Lecture 8

ENERGY AND SCALAR TRASPORT EQUATIONS

8.1 ENERGY EQUATION

Conservation of energy principle states that energy of a system is always conserved. The corresponding physical law is the first law of thermodynamics which states that

$$\begin{pmatrix}\text{Time rate of increase}\\ \text{of the total stored}\\ \text{energy in the system}\end{pmatrix} = \begin{pmatrix}\text{Net rate of}\\ \text{energy addition}\\ \text{by heat transfer}\end{pmatrix} + \begin{pmatrix}\text{Net rate of energy}\\ \text{addition by work}\\ \text{done on the system}\end{pmatrix},$$

i.e.

$$\frac{\mathrm{D}E}{\mathrm{D}t} = \frac{\delta Q}{\delta t} - \frac{\delta W}{\delta t} \tag{8.1}$$

where E denotes the total stored energy which consists of kinetic energy, potential energy and internal energy. The potential energy usually represents the work done by the external force field (i.e. body forces), and hence, this part can be accounted for by the second term on RHS of Eq. (8.1). Hence, we take the stored energy E as the sum of the internal energy and kinetic energy, and its specific measure (i.e. stored energy per unit mass) is given by

$$\eta = e + \frac{1}{2}|\mathbf{v}|^2 \tag{8.2}$$

where e represents internal energy per unit mass. From Reynolds transport theorem, the rate of increase of the stored energy is given by

$$\frac{\mathrm{D}E}{\mathrm{D}t} = \frac{\partial}{\partial t}\int_{\Omega} \rho\eta \ \mathrm{d}\Omega + \int_{S} \eta\rho\mathbf{v}\cdot \mathrm{d}\mathbf{A} \tag{8.3}$$

Energy addition by heat transfer can be due to volumetric heat generation and heat transfer across the surface of the control volume. Thus,

$$\frac{\delta Q}{\delta t} = \int_{\Omega} \dot{q}_g \mathrm{d}\Omega - \int_{S} \mathbf{q}\cdot \mathrm{d}\mathbf{A} \tag{8.4}$$

where $\dot{q}_g$ is the rate of volumetric heat generation and $\mathbf{q}$ is the surface heat flux.

Rate of work done is obtained by taking dot product of force with velocity vector. Note that surface force is given by dot product of stress tensor by area vector. Thus, rate of energy addition due to the work done on the system by body as well as surface forces is given by

$$-\frac{\delta W}{\delta t} = \int_{S} \mathbf{v}\cdot(\boldsymbol{\tau}\cdot \mathrm{d}\mathbf{A}) + \int_{\Omega} \mathbf{v}\cdot(\rho\mathbf{b}) \ \mathrm{d}\Omega \tag{8.5}$$

Substitution of Eqs. (8.3)-(8.5) into Eq. (8.1) yields the integral form of the total energy equation:

$$\frac{\partial}{\partial t}\int_{\Omega} \rho\eta \,\mathrm{d}\Omega + \int_{S} \eta\rho\mathbf{v}\cdot \mathrm{d}\mathbf{A} = \int_{\Omega} \dot{q}_g \mathrm{d}\Omega - \int_{S} \mathbf{q}\cdot \mathrm{d}\mathbf{A} + \int_{S} \mathbf{v}\cdot(\boldsymbol{\tau}\cdot \mathrm{d}\mathbf{A}) + \int_{\Omega} \mathbf{v}\cdot(\rho\mathbf{b})\,\mathrm{d}\Omega \tag{8.6}$$

Using Gauss divergence theorem, the preceding equation leads to the following differential equation for the total energy:

$$\int_{S} \rho\eta\mathbf{v}\cdot \mathrm{d}\mathbf{A} = \int_{\Omega} \nabla\cdot(\rho\eta\mathbf{v})\mathrm{d}\Omega,\ \int_{S} \mathbf{q}\cdot \mathrm{d}\mathbf{A} = \int_{\Omega} \nabla\cdot\mathbf{q}\,\mathrm{d}\Omega \ \text{ and } \int_{S} \mathbf{v}\cdot(\boldsymbol{\tau}\cdot \mathrm{d}\mathbf{A}) = \int_{\Omega} \nabla\cdot(\mathbf{v}\cdot\boldsymbol{\tau})\,\mathrm{d}\Omega \tag{8.7}$$

Substitution of Eq. (8.7) into Eq. (8.6) yields

$$\int_{\Omega}\left[\frac{\partial(\rho\eta)}{\partial t} + \nabla.\left(\rho\eta\mathbf{v}\right) - \dot{q}_g + \nabla.\mathbf{q} - \nabla.(\mathbf{v}\cdot\boldsymbol{\tau}) - \mathbf{v}\cdot(\rho\mathbf{b})\right]\mathrm{d}\Omega = 0 \tag{8.8}$$

Equation (8.8) holds for any control volume which is possible only if the integrand vanishes everywhere leading to the following differential equation for the total energy:

$$\frac{\partial(\rho\eta)}{\partial t} + \nabla.\left(\rho\eta\mathbf{v}\right) = \dot{q}_g - \nabla.\mathbf{q} + \nabla.(\mathbf{v}\cdot\boldsymbol{\tau}) + \mathbf{v}\cdot(\rho\mathbf{b}) \tag{8.9}$$

To obtain the equation of thermal energy, we subtract the equation for mechanical energy (which can be obtained by taking the dot product of velocity $\mathbf{v}$ with momentum equation (6.10)) from the preceding equation. After some algebraic manipulation, we obtain the equation for thermal energy given by

$$\frac{\partial(\rho e)}{\partial t} + \nabla.\left(\rho e\mathbf{v}\right) = \dot{q}_g - \nabla.\mathbf{q} + \nabla\mathbf{v}:\boldsymbol{\tau} \tag{8.10}$$

where the double dot product is defined as

$$\nabla\mathbf{v}:\boldsymbol{\tau} = \frac{\partial u_j}{\partial x_i}\tau_{ij} \tag{8.11}$$

The last term in the thermal energy equation consists of the work done by pressure and viscous dissipation. For the low speed flows (i.e. flow speeds are small as compared to the speed of sound), viscous dissipation part is negligible. Pressure related term has the form $-p\nabla.\mathbf{v}$ and can be combined with the left hand side of Eq. (8.10) to give the following equation:

$$\rho C_p \frac{\mathrm{D}T}{\mathrm{D}t} = \dot{q}_g - \nabla.\mathbf{q} \tag{8.12}$$

If the heat flux obeys the Fourier's law $\mathbf{q} = -k\nabla T$, and thermal conductivity k is assumed constant, Eq. (8.12) simplifies to

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{1}{\rho C_p}\dot{q}_g + \kappa\nabla^2 T \tag{8.13}$$

where $\kappa = k/\rho C_p$ is the thermal diffusivity. Equation (8.13) is commonly referred as the *advection-diffusion* equation for thermal energy.

8.2 GENERALIZED SCALAR TRANSPORT EQUATION

In analogy with the conservation laws for mass, momentum and energy, conservation principle for a generic scalar ϕ can be expressed as

$$\begin{pmatrix}\text{Time rate of increase}\\ \text{of the scalar } \phi\\ \text{in the system}\end{pmatrix} = \begin{pmatrix}\text{Net rate of}\\ \text{energy addition}\\ \text{by heat transfer}\end{pmatrix} + \begin{pmatrix}\text{Net rate of energy}\\ \text{addition by work}\\ \text{done on the system}\end{pmatrix} \tag{8.14}$$

From Reynolds transport theorem,

$$\begin{pmatrix}\text{Time rate of increase}\\ \text{of the scalar } \phi\\ \text{in the system}\end{pmatrix} = \begin{pmatrix}\text{Rate of increase}\\ \text{of } \phi \text{ inside the}\\ \text{control volume}\end{pmatrix} + \begin{pmatrix}\text{Net rate of}\\ \text{decrease } \phi \text{ due}\\ \text{to convection}\\ \text{across CV}\\ \text{boundary}\end{pmatrix} \tag{8.15}$$

Combination of Eq. (8.14) and (8.15) yields the following conservation law for a control volume:

$$\boxed{\begin{pmatrix}\text{Rate of increase}\\ \text{of } \phi \text{ inside the}\\ \text{control volume}\end{pmatrix} + \begin{pmatrix}\text{Net rate of}\\ \text{decrease } \phi \text{ due}\\ \text{to convection}\\ \text{across CV}\\ \text{boundary}\end{pmatrix} = \begin{pmatrix}\text{Net rate of}\\ \text{increase of } \phi\\ \text{due to diffusion}\\ \text{across the CV}\\ \text{boundary}\end{pmatrix} + \begin{pmatrix}\text{Net rate of}\\ \text{creation of}\\ \phi \text{ inside the}\\ \text{control}\\ \text{volume}\end{pmatrix}} \tag{8.16}$$

The preceding statement can be represented by the integral equation

$$\frac{\partial}{\partial t}\int_{\Omega} \rho\phi \, \mathrm{d}\Omega + \int_{S} \phi\rho\mathbf{v}\cdot \mathrm{d}\mathbf{A} = \int_{S} \mathbf{q}_{\phi}\cdot \mathrm{d}\mathbf{A} + \int_{\Omega} \dot{q}_{\phi}\mathrm{d}\Omega \tag{8.17}$$

where $\dot{q}_{\phi}$ represents the volumetric generation (or source) term and $\mathbf{q}_{\phi}$ denotes the surface flux associated with ϕ. Usually, the surface flux is related to ϕ by a gradient law (such as Fourier's law for heat conduction, Fick's law for mass diffusion) which can be expressed as $\mathbf{q}_{\phi} = \Gamma\nabla\phi$. Substituting this expression in Eq. (8.17) and use of Gauss divergence theorem leads to the following differential equation for transport of ϕ:

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla.\left(\rho\phi\mathbf{v}\right) = \dot{q}_{\phi} + \nabla\cdot(\Gamma\nabla\phi) \tag{8.18}$$

In Cartesian coordinates, Eq. (8.18) takes the form

$$\frac{\partial\left(\rho\phi\right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho\phi v_j\right) = \frac{\partial}{\partial x_j}\left(\Gamma\frac{\partial\phi}{\partial x_j}\right) + \dot{q}_{\phi} \tag{8.19}$$

Preceding equation represents the *conservation form* of the transport equation. Expanding terms on the left had side, and using the continuity equation, one can obtain the non-conservation form of the transport equation given by

$$\rho\frac{\mathrm{D}\phi}{\mathrm{D}t} \equiv \rho\left[\frac{\partial\phi}{\partial t} + (\mathbf{v}.\nabla)\phi\right] = \dot{q}_{\phi} + \nabla\cdot(\Gamma\nabla\phi) \tag{8.20}$$

It can be easily observed that Eq. (8.19) [or Eq. (8.20)] can be made to represent the mass, momentum or energy equation by proper choice of the flux and source terms. Thus, in this course on CFD, we will first focus our discussion on the approximation techniques for the generic transport equation (8.19), and then discuss their application and extension to the solution of Navier-Stokes equations.

Let us note that general form of all the conservation equations is similar: each equation contains a temporal derivative, a convective term, a diffusive term and a source term. This commonality is exploited in CFD while developing algorithms and computer programs to solve the fluid flow and scalar transport problems. Further, all of these equations are coupled nonlinear partial differential equations. Thus, analytical solution is very difficult (if not impossible), and approximate numerical solution using techniques of CFD is required for practical problems.

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