right]_0^\pi + \frac{1}{4}\left[\sin 2x\right]_0^\pi = -\frac{1}{2}\pi.$$

Example (2-25): Evalute.

$$\int_0^2 x\ \ x^2+1\ ^3\ dx$$

Solution:

$$\int_0^2 x\,(x^2+1)^3\,dx \ldots\ldots\ldots\ldots.\ 1$$

$$put\ u = x^2+1 \Rightarrow du = 2xdx$$

$$\therefore \int_0^2 x\,(x^2+1)^3\,dx = \frac{1}{2}\int_0^2 \Big(\underbrace{x^2+1}_{u}\Big)^3 \underbrace{2xdx}_{du} = \frac{1}{2}\int_0^2 u^3\,du = \frac{1}{2}\bullet\frac{1}{4}\left[u^4\right]_0^2 =$$

$$= \frac{1}{8}\left[(x^2+1)^4\right]_0^2 = \frac{1}{8}\left[(5)^4 - 1\right] = \frac{624}{8} = 78.$$

Example (2-26): Evalute

$$a)\ \int_0^{\pi/2} \sin^5 x\cos 2x\,dx, \qquad b)\ \int_2^5 (2x-5)(x-3)^9\,dx$$

Solution:

$$a)\ \int_0^{\pi/8} \sin^5 x\cos 2x\,dx,$$

$$put\ u = \sin 2x \Rightarrow du = 2\cos 2x\,dx,$$

$$at\ x = 0 \Rightarrow u = 0, at\ x = \pi/8 \Rightarrow u = \sin \pi/4 = 1/\sqrt{2}$$

$$\int_0^{\pi/8} \sin^5 x\cos 2x\,dx = \frac{1}{2}\int_0^{1/\sqrt{2}} u^5\,du = \left(\frac{u^6}{12}\right)_0^{1/\sqrt{2}} = \frac{\left(1/\sqrt{2}\right)^6}{12} = \frac{1}{96}.$$

$$b)\ \int_2^5 (2x-5)(x-3)^9\,dx$$

$$put\ u = x-3 \Rightarrow x = u+3, du = dx\ and$$

$$at\ x = 2 \Rightarrow u = -1, at\ x = 5 \Rightarrow u = 2,$$

$$\therefore \int_{2}^{5} 2x-5 \quad x-3^{\;9}\, dx = \int_{-1}^{2} 2\;\; u+3 \;-5 \quad u^{\;9}\, du =$$

$$= \int_{-1}^{2} 2u+1 \quad u^{\;9}\, du = \int_{-1}^{2} 2u^{10} + u^{9} \;\; du = \left(\frac{2}{11} u^{11} + \frac{2}{10} u^{10} \right)_{-1}^{2} =$$

$$= \left(\frac{2}{11} \;\; 2^{11} \;\; + \frac{2}{10} \;\; 2^{10} \;\; \right) - \left(\frac{2}{11} \;\; -1^{\;11} + \frac{2}{10} \;\; -1^{\;10} \right) =$$

$$= \left(\frac{2^{12}}{11} + \frac{2^{10}}{10} \right) - \left(-\frac{2}{11} + \frac{2}{10} \right) \approx 474.8.$$

Example (2-27):

Evalute: $a \displaystyle\int_{0}^{3/4} \frac{dx}{1-x}, \quad b \int_{0}^{Ln3} e^{x} \;\; 1+e^{x}\;^{1/2}\, dx$

Solution:

$$a \int_{0}^{3/4} \frac{dx}{1-x}, \quad put\, u = 1-x \Rightarrow du = -dx\, and$$

$$at\; x = 0 \Rightarrow u = 1, at\; x = 3/4 \Rightarrow u = 1/4,$$

$$\therefore \int_{0}^{3/4} \frac{dx}{1-x} = -\int_{1}^{1/4} \frac{du}{u} = - \;\; Ln\, u \;\Big|_{1}^{1/4} = - \;\; Ln\, 1/4 - Ln1 \;\; = Ln\, 4.$$

$$b \int_{0}^{Ln3} e^{x} \;\; 1+e^{x}\;^{1/2}\, dx, \quad put\, u = 1+e^{x} \Rightarrow du = e^{x} dx\, and$$

$$at\; x = 0 \Rightarrow u = 2, at\; x = Ln3 \Rightarrow u = 1 + e^{Ln3} = 4$$

$$\therefore \int_{0}^{Ln3} e^{x} \;\; 1+e^{x}\;^{1/2}\, dx = \int_{2}^{4} u^{1/2} du = \frac{2}{3}\left[u^{3/2} \right] = \frac{2}{3}\left[(4)^{3/2} - (2)^{3/2} \right] =$$

$$= \frac{2}{3}\left[(2)^{3} - (2)^{3/2} \right] = \frac{16}{3} - \frac{4\sqrt{2}}{3}.$$

Exercises (2-3)

Calculate the following integral:

$$(1)\int \frac{x^2}{x^2-4}\,dx \quad (2)\int \frac{x^2+1}{(x+1)^3(x-2)}\,dx \quad (3)\int \frac{x^2+1}{x^3(x+1)^2(x+1)}\,dx$$

$$(4)\int \frac{x}{x^2-5x+6}\,dx \quad (5)\int \frac{x^2+1}{(x-2)(x-1)^2(x^2+4)}\,dx \quad (6)\int \frac{2x^2+3}{x^2(x-1)^2}\,dx$$

$$(7)\int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}\,dx \quad (8)\int \frac{x+1}{x(x-1)^2}\,dx \quad (9)\int \frac{x+1}{x(x-1)^2}\,dx$$

$$(10)\int e^x\sqrt{1-e^{2x}}\,dx \quad (11)\int \frac{dx}{\sqrt{2x^2-6x+4}} \quad (12)\int \frac{2x+3}{4x^2+4x+5}\,dx$$

$$(13)\int \frac{x+1}{x^2-5x+5}dx \quad (14)\int \frac{x+1}{x^2+3x+1}dx, (15)\int_{-2}^{1}(x^2-6x+12)dx,$$

$$(16)\int_{-1}^{2}4x(1-x^2)dx \quad (17)\int_{1}^{4}\frac{4}{x^2}dx. \quad (18)\int_{4}^{9}2x\sqrt{x}\,dx. \quad (19)\int_{1}^{4}\frac{1}{x\sqrt{x}}dx.$$

$$(20)\int_{-\pi/2}^{\pi/2}(\sin\theta+2x-\sec\theta\tan\theta)dx. \quad (21)\int_{0}^{1/\sqrt{2}}\frac{dx}{\sqrt{1-x^2}} \quad (22)\int_{-1}^{1}\frac{dx}{1+x^2},$$

$$(23)\int_{1}^{2}(2x-1)^3dx, \quad (24)\int_{0}^{4}3x\sqrt{25-x^2}dx, \quad (25)\int_{0}^{1}(2x+2)(x+1)^5dx$$

$$(26)\int_{-\pi/}^{2\pi/3}\frac{\sin x}{\sqrt{2+\cos x}}dx, \quad (27)\int_{0}^{\pi/4}\tan^2 x\sec^2 x\,dx \quad (28)\int_{-\ln 3}^{\ln 3}\frac{e^x}{e^x+4}dx.$$

Chapter (3)
Introduction to Differential Equations
(3-1) Introduction to Differential Equations
Definition (3-1) :(Differential Equation)

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a differential equation (DE).we shall classify differential equations according to type, order, and linearity.

(3-1-1) Classification by Type:

If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a single independent variable, it is said to be an ordinary differential equation (ODE). An equation involving partial derivatives of one or more unknown functions of two or more independent variables is called a partial differential equation (PDE). Our first example illustrates several of each type of differential equation.

Definition (3-2):

The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation

Example (3-1):

Determine the order and degree for the D.E:

$$1-\frac{dy}{dx}=\frac{x+\cos x}{y^{2}},$$

$$2-(y'')^{3}+y\,(y')^{2}=0,$$

$$3-2y''''+x\,(y'')^{2}-y'=-\tan x\,.$$

Definition (3-3):

An nth-order ordinary differential equation $F\left(x,y,y',y'',.....y^{(n)}\right)=0.$ **is said to be linear if** F **is linear in** $x,y,y',y'',.....y^{(n)}$ **whe** $F\left(x,y,y',y'',.....y^{(n)}\right)=0$ **is**

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_1(x)y'+a_{\circ}(x)y+g(x)=0$$

Example (3-3):

which of the following differential equations is linear?

$$1-\frac{d^2y}{dx^2}+3\frac{dy}{dx}+6y=0$$

$$2-y'''+x^2y'+3xy=xe^x,$$

$$3-\frac{d^2y}{dx^2}+2y\frac{dy}{dx}+3y^2=0.$$

$$4-\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^3+6y=0$$

Example (3-4):

1- Show that: $y=2x+ce^x$ **is a solution for the D.E** $y'-y=2(x-1).$

2-Show that $y=Ae^x+2$ **is a solution for the D.E** $y'+y=2.$

3-Show that: $y=c_1e^x+c_2e^x+x$ **is a solution for the D.E** $y''-3y'+2y=2x-3.$

Solution:

$$1-y=2x+ce^{x}\ldots\ldots..[1]$$
$$y'-y=2(1-x)\ldots\ldots..[2]$$
$$Diff\ [1]w.r,x\Rightarrow y'=2+ce^{x}\ldots\ldots..[3]$$
$$by\ sub\ from\ [1],[3]in\ [2]we\ get$$
$$L.H.S=2+ce^{x}-\left(2x+ce^{x}\right)=2-2x=2(1-x)=R.H.S$$

2- $\because y=Ae^{x}+2\ldots\ldots\ldots.(1)$
$y'-y=2\ldots\ldots\ldots.(2)$
Diff $(1)w.r.t.x\Rightarrow\ y'=Ae^{x}\ldots\ldots\ldots.(3)$

$$by\ sub\ from\ [1],[3]in\ [2]we\ get$$
$$L.H.S=Ae^{x}-Ae^{x}+2=R.H.s$$

Example (3-4): H. W

In Problems 1 –4 verify that the indicated function is an explicit solution of the given differential equation.
solution

$$1-2y'+y=0,\quad :y=e^{\frac{-x}{2}}$$
$$2-y'+20y=24:\quad y=\frac{6}{5}-\frac{6}{5}e^{-20x}$$
$$3-y''-6y'+13y=0:\quad y=e^{3x}\cos 2x$$
$$4-y''+y=\tan x\quad :\ y=-\cos x\ \ln(\sec x+\tan x).$$

(3-2) Solution of First-Order Differential Equations

(3-2-1) Separable Method

Introduction:

We begin our study of how to solve differential equations with the simplest of all differential equations: first-order equations with separable variables. Because the method in this section and many

techniques for solving differential equations involve integration, you are urged to refresh your memory on important formulas (such as $\int du/u$) and techniques (such as integration by parts) by consulting a calculus text.

Solution by Integration: Consider the first-order differential equation $\frac{dy}{dx} = f(x,y)$. When f does not depend on the variable y ,that is,

$$f(x,y) = g(x) \qquad (1),$$

the differential equation (1) can be solved by integration. If $g(x)$ is a continuous function, then integrating both sides of (1) gives

$y = \int g(x)dx = G(x)+c$, where $G(x)$ is an antiderivative (indefinite integral) of $g(x)$. For example, if $\frac{dy}{dx} = 1+e^{2x}$, then its solution is

$y = \int (1+e^{2x})dx$ or $y = x + \frac{1}{2}e^{2x} + c$.

Definition (3-5) :

A first-order differential equation of the form $\frac{dy}{dx} = g(x)h(y)$ is said to be separable or to have separable variables.

Example (3-6): Solve the differential equation:

$$\frac{dx}{\sin y} = xdy \ .$$

Solution:

$$\because \frac{dx}{\sin y} = x dy \Rightarrow \sin y dy = \frac{dx}{x} \Rightarrow \int \sin y dy = \int \frac{dx}{x} + \ln c$$

$$-\cos y = Lnx + Lnc = Lcx \Rightarrow \cos y = -Lcx .$$

Example (3-6): Solve the differential equation:

$$\left(x^3 + x^2\right) y dx + x^2 \left(y^3 + 2y\right) dy = 0.$$

Solution

$$\because \left(x^3 + x^2\right) y dx + x^2 \left(y^3 + 2y\right) dy = 0 \Rightarrow$$

$$\frac{\left(x^3 + x^2\right)}{yx^2} y dx + \frac{x^2 \left(y^3 + 2y\right)}{yx^2} dy = 0.$$

$$\therefore \frac{\left(x^3 + x^2\right)}{x^2} dx + \frac{\left(y^3 + 2y\right)}{y} dy = 0 \Rightarrow$$

$$(x + 1) dx + (y + 2) dy = 0$$

$$\int (x + 1) dx + \int \left(y^2 + 2\right) dy = c \Rightarrow \frac{x^2}{2} + x + \frac{y^2}{2} + 2y = c.$$

Example (3-7): Solve the differential equation:

$$e^{-y} \frac{dy}{dx} = \cos x .$$

Solution:

$$\because e^{-y} \frac{dy}{dx} = \cos x \Rightarrow e^{-y} dy = \cos x \, dx$$

$$\therefore \int e^{-y} dy = \int \cos x \, dx + c \Rightarrow -e^{-y} = \sin x + c$$

Example (3-8): Solve the differential equation:

$$\frac{1 + x}{x} dx + \frac{1 - y}{y} dy = 0.$$

Solution:

$$\because \frac{1+x}{x}dx + \frac{1-y}{y}dy = 0 \Rightarrow \left(\frac{1}{x}+1\right)dx + \left(\frac{1}{y}-1\right)dy = 0$$

$$\therefore \int\left(\frac{1}{x}+1\right)dx + \int\left(\frac{1}{y}-1\right)dy = Lnc$$

$$\therefore Lnx + x + Lny - y = Lnc \Rightarrow Lnxy + x - y = Lnc$$

$$\therefore Ln\,\frac{xy}{c} = y - x \Rightarrow xy = ce^{y-x}$$

Example (3-9): Solve the differential equation:

$$\tan x\ \sin^2 y\, dx + \cos^2 x\ \cot y\, dy = 0$$

Solution:

$$\because \tan x \sin^2 y\, dx + \cos^2 x \cot y\, dy = 0$$

$$\therefore \tan x \frac{1}{\cos^2 x} dx + \frac{\cot y}{\sin^2 y} dy = 0 \rightarrow$$

$$\therefore \tan x \sec^2 x\, dx + \cot y\, cosec^2 y\ \ dy = 0$$

$$\therefore \int \tan x \sec^2 x\, dx + \int \cot y\, cosec^2 y\ \ dy = C$$

$$\int \tan x\ \ d(\tan x) - \int \cot y\,(-\,cosec^2 y\, dy) = C$$

$$\therefore \left(\frac{1}{2}\right)\tan^2 x - \int \cot y\, d(\cot y) = C$$

$$\left(\frac{1}{2}\right)\tan^2 x - \left(\frac{1}{2}\right)\cot^2 y = C$$

$$\therefore \tan^2 x - \cot^2 y = C'$$

Example (3-10): Solve the differential equation:

$$3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$$

Solution

$$\because 3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$$

$$\therefore \frac{3e^x}{(1-e^x)}dx + sec^2 y \frac{1}{\tan y}dy = 0$$

$$-3\int \frac{-e^x}{(1-e^x)}dx + \int \frac{sec^2 y}{\tan y}dy = C$$

$$\Rightarrow -3\int \frac{d(1-e^x)}{(1-e^x)} + \int \frac{d(\tan y)}{\tan y} = C$$

$$-3\,ln(1-e^x) + ln(\tan y) = ln\,C$$

$$\Rightarrow ln\left[\frac{1}{(1-e^x)^3}\right] + ln(\tan y) = ln\,C$$

$$\therefore ln\left\{\frac{\tan y}{(1-e^x)^3}\right\} = ln\,C \Rightarrow \frac{\tan y}{(1-e^x)^3} = C$$

Example (3-11): Solve the differential equation:

$$(xy+x)dx = (x^2y^2 + x^2 + y^2 + 1)dy$$

Solution

$$\because (xy+x)dx = (x^2y^2 + x^2 + y^2 + 1)dy$$

$$\therefore x(y+1)dx = \left(x^2(y^2+1) + (y^2+1)\right)dy$$

$$x(y+1)dx = (y^2+1)(x^2+1)dy$$

Divide by:$(y+1)(x^2+1)$

$$\therefore \frac{x}{x^2+1}dx = \frac{(y^2+1)}{(y+1)}dy$$

$$\therefore \frac{x}{x^2+1}dx = \left[(y-1) + \frac{2}{y+1}\right]dy \Rightarrow$$

$$\left(\frac{1}{2}\right)\int \frac{2x}{x^2+1}dx = \int \left[(y-1) + \frac{2}{y+1}\right]dy + C$$

$$\therefore \left(\frac{1}{2}\right) ln(x^2+1) = \left(\frac{1}{2}\right)y^2 - y + 2\,ln(y+1) + C$$

Example (3-12): Solve the differential equation:

$$xy' = \cot y$$

Solution:

$$\because xy' = \cot y \Rightarrow x\frac{dy}{dx} = \cot y \Rightarrow xdy = \cot y\, dx \Rightarrow$$

$$\frac{1}{\cot y}dy = \frac{1}{x}dx$$

$$\therefore \int \frac{\sin y}{\cos y}dy = \int \frac{1}{x}dx + C \Rightarrow -\int \frac{-\sin y}{\cos y}dy = \ln x + c$$

$$-\ln\cos y = \ln x + C \Rightarrow -\ln\cos y - \ln x = C$$

$$-(\ln\cos y + \ln x) = C \Rightarrow \ln(x\cos y) = C$$

$$\therefore x\cos y = C'$$

Example (3-13): Solve the differential equation:

$$ydx + \left(x^2 - 4x\right)dy = 0.$$

Solution:

$$\because ydx + \left(x^2 - 4x\right)dy = 0 \Rightarrow \frac{y}{y\left(x^2-4x\right)}dx + \frac{\left(x^2-4x\right)}{y\left(x^2-4x\right)}dy = 0$$

$$\therefore \frac{1}{\left(x^2-4x\right)}dx + \frac{1}{y}dy = 0 \ldots\ldots\ldots\ldots(1)$$

$$\because \frac{1}{\left(x^2-4x\right)} = \frac{1}{x\left(x-4\right)} \Rightarrow \frac{1}{x\left(x-4\right)} = \frac{A}{x} + \frac{B}{\left(x-4\right)}$$

$$\therefore \frac{1}{x\left(x-4\right)} = \frac{A\left(x-4\right)}{x\left(x-4\right)} + \frac{Bx}{\left(x-4\right)} \Rightarrow A\left(x-4\right) + Bx = 1.$$

$$\therefore Ax - 4A + Bx = 1 \Rightarrow \left(A+B\right)x - 4A = 1 \Rightarrow$$

$$-4A = 1 \Rightarrow A = \frac{-1}{4}, A + B = 0 \Rightarrow B = \frac{1}{4}.$$

$$\therefore \frac{1}{x\left(x-4\right)} = \frac{-1}{4x} + \frac{1}{4\left(x-4\right)}.$$

$by\ sub\ in\ (1) we\ get:$

$$\frac{-1}{4x}+\frac{1}{4(x-4)}dx+\frac{1}{y}dy=0$$

$$\therefore \int\left(\frac{-1}{4x}+\frac{1}{4(x-4)}\right)dx+\int\frac{1}{y}dy=Lnc$$

$$\therefore \frac{-1}{4}Lnx+\frac{1}{4}Ln(x-4)+Lny=Lnc \Rightarrow \frac{1}{4}Ln\frac{(x-4)}{x}+Lny=Lnc$$

$$\therefore Ln\frac{(x-4)y}{x}=Lnc \Rightarrow \frac{(x-4)y}{x}=c$$

Example (3-14): Solve the differential equation:

$$(e^v+1)\cos u\,du+e^v(\sec^2 u-1)dv=0$$

Solution:

$$\frac{\cos u}{(\sec^2 u-1)}du=\frac{e^v}{e^v+1}v, \because \sec^2 u-1=\tan^2 u=\frac{\sin^2 u}{\cos^2 u}$$

$$\therefore \int\frac{\cos u}{\sin^2 u}\cos^2 u\ du=-\int\frac{e^v}{e^v+1}dv+C$$

$put\ \sin u=z$

$\therefore \cos u\,du=dz\ ,\qquad \cos^2 u=1-\sin^2 u=1-z^2$

$$\therefore \int\frac{1-z^2}{z^2}dz=-\ln(e^v+1)+C \Rightarrow \int\left(\frac{1}{z^2}-1\right)dz$$
$$=-\ln(e^v+1)+C$$

$$-\frac{1}{z}-z=-\ln(e^v+1)+C$$

$put\ z=\sin u$

$$\therefore \frac{1}{\sin u}+\sin u=\ln(e^v+1)+C$$

Example (3-15): Solve the differential equation:

$$\ln\ x \frac{dx}{dy} = xy$$

Solution:

$$\because \ln x \frac{dx}{dy} = xy \Rightarrow \ln x \frac{dx}{x} = ydy$$

$$\therefore \int \frac{1}{x} \ln x\, dx = \int y\, dy + C \ \Rightarrow \left(\frac{1}{2}\right)(\ln^2 x) = \frac{y^2}{2} + C$$

Example (3-16): Solve the differential equation:

$$\sqrt{1-y^2}dx + (x^2 - 2x + 2)dy = 0$$

Solution:

$$\because \sqrt{1-y^2}dx + (x^2 - 2x + 2)dy = 0$$

$$\therefore \int \frac{dx}{x^2 - 2x + 2} = -\int \frac{dy}{\sqrt{1-y^2}} + C \Rightarrow \int \frac{dx}{(x-1)^2 + 1}$$

$$= -\sin^{-1} y + C$$

$$\tan^{-1}(x-1) = -\sin^{-1} y + C$$

Example (3-17):Solve the differential equation:

$$(1 + x + xy^2 + y^2)dy = \frac{dx}{1-x}$$

Solution:

$$\because (1 + x + xy^2 + y^2)dy = \frac{dx}{1-x}$$

$$\therefore (1+x)(1+y^2)dy = \frac{dx}{1-x} \Rightarrow \int (1+y^2)dy$$

$$= \int \frac{1}{(1+x)(1-x)} dx + C$$

$$\therefore y + \frac{y^3}{3} = \tanh^{-1} x + C$$

Example (3-18): Solve the differential equation:

$$yy' = \sec y^2 \sec^2 x$$

Solution:

$$\because yy' = \sec y^2 \sec^2 x$$

$$\therefore \frac{ydy}{\sec y^2} = \sec^2 x\,dx \Rightarrow \left(\frac{1}{2}\right)\int \cos(y^2)\,2ydy$$
$$= \int \sec^2 x\,dx + C$$

$$\therefore \left(\frac{1}{2}\right)\sin y^2 = \tan x + C$$

Example (3-19): Solve the differential equation:

$$y' = xe^{x-y}\operatorname{cosec} y$$

Solution

$\because y' = xe^{x-y}\operatorname{cosec} y$

$\therefore$

$dy = xe^x e^{-y}\operatorname{cosec} y\,dx$

$$\therefore \int e^y \sin y\,dy = \int xe^x dx + C$$

$$I_1 = \int xe^x dx = xe^x - \int e^x = xe^x - e^x$$

$$I_2 = \int e^y \sin y\,dy = e^y \sin y - \int e^y \cos y\,dy$$
$$= e^y \sin y - \left(e^y \cos y + \int e^y \sin y\,dy\right)$$

$$\therefore I_2 = e^y \sin y - e^y \cos y - I_2 \Rightarrow I_2$$
$$= \left(\frac{1}{2}\right)(e^y \sin y - e^y \cos y)$$

$$\left(\frac{1}{2}\right) e^y(\sin y - \cos y) = (x-1)e^x + C\therefore$$

Example (3-20):

Solve the differential equation: $y' = xe^{x-y}\operatorname{cosec} y$

Solution

$\because y' = xe^{x-y}\operatorname{cosec} y$

$\therefore$

$dy = xe^x e^{-y}\operatorname{cosec} y\,dx$

$$\therefore \int e^y \sin y\, dy = \int xe^x dx + C$$

$$I_1 = \int xe^x dx = xe^x - \int e^x = xe^x - e^x$$

$$I_2 = \int e^y \sin y\, dy = e^y \sin y - \int e^y \cos y\, dy$$

$$= e^y \sin y - \left(e^y \cos y + \int e^y \sin y\, dy\right)$$

$$\therefore I_2 = e^y \sin y - e^y \cos y - I_2 \Rightarrow I_2$$

$$= \left(\frac{1}{2}\right)(e^y \sin y - e^y \cos y)$$

$$\left(\frac{1}{2}\right) e^y(\sin y - \cos y) = (x-1)e^x + C:$$

(3-2-2) Equations reduces to separable:

Let the D.E

$$y' = f(ax + by + C) \,\ldots\ldots\ldots..(1)$$

or

$$x' = f(ax + by + C) \,\ldots\ldots\ldots..(2)$$

Equations$(1),(2)$ **reduces to separable as: :**

$$put\ u = ax + by \Rightarrow du = adx + bdy$$

Example (3-21): Solve the D.E: $y' = e^{(2x+y+1)}$

Solution:

$$\because y' = e^{(2x+y+1)}$$

$$Put\ \ u = (2x + y + 1) \Rightarrow \frac{du}{dx} = 2 + \frac{dy}{dx} \Rightarrow \frac{du}{dx} - 2 = e^u$$

$$\frac{du}{dx} = e^u + 2 \Rightarrow \frac{du}{(2 + e^u)} = dx \Rightarrow \frac{e^{-u}du}{2e^{-u} + 1} = dx$$

$$\frac{-1}{2}\int \frac{-2e^{-u}}{2e^{-u}+1} du = \int dx + C \Rightarrow -\left(\frac{1}{2}\right) \ln(2e^{-u} + 1)$$

$$= x + C$$

$$\therefore \frac{-1}{2} \ln\left(2e^{-(2x+y+1)} + 1\right) = x + C$$

Example (3-23):

Solve the differential equation: $y' = tan^2(x+y-1)$

Solution:

$$\because y' = tan^2(x+y-1)$$

$$put\ u = x+y-1 \Longrightarrow \frac{du}{dx} = 1+\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\therefore \frac{du}{dx} - 1 = tan^2 u \Longrightarrow \frac{du}{dx} = 1 + tan^2 u \Longrightarrow \frac{du}{dx} = sec^2 u$$

$$\int \frac{1}{sec^2 u} du = \int dx + C \Longrightarrow \int cos^2 u\, du = x + C$$

$$\int \frac{1}{2}(cos\,2u+1)du = x + C \Longrightarrow \frac{1}{2}\left[u + \frac{1}{2} sin\,2u\right] - x = C$$

$$\left(\frac{1}{2}\right)\left[x+y-1+\left(\frac{1}{2}\right) sin\,2(x+y-1)\right] - x = C$$

Example (3-24):

Solve the differential equation: $y' = (8x+y+3)^3$

Solution:

$$y' = (8x+y+3)^3$$

$$put\ u = 8x+y+3 \Longrightarrow du = 8dx + dy \Longrightarrow dy = u^3 dx$$

$$du - 8dx = u^3 dx \Rightarrow du = (8+u^3)dx$$

$$\int \frac{du}{u^3+8} = \int dx + C \Longrightarrow \int \frac{du}{(u+2)(u^2-2u+4)} = x + C'$$

By using partial fractions:

$$\frac{1}{(u+2)(u^2-2u+4)} = \frac{A}{u+2} + \frac{Bu+C}{u^2-2u+4}$$

$$A = \left(\frac{1}{12}\right), \qquad Bu + C = \left.\frac{1}{u+2}\right|_{u^2=2u-4}$$

$$(Bu+C)(u+2) = 1 \ \Longrightarrow Bu^2 + cu + 2Bu + 2C = 1$$

$$B(2u-4) + Cu + 2Bu + 2C = 1$$

$$4B + C = 0 \qquad 2C - 4B = 1$$

$$C = \left(\frac{1}{3}\right) \qquad B = -\left(\frac{1}{12}\right)$$

$$\int \frac{\frac{1}{12}}{u+2} du + \int \frac{-\frac{1}{12}u + \frac{1}{3}}{u^2 - 2u + 4} du = x + C'$$

$$\therefore \left(\frac{1}{2}\right) ln(u+2) - \left(\frac{1}{12}\right) \int \frac{u-4}{u^2 + 2u + 4} du = x + C'$$

$$\left(\frac{1}{2}\right) ln(u+2) - \left(\frac{1}{12}\right) \int \frac{u-1}{u^2 + 2u + 4} du + \left(\frac{1}{4}\right) \int \frac{du}{u^2 - 2u + 4} = x + C'$$

$$\left(\frac{1}{2}\right) ln(u+2) - \left(\frac{1}{12}\right)\left(\frac{1}{2}\right) ln(u^2 - 2u + 4) + \left(\frac{1}{4}\right) \int \frac{du}{u^2 - 2u + 4} = x + C'$$

$$\because u^2 - 2u + 4 = (u-1)^2 + 4 - 1 = (u-1)^2 + 3$$

$$\therefore \left(\frac{1}{2}\right) ln(u+2) - \left(\frac{1}{24}\right) ln(u^2 - 2u + 4) + \left(\frac{1}{4}\right) \int \frac{du}{(u-1)^2 + 3} = x + C'$$

$$\therefore \left(\frac{1}{2}\right) ln(u+2) - \left(\frac{1}{24}\right) ln(u^2 - 2u + 4) + \left(\frac{1}{4}\right)\left(\frac{1}{\sqrt{3}}\right) tan^{-1}\left\{\frac{(u-1)}{\sqrt{3}}\right\} = x + C'$$

$$\because (8x + y + 3)$$

$$\therefore \left(\frac{1}{2}\right) ln\left((8x+y+3)+2\right) - \left(\frac{1}{24}\right) ln((8x+y+3)^2 - 2(8x+y+3)+4) + \left(\frac{1}{4}\right)\left(\frac{1}{\sqrt{3}}\right) tan^{-1}\left\{\frac{((8x+y+3)-1))}{\sqrt{3}}\right\} = x + C'$$

$$\therefore \left(\frac{1}{2}\right) ln(8x+y+5) - \left(\frac{1}{24}\right) ln((8x+y+3)^2 - (16x+2y+6)+4) + \left(\frac{1}{4}\right)\left(\frac{1}{\sqrt{3}}\right) tan^{-1}\left\{\frac{(8x+y+2))}{\sqrt{3}}\right\} = x + C'$$

Example (3-25):

Solve the differential equation: $x' = tan^2(x+y+6)$

Solution

$$\because x' = tan^2(x+y+6)$$

$$put\ u = (x+y+6) \Rightarrow \frac{du}{dy} = \frac{dx}{dy} + 1 \Rightarrow \frac{du}{dy} - 1 = tan^2 u$$

$$\frac{du}{dy} = 1 + tan^2 u = sec^2 u$$

$$\int \frac{du}{sec^2 u} = \int dy + C \Rightarrow \int cos^2 u\, du = y + C$$

$$\therefore \left(\frac{1}{2}\right)\int (1 + cos\, 2u) du = y + C \Rightarrow \left(\frac{1}{2}\right)\left[u + \frac{sin\, 2u}{2}\right] = y + C$$

$$\therefore \frac{x+y+6}{2} + \frac{sin\, 2(x+y+6)}{4} = y + C$$

(3-2-3) Homogenous Differential Equations:

Homogenous differential equations has the form $\frac{dy}{dx} = f(x,y)$ ***where*** $f(x,y)$ ***is a homogenous function of degree zero .***

Definition (3-5):

The differential equations $\frac{dy}{dx} = f(x,y)$ ***is said to be a homogenous if it has the form*** $\acute{y} = \frac{g(x,y)}{h(x,y)}$ ***where*** $g(x,y)$ and $h(x,y)$ ***are a homogenous functions of the same degree .the functions*** $g(x,y)$ is said to be ***a homogenous if*** $g(\propto x, \propto y) = \propto^n g(x,y)$***.e.g***

$$(1)\ \acute{y} = \frac{(x-y)}{(x+y)},\ (2)\ \acute{y} = \frac{(x^2+y^2)}{(x^2+3xy-y^2)}\ ,(3)\ \acute{y} = \frac{sinx}{y}$$

To solve the homogenous differential equations put: $y = xv \Rightarrow \acute{y} = v + x\acute{v}$ ***and substitute in differential equation .the differential equation reduces to separable differential equation.***

Definition(3-6):

The function $f(x,y)$ ***is said to be a homogeneous function of degree n if:***

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

This is achieved if both the $f(x,y)$ ***boundaries have the same degree in the variables*** x, y ***for example, the function:*** $f(x,y) = x^3 - x^2y + 2xy^2 + 7y^3$

Is a homogeneous function of degree (3) because:

$f(\lambda x, \lambda y) = (\lambda x)^3 - (\lambda x)^2(\lambda y) + 2(\lambda x)(\lambda y)^2 + 7(\lambda y)^3 = \lambda^3 f(x,y)$

Also:

$$f(x,y) = \frac{x-y}{x+y}$$

Is a homogeneous function of degree zero because:

$$f(\lambda x,\lambda y)=\frac{\lambda x-\lambda y}{\lambda x+\lambda y}=\frac{x-y}{x+y}=f(x,y)=\lambda^0 f(x,y)$$

While the function:

$$f(x,y)=x^3+\sin x^2\cos y$$

Is not a homogeneous function because:

$$f(\lambda x,\lambda y)\neq\lambda^n f(x,y)$$

Also

$$f(x,y)=\frac{x-y+1}{x+y-2}$$

Not homogeneous. Based on the above and as an extension, it is said of the differential equation of the first order:

$$M(x,y)dx+N(x,y)dy=0 \qquad \text{(I)}$$

It is a homogeneous differential equation if each of the functions $M(x,y)$, $N(x,y)$ is a homogeneous function of the same degree. The above equation can be written on the form:

$$\frac{dy}{dx}=-\frac{M(x,y)}{N(x,y)}$$

The right side is clearly a homogeneous function of zero because:

$$\frac{M(\lambda x,y)}{N(\lambda x,\lambda y)}=\frac{\lambda' M(x,y)}{\lambda' N(x,y)}=\lambda^0\frac{M(x,y)}{N(x,y)}$$

Therefore:

$$\frac{dy}{dx}=-\frac{M(x,y)}{N(x,y)}=-\frac{M(x\lambda,\lambda y)}{N(\lambda x,\lambda y)}$$

By giving λ the special value $\lambda = 1/x$ we obtain:

$$\frac{dy}{dx} = -\frac{M(1, y/x)}{N(1, y/x)}$$

That is:

$$\frac{dy}{dx} = g(y/x) \qquad \textit{(II)}$$

To solve the homogenous differential equations put:
$y = xv \Rightarrow y = x\vartheta \Longrightarrow \frac{dy}{dx} = x\frac{d\vartheta}{dx} + \vartheta \Longrightarrow \acute{y} = v + x\acute{v}$ *and substitute in differential equation. the differential equation reduces to separable differential equation. Equation (II) is written on the form:*

$$\frac{1}{g(\vartheta) - \vartheta} d\vartheta = \frac{1}{x} dx$$

$$\therefore \frac{dy}{dx} = x\frac{d\vartheta}{dx} + \vartheta = g(\vartheta)$$

Example (3-25):

Solve the differential equation: $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$

Solution

$$\because \frac{dy}{dx} = \frac{x^3 + y^3}{xy^2} \dots\dots\dots.(1)$$

$Put\ y = xv \Rightarrow \acute{y} = \acute{v} + xv,\ \ \acute{x} = \frac{y}{v}$

by sub in equation (1) we get:

$$x\acute{v} + v = \frac{x^3 + x^3v^3}{x^3v^2} = \frac{1 + v^3}{v^2} = \frac{1}{v^2} + v \Rightarrow x\acute{v} = \frac{1}{v^2}$$

$$\therefore v^2 dv = \frac{dx}{x} \Rightarrow \int v^2 dv = \int \frac{dx}{x} + lnc$$

$$\therefore \frac{v^3}{3} = lnx + lnc \Rightarrow \frac{v^3}{3} = lnxc \Rightarrow \frac{y^3}{x^3} = 3lxc$$

Example (3-26):

Solve the differential equation: $\frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2}$

Solution:

$$\because \frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2} \dots\dots\dots\dots (1)$$

$$Put\ y = xv \Rightarrow \acute{y} = v + x\acute{v}$$

by sub in equation (1) we get:

$$\acute{v}x + v = \frac{xv}{x} \frac{-x^2v^2}{x^2} \Rightarrow \acute{v}x = -v^2 \Rightarrow -\frac{dv}{v^2} = \frac{dx}{x}$$

$$\therefore -\int \frac{dv}{v^2} = \int \frac{dx}{x} + lnc \Rightarrow \frac{1}{v} = lnx + lnc \Rightarrow \frac{1}{v} = lnxc$$

$$\therefore \frac{1}{v} = lnxc \Rightarrow \frac{x}{y} = lnxc \Rightarrow y = \frac{x}{lnxc}$$

Example (3-27): Solve the differential equation:

$$(2x + 3y)dx + (y - x)dy = 0$$

Solution:

$$\because (2x + 3y)dx + (y - x)dy = 0 \Rightarrow \frac{dy}{dx} = \frac{-(2x + 3y)}{(y - x)} \dots\dots\dots\dots (1)$$

$$Put\ y = xv \Rightarrow \acute{y} = \acute{v} + xv,\quad \acute{x} = \frac{y}{v}$$

by sub in equation (1) we get:

$$\acute{v}x + v = \frac{-(2x + 3xv)}{(xv - x)} \Rightarrow \acute{v}x = \frac{-(2 + 3v)}{(v - 1)} - v$$

$$= \frac{-(2 + 3v)}{(v - 1)} - \frac{v}{v - 1} = \frac{-(2 + 3v) - v^2 + 1}{(v - 1)}$$

$$= \frac{-1 - 3v - v^2}{(v - 1)} = \frac{v^2 + 3v + 1}{(1 - v)}$$

$$\therefore \frac{(1 - v)}{v^2 + 3v + 1} dv = \frac{dx}{x}$$

Example (3-28):Solve the differential equation:

$$(x^2 - 3y^2)dx + 2xydy = 0$$

Solution:

$$\because (x^2 - 3y^2)dx + 2xydy = 0 \Rightarrow \frac{dy}{dx} = \frac{(x^2 - 3y^2)}{-2xy} \dots\dots\dots\dots (1)$$

Put $y = xv \Rightarrow \acute{y} = v + x\acute{v}$

by sub in equation (1) we get:

$$v + x\acute{v} = \frac{(x^2 - 3x^2v^2)}{-2x^2v} = \frac{(3v^2 - 1)}{2v}$$

$$\therefore v + x\acute{v} = \frac{(3v^2 - 1)}{2v} \Rightarrow x\acute{v} = \frac{(3v^2 - 1)}{2v} - v = \frac{3v^2 - 1 - 2v^2}{2v}$$

$$\therefore x\acute{v} = \frac{v^2 - 1}{2v} \Longrightarrow \frac{2v}{v^2 - 1}dv = \frac{dx}{x}$$

$$\therefore \int \frac{2v}{v^2 - 1}dv = \int \frac{dx}{x} \Longrightarrow ln(v^2 - 1) = lnx + lnc$$

$$\therefore ln(v^2 - 1) = lnx + lnc \Longrightarrow (v^2 - 1) = xc$$

$$\therefore \frac{y^2}{x^2} - 1 = xc \Longrightarrow y^2 = cx^3 + x^2$$

Example (3-28):

Solve the differential equation: $x^2\frac{dy}{dx} = x^2 + xy + y^2$

Solution:

$$\because x^2\frac{dy}{dx} = x^2 + xy + y^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{(x^2 + xy + y^2)}{x^2} \dots\dots\dots\dots (1)$$

Put $y = xv \Rightarrow \acute{y} = v + x\acute{v}$

by sub in equation (1) we get:

$$v + x\acute{v} = \frac{(x^2 + x^2v^2 + x^2v^2)}{x^2} = (1 + v + v^2)$$

$$\therefore xv' = (1 + v^2) \Rightarrow \int \frac{dv}{1 + v^2} = \int \frac{dx}{x} \Rightarrow tan^{-1} v$$
$$= lnx + lnc$$

$$\therefore tan^{-1}v = lnx + lnc \Rightarrow tan^{-1}\frac{y}{x} = lnxc$$

Example (3-29): Solve the differential equation:

$$(x^3 + y^3)dx - 2xy^2dy = 0$$

Solution

$$\because (x^3 + y^3)dx - 2xy^2dy = 0$$

This is a homogeneous third-degree differential equation, and by using the substitution $y = \vartheta x$ it can be reduced to a separable form.

$$y = x\vartheta \Longrightarrow dy = xd\vartheta + \vartheta dx$$

$$(x^3 + x^3\vartheta^3)dx - 2x(x\vartheta)^2(xd\vartheta + \vartheta dx) = 0$$

Then

$$x^3[(1 + \vartheta^3)dx - 2\vartheta^2(xd\vartheta + \vartheta dx)] = 0$$

$$(1 + \vartheta^3 - 2\vartheta^3)dx - 2\vartheta^2xd\vartheta = 0$$

$$\therefore \frac{1}{x}dx = \frac{2\vartheta^2}{1 - \vartheta^3}d\vartheta$$

By integrating both sides of the equation we get:

$$\ln x = -\frac{2}{3}\ln(1 - \vartheta^3) + \ln A$$

Or

$$x^3 = \frac{A}{(1 - \vartheta^3)^2}$$

$$x^3\left(1-\frac{y^3}{x^3}\right)^2=x^3\left[\frac{x^3-y^3}{x^3}\right]^2=A$$

Then:

$$(x^3-y^3)^2=Ax^3$$

(3-3) Exact ordinary differential equation

In this section we examine first-order equations in differential form: $M(x,y)dx+N(x,y)dy=0$*.*
By applying a simple test to M and N, we can determine whether $M(x,y)dx+N(x,y)dy$ *is a differential of a function f (x, y). If the answer is yes, we can construct f by partial integration*

Definition (3-6) :

A differential expression $M(x,y)dx+N(x,y)dy=0$ *is an exact differential in a region R of the xy-plane if it corresponds to the differential of some function f (x, y) defined in R. A first-order differential equation of the form:*

$$M(x,y)dx+N(x,y)dy=0$$

is said to be an exact equation if the expression on the left-hand side is an exact differential. i .e,

$du(x,y)=M(x,y)dx+N(x,y)dy$*.*

Then if:

$$du(x,y)=0$$

$$u(x,y)=C$$

For example,

$x^2y^3dx+x^3y^2dy=0$ *is an exact equation, because its left-hand side is an exact differential:*

$$d\left(\frac{1}{3}x^3y^3\right)=x^2y^3\,dx+x^3y^2dy=0$$

Notice that if we make the identifications:

$M = x^2y^3$, $N = x^3y^2$, *then* $\frac{\partial M}{\partial y} = 3x^2y^2, \frac{\partial N}{\partial x} = 3x^2y^2 \Rightarrow$
$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Theorem (3-1) :

Let $M(x,y)$and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by a, x, b, c, y, d. Then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential equation is
$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial x}$.

Proof

First: Let $M(x,y)dx + N(x,y)dy$be exact. Then there exist $U(x,y)$such that:

$$dU(x,y) = M(x,y)dx + N(x,y)dy \quad (1)$$

but

$$dU(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad (2)$$

From (1), (2) we get:

$$M = \frac{\partial u}{\partial x}, \qquad N = \frac{\partial u}{\partial y} \qquad (3)$$

Second: Let

$$M = \frac{\partial u}{\partial x}, \qquad N = \frac{\partial u}{\partial y}$$

From (3), we get:

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}, \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y \partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then $M(x,y)dx + N(x,y)dy$ is exact.

Method of Solution : Given an equation in the differential form $M(x,y)dx + N(x,y)dy$=0 .Then there exists a function f for which : $\frac{\partial f}{\partial x} = M(x,y)$
We can find f by integrating $M(x,y)$ with respect to x while holding y constant.

$$f(x,y) = \int M(x,y)dx + g(x)\,dx \qquad (4)$$

Example (3-30)

where the arbitrary function $g(x)$ is the "constant" of integration.
Now differentiate (4) with respect to y and assume that $\frac{\partial f}{\partial y} = N(x,y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + \acute{g}(x) = N(x,y)\,d \Rightarrow$$
$$\Rightarrow x$$

Then:

$$\acute{g}(x) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx + \acute{g}(x) \qquad (5).$$

Finally, integrate (5) with respect to y and substitute the result in (4). The implicit solution of the equation is $f(x,y) = c$.

Example (3-31):

Show t hat the differential equation $(3x^2 + 2xy + 2x)dx + (x^2 + 2y)dy = 0$ is exact and Solve it.

Solution:

$$(3x^2 + 2xy + 2x)dx + (x^2 + 2y)dy = 0$$

$$M = (3x^2 + 2xy + 2x), \qquad \frac{\partial M}{\partial y} = 2x$$ *(1)*

$$N = (x^2 + 2y), \qquad \frac{\partial N}{\partial x} = 2x$$

$$(1),(2) \;\Rightarrow\; \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \textit{the differential equation is exact}$$

The differential equation solved by several methods

Method (1):

$$dU(x,y) = \frac{\partial U}{dx}dx + \frac{\partial U}{\partial y}dy = M(x,y)dx + N(x,y)dy = 0$$

$$\frac{\partial U}{\partial x} = M(x,y) = (3x^2 + 2xy + 2x) \qquad (1)$$

$$\frac{\partial U}{\partial y} = N(x,y) = (x^2 + 2y) \qquad (2)$$

Method (2):Integrating (1) w. r. t. x we get:

$$U(x,y) = x^3 + x^2y + x^2 + \phi(y) \qquad (3)$$

To find ϕddifferentiate(3) w. r. t. y

$$\frac{\partial U}{\partial y} = x^2 + \frac{d\phi}{dy} = N(x,y) = x^2 + 2y$$

Then:

$$\frac{d\phi}{dy} = 2y \Rightarrow \phi(y) = y^2 \Rightarrow$$

$$U(x,y) = x^3 + x^2y + x^2 + y^2$$

So, the solution of differential equation is:

$$x^3 + x^2y + x^2 + y^2 = C$$

where C arbitrary constant.

Method (3): ***We can obtain the function*** $U(x,y)$ ***by integrating*** $M(x,y)$ ***w. r.t.x we get:***

$$U(x,y)=\int M(x,y)dx=\int(3x^2+2xy+2x)dx$$
$$=x^3+x^2y+x^2+\phi(y) \qquad (1)$$

integrating $N(x,y)$ ***w. r. t.*** y ***we get:***

$$U(x,y)=\int N(x,y)dy=\int(x^2+2y)dy$$
$$=x^2y+y^2+\omega(x) \qquad (2)$$

From $(1),(2)\Rightarrow \phi(y)=y^2,\ \omega(x)=x^3+x^2$

Then

$$U(x,y)=x^2y+y^2+x^3+x^2=C:$$

Method (4): ***Using the following formula directly:***

$$U(x,y)=\int M(x,y)dx+\int\left[N-\int\frac{\partial M}{\partial y}\partial x\right]dy$$

So

$$U(x,y)=\int(3x^2+2xy+2x)dx$$
$$+\int\left[(x^2+2y)-\int(2x)\partial x\right]dy$$

$$U(x,y)=x^3+x^2y+x^2+\int(x^2+2y-x^2)dy$$
$$=x^3+x^2y+x^2+y^2$$

$$\int(3x^2+2xy+2x)\partial x+\int 2y\,dy=C$$

$$x^3+x^2y+x^2+y^2=C$$

Method (5): ***Arrange the boundaries of the equation so that each boundary is completely different. The equation given can be written on the image***

$$(3x^2+2x)dx+(2xy)dx+(x^2)dy+(2y)dy=0$$
$$d(x^3+x^2)+d(x^2y)+d(y^2)=0$$
$$x^3+x^2+x^2y+y^2=C$$

Method 6:

$$\int_{x_0}^{x} M(x,y)dx+\int_{y_0}^{y} N(x_0,y)dy=C$$

We choose x_0, y_0 always zeros, but if compensation for $x_0=0$ in $N(x,y)$ gives an infinity we substitute $x_0=1$..

$$\int_{0}^{x}(3x^2+2xy+2x)dx+\int_{0}^{y}(2y)dy=C$$
$$x^3+x^2y+x^2+y^2=C$$

It is the same solution of course. In fact, these ways of solution are linked to each other. It does not mean that the matter is always resolved in more than one way.

Example (3-32): Show hat the differential equation $(3x^2+2xy+2x)dx+(x^2+2y)dy=0$ ***is exact and Solve it.***

Solution

$$\left(\frac{y}{x}+\ln y\right)dx+\left(\frac{x}{y}+\ln x\right)dy=0$$

$$M=\left(\frac{y}{x}+\ln y\right),\qquad \frac{\partial M}{\partial y}=\frac{1}{x}+\frac{1}{y} \qquad (1)$$

$$N=\left(\frac{x}{y}+\ln x\right),\qquad \frac{\partial N}{\partial x}=\frac{1}{y}+\frac{1}{x} \qquad (2)$$

(2)$\Rightarrow$& (1)

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow \textit{the equation is exact}$$

$$du(x,y) = M(x,y)dx + N(x,y)dy$$

By using Method 2

$$U(x,y) = \int \left(\frac{y}{x} + \ln y\right) dx = y\ln x + x\ln y + \phi(y)$$

$$U(x,y) = \int \left(\frac{x}{y} + \ln x\right) dy = x\ln y + y\ln x + \omega(x)$$

So

$$\phi(y) = \omega(x) = 0$$

$$U(x,y) = x\ln y + y\ln x = \ln C \ \Rightarrow y^x x^y = C_1$$

Example (3-33): Show hat the differential equation $(y^2 - 2xy + 6x)dx - (x^2 - 2xy + 2)dy = 0$ ***is exact and Solve it.***

Solution

$$M = (y^2 - 2xy + 6x), \qquad \frac{\partial M}{\partial y} = 2y - 2x \qquad (1)$$

$$N = -(x^2 - 2xy + 2), \qquad \frac{\partial N}{\partial x} = -2x + 2y \qquad (2)$$

(1)&(2)$\Rightarrow$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore U(x,y) = \int (y^2 - 2xy + 6x)dx$$
$$= y^2x - x^2y + 3x^2 + \phi(y)$$

$$U(x,y) = \int -(x^2 - 2xy + 2)dy$$
$$= -x^2y + xy^2 - 2y + \omega(x)$$

Then

$$\phi(y) = -2y, \qquad \omega(x) = 3x^2$$

i.e.

$$U(x,y) = y^2x - x^2y + 3x^2 - 2y = C$$

Example (3-34):

Show hat the differential equation

$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$ **is exact and Solve it.**

Solution

$$\because (3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0 \Rightarrow$$

$$M = (3x^2 + 4xy), \qquad \frac{\partial M}{\partial y} = 4x \qquad (1)$$

$$N = (2x^2 + 2y), \qquad \frac{\partial N}{\partial x} = 4x \qquad (2)$$

(1)&(2)$\Rightarrow$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore U(x,y) = \int (3x^2 + 4xy)dx = x^3 + 2x^2y + \phi(y)$$

$$U(x,y) = \int (2x^2 + 2y)dy = 2x^2y + y^2 + \omega(x)$$

Then:

$$\phi(y) = y^2, \qquad \omega(x) = x^3$$

i.e.

$$U(x,y) = x^3 + 2x^2y + y^2 = C$$

Example (3.-35):

Prove that any differential equation for the separation of variables is a complete differential equation.

Solution

Any differential equation can be separated on the image:$f(x)dx = g(y)dy \Rightarrow f(x)dx - g(y)dy = 0$

$$M(x,y) = f(x) \longrightarrow \frac{\partial M}{\partial y} = 0$$

$$N(x,y)=g(y)\longrightarrow \frac{\partial N}{\partial x}=0$$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Example (3-35): Solve the differential equation

$(3x^2+2y\sin 2x)dx+(2\sin^2 x+3y^2)dy=0$

Solution:

$(3x^2+2y\sin 2x)dx+(2\sin^2 x+3y^2)dy=0\Rightarrow$

$$M=(3x^2+2y\sin 2x),\qquad \frac{\partial M}{\partial y}=2\sin 2x \qquad \textit{(1)}$$

$$N=(2\sin^2 x+3y^2),\qquad \frac{\partial N}{\partial x}=4\sin x\cos x=2\sin 2x \qquad \textit{(2)}$$

(1)&(2)$\Rightarrow$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\therefore U(x,y)=\int(3x^2+2y\sin 2x)dx=x^3-y\cos 2x+\phi(y)$$

$$U(x,y)=\int(2\sin^2 x+3y^2)dy=2y\sin^2 x+y^3+\omega(x)$$

Note here the difficulty of comparison and therefore resort to one of the other ways to reach the solution, we use the equation

$$:\int_{x_0}^{x}M(x,y)dx+\int_{y_0}^{y}N(x_0,y)dy=C$$

We choose:$(x_0,y_0)=(0,0)$

$$\therefore \int_0^x(3x^2+2y\sin 2x)dx+\int_0^y 3y^2dy=C$$

i.e.

$$U(x,y)=\left[x^3+\frac{2y(-\cos 2x)}{2}\right]_0^x+[y^3]_0^y=C$$

Then

$$U(x,y) = x^3 - y\cos 2x + y + y^3 = C$$

Example (3-36): Solve the differential equation:

$$(2xy - 3x^2)dx + (x^2 + 2y)dy = 0$$

Solution:

$$\because (2xy - 3x^2)dx + (x^2 + 2y)dy = 0 \Rightarrow$$

$$M = (2xy - 3x^2), \qquad \frac{\partial M}{\partial y} = 2x \qquad (1)$$

$$N = (x^2 + 2y), \qquad \frac{\partial N}{\partial x} = 2x \qquad (2)$$

(1)&(2)$\Rightarrow$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore U(x,y) = \int (2xy - 3x^2)dx = x^2y - x^3 + \phi(y)$$

$$U(x,y) = \int (x^2 + 2y)dy = x^2y + y^2 + \omega(x)$$

Then

$$\phi(y) = y^2, \qquad \omega(x) = -x^3$$

i.e.

$$U(x,y) = x^2y - x^3 + y^2 = C$$

Example (3-37):

Solve the differential equation: $\frac{dx}{dy} + \frac{x^2\cos y + 3y^2e^x}{2x\sin y + y^3e^x} = 0$

Solution

$$\because \frac{dx}{dy} + \frac{x^2\cos y + 3y^2e^x}{2x\sin y + y^3e^x} = 0 \Rightarrow$$

$$(2x\sin y + y^3e^x)dx + (x^2\cos y + 3y^2e^x)dy = 0$$

Then:

$$M=(2x\sin y+y^3e^x),\qquad \frac{\partial M}{\partial y}=2x\cos y+3y^2e^x \quad (1)$$

$$N=(x^2\cos y+3y^2e^x),\qquad \frac{\partial N}{\partial x}=2x\cos y+3y^2e^x \quad (2)$$

(1)&(2)$\Longrightarrow$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\therefore U(x,y)=\int(2x\sin y+y^3e^x)dx$$
$$=x^2\sin y+y^3e^x+\phi(y)$$

$$U(x,y)=\int(x^2\cos y+3y^2e^x)dy$$
$$=x^2\sin y+y^3e^x+\omega(x)$$

Then:

$$\phi(y)=\ \omega(x)=0$$

i.e.

$$U(x,y)=x^2\sin y+y^3e^x=C$$

Example (3-38):

Solve the differential equation: $\frac{y}{x}dx+(y^2+\ln x)dy=0$

Solution

$$\because \frac{y}{x}dx+(y^2+\ln x)dy=0\Longrightarrow$$

$$M=\left(\frac{y}{x}\right),\qquad \frac{\partial M}{\partial y}=\frac{1}{x} \quad (1)$$

$$N=(y^2+\ln x),\qquad \frac{\partial N}{\partial x}=\frac{1}{x} \quad (2)$$

(1)&(2)$\Longrightarrow$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore U(x,y) = \int \left(\frac{y}{x}\right) dx = y \ln x + \phi(y)$$

$$U(x,y) = \int (y^2 + \ln x) dy = \frac{y^3}{3} + y \ln x + \omega(x)$$

Then:

$$\phi(y) = \frac{y^3}{3}, \qquad \omega(x) = 0$$

i.e.

$$U(x,y) = \frac{y^3}{3} + y \ln x = C$$

(3-5) linear differential equations of the first order

We have already defined the linear differential equation of the n-order. What concerns us in this part is the differential equation of the first order which is written as:

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (1)$$

Where $P(x)$, $Q(x)$ are only in variable n. This equation is not generally a complete differential equation, but an integral factor can be found that converts it into a complete differential equation and says that it is linear and without a second party if $Q(x)$is equal to zero,i.e,

$$\frac{dy}{dx} + P(x)y = 0 \qquad (2)$$

The function $Q(x)$ is called the R.H.S of the equation, and we note that equation (2) written in the form:

$$\frac{dy}{y} + P(x)dx = 0$$

Their integration is: $y = Ae^{-\int P(x)dx}$

Theorem (3-2):

For each linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ *an integral factor is a function in x only as follows:*

$$\mu(x) = e^{\int P(x)dx} \qquad (3)$$

proof

The equation (1) can be written on the image as:

$$[P(x)y - Q(x)]dx + dy = 0 \qquad (4)$$

Where:

$M(x,y) = P(x) - Q(x), \qquad N = 1$

Since $P(x) \neq 0$*, equation (4) is not exact because:*

$$\frac{\partial M}{\partial y} = P(x), \qquad \frac{\partial N}{\partial x} = 0 \Longrightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

On the other hand,

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = P(x)$$

$$\because \frac{d\mu}{\mu} = P(x)dx \Longrightarrow \mu(x) = e^{\int P(x)dx}$$

$$\therefore \mu(x) = e^{\int P(x)dx} \; integrating\ Factor$$

Theorem (3-3): *Each linear differential equation of the first order has a solution that is a linear function for an optional constant and **the converse is true.***

proof:

Let: $y' + P(x)y = Q(x)$

By multiplying the two sides by factor μ (x) we get :

$$\mu y' + \mu P(x)y = \mu Q(x)$$

Which can be written as:

$$\frac{d}{dx}(\mu y) + y\left(\mu P - \frac{d\mu}{dx}\right) = \mu Q$$

Where:

$$\because \frac{d\mu}{dx} = P(x)\mu \quad \Rightarrow \therefore \frac{d}{dx}(\mu y) = \mu Q$$

By integration we get:

$$y' = \mu(x)Q(x)dx + A$$

or

$$y = \frac{1}{\mu}\left[\int \mu(x)Q(x)dx + A\right]$$

By removing brackets and substitute for the value of integration we get:

$$y = e^{-\int P(x)dx}\int Q(x)e^{P(x)dx}dx + Ae^{-P(x)dx}$$

We note that the second side is the sum of two known functions, one of which is multiplied by an optional constant, and the solution becomes as follows:

$$y = Af_1(x) + f_2(x)$$

(3-6) Equations that can be reduced to linear equations:

There are many equations that can be converted to linear equations by either changing the function, changing the variable, or changing both together to a function of two new variables, for example:

$$\frac{1}{y'} + P(y)x = Q(x)$$

It can be written in the form:

$$\frac{dx}{dy} + P(y)x = Q(y)$$

It is linear with respect to x and the variable y.

Example (3-39):

Solve the differential equation: $xy' - 2y - x^3e^x = 0$

Solution: $xy' - 2y - x^3e^x = 0$. *It can be written in the form*

$$y' + \left(-\frac{2}{x}\right)y = x^2e^x$$

It is linear D .E.

$$Q(x) = x^2e^x, \qquad P(x) = -\frac{2}{x}$$

$$\mu = e^{\int P(x)\,dx} = e^{-\int \frac{2}{x}dx} = e^{ln\left(\frac{1}{x^2}\right)} = \frac{1}{x^2}$$

The general solution:

$$\left(\frac{1}{x^2}\right)y = \int x^2e^x\left(\frac{1}{x^2}\right)dx + A = e^x + A$$

Or":

$$y = x^2e^x + Ax^2$$

Example (3-40):Solve the differential equation:

$$x^2dy + (2xy - x + 1)dx = 0, \qquad y(1) = 0$$

Solution: $x^2dy + (2xy - x + 1)dx = 0, \quad y(1) = 0$. *It can be written in the form:*

$$x^2\frac{dy}{dx} + 2xy - x + 1 = 0 \Rightarrow \qquad x^2\frac{dy}{dx} + 2xy = x - 1$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y = \frac{x-1}{x^2}$$

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\,ln\,x} = e^{ln\,x^2} = x^2$$

The general solution:

$$y=\frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx+c\right]=\frac{1}{x^2}\int\frac{x-1}{x^2}x^2dx+C$$

$$y=\frac{1}{x^2}\left[\frac{x^2}{2}-x+c\right]=\frac{1}{2}-\frac{1}{x}+\frac{C}{x^2}$$

To obtain the constant C:

$$y=0 \quad at \quad x=1$$

$$0=\frac{1}{2}-\frac{1}{1}+\frac{C}{1}\Rightarrow \quad C=\frac{1}{2}$$

Then:

$$y=\frac{1}{2}-\frac{1}{x}+\frac{1}{2x^2}$$

Example (3-41): Solve the differential equation:

$$(1+y^2)dx+(2xy+y^2+1)dy=0$$

Solution: $(1+y^2)dx+(2xy+y^2+1)dy=0$***. It can be written in the form***

$$(1+y^2)\frac{dx}{dy}+2xy=-(y^2+1)\Rightarrow \quad \frac{dx}{dy}+\frac{2y}{1+y^2}x=-1$$

Integreating factor:

$$\mu(y)=e^{\int\frac{2y}{1+y^2}dy}=e^{ln(1+y^2)}=(1+y^2)$$

$$x=\frac{1}{\mu(y)}\left[\int \mu(y)Q(y)dy+C\right]$$

$$=\frac{1}{1+y^2}\left[\int -(1+y^2)dy+C\right]$$

$$=\frac{1}{1+y^2}\left[-y-\frac{y^3}{3}+C\right]$$

Example (3-42): Solve the differential equation:

$$(2x + 10y^3)dy + ydx = 0$$

Solution: *The equation can be written in the form:*

$$y\frac{dx}{dy} + 2x + 10y^3 = 0 \Rightarrow \quad \frac{dx}{dy} + \frac{2}{y}x = -10y^2$$

It is a linear equation whose integrating factor is in the form: $\mu(y) = e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$

Thus, the general solution to the equation is:

$$x = \frac{1}{y^2}\left[\int y^2(-10y^2)dy + C\right] = \frac{1}{y^2}[-2y^5 + C]$$

$$= -2y^3 + \frac{C}{y^2}$$

Example (3-43): *Solving the differential equation:*

$$x\frac{dy}{dx} - 2y = x^2 \cos 4x$$

Solution:

The equation can be written in the form:

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \cos 4x$$

It is a linear equation whose integrating factor is in the form:

Thus, the general solution to the equation is:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right] = x^2\left[\int \frac{1}{x^2}x^2 \cos 4x\, dx + C\right]$$

$$y = x^2\left[\frac{1}{4}\sin 4x + C\right] \qquad \Rightarrow y = \frac{x^2}{4}\sin 4x + Cx^2$$

Example (3-44): *Solving the differential equation*

$$\frac{dy}{dx} = \frac{y(y^2 - 1)}{x(xy^3 + 1)}$$

Solution: The equation can be written in the form:

$$x' = \frac{x^2y^3}{y(y^2-1)} + \frac{x}{y(y^2-1)}$$

$$x' - \frac{x}{y(y^2-1)} = x^2\frac{y^2}{y^2-1} \Rightarrow \quad \frac{x'}{x^2} - \frac{1}{xy(y^2-1)} = \frac{y^2}{y^2-1}$$

Bu subsituting

$$\frac{1}{x} = z \qquad \Rightarrow \qquad -\frac{x'}{x^2} = z^1 \quad \Rightarrow -z' - \frac{z}{y(y^2-1)} = \frac{y^2}{y^2-1}$$

We get:

$$z' + \frac{z}{y(y^2-1)} = \frac{-y^2}{y^2-1}$$

It is a linear equation whose integrating factor is in the form:

$$\mu = e^{\int \frac{1}{y(y^2-1)}dy} = e^{\int \frac{-1}{y} + \frac{\frac{1}{2}}{y-1} + \frac{\frac{1}{2}dy}{y+1}}$$

$$\mu = e^{-\ln y + \frac{1}{2}\ln y-1 + \frac{1}{2}\ln y+1} = \frac{1}{y}\sqrt{(y-1)(y+1)}$$

Then the general solution:

$$z = \frac{y}{\sqrt{y^2-1}}\left[\int \frac{\sqrt{y^2-1}}{y}\frac{y^2}{y^2-1}dy + c\right]$$

$$= \frac{y}{\sqrt{y^2-1}}\left[\frac{y}{\sqrt{y^2-1}}dy + c\right]$$

We get:

$$\frac{1}{x} = \frac{y}{\sqrt{y^2-1}}\left[\sqrt{y^2-1} + c\right]$$

Example (3-45): Solving the differential equation

$$y' + \frac{3y}{x} = 6x^2$$

***Solution**: The equation can be written in the form:*

$$y' + \frac{3y}{x} = 6x^2$$

Linear D .E. its integreating factor:

$$\mu = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3$$

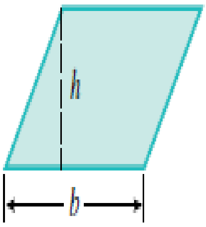
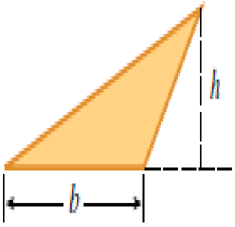
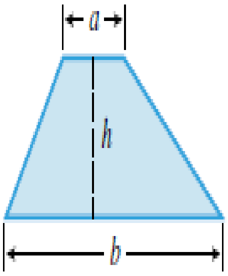
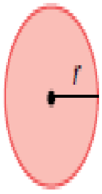
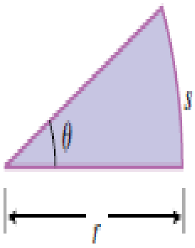
The general solution:

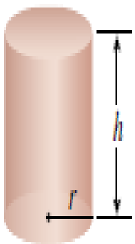
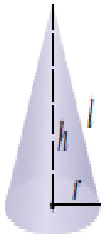
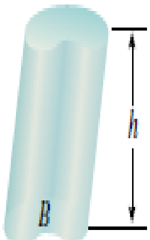
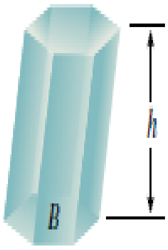
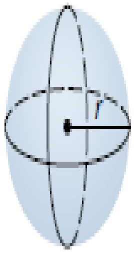
$$\therefore y = \frac{1}{x^3}\left[\int x^3(6x^2)dx + c\right] = \frac{1}{x^3}[x^6 + c]$$

A summary of some geometric and algebraic formulas as well as famous basic integrals

GEOMETRY FORMULAS

A = area, S = lateral surface area, V = volume, h = height, B = area of base, r = radius, l = slant height, C = circumference, s = arc length

Parallelogram	Triangle	Trapezoid	Circle	Sector
				
$A = bh$	$A = \frac{1}{2}bh$	$A = \frac{1}{2}(a+b)h$	$A = \pi r^2,\ C = 2\pi r$	$A = \frac{1}{2}r^2\theta,\ s = r\theta$ (θ in radians)

Right Circular Cylinder	Right Circular Cone	Any Cylinder or Prism with Parallel Bases		Sphere
				
$V = \pi r^2 h,\ S = 2\pi rh$	$V = \frac{1}{3}\pi r^2 h,\ S = \pi rl$	$V = Bh$		$V = \frac{4}{3}\pi r^3,\ S = 4\pi r^2$

ALGEBRA FORMULAS

THE QUADRATIC FORMULA	THE BINOMIAL FORMULA
The solutions of the quadratic equation $ax^2+bx+c=0$ are $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$	$(x+y)^n = x^n + nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 + \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots + nxy^{n-1} + y^n$ $(x-y)^n = x^n - nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 - \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots \pm nxy^{n-1} \mp y^n$

TABLE OF INTEGRALS

BASIC FUNCTIONS

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$
2. $\displaystyle\int \frac{du}{u} = \ln|u| + C$
3. $\displaystyle\int e^u\,du = e^u + C$
4. $\displaystyle\int \sin u\,du = -\cos u + C$
5. $\displaystyle\int \cos u\,du = \sin u + C$
6. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$
7. $\displaystyle\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$
8. $\displaystyle\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$
9. $\displaystyle\int \tan^{-1} u\,du = u\tan^{-1} u - \ln\sqrt{1+u^2} + C$
10. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$
11. $\displaystyle\int \ln u\,du = u\ln u - u + C$
12. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$
13. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C = \ln\left|\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}u\right)\right| + C$
14. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C = \ln\left|\tan \tfrac{1}{2}u\right| + C$
15. $\displaystyle\int \cot^{-1} u\,du = u\cot^{-1} u + \ln\sqrt{1+u^2} + C$
16. $\displaystyle\int \sec^{-1} u\,du = u\sec^{-1} u - \ln\left|u + \sqrt{u^2-1}\right| + C$
17. $\displaystyle\int \csc^{-1} u\,du = u\csc^{-1} u + \ln\left|u + \sqrt{u^2-1}\right| + C$

18. $\int \frac{1}{1 \pm \sin u} du = \tan u \mp \sec u + C$

19. $\int \frac{1}{1 \pm \cos u} du = -\cot u \pm \csc u + C$

20. $\int \frac{1}{1 \pm \tan u} du = \frac{1}{2}(u \pm \ln|\cos u \pm \sin u|) + C$

21. $\int \frac{1}{\sin u \cos u} du = \ln|\tan u| + C$

22. $\int \frac{1}{1 \pm \cot u} du = \frac{1}{2}(u \mp \ln|\sin u \pm \cos u|) + C$

23. $\int \frac{1}{1 \pm \sec u} du = u + \cot u \mp \csc u + C$

24. $\int \frac{1}{1 \pm \csc u} du = u - \tan u \pm \sec u + C$

25. $\int \frac{1}{1 \pm e^u} du = u - \ln(1 \pm e^u) + C$

26. $\int \sin^2 u\, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

27. $\int \cos^2 u\, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

28. $\int \tan^2 u\, du = \tan u - u + C$

29. $\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$

30. $\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\, du$

31. $\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$

32. $\int \cot^2 u\, du = -\cot u - u + C$

33. $\int \sec^2 u\, du = \tan u + C$

34. $\int \csc^2 u\, du = -\cot u + C$

35. $\int \cot^n u\, du = -\frac{1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\, du$

36. $\int \sec^n u\, du = \frac{1}{n-1}\sec^{n-2} u \tan u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du$

37. $\int \csc^n u\, du = -\frac{1}{n-1}\csc^{n-2} u \cot u + \frac{n-2}{n-1}\int \csc^{n-2} u\, du$

38. $\int \sin mu \sin nu\, du = -\frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

39. $\int \cos mu \cos nu\, du = \frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

40. $\int \sin mu \cos nu\, du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{\cos(m-n)u}{2(m-n)} + C$

41. $\int \sin^m u \cos^n u\, du = -\frac{\sin^{m-1} u \cos^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} u \cos^n u\, d$

$$= \frac{\sin^{m+1} u \cos^{n-1} u}{m+n} + \frac{n-1}{m+n}\int \sin^m u \cos^{n-2} u\, du$$

42. $\int e^{au} \sin bu\, du = \frac{e^{au}}{a^2+b^2}(a \sin bu - b \cos bu) + C$

43. $\int e^{au} \cos bu\, du = \frac{e^{au}}{a^2+b^2}(a \cos bu + b \sin bu) + C$

44. $\int u \sin u \, du = \sin u - u \cos u + C$

45. $\int u \cos u \, du = \cos u + u \sin u + C$

46. $\int u^2 \sin u \, du = 2u \sin u + (2 - u^2) \cos u + C$

47. $\int u^2 \cos u \, du = 2u \cos u + (u^2 - 2) \sin u + C$

48. $\int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$

49. $\int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$

50. $\int u^n \ln u \, du = \frac{u^{n+1}}{(n+1)^2}[(n+1) \ln u - 1] + C$

51. $\int ue^u \, du = e^u(u - 1) + C$

52. $\int u^n e^u \, du = u^n e^u - n \int u^{n-1} e^u \, du$

53. $\int u^n a^u \, du = \frac{u^n a^u}{\ln a} - \frac{n}{\ln a} \int u^{n-1} a^u \, du + C$

54. $\int \frac{e^u \, du}{u^n} = -\frac{e^u}{(n-1)u^{n-1}} + \frac{1}{n-1} \int \frac{e^u \, du}{u^{n-1}}$

55. $\int \frac{a^u \, du}{u^n} = -\frac{a^u}{(n-1)u^{n-1}} + \frac{\ln a}{n-1} \int \frac{a^u \, du}{u^{n-1}}$

56. $\int \frac{du}{u \ln u} = \ln |\ln u| + C$

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