Lecture 5

CONTINUITY EQUATION

5.1 MASS CONSERVATION EQUATION: INTEGRAL FORM

For a system, its mass is conserved, i.e. $\mathrm{d}M/\mathrm{d}t = 0$. Since $M = \int_\Omega \rho .1\,\mathrm{d}\Omega$, hence $\phi \equiv 1$, and Reynolds transport theorem (L4.22) yields the following integral form for the mass conservation (or continuity) equation for a stationary control volume:

$$\frac{\partial}{\partial t}\int_\Omega \rho \mathrm{d}\Omega + \int_S \rho \mathbf{v} \cdot \mathrm{d}\mathbf{A} = 0 \tag{5.1}$$

5.2 MASS CONSERVATION EQUATION: DIFFERENTIAL FORM

For a fixed control volume, order of temporal differentiation and integration in Eq. (5.1) can be interchanged. Further, the convective term can be transformed into a volume integral by applying Gauss divergence theorem, i.e.

$$\frac{\partial}{\partial t}\int_\Omega \rho \mathrm{d}\Omega = \int_\Omega \frac{\partial \rho}{\partial t}\mathrm{d}\Omega \quad \text{and} \quad \int_S \rho \mathbf{v} \cdot \mathrm{d}\mathbf{A} = \int_\Omega \nabla \cdot (\rho \mathbf{v})\mathrm{d}\Omega \tag{5.2}$$

Substitution of Eq. (5.2) into Eq. (5.1) yields

$$\int_\Omega \frac{\partial \rho}{\partial t}\mathrm{d}\Omega + \int_\Omega \nabla \cdot (\rho \mathbf{v})\mathrm{d}\Omega = 0 \quad \Rightarrow \int_\Omega \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right]\mathrm{d}\Omega = 0 \tag{5.3}$$

The preceding equation holds for any control volume which is possible only if the integrand vanishes everywhere, i.e.

$$\frac{\partial \rho}{\partial t} + \nabla .\left(\rho \mathbf{v}\right) = 0 \tag{5.4}$$

Equation (5.4) represents the differential form of continuity equation in vector notation. In Cartesian coordinates with usual notation of velocity components (i.e. $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \equiv v_i \mathbf{i}_i$), the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \equiv \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0 \tag{5.5}$$

The differential form (5.4) or (5.5) can be also derived by considering mass conservation for an infinitesimal differential control volume (see Example 5.1 below). Expanded form of continuity equation in cylindrical polar and spherical polar coordinates can be found in any text on fluid mechanics, e.g. Kundu and Cohen (2008) and Panton (2005).

Example 5.1
Derive the differential form of continuity equation using an infinitesimal differential control volume in Cartesian coordinates.

Solution
Let us consider flow of a fluid through an infinitesimal differential control volume of dimensions dx, dy and dz. For the sake of clarity, Figure 5.1 depicts the flow through a two-dimensional control volume. The mass flow rate of fluid entering from the left face (negative x-face) of the CV is $\rho u \mathrm{d}y\mathrm{d}z$ and the mass flow rate leaving the positive x-face of the CV is $\left[\rho u + \frac{\partial(\rho u)}{\partial x}\mathrm{d}x\right]\mathrm{d}y\mathrm{d}z.$ Further, the mass flow rate entering from the bottom face (negative y-face) of the CV is $\rho v \mathrm{d}x\mathrm{d}z$ and the mass flow rate leaving the top face (positive y-face) of the CV is $\left[\rho v + \frac{\partial(\rho v)}{\partial y}\mathrm{d}y\right]\mathrm{d}x\mathrm{d}z.$ Therefore,

$$\text{Net mass efflux rate through } x\text{-faces} = \left[\rho u + \frac{\partial(\rho u)}{\partial x}\mathrm{d}x\right]\mathrm{d}y\mathrm{d}z - \rho u\mathrm{d}y\mathrm{d}z$$
$$= \frac{\partial(\rho u)}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\text{Net mass efflux rate through } y\text{-faces} = \left[\rho v + \frac{\partial(\rho v)}{\partial y}\mathrm{d}y\right]\mathrm{d}x\mathrm{d}z - \rho v\mathrm{d}x\mathrm{d}z$$
$$= \frac{\partial(\rho v)}{\partial y}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

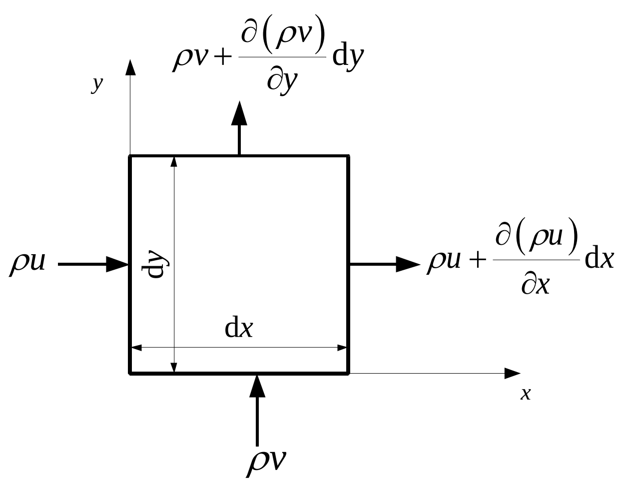

Figure 5.1 Mass fluxes through two-dimensional differential control volume

Similarly, for a three-dimensional control volume

$$\text{Net mass efflux rate through } z\text{-faces} = \frac{\partial(\rho w)}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\text{Hence, the net mass efflux rate} = \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Rate of accumulation of mass inside the CV = $\frac{\partial \rho}{\partial t}\mathrm{d}x\mathrm{d}y\mathrm{d}z$

For mass conservation, the rate of mass accumulation in the control volume must be negative of the net mass efflux rate, i.e.

$$\frac{\partial \rho}{\partial t}\mathrm{d}x\mathrm{d}y\mathrm{d}z = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Dividing both sides by the differential volume dxdydz and transferring all the terms on one side gives the following equation for mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

which is same as the continuity equation (5.5) derived from the integral form of the continuity equation.

Exercise 5.1: Derive the differential form of continuity equation in polar coordinates by take an infinitesimal control volume in (a) cylindrical polar coordinates and (b) spherical polar coordinates.

REFERENCES

Kundu, P. K. and Cohen, I. M. (2008). *Fluid Mechanics*, 4th Ed., Academic Press.

Panton, R. L. (2005). *Incompressible Flow*, 3rd Ed., Wiley.