Lecture 19
APPLICATIONS OF FINITE VOLUME METHOD TO SCALAR TRANSPORT PROBLEM

19.1 APPLICATION OF FVM TO 1-D DIFFUSION PROBLEM

Consider the steady state diffusion of a property ϕ in a one-dimensional domain defined in Figure 19.1. The process is governed by the differential equation

$$\frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) + S = 0 \tag{19.1}$$

where Γ is the diffusion coefficient and S is the source term. Boundary values of ϕ at points A and B are given.

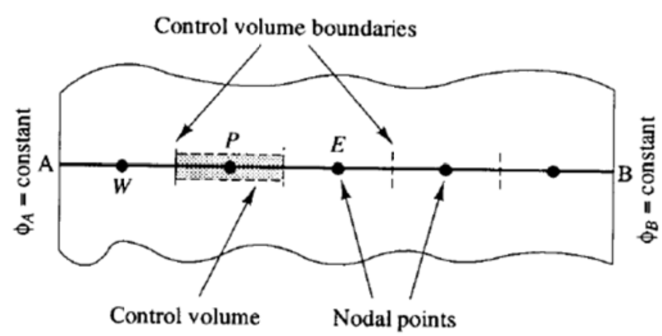


Figure 19.1 Schematic of the 1-D diffusion problem.

19.2 GRID GENERATION

The first step in the finite volume method is to divide the domain into discrete control volumes. The schematic of grid generation is shown in Fig. 19.1. The usual notation for the CV is shown in Fig. 19.2. A general nodal point is identified by P and its neighbors in a one-dimensional geometry, the nodes to the west and east, are identified by W and E respectively.

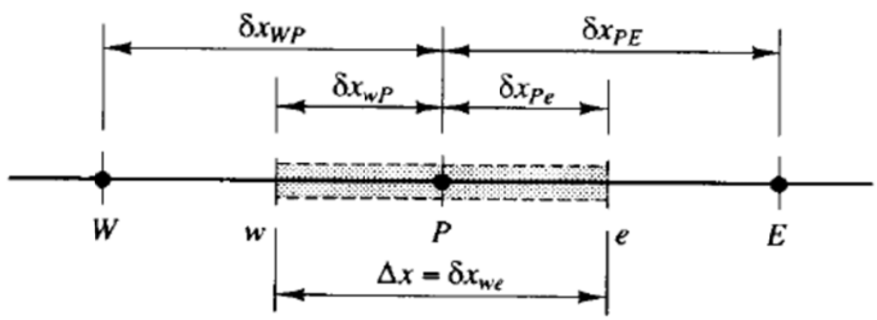


Figure 19.2 Notations used for the 1-D control volume

19.3 DISCRETIZATION

The key step of the FVM is the integration of the governing equation(s) over a control volume to yield a discretized equation at its nodal point P. For the control volume defined above this gives

$$\int_{\Delta V} \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) dV + \int_{\Delta V} S dV = \left(\Gamma A \frac{d\phi}{dx}\right)_e - \left(\Gamma A \frac{d\phi}{dx}\right)_w + \bar{S} dV = 0 \tag{19.2}$$

Here A is the cross-sectional area of the control volume face, dV is the volume and $\bar{S}$ is the average value of source S over the control volume.

In order to derive useful forms of the discretized equations, the interface diffusion coefficient Γ and the gradient $d\phi/dx$ at east face (e) and west face (w) are required. To calculate gradients (and hence fluxes) at the control volume faces, an approximate distribution of properties between nodal points is used. Using simple averaging, one can write

$$\begin{aligned} \Gamma_w &= \frac{\Gamma_W + \Gamma_P}{2} \\ \Gamma_e &= \frac{\Gamma_P + \Gamma_E}{2} \end{aligned} \tag{19.3}$$

Using central difference method, the diffusive fluxes can be written as

$$\begin{aligned} \left(\Gamma A \frac{d\phi}{dx}\right)_e &= \Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}}\right) \\ \left(\Gamma A \frac{d\phi}{dx}\right)_w &= \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}}\right) \end{aligned} \tag{19.4}$$

The source term S may also depend on the dependent variable. In such cases the finite volume method approximates the source term by means of a linear form expressed as

$$\bar{S} dV = S_u + S_P \phi_P \tag{19.5}$$

where S_u and S_P are independent of ϕ. Use of the preceding approximations leads to

$$\Gamma_e A_e \left(\frac{\phi_E - \phi_P}{\delta x_{PE}}\right) - \Gamma_w A_w \left(\frac{\phi_P - \phi_W}{\delta x_{WP}}\right) + \left(S_u + S_P \phi_P\right) = 0$$

Rearrangement of the preceding equation yields

$$\left(\frac{\Gamma_e}{\delta x_{PE}} A_e + \frac{\Gamma_w}{\delta x_{WP}} A_w - S_P\right)\phi_P = \left(\frac{\Gamma_w}{\delta x_{WP}} A_w\right)\phi_W + \left(\frac{\Gamma_e}{\delta x_{PE}} A_e\right)\phi_E + S_u \tag{19.6}$$

The above equation can be written in a short form as

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u \tag{19.7}$$

where

a_W	a_E	a_P
$\dfrac{\Gamma_W}{\delta x_{WP}} A_W$	$\dfrac{\Gamma_e}{\delta x_{PE}} A_e$	$a_W + a_P - S_P$

19.4 SOLUTION OF EQUATIONS

Discretized equations for each of the control volume are formulated and suitable modifications are incorporated for the boundary control volumes. The resulting set of linear equations is solved using suitable linear solver (e.g. TDMA would be the best solver for the present example).

Example 19.1

Consider the steady state heat conduction in a slab of width l = 0.5 m with heat generation. The left end of the slab (x = 0) is maintained at T = 373 K. The right end of the slab (x = 0.5 m) is being heated by a heater for which the heat flux is 1 kW/m^2. The heat generation in the slab is temperature dependent and is given by Q = (1273 – T) W/m^3. Thermal conductivity is constant at k = 1 W/(m-K).Write down the governing equation and boundary conditions for the problem. Use the finite difference method (central difference scheme) to obtain an approximate numerical solution of the problem. For the first order derivative, use forward or backward difference approximation of first order. Choose Δx = 0.1, and **use the TDMA.** (We have chosen the same as the one solved previously in Example 15.1 using FDM to illustrate comparison between FVM and FDM.)

Solution

Let us recall that governing equation for the steady state heat conduction with constant heat generation is the slab is

$$k\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + Q = 0 \qquad \text{(i)}$$

Given: Q = 1273 – T. Thus, Eq. (i) becomes

$$k\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + (1273 - T) = 0 \qquad \text{(ii)}$$

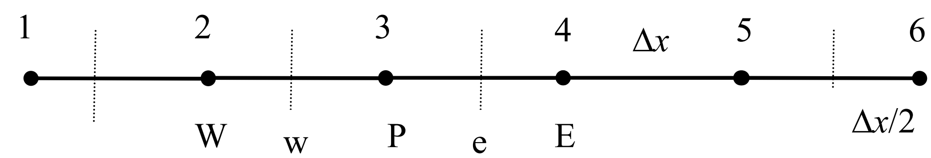
Left end of the slab is maintained at constant temperature; hence boundary condition at this end is given by

$$T(0) = 373 \qquad \text{(iii)}$$

At the right end, heat influx is specified. Thus, boundary condition at this end is

$$k\frac{\mathrm{d}T}{\mathrm{d}x} = 1000 \qquad \text{(iv)}$$

For discretization, let us use a face centered grid. Computational nodes are 1 (x=0), 2 (x=0.1), 3(x=0.2) , 4(x=0.3), 5(x=0.4) and 6 (x=0.5). Each interior finite volume is of width Δx = 0.1, whereas finite volumes around boundary nodes 1 and 6 are of width 0.05.



Let us assume that standard discrete equation can be written as

$$A_W^i T_{i-1} + A_P^i T_i + A_E^i T_{i+1} = b_i \tag{v}$$

Temperature is specified at node 1, hence, we have

$$T_1 = 373, \text{ i.e. } A_W^1 = 0,\ A_P^1 = 1, \quad A_E^1 = 0, \quad b_1 = 373 \tag{vi}$$

For a finite volume around computational node P, finite volume integral equation yields

$$\int_{\Delta V} \frac{d}{dx}\left(k\frac{dT}{dx}\right)dV + \int_{\Delta V}(1273 - T)dV = \left(kA\frac{dT}{dx}\right)_e - \left(kA\frac{dT}{dx}\right)_w + (1273 - T_{\mathrm{P}})\Delta V = 0 \tag{vii}$$

where we have used mid-point rule for evaluation of the second volume integral. For an interior finite volume, using central difference scheme for the derivative, and assuming unit area of CV surfaces and substituting k = 1, the preceding equation can be re-written as

$$\frac{T_{\mathrm{E}} - T_{\mathrm{P}}}{\Delta x} - \frac{T_{\mathrm{P}} - T_{\mathrm{W}}}{\Delta x} + (1273 - T_{\mathrm{P}})\Delta x = 0 \tag{viii}$$

Using Δx = 0.1, the preceding equation simplifies to

$$-T_{i-1} + 2.01T_i - T_{i+1} = 12.73, \quad i = 2,3,4,5 \tag{ix}$$

where we have used indices i for P, (i-1) for W and (i+1) for node E. Hence, the coefficients in the standard equation $A_W^i T_{i-1} + A_P^i T_i + A_E^i T_{i+1} = b_i$ are:

$$A_W^i = -1,\ A_P^i = 2.01, \quad A_E^i = -1, \quad b_i = 12.73, \quad i = 2,3,4,5 \tag{x}$$

For the last FV around boundary node 6, terms in finite volume equation (vii) are

$$\left(kA\frac{dT}{dx}\right)_e = 1000, \quad \left(kA\frac{dT}{dx}\right)_w = \frac{T_6 - T_5}{\Delta x}$$

$$\int_{\Delta V}(1273 - T)dV = (1273 - T_m)\Delta x / 2 \tag{xi}$$

Using linear interpolation, temperature at the centroid of this FV can be expressed as

$$T_m = (T_6 + T_w)/2 = \frac{(T_6 + (T_6 + T_5)/2)}{2} \tag{xii}$$

From Eq. (vii), (xi) and (xii), discrete equation for the boundary finite volume becomes

$$1000 - \frac{T_6 - T_5}{\Delta x} + \left(1273 - \frac{(3T_6 + T_5)}{4}\right)\frac{\Delta x}{2} = 0 \tag{xiii}$$

Using Δx = 0.1, the preceding equation simplifies to

$$1.00375T_6 - 0.99875T_5 = 106.365 \tag{xiv}$$

Thus, $A_W^6 = -0.99875,\ A_P^6 = 1.00375, \quad A_E^6 = 0, \quad b_6 = 106.365$.

Therefore, the discrete system obtained from finite volume discretization is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2.01 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2.01 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2.01 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2.01 & -1 \\ 0 & 0 & 0 & 0 & -0.99875 & 1.00375 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 373 \\ 12.73 \\ 12.73 \\ 12.73 \\ 12.73 \\ 106.365 \end{Bmatrix} \qquad \text{(xv)}$$

The preceding system is very similar to the discrete system (x) obtained using finite difference discretization in Example 15.1 (only coefficients in last row differ). Numerical calculations using TDMA are given in the following table:

i	A_W^i	A_P^i	A_E^i	b_i	$A_P^i \leftarrow A_P^i - \frac{A_W^i A_E^{i-1}}{A_P^{i-1}}$	$b_i^* = b_i - \frac{A_W^i b_{i-1}^*}{A_P^{i-1}}$	$T_i = \frac{b_i^* - A_E^i T_{i+1}}{A_P^i}$	T_{ex}
1	0	1.00	0	373	1	373	373.000	373.00
2	-1	2.01	-1	12.73	2.01	385.73	498.931	498.95
3	-1	2.01	-1	12.73	1.512487	204.6355	617.121	617.18
4	-1	2.01	-1	12.73	1.34883754	148.02734	728.753	728.85
5	-1	2.01	-1	12.73	1.26862087	122.47438	834.942	835.08
6	*W6*	1.00	0	100.36	0.21174243	196.54136	936.751	936.92

where *W6*=-0.99875. Comparison with results obtained in Example 15.1 using finite difference method, we can clearly observe that the FV results using identical grid spacing are more accurate. This is simply due to the fact that all interpolation or integration formulae used in finite volume formulation were second order accurate. On the other hand, first order backward order method was used in finite difference solution for incorporation of flux boundary condition.

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