Lecture 21
VARIATIONAL FORMULATIONS

21.1 VARIATIONAL PRINCIPLES

For certain class of problems in physics, it is possible to define a scalar quantity (for example, potential energy for a problem in mechanics) whose stationarity leads to the solution of the problem. Formally, a variational principle specifies a scalar functional Π defined as an integral given by

$$\Pi = \int_{\Omega} F(\mathbf{x}, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, ...)\mathrm{d}\Omega + \int_{\Gamma} G(\mathbf{x}, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, ...)\mathrm{d}\Gamma \tag{21.1}$$

where u is the dependent variable, and F and G are differential operators. The solution of the continuum problem is the function u which makes this functional Π stationary with respect to arbitrary changes δu, i.e. $\delta\Pi = 0$.

Thus, if a variational principle can be found, then it is straightforward to establish an integral form suitable for finite element analysis. However, in practice, it is very difficult to obtain an expressions for functions F and G from the differential equation of the problem, except for linear differential equations. For instance, if L were a linear self-adjoint operator in the differential equation

$$L(u) + f = 0 \tag{21.2}$$

then $F = u(Lu - 2f)$, i.e. functional Π can defined as

$$\Pi = \int_{\Omega} u(L(u) - 2f)\mathrm{d}\Omega \tag{21.3}$$

21.2 VARIATIONAL FINITE ELEMENT FORMULATION

Let us assume that an approximate solution $\tilde{u}$ of Eq. (21.2) expressed in the form

$$u(\mathbf{x}) \approx \tilde{u}(\mathbf{x}) = \sum_i N_i(\mathbf{x}) u(\mathbf{x}_i) \tag{21.4}$$

where N_i are prescribed functions, called interpolation (shape or trial) functions and u_i is still unknown value of variable u at a discrete spatial point $\mathbf{x}_i$. We can insert this approximation in Eq. (21.1) and express the stationarity condition as

$$\delta\Pi = \sum_i \frac{\partial \Pi}{\partial u_i}\delta u_i = 0 \tag{21.5}$$

The preceding equation holds for any variation δu, which is possible only if

$$\frac{\partial \Pi}{\partial u_i} = 0 \text{ for all } i. \tag{21.6}$$

Nodal values u_i can be determined by solving the system of equations (21.6).

Note that if the functional Π is *quadratic* (in function u and its derivatives), then Eq. (21.6) results in a linear system

$$\mathbf{Ku} = \mathbf{b} \tag{21.7}$$

where **K** is a symmetric matrix. In fact, if a variational principle exists, variational finite element formulation would result in a symmetric system matrix. In such problems, even a Galerkin finite element formulation would result in a symmetric system.

Further generalizations of the variational formulation include constrained variational forms based on (i) Lagrange multipliers and (ii) penalty functions. These forms are used to enforce additional constraints including boundary constraints. For further details, see Zienkiewicz, Taylor and Zhu (2005).

20.3 LEAST SQUARES FORMULATION

Least squares approach can be used to define a scalar functional for any PDE to circumvent the difficulty associated with derivation of a variational principle. Thus, for the differential equation $L(u)+f=0,$ the least squares functional can be defined as

$$\bar{\Pi}=\int_{\Omega}(L(u)+f)(L(u)+f)\mathrm{d}\Omega \tag{21.8}$$

Use of usual finite element approximation in conjunction with minimization of the above functional again leads to a system of equation (21.7) with a symmetric matrix **K.**

REFERENCES/FURTHER READING

Muralidhar, K. and Sundararajan, T. (2003). *Computational Fluid Dynamics and Heat Transfer*, Narosa Publishing House.

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Zienkiewicz, O. C., Taylor, R. L., Zhu, J. Z. (2005). *The Finite Element Method: Its Basis and Fundamentals*, 6th Ed., Butterworth-Heinemann (Elsevier).