IB Math Analysis and Approaches HL Exploration

Optimizing the force and angle needed to score a ball in a net accurately

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Introduction:

In football, a player's ability to kick a ball accurately into the goal involves understanding and optimizing key factors, primarily the angle and force of the kick. This investigation aims to use mathematical principles, specifically triangulation and calculus, to determine the optimal angle and force required for a player to maximize accuracy when attempting to score.

Understanding this optimization could be valuable for training, allowing players to develop muscle memory for ideal angles and forces under different conditions. This investigation aligns with both practical and theoretical aspects of sports mathematics, showcasing the relevance of trigonometry, calculus, and physics in a sports context.

Rationale:

I have been playing football since I was a kid and the one area that I lacked was scoring the ball, due to the size of my legs I would always end up applying too much force on the ball which would send it flying away from the net, and when I applied the appropriate amount of force I would have kicked the ball from the wrong angle.

So I decided to solve this problem using the skills and methods I have learned during my time in my Math AA HL classes, despite it being that kicking a ball with an exact amount of force and an exact angle is difficult I believe that it can be incorporated into my muscle memory with enough practice on the angle and force calculated in the exploration

Assumptions and Foundations:

The flight of a football follows a parabolic trajectory that can be analyzed using projectile motion equations. These equations account for the horizontal and vertical displacement of the ball over time. The basic equations are:

1. Horizontal displacement:

$$x = v_0 \cdot cos(\theta) \cdot t$$ [1]

2. Vertical displacement:

$$y = v_0 \cdot sin(\theta) \cdot t - \frac{1}{2}gt^2$$ [2]

Where:

- v_0 is the initial velocity,
- θ is the angle of projection,
- g is the gravitational constant ($9.8ms^{-2}$), and
- t is time.

These equations will allow us to set up relationships between angle, distance, and velocity for the kick.

To ensure the ball is kicked directly toward the goal, we use triangulation to calculate the necessary angle. Given the field's geometry, we treat the player's position, the left goalpost, and the right goalpost as vertices of a triangle.

Using the Law of Cosines:

$$cos(A) = \frac{b^2+c^2-a^2}{2bc}$$ [3]

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[1] https://problemsphysics.com/mechanics/projectile/projectile_equation.html

[2] https://problemsphysics.com/mechanics/projectile/projectile_equation.html

[3] https://byjus.com/maths/cosine-rule/

where A is the angle needed for the ball to reach the goal. This triangulation gives us an accurate angle to optimize the force needed for scoring.

Determining the Optimal Angle

Using the range formula for projectile motion:

$$R = \frac{v_0^{\ 2} sin(2\theta)}{g}$$ [4]

we set R to the known distance from the player to the goal, then solve for θ by maximizing $sin(2\theta)$. This approach yields the angle at which the range of the ball is optimized.

Let's assume:
- Distance to the goal = 20 meters
- Initial velocity v_0 = 15ms^{-1}
- Gravitational constant $g = 9.8 ms^{-2}$

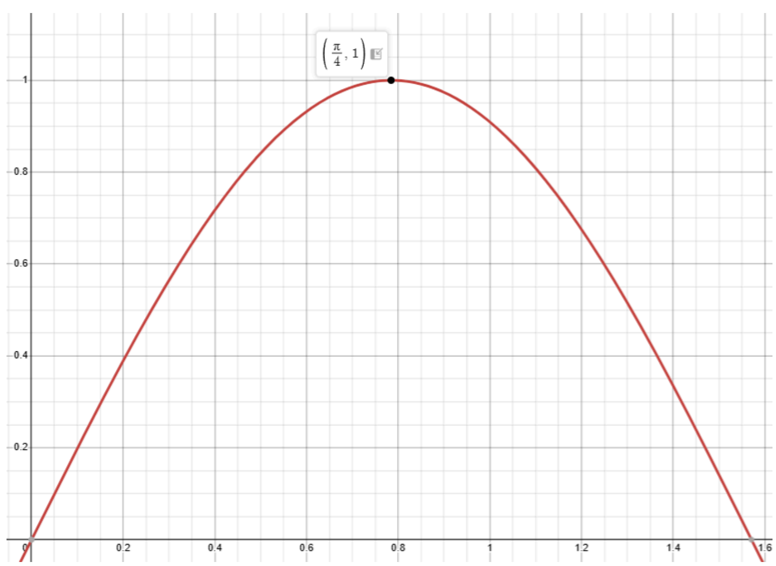

Fig 1: Graph of relationship between range and angle

If we remove the initial velocity and gravity constants and plot a graph to display the relationship between range and angle, we can see that the graph peaks at $\frac{\pi}{4}$ or 45 degrees. This is our optimal angle as it maximizes the range. This can also be verified numerically by:

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[4] https://www.homeworkhelpr.com/study-guides/physics/motion-in-a-plane/projectile-motion/

We solve for $sin(2\theta)$ to maximize R for a shot taken at this distance.

Setting up the derivative:

$$\frac{dR}{d\theta}\left(\frac{v_0^{\ 2}sin(2\theta)}{g}\right) = \frac{2v_0^{\ 2}cos(2\theta)}{g} = 0$$

Plug in the values for velocity and gravity.

$$\frac{2\times15^2cos(2\theta)}{9.81} = 0$$

Solving for θ

$$\frac{2\times15^2cos(2\theta)}{9.81} = 0$$

$$2 \times 15^2cos(2\theta) = 0$$

$$cos(2\theta) = 0$$

$$\theta = \frac{cos^{-1}(0)}{2}$$

$$\theta = 45$$

Calculating the Required Force

To determine the necessary force, we use:

$$F = ma$$ [5]

where m is the mass of the ball and a is the required acceleration for v_0. Assuming a standard football mass of 0.45 kg, we calculate the required force for our determined angle.

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5 https://www.bbc.co.uk/bitesize/guides/zgv797h/revision/3#:~:text=Force%2C%20mass%20and%20acceleration.is%20measured%20in%20newtons%20(N)

Using all our findings, let's apply the principles in a full example:

Position: Player at 20 meters from the goal
Goal Width: 7.32 meters
Ball Mass: 0.45 kg

Force Calculation: For this angle, and with initial velocity $v_0 = 15ms^{-1}$, the required force is:

$$F = ma = 0.45 \times 15 = 6.75\ N$$

Factoring in Air resistance

To close the gap between this investigation and real life I will attempt to factor air resistance into my calculations, for that it would be important that we consider the drag force because it is proportional to the square of the ball's velocity, the formula is given by:

$$F_d = -\frac{1}{2}\rho C_d A v^2$$ [6]

Where:

- ρ is the air density

- C_d is the drag coefficient

- A is the cross-sectional area of the ball

- v is the velocity of the ball

The drag equation is negative due to the drag force acting opposing to the direction of the ball

Let the initial velocity v_0 have a vertical component $v_{0y} = v_0 sin(\theta)$ and a horizontal component $v_{0x} = v_0 cos(\theta)$

[6] https://www.grc.nasa.gov/www/k-12/VirtualAero/BottleRocket/airplane/drageq.html

Factoring in the horizontal component the equation becomes:

$$F_{d,x} = -\frac{1}{2}\rho C_d A v_x^2$$

Factoring in the vertical component the equation becomes:

$$F_{d,y} = -\frac{1}{2}\rho C_d A v_y^2$$

Setting up the derivatives

Horizontal motion through air

Using Newton's Second law $F = ma$ we can differentiate acceleration so we get:

$$m\frac{dv_x}{dt} = -\frac{1}{2}\rho C_d A v_x^2$$

$$\frac{dv_x}{dt} = -\frac{\rho C_d A v_x^2}{2m}$$

Vertical motion through air

When looking at the vertical side of motion both gravity and air resistance would be acting on the ball so they would also have to be included in our equation using the equation for weight $W = mg$ we get:

$$m\frac{dv_y}{dt} = -mg - \frac{1}{2}\rho C_d A v_y^2$$

$$\frac{dv_y}{dt} = -g - \frac{\rho C_d A v_y^2}{2m}$$

Solving the derivatives

Due to the non-linearity of the differential equations we have produced it would be best to solve them numerically for that, I will be utilizing Euler's method [7] which is:

$$\frac{dy}{dx} = f(x, y),\ h$$

$$x_{n+1} = x_n + h \qquad\qquad y_{n+1} = y_n + hf(x_n, y_n)$$

To approximate the solution over a small time step we let $h = \Delta t$, applying Euler's method we get:

For Horizontal Velocity:

$$x_{n+1} = x_n + \ \Delta t$$

$$y_{n+1} = y_n + \ \Delta t(-\frac{\rho C_d A v_{x,n}^2}{2m})$$

For Vertical Velocity:

$$x_{n+1} = x_n + \ \Delta t$$

$$y_{n+1} = y_n + \ \Delta t(-\ g\ -\frac{\rho C_d A v_{y,n}^2}{2m})$$

For horizontal position

$$x_{n+1} = x_n + v_{x,n}\Delta t$$

For Vertical position

$$y_{n+1} = y_n + v_{y,n}\Delta t$$

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[7] https://tutorial.math.lamar.edu/classes/de/eulersmethod.aspx

Let's assume that:

$v_{x_0} = 20ms^{-1}$

$v_{y_0} = 20ms^{-1}$

$g = 9.81ms^{-2}$

$m = 0.45kg$

$\rho = 1.225kgm^{-3}$

$C_d = 0.12$

$A = 0.166m^2$

$\Delta t = 0.01$

Substituting these values into our equations gives us:

For the horizontal Velocity:

$$x_{n+1} = x_n + 0.01$$

$$y_{n+1} = y_n + 0.01\left(- \frac{1.225\times0.12\times0.166y_n^2}{2\times0.45}\right)$$

$$y_{n+1} = v_n + 0.1(- 0.0271y_n^2)$$

For the vertical velocity:

$$x_{n+1} = x_n + 0.01$$

$$y_{n+1} = y_n + 0.1\left(- 9.81 - \frac{1.225\times0.12\times0.166y_n^2}{2\times0.45}\right)$$

$$y_{n+1} = y_n + 0.1(- 9.81 - 0.0271y_n^2)$$

Results:

A simulation was run to calculate the results. The following is a table of the initial and final vertical and horizontal velocity, the horizontal range, and the maximum vertical Range. The calculated data is also displayed in a graph [Fig 2].

Initial Horizontal Velocity	14.1
Initial Vertical Velocity	14.1
Final Horizontal Velocity	7.13
Final Vertical Velocity	-11.3
Horizontal Range	25.2
Maximum Vertical Height	8.02

Table 1: Parameters

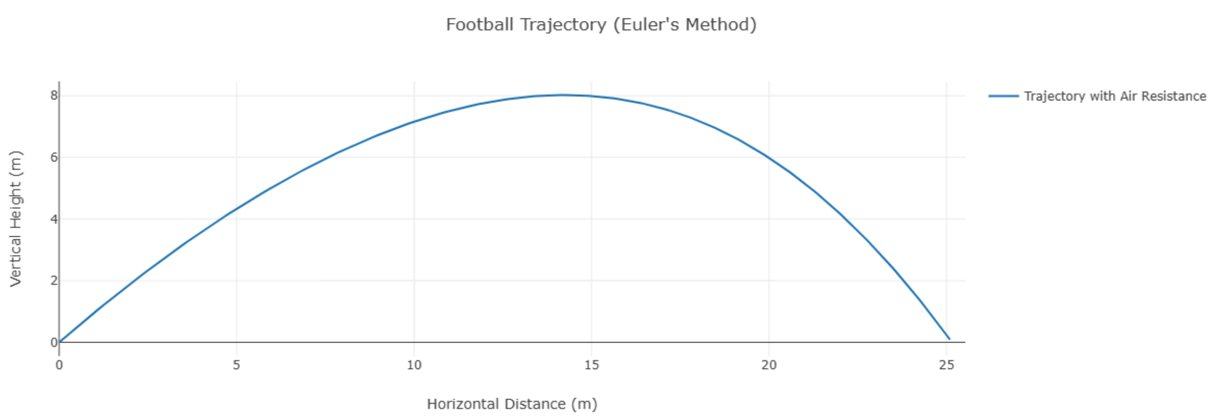

Fig 2: Graph of Vertical height vs Horizontal Distance Results

Calculating the Range Reduction Due to Air Resistance

Now that we have found the numerical values we can use x_{max} which is the maximum horizontal distance we obtain from the values and compare it with x_{ideal} which is the range we calculated when we didn't factor in air resistance, so our range reduction would be:

$$Range\ Reduction\ =\ (1-\frac{x_{max}}{x_{ideal}})\ \times\ 100\%$$

$$x_{ideal}=\frac{v_0^{\ 2}sin(2\theta)}{g}$$

$$x_{ideal}=\frac{20^2\times sin(2\times 45)}{9.81}$$

$$x_{ideal}\ =\ 40.77m$$

$$x_{max}\ =\ 25.2m$$

$$Range\ Reduction\ =\ (1-\frac{25,2}{40.77})\ \times\ 100\ =\ 38.2\%$$

From these findings, we can infer that the horizontal and vertical velocity decrease significantly due to air resistance; the final horizontal velocity reaches 7.13ms^{-1} , and the final vertical velocity reaches -11.3ms^{-1}, which indicates the ball's rapid descent; furthermore, the 38.2% range reduction when comparing ideal range and maximum range shows the effect of air resistance on the distance travelled by the ball, meaning that in real scenarios a force higher than 6.75N must be used for the ball to travel 20 meters, this means that during training its best that players experience and experiment with attempting long range shots, and under different conditions as well as repeating those shots this allows them to build muscle memory which makes them able to consistently perform accurate shots that are delivered with the appropriate amount of force to score.

Curved shots and ball spin

When a football player attempts to score a ball, he is often met with opposition in the form of the enemy team's defenders. Thus, he no longer can shoot the ball in a straightforward line towards the goal, and it would be relatively easy to block, so most players go for curved shots. This is achieved by giving the ball spin, causing to curve through the air this is known as the Magnus effect, which is a physical phenomenon where generation of a sidewise force on a spinning cylindrical or spherical solid immersed in a fluid (liquid or gas) when there is relative motion between the spinning body and the fluid [8].

It is given by

$$F_m = S(w \times v) \quad [9]$$

Where

- S is a proportionality constant depending on air properties and ball characteristics
- w is the angular velocity of the ball
- v is the velocity of the ball
- The cross product $\times$ indicates that the Magnus force is perpendicular to both v and w

To model this effect and account for other forces such as gravity and resistance, we use numerical integration with the Runge-Kutta method [10], RK4 for short.

First, we identify the forces acting on the ball.l

Gravity, given by

$$F_g = mg$$

Acceleration due to gravity

$$a_y = -g$$

[8] https://www.britannica.com/science/Magnus-effect

[9] https://live.stemfellowship.org/physics-of-free-kicks/#:~:text=The%20equation%20for%20the%20Magnus,of%20the%20fluid%20(air).

[10] https://web.mit.edu/10.001/Web/Course_Notes/Differential_Equations_Notes/node5.html

Air resistance, given by

$$F_d = \frac{1}{2}\rho C_d A v^2$$

The drag components in the x, y, and z directions

$$F_d = \frac{1}{2}\rho C_d A v v_x$$

$$F_d = \frac{1}{2}\rho C_d A v v_y$$

$$F_d = \frac{1}{2}\rho C_d A v v_z$$

The Magnus effect

$$F_m = S(w \times v)$$

Now, we have to set up the differential equations
Using Newton's Second law:

$$m\frac{dv_x}{dt} = F_{d_x} + F_{m_x}$$

$$m\frac{dv_y}{dt} = F_{d_y} + F_{m_y} - mg$$

$$m\frac{dv_z}{dt} = F_{d_z} + F_{m_z}$$

Rearranging to obtain the velocity components as

$$\frac{dv_x}{dt} = \frac{F_{d_x} + F_{m_x}}{m}$$

$$\frac{dv_y}{dt} = \frac{F_{d_y} + F_{m_y}}{m} - g$$

$$\frac{dv_z}{dt} = \frac{F_{d_z} + F_{m_z}}{m}$$

RK4 uses four weighted steps to improve accuracy over Euler's method

$$k_1 = dt \cdot f(t, v)$$

$$k_2 = dt \cdot f(t + \frac{dt}{2}, v + \frac{k_1}{2})$$

$$k_3 = dt \cdot f(t + \frac{dt}{2}, v + \frac{k_2}{2})$$

$$k_4 = dt \cdot f(t + dt, v + k_3)$$

Thus, we get the new velocity computed as

$$v_{new} = v + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

The positions are updated using.

$$x_{new} = x + v_x \cdot dt$$

$$y_{new} = y + v_y \cdot dt$$

$$z_{new} = x + v_z \cdot dt$$

Now, we compute the solution graphically, we set the constants:
mass = 0.45kg

Air density = 1.225kgm^{-3}

Drag Coefficient = 0.12

Radius = 0.115m

Surface Area = 0.166m^2

Gravity = 9.81ms^{-2}

Magnus proportionality constant = 0.004152

Spin vector = [0, 0, 10]

Initial Vertical Velocity = 20ms^{-1}

Initial Horizontal Velocity = 20ms^{-1}

Results:

The following is a graphical display of the results as well as a numerical display for the values found in the 2 lines:

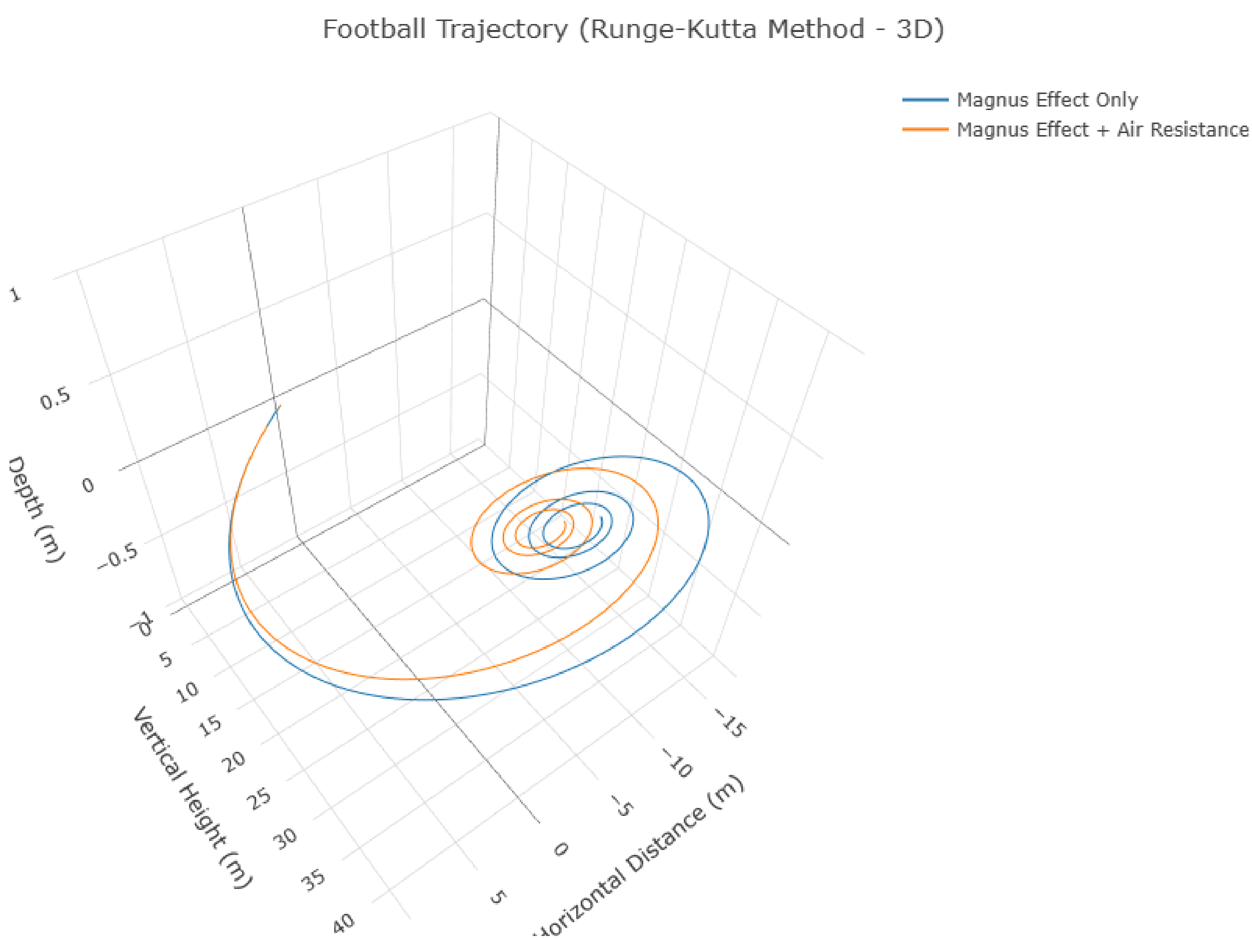

Fig 2: 3D graph for football trajectory

Initial Horizontal Velocity	14.1
Initial Vertical Velocity	14.1
Final Horizontal Velocity	2.48
Final Vertical Velocity	-2.47
Final Horizontal Position	-7.05
Final Vertical Position	12.4

Table 2: Magnus Effect Line Values

Initial Horizontal Velocity	14.1
Initial Vertical Velocity	14.1
Final Horizontal Velocity	-0.11
Final Vertical Velocity	-0.70
Final Horizontal Position	-8.50
Final Vertical Position	29.0

Table 3: Magnus Effect + Air resistance Line Values

From the results, graphically, we can see the curve that the ball undergoes during its motion as the 3D graph shows a curved path since the ball's spin causes it to deviate from a parabolic path and it forms a curve instead; this is shown as well in the final horizontal position of -8.50m, which is a deviation from the straight-line trajectory, aligning with the results found in Euler's method we can observe the decrease in both the horizontal and vertical velocities due to air resistance, which indicate that the ball is coming a stop, these results imply how volatile a balls trajectory can be when shooting a curved shot as preicse control over the ball's spin and velocity is required to successfuly excute a curved shot with the intended results, similary to what was stated earlier consistent practice to build up muscle memory can help in shooting better curved shots.

Conclusion

This investigation ventured into optimizing the force and angle required to score a football accurately, considering factors such as air resistance and the Magnus effect. It concludes that the optimal angle is 45 degrees, and the optimal force is 6.75N assuming no air resistance, it also found that air resistance had an 38.2% reduction in the range of motion of the ball, furthermore it explored the Magnus effect and how it is utilized executing curved shots which are harder to block by defenders.

Some improvements for further research include:

- Real-world testing: Despite the valuable insight that models and simulations provide, the simple model used here is inadequate when compared to a real-world scenario as many confounding variables can't be controlled nor anticipated when making these models, and thus, real-world testing is required to provide more accurate results
- Varying Conditions: Further research can explore the impact of variables that were not considered, such as wind speed, temperature, altitude, humidity
- Player biomechanics, including data on how players generate force and spin, can provide a better understanding of the factors that influence the trajectory of a ball.

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