Point Process Simulation

Estimated Parameters

```
set.seed(1899)
ad_types <- c("search", "display", "other", "purchase")
K <- length(ad_types)
alpha_mean <- matrix(c(2.817, 0.0860, 0.5381, 0.6167,
                       0.1614, 1.7818, 0.2055, 0.0845,
                       0.4647, 0.1270, 8.0526, 0.8384),
                     nrow = 3, ncol = 4, byrow = T)
row.names(alpha_mean) <- ad_types[1:3]
colnames(alpha_mean) <- ad_types

alpha_sd <- matrix(c(0.1765, 0.0214, 0.0562, 0.0633,
                     0.0496, 0.2314, 0.0572, 0.0347,
                     0.0654, 0.0367, 0.4117, 0.0867),
                   nrow = 3, ncol = 4, byrow = T)
row.names(alpha_sd) <- ad_types[1:3]
colnames(alpha_sd) <- ad_types

beta_mean <- c(34.0188, 46.8854, 51.5114)
beta_sd <- c(1.7426, 4.9370, 2.3241)
names(beta_mean) <- ad_types[1:3]

theta_mu_mean <- c(-5.3926, -6.1027, -5.8063, -9.7704)
theta_mu_sd <- c(0.0166, 0.0212, 0.0221, 0.0762)
names(theta_mu_mean) <- ad_types
names(theta_mu_sd) <- ad_types

Sigma_mu_mean <- matrix(c(0.4584, -0.1197, -0.4942, 0.2256,
                     -0.1197, 0.5934, -0.3380, -0.4157,
                     -0.4942, -0.3380, 1.0014, 0.0762,
                     0.2256, -0.4157, 0.0762, 2.3914), nrow = 4, ncol = 4)

Sigma_mu_sd <- matrix(c(0.0246, -0.0212, 0.0257, 0.1228,
                     0.0212, 0.0335, 0.0304, 0.1665,
                     0.0257, 0.0304, 0.0365, 0.2440,
                     0.1228, 0.1665, 0.2440, 0.2575), nrow = 4, ncol = 4)

row.names(Sigma_mu_mean) <- ad_types
colnames(Sigma_mu_mean) <- ad_types

n <- 12000
#inv_df <- n - 1
inv_df <- 7
psi_mean <- c(-0.5664, -0.7556, -0.6235, 0.2787)
psi_sd <- c(0.1228, 0.2348, 0.1229, 0.2160)
names(psi_mean) <- ad_types
names(psi_sd) <- ad_types
```

```r
first_click_prob <- c(1199, 418, 811) %>% prop.table()
```

Table 1: alpha

	search	display	other	purchase
search	2.8170	0.0860	0.5381	0.6167
display	0.1614	1.7818	0.2055	0.0845
other	0.4647	0.1270	3.0000	0.8384

Table 2: beta

	x
search	34.0188
display	46.8854
other	51.5114

Table 3: theta

	x
search	-5.3926
display	-6.1027
other	-5.8063
purchase	-9.7704

Table 4: Sigma

	search	display	other	purchase
search	0.4584	-0.1197	-0.4942	0.2256
display	-0.1197	0.5934	-0.3380	-0.4157
other	-0.4942	-0.3380	1.0014	0.0762
purchase	0.2256	-0.4157	0.0762	2.3914

Table 5: psi

	x
search	-0.5664
display	-0.7556
other	-0.6235
purchase	0.2787

Simulate Data

Draw Parameters

1. Draw $\alpha, \beta, \psi, \theta_\mu, \Sigma_\mu$ and generate $\mu^i \sim MVN_K(\theta_\mu, \Sigma_\mu)$ to simulate the behaviour of a representative customer. Draw $\mu^i \sim log\text{-}MVN_K(\theta_\mu, \Sigma_\mu)$.

```
# calculate gamma and inv wishart params from means and SDs
gammaShRaFromMeanSD <- function( mean , sd ) {
  if ( mean <=0 ) stop("mean must be > 0")
  if ( sd <=0 ) stop("sd must be > 0")
  shape = mean^2/sd^2
  rate = mean/sd^2
  return( list( shape=shape , rate=rate ) )
}

invParamsWishartFromMeanPAndDF <- function(Sigma_mu_mean, df, dim){
  scale_matrix <- Sigma_mu_mean*df - Sigma_mu_mean*dim - Sigma_mu_mean
  return(list(scale_matrix=scale_matrix, df=df, dim=dim))
}

draw_params <- function() {
  alpha <- matrix(nrow = nrow(alpha_mean), ncol = ncol(alpha_mean))
  for(j in c(1:nrow(alpha_mean))) {
    for(k in c(1:ncol(alpha_mean))) {
      gam <- gammaShRaFromMeanSD(mean = alpha_mean[j,k], sd = alpha_sd[j,k])
      alpha[j,k] <- rgamma(n=1, shape = gam$shape, rate = gam$rate)
    }
  }

  beta <- numeric()
  for(j in c(1:length(beta_mean))) {
      gam <- gammaShRaFromMeanSD(mean = beta_mean[j], sd = beta_sd[j])
      beta[j] <- rgamma(n=1, shape = gam$shape, rate = gam$rate)
  }

  # assumption that all psi_ks are independet we can sample with the diagonal of sds,
  # in appendix they calculate psi with the Identity matrix which assumes independence!
  psi <- mvrnorm(mu = psi_mean,
                Sigma = psi_sd^2 * diag(nrow = length(psi_sd)))
  theta_mu <- mvrnorm(mu = theta_mu_mean,
                     Sigma = theta_mu_sd^2 * diag(nrow = length(theta_mu_sd)))

  inv_wishart_params <- invParamsWishartFromMeanPAndDF(Sigma_mu_mean = Sigma_mu_mean,
                                                      df = inv_df, dim = length(ad_types))
  Sigma_mu <- riwish(v = inv_wishart_params$df, S = inv_wishart_params$scale_matrix)
  mu <- exp(mvrnorm(mu = theta_mu, Sigma = Sigma_mu))
  list(alpha = alpha, beta = beta, psi = psi,
       theta_mu = theta_mu, Sigma_mu = Sigma_mu, mu = mu)
}
```

2. Simulate point process in $[0, T]$ given $\alpha, \beta, \psi, \mu^i$ and realized type j_0 at $t_0 = 0,(j_0 = 1, \ldots, K-1)$.
   a. initialize $t = 0, n = 0, n_K = 0, m = \sum_{k=1}^{K}(\mu_k^i + \alpha_{j_o k})$
   b. Repeat until $t > T$

i) Simulate $s \sim Exp(m)$
ii) Set $t = t + s$
iii) If $t < T$, calculate

$$\lambda_k = \mu_k^i \exp(\psi_k n_K) + \alpha_{j_0 k} \exp(-\beta_{j_0} t) + \sum_{l=1, j_l \neq K}^{n} \alpha_{j_l k} \exp(-\beta_{j_l}(t - t_l))$$

and let $\lambda = \sum_{k=1}^{K} \lambda_k$ and generate $U \sim Unif(0,1)$.

1. If $U \leq \frac{\lambda}{m}$, $n = n + 1$, $t_n = t$. Simulate $j_n \sim multinomial(1, \lambda_1/\lambda, \ldots, \lambda_K/\lambda)$.
   - If $j_n = K$ then

$$n_K = n_K + 1$$

$$m = \lambda - \mu_k^i \exp(\psi_k(n_K - 1)) + \mu_k^i \exp(\psi_k n_K)$$

   - else

$$m = \lambda + \alpha_{j_n k}$$

2. If $U > \frac{\lambda}{m}$ then

$$m = \lambda$$

c. Simulation output is $\{t_1, \ldots, t_n\}$ and $\{j_1, \ldots, j_n\}$

3. Repeat Step 1 and 2 R times.

```r
T <- 120
```

```r
simulate_user_clickstream <- function(alpha, beta, psi, theta_mu, Sigma_mu, mu) {
  t = 0; n = 0; n_K = 0; m = c(alpha[j_0,] + mu) %>% sum; j = integer(); t_j = numeric()

  hist_intensity_decay <- function(a, b, events, current_time, event_times) {
    result <- numeric(K)
    names(result) <- ad_types
    if(length(events) == 0) return(result)

    for(k in 1:K){
      for(idx in 1:length(events)){
        j_l <- events[idx]
        if(j_l != K) { # do not consider purchases (K) here
          result[k] <- result[k] +
            alpha[j_l, k] * exp(-1 * beta[j_l] * (t - event_times[idx]))
        }
      }
    }
    return(result)
  }

  repeat {
    s <- rexp(n = 1, rate = m)
    t <- t + s
    if(!is.nan(t) && t < T){
      lambda <- mu * exp(psi*n_K) + alpha[j_0, ] * exp(-1 * beta[j_0] * t)
      lambda <- lambda + hist_intensity_decay(alpha, beta, j, t, t_j)
      U <- runif(n = 1)
      if(is.na(m) || is.na(U)  || NA %in% lambda || is.infinite(m) || Inf %in% lambda) {
        print(paste("Value is NA ", c(m, U, lambda)))
      }
```

```r
      if(U <= sum(lambda) / m){
        n <- n + 1
        t_j[n] <- t
        j[n] <- which(rmultinom(n = 1, size = 1,
                                prob = lambda/sum(lambda))[,1] %in% c(1))
        if(j[n] == K) { # if purchase
          n_K <- n_K + 1
          m <- sum(lambda - mu * exp(psi * (n_K -1)) + mu * exp(psi * n_K))
        } else {
          m <- sum(lambda + alpha[j[n], ])
        }
      } else {
        m <- sum(lambda)
      }
    }
    # in some cases n_K becomes greater then a few thousand.
    # hence, m tends so go to Inf as s becomes 0.
    # this breaks the code and so we break at a certain purchase count.
    if(t > T || n_K >= 10) return(list(timestamp = t_j, event = ad_types[j]))
  }
}
```

Simulate click stream data

Simulate for one user

```r
j_0 <- sample(length(ad_types) - 1, size = 1, prob = first_click_prob)
params <- draw_params()
data <- do.call(simulate_user_clickstream, params)
```

j_0 is set to type “other” with $T = 120$.

	time	event
1	95.01	other

Table 6: Simulated click stream

Simulate for multiple users and plot their click streams.

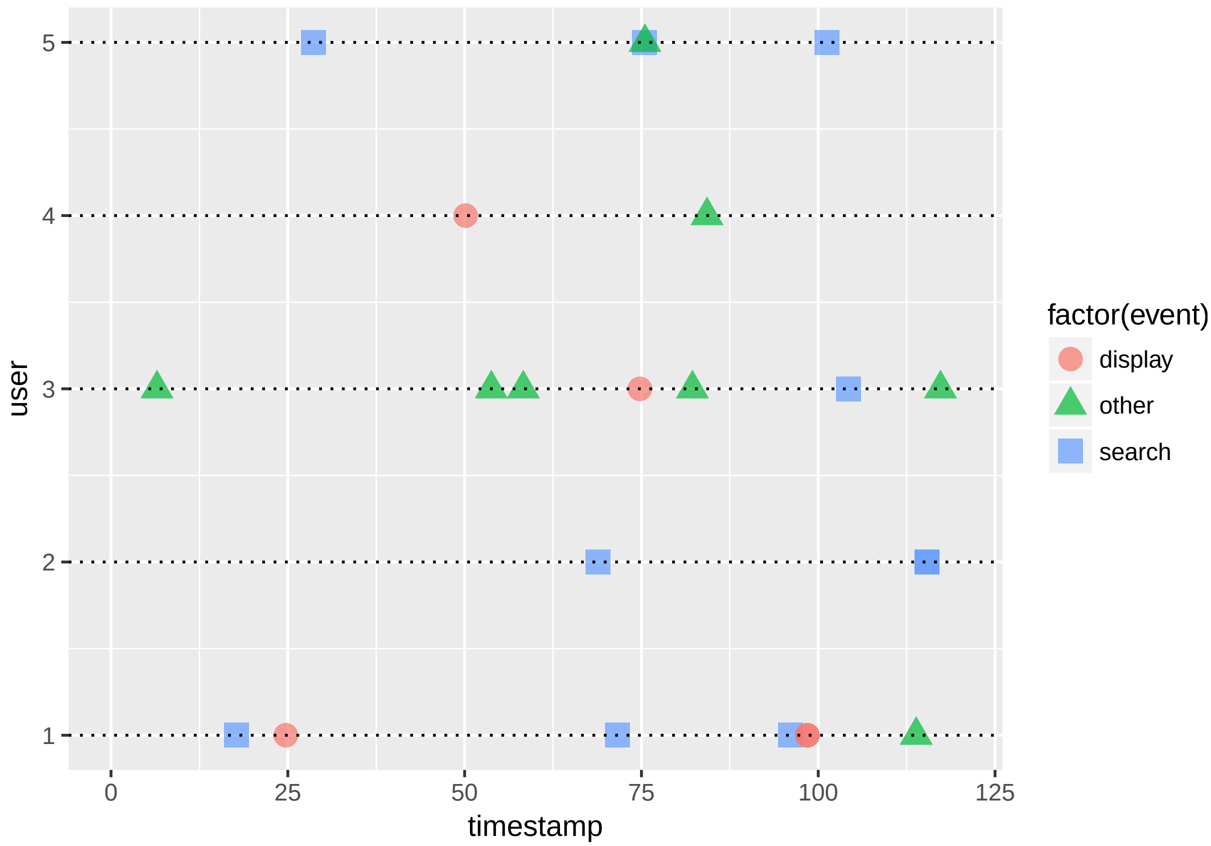
```r
sim_cnt <- 5
```

Simulate for 5 users.

```r
sim_data <- function(count) {
  users <- data.frame()
  for(i in 1:count) {
    j_0 <- sample(length(ad_types) - 1, size = 1, prob = first_click_prob)
    params <- draw_params()
    data <- do.call(simulate_user_clickstream, params)
    users <- users %>%
      rbind(data_frame(user = rep.int(i, length(data$event)),
```

```
                        event = data$event,
                        timestamp=data$timestamp))
  }
  users
}
```

```
users <- sim_data(sim_cnt)
ggplot(users, aes(x = timestamp, y = user,
                  label = event, color = factor(event),
                  shape = factor(event),
                  xmax = T)) +
  geom_point(size = 4, alpha = .7) +
  geom_hline(aes(yintercept = user), lty="dotted") +
  xlim(0, T) +
  ylim(1, sim_cnt)
```



##

Simulate 12,000 users

```
data <- sim_data(n)
```

```
data %>%
  filter(event != 'purchase') %>%
  group_by(event) %>%
  summarise(n = n()) %>%
  mutate(ratio = n / sum(n)) %>%
  knitr::kable(caption = "Ad click ratio")
```

Table 7: Ad click ratio

event	n	ratio
display	7949	0.2990257
other	9588	0.3606816
search	9046	0.3402927

```r
data %>%
  mutate(event_short = recode(event, 'purchase' = 'P', 'other' = 'O',
                              'search' = 'S', 'display' = 'D')) %>%
  group_by(user) %>%
  # concatenate all event abbr. into one string
  summarise(stream = paste(event_short, collapse = "")) %>%
  # only include streams until first P
  mutate(stream = stream %>% gsub(x = ., "(P).*","\\1")) %>%
  # only display click streams with more than 2 events ending with a purchase
  filter(stream %>% nchar() > 2, stream %>% endsWith('P')) %>%
  group_by(stream) %>%
  # count distinct click patterns
  summarise(n = n()) %>%
  arrange(desc(n)) %>%
  head(n=20) %>%
  knitr::kable(caption = "Most frequent click combinations ending with a purchase")
```

Table 8: Most frequent click combinations ending with a purchase

stream	n
SSP	33
OOP	30
OOOP	12
OSP	12
SOP	12
SSSSP	10
DSP	7
SSSP	6
DOP	5
ODSP	4
OOOOOOP	4
OOOOP	3
OOSP	3
OSOP	3
SDP	3
SDSP	3
SSOP	3
DDOP	2
DDSP	2
DOOP	2

```r
data %>%
  mutate(event_short = recode(event, 'purchase' = 'P', 'other' = 'O',
                              'search' = 'S', 'display' = 'D')) %>%
```

```r
  group_by(user) %>%
  # concatenate all event abbr. into one string
  summarise(stream = paste(event_short, collapse = "")) %>%
  mutate(ev_cnt = stream %>% nchar()) %>%
  # cut streams longer than 25 chars
  mutate(stream = ifelse(stream %>% nchar() > 25,
                         stream %>% strtrim(25) %>% paste("...", sep = ""), stream)) %>%
  group_by(stream) %>%
  #summarise(n = nchar()) %>%
  arrange(desc(ev_cnt)) %>%
  head(n=20) %>%
  knitr::kable(caption = "Longest click streams")
```

Table 9: Longest click streams

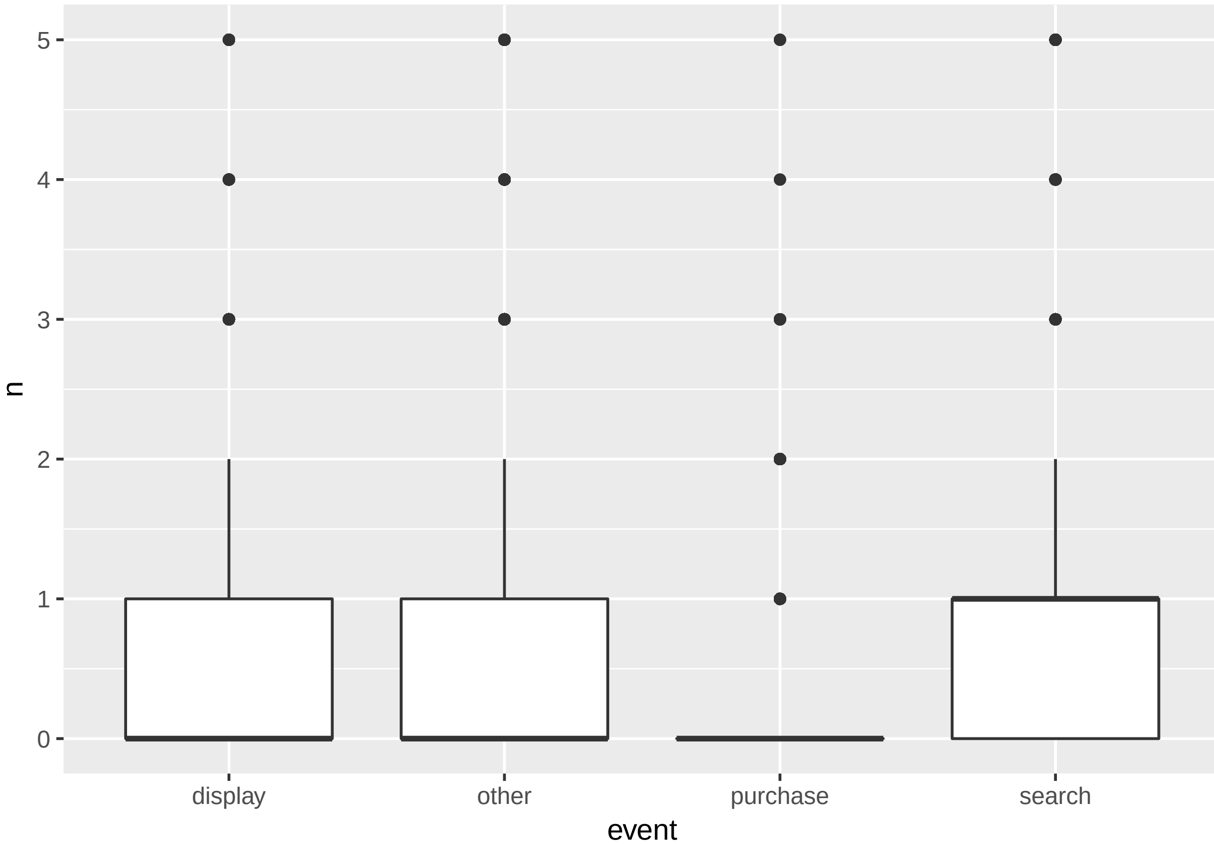
user	stream	ev_cnt
619	DDSODDDDDDDDDSDDDDDDDDDDD...	2180
4322	DDDDDDDDDDDDDDDDDDDDDDDDD...	662
3799	OOOOOOOOOSOOOOOOOOOOOOOOO...	399
7140	DDDDDDDDDDDDDDDDDDDDDDDDD...	193
9470	OOOOOOOOOOOOOOOOOOOOOSOOO...	183
3735	OOOOOOOOOOOOOPOOOOOOOOOOO...	147
4365	OOOOOOOOOOOOOOPOOOOOOOOOO...	131
4880	PSSSSSSSSSSSSPSSSSSSSSSSS...	106
4784	OOOOOOOOOOOOOOOOOOOOOOOOO...	105
7017	OOOOOOOOOOOOOOPOOOOOOOOOO...	79
9461	OOOSOOOOOOOOOOOOOOOOOOOOO...	79
1050	DDDDDDDDDDDDDDDDDDDDDDDDD...	74
3454	OOOOOOOOOOOOOOOOOOOOOOOOO...	71
5401	OOPOOOOOOOOOOOOOOOOOOOOOO...	60
9908	OOOOOOOOOOOOOOOOOOOOOOOOO...	58
718	OOOOOOOOOOOOOOOOOOOOOOOOO...	51
7377	SSDSSDSSDDDDSSDDODSSSDSSS...	51
3416	OOOOOOOOOOOOOOOOOOOOOOOOO...	50
8099	OOOOOOOOOOOOOOOOOOOOOOOOO...	48
3215	OOOOOOOOOOOOOOOOOOOOOOOOO...	46

```r
all_comb <- expand.grid(user = data$user %>% unique, event = ad_types)
event_counts <- data %>%
  group_by(user, event) %>%
  summarise(n = n()) %>%
  # join all combinations to get zero count for non-present events
  left_join(all_comb, ., by = c('user', 'event')) %>%
  mutate_if(is.integer, funs(replace(., is.na(.), 0)))
```

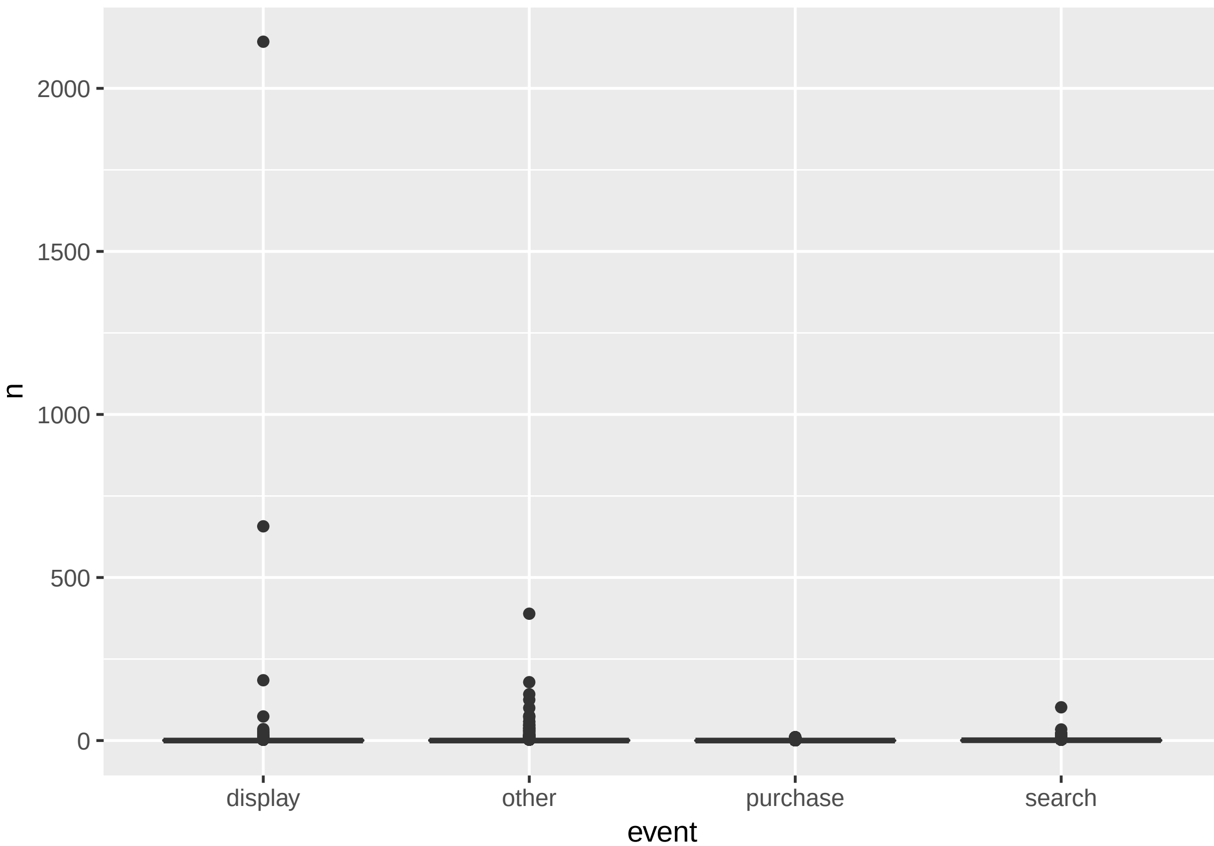
```
## Warning: Column `event` joining factor and character vector, coercing into
## character vector
```

```r
rm(all_comb)
event_counts %>%
  ggplot(aes(y = n, x = event)) +
  geom_boxplot() +
  ylim(c(0, 5))
```

```
## Warning: Removed 318 rows containing non-finite values (stat_boxplot).
```




```
event_counts %>%
  ggplot(aes(y = n, x = event)) +
  geom_boxplot()
```


2000
1500
1000
500
0
n
display
other
purchase
search
event