Lecture 20

FINITE ELEMENT APPROXIMATION AND WEIGHTED-RESIDUAL METHOD

20.1 FINITE ELEMENT METHOD

The finite element method (FEM) originated from the matrix methods in structural mechanics in early 1950. Its initial applications were focussed on stress analysis problems using variational principles. Variational principles are not available for all class of continuum problems. Hence, later research focussed on generalization of finite element formulation based on weighted residual approach. This latter approach has become extremely popular since it has allowed application of finite element method to a wide range of problems across different disciplines in physical sciences and engineering --- from continuum mechanics to electrodynamics to nuclear physics.

20.2 FINITE ELEMENT METHODOLOGY

Starting point for the finite element method is the differential form of conservation equations of a continuum problem. A variational or weighted-residual formulation is used to transform the governing equations into an integral equation (usually called the global *weak form*). The problem domain is discretized into a set of non-overlapping finite elements, and the weak form is applied to each finite element resulting into a set of discrete equations in terms of nodal unknowns. Thus, the finite element solution of a continuum problem consists of the following steps:

- Discretize the solution domain by a mesh (i.e. a set of non-overlapping finite elements).
- Choose an appropriate set of interpolation (shape) functions to approximate the unknown function in terms of nodal values of the unknown.
- Apply the weak form of governing equation to each element, and evaluate the required integrals numerically. This procedure leads to a set of discrete equations for the unknown function values at the nodes of the element.
- Assemble all the elemental equations to form a global discrete system for the nodal unknowns.
- Apply the boundary conditions and solve the global system of algebraic equations to obtain values of the variable at each node.

The solution obtained at the nodes can be interpolated and processed to obtain the desired physical quantities in the entire domain in the post-processing step of the simulation.

20.3 FINITE ELEMENT FORMULATIONS

Finite element method requires transformation of the governing differential equation into an appropriate integral equation. This process accomplished either through (a) variational formulation, or (b) weighted residual formulation. Variational formulation is possible only for a limited class of problems. Weighted residual formulation can be applied to any problem, and hence, it is the most widely used.

20.4 WEIGHTED RESIDUAL FORMULATION

Let the governing equation for a physical problem be expressed as

$$L(u) = 0 \tag{20.1}$$

where L is a differential operator, with associated boundary (and initial) conditions. We seek an approximate solution $\tilde{u}$ of Eq. (20.1) expressed in the form

$$u(\mathbf{x}) \approx \tilde{u}(\mathbf{x}) = \sum_i N_i(\mathbf{x}) u(\mathbf{x}_i) \tag{20.2}$$

where N_i are prescribed functions, called interpolation (shape or trial) functions and u_i is still unknown value of variable u at a discrete spatial point $\mathbf{x}_i$. Substitution of the preceding approximation in Eq. (20.1) would lead to the residual (error) function R given by

$$R = L(\tilde{u}) \tag{20.3}$$

To determine nodal values u_i, the inner product of the residual R with a prescribed weight function w_i is set to zero, i.e.

$$\int_\Omega R w_i \, \mathrm{d}\Omega \equiv \int_\Omega L(\tilde{u}) w_i \, \mathrm{d}\Omega = 0 \tag{20.4}$$

Weight function w_i is also referred as test function. If the functions w_i belong to a complete set of functions, then weighted residual statement (20.4) implies that R must be orthogonal to every member of a complete set of functions. Therefore, R converges to zero in the mean, and thus, $\tilde{u}$ converges to the exact solution u of Eq. (20.1) in the mean.

Different solution methods can be obtained with different choices of weight functions w_i. Some of the common choices are:

- ***Point collocation*:** $w_i(\mathbf{x}) = \delta(\mathbf{x}, \mathbf{x}_i)$ where δ is Dirac-delta function. With this choice, Eq. (20.4) simplifies to

  $$R(\mathbf{x}_k) = 0, \qquad k = 1, 2, \ldots, n \tag{20.5}$$

  Thus, this choice simply involves setting the residual to zero at n points in the problem domain. This procedure is analogous to finite difference method.
- ***Subdomain collocation*:** Problem domain is subdivided into a set of non-overlapping subdomains, and weight function w is a piecewise function such that $w_k(\mathbf{x}) = 1$ in the k^{th} subdomain and zero elsewhere. This choice leads to a finite volume formulation.
- ***Galerkin (Bubnov-Galerkin) method*:** Choice of $w_i(\mathbf{x}) = N_i(\mathbf{x})$ is referred to as the Galerkin formulation. In case of Galerkin finite element method, trial functions N_i are local functions defined over a finite element.
- ***Petrov-Galerkin method*:** Any choice of weight function such that $w_i(\mathbf{x}) \neq N_i(\mathbf{x})$ is referred to as the Petrov-Galerkin method. Thus, it represents a generalization of all formulations except Galerkin method.
- ***Boundary Element Method*:** If the weight function w is chosen as the particular solution (also called free space Green's function or fundamental solution) of the following equation:

  $$L^*(w) + \delta(\boldsymbol{\xi}, \mathbf{x}) = 0 \tag{20.6}$$

where L^* is the adjoint operator of L, then Eq. (20.4) leads to an integral equation given by

$$c(\xi)u(\xi) + \int_{\Gamma} F(u)G(w)\mathrm{d}\Gamma = 0 \tag{20.7}$$

and the resulting method is popularly known as boundary element method (BEM).

Weak Form

Equation (20.4) is called the *strong form* of weighted residual statement. In many cases, it is possible to perform integration by parts and obtain the following form:

$$\int_{\Gamma} A(\tilde{u})w_i \,\mathrm{d}\Gamma + \int_{\Omega} B(\tilde{u})C(w_i)\,\mathrm{d}\Omega = 0 \tag{20.8}$$

where A, B and C are differential operators which contain lower order derivatives than those occurring in L, and hence, Eq. (20.8) is called the *weak form* of weighted residual statement. Due to reduced continuity requirement of trial functions, *weak form* is preferred in most finite element formulations.

20.5 WEIGHTED RESIDUAL FORMULATION FOR POISSON EQUATION

To illustrate the discussion of weighted residual formulation in preceding section, let us consider linear Poisson equation given by

$$\nabla^2 u - p = 0 \tag{20.9}$$

where p is the source function. *Strong form* of the weighted residual statement is given by

$$\int_{\Omega} (\nabla^2 \tilde{u} - p) w_i \mathrm{d}\Omega = 0 \tag{20.10}$$

which requires trial functions N_i to be twice differentiable i.e. it should be at least C^1 continuous. On the other hand, there is no continuity requirement on the test function w_i. Performing integration by parts, we get

$$\int_{\Gamma} q w_i \mathrm{d}\Gamma - \int_{\Omega} (\tilde{u}_{,k} w_{i,k} + p w_i)\,\mathrm{d}\Omega = 0 \tag{20.11}$$

Clearly, the differentiability requirement of the trial function N_i has reduced in Eq. (20.11), and hence, it referred to as the *weak form*. In this case it is a *symmetric weak form* as the continuity requirements for trial and test functions in Eq. (20.11) are the same.

Let us discretized the problem domain Ω into a set of non-overlapping elements Ω^e, i.e. $\Omega = \bigcup_e \Omega^e$. Using the Galerkin formulation, and assuming that the trial functions w_i vanish on the element boundaries, Eq. (20.11) yields the discrete algebraic system

$$\mathbf{Ku} = \mathbf{f} \tag{20.12}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{u}$ is the vector of nodal unknowns u_i and $\mathbf{f}$ is called the load vector. Elements of the matrices in the preceding equation are given by

$$K_{mn} = \sum_e K^e_{mn}, \quad \text{and} \quad f_m = \sum_e f^e_m \tag{20.13}$$

where

$$K_{mn}^{e} = \int_{\Omega_e} N_{m,k} N_{n,k} \mathrm{d}\Omega, \quad \text{and} \quad \mathrm{f}_{m}^{e} = -\int_{\Omega_e} N_m p \; \mathrm{d}\Omega. \tag{20.14}$$

Finite element solution of Eq. (20.9) is obtained by solving the linear system (20.12).

REFERENCES/FURTHER READING

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