# Lecture 38

# TURBULENT FLOWS: RANS TURBULENCE MODELS

## 38.1 TURBULENCE MODELS

Turbulence models used in RANS simulations can be broadly classified into two categories: (a) eddy viscosity models, and (b) Reynolds stress/flux models. The former models employ an eddy viscosity assumption based on Boussinesq proposition, whereas the latter employ transport equations for Reynolds stress tensor and turbulence flux. Computation of eddy viscosity may also require solution of a set of transport equations (depending on the model employed). Thus, RANS turbulence models are also categorized as follows based on number of additional transport equations (PDEs) which must be solved to enforce closure:

- Zero equation model (e.g. Prandtl's mixing length model)
- One equation model (e.g. Spalart-Allmaras model)
- Two equation model (e.g. $k$-$\varepsilon$ model, $k$-$\omega$ model, algebraic stress model)
- Seven equation model (Reynolds stress model)

Various version of the preceding model have been suggested in the literature to take care of specific physical problems. All the models involve empirical numerical constants which have been obtained by validation with experimental data. However, these constants are not universal in the sense that the suggested values may not yield correct results for all turbulent flows, and hence, care must be exercised in choice of model constants. Fine-tuning of these turbulence models with extensive experiments and DNS data is an active area of current research on turbulent flows. In this lecture, we provide a very brief outline of some of the most-widely used models in industrial CFD analysis. For details, refer to Pope (2000) and Versteeg and Malalasekera (2007).

## 38.2 EDDY VISCOSITY MODELS

**Boussinesq Proposition**

Experimental evidence suggests that turbulent stresses increase with mean rate of deformation. Further, in analogy with laminar flows in which viscous stresses are proportional to velocity gradients, Boussinesq proposed that Reynolds stresses are proportional to mean velocity gradients, i.e. deviatoric Reynolds stress is proportional to mean rate of strain. Thus,

$$\tau_{ij}^R \equiv -\rho\overline{v_i'v_j'} = \mu_T\left(\frac{\partial\overline{v}_i}{\partial x_j}+\frac{\partial\overline{v}_j}{\partial x_i}\right)-\frac{2}{3}\rho k\delta_{ij} \qquad (38.1)$$

where $\mu_T$ is the dynamic turbulent (or eddy) viscosity and $k=\left(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right)/2$ is the turbulent kinetic energy per unit mass. Similarly, turbulent flux of scalar $\phi$ can also be related to gradient of its mean value, i.e.

$$q_j^R \equiv -\rho\overline{v_j'\phi'} = \Gamma_T\frac{\partial\overline{\phi}}{\partial x_j} \qquad (38.2)$$

where $\Gamma_T$ is the turbulent or eddy diffusivity. Relations (38.1) and (38.2) are referred to as the eddy viscosity model and eddy diffusivity model respectively. Note that the turbulent viscosity and turbulent diffusivity are not material properties; these depend on the flow field, and would have their own spatial and temporal variation.

In its simplest form, turbulence can be characterized by two parameters: its kinetic energy, $k$ (or a velocity scale $q=\sqrt{k}$ ) and a length scale $L$. From dimensional analysis, turbulent (eddy) viscosity can be expressed in terms of these scales as

$$\mu_T = C_\mu \rho q L \tag{38.3}$$

where $C_\mu$ is a dimensionless constant. Most of the eddy viscosity turbulence models obtain estimates for these length scales and thereafter compute turbulent viscosity using the preceding relation. We outline a representative, namely $k$-$\varepsilon$ model, in this section. For estimation of turbulent diffusivity, most of the CFD procedures assume a constant value of unity for turbulent Schmidt number (Versteeg and Malalasekera, 2007) and thus, the numerical value of turbulent diffusivity is taken to be the same as that of the kinematic eddy viscosity $\nu_T (= \mu_T / \rho)$.

**The $k$-$\varepsilon$ model**

The standard $k$-$\varepsilon$ model proposed by Launder and Spalding (1974) makes use of two model equations, one for the turbulent kinetic energy $k$ and one for the rate of dissipation of turbulent kinetic energy per unit mass, $\varepsilon$. These equations are

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho k \overline{v}_j\right) = \frac{\partial}{\partial x_j}\left[\frac{\mu_T}{\sigma_k}\frac{\partial k}{\partial x_j}\right] + 2\mu_T S_{ij} S_{ij} - \rho\varepsilon \tag{38.4}$$

$$\frac{\partial(\rho \varepsilon)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho \varepsilon \overline{v}_j\right) = \frac{\partial}{\partial x_j}\left[\frac{\mu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right] + C_{1\varepsilon}\frac{\varepsilon}{k} 2\mu_T S_{ij} S_{ij} - 2C_{2\varepsilon}\rho\frac{\varepsilon^2}{k} \tag{38.5}$$

Transport equation (38.4) and (38.5) are solved iteratively along with Reynolds averaged Navier-Stokes equations. Using $k$ and $\varepsilon$, velocity and length scales are defined as follows:

$$q = \sqrt{k}, \qquad L = \frac{k^{3/2}}{\varepsilon} \tag{38.6}$$

The preceding velocity and length scales are used in Eq. (38.3) to determine the eddy viscosity $\mu_T$. The preceding equations contain five adjustable constants ($C_\mu, \sigma_k, \sigma_\varepsilon$, $C_{1\varepsilon}$ and $C_{2\varepsilon}$) which have been determined by data fitting for a wide range of turbulent flow. Values of these constants are

$$C_\mu = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\varepsilon = 1.3, \quad C_{1\varepsilon} = 1.44, \; C_{2\varepsilon} = 1.92 \tag{38.7}$$

The standard $k$-$\varepsilon$ model is well established and the most widely validated and used eddy viscosity turbulence model. Various modifications of the standard model have been suggested to account for the near-wall region and low Reynolds number turbulent flows. In addition, various other one and two equations models have also been suggested, each having its own advantages and disadvantages. For further details on these models, refer to Versteeg and Malalasekera (2007) or Pope (2000).

## 38.3 REYNOLDS STRESS MODELS (RSM)

Eddy viscosity models have significant deficiencies some of which are consequences of eddy viscosity assumption. Measurements and simulations indicate that in 3-D turbulent flow, eddy viscosity becomes a tensor quantity. Hence, use of a scalar eddy viscosity for computing Reynolds stresses is not really appropriate. It would instead be desirable to compute Reynolds stresses directly using their own dynamic (transport) equations. This idea forms the basis of Reynolds stress model. It is, however, the most expensive of the turbulence models in use for RANS simulations.

Let us define the (kinematic) Reynolds stress tensor $R_{ij} = -\tau^R_{ij} / \rho = \overline{v'_i v'_j}$. Transport equation for $R_{ij}$ can be derived from Navier-Stokes equations, and can be written in the following form:

$$\frac{\partial\left(\rho R_{ij}\right)}{\partial t} + \frac{\partial\left(\rho \overline{v}_k R_{ij}\right)}{\partial x_k} = P_{ij} + D_{ij} - \varepsilon_{ij} + \Pi_{ij} + E_{ij} \tag{38.8}$$

where $P_{ij}$ is the production term, $D_{ij}$ is the diffusion term, $\varepsilon_{ij}$ is the dissipation rate tensor, $\Pi_{ij}$ is the pressure-strain term, and $E_{ij}$ represents turbulent diffusion. These terms are given by

$$\begin{aligned}
&P_{ij} = -\left(\rho R_{ik} \frac{\partial \overline{v}_j}{\partial x_k} + \rho R_{jk} \frac{\partial \overline{v}_i}{\partial x_k}\right), \qquad D_{ij} = \frac{\partial}{\partial x_k}\left(\mu \frac{\partial R_{ij}}{\partial x_k}\right) \\
&\Pi_{ij} = \overline{p'\left(\frac{\partial v'_i}{\partial x_j} + \frac{\partial v'_j}{\partial x_i}\right)}, \qquad \varepsilon_{ij} = 2\mu \overline{\frac{\partial v'_i}{\partial x_k}\frac{\partial v'_j}{\partial x_k}} \\
&E_{ij} = \frac{\partial}{\partial x_k}\left(\rho \overline{v'_i v'_j v'_k} + \overline{p' v'_i}\delta_{jk} + \overline{p' v'_j}\delta_{ik}\right)
\end{aligned} \tag{38.9}$$

The first two terms on RHS of Eq. (38.8) are exact. However, the last three (i.e., the dissipation rate tensor, pressure-strain term and turbulent diffusion) cannot be computed exactly and hence, must be modelled. For further details on modelling of these terms, see Launder *et al*. (1975) and Versteeg and Malalasekera (2007).

In comparison to eddy diffusivity models, Reynolds stress models require fairly large computational cost (we need to solve seven additional PDEs in this case, in contrast to two additional PDEs required in $k$-$\varepsilon$ model). Thus, these models have not been as popular as the eddy diffusivity models in industrial CFD analyses.

## REFERENCES/FURTHER READING

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