Lecture 34

NAVIER-STOKES EQUATIONS: IMPLICIT PRESSURE CORRECTION METHODS

34.1 IMPLICIT PRESSURE CORRECTION METHODS

Implicit pressure correction methods are primarily used for steady and slow transient flows. These employ an implicit time integration approach along with a pressure (or pressure correction) equation to enforce a mass continuity. To look at the algorithmic aspects of these methods, let us write the discretized equation obtained using an implicit time integration method (say, Eq. (32. 7)) in a quasi-linearized form as

$$A_{\mathrm{P}}^{v_i} v_{i,\mathrm{P}}^{n+1} + \sum_l A_l^{v_i} v_{i,l}^{n+1} = Q_{v_i}^{n+1} - \left(\frac{\delta p^{n+1}}{\delta x_i} \right)_{\mathrm{P}} \qquad (34.1)$$

where P is the index of an arbitrary velocity node, and l denotes the neighbouring points that appear in the momentum equation. Source term Q and coefficients A depend on unknown solution, v_i^{n+1}. Thus, Eq. (34.1) represents a set of coupled nonlinear equations which must be solved iteratively. For accurate time history, iteration must be continued at each time step to specified tolerance.

Terminology

- ***Outer iterations*:** The iterations at each time step, in which coefficients A and source term Q are updated, are called ***outer iterations***.
- distinguish them from the ***inner iterations*** performed on linear systems with fixed coefficients.

Let m denote the counter for outer iterations. We replace superscript n+1 with iteration counter m in Eq. (34.0) to obtain the following equation which must be solved at each outer iteration:

$$A_{\mathrm{P}}^{v_i} v_{i,\mathrm{P}}^{m} + \sum_l A_l^{v_i} v_{i,l}^{m} = Q_{v_i}^{m} - \left(\frac{\delta p^{m}}{\delta x_i} \right)_{\mathrm{P}} \qquad (34.1)$$

Momentum equations are solved sequentially. In the iteration process, we actually solve the linearized form of (34.1) using values of Q_{v_i} and p from previous iteration. Thus, we actually obtain an intermediate estimate of v_i^m denoted by which v_i^{m*} satisfies

$$A_{\mathrm{P}}^{v_i} v_{i,\mathrm{P}}^{m*} + \sum_l A_l^{v_i} v_{i,l}^{m*} = Q_{v_i}^{m-1} - \left(\frac{\delta p^{m-1}}{\delta x_i} \right)_{\mathrm{P}} \qquad (34.2)$$

Velocity field v_i^{m*} obtained as solution of (34.2) does not satisfy the continuity equation. Hence, it is not the final value of velocity for iteration m; it is a predicted value which must be corrected to ensure continuity. Let rewrite Eq. (34.2) as

$$v_{i,\mathrm{P}}^{m*} = \frac{Q_{v_i}^{m-1} - \sum_l A_l^{v_i} v_{i,l}^{m*}}{A_\mathrm{P}^{v_i}} - \frac{1}{A_\mathrm{P}^{v_i}} \left(\frac{\delta p^{m-1}}{\delta x_i} \right)_\mathrm{P} \tag{34.3}$$

For sake of convenience, let us use symbol $\tilde{v}_{i,P}^{m*}$ to denote the first term on RHS of the preceding equation, and rewrite it as

$$v_{i,\mathrm{P}}^{m*} = \tilde{v}_{i,\mathrm{P}}^{m*} - \frac{1}{A_\mathrm{P}^{v_i}} \left(\frac{\delta p^{m-1}}{\delta x_i} \right)_\mathrm{P} \quad \text{where} \quad \tilde{v}_{i,\mathrm{P}}^{m*} = \frac{1}{A_\mathrm{P}^{v_i}} \left(Q_{v_i}^{m-1} - \sum_l A_l^{v_i} v_{i,l}^{m*} \right) \tag{34.4}$$

Velocity field $\tilde{v}_{i,P}^{m*}$ can be thought of as one from which contribution of pressure field has been removed. Equation (34.4) represents a link between the predicted velocity and pressure field and can be extended to obtain the following relation between the corrected velocity field and pressure at the outer iteration *m*:

$$v_{i,\mathrm{P}}^{m} = \tilde{v}_{i,\mathrm{P}}^{m*} - \frac{1}{A_\mathrm{P}^{v_i}} \left(\frac{\delta p^{m}}{\delta x_i} \right)_\mathrm{P} \tag{34.5}$$

Corrected velocity v_i^m obtained from Eq. (34.5) must satisfy the continuity equation, i.e.

$$\frac{\delta\left(\rho v_i^m\right)}{\delta x_i} = 0 \tag{34.6}$$

Substituting v_i^m obtained from Eq. (34.5) into continuity equation (34.6) yields the Poisson equation for pressure given by

$$\frac{\delta}{\delta x_i}\left[\frac{\rho}{A_\mathrm{P}^{v_i}} \left(\frac{\delta p^m}{\delta x_i} \right) \right]_\mathrm{P} = \frac{\delta}{\delta x_i}\left[\frac{\delta\left(\rho \tilde{v}_i^{m*}\right)}{\delta x_i} \right]_\mathrm{P} \tag{34.7}$$

Using equations (34.2)-(34.7), we can formulate an iterative algorithm (Algorithm 9.3) based on an implicit time discretization, linearization and pressure correction at each time step (Figure 9.14).

Algorithm 9.3: Semi-implicit Pressure Correction based N-S Solver

Step 1. Using values of velocity and pressure at previous iteration level (v_i^{m-1} and p_i^{m-1}) or previous time step for *m*=1, solve the linearized equation (34.2) to obtain $v_i^{m_*}$.

Step 2. Compute $\tilde{v}_{i,P}^{m*}$ and its derivatives, and solve pressure Poisson equation (34.7) to obtain p^m.

Step 3. Compute corrected velocity field v_i^m using Eq. (34.5) which would satisfy continuity equation.

Step 4. Check if velocity field v_i^m and pressure field p^m satisfy the momentum equation (34.1).

- If yes, set these as values at time level *n*+1, i.e. $v_i^{n+1} \leftarrow v_i^m$, $p^{n+1} \leftarrow p^m$ and proceed to computations at next time level.
- If no, set *m* = *m*+1, go to Step 1.

Figure 9.14 A semi-implicit algorithm based on pressure correction for incompressible Navier-Stokes equation.

SIMPLE, SIMPLEC, SIMPLER

Many variants of the implicit pressure correction method (Algorithm 9.3) have been proposed and used in CFD literature. These methods assume that velocity field computed from linearized momentum equations and pressure p^{m-1} can be taken as provisional values to which small corrections must be added to obtain the final values for current outer iteration, i.e.

$$v_i^m = v_i^{m_*} + v_i' \quad \text{and} \quad p^m = p^{m-1} + p' \tag{34.8}$$

The first such algorithm, called SIMPLE (Semi-Implicit Method for Pressure Linked Equations), was proposed by Patankar and Spalding (1972). Subsequently, various improvements of SIMPLE such as SIMPLER, SIMPLEC, and PISO have been put forth. We present here a brief description of SIMPLE, SIMPLEC and SIMPLER. For further details, see Ferziger and Peric (2003) and Versteeg and Malalasekera (2007).

Substitution of Eq. (34.8) in momentum equation (34.1) and use of Eq. (34.2) yield the following relation between velocity and pressure corrections:

$$v'_{i,\mathrm{P}} = \tilde{v}'_{i,\mathrm{P}} - \frac{1}{A_\mathrm{P}^{v_i}}\left(\frac{\delta p'}{\delta x_i}\right)_\mathrm{P} \tag{34.9}$$

where $\tilde{v}_i'$ is defined by

$$\tilde{v}'_{i,\mathrm{P}} = -\frac{1}{A_\mathrm{P}^{v_i}}\sum_l A_l^{v_i} v'_{i,l} \tag{9.220}$$

Let us apply discretized continuity equation (34.6) to the corrected velocity v_i^m, and use Eq. (34.8) and (34.9) to obtain the following Poisson equation for pressure correction:

$$\frac{\delta}{\delta x_i}\left[\frac{\rho}{A_{\mathrm{P}}^{v_i}}\left(\frac{\delta p'}{\delta x_i}\right)\right]_{\mathrm{P}} = \left[\frac{\delta\left(\rho v_i^{m*}\right)}{\delta x_i}\right]_{\mathrm{P}} + \left[\frac{\delta\left(\rho \tilde{v}_i'\right)}{\delta x_i}\right]_{\mathrm{P}} \tag{9.221}$$

Velocity correction $\tilde{v}_i'$ are unknown at this point. If these are neglected, we get the following equations for pressure and velocity corrections:

$$\frac{\delta}{\delta x_i}\left[\frac{\rho}{A_{\mathrm{P}}^{v_i}}\left(\frac{\delta p'}{\delta x_i}\right)\right]_{\mathrm{P}} = \left[\frac{\delta\left(\rho v_i^{m*}\right)}{\delta x_i}\right]_{\mathrm{P}} \tag{9.222}$$

$$v_{i,\mathrm{P}}' = -\frac{1}{A_{\mathrm{P}}^{v_i}}\left(\frac{\delta p'}{\delta x_i}\right)_{\mathrm{P}} \tag{9.223}$$

We can solve Eq. (9.222) for pressure corrections, and thereafter compute velocity corrections using Eq. (9.223). Omission of $\tilde{v}_i'$ term is the main approximation of the SIMPLE algorithm which is outlined as Algorithm 9.4 in Figure 9.15.

Algorithm 9.4: SIMPLE Algorithm

Step 1. Using values of velocity and pressure at previous iteration level (v_i^{m-1} and p_i^{m-1}) or previous time step for m=1, solve the linearized equation (34.2) to obtain $v_i^{m_*}$.

Step 2. Solve Poisson equation for pressure correction (9.222) to obtain p' .

Step 3. Compute velocity correction v_i' using Eq. (9.223).

Step 4. Compute corrected velocity field v_i^m and pressure field p^m using Eq. (34.8).

Step 5. Check if velocity field v_i^m and pressure field p^m satisfy the momentum equation (34.1).

a. If yes, set these as values at time level n+1, i.e. $v_i^{n+1} \leftarrow v_i^m$, $p^{n+1} \leftarrow p^m$ and proceed to computations at next time level.

b. If no, set $m = m$+1, go to Step 1.

Figure 9.15 SIMPLE algorithm (Patankar and Spalding, 1972) for incompressible Navier-Stokes equation.

To improve the convergence of SIMPLE algorithm, an under-relaxation of pressure and velocity correction is normally used:

$$\begin{aligned} p^m &= p^{m-1} + \alpha_p\, p' \\ v_i^m &= v_i^{m*} + \alpha_v v_i' \end{aligned} \tag{9.224}$$

Patankar (1980) recommends $\alpha_p = 0.8$ and $\alpha_v = 0.5$ with the note that these have been satisfactory in a large number of flow computations, but these are neither optimum nor guarantee convergence for all problems.

In SIMPLE algorithm, contribution of unknown velocity correction $\tilde{v}'_i$ was neglected in pressure correction equation (9.221). To improve the convergence behaviour, it would better to approximate $\tilde{v}'_i$ instead of neglecting it. One can approximate velocity correction at any node $v'_{i,\mathrm{P}}$ by a weighted mean of neighbour values of v'_i, i.e.

$$v'_{i,\mathrm{P}} \approx \left(\sum_l A_l^{v_i} v'_{i,l}\right) / \left(\sum_l A_l^{v_i}\right) \tag{9.225}$$

Substitution of this approximation in Eq. (9.220) yields

$$\tilde{v}'_{i,\mathrm{P}} \approx -\frac{1}{A_\mathrm{P}^{v_i}} v'_{i,\mathrm{P}} \sum_l A_l^{v_i} \tag{9.226}$$

Substitution Eq. (9.226) into Eq. (34.9) leads to the following relation between velocity and pressure corrections:

$$v'_{i,\mathrm{P}} = -\frac{1}{A_\mathrm{P}^{v_i} + \sum_l A_l^{v_i}} \left(\frac{\delta p'}{\delta x_i}\right)_\mathrm{P} \tag{9.227}$$

Use of the preceding expression for velocity correction in discretized continuity equation (34.6) yields the following Poisson equation for pressure correction:

$$\frac{\delta}{\delta x_i}\left[\frac{\rho}{A_\mathrm{P}^{v_i} + \sum_l A_l^{v_i}}\left(\frac{\delta p'}{\delta x_i}\right)\right]_\mathrm{P} = \left[\frac{\delta\left(\rho v_l^{m*}\right)}{\delta x_i}\right]_\mathrm{P} \tag{9.228}$$

which leads the SIMPLEC (SIMPLE-Consistent) algorithm of van Doormal and Raithby (1984). Solution steps in this algorithm are similar to SIMPLE (Algorithm 9.4) with the following two modifications:

- In Step 2, solve Eq. (9.228) instead of Eq. (9.222) to obtain pressure corrections.
- In Step 3, use Eq. (9.227) instead of Eq. (9.223) to obtain velocity corrections.

Many other attempts have been made to improve SIMPLE algorithm. One such method called SIMPLER (SIMPLE Revised) was suggested by Patankar (1980). In this method, pressure correction is computed in the same way as in SIMPLE, but it used only to obtain velocity corrections. New pressure field is computed separately using Eq. (34.7) in which corrected velocity v_i^m is used in place of v_i^{m*}. This algorithm is outlined in Figure 9.16 as Algorithm 9.5 and leads to significant improvement in convergence of the iterative solution process at the cost of the solution of an additional Poisson equation (i.e. a system of linear equations).

Algorithm 9.5: SIMPLER Algorithm

Step 1. Using values of velocity and pressure at previous iteration level (v_i^{m-1} and p_i^{m-1}) or previous time step for m=1, solve the linearized equation (34.2) to obtain v_i^{m*} .

Step 2. Solve Poisson equation for pressure correction (9.222) to obtain p' .

Step 3. Compute velocity correction v_i' using Eq. (9.223).

Step 4. Compute corrected velocity field v_i^m using Eq. (34.8).

Step 5. Solve pressure Poisson equation (34.7) by using corrected velocity field v_i^m

Step 6. Check if velocity field v_i^m and pressure field p^m satisfy the momentum equation (34.1).

a. If yes, set these as values at time level n+1, i.e. $v_i^{n+1} \leftarrow v_i^m$, $p^{n+1} \leftarrow p^m$ and proceed to computations at next time level.

b. If no, set $m = m$+1, go to Step 1.

Figure 9.16 SIMPLER algorithm (Patankar, 1980) for incompressible Navier-Stokes equation.

There are numerous other methods proposed in literature for unsteady as well as steady incompressible flows. Prominent approaches include fractional step methods based on approximate factorization (Kim and Moin, 1985; Choi and Moin, 1994), and artificial compressibility method (Chorin, 1967). For further details, refer to Ferziger and Peric (2003) and Chung (2010).

FURTHER READING

Ferziger, J. H. And Perić, M. (2003). *Computational Methods for Fluid Dynamics*. Springer.

Versteeg, H. K. and Malalasekera, W. M. G. (2007). *Introduction to Computational Fluid Dynamics: The Finite Volume Method*. Second Edition (Indian Reprint) Pearson Education.