Lecture 17
APPROXIMATION OF FINITE VOLUME INTEGRALS

17.1 INTRODUCTION

The integral form of conservation law for a control volume involves both surface and volume integrals. These integrals involve unknown function, and hence, can't be evaluated exactly. For finite volume solution, we must approximate these surface and volume integrals using appropriate quadrature formulae in terms of the function values at computational nodes.

17.2 APPROXIMATION OF SURFACE INTEGRALS

The net flux through the CV boundary is sum of the integrals over CV faces, i.e.

$$\int_S f \, \mathrm{d}S = \sum_k \int_{S_k} f \, \mathrm{d}S \tag{17.1}$$

where f is the component of convective $\left(\rho\phi\mathbf{v}\cdot\mathbf{n}\right)$ or diffusive $\left(\Gamma\nabla\mathbf{\Phi}\cdot\mathbf{n}\right)$ flux in the direction normal to the CV faces. To obtain approximate quadrature formulae, let us consider a typical CV face (say east face 'e'). Analogous expressions can be easily obtained for all faces by making appropriate index substitutions.

To calculate the integral exactly, f is required everywhere on the surface S_e. However, this information is not available, and must be approximated in terms of nodal values. To obtain an approximate value of the integral, two levels of approximations are used:

1. Approximate the integral in terms of variable values at one or more locations on the cell face.
2. Approximate the cell face value in terms of nodal (CV centre) values.

Integrals can be evaluated using (a) mid-point, (b) trapezoid or (c) Simpson's rule. The mid-point rule is the simplest and yields the following expression:

$$\int_{S_e} f \, \mathrm{d}S \approx f_e S_e \tag{17.2}$$

The preceding approximation is second order accurate if the value of f is known at location 'e'. Trapezoid rule requires the values of f at the CV corners and is given by

$$\int_{S_e} f \, \mathrm{d}S \approx \frac{S_e}{2}\left(f_{ne} + f_{se}\right) \tag{17.3}$$

Simpson's rule provides a fourth order approximation, and is given by

$$\int_{S_e} f \, \mathrm{d}S \approx \frac{S_e}{6}\left(f_{ne} + 4f_e + f_{se}\right) \tag{17.4}$$

The fourth order accuracy of the preceding formula can be retained if and only if the corner values are approximated with the accuracy of the same order. Of the preceding quadrature formulae, the mid-point rule is the most widely used, especially in 3-D applications.

17.3 APPROXIMATION OF VOLUME INTEGRALS

The most popular and the simplest method for evaluation of volume integrals is again the mid-point rule and is given by

$$Q_{\mathrm{P}} = \int_{\Omega} q\, d\Omega \equiv \bar{q}\Delta\Omega \approx q_{\mathrm{P}}\Delta\Omega\,, \qquad q_{\mathrm{P}} = \text{value at the cell centre} \tag{17.5}$$

The preceding formula is second order accurate (it is exact if q is constant or linear).

Higher order approximations of the volume integral require values of q at more location than just the centre of the CV. These values are obtained by interpolating nodal values (or using shape function). For instance, a fourth order approximation of 2D volume integral (which is essentially an area integral) can be obtained using bi-quadratic shape function given by

$$q(x,y) = a_0 + a_1x + a_2y + a_3x^2 + a_4y^2 + a_5xy + a_6x^2y + a_7xy^2 + a_8x^2y^2 \tag{17.6}$$

FURTHER READING

Chung, T. J. (2010). *Computational Fluid Dynamics*. 2nd Ed., Cambridge University Press, Cambridge, UK.

Ferziger, J. H. And Perić, M. (2003). *Computational Methods for Fluid Dynamics*. Springer.

Versteeg, H. K. and Malalasekera, W. M. G. (2007). *Introduction to Computational Fluid Dynamics: The Finite Volume Method*. Second Edition (Indian Reprint) Pearson Education.