Lecture 7

NAVIER-STOKES EQUATIONS

7.1 NAVIER-STOKES EQUATIONS

Momentum (Cauchy) equation

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla.(\rho \mathbf{v}\mathbf{v}) = \nabla.\boldsymbol{\tau} + \rho \mathbf{b} \tag{6.10}$$

contains nine additional unknowns (the elements of the stress tensor $\boldsymbol{\tau}$). Hence, we need a relation between stress tensor $\boldsymbol{\tau}$ and velocity field $\mathbf{v}$ to reduce the momentum equation in terms of primary unknowns of the flow field. For a fluid, this relationship (which is called ***constitutive relation***) relates the stress tensor to the rate of strain tensor $\mathbf{S}$ by the following general functional form:

$$\boldsymbol{\tau} = f(\mathbf{S}, \dot{\mathbf{S}}, \ddot{\mathbf{S}}, \ldots.), \quad \mathbf{S} = \frac{1}{2}\left[(\nabla \mathbf{v}) + (\nabla \mathbf{v})^T\right] \tag{7.1}$$

For a general fluid, number of tensor parameters of the function *f*(..) would depend on the memory of the fluid. For example, for a second order fluid with memory (a visco-elastic fluid):

$$\boldsymbol{\tau} = -p\mathbf{I} + \alpha_1 \mathbf{S} + \alpha_2 \mathbf{S}^2 + \alpha_3 \dot{\mathbf{S}} \tag{7.2}$$

where p is thermodynamic pressure and α's are material properties dependent on thermodynamic state of the fluid.

Most of the common fluids (e.g. air, water, and gases) have little or no memory, and the stress-strain rate relationship is linear. Such fluids are called Newtonian fluids for which constitutive relationship becomes (Panton, 2005)

$$\boldsymbol{\tau} = -p\mathbf{I} + \lambda(\nabla \cdot \mathbf{v})\mathbf{I} + 2\mu \mathbf{S} \tag{7.3}$$

where λ and μ are fluid properties (called viscosities). For a wide class of Newtonian fluids, the bulk viscosity, $(3\lambda + 2\mu)$, is quite small, i.e. $(3\lambda + 2\mu) \approx 0$. This is called the *Stokes hypothesis*, and leads to the following constitutive relation for Newtonian fluids:

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\mu\left(\mathbf{S} - \frac{1}{3}(\nabla \cdot \mathbf{v})\mathbf{I}\right) \tag{7.4}$$

where μ is the (dynamic) viscosity of the fluid. Substitution of relation (7.4) into momentum equation (6.10) leads to the so-called Navier-Stokes equation in *conservation form* given by

$$.\ \frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla.(\rho \mathbf{v}\mathbf{v}) = \rho \mathbf{b} - \nabla p + 2\nabla.\left[\mu\left(\mathbf{S} - \frac{1}{3}(\nabla \cdot \mathbf{v})\mathbf{I}\right)\right] \tag{7.5}$$

Non-conservation form of Navier-Stokes equation can be obtained by substituting Eq. (7.4) into Eq. (6.12) and is given by

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right] = \rho \mathbf{b} - \nabla p + 2\nabla.\left[\mu\left(\mathbf{S} - \frac{1}{3}(\nabla \cdot \mathbf{v})\mathbf{I}\right)\right] \tag{7.6}$$

Cartesian form of Navier-Stokes equations (7.5):

$$\frac{\partial(\rho v_i)}{\partial t}+\frac{\partial(\rho v_j v_i)}{\partial x_j}=\rho b_i-\frac{\partial}{\partial x_i}\left(p+\frac{2}{3}\mu\frac{\partial v_j}{\partial x_j}\right)+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right] \qquad (7.7)$$

For incompressible flow, $\nabla\cdot\mathbf{v}=0$. Hence, Navier -Stokes equation reduces to

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla.(\mathbf{vv})=\mathbf{b}-\frac{1}{\rho}\nabla p+\frac{2}{\rho}\nabla.[\mu\mathbf{S}] \qquad (7.8)$$

In addition, if the temperature variations in an incompressible flow are small, the viscosity of the fluid μ can be assumed constant, and Eq. (7.8) simplifies to

$$\frac{\partial \mathbf{v}}{\partial t}+\nabla.(\mathbf{vv})=\mathbf{b}-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{v} \qquad (7.9)$$

where $\nu\ (=\mu/\rho)$ is the kinematic viscosity. In Cartesian coordinates, Eq. (7.9) takes the form

$$\frac{\partial v_i}{\partial t}+\frac{\partial(v_j v_i)}{\partial x_j}=b_i-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 v_i}{\partial x_j x_j} \qquad (7.10)$$

7.2 EULER EQUATION

For high speed flows (e.g. flow over an aircraft), effects of viscosity are usually very small away from solid boundaries. If effects of viscosity are neglected altogether, then $\boldsymbol{\tau}=-p\mathbf{I}$ and momentum equation reduces to Euler equation

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla.(\rho\mathbf{vv})=\rho\mathbf{b}-\nabla p \qquad (7.11)$$

Euler equation is often used to study compressible flow at high Mach numbers. Neglect of viscosity permits use of coarser grids near the solid surfaces. Thus, use of Euler equation allows the simulation of flow over the whole aircraft.

7.3 CREEPING FLOW: STOKES EQUATION

If the Reynolds number is very small (i.e. flow velocity is very small, the fluid is very viscous, or the geometric dimensions are very small), the nonlinear convective terms in the Navier-Stokes equation can be neglected. Such flows, which are dominated by pressure, body force and viscous forces, are called creeping flow. Because of low velocity, unsteady terms can also be neglected and the momentum equation takes the following form:

$$\mu\nabla^2\mathbf{v}+\rho\mathbf{b}-\nabla p=0 \qquad (7.12)$$

REFERENCES

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