FEM with Adjoint Method for Inverse Heat Conduction Problem

1 An Iterative Finite-Element Algorithm for Solving Two-Dimensional Nonlinear Inverse Heat Conduction Problems

1.1 Primal Problem

We have the primal problem which is the following system

$$\begin{cases} \rho c(T)\frac{\partial T}{\partial t} = \nabla \cdot (\lambda(T)\nabla T) & (\boldsymbol{x}, t) \in (\Omega, \mathcal{T}) \quad \text{(a)} \\ T(\boldsymbol{x}, 0) = \tilde{Y}^{\xi} & \boldsymbol{x} \in \Omega \quad \text{(b)} \\ \lambda(T)\nabla T \cdot \boldsymbol{n} = q_u(\boldsymbol{x}, t) & (\boldsymbol{x}, t) \in (\Gamma_u, \mathcal{T}) \quad \text{(c)} \\ \lambda(T)\nabla T \cdot \boldsymbol{n} = q_g(\boldsymbol{x}, t) & (\boldsymbol{x}, t) \in (\Gamma_g, \mathcal{T}) \quad \text{(d)} \end{cases} \tag{1}$$

In Eq.42, q_u is the unknown Neumann boundary condition on Γ_u while q_g is known on Γ_g.

1.2 Objective Function

From the primal problem, we define the objective function

$$\mathcal{J}(q_u) = \frac{1}{2}\int_0^{t_f}\int_{\Gamma_g} \left(T - Y^{\xi}\right)^2 d\Gamma dt + \frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u} (q_u)^2 \, d\Gamma dt. \tag{2}$$

1.3 Sensitivity Problem

Next, we are going to derive the sensitivity problem. Firstly, we have the following definition

$$\theta(\boldsymbol{x}, t; q_u, \Delta q_u) = \lim_{\epsilon \to 0} \frac{T(\boldsymbol{x}, t; q_u + \epsilon \Delta q_u) - T(\boldsymbol{x}, t; q_u)}{\epsilon}, \tag{3}$$

it is clear that

$$T(\boldsymbol{x}, t; q_u + \epsilon\Delta q_u) \approx T(\boldsymbol{x}, t; q_u) + \epsilon\theta(\boldsymbol{x}, t; q_u, \Delta q_u). \tag{4}$$

For simplicity,

$$\begin{aligned} T^{+} &:= T(q_u + \epsilon\Delta q_u), \\ T &:= T(q_u). \end{aligned}$$

From Eq.42(a), we have

$$\rho c(T^+)\frac{\partial T^+}{\partial t} - \rho c(T)\frac{\partial T}{\partial t} = \nabla\left(\lambda(T^+)\nabla T^+\right) - \nabla\left(\lambda(T)\nabla T\right).$$

Then

$$\rho c(T^+)\frac{\partial T}{\partial t} - \rho c(T)\frac{\partial T}{\partial t} + \epsilon\rho c(T^+)\frac{\partial \theta}{\partial t} = \nabla\left[\left(\lambda(T^+) - \lambda(T)\right)\nabla T + \epsilon\lambda(T^+)\nabla\theta\right]. \tag{5}$$

Because of the next two equations

$$\lim_{\epsilon\to 0}\frac{\rho c(T^+) - \rho c(T)}{\epsilon} = \frac{\partial \rho c}{\partial T}\theta, \tag{6}$$

$$\lim_{\epsilon\to 0}\frac{\lambda(T^+) - \lambda(T)}{\epsilon} = \frac{\partial \lambda}{\partial T}\theta, \tag{7}$$

divide ϵ on both sides of Eq.5 and let ϵ goes to 0. We have

$$\begin{aligned}
\frac{\partial \rho c(T)}{\partial T}\theta\frac{\partial T}{\partial t} + \rho c(T)\frac{\partial \theta}{\partial t} &= \nabla\left[\frac{\partial \lambda}{\partial T}\theta\nabla T + \lambda(T)\nabla\theta\right] \\
\frac{\rho c(T)}{\partial t}\theta + \rho c(T)\frac{\partial \theta}{\partial t} &= \nabla\left(\theta\frac{\partial \lambda}{\partial T}\nabla T\right) + \nabla\lambda(T)\nabla\theta \\
\frac{\partial \rho c(T)\theta}{\partial t} &= \nabla^2\left(\lambda(T)\theta\right).
\end{aligned} \tag{8}$$

Next, from Eq.42(b), we have

$$\begin{aligned}
T^+(\boldsymbol{x}, 0) &= Y^\xi \\
T(\boldsymbol{x}, 0) &= Y^\xi.
\end{aligned}$$

It is easy to get

$$\theta(\boldsymbol{x}, 0) = 0. \tag{9}$$

At last, we will derive the boundary conditions of the sensitivity problem. From Eq.42(c), we can get

$$\begin{aligned}
\lambda(T^+)\nabla T^+ \cdot \boldsymbol{n} &= q_u + \epsilon\Delta q_u, \\
\lambda(T)\nabla T \cdot \boldsymbol{n} &= q_u.
\end{aligned}$$

Subtract them and we have

$$\begin{aligned}
\left[\lambda(T^+)\nabla T^+ - \lambda(T)\nabla T\right]\cdot\boldsymbol{n} &= \epsilon\Delta q_u \\
\left[\lambda(T^+)\nabla T + \lambda(T^+)\epsilon\nabla\theta - \lambda(T)\nabla T\right]\cdot\boldsymbol{n} &= \epsilon\Delta q_u.
\end{aligned}$$

Divide ϵ on both sides of the equation above and let ϵ goes to 0. Then, we have

$$\begin{aligned}
\left[\frac{\partial \lambda}{\partial T}\theta\nabla T + \lambda(T)\nabla\theta\right]\cdot\boldsymbol{n} &= \Delta q_u \\
\nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n} &= \Delta q_u.
\end{aligned} \tag{10}$$

Notice that, Eq.10 is satisfied on boundary Γ_u.

Similarly, we can also get

$$\nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}=0 \tag{11}$$

on boundary Γ_g.

With Eq.8, Eq.9, Eq.10 and Eq.11, we can define the sensitivity problem

$$\begin{cases} \rho\frac{\partial(c(T)\theta)}{\partial t}=\nabla^2\left(\lambda(T)\theta\right) & (\boldsymbol{x},t)\in(\Omega,\mathcal{T}) & (a)\\ \theta(\boldsymbol{x},t;q_u,\Delta q_u)=0 & \boldsymbol{x}\in\Omega & (b)\\ \nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}=\Delta q_u(\boldsymbol{x},t) & (\boldsymbol{x},t)\in(\Gamma_u,\mathcal{T}) & (c)\\ \nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}=0 & (\boldsymbol{x},t)\in(\Gamma_g,\mathcal{T}) & (d) \end{cases} \tag{12}$$

1.4 Dual Problem

Now, let's derive the dual problem. From Eq.12(a), we can get

$$\rho\frac{\partial\left(c(T)\theta\right)}{\partial t}-\nabla^2\left(\lambda(T)\theta\right)=0.$$

Multiply both sides of the equation above by ϕ and integrate on spatial domain Ω and time domain $\mathcal{T}$

$$\begin{aligned}
0=&\int_0^{t_f}\int_\Omega\left[\rho\frac{\partial\left(c(T)\theta\right)}{\partial t}-\nabla^2\left(\lambda(T)\theta\right)\right]\phi d\Omega dt\\
=&\int_\Omega\int_0^{t_f}\rho\phi d\left(c(T)\theta\right)d\Omega-\int_0^{t_f}\int_{\partial\Omega}\nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}\phi d\Gamma dt+\int_0^{t_f}\int_\Omega\nabla\left(\lambda(T)\theta\right)\nabla\phi d\Omega dt\\
=&\int_\Omega\rho\phi c(T)\theta\big|_0^{t_f}d\Omega-\int_0^{t_f}\int_\Omega\rho c(T)\theta\frac{\partial\phi}{\partial t}d\Omega dt-\int_0^{t_f}\int_{\partial\Omega}\nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}\phi d\Gamma dt+\int_0^{t_f}\int_\Omega\nabla\left(\lambda(T)\theta\right)\nabla\phi d\Omega dt\\
=&\int_\Omega\rho\phi c(T)\theta\big|_{t_f}d\Omega-\int_0^{t_f}\int_{\Gamma_u}\nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}\phi d\Gamma dt+\int_0^{t_f}\int_\Omega\nabla\left(\lambda(T)\theta\right)\nabla\phi d\Omega dt-\int_0^{t_f}\int_\Omega\rho c(T)\theta\frac{\partial\phi}{\partial t}d\Omega dt\\
=&\int_\Omega\rho\phi c(T)\theta\big|_{t_f}d\Omega-\int_0^{t_f}\int_{\Gamma_u}\nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}\phi d\Gamma dt-\int_0^{t_f}\int_\Omega\nabla^2\phi\left(\lambda(T)\theta\right)d\Omega dt\\
&+\int_0^{t_f}\int_{\partial\Omega}\nabla\phi\cdot\boldsymbol{n}\left(\lambda(T)\theta\right)d\Omega dt-\int_0^{t_f}\int_\Omega\rho c(T)\theta\frac{\partial\phi}{\partial t}d\Omega dt\\
=&\int_\Omega\rho c(T)\theta\phi\big|_{t_f}d\Omega-\int_0^{t_f}\int_{\Gamma_u}\Delta q_u\phi d\Gamma dt-\int_0^{t_f}\int_\Omega\left[\rho c(T)\frac{\partial\phi}{\partial t}+\lambda(T)\nabla^2\phi\right]\theta d\Omega dt\\
&+\int_0^{t_f}\int_{\Gamma_u}\lambda(T)\nabla\phi\cdot\boldsymbol{n}\theta d\Omega dt+\int_0^{t_f}\int_{\Gamma_g}\lambda(T)\nabla\phi\cdot\boldsymbol{n}\theta d\Omega dt.
\end{aligned}\tag{13}$$

From the definition of Eq.42, we can get

$$\begin{aligned}
\mathcal{J}(q_u+\epsilon\Delta q_u)-\mathcal{J}(q_u)=&\frac{1}{2}\int_0^{t_f}\int_{\Gamma_g}\left(T+\epsilon\theta-Y^\xi\right)^2d\Gamma dt+\frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u}\left(q_u+\epsilon\Delta q_u\right)^2d\Gamma dt\\
&-\frac{1}{2}\int_0^{t_f}\int_{\Gamma_g}\left(T-Y^\xi\right)^2d\Gamma dt-\frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u}\left(q_u\right)^2d\Gamma dt\\
=&\frac{1}{2}\int_0^{t_f}\int_{\Gamma_g}\left(2\epsilon\theta T-2\epsilon\theta Y^\xi+\epsilon^2\theta^2\right)d\Gamma dt+\frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u}\left(2\epsilon q_u\Delta q_u+\epsilon^2\Delta q_u^2\right)d\Gamma dt
\end{aligned}$$

Then

$$
\begin{aligned}
(\mathcal{J}(q_u), \Delta q_u) &= \lim_{\epsilon\to 0} \frac{\mathcal{J}(q_u+\epsilon\Delta q_u)-\mathcal{J}(q_u)}{\epsilon} \\
&= \int_0^{t_f}\int_{\Gamma_g} \theta\left(T-Y^{\xi}\right) d\Gamma dt + \gamma\int_0^{t_f}\int_{\Gamma_u} q_u \Delta q_u d\Gamma dt
\end{aligned}
\tag{14}
$$

Based on Eq.14 and Eq.13, we can get the dual problem

$$
\begin{cases}
\rho c(T)\frac{\partial\phi}{\partial t} = -\lambda(T)\nabla^2\phi & (\boldsymbol{x},t)\in(\Omega,\mathcal{T}) \quad (a) \\
\phi(\boldsymbol{x},t_{\max}) = 0 & \boldsymbol{x}\in\Omega \quad (b) \\
\lambda(T)\nabla\phi\cdot\boldsymbol{n} = 0 & (\boldsymbol{x},t)\in(\Gamma_u,\mathcal{T}) \quad (c) \\
\lambda(T)\nabla\phi\cdot\boldsymbol{n} = T(\boldsymbol{x},t;q_u)-Y^{\xi} & (\boldsymbol{x},t)\in(\Gamma_g,\mathcal{T}) \quad (d)
\end{cases}
\tag{15}
$$

1.5 Derivative of Objective Function

Multiply both sides of Eq.12(a) by ϕ and integrate on Ω

$$
\int_0^{t_f}\int_\Omega \rho\frac{\partial\left(c(T)\theta\right)}{\partial t}\phi d\Gamma dt = \int_0^{t_f}\int_\Omega \nabla\cdot\nabla\left(\lambda(T)\theta\right)\phi d\Gamma dt. \tag{16}
$$

Left hand side of Eq.16 is

$$
\begin{aligned}
LHS &= \int_\Omega\int_0^{t_f} \rho\phi d\left(c(T)\right) d\Omega \\
&= \int_\Omega\left[\rho\phi c(T)\theta\Big|_0^{t_f} - \int_0^{t_f}\rho c(T)\theta\frac{\partial\phi}{\partial t}dt\right] d\Omega \\
&= \int_\Omega\int_0^{t_f}\lambda(T)\nabla^2\phi\theta dt d\Omega \\
&= \int_0^{t_f}\int_{\partial\Omega}\lambda(T)\nabla\phi\cdot\boldsymbol{n}\theta d\Gamma dt - \int_0^{t_f}\int_\Omega\nabla\left(\lambda(T)\theta\right)\nabla\phi d\Omega dt \\
&= \int_0^{t_f}\int_{\Gamma_g}\lambda(T)\nabla\phi\cdot\boldsymbol{n}\theta d\Gamma dt - \int_0^{t_f}\int_\Omega\nabla\left(\lambda(T)\theta\right)\nabla\phi d\Omega dt.
\end{aligned}
\tag{17}
$$

In Eq.17, the third equality comes from Eq.12(b) and Eq.15(b) and the last equality comes from Eq.15(c). Right hand side of Eq.16 is

$$
\begin{aligned}
RHS &= \int_0^{t_f}\int_{\partial\Omega}\nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}\phi d\Omega dt - \int_0^{t_f}\int_\Omega\nabla\left(\lambda(T)\theta\right)\nabla\phi d\Omega dt \\
&= \int_0^{t_f}\int_{\Gamma_u}\nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}\phi d\Omega dt - \int_0^{t_f}\int_\Omega\nabla\left(\lambda(T)\theta\right)\nabla\phi d\Omega dt
\end{aligned}
\tag{18}
$$

The second equality of Eq.18 comes from Eq.12(d). Combine Eq.17 and Eq.18, we can get

$$
\int_0^{t_f}\int_{\Gamma_g}\lambda(T)\nabla\phi\cdot\boldsymbol{n}\theta d\Omega dt = \int_0^{t_f}\int_{\Gamma_u}\nabla\left(\lambda(T)\theta\right)\cdot\boldsymbol{n}\phi d\Omega dt.
$$

With Eq.12(c) and Eq.15(d), the equation above becomes

$$\int_0^{t_f} \int_{\Gamma_g} \left(T - Y^\xi\right) \theta d\Omega dt = \int_0^{t_f} \int_{\Gamma_u} \Delta q_u \phi d\Omega dt. \tag{19}$$

Combining Eq.14 and Eq.19, here comes

$$\begin{aligned}
(\mathcal{J}(q_u), \Delta q_u) &= \int_0^{t_f} \int_{\Gamma_g} \theta \left(T - Y^\xi\right) d\Gamma dt + \gamma \int_0^{t_f} \int_{\Gamma_u} q_u \Delta q_u d\Gamma dt \\
&= \int_0^{t_f} \int_{\Gamma_u} \phi \Delta q_u d\Gamma dt + \gamma \int_0^{t_f} \int_{\Gamma_u} q_u \Delta q_u d\Gamma dt \\
&= \int_0^{t_f} \int_{\Gamma_u} \left(\phi + \gamma q_u\right) \Delta q_u d\Gamma dt.
\end{aligned} \tag{20}$$

So it proves that Eq.21

$$\mathcal{J}'(q_u) = \phi + \gamma q_u. \tag{21}$$

1.6 Optimal Step Size

The expression for the optimal step size α_k is

$$\alpha_k = -\frac{\int_0^{t_f} \int_{\Gamma_g} \left(T_k - Y^\xi\right) \theta_k d\Gamma dt + \gamma_{k+1} \int_0^{t_f} \int_{\Gamma_u} \Delta q_{u,k} q_{u,k} d\Gamma dt}{\int_0^{t_f} \int_{\Gamma_g} \theta_k^2 d\Gamma dt + \gamma_{k+1} \int_0^{t_f} \int_{\Gamma_u} \Delta q_{u,k}^2 d\Gamma dt}. \tag{22}$$

In Eq.4, we replace ϵ with α, then

$$\begin{aligned}
\mathcal{J}(q_{u,k+1}) =& \mathcal{J}(q_{u,k} + \alpha_k \Delta q_{u,k}) \\
=& \frac{1}{2} \int_0^{t_f} \int_{\Gamma_g} (T_{k+1} - Y^\xi)^2 d\Gamma dt + \frac{\gamma_{k+1}}{2} \int_0^{t_f} \int_{\Gamma_u} (q_{u,k+1})^2 d\Gamma dt \\
=& \frac{1}{2} \int_0^{t_f} T_k^2 + \alpha_k^2 \theta_k^2 + (Y^\xi)^2 - 2T_k Y^\xi - 2\alpha_k \theta_k Y^\xi + 2T_k \alpha_k \theta_k d\Gamma dt \\
&+ \frac{\gamma_{k+1}}{2} \int_0^{t_f} \int_{\Gamma_u} q_{u,k}^2 + \alpha_k^2 \Delta q_{u,k}^2 + 2\alpha_k q_{u,k} \Delta q_{u,k} d\Gamma dt
\end{aligned}$$

Taking derivative of $\mathcal{J}(q_{u,k+1})$ with α_k, we have

$$\frac{\partial \mathcal{J}(q_{u,k+1})}{\partial \alpha_k} = \int_0^{t_f} \int_{\Gamma_g} \alpha_k \theta_k^2 - \theta_k Y^\xi + \theta_k T_k d\Gamma dt + \gamma_{k+1} \int_0^{t_f} \int_{\Gamma_u} \alpha_k \Delta q_{u,k}^2 + q_{u,k} \Delta q_{u,k} d\Gamma dt. \tag{23}$$

Let Eq.23 equals to 0 and we can get Eq.22.

2 FEM with Adjoint Method for Solving Multi-layers Inverse Heat Conduction Problem

2.1 Primal Problem

We study on a multi-layers inverse heat condution problem which is defined on the domain showed in Fig.1. The heat flux on boundary Γ_g is known while it on Γ_u is unknown. $\Gamma_{ci} (i = 1, 2, \ldots, N-1)$

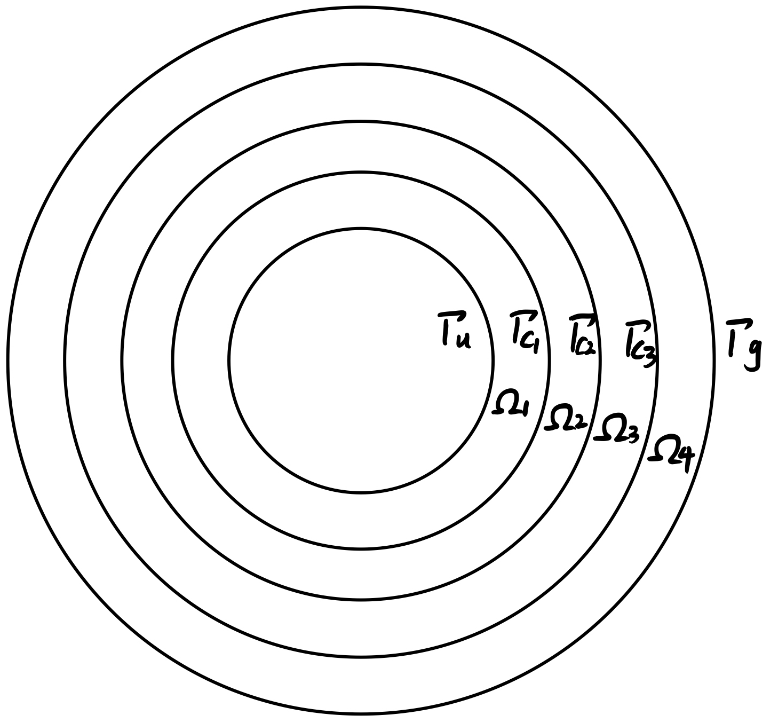

Figure 1: domain of the multi-layers heat conduction problem

is the interface of subdomain Ω_i and Ω_{i+1}.

The primal problem is discribed as below

$$\begin{cases} \rho_i c_i(T_i)\frac{\partial T_i}{\partial t} = \nabla\cdot(\lambda_i(T_i)\nabla T_i) & (\boldsymbol{x},t)\in(\Omega_i,\mathcal{T}) & \text{(a)} \\ T_i(\boldsymbol{x},0) = T_{i0} & \boldsymbol{x}\in\Omega_i & \text{(b)} \\ \lambda_1(T_1)\nabla T_1\cdot\boldsymbol{n} = q_u(\boldsymbol{x},t) & (\boldsymbol{x},t)\in(\Gamma_u,\mathcal{T}) & \text{(c)} \\ \lambda_N(T_N)\nabla T_N\cdot\boldsymbol{n} = q_g(\boldsymbol{x},t) & (\boldsymbol{x},t)\in(\Gamma_g,\mathcal{T}) & \text{(d)} \\ T_i(\boldsymbol{x},t) = T_{i+1}(\boldsymbol{x},t) & (\boldsymbol{x},t)\in(\Gamma_i,\mathcal{T}) & \text{(e)} \\ \lambda_i(T_i)\nabla T_i\cdot\boldsymbol{n}_i + \lambda_{i+1}(T_{i+1})\nabla T_{i+1}\cdot\boldsymbol{n}_{i+1} = 0 & (\boldsymbol{x},t)\in(\Gamma_i,\mathcal{T}) & \text{(f)} \end{cases} \tag{24}$$

where $i = 1, 2, \dots, N$. To inverse the unknown heat flux on boundary Γ_u. We can set sensors on boundary Γ_g to detect the temperature distribution Y on it.

2.2 Object Function

The object function is

$$\mathcal{J}(q_u) = \frac{1}{2}\int_0^{t_f}\int_{\Gamma_g}(T_N - Y)^2 d\Gamma dt + \frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u} q_u^2 d\Gamma dt. \tag{25}$$

Firstly, define

$$\theta_i = \lim_{\epsilon\to 0} \frac{T_i(q_u + \epsilon\Delta q_u) - T_i(q_u)}{\epsilon}. \tag{26}$$

So

$$T_i(q_u + \epsilon\Delta q_u) \approx T_i(q_u) + \epsilon\theta_i. \tag{27}$$

For simplicity, we note $T_i(q_u + \epsilon\Delta q_u)$ as T_i^+ and $T_i(q_u)$ as T_i.

Then from Eq.25,

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{\mathcal{J}(q_u + \epsilon\Delta q_u) - \mathcal{J}(q_u)}{\epsilon} &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[\frac{1}{2}\int_0^{t_f}\int_{\Gamma_g} (T_N^+ - Y)^2 d\Gamma dt + \frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u} (q_u + \epsilon\Delta q_u)^2 d\Gamma dt \\
&\quad - \frac{1}{2}\int_0^{t_f}\int_{\Gamma_g} (T_N - Y)^2 d\Gamma dt - \frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u} q_u^2 d\Gamma dt\Big] \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[\frac{1}{2}\int_0^{t_f}\int_{\Gamma_g} (T_N + \epsilon\theta_N - Y)^2 d\Gamma dt + \frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u} (q_u + \epsilon\Delta q_u)^2 d\Gamma dt \\
&\quad - \frac{1}{2}\int_0^{t_f}\int_{\Gamma_g} (T_N - Y)^2 d\Gamma dt - \frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u} q_u^2 d\Gamma dt\Big] \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[\frac{1}{2}\int_0^{t_f}\int_{\Gamma_g} \epsilon^2\theta_N^2 + 2\epsilon\theta_N(T_N - Y) d\Gamma dt + \frac{\gamma}{2}\int_0^{t_f}\int_{\Gamma_u} \epsilon^2\Delta q_u^2 + 2\epsilon q_u\Delta q_u d\Gamma dt\Big] \\
&= \int_0^{t_f}\int_{\Gamma_g} \theta_N(T_N - Y) d\Gamma dt + \gamma\int_0^{t_f}\int_{\Gamma_u} q_u\Delta q_u d\Gamma dt. \qquad (28)
\end{aligned}$$

We take the note that

$$(\mathcal{J}'(q_u), \Delta q_u) \coloneqq \int_0^{t_f}\int_{\Gamma_g} \theta_N(T_N - Y) d\Gamma dt + \gamma\int_0^{t_f}\int_{\Gamma_u} q_u\Delta q_u d\Gamma dt. \tag{29}$$

2.3 Sensitivity Problem

From Eq.24(a), we have

$$\begin{aligned}
\rho_i c_i(T_i^+)\frac{\partial T_i^+}{\partial t} &= \nabla\cdot\left(\lambda_i(T_i^+)\nabla T_i^+\right), \\
\rho_i c_i(T_i)\frac{\partial T_i}{\partial t} &= \nabla\cdot\left(\lambda_i(T_i)\nabla T_i\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\rho_i c_i(T_i^+)\frac{\partial T_i^+}{\partial t} - \rho_i c_i(T_i)\frac{\partial T_i}{\partial t} &= \nabla\cdot\left(\lambda_i(T_i^+)\nabla T_i^+ - \lambda_i(T_i)\nabla T_i\right) \\
\rho_i c_i(T_i^+)\frac{\partial T_i}{\partial t} + \rho_i c_i(T_i^+)\epsilon\frac{\partial \theta_i}{\partial t} - \rho_i c_i(T_i)\frac{\partial T_i}{\partial t} &= \nabla\cdot\left(\lambda_i(T_i^+)\nabla T_i + \lambda_i(T_i^+)\epsilon\nabla\theta_i - \lambda_i(T_i)\nabla T_i\right).
\end{aligned}$$

Deviding ϵ on both sides and let $\epsilon \to 0$, we have

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{\rho_i c_i(T_i^+) - \rho_i c_i(T_i)}{\epsilon} \frac{\partial T_i}{\partial t} + \rho_i c_i(T_i^+) \frac{\partial \theta_i}{\partial t} &= \nabla \cdot \left(\lim_{\epsilon\to 0} \frac{\lambda_i(T_i^+) - \lambda_i(T_i)}{\epsilon} \nabla T_i + \lambda_i(T_i^+) \nabla \theta_i \right) \\
\frac{\partial \rho_i c_i(T_i)}{\partial T_i} \theta_i \frac{\partial T_i}{\partial t} + \rho_i c_i(T_i) \frac{\partial \theta_i}{\partial t} &= \nabla \cdot \left(\frac{\partial \lambda_i(T_i)}{\partial T_i} \theta_i \nabla T_i + \lambda_i(T_i) \nabla \theta_i \right) \\
\rho_i \frac{\partial c_i(T_i)\theta_i}{\partial t} &= \nabla \cdot \left(\nabla \lambda_i(T_i) \theta_i \right). \qquad (30)
\end{aligned}$$

From Eq.24(b), we have

$$T_i^+(t=0) - T_i(t=0) = T_{i0} - T_{i0}.$$

It is easy to get that

$$\theta_i(t=0) = 0. \qquad (31)$$

Then, on Γ_u, based on Eq.24(c)

$$\begin{aligned}
\left(\lambda_1(T_1^+) \nabla T_1^+ - \lambda_1(T_1) \nabla T_1 \right) \cdot \boldsymbol{n} &= q_u + \epsilon \Delta q_u - q_u \\
\left(\lambda_1(T_1^+) \nabla T_1 + \epsilon \lambda_1(T_1^+) \nabla \theta_1 - \lambda_1(T_1) \nabla T_1 \right) \cdot \boldsymbol{n} &= \epsilon \Delta q_u.
\end{aligned}$$

Deviding ϵ on both sides and let $\epsilon \to 0$

$$\begin{aligned}
\lim_{\epsilon\to 0} \left(\frac{\lambda_1(T_1^+) - \lambda_1(T_1)}{\epsilon} \nabla T_1 + \lambda_1(T_1^+) \nabla \theta_1 \right) \cdot \boldsymbol{n} &= \delta q_u \\
\left(\frac{\partial \lambda_1(T_1)}{\partial T_1} \theta_1 \nabla T_1 + \lambda_1(T_1) \nabla \theta_1 \right) \cdot \boldsymbol{n} &= \Delta q_u \\
\nabla \left(\lambda_1(T_1) \theta_1 \right) \cdot \boldsymbol{n} &= \Delta q_u. \qquad (32)
\end{aligned}$$

Similarly, we can get

$$\nabla \left(\lambda_N(T_N) \theta_N \right) \cdot \boldsymbol{n} = 0 \qquad (33)$$

on Γ_g.

Next, from Eq.24(e), we can get

$$\begin{aligned}
T_i^+ - T_i &= T_{i+1}^+ - T_{i+1} \\
\epsilon \theta_i &= \epsilon \theta_{i+1} \\
\theta_i &= \theta_{i+1} \qquad (34)
\end{aligned}$$

on Γ_i.

At last, from Eq.24(f),

$$\left(\lambda_i(T_i^+) \nabla T_i^+ - \lambda_i(T_i) \nabla T_i \right) \cdot \boldsymbol{n}_i + \left(\lambda_{i+1}(T_{i+1}^+) \nabla T_{i+1}^+ - \lambda_{i+1}(T_{i+1}) \nabla T_{i+1} \right) \cdot \boldsymbol{n}_{i+1} = 0.$$

It is easy to derive that

$$\nabla(\lambda_i(T_i)\theta_i) \cdot \boldsymbol{n}_i + \nabla(\lambda_{i+1}(T_{i+1}) \nabla \theta_{i+1}) \cdot \boldsymbol{n}_{i+1} = 0. \qquad (35)$$

Combining Eq.30, Eq.31, Eq.32, Eq.33, Eq.34 and Eq.35, we have the sensitivity problem

$$\begin{cases} \rho_i \frac{\partial c_i(T_i)\theta_i}{\partial t} = \nabla \cdot (\nabla \lambda_i(T_i)\theta_i) & (\boldsymbol{x}, t) \in (\Omega_i, \mathcal{T}) & \text{(a)} \\ \theta_i(t=0) = 0 & \boldsymbol{x} \in \Omega_i & \text{(b)} \\ \nabla\left(\lambda_1(T_1)\theta_1\right) \cdot \boldsymbol{n} = \Delta q_u & (\boldsymbol{x}, t) \in (\Gamma_u, \mathcal{T}) & \text{(c)} \\ \nabla\left(\lambda_N(T_N)\theta_N\right) \cdot \boldsymbol{n} = 0 & (\boldsymbol{x}, t) \in (\Gamma_g, \mathcal{T}) & \text{(d)} \\ \theta_i = \theta_{i+1} & (\boldsymbol{x}, t) \in (\Gamma_i, \mathcal{T}) & \text{(e)} \\ \nabla(\lambda_i(T_i)\theta_i) \cdot \boldsymbol{n}_i + \nabla(\lambda_{i+1}(T_{i+1})\nabla\theta_{i+1}) \cdot \boldsymbol{n}_{i+1} = 0 & (\boldsymbol{x}, t) \in (\Gamma_i, \mathcal{T}) & \text{(f)} \end{cases} \tag{36}$$

where $i = 1, 2, \ldots, N$.

2.4 Dual Problem

According to Eq.36(a), one can get

$$0 = \sum_{i=1}^{N} \int_0^{t_f} \int_{\Omega_i} \left(\rho_i \frac{\partial c_i(T_i)\theta_i}{\partial t} - \nabla \cdot (\nabla \lambda_i(T_i)\theta_i) \right) \phi_i d\Omega dt.$$

Then,

$$\begin{aligned} 0 =& \sum_{i=1}^{N} \left(\int_0^{t_f} \int_{\Omega_i} \rho_i \frac{\partial c_i(T_i)\theta_i}{\partial t} \phi_i d\Omega dt - \int_0^{t_f} \int_{\Omega_i} \nabla \cdot \big(\nabla \lambda_i(T_i)\theta_i\big) \phi_i d\Omega dt \right) \\ =& \sum_{i=1}^{N} \left(\int_{\Omega_i} \int_0^{t_f} \rho_i \phi_i d\big(c_i(T_i)\theta_i\big) d\Omega - \int_0^{t_f} \int_{\Omega_i} \nabla \cdot \big(\nabla \lambda_i(T_i)\theta_i\big) \phi_i d\Omega dt \right) \\ =& \sum_{i=1}^{N} \left(\int_{\Omega_i} \rho_i c_i(T_i)\theta_i \phi_i \big|_0^{t_f} d\Omega - \int_0^{t_f} \int_{\Omega_i} \rho_i c_i(T_i)\theta_i \frac{\partial \phi_i}{\partial t} d\Omega dt \right. \\ & \left. - \int_0^{t_f} \int_{\partial\Omega_i} \nabla \lambda_i(T_i)\theta_i \cdot \boldsymbol{n} \phi_i d\Gamma dt + \int_0^{t_f} \int_{\Omega_i} \nabla \lambda_i(T_i)\theta_i \nabla \phi_i d\Omega dt \right) \\ =& \sum_{i=1}^{N} \left(\int_{\Omega_i} \rho_i c_i(T_i)\theta_i \phi_i \big|_{t_f} d\Omega - \int_0^{t_f} \int_{\Omega_i} \rho_i c_i(T_i)\theta_i \frac{\partial \phi_i}{\partial t} d\Omega dt \right. \\ & - \int_0^{t_f} \int_{\partial\Omega_i} \nabla \lambda_i(T_i)\theta_i \cdot \boldsymbol{n} \phi_i d\Gamma dt + \int_0^{t_f} \int_{\partial\Omega_i} \lambda_i(T_i)\theta_i \nabla \phi_i \cdot \boldsymbol{n} d\Gamma dt \\ & \left. - \int_0^{t_f} \int_{\Omega_i} \lambda_i(T_i)\theta_i \nabla \cdot \nabla \phi_i d\Omega dt \right) \end{aligned}$$

$$
\begin{aligned}
=&\sum_{i=1}^{N}\Bigg(\int_{\Omega_i}\rho_i c_i(T_i)\theta_i\phi_i\big|_{t_f}d\Omega-\int_0^{t_f}\int_{\Omega_i}\Big(\rho_i c_i(T_i)\frac{\partial\phi_i}{\partial t}+\lambda_i(T_i)\nabla\cdot\nabla\phi_i\Big)\theta_i d\Omega dt\\
&-\Big(\int_0^{t_f}\int_{\Gamma_g}\nabla\lambda_N(T_N)\theta_N\cdot\boldsymbol{n}\phi_N d\Gamma dt+\int_0^{t_f}\int_{\Gamma_u}\nabla\lambda_1(T_1)\theta_1\cdot\boldsymbol{n}\phi_1 d\Gamma dt\\
&+\int_0^{t_f}\int_{\Gamma_i}\nabla\lambda_i(T_i)\theta_i\cdot\boldsymbol{n}\phi_i+\nabla\lambda_{i+1}(T_{i+1})\theta_{i+1}\cdot\boldsymbol{n}\phi_{i+1}d\Gamma dt\Big)\\
&+\Big(\int_0^{t_f}\int_{\Gamma_g}\lambda_N(T_N)\theta_N\nabla\phi_N\cdot\boldsymbol{n}d\Gamma dt+\int_0^{t_f}\int_{\Gamma_u}\lambda_1(T_1)\theta_1\nabla\phi_1\cdot\boldsymbol{n}d\Gamma dt\\
&+\int_0^{t_f}\int_{\Gamma_i}\lambda_i(T_i)\theta_i\nabla\phi_i\cdot\boldsymbol{n}+\lambda_{i+1}(T_{i+1})\theta_{i+1}\nabla\phi_{i+1}\cdot\boldsymbol{n}d\Gamma dt\Big)\Bigg)\\
=&\sum_{i=1}^{N}\Bigg(\int_{\Omega_i}\rho_i c_i(T_i)\theta_i\phi_i\big|_{t_f}d\Omega-\int_0^{t_f}\int_{\Omega_i}\Big(\rho_i c_i(T_i)\frac{\partial\phi_i}{\partial t}+\lambda_i(T_i)\nabla\cdot\nabla\phi_i\Big)\theta_i d\Omega dt\\
&-\Big(\int_0^{t_f}\int_{\Gamma_u}\Delta q_u\phi_1 d\Gamma dt+\int_0^{t_f}\int_{\Gamma_i}\nabla\lambda_i(T_i)\theta_i\cdot\boldsymbol{n}\phi_i+\nabla\lambda_{i+1}(T_{i+1})\theta_{i+1}\cdot\boldsymbol{n}\phi_{i+1}d\Gamma dt\Big)\\
&+\Big(\int_0^{t_f}\int_{\Gamma_g}\lambda_N(T_N)\theta_N\nabla\phi_N\cdot\boldsymbol{n}d\Gamma dt+\int_0^{t_f}\int_{\Gamma_u}\lambda_1(T_1)\theta_1\nabla\phi_1\cdot\boldsymbol{n}d\Gamma dt\\
&+\int_0^{t_f}\int_{\Gamma_i}\lambda_i(T_i)\theta_i\nabla\phi_i\cdot\boldsymbol{n}+\lambda_{i+1}(T_{i+1})\theta_{i+1}\nabla\phi_{i+1}\cdot\boldsymbol{n}d\Gamma dt\Big)\Bigg).
\end{aligned}
\tag{37}
$$

On the basis of Eq.36, dual problem is raised as below

$$
\begin{cases}
\rho_i c_i(T_i)\frac{\partial\phi_i}{\partial t}=-\lambda_i(T_i)\nabla\cdot\nabla\phi_i & (\boldsymbol{x},t)\in(\Omega_i,\mathcal{T}) & \text{(a)}\\
\phi_i(t=t_f)=0 & \boldsymbol{x}\in\Omega_i & \text{(b)}\\
\lambda_1(T_1)\nabla\phi_1\cdot\boldsymbol{n}=0 & (\boldsymbol{x},t)\in(\Gamma_u,\mathcal{T}) & \text{(c)}\\
\lambda_N(T_N)\nabla\phi_N\cdot\boldsymbol{n}=T_N-Y & (\boldsymbol{x},t)\in(\Gamma_g,\mathcal{T}) & \text{(d)}\\
\phi_i=\phi_{i+1} & (\boldsymbol{x},t)\in(\Gamma_i,\mathcal{T}) & \text{(e)}\\
\lambda_i(T_i)\nabla\phi_i\cdot\boldsymbol{n}_i+\lambda_{i+1}(T_{i+1})\nabla\phi_{i+1}\cdot\boldsymbol{n}_{i+1}=0 & (\boldsymbol{x},t)\in(\Gamma_i,\mathcal{T}) & \text{(f)}
\end{cases}
\tag{38}
$$

where $i=1,2,\ldots,N$.

So Eq.37 can be written as

$$
0=-\int_0^{t_f}\int_{\Gamma_u}\Delta q_u\phi_1 d\Gamma dt+\int_0^{t_f}\int_{\Gamma_g}\theta_N(T_N-Y)d\Gamma dt. \tag{39}
$$

Then, combine Eq.29 and Eq.39, we can get the derivative of object function $\mathcal{J}(q_u)$ is

$$
\mathcal{J}'(q_u)=\phi_1+\gamma q_u. \tag{40}
$$

2.5 Optimal Step Size

Replacing ϵ with α_k, then

$$\begin{aligned}
\mathcal{J}(q_{u,k+1}) =&\mathcal{J}(q_{u,k}+\alpha_k\Delta q_{u,k}) \\
=&\frac{1}{2}\int_0^{t_f}\int_{\Gamma_g}(T_{N,k+1}-Y)^2 d\Gamma dt+\frac{\gamma_{k+1}}{2}\int_0^{t_f}\int_{\Gamma_u}(q_{u,k+1})^2 d\Gamma dt \\
=&\frac{1}{2}\int_0^{t_f}\int_{\Gamma_g}(T_{N,k}+\alpha_k\theta_{N,k}-Y)^2 d\Gamma dt+\frac{\gamma_{k+1}}{2}\int_0^{t_f}\int_{\Gamma_u}(q_{u,k}+\alpha_k\Delta q_{u,k})^2 d\Gamma dt \\
=&\frac{1}{2}\int_0^{t_f}\int_{\Gamma_g}(T_{N,k}-Y)^2+\alpha_k^2\theta_{N,k}^2+2\alpha_k\theta_{N,k}(T_{N,k}-Y)d\Gamma dt \\
&+\frac{\gamma_{k+1}}{2}\int_0^{t_f}\int_{\Gamma_u}q_{u,k}^2+\alpha_k^2\Delta q_{u,k}^2+2\alpha_k q_{u,k}\Delta q_{u,k}d\Gamma dt
\end{aligned}$$

Taking derivative of $\mathcal{J}(q_{u,k+1})$ with α_k

$$\frac{\partial\mathcal{J}(q_{u,k+1})}{\partial\alpha_k}=\int_0^{t_f}\int_{\Gamma_g}\alpha_k\theta_{N,k}^2+\theta_{N,k}(T_{N,k}-Y)d\Gamma dt+\gamma_{k+1}\int_0^{t_f}\int_{\Gamma_u}\alpha_k\Delta q_{u,k}^2+q_{u,k}\Delta q_{u,k}d\Gamma dt.$$

Letting $\partial\mathcal{J}(q_{u,k+1})/\partial\alpha_k$ equals to 0, then we can get the optimal step

$$\alpha_k=-\frac{\int_0^{t_f}\int_{\Gamma_g}(T_{N,k}-Y)\theta_{N,k}d\Gamma dt+\gamma_{k+1}\int_0^{t_f}\int_{\Gamma_u}q_{u,k}\Delta q_{u,k}d\Gamma dt}{\int_0^{t_f}\int_{\Gamma_g}\theta_{N,k}^2 d\Gamma dt+\gamma_{k+1}\int_0^{t_f}\int_{\Gamma_u}\Delta q_{u,k}^2 d\Gamma dt}. \tag{41}$$

3 An Iterative Finite-Element Algorithm for Solving Nonlinear Inverse Heat Conduction Problems With Unknown Conduction Coefficient

In this problem, the coefficient of heat conduction $\lambda(T)$ is unknown. Then we consider λ is function of $\boldsymbol{x}$ and t which is $\lambda(\boldsymbol{x},t)$.

3.1 Primal Problem

$$\begin{cases}
\rho c(T)\frac{\partial T}{\partial t}=\nabla\cdot(\lambda\nabla T) & (\boldsymbol{x},t)\in(\Omega,\mathcal{T}) \quad \text{(a)}\\
T(\boldsymbol{x},0)=\tilde{Y}^{\xi} & \boldsymbol{x}\in\Omega \quad \text{(b)}\\
\lambda\nabla T\cdot\boldsymbol{n}=q(\boldsymbol{x},t) & (\boldsymbol{x},t)\in(\partial\Omega,\mathcal{T}) \quad \text{(c)}
\end{cases} \tag{42}$$

3.2 Objective Function

From the primal problem, we define the objective function

$$\mathcal{J}(\lambda)=\frac{1}{2}\int_0^{t_f}\int_{\partial\Omega}\left(T-Y^{\xi}\right)^2 d\Gamma dt+\frac{\gamma}{2}\int_0^{t_f}\int_{\Omega}\lambda^2 d\Omega dt. \tag{43}$$

3.3 Sensitivity Problem

First

$$\theta(\boldsymbol{x},t;\lambda,\Delta\lambda)=\lim_{\epsilon\to 0}\frac{T(\boldsymbol{x},t;\lambda+\epsilon\Delta\lambda)-T(\boldsymbol{x},t;\lambda)}{\epsilon}, \tag{44}$$

Then

$$T(\boldsymbol{x},t;\lambda+\epsilon\Delta\lambda)\approx T(\boldsymbol{x},t;\lambda)+\epsilon\theta(\boldsymbol{x},t;\lambda,\Delta\lambda). \tag{45}$$

Note that

$$\begin{aligned}T^+ &\coloneqq T(\lambda+\epsilon\Delta\lambda),\\ T &\coloneqq T(\lambda).\end{aligned}$$

From

$$\rho c(T^+)\frac{\partial T^+}{\partial t}-\rho c(T)\frac{\partial T}{\partial t}=\nabla\left((\lambda+\epsilon\Delta\lambda)\nabla T^+\right)-\nabla\left(\lambda\nabla T\right) \tag{46}$$

we can get

$$\rho c(T^+)\frac{\partial T}{\partial t}-\rho c(T)\frac{\partial T}{\partial t}+\epsilon\rho c(T^+)\frac{\partial\theta}{\partial t}=\nabla\left(\epsilon\Delta\lambda\nabla T+\epsilon\lambda\nabla\theta+\epsilon^2\Delta\lambda\nabla\theta\right) \tag{47}$$

Divide ϵ on both sides of equation above and let ϵ goes to 0. We have

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\rho c(T^+)-\rho c(T)}{\epsilon}\frac{\partial T}{\partial t}+\rho c(T)\frac{\partial\theta}{\partial t}&=\nabla\cdot\left(\Delta\lambda\nabla T+\lambda\nabla\theta\right)\\ \frac{\partial\rho c(T)}{\partial T}\theta\frac{\partial T}{\partial t}+\rho c(T)\frac{\partial\theta}{\partial t}&=\nabla\cdot\left(\Delta\lambda\nabla T+\lambda\nabla\theta\right)\end{aligned}$$

It is easy to get

$$\theta(\boldsymbol{x},0)=0. \tag{48}$$

$$\begin{aligned}(\lambda+\epsilon\Delta\lambda)\,\nabla T^+\cdot\boldsymbol{n}&=q,\\ \lambda\nabla T\cdot\boldsymbol{n}&=q.\end{aligned} \tag{49}$$

$$\begin{aligned}\left(\epsilon\Delta\lambda\nabla T+\epsilon\lambda\nabla\theta+\epsilon^2\Delta\lambda\nabla\theta\right)\cdot\boldsymbol{n}&=0\\ \left(\Delta\lambda\nabla T+\lambda\nabla\theta\right)\cdot\boldsymbol{n}&=0\end{aligned} \tag{50}$$

So, the sensitivity problem is

$$\begin{cases}\frac{\partial\rho c(T)\theta}{\partial t}=\nabla\cdot\left(\lambda\nabla\theta+\Delta\lambda\nabla T\right) & (\boldsymbol{x},t)\in(\Omega,\mathcal{T}) & (a)\\ \theta(\boldsymbol{x},0)=0 & \boldsymbol{x}\in\Omega & (b)\\ \left(\lambda\nabla\theta+\Delta\lambda\nabla T\right)\cdot\boldsymbol{n}=0 & (\boldsymbol{x},t)\in(\partial\Omega,\mathcal{T}) & (c)\end{cases} \tag{51}$$

3.4 Dual Problem

Based on the first equation of sensitivity problem

$$
\begin{aligned}
0 &= \int_0^{t_f}\int_\Omega \left[\rho\frac{\partial c(T)\theta}{\partial t} - \nabla\cdot(\lambda\nabla\theta + \Delta\lambda\nabla T)\right]\phi d\Omega dt\\
&= \int_\Omega \rho\phi d\left(c(T)\theta\right)d\Omega - \int_0^{t_f}\int_{\partial\Omega}(\Delta\lambda\nabla T + \lambda\nabla\theta)\cdot\boldsymbol{n}\phi d\Gamma dt + \int_0^{t_f}\int_\Omega(\Delta\lambda\nabla T + \lambda\nabla\theta)\nabla\phi d\Omega dt\\
&= \int_\Omega \rho\phi c(T)\theta\Big|_0^{t_f} d\Omega - \int_0^{t_f}\int_\Omega \rho c(T)\theta\frac{\partial\phi}{\partial t}d\Omega dt + \int_0^{t_f}\int_\Omega \Delta\lambda\nabla T\nabla\phi d\Omega dt + \int_0^{t_f}\int_\Omega \lambda\nabla\theta\nabla\phi d\Omega dt\\
&= \int_\Omega \rho\phi c(T)\theta\Big|_{t_f} d\Omega + \int_0^{t_f}\int_\Omega \Delta\lambda\nabla T\nabla\phi d\Omega dt + \int_0^{t_f}\int_{\partial\Omega}\nabla\phi\cdot\boldsymbol{n}(\lambda\theta)d\Gamma dt - \int_0^{t_f}\int_\Omega \nabla^2\phi\lambda\theta d\Omega dt\\
&\quad - \int_0^{t_f}\int_\Omega \rho c(T)\theta\frac{\partial\phi}{\partial t}d\Omega dt\\
&= \int_\Omega \rho\phi c(T)\theta\Big|_{t_f} d\Omega + \int_0^{t_f}\int_\Omega \Delta\lambda\nabla T\nabla\phi d\Omega dt + \int_0^{t_f}\int_{\partial\Omega}\lambda\nabla\phi\cdot\boldsymbol{n}\theta d\Gamma dt - \int_0^{t_f}\int_\Omega\left(\rho c(T)\frac{\partial\phi}{\partial t} + \lambda\nabla^2\phi\right)\theta d\Omega dt
\end{aligned}
$$

According to object fucntion, we can get

$$
\begin{aligned}
\mathcal{J}(\lambda+\epsilon\Delta\lambda) - \mathcal{J}(\lambda) =& \frac{1}{2}\int_0^{t_f}\int_{\partial\Omega}\left(T+\epsilon\theta - Y^\xi\right)^2 d\Gamma dt + \frac{\gamma}{2}\int_0^{t_f}\int_\Omega(\lambda+\epsilon\Delta\lambda)^2 d\Omega dt\\
&- \frac{1}{2}\int_0^{t_f}\int_{\partial\Omega}\left(T - Y^\xi\right)^2 d\Gamma dt - \frac{\gamma}{2}\int_0^{t_f}\int_\Omega \lambda^2 d\Omega dt\\
=& \frac{1}{2}\int_0^{t_f}\int_{\partial\Omega}\left(2\epsilon\theta T - 2\epsilon\theta Y^\xi + \epsilon^2\theta^2\right)d\Gamma dt + \frac{\gamma}{2}\int_0^{t_f}\int_\Omega\left(2\epsilon\lambda\Delta\lambda + \epsilon^2\Delta\lambda^2\right)d\Omega dt
\end{aligned} \tag{52}
$$

Divide ϵ on both sides of the eqution above and we can get

$$
\lim_{\epsilon\to 0}\frac{\mathcal{J}(\lambda+\epsilon\Delta\lambda) - \mathcal{J}(\lambda)}{\epsilon} = \int_0^{t_f}\int_{\partial\Omega}\theta\left(T - Y^\xi\right)d\Gamma dt + \gamma\int_0^{t_f}\int_\Omega \lambda\Delta\lambda d\Omega dt \tag{53}
$$

So dual problem is

$$
\left\{\begin{array}{lll}
\rho c(T)\frac{\partial\phi}{\partial t} = -\lambda\nabla^2\phi & (\boldsymbol{x},t)\in(\Omega,\mathcal{T}) & (a)\\
\phi(\boldsymbol{x},t_f) = 0 & x\in\Omega & (b)\\
\lambda\nabla\phi\cdot\boldsymbol{n} = T - Y^\xi & (\boldsymbol{x},t)\in(\partial\Omega,\mathcal{T}) & (c)
\end{array}\right. \tag{54}
$$

Let α_k represents ϵ and p_k represents $\Delta\lambda$. Then we get

$$
\alpha_k = -\frac{\int_0^{t_f}\int_{\partial\Omega}\left(T_k - Y^\xi\right)\theta_k d\Gamma dt + \gamma_{k+1}\int_0^{t_f}\int_\Omega \lambda_k p_k d\Omega dt}{\int_0^{t_f}\int_{\partial\Omega}\theta_k^2 d\Gamma dt + \gamma_{k+1}\int_0^{t_f}\int_\Omega p_k^2 d\Omega dt} \tag{55}
$$

$$
\beta_k = \left\{\begin{array}{l} 0\\ \max(0,\beta^{PR})\end{array}\right. \tag{56}
$$

$$\beta^{PR} = \frac{\int_0^{t_f}\int_\Omega \mathcal{J}'(\lambda_k)\left[\mathcal{J}'(\lambda_k) - \mathcal{J}'(\lambda_{k-1})\right] d\Omega dt}{\int_0^{t_f}\int_\Omega \left[\mathcal{J}'(\lambda_{k-1})\right]^2 d\Omega dt} \tag{57}$$

$$p_k = \begin{cases} -\mathcal{J}'(\lambda_k) \\ -\mathcal{J}'(\lambda_k) + \beta_k p_{k-1} \end{cases} \tag{58}$$

And the derivative of object function is

$$\mathcal{J}'(\lambda) = \nabla T \nabla \phi - \gamma\lambda \tag{59}$$

3.5 Discretization of One Dimensional Primal Problem

$$\begin{cases} \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\lambda\frac{\partial T}{\partial x} \\ T(x,0) = Y^\xi \\ -\lambda\frac{\partial T}{\partial x}(0,t) = q_1(t) \\ \lambda\frac{\partial T}{\partial x}(1,t) = q_2(t) \end{cases} \tag{60}$$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{(\lambda_{i+1}^n + \lambda_i^n)T_{i+1}^n - (\lambda_{i+1}^n + 2\lambda_i^n + \lambda_{i-1}^n)T_i^n + (\lambda_i^n + \lambda_{i-1}^n)T_{i-1}^n}{4\Delta x^2} \tag{61}$$

$$+ \frac{(\lambda_{i+1}^{n+1} + \lambda_i^{n+1})T_{i+1}^{n+1} - (\lambda_{i+1}^{n+1} + 2\lambda_i^{n+1} + \lambda_{i-1}^{n+1})T_i^{n+1} + (\lambda_i^{n+1} + \lambda_{i-1}^{n+1})T_{i-1}^{n+1}}{4\Delta x^2} \tag{62}$$

3.6 Discretization of One Dimensional Dual Problem

$$\begin{cases} \frac{\partial \phi}{\partial t} = -\lambda\nabla^2\phi \\ \phi(x,t_f) = 0 \\ \lambda\nabla\phi\cdot\boldsymbol{n} = T - Y^\xi \end{cases} \tag{63}$$

It is clear that dual problem is a backward problem which has a 'final condition'. Let $t = t_f - \tau$ and we can get a forward problem.

$$\begin{cases} \frac{\partial \phi}{\partial \tau} = \lambda\nabla^2\phi \\ \phi(x,0) = 0 \\ \lambda\nabla\phi\cdot\boldsymbol{n} = T - Y^\xi \end{cases} \tag{64}$$

3.7 Discretization of One Dimensional Sensitivity Problem

$$\begin{cases} \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\lambda\frac{\partial\theta}{\partial x} + \Delta\lambda\frac{\partial T}{\partial x}\right) \\ \theta(x,0) = 0 \\ \lambda\frac{\partial\theta}{\partial x}(0,t) + \Delta\lambda\frac{\partial T}{\partial x}(0,t) = 0 \\ \lambda\frac{\partial\theta}{\partial x}(1,t) + \Delta\lambda\frac{\partial T}{\partial x}(1,t) = 0 \end{cases} \tag{65}$$

3.7.1 Calcuate $\Delta\lambda$ (p_k)

$$p_k = \begin{cases} -\mathcal{J}'(\lambda_k) \\ -\mathcal{J}'(\lambda_k) + \beta_k p_{k-1} \end{cases} \tag{66}$$

3.7.2 Discretization

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial \theta}{\partial x} + p\frac{\partial T}{\partial x}\right) \tag{67}$$

$$\begin{aligned}
\frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} =& \frac{(\lambda_{i+1}^n + \lambda_i^n)\theta_{i+1}^n - (\lambda_{i+1}^n + 2\lambda_i^n + \lambda_{i-1}^n)\theta_i^n + (\lambda_i^n + \lambda_{i-1}^n)\theta_{i-1}^n}{4\Delta x^2} \\
&+ \frac{(\lambda_{i+1}^{n+1} + \lambda_i^{n+1})\theta_{i+1}^{n+1} - (\lambda_{i+1}^{n+1} + 2\lambda_i^{n+1} + \lambda_{i-1}^{n+1})\theta_i^{n+1} + (\lambda_i^{n+1} + \lambda_{i-1}^{n+1})\theta_{i-1}^{n+1}}{4\Delta x^2} \\
&+ \frac{(p_{i+1}^n + p_i^n)T_{i+1}^n - (p_{i+1}^n + 2p_i^n + p_{i-1}^n)T_i^n + (p_i^n + p_{i-1}^n)T_{i-1}^n}{4\Delta x^2} \\
&+ \frac{(p_{i+1}^{n+1} + p_i^{n+1})T_{i+1}^{n+1} - (p_{i+1}^{n+1} + 2p_i^{n+1} + p_{i-1}^{n+1})T_i^{n+1} + (p_i^{n+1} + p_{i-1}^{n+1})T_{i-1}^{n+1}}{4\Delta x^2}
\end{aligned}$$