Outcomes of an 8-lane swimming race and any n-laned sport

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1 Abstract

Competitive swimming is one of the most data rich sports out there. Just within a single race, all the splits, times, points, and places are automatically calculated and stored by the computer running the meet. Even with so much data the problem: "how many outcomes are there for a swimming race?" still has an unsatisfactory answer. Most solutions currently published only mention that for an 8-lane pool (standard for the Olympics or World Championships), the result would be $8! = 40320$. The reason being that the 1$^{\text{st}}$ lane has 8 available places to finish in, the 2$^{\text{nd}}$ lane has 7, 3$^{\text{rd}}$ has 6 and so forth. However, this particular analysis doesn't take into account the possibility of athletes being disqualified, or more importantly, tying for the same place.

This paper will cover some basic swimming rules that provide context and clarification and a solution to the original question of a typical 8-laned event. Then from the example, several general solutions will be listed for when ties or DQs don't occur.

2 Introduction

2.1 Meet formats

Smaller swim meets typically run events as timed finals, where every athlete swims their event once, and the results are compiled at the end to decide who got 1$^{\text{st}}$, 2$^{\text{nd}}$, 3$^{\text{rd}}$ and so on.

Medium and large meets often have swimmers swim their event two times, once in the preliminary session and once in the finals session (if they are fast enough to make it back). Prelims are swam as a timed final, allowing every athlete to post a seed time. The seed times are then compiled and depending on how many lanes and heats will be swum at finals, swimmers are assigned into a heat by seed time. Most meets use an 8-lane pool and three heats at finals: A, B, and C (Championship, Consolation, and Bonus respectively) allowing for 24 swimmers per event to swim in the finals session.

The focus of this paper will be on heats where n swimmers are *supposed* to compete for in n-laned heat. Occasionally there are not enough swimmers to fill all n lanes for a particular heat, but these heats will be disregarded as they are often non-standard and can have varying lane arrangements.

2.2 Ties and Disqualifications

To understand why ties and disqualifications change the number of outcomes of a typical finals heat, one must understand how they work.

I chose to cite the governing body of swimming in the US (USA Swimming) as it is one of the largest and most developed in the world. Additionally, many rules are written in conjunction with World Swimming (the world governing body, formerly FINA) as their goal is to have swimming be standardized at every meet.

2.2.1 Ties

From the 2023 USA Swimming Handbook, section 102.25:

> AWARDS - When two or more swimmers tie for any place, duplicate awards shall be given to each of such tied swimmers. In such cases, no awards shall be given for the place or places immediately following the tied positions. If two tie for 1$^{\text{st}}$ place, no award for 2$^{\text{nd}}$ place; if three tie for 1$^{\text{st}}$ place, no awards for 2$^{\text{nd}}$ or 3$^{\text{rd}}$, and so on.

This rule extends to places beyond 1$^{\text{st}}$ as well. If two people tie for 2$^{\text{nd}}$, the next available place is 4$^{\text{th}}$ and so on.

2.2.2 Disqualifications

From the 2023 USA Swimming Handbook, section 102.24.6:

> Disqualifications - When a relay team or individual swimmer is disqualified, the subsequent places will move up accordingly and points shall be awarded to conform to the new places. Consolation finalists shall not receive championship final placing. Alternates shall not receive consolation final placing.

Similar to the ties, n-disqualifications remove the last n places from being awarded. For example with two disqualifications swimmers can only place 1$^{\text{st}}$-6$^{\text{th}}$. The rest of the quote pertains to finals heats being "locked"; swimmers in the B heat are unable to place better than 9th place overall (1$^{\text{st}}$ in their heat) even if they swim faster than individuals in the A heat.

3 8-laned example

3.1 Notation

Disqualifications are almost always abbreviated as ”DQs” in swimming, even by USA Swimming on official meet reports. From here on, a disqualification will be referred to as a DQ.

Additionally, a standardized way must be used when conveying results of a race. By using an ordered 8-tuple, both lane number and final placing can easily be conveyed. For example, the ordered 8-tuple:

$$(2, 3, 1, 5, 4, 5, DQ, 7)$$

represents the following information:

Lane Number	Final Result
1	2^{nd}
2	3^{rd}
3	1^{st}
4	5^{th}
5	4^{th}
6	5^{th}
7	DQ
8	7^{th}

Table 1: Example result for an 8-lane heat.

This labeling convention will be used for the rest of the document. Additionally, unless specified otherwise, n is a positive integer and k is a non-negative integer.

3.2 DQ Combinations

The easiest way to think about DQs is to break the problem into 9 cases, one for each number of non-DQed swimmers. By doing this, the number of ways a certain number of DQs could be arranged can be calculated separately and then multiplied by the result found for each case in a later step.

Calculating this is fairly simple. When n is the number of lanes and k is the number of non-DQed swimmers, the number of combinations is:

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

Below is a table of combinations for $n = 8$ and $0 \leq k \leq 8$:

k	Combinations	Example Result
0	$\binom{8}{0} = 1$	$(DQ, ..., DQ)$
1	$\binom{8}{1} = 8$	$(x, DQ, ..., DQ)$
2	$\binom{8}{2} = 28$	$(x, x, DQ, ..., DQ)$
3	$\binom{8}{3} = 56$	$(x, x, x, DQ, ..., DQ)$
4	$\binom{8}{4} = 70$	$(x, x, x, x, DQ, ..., DQ)$
5	$\binom{8}{5} = 56$	$(x, x, x, x, x, DQ, ..., DQ)$
6	$\binom{8}{6} = 28$	$(x, x, x, x, x, x, DQ, DQ)$
7	$\binom{8}{7} = 8$	$(x, x, x, x, x, x, x, DQ)$
8	$\binom{8}{8} = 1$	(x, x, x, x, x, x, x, x)

Table 2: Combinations of non-DQed swimmers for an 8-lane heat.

For those with a keen eye, this is simply the 8^{th} row of Pascal's triangle.

3.3 Tie Compositions

The remaining arrangements come from the number of ways to assign the available places into the available lane arrangements found above. A convenient way to visualize this step is through the use of compositions. A composition of a number is simply a particular arrangement of positive integers that sum to that number.

Why compositions? Take for example a 4-lane race where there is a two-way tie and two solo swimmers. This can be represented as the multiset (unordered, non-unique) $\{a, a, b, c\}$. Substituting the variables for legal finish places, we get 3 variations and their comparable composition:

Variation	Composition
$\{1, 1, 3, 4\}$	$2 + 1 + 1$
$\{1, 2, 2, 4\}$	$1 + 2 + 1$
$\{1, 2, 4, 4\}$	$1 + 1 + 2$

Table 3: Example compositions for $k = 4$.

It is important to note that since summand order matters, compositions are *permutations* of the summands. These are not to be confused with partitions - a similar concept but where only the *combinations* of summands are of importance.

The next step is to calculate the compositions for each $0 \leq k \leq 8$. Below is an exhaustive list for $0 \leq k \leq 4$ and the first 8 compositions for $5 \leq k \leq 8$.

$k = 0$	1	2	3	4	5	6	7	$k = 8$
0	1	2	3	4	5	6	7	8
		$1+1$	$2+1$	$3+1$	$4+1$	$5+1$	$6+1$	$7+1$
			$1+2$	$1+3$	$1+4$	$1+5$	$1+6$	$1+7$
			$1+1+1$	$2+2$	$3+2$	$4+2$	$5+2$	$6+2$
				$2+1+1$	$3+1+1$	$4+1+1$	$5+1+1$	$6+1+1$
				$1+2+1$	$1+3+1$	$1+4+1$	$1+5+1$	$1+6+1$
				$1+1+2$	$1+1+3$	$1+1+4$	$1+1+5$	$1+1+6$
				$1+1+1+1$	$2+2+1$	$3+2+1$	$4+2+1$	$5+3$
					$\vdots$	$\vdots$	$\vdots$	$\vdots$

Additionally, the number of compositions for each $k \geq 1$ can be calculated with 2^{k-1}. This fact will be used later.

3.4 Composition Permutations

Currently the compositions only represent the relative placing of the non-DQed swimmers. For example the composition $2 + 1 + 1$ represents the relative placing of two 1^{st} place finishes, one 3^{rd} place finish and one 4^{th} place finish as described above.

The number of ways this relative positioning can be assigned to the available lanes can be calculated by using the permutations with groupings formula:

$$\binom{k}{a_1, a_2, ..., a_r} = \frac{k!}{a_1! \cdot a_2! \cdot ... \cdot a_r!}$$

where k is the number of non-DQed swimmers and $a_1, a_2, ..., a_r$ are the summands of each composition. Continuing the example from above, the number of assignments of $2 + 1 + 1$ would be:

$$\binom{4}{2, 1, 1} = \frac{4!}{2! \cdot 1! \cdot 1!} = 12.$$

Doing this for the values in Table 3 yields:

$k = 0$	1	2	3	4	5	6	7	$k = 8$
1	1	1	1	1	1	1	1	1
		2	3	4	5	6	7	8
			3	4	5	6	7	8
			1	3	10	15	21	28
				12	20	30	42	56
				12	20	30	42	56
				12	20	30	42	56
				1	5	10	15	7
					$\vdots$	$\vdots$	$\vdots$	$\vdots$

Before defining a function $S(k)$ for the sum of each column in the above table, the corner case of $k = 0$ must be addressed. Since 0 has only one composition: the empty set, and there is only 1 way to permute this, the sum will always be 1. However, for values of $k \geq 1$, a generalized formula $s(k)$ can be defined. So far we have:

$$S(k) = \begin{cases} 1 & \text{if } k = 0 \\ s(k) & \text{if } k \geq 1 \end{cases}$$

To define $s(k)$ for values of $k \geq 1$, we must first be able to iterate through each composition. We will define $C(k)$ to be the set of compositions for a particular k, and $C_j(k)$ to be the j^{th} composition of $C(k)$. Note: since $C(k)$ is a set, the order of the compositions included does not matter, as long as all are present.

Now, to iterate through each summand of each composition, we define $c_j(k)$ to be the multiset of summands for a particular j and k, and $c_j(k)_m$ to be the m^{th} summand of $c_j(k)$. Note: since $c_j(k)$ is a multiset, the order of the summands included does not matter, as long as all are present.

With the definitions above and using the fact that $|C(k)| = 2^{k-1}$ for $k \geq 1$ and definining $|c_j(k)| = r$ for $k \geq 1$ we get the following:

$$s(k) = \sum_{j=1}^{2^{k-1}} \frac{k!}{c_j(k)_1! \cdot c_j(k)_2! \cdot \ldots \cdot c_j(k)_r!}$$

A table providing the result of $S(k)$ for each value of k (including the omitted values for $5 \leq k \leq 8$) is below.

k	0	1	2	3	4	5	6	7	8
$S(k)$	1	1	3	13	75	541	4683	47293	545835

3.5 Product of DQs & Ties

We now have the required combinations of DQs and permutations of ties shown below.

k	0	1	2	3	4	5	6	7	8
$\binom{n}{k}$	1	8	28	56	70	56	28	8	1
$S(k)$	1	1	3	13	75	541	4683	47293	545835

Multiplying $\binom{n}{k}$ by $S(k)$ for each value of k we get:

$1 + 8 + 84 + 728 + 5250 + 30296 + 131124 + 378344 + 545835 = \boxed{1{,}091{,}670}$.

At last, a proper answer for the number of outcomes for an 8-laned swimming race.

4 Generalizations and more

4.1 Other considerations

In section *3.3*, an alternative solution was available. Rather than calcualte the exhaustive list of compositions, the partitions of each k could be calculated but there would be another step required to calculate the combinations excluded by the partitions. For example, $2 + 1 + 1$ is a partion of 4 but $1 + 2 + 1$ and $1 + 1 + 2$ are not (but they are compositions). To make up for the missing arrangements, each $S(k)$ from section *3.4* would need to be multiplied by the sum of

$$\frac{a!}{b_1! \cdot b_2! \cdot \ldots \cdot b_r!}$$

where a is the number of summands of the partition and b_m is the sum of the summands for each unique summand. For example $2 + 1 + 1$ would yield

$$\frac{3!}{1! \cdot 2!} = 3.$$

This matches the three compositions consisting of $\{2, 1, 1\}$ for $k = 4$ listed above.

4.2 Bound estimation

Before starting to solve this problem, estimating an upper and lower bound seemed like a good idea.

The most sensible upper bound was $9^8 = 43{,}046{,}721$ (8 lanes each having 9 places 1^{st}-8^{th} and DQ). For the lower bound, $8! = 40{,}320$ seemed reasonable as it is the total number of outcomes without DQs or ties.

Interestingly, averaging the log of these bounds results in 14.091, not so far off of the log of our result of ~ 1.1 million (13.903).

4.3 Generalizations

4.3.1 Definitions

The following definitions are simply restated for the reader's convienence:

- $n \in \mathbb{Z}^+$

  the number of lanes

- $k \in \mathbb{Z}^+ \cup \{0\}$

  the number of non-DQed swimmers

- $C(k)$

  the set of compositions for a particular k

- $C_j(k)$

  the j^{th} composition of $C(k)$

- $c_j(k)$

  the multiset of summands for a particular $C_j(k)$

- $c_j(k)_m$

  the m^{th} summand of $c_j(k)$

- $r = |c_j(k)|$

  the number of summands for a particular $c_j(k)$

- $s(k) = \sum_{j=1}^{2^{k-1}} \frac{k!}{c_j(k)_1! \cdot c_j(k)_2! \cdot \ldots \cdot c_j(k)_r!}$

  the sum of the permutations of a particular $C(k)$ for $k \geq 1$

- $S(k) = \begin{cases} 1 & \text{if } k = 0 \\ s(k) & \text{if } k \geq 1 \end{cases}$

  the sum of the permutations of a particular $C(k)$ for any $k \geq 0$

4.3.2 Neither DQs nor Ties ("Normal")

As mentioned at the beginning, when both DQs and ties are disregarded, the number of outcomes is simply

$$\boxed{n!}$$

4.3.3 DQs and not Ties ("DQs")

When ties are not considered but DQs are, only the number of choices of non-DQed athletes as well as permutations they can be arranged must be included:

$$\sum_{k=0}^{n} \binom{n}{k} \cdot k! = \sum_{k=0}^{n} \frac{n!}{(n-k)!} = \boxed{\sum_{k=0}^{n} \frac{n!}{k!}}$$

4.3.4 Ties and not DQs ("Ties")

When DQs are not considered but ties are, the formula in 4.3.1 drops the $\binom{n}{k}$ multiplier but keeps the rest:

$$\boxed{\sum_{k=0}^{n} S(k)}$$

4.3.5 Both DQs and Ties ("Both")

As stated above, the number of outcomes when considering both DQs and ties for an n-laned sporting event is:

$$\boxed{\sum_{k=0}^{n} \left[\binom{n}{k} \cdot S(k) \right]}$$

4.3.6 Types of DQs

While not formally covered by the scope of this paper, a DQ is actually an umbrella term used to describe a swimmer who didn't complete the race legally. A DQ is really for when a swimmer breaks a rule during the event. A false start (FS) happens when a swimmer starts the race before the signal is given. A declared false start (DFS) happens when a swimmer is supposed to compete in an event but legally withdraws. A no swim (NS) is similar to a DFS but happens when a swimmer illegally withdraws. Only a small change must be made to the equation in 4.3.4:

$$\sum_{k=0}^{n} \left[\binom{n}{k} \cdot (n-k)^d \cdot S(k) \right]$$

where d is the number of types of DQs.

4.4 Results

Below is a table for 2 through 10 lanes for heats that are: Normal, DQs, Ties, or Both as described in the section previous.

n	Normal	DQs	Ties	Both
2	2	5	3	6
3	6	16	13	26
4	24	65	75	150
5	120	326	541	1082
6	720	1957	4683	9366
7	5040	13700	47293	94586
8	40320	109601	545835	1091670
9	362880	986410	7087261	14174522
10	3628800	9864101	102247563	204495126