Lecture 4

CONSERVATION LAWS AND MATHEMATICAL PRELIMINARIES

4.1 CONSERVATION LAWS

CFD is based on fundamental governing equations of fluid dynamics which are essentially mathematical models of conservation laws of physics. Assuming a fluid to be a continuum, these conservation laws are

1. Conservation of mass
2. Conservation of momentum (Newton's second law)
3. Conservation of energy (first law of thermodynamics)

These conservation laws are supplemented with constitutive relations (e.g. stress-strain rate relation, heat diffusion law, etc.) for a specific material.

In this lecture, we provide a brief overview mathematical notation and a few important theorems which are used to obtain mathematical statements of the conservation laws, in integral as well as differential forms. The integral forms provide the starting point for the finite volume method whereas the differential form of conservation equations is used by the finite difference and the finite element methods. This lecture closely follows the approach of Kundu and Cohen (2008) and Panton (2005) which should be consulted for further details and supplemental reading.

4.2 MATHEMATICAL NOTATIONS

Conservation laws of a continuum medium involve vector and tensor quantities. Following three different types of notations are usually employed in continuum mechanics:

- Dyadic or vector notation
- Expanded or component form
- Cartesian tensor notation

Dyadic or vector notation

Dyadic notation is usually preferred for clear enunciation of the underlying physical principles in a compact form. In this notation, form of governing equations is independent of the choice of coordinate axes. Hence, this is also known as *coordinate-free form*. We would use bold face letters to denote vectors or tensor quantities (e.g. velocity **v**), whereas simple italics symbols are used for scalars (e.g., temperature T, pressure p).

Expanded or component form

Component form of governing equations is dependent on the choice of coordinate axes. Algebraic manipulations are a lot simpler with an expanded form of conservation equations (say, in Cartesian coordinates).

Cartesian tensor notation

Cartesian tensor notation provides compactness of the dyadic notation and ease of algebraic manipulation of Cartesian coordinate representation. In this notation, $O_{x_1x_2x_3}$ represents the Cartesian reference frame O_{xyz}, and n subscripts are used for an nth order tensor. Thus,

- one subscript is used to denote a vector (e.g., v_i denotes the vector **v**);
- two subscripts are used to denote a second order tensor (e.g. τ_{ij} represents stress tensor $\boldsymbol{\tau}$).

The summation convention given below is widely used in Cartesian tensor notation.

Summation convention

A repeated index in a term implies summation over the range of that index. For example

$$a_ib_i \equiv \sum_i a_ib_i \qquad \text{(Dot product of two vectors } \mathbf{a} \text{ and } \mathbf{b}) \tag{4.1}$$

$$\frac{\partial v_i}{\partial x_i} \equiv \sum \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \qquad \text{(Divergence of vector } \mathbf{v}) \tag{4.2}$$

Kronecker delta

The Kronecker delta, δ_{ij}, is a second order isotropic tensor defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{4.3}$$

Substitution property of Kronecker delta:

$$\delta_{ij}u_j = u_i \tag{4.4}$$

Alternating tensor (permutation symbol)

The alternating tensor, ε_{ijk} is isotropic tensor of third order defined as

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231 \text{ or } 312 \text{ (cyclic order)} \\ 0 & \text{if any two indices are equal} \\ -1 & \text{if } ijk = 321, 213 \text{ or } 132 \text{ (anticyclic order)} \end{cases} \tag{4.5}$$

Products of Two Vectors a and b

- **Scalar or dot product**

  $$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = a_1b_1 + a_2b_2 + a_3b_3 = a_ib_i \tag{4.6}$$

- **Vector or cross product**

  $$\mathbf{c} = \mathbf{a}\times\mathbf{b} \Rightarrow c_i = \varepsilon_{ijk}a_jb_k \tag{4.7}$$

- **Tensor product**

  $$\mathbf{C} = \mathbf{ab} \Rightarrow C_{ij} = a_ib_j \tag{4.8}$$

Products of Two Second Order Tensors A and B

- **Simple tensor product**

  $$\mathbf{C} = \mathbf{AB} \Rightarrow C_{ijkl} = A_{ij}B_{kl} \tag{4.9}$$

- **Singly contracted product (dot product)**

  $$(\mathbf{A}\cdot\mathbf{B})_{ij} = A_{ik}B_{kj} \tag{4.10}$$

- **Doubly contracted product (scalar product)**
  $$c = \mathbf{A} : \mathbf{B} \Rightarrow c = A_{ij} B_{ji} \tag{4.11}$$

Products of a Second Order Tensors A and vector u

- **Simple tensor product**
  $$\mathbf{C} = \mathbf{A}\mathbf{u} \Rightarrow C_{ijk} = A_{ij} u_k \tag{4.12}$$
- **Singly contracted product (dot product)**
  $$(\mathbf{A} \cdot \mathbf{u})_i = A_{ij} u_j \tag{4.13}$$

Gradient operator (∇)

The gradient operator, ∇ ("del") is the vector operator defined as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial w} \equiv \mathbf{i}_i \frac{\partial}{\partial x_i} \tag{4.14}$$

When operated on a scalar function ϕ, it generates a vector whose i^{th} component is $\partial\phi / \partial x_i$.

Divergence operator ($\nabla.$)

The divergence of a vector field is defined as the scalar quantity given by

$$\nabla.\mathbf{v} \equiv \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \tag{4.15}$$

Divergence of a second order tensor $\boldsymbol{\tau}$ yields a vector whose i^{th} component is given by

$$\left(\nabla.\boldsymbol{\tau}\right)_i = \frac{\partial \tau_{ij}}{\partial x_j} \tag{4.16}$$

Thus, divergence operator decreases the order of a tensor by 1, whereas gradient operator increases the order of a tensor by 1.

4.3 GAUSS DIVERGENCE THEOREM

Let V be a volume bounded by a closed surface A. Let $F(x)$ be any scalar, vector or tensor field. Gauss theorem states that

$$\int_V \frac{\partial F}{\partial x_i} \,\mathrm{d}V = \int_A F \mathrm{d}A_i \tag{4.17}$$

If $\mathbf{F}$ is a vector, then Gauss theorem becomes

$$\int_V \frac{\partial F_i}{\partial x_i} \,\mathrm{d}V = \int_A \mathrm{d}A_i F_i \qquad \text{or} \qquad \int_V \nabla \cdot \mathbf{F} \mathrm{d}V = \int_A \mathbf{F} \cdot \mathrm{d}\mathbf{A} \tag{4.18}$$

which is popularly known as Gauss divergence theorem. Gauss divergence theorem is used to convert a surface integral to a volume integral (or vice-versa).

4.4 REYNOLDS TRANSPORT THEOREM (RTT)

Conservation laws are defined for a control mass system whereas a control volume based (Eulerian) description is usually preferred for a fluid medium. Reynolds transport theorem provides a relation between the time rates of change in two descriptions, and is used to obtain the integral form of the conservation laws for a fluid medium.

Let ϕ be an intensive property, then corresponding extensive property Φ for a given system (or control mass) can be expressed as

$$\Phi = \int_{\Omega_{CM}} \rho\phi \, d\Omega \tag{4.19}$$

where Ω_{CM} represents the volume of the system which occupies the control volume Ω at a given instant of time and ρ is mass density. Reynolds transport theorem states that *the time rate of change of Φ for the system is equal to the rate of change of ϕ in control volume plus net flux of ϕ through boundaries of the control volume*, i.e.

$$\left(\frac{d\Phi}{dt}\right)_{CM} = \frac{\partial}{\partial t}\int_{\Omega} \rho\phi d\Omega + \int_{S} \rho\phi\left(\mathbf{v} - \mathbf{v}_c\right)\cdot d\mathbf{A} \tag{4.20}$$

where $\mathbf{v}_c$ is velocity of the control volume with respect to the fixed inertial reference frame in which $\mathbf{v}$ is defined, and S denotes the boundary surface of the control volume. The second term on RHS is usually called the convective (or advective) term.

We would employ the preceding notations and theorems to derive the integral as well as the differential forms of the mass, momentum and energy equations in next few lectures.

REFERENCES

Kundu, P. K. and Cohen, I. M. (2008). *Fluid Mechanics*, 4th Ed., Academic Press.

Panton, R. L. (2005). *Incompressible Flow*, 3rd Ed., Wiley.