Lecture 9

CLASSIFICATION OF GOVERNING EQUATIONS

9.1 INTRODUCTION

The governing equations of fluid flow (momentum, energy or scalar transport) are second order partial differential equations (PDEs). Choice of the numerical solution technique and number of initial/boundary conditions required for their solution depends on their mathematical character. However, since these equations are system of coupled nonlinear equations in four independent variables (time + three space coordinates), their mathematical classification is rather difficult. Nevertheless, a classification of these equations is usually applied to the linearized form of Navier-Stokes equations.

9.2 CLASSIFICATION OF QUASI-LINEAR PDES

Mathematical classification of quasi-linear PDEs is usually based on the classification procedure evolved for a general second order PDE in two independent variables (say, x and y) given by

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu+G=0 \tag{9.1}$$

where A, B, C, D, E, F and G are functions of x and y. Let us assume that $A\neq 0$ and B^2-4AC is of uniform sign for the range of values of x and y of interest. Classification of PDE (9.1) is based on the type of roots of the associated characteristic equation

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+B\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+C=0 \tag{9.2}$$

Let $r(x,y)$ and $s(x,y)$ be the solutions of the preceding equation. Each of these is called a characteristic. Existence of these roots depends on the value of the discriminant B^2-4AC. We list in Table 9.1 the type of the PDE for each of the three possible cases.

Table 9.1: Classification of quasi-linear second order PDEs

Discriminant	**Nature of characteristics**	**Type of PDE**
$B^2-4AC>0$	Two real characteristics	Hyperbolic
$B^2-4AC=0$	One real characteristic	Parabolic
$B^2-4AC<0$	No real characteristics	Elliptic

The type name for the PDE has been assigned in analogy with conic sections from analytical geometry wherein the quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents a hyperbola if $B^2 - 4AC > 0$, a parabola if $B^2 - 4AC = 0$ and an ellipse if $B^2 - 4AC < 0$.

The classification of PDE tells us about physical behaviour of the problem. The propagation of information (e.g. effect of disturbance introduced in a flow field) takes place along the characteristics. It also sets the required number of initial/boundary conditions for the given problem.

9.2.1 Hyperbolic Equations

In this case, information propagates at a finite speed along the characteristics. Further, any disturbance introduced in the problem domain at a specific point affects the solution in a limited region enclosed within the characteristics through that point. Equations of this type usually appear in time-dependent processes with negligible energy dissipation and lead to wave-type solutions. A typical hyperbolic equation is the wave equation given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{9.3}$$

where c denotes the finite speed of wave propagation. The preceding equation requires specification of two sets of initial conditions.

9.2.2 Parabolic Equations

Parabolic equations have only one real characteristic. Hence, these require only one set of initial conditions. Further, information travels in only one direction along the characteristic. Thus, these equations lend themselves to marching type solution schemes starting with the given initial data. Parabolic equations usually describe time dependent problems which involve significant amount of diffusion. The typical example is transient heat conduction in a plane wall governed by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{9.4}$$

For the heat diffusion problem (9.4), initial conditions are required at time $t = 0$. In addition, boundary conditions must be specified at both ends of the wall for all time $t > 0$. Hence, this problem is also called an ***initial-boundary value*** problem. To emphasize the requirement of the boundary conditions, some prefer to classify the transient heat conduction equation (9.4) as **parabolic-in-time and elliptic in space.**

9.2.3 Elliptic Equations

Elliptic equations have no real characteristics. Thus, there is no preferred direction for information propagation, and information travels equally well in all the directions. As a consequence, effect of a disturbance introduced anywhere in the problem domain will be felt everywhere. Thus, one boundary condition is required at all the points on the boundary of the problem domain. Hence, these equations are called **boundary-value** problems.

Elliptic equations usually represent steady state or equilibrium problems (unsteady problems are never elliptic). The representative of the elliptic equation is the Laplace equation given

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{9.5}$$

which may describe steady state heat conduction or potential flow problem.

9.3 CLASSIFICATION OF NAVIER-STOKES EQUATIONS

Navier-Stokes equations are coupled nonlinear partial differential equations in four variables. For mathematical classification, we can look at their linearized form. The formal classification for incompressible flows is as follows:

- Steady viscous flows: elliptic.
- Unsteady viscous flows: parabolic.

Energy equation has the same behaviour, i.e. it is elliptic for steady flows, and parabolic, for time dependent problems. Unsteady Navier-Stokes and energy equations are in fact parabolic in time and elliptic in space. Hence, solution of these equations requires (a) one set of initial conditions, and (b) boundary conditions at all the boundary points for all values to time $t > 0$. Compressible Navier-Stokes equations may be considered as mixed hyperbolic, parabolic and elliptic (or incompletely parabolic) equations.

Note that elliptic equations are more difficult to solve than parabolic equations, which lend themselves to marching type solution procedure. Thus, in practice, steady viscous flows are usually converted to unsteady problems, and solved using a time marching scheme. The solution obtained for large values of time t provides the desired solution of the steady viscous flow problem.

9.3 CLASSIFICATION OF EULER EQUATION

Classification of inviscid flow equations governed by Euler equation is different from that of Navier-Stokes or energy equations due to complete absence of the second order terms. The classification of these equations depends on the extent of compressibility. Further, for compressible flows, flow speed (Mach number, Ma) has a significant role in determining the behaviour of the problem. The formal classification of the inviscid flows is as follows (Anderson, 1995, Versteeg and Malalasekera, 2007):

- Inviscid incompressible flows
  - Steady flows: elliptic.
  - Unsteady flows: parabolic.
- Inviscid compressible flows
  - Steady subsonic flows (Ma < 1): elliptic.
  - Steady supersonic flows (Ma > 1): hyperbolic.
  - Unsteady flows: hyperbolic.

Dependence of the flow behaviour on local Mach number makes the solution of steady inviscid compressible flows pretty complicated. For example, let us consider high speed flow around a bluff body. Even if the upstream Mach number Ma > 1, in the vicinity of the solid surface, there exists a subsonic zone as the flow velocity goes to zero at the stagnation point. Therefore, separate algorithms would be required for numerical simulation of the problem in subsonic and supersonic zones (whose extent and boundaries are themselves unknown). This situation has baffled aerodynamicist for a while in early 1960s. A way out was provided by the time dependent approach to solve a steady problem (Moretti and

Abbett, 1966), since the unsteady Euler equation for compressible flows is hyperbolic everywhere (irrespective of the local Mach number). In fact this time dependent approach to steady state is also widely used now for solution of steady state viscous flows.

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