Lecture 11

FINITE DIFFERENCE METHOD: METHODOLOGY AND GRID NOTATION

11.1 FINITE DIFFERENCE METHOD

The finite difference method (FDM) is the oldest method for numerical solution of partial differential equations. Its first suggested application dates back to 18th century by Leonhard Euler for solution of initial value problems (two suggested methods are popularly known as forward Euler and backward Euler method). FDM has been very popular in CFD and its development, analysis and applications are well documented in many text books such as Anderson (1995), Ferziger and Peric (2003), Chung (2010).

Attractive features of FDM are:

- It is the easiest method to use for simple geometries, both in terms of formulation and programming.
- It can also be adapted for problems in complex geometries using boundary fitted grids or the concept of immersed boundaries.
- Domain decomposition based solvers can be easily adapted for solution of algebraic systems obtained from FDM, and thus, this method is uniquely suited for massively parallel architectures.

These features have made FDM very popular in CFD. In this and next few lectures, we will have a brief overview of the method and its applications to the generic transport problem, which will enable the reader to understand the method, and apply it for solution of heat transfer and fluid flow problems.

11.2 FDM: BASIC METHODOLOGY

Starting point for the finite difference method is the differential form of a conservation equation of a continuum problem. The continuum problem is discretized by using finite difference approximation of the derivatives in the governing differential equation(s) at a set of finite discrete points (which are called nodes or grid points). The simulation procedure based on finite difference method consists of the following steps:

- Discretize the solution domain by a grid (i.e. a set of discrete points)
- At each grid point, approximate the differential equation using finite difference approximation of derivatives, and thus convert it into an algebraic equation.
- At the boundary grid points, apply the boundary conditions. Any derivatives in boundary conditions are replaced by one sided finite difference approximation involving values at boundary and interior nodes.
- Collect the algebraic equations at all the grid points --- interior as well as boundary --- to obtain a system of algebraic equations in terms of unknown values of the variable at these nodes.
- Solve the resulting system of algebraic equation to obtain values of the variable at each grid point.

The solution obtained at the grid points can be interpolated and processed to obtain the desired physical quantities in the so-called post-processing step of the simulation.

11.3 FINITE DIFFERENCE GRID, NOTATION AND TERMINOLOGY

Finite difference formulation employs a structured grid in which each grid point (node) may be considered as the origin of a local coordinate system whose axes coincide with grid lines. Thus,

- Two grid lines of the same family (say, x_i) do not intersect ,
- Any pair of grid lines of two different families (say, x = constant, y = constant) intersect only once.

Figures 11.1 and 11.2 depict typical finite difference grids for one and two-dimensional problems respectively. Each grid line is identified by an integer index. The most commonly used indices are i, j and k for grid lines $x = x_i$, $y = y_j$ and $z = z_k$ respectively.

Each node is identified by a set of indices which are the indices of the grid lines that intersect at it. Thus,

- Node i represents the grid point $x = x_i$ in one dimension.
- Node (i,j) represents the intersection point of grid lines $x = x_i$ and $y = y_j$ in 2-D.
- Node (i,j,k) represents the intersection point of grid lines $x = x_i$, $y = y_j$ and $z = z_k$ in 3D.

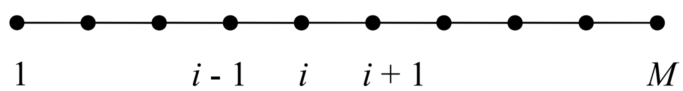

Figure 11.1 One dimensional finite difference grid

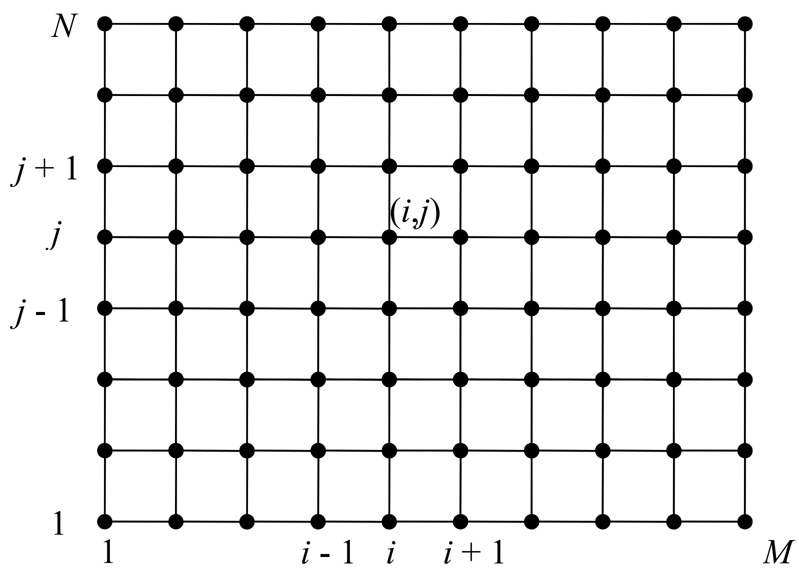

Figure 11.2 Two dimensional finite difference grid

A finite difference grid is called a uniform grid if the spacing between the grid lines of the same family is constant, i.e. $x_{i+1} - x_i = x_i - x_{i-1} = \Delta x$, for all i. Similarly, for multi-dimensional problems, $y_{j+1}-y_j=y_j-y_{j-1}=\Delta y,\ \forall j;\ \ z_{k+1}-z_k=z_k-z_{k-1}=\Delta z,\ \forall k$. The grid is otherwise called non-uniform. In the latter case, the following notation is used to denote the grid spacing in respective directions:

$$\Delta x_i = x_{i+1}-x_i,\ \Delta y_j = y_{j+1}-y_j,\ \Delta z_k = z_{k+1}-z_k. \tag{11.1}$$

In view of the index notation for the grid points, the following short notations are commonly used in finite difference equations for the function values at grid points:

- For one dimensional problems, $f_i \equiv f(x_i)$.
- For two-dimensional problems, $f_{i,j} \equiv f_{ij} \equiv f(x_i, y_j)$.
- For three-dimensional problems, $f_{i,j,k} \equiv f_{ijk} \equiv f(x_i, y_j, z_k)$.

Definition (Order of Magnitude)

A quantity *g(x)* is said to be *of order* h^m $(m>0)$ denoted by $g(x)\sim O(h^m)$ if $\lim\limits_{h\to 0}\dfrac{f(x)}{h^m}=L$ where L is a finite quantity (Niyogi et al., 2005).

11.4 FINITE DIFFERENCE APPROXIMATION

Basic idea of finite difference approximation comes from the definition of the derivative

$$\left(\frac{\partial f}{\partial x}\right)_{x_i} = \lim_{\Delta x\to 0}\frac{f(x_i+\Delta x)-f(x_i)}{\Delta x} \tag{11.2}$$

which clearly indicates that an approximate value of the derivative can be obtained from the finite difference expression given by

$$\left(\frac{\partial f}{\partial x}\right)_{x_i} \approx \frac{f(x_i+\Delta x)-f(x_i)}{\Delta x} \equiv \frac{f_{i+1}-f_i}{\Delta x} \tag{11.3}$$

The preceding equation is commonly referred to as the forward difference approximation. More refined finite difference approximations for the derivatives can be obtained using a number of formal approaches. The two most popular approaches are:

1. Taylor series expansion
2. Polynomial fitting

Pade approximants offer another method, and are used to obtain higher order finite difference approximations (Chung, 2010).

The finite difference approximation of a derivative at point is expressed in terms of function values at the neighbouring points and spacing between these points (i.e. the grid spacing). The error involved in the approximation is usually referred to as the truncation error.

Definition (Truncation Error)

Truncation error (TE) represents the sum of the terms in a Taylor series expansion which were deleted in obtaining the approximation of the derivative.

Thus, the truncation error represents a correction term which if added to the finite difference approximation can yield an exact value of the derivative at the point. On a uniform grid with spacing Δx, the truncation error ε_τ can be expressed as (Ferziger and Peric, 2003):

$$\varepsilon_\tau = (\Delta x)^m \alpha_{m+1} + (\Delta x)^{m+1} \alpha_{m+2} + (\Delta x)^{m+2} \alpha_{m+3} + ... \qquad (11.4)$$

where α's are higher order derivatives multiplied by constant factors. The leading power m of Δx determines the order of accuracy of a finite difference approximation.

Definition (Order of Accuracy)

If the truncation error of a finite difference approximation is $O\left(\Delta x^m\right)$, then it is said to have the accuracy of order m (or be m^{th} order accurate).

Conservation equations in fluid dynamics and heat transfer involve first and second order derivatives. We, therefore, focus on the finite difference approximation of these derivatives in the next two lectures.

REFERENCES

Anderson, J. D., Jr. (1995). *Computational Fluid Dynamics: The Basics with Applications*. McGraw Hill, New York.

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Ferziger, J. H. And Perić, M. (2003). *Computational Methods for Fluid Dynamics*. Springer.