Lecture 28

TIME INTEGRATION: PREDICTOR-CORRECTOR AND RUNGE-KUTTA METHODS

28.1 INTRODUCTION

Let us consider the first order ordinary differential equation for the dependent variable ϕ given by

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = f\left(t, \phi(t)\right), \tag{27.1}$$

with an initial condition $\phi(t_0) = \phi^0$. Explicit methods for time integration of Eq. (27.1) discussed in previous lecture are easy to program and use, but are conditionally stable. Implicit methods (such as backward Euler method) offer better stability but are computationally expensive. Predictor-corrector methods offer a compromise between these choices. There exists a wide-variety of such methods based on the choice of the base methods and time instants used in predictor and corrector steps. In this lecture, we discuss two of the most widely used family of such methods.

28.2 MUTI-LEVEL PREDICTOR-CORRECTOR METHODS

A multi-level predictor-corrector method can be constructed based on an Adams-Bashforth method as predictor and an Adams-Moulton method as corrector. This composite technique is also called Adams-Bashforth-Moulton scheme. For example, a fourth order Adams-Bashforth-Moulton scheme is given by

$$\text{Predictor: } \phi_*^{n+1} = \phi^n + \frac{\Delta t}{24}\left[55 f^n - 59 f^{n-1} + 37 f^{n-2} - 9 f^{n-3}\right] \tag{28.1}$$

$$\text{Corrector: } f_*^{n+1} = f\left(t_{n+1}, \phi_*^{n+1}\right)$$

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{24}\left[9 f_*^{n+1} + 19 f^n - 5 f^{n-1} + f^{n-2}\right] \tag{28.2}$$

Advantage: Only two function evaluations are required per time step.

Disadvantage: Not self-starting.

28.3 RUNGE-KUTTA METHODS

These methods are two level multi-point methods which are easy to use and self-starting, but require more computational effort per time step as compared to multi-point methods. These are essentially predictor-corrector methods which employ function values at points between t_n and t_{n+1}. These methods are generally more expensive but are more accurate and stable than multi-point methods of the same order. Two most popular methods of this family are given below.

Second order Runge-Kutta Method

It consists of two steps: a half-step predictor based on explicit Euler method followed by a mid-point rule corrector, i.e.

Predictor: $$\phi_*^{n+\frac{1}{2}} = \phi^n + \frac{\Delta t}{2} f\left(t_n, \phi^n\right) \tag{28.3}$$

Corrector: $$\phi^{n+1} = \phi^n + \Delta t\, f\left(t_{n+\frac{1}{2}}, \phi_*^{n+\frac{1}{2}}\right) \tag{28.4}$$

Fourth order Range-Kutta Method

This method consists of four steps: explicit Euler predictor and implicit Euler corrector at $t_{n+\frac{1}{2}}$ followed by mid-point rule predictor for full step and Simpson's rule corrector for full step, i.e.

- Explicit Euler Predictor: $$\phi_*^{n+\frac{1}{2}} = \phi^n + \frac{\Delta t}{2} f\left(t_n, \phi^n\right) \tag{28.5}$$
- Implicit Euler corrector: $$\phi_{**}^{n+\frac{1}{2}} = \phi^n + \frac{\Delta t}{2} f\left(t_{n+\frac{1}{2}}, \phi_*^{n+\frac{1}{2}}\right) \tag{28.6}$$
- Mid-point rule predictor: $$\phi_*^{n+1} = \phi^n + \Delta t f\left(t_{n+\frac{1}{2}}, \phi_{**}^{n+\frac{1}{2}}\right) \tag{28.7}$$
- Simpson's rule Corrector:

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{6}\left[f\left(t_n, \phi^n\right) + 2 f\left(t_{n+\frac{1}{2}}, \phi_*^{n+\frac{1}{2}}\right) + 2 f\left(t_{n+\frac{1}{2}}, \phi_{**}^{n+\frac{1}{2}}\right) + f\left(t_{n+1}, \phi_*^{n+1}\right)\right] \tag{28.8}$$

28.4 FINITE DIFFERENCE SCHEMES

Finite difference approximation of the time derivative can be used to construct time-marching schemes similar to two-level and multi-level methods. For example, forward difference approximation of $\frac{d\phi}{dt}$ at $t = t_n$ leads to:

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = f\left(t^n, \phi^n\right) \qquad \Rightarrow \phi^{n+1} = \phi^n + \Delta t f\left(t^n, \phi^n\right) \tag{28.9}$$

which is same as the explicit Euler method obtained from integral approach considered earlier. Similarly, a three-point backward difference approximation for the time derivative

$$\left(\frac{d\phi}{dt}\right)_{t_{n+1}} \approx \frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\Delta t} \tag{28.10}$$

leads to the second order accurate implicit scheme given by

$$\phi^{n+1} = \frac{4}{3}\phi^n - \frac{1}{3}\phi^{n-1} + \frac{2}{3}\phi^n f\left(t_{n+1}, \phi^{n+1}\right)\Delta t \tag{28.11}$$

Similar schemes can also be derived using weighted residual finite element formulation. For further discussions on various time integration algorithms, refer Wood (1990) and Chung (2010).

FURTHER READING

Chung, T. J. (2010). *Computational Fluid Dynamics*. 2nd Ed., Cambridge University Press, Cambridge, UK.

Ferziger, J. H. And Perić, M. (2003). *Computational Methods for Fluid Dynamics*. Springer.

Wood, W. L. (1990). *Practical Time-stepping Schemes*. Clarendon Press, Oxford.