$L_+L_- = (L_x+iL_y)(L_x-iL_y)$
$= L_xL_x - iL_xL_y + iL_yL_x + L_y^2$
$= L_x^2+L_y^2 - i[L_x, L_y]$
$= L_x^2+L_y^2 - i[i\hbar L_z]$
$= L_z^2+L_y^2 \underline{+\hbar L_z}$

$L^2 = L^+L_- - \hbar L_z + L_z^2$

$L_-\psi_{l'} = 0 \cdot L_z\psi_{l'} = \hbar l'\psi_{l'}$

$L^2\psi_{l'} = \underbrace{(L^+L_-)\psi_{l'}}_{0} - \hbar^2 l' + \hbar^2 l'^2 = \hbar^2 l'(l'-1)$

$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$

$\psi_l \rightarrow L_i\ (i=1,2,3) \cdots \rightarrow \psi_{l'}$

L^2

$\underline{l'(l'-1) = l(l+1)}$

$k(k-1) = l(l+1)$

$k(k-1) - l(l+1) \leftarrow k = l+1\ 2\ 0$

$\overset{\shortparallel}{(k-l-1)(k+l)}$

$k = -l,\ l+1$
$\underline{l' = -l}$ or $\underline{l' = l+1}$
$\times$

$$\begin{array}{r} k+l \\ k-l-1 \ \overline{)\,k^2-k-l^2-l} \\ \underline{k^2-kl-k} \\ kl-l^2-l \\ \underline{kl-l^2-l} \\ 0 \end{array}$$

$L^2 = \hbar^2 l(l+1)$

$|l, l\rangle$, $|l, l-1\rangle$, $|l, l-2\rangle \cdots$ $|l, -l\rangle$
$\downarrow$
L_z

$l=1$ (example)

$|1,1\rangle$ $|1,0\rangle$, $|1,-1\rangle$

$L_-|l,m\rangle = \hbar\sqrt{l(l+1)-m(m-1)}\,|l, m-1\rangle$

$L_-|1,1\rangle = \hbar\sqrt{2}\,|1,0\rangle$

$L_-|1,0\rangle = \hbar\sqrt{2}\,|1,-1\rangle$

$\underline{-\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} - \frac{m}{\sin^2\theta}\right)\Theta(\theta) = \lambda\Theta}$ (12.65)

$x = \cos\theta$ $\quad dx = -\sin\theta\, d\theta$ $\quad \frac{d\theta}{dx} = -\frac{1}{\sin\theta}$

$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}$

$-\left\{\boxed{\sin\theta\frac{d}{dx}}\left(\sin\theta\frac{d}{dx}\Theta\right) - \cos\theta\frac{d}{dx}\Theta - \frac{m}{\sin^2\theta}\Theta\right\} = \lambda\Theta$

$\sin\theta\frac{d}{dx}f = \frac{d}{dx}(\sin\theta f) - \cos\theta f$

$-\left\{\ldots\sin\theta\frac{d^2}{dx^2}\Theta\right\} - \cos\theta\frac{d}{dx}\Theta - \frac{m}{\sin^2\theta}\Theta\} = \lambda\Theta$

$-\left[\sin\theta\left\{\cos\theta\left(-\frac{1}{\sin\theta}\right)\frac{d}{dx}\Theta + \sin\theta\frac{d^2}{dx^2}\Theta\right\} - \cos\theta\frac{d}{dx}\Theta - \frac{m}{\sin^2\theta}\Theta\right] = \lambda\Theta$

$-\left[\sin^2\theta\frac{d^2}{dx^2}\Theta - 2\cos\theta\frac{d}{dx}\Theta - \frac{m}{\sin^2\theta}\Theta\right] = \lambda\Theta$

$\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)e^{il\varphi}\,\Theta_l^l(\theta)$ $\quad e^{il\varphi}$

$= \hbar e^{i\varphi}\left\{e^{il\varphi}\frac{\partial}{\partial\theta}\Theta_l^l(\theta) + i\cot\theta\,(il)\,\Theta_l^l(\theta)\right\}$

$= \hbar e^{i(l+1)\varphi}\underline{\left\{\frac{\partial}{\partial\theta} - l\cot\theta\right\}\Theta_l^l(\theta) = 0}$

$\frac{d}{d\theta}f = l\cot\theta f$ $\quad \frac{f'}{f} = l\cot\theta$

$(\log f)' = l\cot\theta$

$\log f = l\int d\theta\,\cot\theta$ $\quad \sin\theta = u$, $du = \cos\theta\, d\theta$

$= l\int\frac{\cos\theta\, d\theta}{\sin\theta}$

$= l\int\frac{du}{u} = l\log u + C$

$f = C(e^{\log u})^l = Cu^l = C\sin^l\theta$

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