Lecture 29

TIME INTEGRATION: APPLICATION TO UNSTEADY TRANSPORT PROBLEMS

29.1 INTRODUCTION

The generic conservation equation for transport of scalar ϕ

$$\frac{\partial(\rho\phi)}{\partial t}+\nabla.(\rho\phi\mathbf{v})=\nabla.(\Gamma\nabla\phi)+\dot{q}_\phi$$

can be re-written in a form which resembles the ordinary differential equation (27.1), i.e.

$$\frac{\partial(\rho\phi)}{\partial t}=-\nabla.(\rho\phi\mathbf{v})+\nabla.(\Gamma\nabla\phi)+\dot{q}_\phi\equiv f\left(t,\phi(t)\right) \tag{29.1}$$

Now any method of time integration discussed earlier can be used by taking due care for evaluation of terms in RHS at appropriate time. For an explicit method, RHS is evaluated only at times for which solution is already known. With an implicit method, the discretized RHS involves values at the new time level resulting in a system of algebraic equations which must be solved to obtain the solution at the new time level. The choice of an explicit or implicit method depends on the objectives of the numerical simulation and nature of the problem (which dictates the stability requirements).

- If obtaining steady state solution is the primary objective, the implicit methods, which allow large Δt, are preferred.
- If accurate time history is required, then choice of Δt is dictated by accuracy requirements and it may be small enough to meet the stability condition for explicit method. In this case, it may be preferable to use an explicit method which would be computationally more efficient than an implicit scheme.

Thus, explicit methods of Adams-Bashforth or Runge-Kutta family are preferred in LES and DNS of turbulent flows, whereas implicit Euler method is frequently used for obtaining steady state solution.

29.2 APPLICATION OF EXPLICIT METHOD

In the explicit Euler method, all fluxes and source terms are evaluated using known values at t_n. The only unknown at new time is the value at a grid point, which is explicitly computed. Let us consider 1-D unsteady advection, diffusion problem with constant velocity, constant fluid properties and no source terms. The governing equation (29.1) has the following form:

$$\frac{\partial\phi}{\partial t}=-u\frac{\partial\phi}{\partial x}+\frac{\Gamma}{\rho}\frac{\partial^2\phi}{\partial x^2} \tag{29.2}$$

Application of explicit Euler method yields

$$\phi^{n+1} = \phi^n + \Delta t\left[-u\frac{\partial \phi}{\partial x} + \frac{\Gamma}{\rho}\frac{\partial^2 \phi}{\partial x^2}\right]^n \tag{29.3}$$

Let us employ a uniform spatial grid of size Δx and use CDS for approximation of spatial derivatives. Then, the discrete equation for value of ϕ at node *i* is

$$\phi_i^{n+1} = \phi_i^n + \Delta t\left[-u\frac{\phi_{i+1}^n - \phi_{i-1}^n}{2\Delta x} + \frac{\Gamma}{\rho}\frac{\phi_{i+1}^n + \phi_{i-1}^n - 2\phi_i^n}{\left(\Delta x\right)^2}\right] \tag{29.4}$$

Define non-dimensional parameters *c* and *d* as

$$c = \frac{u\Delta t}{\Delta x}; \quad d = \frac{\Gamma \Delta t}{\rho\left(\Delta x\right)^2} \tag{29.5}$$

c is the ratio of time step to convection time, ($u/\Delta x$), which represents the time required by a disturbance to be convected a distance Δx. This ratio is called *Courant number*. Parameter *d* is the ratio of time step to characteristic diffusion time (i.e. the time required for transmission of a disturbance by diffusion). In terms of these parameters, Eq. (29.4) can be re-written as

$$\phi_i^{n+1} = \left(1-2d\right)\phi_i^n + \left(d - \frac{c}{2}\right)\phi_{i+1}^n + \left(d + \frac{c}{2}\right)\phi_{i-1}^n \tag{29.6}$$

Stability requires that coefficients of old nodal values must be positive, i.e.

$$\left(1-2d\right) > 0 \quad \Rightarrow \quad d < \frac{1}{2} \quad \text{i.e.} \quad \Delta t < \frac{\rho\left(\Delta x\right)^2}{2\Gamma} \tag{29.7}$$

$$d - c/2 > 0 \Rightarrow d > c/2 \Rightarrow \frac{\rho u \Delta x}{\Gamma} < 2 \quad \text{or} \quad \mathrm{Pe}_{\mathrm{cell}} < 2 \tag{29.8}$$

Stability condition (29.7) requires that Δt must be reduced by a factor of 4 if spatial mesh is refined by a factor of 2. This feature makes this method unsuitable for problems which do not require high temporal resolution. Method (29.6) being a combination of forward Euler and CDS is first order accurate in time and second order in space. Note that in absence of diffusion (d=0), condition (29.7) can never be satisfied, which makes this method *unconditionally unstable* for pure convective problems.

FURTHER READING

Chung, T. J. (2010). *Computational Fluid Dynamics*. 2nd Ed., Cambridge University Press, Cambridge, UK.

Ferziger, J. H. And Perić, M. (2003). *Computational Methods for Fluid Dynamics*. Springer.

Wood, W. L. (1990). *Practical Time-stepping Schemes*. Clarendon Press, Oxford.