Lecture 23
APPLICATIONS OF FINITE ELEMENT METHOD TO SCALAR TRANSPORT PROBLEMS

23.1 APPLICATION OF FEM TO 1-D DIFFUSION PROBLEM

Consider the steady state diffusion of a property ϕ in a one-dimensional domain defined in Figure 23.1. The process is governed by the differential equation

$$\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S=0 \tag{23.1}$$

where Γ is the diffusion coefficient and S is the source term. Boundary values of ϕ at end-points 0 and L are specified as

$$\phi(0)=\phi_A, \qquad \phi(L)=\phi_B, \tag{23.2}$$

Let us divide the domain in two node linear finite elements. In any element, the unknown function ϕ can be approximated as

$$\phi^e(x)=N_1(x)\phi_1^e+N_2(x)\phi_2^e \tag{23.3}$$

where N_1 and N_2 are linear shape functions, and ϕ_1^e, ϕ_2^e are values of ϕ at local nodes 1 and 2 of the element. *Strong form* of the weighted residual statement is given by

$$\int_0^L w_i\left[\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S\right]\mathrm{d}x=0 \tag{23.4}$$

Performing integration by parts, and requiring that the weight function vanishes at the end-points, we get the following *weak form*:

$$\int_0^L\left[\frac{dw_i}{dx}\left(\Gamma\frac{d\phi}{dx}\right)-w_iS\right]\mathrm{d}x=0 \tag{23.5}$$

For Galerkin formulation, $w_i=N_i$. Substitution of approximation (23.3) in Eq. (23.5) yields the discrete algebraic system

$$\mathbf{Ku}=\mathbf{f} \tag{23.6}$$

where **K** is the stiffness matrix, **u** is the vector of nodal unknowns ϕ_i and **f** is called the load vector. Elements of the matrices in the preceding equation are given by

$$K_{mn}=\sum_e K_{mn}^e, \quad \text{and} \quad f_m=\sum_e f_m^e \tag{23.7}$$

where

$$K_{mn}^e=\int_{\Omega_e} N_{m,k}N_{n,k}\mathrm{d}x, \quad \text{and} \quad \mathrm{f}_m^e=\int_{\Omega_e} N_m S\,\mathrm{d}x. \tag{23.8}$$

System of linear equations is solved using suitable linear solver (e.g. TDMA would be the best solver for the present example as **K** would be tri-diagonal for linear finite elements). Next example illustrates the preceding process explicitly for a sample heat conduction problem.

Example 23.1
Consider the steady state heat conduction in a slab of width l = 0.5 m with heat generation. The left end of the slab (x = 0) is maintained at T = 373 K. The right end of the slab (x = 0.5 m) is being heated by a heater for which the heat flux is 1 kW/m^2. The heat generation in the slab is temperature dependent and is given by Q = (1273 – T) W/m^3. Thermal conductivity is constant at k = 1 W/(m-K).Write down the governing equation and boundary conditions for the problem. Use the finite difference method (central difference scheme) to obtain an approximate numerical solution of the problem. For the first order derivative, use forward or backward difference approximation of first order. Choose element size h = 0.1, and **use the TDMA.** (We have chosen the same as the one solved previously in Example 15.1 using FDM to illustrate comparison between FEM, FVM and FDM.)

Solution
Let us recall that governing equation for the steady state heat conduction with constant heat generation is the slab is

$$k\frac{\mathrm{d}^2T}{\mathrm{d}x^2}+Q=0 \qquad \text{(i)}$$

Given: Q = 1273 – T. Thus, Eq. (i) becomes

$$k\frac{\mathrm{d}^2T}{\mathrm{d}x^2}+(f-T)=0 \qquad \text{(ii)}$$

where f = 1273. Left end of the slab is maintained at constant temperature; hence boundary condition at this end is given by

$$T(0)=373 \qquad \text{(iii)}$$

At the right end, heat influx is specified. Thus, boundary condition at this end is

$$k\frac{\mathrm{d}T}{\mathrm{d}x}(L)=1000 \qquad \text{(iv)}$$

For discretization, let us use a finite element mesh of linear elements. Global nodes are 1 (x=0), 2 (x=0.1), 3(x=0.2) , 4(x=0.3), 5(x=0.4) and 6 (x=0.5). Each finite element is of width h = 0.1, whereas finite volumes around boundary nodes 1 and 6 are of width 0.05.

Galerkin weighted residual statement (23.4) for this problem takes the form

$$\int_0^L w_i\left[k\frac{d^2T}{dx^2}+(f-T)\right]\mathrm{d}x=0 \qquad \text{(v)}$$

Integration by parts yields the following weak form:

$$\left[k\frac{\mathrm{d}T}{\mathrm{d}x}w_i\right]_0^L-\int_0^L\left[k\frac{\mathrm{d}w_i}{\mathrm{d}x}\frac{\mathrm{d}T}{\mathrm{d}x}-w_i(f-T)\right]\mathrm{d}x=0 \qquad \text{(vi)}$$

The first term in the preceding equation would be non-zero only for the elements at the boundary of the domain. Now, let us use linear shape functions for interpolation in each element, i.e.

$$T^e(x)=N_1(x)T_1^e+N_2(x)T_2^e \qquad \text{(vii)}$$

For an element of length h with node 1 (located at x_1) and node 2 (at x_2), linear shape functions are

$$N_1(x)=\frac{x_2-x}{h}, N_2(x)=\frac{x-x_1}{h} \Rightarrow \frac{\mathrm{d}N_1}{\mathrm{d}x}=-\frac{1}{h}, \quad \frac{\mathrm{d}N_2}{\mathrm{d}x}=\frac{1}{h} \qquad \text{(viii)}$$

Inserting the approximation (vii), weighted residual formulation for an element can be written as

$$\int_{\Omega^e}\left[k\frac{\mathrm{d}w_i}{\mathrm{d}x}\left(\frac{\mathrm{d}N_1}{\mathrm{d}x}T_1^e+\frac{\mathrm{d}N_2}{\mathrm{d}x}T_2^e\right)+w_i(N_1(x)T_1^e+N_2(x)T_2^e)\right]\mathrm{d}x=\left[k\frac{\mathrm{d}T}{\mathrm{d}x}w_i\right]_0^L+\int_{\Omega^e} f w_i \mathrm{d}x \qquad \text{(ix)}$$

For Galerkin formulation, weight function w_i will be taken as one of the shape functions, N_i. Hence, the preceding equation can be written as in matrix form as

$$\begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{e1}^e & K_{22}^e \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix}=\begin{Bmatrix} b_1^e \\ b_2^e \end{Bmatrix} \qquad \text{(x)}$$

where

$$K_{mn}^e=\int_{\Omega_e}\left(\frac{\mathrm{d}N_m}{\mathrm{d}x}\frac{\mathrm{d}N_n}{\mathrm{d}x}+N_mN_n\right)\mathrm{d}x \quad \text{and} \quad b_m^e=\int_{\Omega_e} N_m f\ \mathrm{d}x, \qquad \text{(xi)}$$

with an additional contribution to **b** for the first and last element coming from specified flux. Thus,

$$K_{ii}^e=\int_{\Omega_e}\left(\frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}N_i}{\mathrm{d}x}+N_iN_i\right)\mathrm{d}x=\frac{1}{h}+\frac{h}{3}, \qquad i=1,2$$

$$K_{ij}^e=\int_{\Omega_e}\left(\frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}N_j}{\mathrm{d}x}+N_iN_j\right)\mathrm{d}x=-\frac{1}{h}+\frac{h}{6}, \qquad i,j=1,2$$

$$b_1^e=b_2^e=\frac{h}{2}f \qquad \text{(for interior elements)} \qquad \text{(xii)}$$

$$b_1^1=\frac{h}{2}f-k\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=0}, b_2^1=\frac{h}{2}f,$$

$$b_1^N=\frac{h}{2}f, \quad b_2^N=\frac{h}{2}f+k\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=L}$$

Assembling all elemental equations, we get

$$\mathbf{KT=b}, \text{ where } K_{mn}=\sum_e K_{mn}^e, \quad \text{and} \quad f_m=\sum_e f_m^e \qquad \text{(xiii)}$$

$$\begin{bmatrix}
\frac{1}{h}+\frac{h}{3} & -\frac{1}{h}+\frac{h}{6} & 0 & 0 & 0 & 0 \\
-\frac{1}{h}+\frac{h}{6} & 2\left(\frac{1}{h}+\frac{h}{3}\right) & -\frac{1}{h}+\frac{h}{6} & 0 & 0 & 0 \\
0 & -\frac{1}{h}+\frac{h}{6} & 2\left(\frac{1}{h}+\frac{h}{3}\right) & -\frac{1}{h}+\frac{h}{6} & 0 & 0 \\
0 & 0 & 0 & 2\left(\frac{1}{h}+\frac{h}{3}\right) & -\frac{1}{h}+\frac{h}{6} & 0 \\
0 & 0 & 0 & -\frac{1}{h}+\frac{h}{6} & 2\left(\frac{1}{h}+\frac{h}{3}\right) & -\frac{1}{h}+\frac{h}{6} \\
0 & 0 & 0 & 0 & -\frac{1}{h}+\frac{h}{6} & \frac{1}{h}+\frac{h}{3}
\end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}=\begin{Bmatrix} \frac{h}{2}f-k\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=0} \\ hf \\ hf \\ hf \\ hf \\ \frac{h}{2}f+k\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x=L} \end{Bmatrix} \qquad \text{(xiv)}$$

Let us note that temperature is specified at x = 0 (flux kdT/dx is unknown). Hence, we modify the first equation in Eq. (xiv) by incorporating known temperature value. Further, substitute $h = 0.1$, $f = 1273$, value of flux at right-end of the domain, and divide last five equations by 10 (to make all the diagonal elements to be of same order of magnitude). The resulting discrete system is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -a & d & -a & 0 & 0 & 0 \\ 0 & -a & d & -a & 0 & 0 \\ 0 & 0 & 0 & d & -a & 0 \\ 0 & 0 & 0 & -a & d & -a \\ 0 & 0 & 0 & 0 & -a & d/2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 373 \\ 12.73 \\ 12.73 \\ 12.73 \\ 12.73 \\ 106.365 \end{Bmatrix} \qquad \text{(xv)}$$

where $a = 0.9983333$, $d = 2.0066667$.

The preceding system is very similar to the discrete system (x) obtained using finite difference discretization in Example 15.1 (only coefficients in last row differ). Numerical calculations using TDMA are given in the following table:

i	A_W^i	A_P^i	A_E^i	b_i	$A_P^i \leftarrow A_P^i - \frac{A_W^i A_E^{i-1}}{A_P^{i-1}}$	$b_i^* = b_i - \frac{A_W^i b_{i-1}^*}{A_P^{i-1}}$	$T_i = \frac{b_i^* - A_E^i T_{i+1}}{A_P^i}$	T_{ex}
1	0	1.00	0	373	1	373	373.000	373.00
2	-*a*	*d*	-*a*	12.73	2.006666667	385.1083333	499.006	498.95
3	-*a*	*d*	-*a*	12.73	1.509987542	204.3245944	617.259	617.18
4	-*a*	*d*	-*a*	12.73	1.346615231	147.8198917	728.944	728.85
5	-*a*	*d*	-*a*	12.73	1.266537325	122.3184866	835.179	835.08
6	-*a*	*d*/2	0	100.36	0.216408681	202.7811261	937.029	936.92

Comparison of results obtained using FEM, FVM and FDM using the same grid size is given below:

Exact	FEM	FVM	FDM	%Error FEM	%Error FVM	%Error FDM
373.000	373.000	373.000	373.000	0.000	0.000	0.000
498.950	499.006	498.931	497.289	0.011	0.004	0.333
617.180	617.259	617.121	613.821	0.013	0.009	0.544
728.850	728.944	728.753	723.762	0.013	0.013	0.698
835.080	835.179	834.942	828.209	0.012	0.017	0.823
936.920	937.029	936.751	928.209	0.012	0.018	0.930

We can clearly observe that the finite element and finite volume results using identical grid spacing are more accurate than those obtained using FDM. The primary reason is use of first order backward difference method used in finite difference solution for incorporation of flux boundary condition.

REFERENCES/FURTHER READING

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