IB Diploma Programme
Extended Essay

Research Question:

How can we utilize the superposition principle and Fourier series to solve real-world problems?

Word count: [3999]

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1 Introduction

Mathematics is full of potent methods that allow for solutions, understanding, and modeling of complex problems; amongst these, the principle of superposition and Fourier series hold a special place because they have played a very fundamental role in simplifying and representing reality. The principle of superposition uses linear algebra and differential equations to allow the addition of individual solutions in order to get a complex one, this itself is very handy when working with linear systems. Meanwhile, the Fourier series represents an advanced tool for expressing periodic functions as a sum of sine and cosine curves, therefore shedding light on the very process by means of which complex signals can be decomposed and analyzed.

This essay will discuss how the superposition principle and Fourier series can be used in solving practical problems in fields such as physics, engineering, and signal processing. The work will focus on the detailed mathematical explanation of these two areas of science and will seek to apply them using their practical applicability, for example, vibration analysis in mechanical systems and the decomposition of sound waves in audio engineering.

The essay will try to bridge the gap between pure mathematics and its applications, bringing into focus the versatility and power of superposition and Fourier series principles in solving real-world problems. The essay, therefore, through a careful analysis of case studies, will attempt to answer the essential question: How can we utilize the superposition principle and Fourier series to solve real-world problems?

2 Complex analysis

2.1 Introduction

Complex analysis is the study of functions of complex numbers, a branch that extends calculus and real analysis into the complex plane. It has applications, beginning with engineering and physics, right to number theory; it acts as a tool to integrate complicated integrals, model wave phenomena, and carry out an analysis of electrical circuits.

The imaginary number i is shown as

$$i = \sqrt{-1}$$

With this information, we can establish that:

$$i^2 = -1$$
$$i^3 = -\sqrt{-1} = -i$$
$$i^4 = 1$$

A complex number is represented throught the following notation

$$z = a + bi$$

Where a is the real part $a = Re\{z\}$ and b is the imaginary part $b = Im\{z\}$

The conjugate of $z = a + bi$ is represented as

$$z^* = a - bi$$

2.2 Eulers formula

We will use this formula to be able to establish a relationship between trigonometric functions and exponential functions

$$e^{ix} = cos(x) + isin(x)$$

x is a real number
i is the imaginary unit
e is the base of natural logarithms

Proof

We can define e^{ix} as a complex function z such that

$$z = e^{ix}$$

$$z = cos(x) + isin(x)$$

Now we find the derivative of the equation with respect to dx

$$\frac{dz}{dx} = - sin(x) + icos(x)$$

Rearranging the equation we get

$$\frac{dz}{dx} = icos(x) + i^2 sin(x)$$
$$\frac{dz}{dx} = i(cos(x) + isin(x))$$

Since $z = cos(x) + isin(x)$ we can state that

$$\frac{dz}{dx} = iz$$

Now we can rearrange the equation so that

$$\frac{dz}{z} = idx$$

Now we integrate both sides

$$\int \frac{dz}{z} = \int idx$$

$$ln(z) = ix + C$$

To eliminate the natural log we use its base e

$$e^{ln(z)} = e^{ix + C}$$

$$z = e^{ix} \times C$$

Now we need to prove that the constant C is equal to 1 to achieve that we can substitute the value of x as 0

$$cos(0) + isin(0) = e^{i0} \times C$$

$$1 = 1 \times C$$

$$1 = C$$

Hence proven that

$$e^{ix} = cos(x) + isin(x)$$

3 Waves in Mathematics

Waves in math are primarily expressed using trigonometric functions: sin(x),cos(x), and tan(x) these trigonometric functions when graphed create a series of repeating waves that share the same properties from negative infinity to positive infinity.

There are four factors present that manipulate these waves they are:

$$y = Asin(B(x + C)) + D$$

- **Amplitude (A)**

  The amplitude scales the sin function vertically on the y-axis, initially a sin function with an amplitude of 1 would have

  $$y_{max} = 1 \, , \, y_{min} = -1$$

  with amplitude A, and the range of the sin function becomes

  $$-A \leq y \leq A$$

- **Period (C)**

  The period of a trigonometric function graph is the horizontal length over which the function completes one full cycle before repeating.
  For a sin(x) and cos(x) graph, the period is calculated as:

  $$T = \frac{2\pi}{|C|}$$

  For a tan(x) graph, the period is calculated as:

  $$T = \frac{\pi}{|C|}$$

  This is because in an unaltered sin(x) and cos(x) graph, the function oscillates (repeats itself) at every 2π whilst in a tan(x) graph the function oscillates at every π

- **Phase shift (B)**

  The phase shift refers to the horizontal shift across the x-axis and is calculated by:

$$phase\ shift = -\frac{C}{B}$$

When the phase shift is positive such as in

$$sin(x + B)$$

The sin function shifts to the left, and the opposite holds true when the phase shift is negative the sin function shifts to the right when

$$sin(x - B)$$

- **Vertical shift (D)**

The vertical shift moves the entire wave along the y-axis, changing the baseline of which it oscillates

For a sin(x) function the baseline for the wave is at $y = 0$
For a cos(x) function the baseline for the wave is at $y = 1$
For a tan(x) function the baseline for the wave is at $y = 0$

It is calculated by:

$$\frac{y_{max} + y_{min}}{2} = D$$

Unlike in a phase shift when the Vertical shift is positive, it would shift the graph up across the y-axis and when it is negative it shifts the graph down across the y-axis

This is important, because Fourier series are based on the decomposition of complex periodic signals into the sum of sinusoidal functions of varied amplitudes, frequencies, phase shifts, and vertical shifts: y=csin(ax+b)+d. Each one of these terms represents either a sine or cosine function in a Fourier series, each one of which will correspond with a different aspect of the overall signal, understanding how these individuals work alone can enable us to appreciate how they interact to recreate complex waveforms.

4 Super Position Principle

4.1 Definition and Explanation

The Superposition Principle in general form states that the total response at a particular place and time to two or more stimuli is the *sum* of the responses from each individual stimulus.[1]

Mathematically it can be expressed as:

$$f(x_1) = y_1$$
$$f(x_2) = y_2$$
$$f(x_1 + x_2) = y_1 + y_2$$

If time were to be taken as a variable then it would be:

$$f(x_1(t)) = y_1(t)$$
$$f(x_2(t)) = y_2(t)$$
$$f(x_1(t) + x_2(t)) = y_1(t) + y_2(x)$$

From these expressions we can say that:
Any system that adheres to the superposition principle is a linear system
The superposition principle applies to all linear systems due to its inputs being directly in variation with the outputs and the presence of multiple external forces would cause the inputs to be altered and affected. The initial conditions would be affected as well.

A system is linear only when it exhibits Additivity and Homogeneity:

Additivity: $f(x_1 + x_2) = f(x_1) + f(x_2)$

Homogeneity: $f(c \times x_1) = c \times f(x_1)$ (c is a constant)

1 https://practicalee.com/superposition/

5. Fourier series

The Fourier series refers to the breakdown of waves into a series of sinusoidal periodic functions,
The general formula for a Fourier series can be expressed as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \times cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \times sin(\frac{n\pi x}{L})$$

Where the Fourier coefficients are:

$$a_0 = \frac{1}{2L} \times \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \times \int_{-L}^{L} f(x)\, cos(\frac{n\pi x}{L})dx, \ \ n > 0$$

$$b_n = \frac{1}{L} \times \int_{-L}^{L} f(x)\, sin(\frac{n\pi x}{L})dx, \ \ n > 0$$

Where L is half the period of the function if the function is periodic

The Fourier series can be made up of only sin or cosine terms this depends on whether the function is odd or even,

In the case in which the function is even the Fourier series has only cosine terms and is expressed as:

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \times cos(\frac{\pi n x}{L})$$

In the opposite case where the function is odd, the Fourier series has only sine terms and is expressed as:

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \times sin(\frac{\pi n x}{L})$$

We can also express a Fourier series in a complex exponential form using Euler's formula $e^{ix} = cos(x) + isin(x)$ and it is expressed as:

$$f(x) = \sum_{-\infty}^{+\infty} C_n e^{i\frac{\pi n x}{L}}$$

Where the Fourier coefficients can be expressed as:

$$C_n = \frac{1}{L} \times \int_{-L}^{L} f(x) \times e^{-i\frac{\pi n x}{L}} dx$$

Calculating the Fourier series of a function

Let's set our function and parameters to be:

$$f(x) = 5x + 2, \ - 3 < x < 3, \ f(x) = f(x + 6)$$

The period of the function is 6
Thus $L = 3$

Now we need to find our coefficients a_0, a_n, b_n using our previously established definitions

$$a_0 = \frac{1}{6} \int_{-3}^{3} 5x + 2 \, dx$$

$$a_0 = \frac{1}{6}\left[\frac{5x^2}{2} + 2x\right]_{-3}^{3}$$

$$a_0 = 2$$

$$a_n = \frac{1}{3}\int_{-3}^{3}(5x + 2)\left(\cos\left(\frac{\pi n x}{3}\right)\right) dx$$

$$a_n = \frac{4\sin(\pi n)}{\pi n}$$

Since $\sin(\pi n) = 0$

$$a_n = 0$$

$$b_n = \frac{1}{3}\int_{-3}^{3}(5x + 2)\left(\sin\left(\frac{\pi n x}{3}\right)\right) dx$$

$$b_n = \frac{30(\sin(\pi n) - \pi n\cos(\pi n))}{\pi^2 n^2}$$

Since $\cos(\pi n) = (-1)^n$

$$b_n = \frac{30(0 - \pi n(-1)^n)}{\pi^2 n^2}$$

$$b_n = \frac{-30\pi n(-1)^n}{\pi^2 n^2}$$

$$b_n = \frac{-30(-1)^n}{\pi n}$$

now that we have computed all the coefficients the Fourier series of the function would be:

$$f(x) = 2 + \sum_{n=1}^{\infty} \frac{-30(-1)^n}{\pi n} \times sin(\frac{\pi n x}{3})$$

In the complex exponential form, the Fourier series would be:

$$C_n = \frac{1}{3} \times \int_{-3}^{3} 5x + 2 \times e^{-i\frac{\pi n x}{3}} dx$$

$$C_n = \frac{(12\pi n + 90i) sin(\pi n) - 90 i \pi n cos(\pi n)}{3\pi^2 n^2}$$

$$C_n = \frac{(12\pi n + 90i)(0) - 90 i \pi n (-1)^n}{3\pi^2 n^2}$$

$$C_n = \frac{-90i(-1)^n}{3\pi n}$$

$$C_n = \frac{-30i(-1)^n}{\pi n}$$

$$f(x) = \sum_{-\infty}^{+\infty} \frac{-30i(-1)^n}{\pi n} \times e^{i\frac{\pi n x}{3}}$$

Graph for the first term:

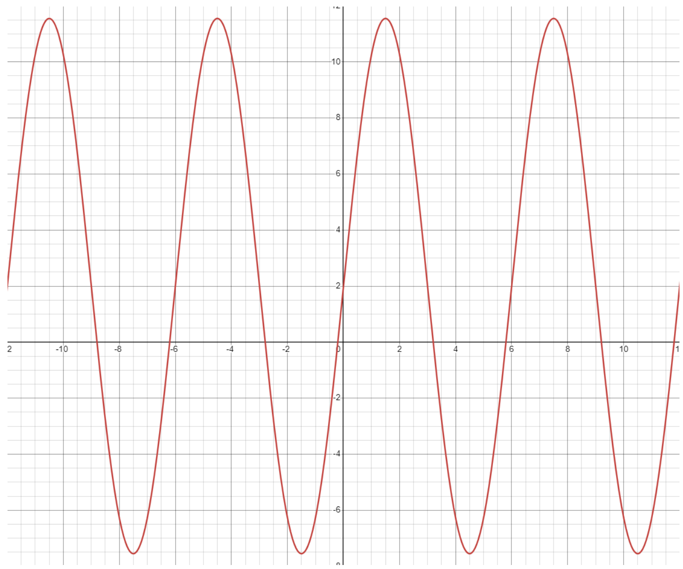


Fig 1.1: Graph of the first term of the Fourier series

Graph for the first 10 terms:

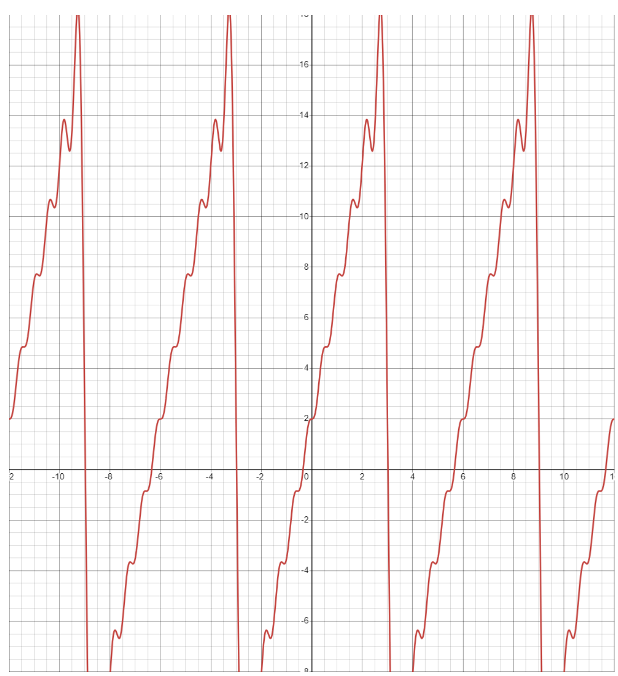


Fig 1.2: Graph of the first 10 terms of the Fourier series

Graph for the first 100 terms:

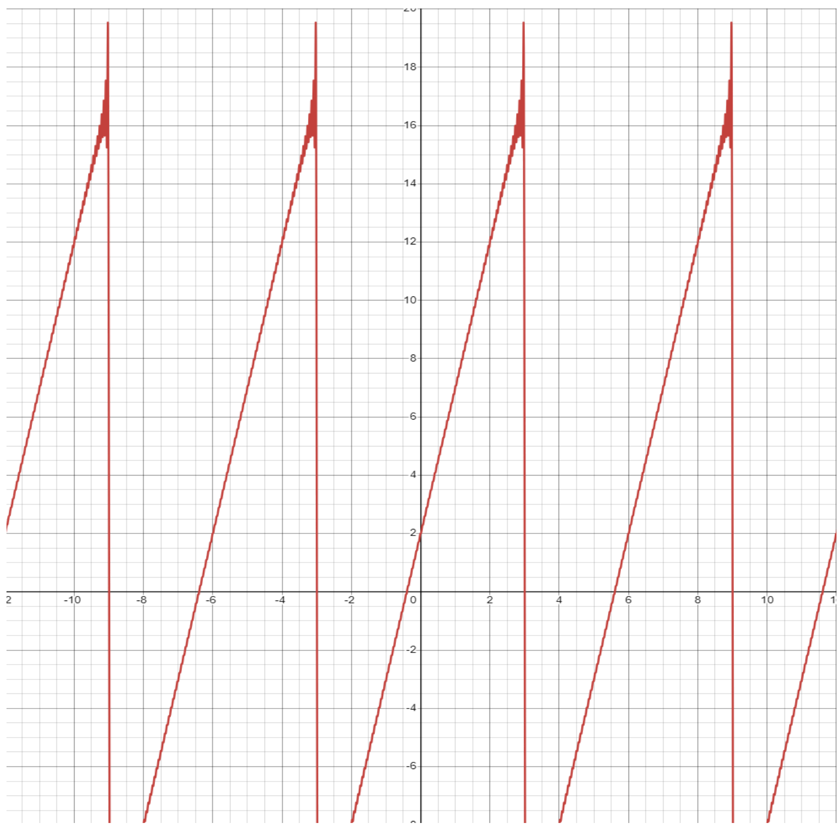


Fig 1.3: Graph of the first 100 term of the Fourier series

Applications

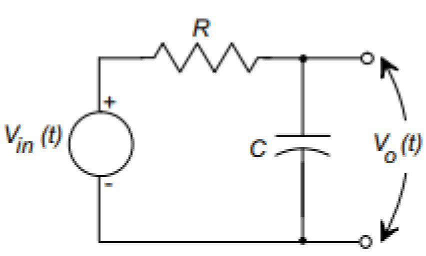
Electrical Engineering and Processing of Signals

The most common use of the superposition principle and the Fourier series comes in electrical engineering, voltages and currents are often depicted as sinusoidal waveforms which then in turn are turned into a Fourier series, and utilizing the superposition principle engineers can investigate a circuit's responses to different sinusoidal inputs, which is a frequent occurrence in power systems where multiple AC signals that produce different frequencies interact

An example of this could be present in harmonics which are known as unwanted frequencies in an electric grid that may cause interference with the fundamental frequency, the Fourier series allows engineers to identify and breakdown the current or voltage waveforms into fundamental and harmonic components. Using the principle of superposition, the current or voltage in the circuit as a whole can be represented by the sum of the components above and allows the engineer to identify and filter unwanted harmonics, improving the power supply in terms of both efficiency and stability.

This can be demonstrated with the following example:

A first-order electrical network is shown in the figures below, it is excited with a saw-tooth function, we can find an expression for the series representing the output $V_o(t)$



[2]

Fig 2: first-order electrical system

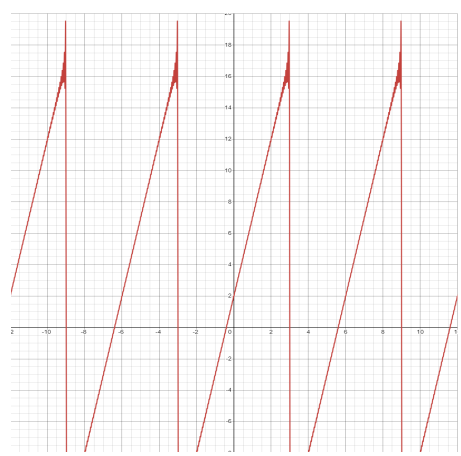


Fig 2.1: saw-tooth input waveform

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2 https://ocw.mit.edu/courses/2-161-signal-processing-continuous-and-discrete-fall-2008/3ab918dbe6a0376dbd9216e404fee31b_fourier.pdf

The network has a transfer function which can be defined as

$$H(s) \ = \ \frac{1}{RCs + 1}$$

And its frequency response function is

$$|H(i\Omega)| = \frac{1}{\sqrt{(\Omega RC)^2 + 1}}$$

$$\angle H(j\Omega) \ = \ tan^{-1}(- \ \Omega RC)$$

Using the fourier series we calculated for the saw tooth wave previously the input function u(t) can be represented as:

$$u(t) \ = 2 \ + \sum_{n=1}^{\infty} \frac{30(-1)^n}{\pi n} \times \ sin(\frac{\pi n x}{3})$$

The harmonic components:

- The frequency is $\Omega_n = \frac{n\pi}{3}$
- The magnitude response can be shown as:

$$\left|H(i\Omega_n)\right| = \frac{1}{\sqrt{(\Omega_n RC)^2 + 1}}$$

- The phase response can be shown as:

$$\angle H(i\Omega_n) \ = \ tan^{-1}(- \ RC\Omega_n)$$

With these components we can find the output for the nth term:

$$\frac{30(-1)^n}{\pi n} \times \left|H(i\Omega_n)\right| \times \ sin(\frac{n\pi}{3} \ + \ \angle H(i\Omega_n))$$

The output of the series can be represented as:

$$y(t) \ = 2 \ + \sum_{n=1}^{\infty} \frac{30(-1)^{n}}{\pi n} \times \frac{1}{\sqrt{(\frac{n\pi}{3}RC)^{2} + 1}} \times \ sin(\frac{n\pi x}{3} + \ tan^{-1}(- \ RC\frac{n\pi}{3}))$$

To graph this function let’s set the values of R and C to 1, and graph the first 100 terms:

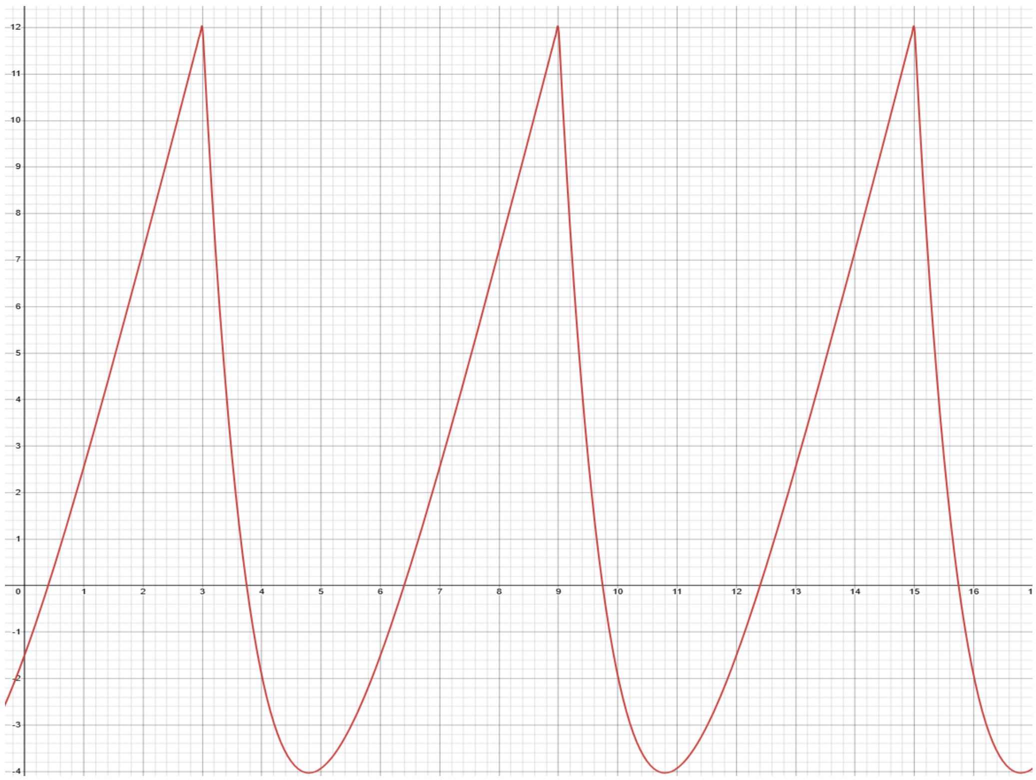

Fig 3: Graph of the output function for the first 100 terms

By finding graphing the output function, we can get a visualization of how the output differs from the input due to the system’s frequency response, these frequency responses modify the Fourier coefficients of the input, this modification alters the shape of the input signal with low pass-systems smoothing out high frequency components (harmonics) and high pass systems amplifying them, this visualization also allows for the investigation of the behavior of the system, for example, the attenuation of high frequencies or the resonance at specific frequencies.

Furthermore, by graphing we get a better understanding of the system’s processes and transform its inputs in terms of frequency content, and it aids in design and troubleshooting.

Heat Conduction and Diffusion

The heat equation can be readily solved by Fourier series; it is a partial differential equation describing the distribution of heat in a given region over time. Owing to the superposition principle, the solution of complicated heat problems can be viewed as a superposition of solutions of simpler ones.

Thermal Equation in a Rod

Think of a rod with some non-uniform initial temperature distribution. With the use of Fourier series, this temperature profile can be decomposed into sinusoidal components. Each of these components will often decay, depending on its frequency, and hence the temperature should become a superposition of sine waves that decay over time. Engineers and physicist use this method to predict the temperature gradient in materials, which is vital to the cooling or thermal insulation processes

We can demonstrate this with the following scenario:

A **thin metal rod** of length L is thermally insulated along its length and initially at a uniform temperature T_0. At time $t = 0$, the boundary at $x = 0$ is suddenly fixed at a constant temperature T_1, while the other end at $x = L$ is insulated.

We want to find the temperature distribution $T(x, t)$ at any time t and determine:

1. The time at which the temperature at $x = L$ to reach within 5% of its steady-state value.
2. The impact of increasing the rod's length on heat diffusion time.

First, the heat conduction equation for a 1D rod with thermal diffusivity α is:

$$\frac{dT}{dt} = \alpha\frac{d^2T}{dt^2}$$

The boundary conditions are:

$$T(0, t) = T_1\text{, fixed temperature at } x = 0$$

$$\frac{dT}{dt}(L, t) = 0\text{, fixed temperature at } x = L$$

The initial condition is:

$$T(x, 0) = T_0$$

Using superposition, we can write the solution as

$$T(x,t) = T_s(x) + T_t(x,t)$$

$T_s(x)$ is the steady-state solution, which is the equilibrium temperature distribution; solving for it allows us to determine the transient component
$T_t(x,t)$ is the transient solution, the time-dependent part that vanishes as $t \to \infty$

From the above information, we know that at the steady state, $\frac{dT}{dt} = 0$ which makes the heat equation:

$$\frac{d^2T_s}{dx^2} = 0$$

Now we integrate twice:

$$\int \frac{d^2T_s}{dx^2} = \int 0\, dx$$

$$\frac{dT_s}{dx} = C$$

$$\int \frac{dT_s}{dx} = \int C\, dx$$

$$T_s(x) = C_1 x + C_2$$

Utilizing the boundary conditions from earlier:

$$T_s(0) = T_1 \Rightarrow C_2 = T_1$$

$$\frac{dT_s}{dx}(L) = 0 \Rightarrow C_1 = 0$$

From this, the steady-state solution can be expressed as:

$$T_s(x) = T_1$$

What this shows is that after an infinite amount of time, the entire rod will be at a temperature T_1 this is due to heat continuously being supplied at $x = 0$

To determine, $T_t(x, t)$ we shall use a Fourier series:

The transient equation satisfies:

$$\frac{dT_t}{dt} = \alpha \frac{d^2T_t}{dx^2}$$

Applying the results from earlier on the boundary conditions, we get:

$$T_s(0, t) = 0$$

$$\frac{dT_t}{dx}(L, t) = 0$$

Expanding $T_t(x, t)$ as a Fourier cosine series:

In this scenario, we expand as a Fourier cosine series because the boundary condition at $x = 0$ is that the temperature is fixed at T_1, this means that the solution must match this at $x = 0$, additionally the insulated end at $x = L$ implies a Neumann boundary condition, which is a type of boundary condition that specifies the derivative of a function rather than its value at the boundary, in this scenario it means that heat flux at the boundary is controlled and it is 0, $\frac{dT}{dx}$ at $x = L$ is 0 meaning there is no heat flux through the boundary, and this is a characteristic of cosine expansions

Due the Neumann boundary condition at $x = L$ the allowed modes must satisfy

$$\frac{d}{dx}cos(k_n x) = 0 \text{ at } x = L$$

The solution that satisfies this is

$$k_n = \frac{(2n-1)\pi}{2L},\ n = 1, 2, 3, ...$$

Thus we get

$$cos(\frac{(2n-1)\pi x}{2L})$$

Now that we have found the spatial part we need to find the time part to do this we need to:

Assume a solution of the form

$$T(x,t) = X(x)T(t)$$

Plug this into the heat equation and separate the variables to obtain:

$$\frac{1}{T} \times \frac{dT}{dt} = \alpha \frac{1}{X} \times \frac{d^2X}{dx^2} = -\lambda$$

Since we have already found the spatial part of the fourier series, we can simply write that:

$$X_n(x) = cos(\frac{(2n-1)\pi x}{2L})$$

With eigenvalues:

$$\lambda_n = (\frac{(2n-1)\pi x}{2L})^2$$

Now we can start solving for the time part
The time equation is:

$$\frac{dT_n}{dt} + \alpha\lambda_n T_n = 0$$

This has a general solution which is:

$$T_n = C_n e^{-\alpha\lambda_n t}$$

Substituting λ_n:

$$T_n = C_n e^{-\alpha(\frac{(2n-1)\pi x}{2L})^2 t}$$

All we need is the term, $e^{-\alpha(\frac{(2n-1)\pi x}{2L})^2 t}$ and this leads the Fourier series to be:

$$T_t(x,t) = \sum_{n=1}^{\infty} B_n \times cos(\frac{(2n-1)\pi x}{2L}) \times e^{-\alpha(\frac{(2n-1)\pi}{2L})^2 t}$$

We can find the B_n coefficient using the initial condition

$$T_t(x,0) = T_0 - T_1$$

$$B_n = \frac{2}{L} \times \int_0^L (T_0 - T_1) \times cos(\frac{(2n-1)\pi x}{2L})dx$$

$$B_n = \frac{4(T_0 - T_1)}{(2n-1)\pi} \times sin(\frac{(2n-1)\pi L}{2L})$$

Since cosine series are even functions, the term $sin(\frac{(2n-1)\pi L}{2L})$ will naturally be absorbed so we can eliminate it and B_n becomes:

$$B_n = \frac{4(T_0 - T_1)}{(2n-1)\pi}$$

Now the Fourier series can be expressed as:

$$T_t(x,t) = T_1 + \sum_{n=1}^{\infty} \frac{4(T_0 - T_1)}{(2n-1)\pi} \times cos(\frac{(2n-1)\pi x}{2L}) \times e^{-\alpha(\frac{(2n-1)\pi}{2L})^2 t}$$

Now to find the time at which the temperature at $x = L$ to reach within 5% of its steady-state value, we need to:

The temperature at $x = L$ is:

$$T_t(L,t) = T_1 + \sum_{n=1}^{\infty} \frac{4(T_0 - T_1)}{(2n-1)\pi} \times cos(\frac{(2n-1)\pi L}{2L}) \times e^{-\alpha(\frac{(2n-1)\pi}{2L})^2 t}$$

Since $cos(\frac{(2n-1)\pi}{2}) = 0$ for all even n, despite that, the rest of the terms except T_1 don't get eliminated, even though they are multiplied by 0. This is because even though the steady-state solution does reduce to T_1 everywhere after infinite time, the transient terms in the equation are still essential in determining how heat distributes over time, so they are kept. The first term dominates:

$$T_t(L,t) \approx T_1 + \frac{4(T_0-T_1)}{\pi} \times e^{-\alpha(\frac{\pi}{2L})^2 t}$$

We solve for t when:

$$T_t(L,t) - T_1 \leq 0.05(T_0 - T_1)$$

$$\frac{4}{\pi} e^{-\alpha(\frac{\pi}{2L})^2 \times t} \leq 0.05$$

$$-\alpha(\frac{\pi}{2L})^2 \times t \leq ln(0.05 \times \frac{\pi}{4})$$

Solving for t we get:

$$t \geq \frac{-ln(0.05 \times \frac{\pi}{4})}{\alpha(\frac{\pi}{2L})^2}$$

From this equation, we can deduce that:

$$t \propto L^2$$

This relationship shows how the length of the rod affects the time required for the temperature at $x = L$ to reach 95% of steady-state. This finding is essential when designing thermal systems, since the length of the chosen conductive material can have significant impact on the time taken for the material to reach a temperature equilibrium. This method is very useful in fields of thermal engineering

Other applications include:

Wave Analysis and Dynamics of Vibration

Probably one of the biggest uses of the superposition principle and Fourier series is for wave and vibration analysis. In the systems in which the dynamics are oscillatory in nature, the integration of such principles allows complex waveforms to be decomposed into simpler sinusoidal components.

Vibrating Strings and Sound Waves

For a vibrating string- as with a guitar string harmonic frequency represents a specific frequency over a set number of nodes across the string. Using Fourier series, one can identify the wave configuration on a string at any time as a sum of a variety of sine and cosine functions, each representing a particular harmonic. From the principle of superposition, the entire vibration of the string is simply a sum of each individual frequency component.

Acoustic waves work in a similar way. The sound of a musical instrument can be represented as a complex waveform, which is decomposed into a set of simpler harmonic frequencies by Fourier analysis. Here, the Fourier series enables the analysis and production of the sound, an important task in music studies, acoustic engineering, and audio engineering.

Image Analysis and Data Compression

The Fourier series, combined with the principle of superposition, also allows the processing of images, particularly in the field of image compression where complex visual information must be reduced in size but still retain the order of magnitude necessary to perceive important details. One common image format, JPEG compression, uses a form of Fourier analysis called the Discrete Cosine Transform (DCT), related to the Fourier series but modified for real-valued data.

Image Compression by DCT A picture, in JPEG compression, is first divided into small blocks. The DCT then converts each block into its frequency components. These are combined using the superposition principle to represent the original image block as a sum of such frequency components. The data from the picture can then be compressed substantially by discarding some frequencies, such as high-frequency components that do not contribute much to visual quality.

Fourier series allows the transformation that will make the process of image compression easier, while the superposition principle enables the reconstruction process during decompression.

Conclusion:

To sum up, this essay showed the power of the Fourier series and the superposition principle, it displayed how the Fourier series can be used to turn a simple function into a series of sinusoidal waves giving insight into its patterns and showing its individual components for further analysis, it also showed the usefulness of the superposition principle in this analysis.

The benefits of the Fourier series and the superposition principle as a method of analysis of systems is shown in this essay, it is quick and easy and, provides a visualization of the system, and visually displays, the effect of each component on the whole system

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