Lecture 22
SHAPE FUNCTIONS AND NUMERICAL INTEGRATION

22.1 SHAPE FUNCTIONS

Shape functions in finite element analysis depend on the dimensionality of the problem and type of elements used for discretization of the problem domain. These must also satisfy continuity requirements depending on the underlying PDE and form (strong or weak) of finite element formulation. Normally these are chosen as polynomial functions. In addition, the shape functions must satisfy/possess the following properties:

- *Partition of unity*:

$$\sum_{k=1}^{n} N_k = 1 \tag{22.1}$$

- *Kronecker-delta property*:

$$N_i(\mathbf{x}_k) = \delta_{ik} \tag{22.2}$$

where n is number of nodes in an element, and $\mathbf{x}_k$ represents the coordinate of k_{th} node.

22.2 SHAPE FUNCTIONS FOR ONE DIMENSIONAL ELEMENTS

Shape functions for a two-node (linear) one dimensional element are given by

$$N_1 = \frac{1}{2}(1-\xi), \qquad N_2 = \frac{1}{2}(1+\xi) \tag{22.3}$$

where ξ is the local coordinate defined as $\xi = 2(x - x_c)/l$ in which l is length of the element and x_c represents its mid-point. Here, nodes 1 and 2 correspond to $\xi = -1$ and $\xi = +1$ respectively. Similarly, shape functions for a quadratic (three-node) element are given by

$$N_1 = \frac{1}{2}\xi(\xi - 1), \;\; N_2 = (1-\xi^2) \qquad N_3 = \frac{1}{2}\xi(1+\xi) \tag{22.4}$$

where nodes 1, 2 and 3 correspond to $\xi = -1$, $\xi = 0$ and $\xi = +1$ respectively.

22.2 SHAPE FUNCTIONS FOR TWO DIMENSIONAL ELEMENTS

In two-dimensions, one can opt for rectangular or triangular elements. In each case, shape functions are expressed in corresponding natural coordinates. These shape functions can also be used to define the physical coordinates of any point as follows:

$$x(\xi,\eta) = \sum_{\alpha=1}^{n} N_\alpha(\xi,\eta) x_\alpha, \quad y(\xi,\eta) = \sum_{\alpha=1}^{n} N_\alpha(\xi,\eta) y_\alpha \tag{22.5}$$

where n denotes the number of nodes in the element. Elements of preceding type for which the shape functions can be used to represent the geometry as well as function approximation are called *iso-parametric* elements.

22.2.1 Rectangular Elements

Shape functions for rectangular elements are expressed in terms of natural coordinates ξ and η defined as

$$\xi = \frac{2(x - x_c)}{l_x}, \qquad \eta = \frac{2(y - y_c)}{l_y} \tag{22.6}$$

where (x_c , y_c) is the centroid of the element, and l_x, l_y represent its extents in x and y-direction. Two-dimensional bilinear rectangular elements are given by

$$N_\alpha = \frac{1}{4}(1 + \xi\xi_\alpha)(1 + \eta\eta_\alpha), \qquad \alpha = 1,..,4 \tag{22.7}$$

where ξ and η are natural coordinates, and coordinates of the four nodes in natural coordinates are (-1,-1), (1,-1), (1, 1) and (-1, 1). For further detailed discussion of different types of rectangular element employed in finite element analysis, see Zienkiewicz, Taylor and Zhu (2005).

22.2.1 Triangular Elements

Shape functions for triangular elements are expressed in terms of area coordinates ξ_1, ξ_2 and ξ_3 defined as

$$\xi_1 = \frac{a_1 + b_1 x + c_1 y}{2\Delta}, \qquad \xi_2 = \frac{a_2 + b_2 x + c_2 y}{2\Delta}, \qquad \xi_3 = 1 - (\xi_1 + \xi_2) \tag{22.8}$$

where Δ is the area of the triangle with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3), and coefficients a_k, b_k and c_k are given by

$$\begin{aligned} a_1 &= x_2 y_3 - x_3 y_2, \quad b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2 \\ a_2 &= x_3 y_1 - x_1 y_3, \quad b_2 = y_3 - y_1, \quad c_2 = x_1 - x_3 \end{aligned} \tag{22.9}$$

In terms of area coordinates, shape functions for a linear triangular element are defined as

$$N_1 = \xi_1, \qquad N_2 = \xi_2, \qquad N_3 = \xi_3 \tag{22.10}$$

Shape functions for a quadratic triangular element are given by

- Corner nodes (numbered as 1, 2, 3)

  $$N_1 = \xi_1(2\xi_1 - 1), \qquad N_2 = \xi_2(2\xi_2 - 1), \qquad N_3 = \xi_3(2\xi_3 - 1)$$

- Mid-side nodes (numbered as 4,5,6)

  $$N_4 = 4\xi_1\xi_2, \qquad N_5 = 4\xi_2\xi_3, \qquad N_6 = 4\xi_3\xi_1 \tag{22.11}$$

For further details of two-dimensional rectangular and triangular elements, see Zienkiewicz, Taylor and Zhu (2005). You can also find shape functions for three dimensional elements, for example, prism or tetrahedral elements in this reference.

22.2 EVALUATION OF INTEGRALS

Formation of discrete finite element system (20.12) requires evaluation of domain integrals given in Eq. (20.14). Except for the simplest of element geometries, these integrals cannot be evaluated analytically. Hence, numerical integration is the only alternative. For line elements (1-D), quadrilateral elements (in two dimensions) and brick elements (in three dimensions), Gaussian quadrature is most commonly employed (Zienkiewicz, Taylor and Zhu, 2005). For example, a typical integral for a two dimensional quadrilateral element can be evaluated as

$$
\begin{aligned}
I &= \int_{\Omega_e} G(\mathbf{x})\mathrm{d}\Omega = \int_{-1}^{1}\int_{-1}^{1} G(\mathbf{x}(\xi,\eta))\,|\,\mathbf{J}\,|\,\mathrm{d}\xi\mathrm{d}\eta \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{G}(\xi_i,\eta_j)w_i w_j, \qquad \bar{G}(\xi_i,\eta_j) \equiv G(\mathbf{x}(\xi,\eta))\,|\,\mathbf{J}\,|
\end{aligned}
\tag{22.4}
$$

where ξ and η are natural coordinates, $|\,\mathbf{J}\,|$ is the determinant of the Jacobian matrix **J** given by

$$
\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{22.5}
$$

ξ_i, η_j are Gaussian quadrature abscissa, and w_i and w_j are corresponding weights. Special quadrature formulae are available for triangle and tetrahedral elements. For further details and guidelines for evaluation of finite element integrals, see Zienkiewicz, Taylor and Zhu (2005).

REFERENCES/FURTHER READING

Chung, T. J. (2010). *Computational Fluid Dynamics*. 2nd Ed., Cambridge University Press, Cambridge, UK.

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