Lecture 27

TIME INTEGRATION: TWO LEVEL AND MULTI-LEVEL METHODS FOR FIRST ORDER IVPS

27.1 INTRODUCTION

For time-dependent problems, apart from the spatial domain, the temporal domain must also be discretized. Advantage can be taken of the special nature of time coordinate for which direction of influence is only in future. Thus, all solution methods for an unsteady problem advance in time in a step by-step or "marching" manner starting from given initial data.

Time marching schemes for fluid flow or heat transfer problems can be obtained either by adapting methods for initial value problems (IVP) or by directly applying finite difference or finite element method in time domain. In this lecture, we take a brief look at two and multilevel methods for IVPs.

27.2 TWO LEVEL METHODS FOR FIRST ORDER IVPS

Let us consider the first order ordinary differential equation for the dependent variable ϕ given by

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = f\left(t,\phi(t)\right), \tag{27.1}$$

with an initial condition $\phi(t_0)=\phi^0$. In this lecture, we use a superscript to denote the value of the variable at given time instant, i.e. $\phi^n \equiv \phi(t_n)$.

If we can find the solution ϕ^1 at $t_1 = t_0 + \Delta t$, then ϕ^1 can be regarded as the new initial condition to obtained solution at $t_2 = t_1 + \Delta t$ and so on. The simplest way to construct a marching algorithm would be to integrate (27.1) from t_n to $t_{n+1} = t_n + \Delta t$

$$\int_{t_n}^{t_{n+1}} \frac{\mathrm{d}\phi}{\mathrm{d}t} dt = \phi^{n+1} - \phi^n = \int_{t_n}^{t_{n+1}} f\left(t,\phi(t)\right)\,\mathrm{d}t \tag{27.2}$$

Integral on RHS in Eq. (27.2) involves the unknown variable, and hence, it cannot be evaluated exactly and must be approximated. If it is evaluated using the value of integrand at the initial point, we get

$$\phi^{n+1} = \phi^n + \Delta t f\left(t_n,\phi_n\right) \tag{27.3}$$

The preceding formula is called the *explicit or forward Euler method*, since it could have been obtained using forward difference approximation of the temporal derivative, i. e.

$$\frac{d\phi}{dt} = \frac{\phi^{n+1} - \phi^n}{\Delta t} = f\left(t_n, \phi^n\right) \tag{27.4}$$

If we instead use value of the function at the final point t_{n+1}, we obtain the *backward Euler or implicit Euler method* given by

$$\phi^{n+1} = \phi^n + \Delta t\, f\left(t_{n+1}, \phi^{n+1}\right) \tag{27.5}$$

Similarly, using the value of the function at mid-point of the internal, we get the mid-point rule

$$\phi^{n+1} = \phi^n + \Delta t\, f\left(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}}\right) \tag{27.6}$$

Further, straight line interpolation between t_n and t_{n+1} leads to the trapezoid rule or the Crank-Nicolson method given by

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2}\left[f\left(t_n, \phi^n\right) + f\left(t_{n+1}, \phi^{n+1}\right)\right] \tag{27.7}$$

A generalized scheme, called θ-method, can be obtained using a weighted average value for approximation of the integral in (27.2) given by:

$$\phi^{n+1} = \phi^n + \Delta t\left[\theta f\left(t_{n+1}, \phi^{n+1}\right) + \left(1-\theta\right) f\left(t_n, \phi^n\right)\right] \tag{27.8}$$

Note that $\theta = 0, 1/2$ and 1 correspond to the forward Euler, Crank-Nicolson and the backward Euler method respectively.

Accuracy

The preceding two level methods are first order accurate except for the Crank-Nicolson and the mid-point rule which are second order accurate.

Stability

The forward Euler method is conditionally stable. All the implicit methods defined above are unconditionally stable if $\theta \geq 1/2$. Moreover, the implicit Euler method tends to produce smooth solutions even with large Δt. The trapezoidal rule, on the other hand, frequently yields oscillatory solution. Consequently, backward Euler method is preferred for nonlinear problems over the Crank- Nicolson method.

Computational Aspects

Explicit methods are easy to program, use little memory and computational time per step; but are unstable for large Δt. Implicit methods are much more stable, but require iterative solution (at least, solution of a linear system for a linear problem) at each time step.

27.3 MULTI-POINT METHODS

Multi-point methods involve function values at more than two time instants. Most-popular multi-point methods are the Adams method which are derived by fitting a polynomial to the derivative i.e. $f(\phi, t)$ at a number of points in time.

Adams –Bashforth Methods

Adams–Bashforth methods are explicit methods of order (m+1) which are obtained by fitting a Lagrange polynomial to $f(t, \phi(t))$ using values at points $t_{n-m}, t_{n-m+1}, \ldots\ldots\ldots t_n$.

- The *first order method* is the explicit Euler method.
- *Second order method* (m=1) is

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2}\left[3 f\left(t_n, \phi^n\right) - f\left(t_{n-1}, \phi^{n-1}\right)\right] \tag{27.9}$$

- *Third order Adams-Bashforth method* (m=2) is

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{12}\left[23 f\left(t_n, \phi^n\right) - 16 f\left(t_{n-1}, \phi^{n-1}\right) + 5 f\left(t_{n-2}, \phi^{n-2}\right)\right] \tag{27.10}$$

Adams-Moulton Methods

If we include the data at t_{n+1} in interpolation polynomial, we obtain implicit methods called Adams-Moulton method. The first order method is the implicit Euler method, second order method is the Crank-Nicolson method and third order method is

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2}\left[5 f^{n+1} + 8 f^n - f^{n-1}\right] \tag{27.11}$$

Advantages of Adams Methods

- It is easy to construct and program method of any order.
- Only one evaluation of $f(t, \phi)$ is required per time step.

Disadvantage

These methods require initial data at many points. Hence, these are not self-starting. Thus, at the first time step, we have to use a lower order Adams method or a Runge-Kutta method.

REFERENCES/FURTHER READING

Chung, T. J. (2010). *Computational Fluid Dynamics*. 2nd Ed., Cambridge University Press, Cambridge, UK.

Ferziger, J. H. And Perić, M. (2003). *Computational Methods for Fluid Dynamics*. Springer.

Wood, W. L. (1990). *Practical Time-stepping Schemes*. Clarendon Press, Oxford.