Lecture 33

NAVIER-STOKES EQUATIONS: TIME INTEGRATION

33.1 SOLUTION TECHNIQUES FOR INCOMPRESSIBLE FLOWS

Most of the primary variable solution techniques for incompressible flows are built on either an explicit or an implicit time marching scheme. Majority of these methods can be classified as *projection methods* which first construct a velocity field that does not satisfy continuity equation, and then correct it by subtracting something (e.g. a pressure gradient) to enforce continuity.

A typical numerical simulation algorithm for Navier-Stokes equations consists of two separate discretization steps:

(a) spatial discretization and
(b) temporal discretization.

The spatial discretization using finite difference, finite volume or finite element method results in a system of semi-discrete coupled non-linear ordinary equations in time which are solved using an appropriate time marching scheme (implicit or explicit). Further, at each time step, we may need to solve a set of linear or non-linear algebraic equations depending on the choice of time integration algorithm. In what follows in this lecture, we tacitly assume that Navier-Stokes equations have already been discretized in space by the analyst using her/his favourite discretization method (namely FDM, FVM or FEM), and we present the overall solution algorithm following this step.

33.2 EXPLICIT TIME INTEGRATION TECHNIQUES

Navier-Stokes solvers based on explicit time integration techniques are the simplest to implement in a computer code. However, stability requirements of the explicit time integration techniques impose severe limitations on time step. Hence, these techniques are primarily used for the flow problems in which accuracy requirements demand use of very small time steps (which also satisfy the stability conditions). Typical examples include simulation of flow transients and LES/DNS of turbulent flows which require accurate time history.

To obtain an explicit time integration algorithm for Navier-Stokes equations, let us express the semi-discretized (i.e. discrete in space but not in time) momentum equation symbolically as

$$\frac{\partial\left(\rho v_i\right)}{\partial t} = -\frac{\delta\left(\rho v_i v_j\right)}{\delta x_j} - \frac{\delta p}{\delta x_i} + \frac{\delta \tau_{ij}}{\delta x_j} \equiv H_i - \frac{\delta p}{\delta x_i} \tag{33.1}$$

where $\delta()/\delta x_i$ represents a discretized spatial derivative (which could be different for each term) and H_i is short-hand notation for advective and viscous terms. Any explicit time integration scheme can be used for solution of Eq. (33.1). For sake of simplicity, let us apply explicit Euler method (procedure for higher order methods such as Adams-Bashforth and Range-Kutta methods would be very similar) to Eq. (33.1) which yields

$$\left(\rho v_i\right)^{n+1} = \left(\rho v_i\right)^n + \Delta t\left(H_i^n - \frac{\delta p^n}{\delta x_i}\right) \tag{33.2}$$

In the preceding equation, all quantities on RHS except pressure p^n are known. Pressure field p^n must be computed by solving a pressure Poisson equation which would ensure satisfaction of mass conservation. To derive this discrete pressure Poisson equation, let us note that the computed velocity field must satisfy the continuity equation, i.e.

$$\frac{\delta\left(\rho v_i\right)^{n+1}}{\delta x_i} = 0 \tag{33.3}$$

Taking numerical divergence of Eq. (33.2), we get

$$\frac{\delta\left(\rho v_i\right)^{n+1}}{\delta x_i} = \frac{\delta\left(\rho v_i\right)^{n}}{\delta x_i} + \Delta t\left(\frac{\delta}{\delta x_i}\left(H_i^n - \frac{\delta p^n}{\delta x_i}\right)\right) \tag{33.4}$$

For mass conservation, LHS of preceding equation must be zero. First term on RHS should also be zero if continuity was enforced at time step *n,* otherwise this term must be retained. Assuming satisfaction of continuity at time step *n*, we get the following discrete Poisson equation for pressure p^n:

$$\frac{\delta}{\delta x_i}\left(\frac{\delta p^n}{\delta x_i}\right) = \frac{\delta H_i^n}{\delta x_i} \tag{33.5}$$

In Eq.(33.5), the outer operator $\delta()/\delta x_i$ is the numerical divergence operator inherited from the continuity equation, whereas $\delta p/\delta x_i$ is the pressure gradient from the momentum equations. From the derivation of Eq. (33.5), it is clear that if the pressure p^n satisfies this discrete Poisson equation, the velocity field at time step (n+1) will be divergence free (i.e. it will be satisfy the continuity equation). We now have all the desired discrete equations at a given time step to obtain all the flow variables, and the solution algorithm based on explicit time integration can be formalized as Algorithm 33.1 for a given time instant (time level n+1).

Algorithm 33.1: Explicit Navier-Stokes Solver

Step 1. Starting with velocity field v_i^n at time t_n, compute H_i^n (which contains the advective and viscous terms of the momentum equation) and its divergence.

Step 2. Solve the pressure Poisson equation (33.5) to obtain p^n.

Step 3. Compute the velocity field at new time step t_{n+1} using Eq. (33.2).

The biggest advantage of the explicit solver given by Algorithm 33.1 is its simplicity and computational requirements at each time step: we do not have to solve a non-linear system of equations as velocity field is obtained explicitly in terms of values at previous time step; we only need to solve one linear system resulting from pressure Poisson equation. In fact, the most demanding step in Algorithm 33.1 is the solution of pressure Poisson equation. The disadvantage is requirement of small time step, which makes these methods unsuitable for steady state or slow transient problems wherein large time steps are preferred. For these cases, implicit or semi-implicit solvers discussed next are preferred.

33.3 IMPLICIT TIME INTEGRATION TECHNIQUES

Implicit methods are normally used for solution of steady state or slow-transient flows for which we need to employ large time steps. These methods involve additional complications which are illustrated below. Let us apply the simplest implicit scheme, the backward Euler method, for time integration of the semi-discrete momentum equation (33.1) which leads to the discretized momentum equation

$$\left(\rho v_i\right)^{n+1} - \left(\rho v_i\right)^n = \Delta t\left(-\frac{\delta\left(\rho v_i v_j\right)^{n+1}}{\delta x_j} - \frac{\delta p^{n+1}}{\delta x_i} + \frac{\delta \tau_{ij}^{n+1}}{\delta x_j}\right) \tag{33.6}$$

Equation (33.6) represents a system of coupled nonlinear equations in velocity components and has an additional unknown p^{n+1}. To derive the discrete equation for pressure, let us take divergence of the preceding equation and apply the continuity condition. If viscosity is assumed constant, only terms left are the once related to pressure and advective part in the preceding equation, and lead to the Poisson equation for pressure given by

$$\frac{\delta}{\delta x_i}\left(\frac{\delta p^{n+1}}{\delta x_i}\right) = \frac{\delta}{\delta x_i}\left[\frac{\delta\left(\rho v_i v_j\right)^{n+1}}{\delta x_j}\right] \tag{33.7}$$

Equations (33.6) and (33.7) represent a set of coupled nonlinear equations which must be solved simultaneously using a Newton-Raphson type iterative scheme. Results for the preceding time step are used as the initial guess in this iterative process. However, we must solve a very large system of non-linear equations at each time step. This process can be formalized as Algorithm 33.2.

Algorithm 33.2: Implicit Navier-Stokes Solver

Step 1. Take the velocity and pressure field at time level *n* as the starting guess for values at time level *n*+1.

Step 2. Solve the coupled non-linear system of equations (33.6) and (33.7) simultaneously to obtain velocity and pressure field at time level

Algorithm 33.2 is extremely demanding in terms of computation time and memory requirements as we need to solve a very large non-linear system at each time step. Usually, a quasi-linearization is attempted so that we need to solve only a sequence of linear systems especially for steady state problems. There are varieties of such options, some which will be discussed in the next lecture.

FURTHER READING

Ferziger, J. H. And Perić, M. (2003). *Computational Methods for Fluid Dynamics*. Springer.

Versteeg, H. K. and Malalasekera, W. M. G. (2007). *Introduction to Computational Fluid Dynamics: The Finite Volume Method*. Second Edition (Indian Reprint) Pearson Education.