Some Helper Function:

Softmax Function:

```
import numpy as np

def softmax(z):
    """
    Compute the softmax probabilities for a given input matrix.

    Parameters:
    z (numpy.ndarray): Logits (raw scores) of shape (m, n), where
                       - m is the number of samples.
                       - n is the number of classes.

    Returns:
    numpy.ndarray: Softmax probability matrix of shape (m, n), where
                   each row sums to 1 and represents the probability
                   distribution over classes.

    Notes:
    - The input to softmax is typically computed as: z = XW + b.
    - Uses numerical stabilization by subtracting the max value per
row.
    """

    z_max = np.max(z, axis=1, keepdims=True)
    exp_z = np.exp(z - z_max)
    softmax_probs = exp_z / np.sum(exp_z, axis=1, keepdims=True)

    return softmax_probs
```

Softmax Test Case:

This test case checks that each row in the resulting softmax probabilities sums to 1, which is the fundamental property of softmax.

```
# Example test case
z_test = np.array([[2.0, 1.0, 0.1], [1.0, 1.0, 1.0]])
softmax_output = softmax(z_test)

# Verify if the sum of probabilities for each row is 1 using assert
row_sums = np.sum(softmax_output, axis=1)

# Assert that the sum of each row is 1
assert np.allclose(row_sums, 1), f"Test failed: Row sums are
{row_sums}"

print("Softmax function passed the test case!")
```

```
Softmax function passed the test case!
```

Prediction Function:

```python
def predict_softmax(X, W, b):
    """
    Predict the class labels for a set of samples using the trained softmax model.

    Parameters:
    X (numpy.ndarray): Feature matrix of shape (n, d), where n is the number of samples and d is the number of features.
    W (numpy.ndarray): Weight matrix of shape (d, c), where c is the number of classes.
    b (numpy.ndarray): Bias vector of shape (c,).

    Returns:
    numpy.ndarray: Predicted class labels of shape (n,), where each value is the index of the predicted class.
    """
    logits = np.dot(X, W) + b
    probabilities = softmax(logits)
    predicted_classes = np.argmax(probabilities, axis=1)

    return predicted_classes
```

Test Function for Prediction Function:

The test function ensures that the predicted class labels have the same number of elements as the input samples, verifying that the model produces a valid output shape.

```python
# Define test case
X_test = np.array([[0.2, 0.8], [0.5, 0.5], [0.9, 0.1]])  # Feature matrix (3 samples, 2 features)
W_test = np.array([[0.4, 0.2, 0.1], [0.3, 0.7, 0.5]])  # Weights (2 features, 3 classes)
b_test = np.array([0.1, 0.2, 0.3])  # Bias (3 classes)

# Expected Output:
# The function should return an array with class labels (0, 1, or 2)

y_pred_test = predict_softmax(X_test, W_test, b_test)

# Validate output shape
assert y_pred_test.shape == (3,), f"Test failed: Expected shape (3,), got {y_pred_test.shape}"

# Print the predicted labels
print("Predicted class labels:", y_pred_test)
```

```
Predicted class labels: [1 1 0]
```

Loss Function:

```python
def loss_softmax(y_pred, y):
    """
    Compute the cross-entropy loss for a single sample.

    Parameters:
    y_pred (numpy.ndarray): Predicted probabilities of shape (c,) for
a single sample,
                             where c is the number of classes.
    y (numpy.ndarray): True labels (one-hot encoded) of shape (c,),
where c is the number of classes.

    Returns:
    float: Cross-entropy loss for the given sample.
    """

    epsilon = 1e-12
    y_pred = np.clip(y_pred, epsilon, 1.0 - epsilon)
    loss = -np.sum(y * np.log(y_pred))

    return loss
```

Test case for Loss Function:

This test case Compares loss for correct vs. incorrect predictions.

- Expects low loss for correct predictions.
- Expects high loss for incorrect predictions.

```python
import numpy as np

# Define correct predictions (low loss scenario)
y_true_correct = np.array([[1, 0, 0], [0, 1, 0], [0, 0, 1]])  # True
one-hot labels
y_pred_correct = np.array([[0.9, 0.05, 0.05],
                           [0.1, 0.85, 0.05],
                           [0.05, 0.1, 0.85]])  # High confidence in
the correct class

# Define incorrect predictions (high loss scenario)
y_pred_incorrect = np.array([[0.05, 0.05, 0.9],  # Highly confident in
the wrong class
                              [0.1, 0.05, 0.85],
                              [0.85, 0.1, 0.05]])

# Compute loss for both cases
loss_correct = loss_softmax(y_pred_correct, y_true_correct)
```

```
loss_incorrect = loss_softmax(y_pred_incorrect, y_true_correct)

# Validate that incorrect predictions lead to a higher loss
assert loss_correct < loss_incorrect, f"Test failed: Expected
loss_correct < loss_incorrect, but got {loss_correct:.4f} >=
{loss_incorrect:.4f}"

# Print results
print(f"Cross-Entropy Loss (Correct Predictions): {loss_correct:.4f}")
print(f"Cross-Entropy Loss (Incorrect Predictions):
{loss_incorrect:.4f}")
```

```
Cross-Entropy Loss (Correct Predictions): 0.4304
Cross-Entropy Loss (Incorrect Predictions): 8.9872
```

Cost Function:

```
def cost_softmax(X, y, W, b):
    """
    Compute the average softmax regression cost (cross-entropy loss)
over all samples.

    Parameters:
    X (numpy.ndarray): Feature matrix of shape (n, d), where n is the
number of samples and d is the number of features.
    y (numpy.ndarray): True labels (one-hot encoded) of shape (n, c),
where n is the number of samples and c is the number of classes.
    W (numpy.ndarray): Weight matrix of shape (d, c).
    b (numpy.ndarray): Bias vector of shape (c,).

    Returns:
    float: Average softmax cost (cross-entropy loss) over all samples.
    """

    logits = np.dot(X, W) + b
    probabilities = softmax(logits)
    epsilon = 1e-12
    probabilities = np.clip(probabilities, epsilon, 1.0 - epsilon)
    total_loss = -np.sum(y * np.log(probabilities))
    n = X.shape[0]

    return total_loss / n
```

Test Case for Cost Function:

The test case assures that the cost for the incorrect prediction should be higher than for the correct prediction, confirming that the cost function behaves as expected.

```
import numpy as np
```

```python
# Example 1: Correct Prediction (Closer predictions)
X_correct = np.array([[1.0, 0.0], [0.0, 1.0]])  # Feature matrix for correct predictions
y_correct = np.array([[1, 0], [0, 1]])  # True labels (one-hot encoded, matching predictions)
W_correct = np.array([[5.0, -2.0], [-3.0, 5.0]])  # Weights for correct prediction
b_correct = np.array([0.1, 0.1])  # Bias for correct prediction

# Example 2: Incorrect Prediction (Far off predictions)
X_incorrect = np.array([[0.1, 0.9], [0.8, 0.2]])  # Feature matrix for incorrect predictions
y_incorrect = np.array([[1, 0], [0, 1]])  # True labels (one-hot encoded, incorrect predictions)
W_incorrect = np.array([[0.1, 2.0], [1.5, 0.3]])  # Weights for incorrect prediction
b_incorrect = np.array([0.5, 0.6])  # Bias for incorrect prediction

# Compute cost for correct predictions
cost_correct = cost_softmax(X_correct, y_correct, W_correct, b_correct)

# Compute cost for incorrect predictions
cost_incorrect = cost_softmax(X_incorrect, y_incorrect, W_incorrect, b_incorrect)

# Check if the cost for incorrect predictions is greater than for correct predictions
assert cost_incorrect > cost_correct, f"Test failed: Incorrect cost {cost_incorrect} is not greater than correct cost {cost_correct}"

# Print the costs for verification
print("Cost for correct prediction:", cost_correct)
print("Cost for incorrect prediction:", cost_incorrect)

print("Test passed!")
```

```
Cost for correct prediction: 0.0006234364133349324
Cost for incorrect prediction: 0.29930861359446115
Test passed!
```

Computing Gradients:

```python
def compute_gradient_softmax(X, y, W, b):
    """
    Compute the gradients of the cost function with respect to weights and biases.

    Parameters:
    X (numpy.ndarray): Feature matrix of shape (n, d).
```

```
    y (numpy.ndarray): True labels (one-hot encoded) of shape (n, c).
    W (numpy.ndarray): Weight matrix of shape (d, c).
    b (numpy.ndarray): Bias vector of shape (c,).

    Returns:
    tuple: Gradients with respect to weights (d, c) and biases (c,).
    """

    logits = np.dot(X, W) + b
    probabilities = softmax(logits)
    error = probabilities - y
    grad_W = np.dot(X.T, error) / X.shape[0]
    grad_b = np.sum(error, axis=0) / X.shape[0]

    return grad_W, grad_b
```

Test case for compute_gradient function:

The test checks if the gradients from the function are close enough to the manually computed gradients using np.allclose, which accounts for potential floating-point discrepancies.

```
import numpy as np

# Define a simple feature matrix and true labels
X_test = np.array([[0.2, 0.8], [0.5, 0.5], [0.9, 0.1]])  # Feature
matrix (3 samples, 2 features)
y_test = np.array([[1, 0, 0], [0, 1, 0], [0, 0, 1]])  # True labels
(one-hot encoded, 3 classes)

# Define weight matrix and bias vector
W_test = np.array([[0.4, 0.2, 0.1], [0.3, 0.7, 0.5]])  # Weights (2
features, 3 classes)
b_test = np.array([0.1, 0.2, 0.3])  # Bias (3 classes)

# Compute the gradients using the function
grad_W, grad_b = compute_gradient_softmax(X_test, y_test, W_test,
b_test)

# Manually compute the predicted probabilities (using softmax
function)
z_test = np.dot(X_test, W_test) + b_test
y_pred_test = softmax(z_test)

# Compute the manually computed gradients
grad_W_manual = np.dot(X_test.T, (y_pred_test - y_test)) /
X_test.shape[0]
grad_b_manual = np.sum(y_pred_test - y_test, axis=0) / X_test.shape[0]

# Assert that the gradients computed by the function match the
manually computed gradients
```

```python
assert np.allclose(grad_W, grad_W_manual), f"Test failed: Gradients w.r.t. W are not equal.\nExpected: {grad_W_manual}\nGot: {grad_W}"
assert np.allclose(grad_b, grad_b_manual), f"Test failed: Gradients w.r.t. b are not equal.\nExpected: {grad_b_manual}\nGot: {grad_b}"

# Print the gradients for verification
print("Gradient w.r.t. W:", grad_W)
print("Gradient w.r.t. b:", grad_b)

print("Test passed!")
```

```
Gradient w.r.t. W: [[ 0.1031051   0.01805685 -0.12116196]
 [-0.13600547  0.00679023  0.12921524]]
Gradient w.r.t. b: [-0.03290036  0.02484708  0.00805328]
Test passed!
```

Implementing Gradient Descent:

```python
def gradient_descent_softmax(X, y, W, b, alpha, n_iter, show_cost=False):
    """
    Perform gradient descent to optimize the weights and biases.

    Parameters:
    X (numpy.ndarray): Feature matrix of shape (n, d).
    y (numpy.ndarray): True labels (one-hot encoded) of shape (n, c).
    W (numpy.ndarray): Weight matrix of shape (d, c).
    b (numpy.ndarray): Bias vector of shape (c,).
    alpha (float): Learning rate.
    n_iter (int): Number of iterations.
    show_cost (bool): Whether to display the cost at intervals.

    Returns:
    tuple: Optimized weights, biases, and cost history.
    """
    cost_history = []

    for i in range(n_iter):
        grad_W, grad_b = compute_gradient_softmax(X, y, W, b)

        W -= alpha * grad_W
        b -= alpha * grad_b

        cost = cost_softmax(X, y, W, b)
        cost_history.append(cost)

        if show_cost and (i % (n_iter // 10) == 0 or i == n_iter - 1):
            print(f"Iteration {i + 1}/{n_iter}, Cost: {cost:.6f}")

    return W, b, cost_history
```

Preparing Dataset:

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from sklearn.model_selection import train_test_split

def load_and_prepare_mnist(csv_file, test_size=0.2, random_state=42):
    """
    Reads the MNIST CSV file, splits data into train/test sets, and
plots one image per class.

    Arguments:
    csv_file (str)       : Path to the CSV file containing MNIST data.
    test_size (float)    : Proportion of the data to use as the test
set (default: 0.2).
    random_state (int)   : Random seed for reproducibility (default:
42).

    Returns:
    X_train, X_test, y_train, y_test : Split dataset.
    """

    # Load dataset
    df = pd.read_csv(csv_file)

    # Separate labels and features
    y = df.iloc[:, 0].values  # First column is the label
    X = df.iloc[:, 1:].values  # Remaining columns are pixel values

    # Normalize pixel values (optional but recommended)
    X = X / 255.0  # Scale values between 0 and 1

    # Split data into train and test sets
    X_train, X_test, y_train, y_test = train_test_split(X, y,
test_size=test_size, random_state=random_state)

    # Plot one sample image per class
    plot_sample_images(X, y)

    return X_train, X_test, y_train, y_test

def plot_sample_images(X, y):
    """
    Plots one sample image for each digit class (0-9).

    Arguments:
    X (np.ndarray): Feature matrix containing pixel values.
    y (np.ndarray): Labels corresponding to images.
    """
```

```
    plt.figure(figsize=(10, 4))
    unique_classes = np.unique(y)  # Get unique class labels

    for i, digit in enumerate(unique_classes):
        index = np.where(y == digit)[0][0]  # Find first occurrence of
the class
        image = X[index].reshape(28, 28)  # Reshape 1D array to 28x28

        plt.subplot(2, 5, i + 1)
        plt.imshow(image, cmap='gray')
        plt.title(f"Digit: {digit}")
        plt.axis('off')

    plt.tight_layout()
    plt.show()

csv_file_path = "mnist_dataset.csv"  # Path to saved dataset
X_train, X_test, y_train, y_test =
load_and_prepare_mnist(csv_file_path)

----------------------------------------------------------------------
-----
FileNotFoundError                         Traceback (most recent call
last)
<ipython-input-14-26a8d649d717> in <cell line: 0>()
      1 csv_file_path = "mnist_dataset.csv"  # Path to saved dataset
----> 2 X_train, X_test, y_train, y_test =
load_and_prepare_mnist(csv_file_path)

<ipython-input-12-8e50d2cd7b7f> in load_and_prepare_mnist(csv_file,
test_size, random_state)
     18
     19     # Load dataset
---> 20     df = pd.read_csv(csv_file)
     21
     22     # Separate labels and features

/usr/local/lib/python3.11/dist-packages/pandas/io/parsers/readers.py
in read_csv(filepath_or_buffer, sep, delimiter, header, names,
index_col, usecols, dtype, engine, converters, true_values,
false_values, skipinitialspace, skiprows, skipfooter, nrows,
na_values, keep_default_na, na_filter, verbose, skip_blank_lines,
parse_dates, infer_datetime_format, keep_date_col, date_parser,
date_format, dayfirst, cache_dates, iterator, chunksize, compression,
thousands, decimal, lineterminator, quotechar, quoting, doublequote,
escapechar, comment, encoding, encoding_errors, dialect, on_bad_lines,
delim_whitespace, low_memory, memory_map, float_precision,
storage_options, dtype_backend)
   1024     kwds.update(kwds_defaults)
   1025
```

```
-> 1026     return _read(filepath_or_buffer, kwds)
   1027
   1028

/usr/local/lib/python3.11/dist-packages/pandas/io/parsers/readers.py
in _read(filepath_or_buffer, kwds)
    618
    619     # Create the parser.
--> 620     parser = TextFileReader(filepath_or_buffer, **kwds)
    621
    622     if chunksize or iterator:

/usr/local/lib/python3.11/dist-packages/pandas/io/parsers/readers.py
in __init__(self, f, engine, **kwds)
   1618
   1619         self.handles: IOHandles | None = None
-> 1620         self._engine = self._make_engine(f, self.engine)
   1621
   1622     def close(self) -> None:

/usr/local/lib/python3.11/dist-packages/pandas/io/parsers/readers.py
in _make_engine(self, f, engine)
   1878                 if "b" not in mode:
   1879                     mode += "b"
-> 1880             self.handles = get_handle(
   1881                 f,
   1882                 mode,

/usr/local/lib/python3.11/dist-packages/pandas/io/common.py in
get_handle(path_or_buf, mode, encoding, compression, memory_map,
is_text, errors, storage_options)
    871         if ioargs.encoding and "b" not in ioargs.mode:
    872             # Encoding
--> 873             handle = open(
    874                 handle,
    875                 ioargs.mode,

FileNotFoundError: [Errno 2] No such file or directory:
'mnist_dataset.csv'
```

A Quick debugging Step:

```python
# Assert that X and y have matching lengths
assert len(X_train) == len(y_train), f"Error: X and y have different
lengths! X={len(X_train)}, y={len(y_train)}"
print("Move forward: Dimension of Feture Matrix X and label vector y
matched.")

----------------------------------------------------------------------
-----
```

```
NameError                                 Traceback (most recent call
last)
<ipython-input-17-c0bddc994d13> in <cell line: 0>()
      1 # Assert that X and y have matching lengths
----> 2 assert len(X_train) == len(y_train), f"Error: X and y have
different lengths! X={len(X_train)}, y={len(y_train)}"
      3 print("Move forward: Dimension of Feture Matrix X and label
vector y matched.")

NameError: name 'X_train' is not defined
```

```
from google.colab import drive
drive.mount('/content/drive')
```

Train the Model:

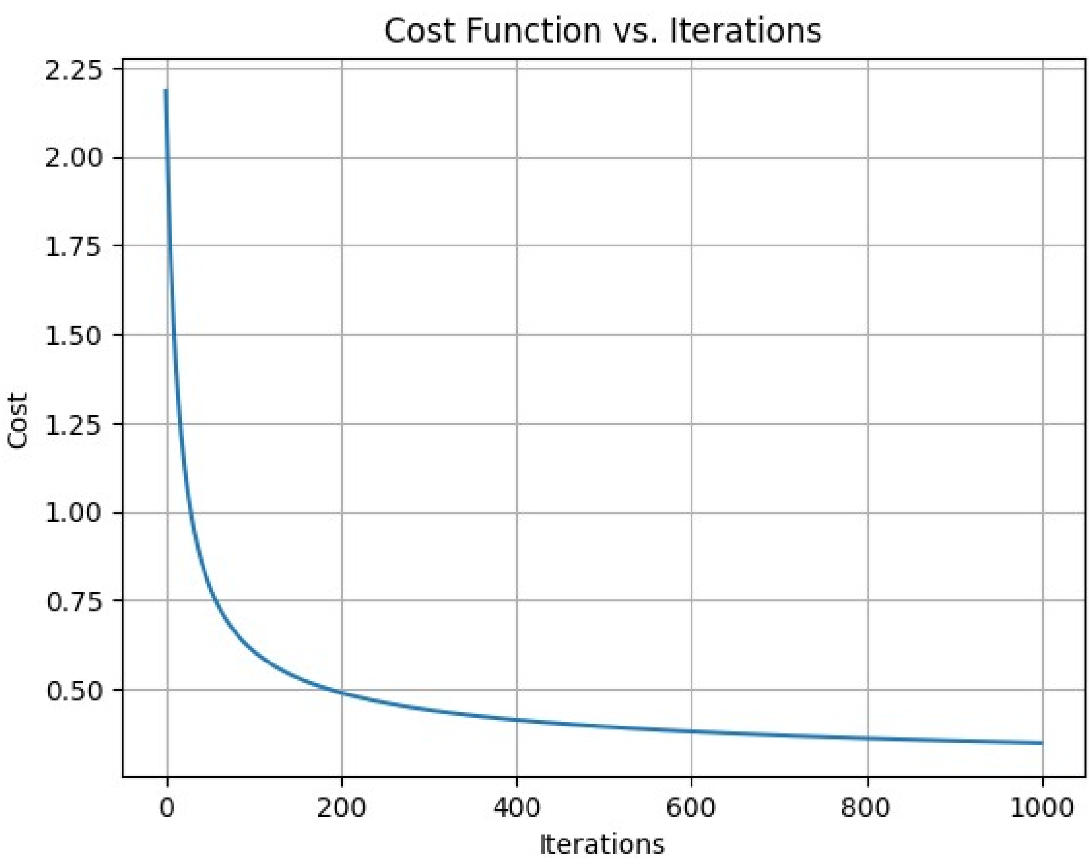
```
print(f"Training data shape: {X_train.shape}")
print(f"Test data shape: {X_test.shape}")
```

```
Training data shape: (48000, 784)
Test data shape: (12000, 784)
```

```
from sklearn.preprocessing import OneHotEncoder

# Check if y_train is one-hot encoded
if len(y_train.shape) == 1:
    encoder = OneHotEncoder(sparse_output=False)  # Use
sparse_output=False for newer versions of sklearn
    y_train = encoder.fit_transform(y_train.reshape(-1, 1))  # One-hot
encode labels
    y_test = encoder.transform(y_test.reshape(-1, 1))  # One-hot
encode test labels

# Now y_train is one-hot encoded, and we can proceed to use it
d = X_train.shape[1]  # Number of features (columns in X_train)
c = y_train.shape[1]  # Number of classes (columns in y_train after
one-hot encoding)

# Initialize weights with small random values and biases with zeros
W = np.random.randn(d, c) * 0.01  # Small random weights initialized
b = np.zeros(c)  # Bias initialized to 0

# Set hyperparameters for gradient descent
alpha = 0.1  # Learning rate
n_iter = 1000  # Number of iterations to run gradient descent

# Train the model using gradient descent
W_opt, b_opt, cost_history = gradient_descent_softmax(X_train,
y_train, W, b, alpha, n_iter, show_cost=True)
```

```python
# Plot the cost history to visualize the convergence
plt.plot(cost_history)
plt.title('Cost Function vs. Iterations')
plt.xlabel('Iterations')
plt.ylabel('Cost')
plt.grid(True)
plt.show()
```

```
Iteration 1/1000, Cost: 2.184226
Iteration 101/1000, Cost: 0.607047
Iteration 201/1000, Cost: 0.489254
Iteration 301/1000, Cost: 0.440654
Iteration 401/1000, Cost: 0.412597
Iteration 501/1000, Cost: 0.393753
Iteration 601/1000, Cost: 0.379953
Iteration 701/1000, Cost: 0.369266
Iteration 801/1000, Cost: 0.360659
Iteration 901/1000, Cost: 0.353524
Iteration 1000/1000, Cost: 0.347534
```

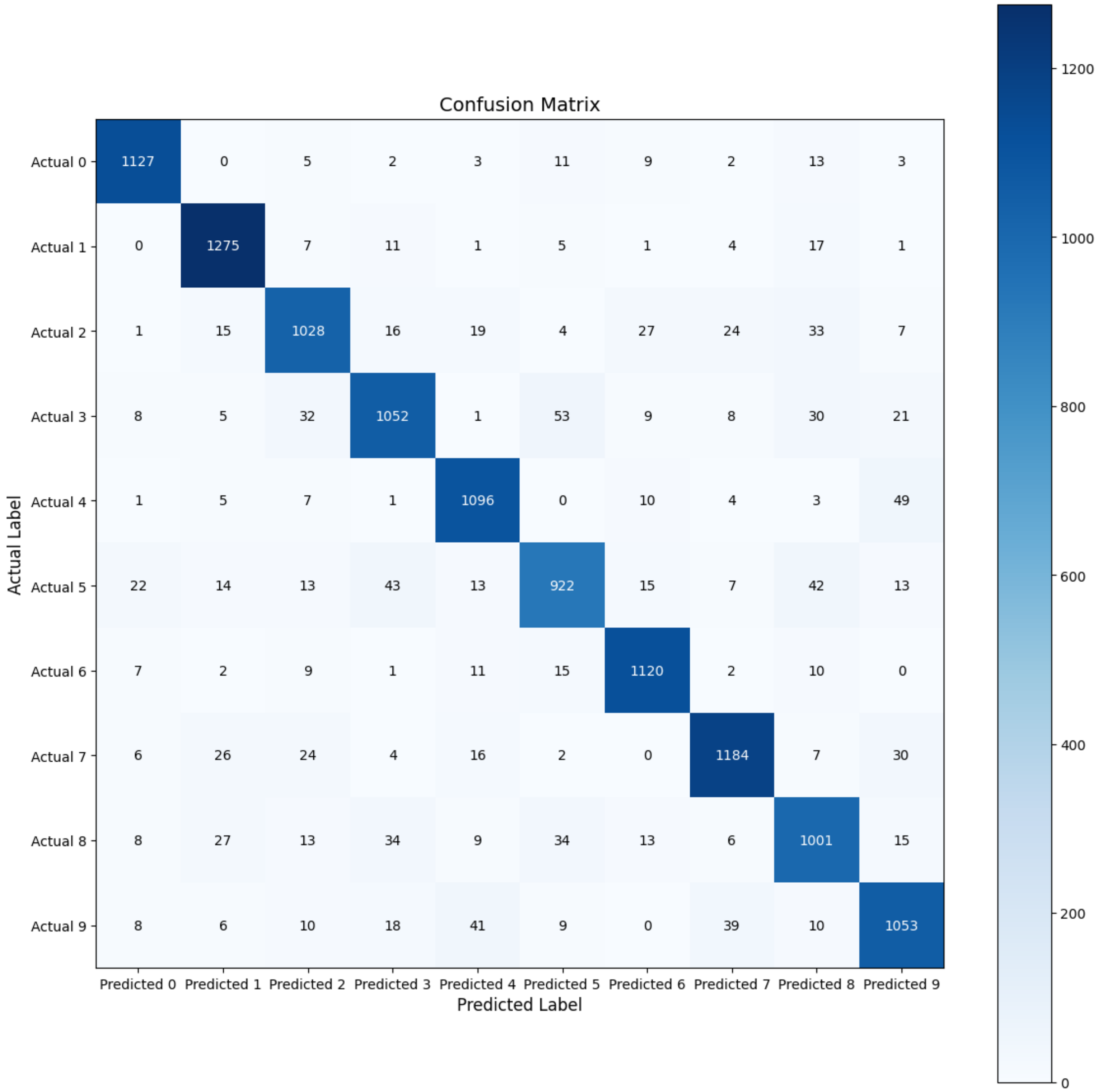


Evaluating the Model:

```
import numpy as np
import matplotlib.pyplot as plt
from sklearn.metrics import confusion_matrix, precision_score,
recall_score, f1_score

# Evaluation Function
def evaluate_classification(y_true, y_pred):
    """
    Evaluate classification performance using confusion matrix,
precision, recall, and F1-score.

    Parameters:
    y_true (numpy.ndarray): True labels
    y_pred (numpy.ndarray): Predicted labels

    Returns:
    tuple: Confusion matrix, precision, recall, F1 score
    """
    # Compute confusion matrix
    cm = confusion_matrix(y_true, y_pred)

    # Compute precision, recall, and F1-score
    precision = precision_score(y_true, y_pred, average='weighted')
    recall = recall_score(y_true, y_pred, average='weighted')
    f1 = f1_score(y_true, y_pred, average='weighted')

    return cm, precision, recall, f1

# Predict on the test set
y_pred_test = predict_softmax(X_test, W_opt, b_opt)

# Evaluate accuracy
y_test_labels = np.argmax(y_test, axis=1)  # True labels in numeric
form

# Evaluate the model
cm, precision, recall, f1 = evaluate_classification(y_test_labels,
y_pred_test)

# Print the evaluation metrics
print("\nConfusion Matrix:")
print(cm)
print(f"Precision: {precision:.2f}")
print(f"Recall: {recall:.2f}")
print(f"F1-Score: {f1:.2f}")

# Visualizing the Confusion Matrix
fig, ax = plt.subplots(figsize=(12, 12))
cax = ax.imshow(cm, cmap='Blues')  # Use a color map for better
```

```python
visualization

# Dynamic number of classes
num_classes = cm.shape[0]
ax.set_xticks(range(num_classes))
ax.set_yticks(range(num_classes))
ax.set_xticklabels([f'Predicted {i}' for i in range(num_classes)])
ax.set_yticklabels([f'Actual {i}' for i in range(num_classes)])

# Add labels to each cell in the confusion matrix
for i in range(cm.shape[0]):
    for j in range(cm.shape[1]):
        ax.text(j, i, cm[i, j], ha='center', va='center',
color='white' if cm[i, j] > np.max(cm) / 2 else 'black')

# Add grid lines and axis labels
ax.grid(False)
plt.title('Confusion Matrix', fontsize=14)
plt.xlabel('Predicted Label', fontsize=12)
plt.ylabel('Actual Label', fontsize=12)

# Adjust layout
plt.tight_layout()
plt.colorbar(cax)
plt.show()
```

```
Confusion Matrix:
[[1127    0    5    2    3   11    9    2   13    3]
 [   0 1275    7   11    1    5    1    4   17    1]
 [   1   15 1028   16   19    4   27   24   33    7]
 [   8    5   32 1052    1   53    9    8   30   21]
 [   1    5    7    1 1096    0   10    4    3   49]
 [  22   14   13   43   13  922   15    7   42   13]
 [   7    2    9    1   11   15 1120    2   10    0]
 [   6   26   24    4   16    2    0 1184    7   30]
 [   8   27   13   34    9   34   13    6 1001   15]
 [   8    6   10   18   41    9    0   39   10 1053]]
Precision: 0.90
Recall: 0.90
F1-Score: 0.90
```

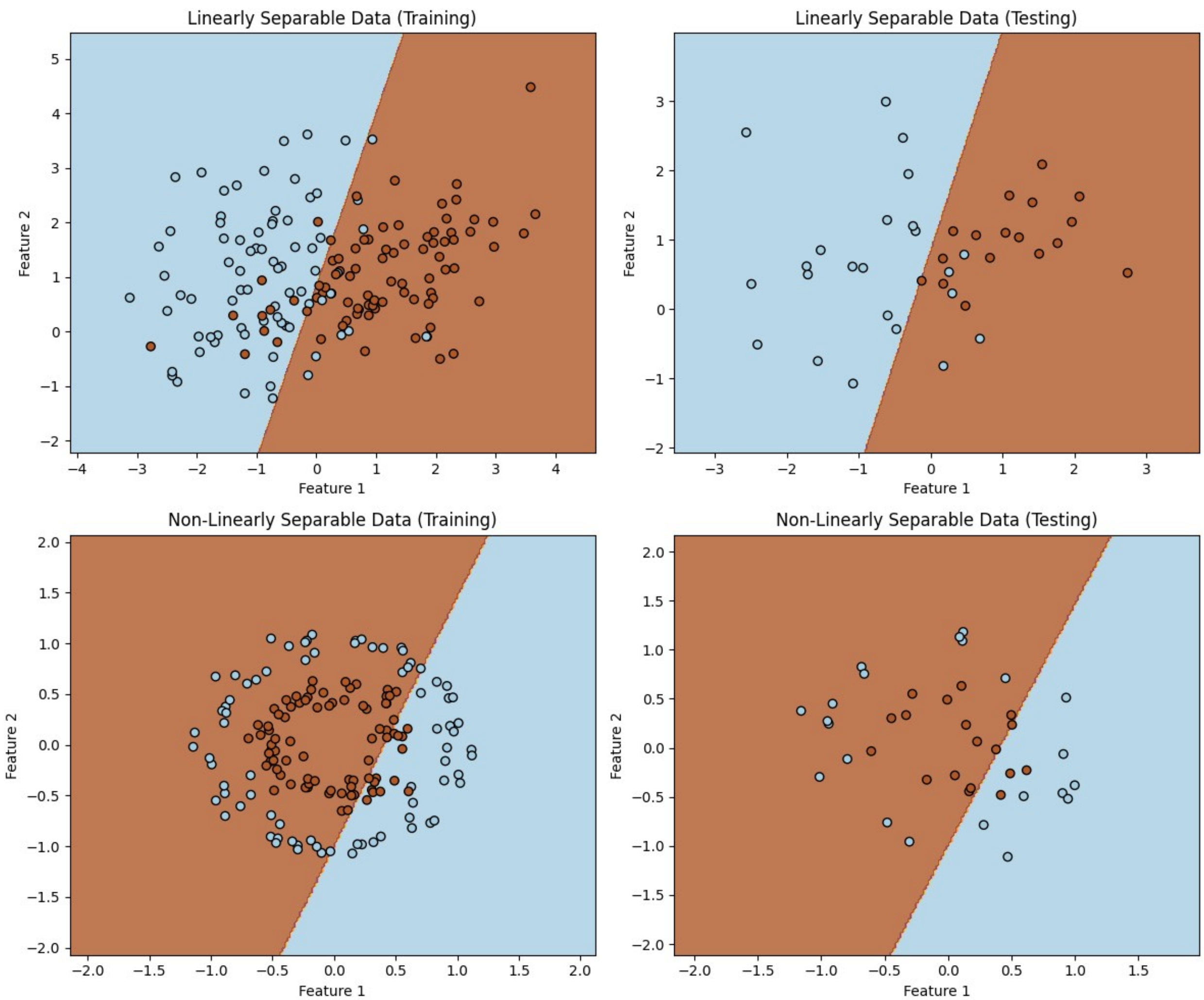


Linear Seperability and Logistic Regression:

```python
import numpy as np
import matplotlib.pyplot as plt
from sklearn.datasets import make_classification, make_circles
from sklearn.model_selection import train_test_split
from sklearn.linear_model import LogisticRegression

# Set random seed for reproducibility
np.random.seed(42)
```

```
# Generate linearly separable dataset
X_linear_separable, y_linear_separable =
make_classification(n_samples=200, n_features=2,

n_informative=2, n_redundant=0,

n_clusters_per_class=1, random_state=42)

# Split the data into training and testing sets
X_train_linear, X_test_linear, y_train_linear, y_test_linear =
train_test_split(
    X_linear_separable, y_linear_separable, test_size=0.2,
random_state=42
)

# Train logistic regression model on linearly separable data
logistic_model_linear_separable = LogisticRegression()
logistic_model_linear_separable.fit(X_train_linear, y_train_linear)

# Generate non-linearly separable dataset (circles)
X_non_linear_separable, y_non_linear_separable =
make_circles(n_samples=200, noise=0.1, factor=0.5,

random_state=42)

# Split the data into training and testing sets
X_train_non_linear, X_test_non_linear, y_train_non_linear,
y_test_non_linear = train_test_split(
    X_non_linear_separable, y_non_linear_separable, test_size=0.2,
random_state=42
)

# Train logistic regression model on non-linearly separable data
logistic_model_non_linear_separable = LogisticRegression()
logistic_model_non_linear_separable.fit(X_train_non_linear,
y_train_non_linear)

# Plot decision boundaries for linearly and non-linearly separable
data
def plot_decision_boundary(ax, model, X, y, title):
    h = .02  # step size in the mesh
    x_min, x_max = X[:, 0].min() - 1, X[:, 0].max() + 1
    y_min, y_max = X[:, 1].min() - 1, X[:, 1].max() + 1
    xx, yy = np.meshgrid(np.arange(x_min, x_max, h), np.arange(y_min,
y_max, h))
    Z = model.predict(np.c_[xx.ravel(), yy.ravel()])
    Z = Z.reshape(xx.shape)
    ax.contourf(xx, yy, Z, alpha=0.8, cmap=plt.cm.Paired)
    ax.scatter(X[:, 0], X[:, 1], c=y, edgecolors='k',
cmap=plt.cm.Paired)
```

```
    ax.set_title(title)
    ax.set_xlabel('Feature 1')
    ax.set_ylabel('Feature 2')

# Create subplots
fig, axes = plt.subplots(2, 2, figsize=(12, 10))

# Plot decision boundary for linearly separable data (Training)
plot_decision_boundary(axes[0, 0], logistic_model_linear_separable,
X_train_linear, y_train_linear,
                       'Linearly Separable Data (Training)')

# Plot decision boundary for linearly separable data (Testing)
plot_decision_boundary(axes[0, 1], logistic_model_linear_separable,
X_test_linear, y_test_linear,
                       'Linearly Separable Data (Testing)')

# Plot decision boundary for non-linearly separable data (Training)
plot_decision_boundary(axes[1, 0],
logistic_model_non_linear_separable, X_train_non_linear,
                       y_train_non_linear, 'Non-Linearly Separable
Data (Training)')

# Plot decision boundary for non-linearly separable data (Testing)
plot_decision_boundary(axes[1, 1],
logistic_model_non_linear_separable, X_test_non_linear,
                       y_test_non_linear, 'Non-Linearly Separable Data
(Testing)')

plt.tight_layout()

# Save the plots as PNG files
plt.savefig('decision_boundaries.png')
plt.show()
```


Linearly Separable Data (Training)
Linearly Separable Data (Testing)
Non-Linearly Separable Data (Training)
Non-Linearly Separable Data (Testing)
Feature 1
Feature 2