Solutions Manual

for

Communication Systems

4th Edition

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Preface

This Manual is written to accompany the fourth edition of my book on Communication Systems. It consists of the following:

- Detailed solutions to all the problems in Chapters 1 to 10 of the book

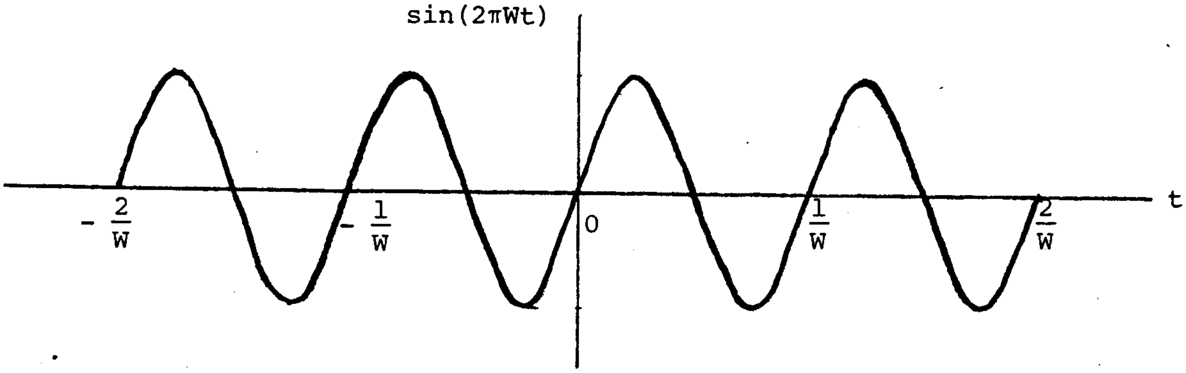
- MATLAB codes and representative results for the computer experiments in Chapters 1, 2, 3, 4, 6, 7, 9 and 10

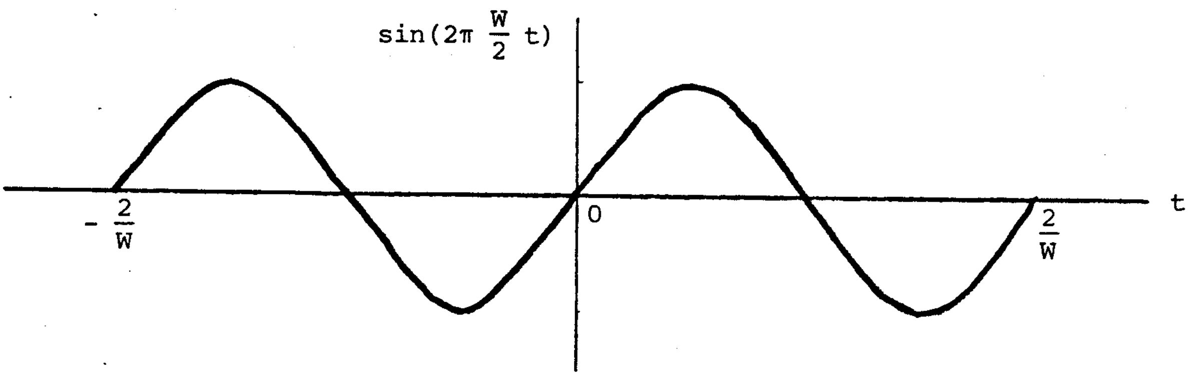
I would like to express my thanks to my graduate student, Mathini Sellathurai, for her help in solving some of the problems and writing the above-mentioned MATLAB codes. I am also grateful to my Technical coordinator, Lola Brooks for typing the solutions to new problems and preparing the manuscript for the Manual.

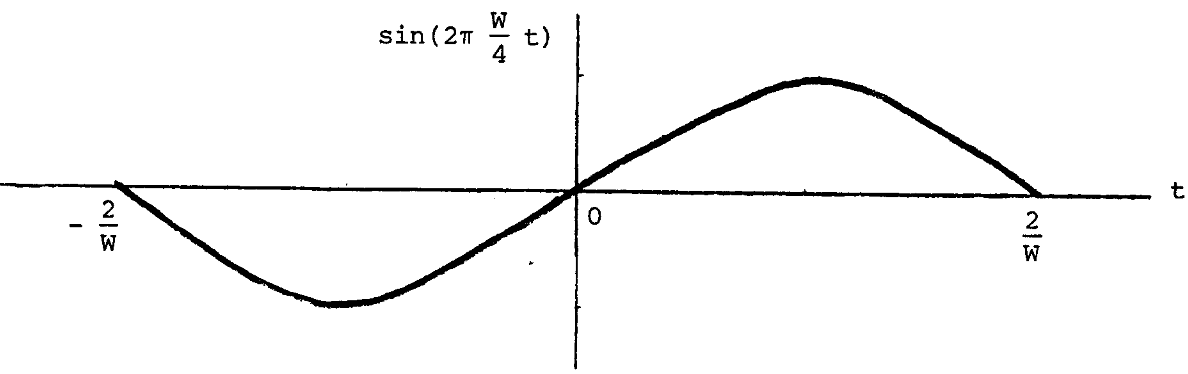
Simon Haykin
Ancaster
April 29, 2000

Problem 1.1

As an illustration, three particular sample functions of the random process X(t), corresponding to F = W/4, W/2, and W, are plotted below:







To show that X(t) is nonstationary, we need only observe that every waveform illustrated above is zero at t = 0, positive for $0 < t < 1/2W$, and negative for $-1/2W < t < 0$. Thus, the probability density function of the random variable $X(t_1)$ obtained by sampling X(t) at $t_1 = 1/4W$ is identically zero for negative argument, whereas the probability density function of the random variable $X(t_2)$ obtained by sampling X(t) at $t = -1/4W$ is nonzero only for negative arguments. Clearly, therefore,

$f_{X(t_1)}(x_1) \neq f_{X(t_2)}(x_2)$, and the random process X(t) is nonstationary.

Problem 1.2

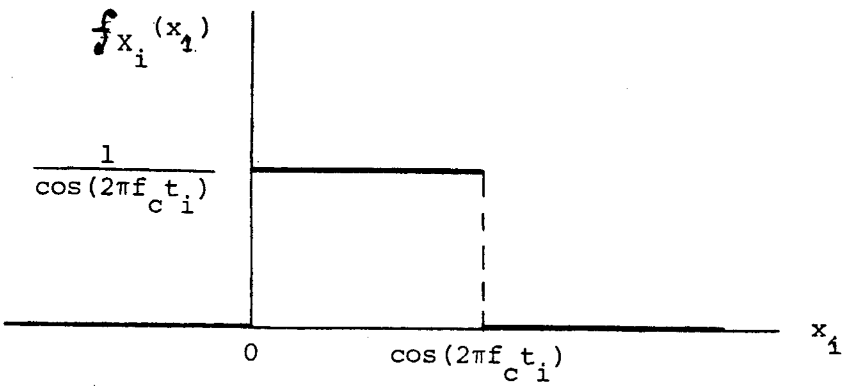
$$X(t) = A\cos(2\pi f_c t)$$

Therefore,

$$X_i = A\cos(2\pi f_c t_i)$$

Since the amplitude A is uniformly distributed, we may write

$$f_{X_i}(x_1) = \begin{cases} \dfrac{1}{\cos(2\pi f_c t_i)}, & 0 \le x_1 \le \cos(2\pi f_c t_i) \\ 0, & \text{otherwise} \end{cases}$$



Similarly, we may write

$$X_{i+\tau} = A\cos[2\pi f_c(t_i+\tau)]$$

and

$$f_{X_{i+\tau}}(x_2) = \begin{cases} \dfrac{1}{\cos[2\pi f_c(t_i+\tau)]}, & 0 \le x_2 \le \cos[2\pi f_c(t_i+\tau)] \\ 0, & \text{otherwise} \end{cases}$$

We thus see that $f_{X_i}(x_1) \neq f_{X_{i+\tau}}(x_2)$, and so the process X(t) is nonstationary.

Problem 1.3

(a) The integrator output at time t is

$$Y(t) = \int_0^t X(\tau)\, d\tau$$

$$= A\int_0^t \cos(2\pi f_c \tau)\, d\tau$$

$$= \frac{A}{2\pi f_c} \sin(2\pi f_c t)$$

Therefore,

$$E[Y(t)] = \frac{\sin(2\pi f_c t)}{2\pi f_c} E[A] = 0$$

$$Var[Y(t)] = \frac{\sin^2(2\pi f_c t)}{(2\pi f_c)^2} Var[A]$$

$$= \frac{\sin^2(2\pi f_c t)}{(2\pi f_c)^2} \sigma_A^2 \tag{1}$$

Y(t) is Gaussian-distributed, and so we may express its probability density function as

$$f_{Y(t)}(y) = \frac{\sqrt{2\pi}\, f_c}{\sigma_A \sin(2\pi f_c t)} \exp[- \frac{2\pi^2 f_c^2}{\sin^2(2\pi f_c t)\, \sigma_A^2} y^2]$$

(b) From Eq. (1) we note that the variance of Y(t) depends on time t, and so Y(t) is nonstationary.

(c) For a random process to be ergodic it has to be stationary. Since Y(t) is nonstationary, it follows that it is not ergodic.

Problem 1.4

(a) The expected value of $Z(t_1)$ is

$$E[Z(t_1)] = \cos(2\pi t_1)\, E[X] + \sin(2\pi t_1)\, E[Y]$$

Since E[X] = E[Y] = 0, we deduce that

$$E[Z(t_1)] = 0$$

Similarly, we find that

$$E[Z(t_2)] = 0$$

Next, we note that

$$\begin{aligned} Cov[Z(t_1)Z(t_2)] &= E[Z(t_1)Z(t_2)] \\ &= E\{[X\cos(2\pi t_1) + Y\sin(2\pi t_1)][X\cos(2\pi t_2) + Y\sin(2\pi t_2)]\} \\ &= \cos(2\pi t_1)\cos(2\pi t_2)\, E[X^2] \\ &\quad + [\cos(2\pi t_1)\sin(2\pi t_2)+\sin(2\pi t_1)\cos(2\pi t_2)]E[XY] \\ &\quad + \sin(2\pi t_1)\sin(2\pi t_2)E[Y^2] \end{aligned}$$

Noting that

$$E[X^2] = \sigma_X^2 + \{E[X]\}^2 = 1$$

$$E[Y^2] = \sigma_Y^2 + \{E[Y]\}^2 = 1$$

$$E[XY] = 0$$

we obtain

$$\mathrm{Cov}[Z(t_1)Z(t_2)] = \cos(2\pi t_1)\cos(2\pi t_2)+\sin(2\pi t_1)\sin(2\pi t_2)$$

$$= \cos[2\pi(t_1-t_2)] \tag{1}$$

Since every weighted sum of the samples of the process Z(t) is Gaussian, it follows that Z(t) is a Gaussian process. Furthermore, we note that

$$\sigma^2_{Z(t_1)} = E[Z^2(t_1)] = 1$$

This result is obtained by putting $t_1=t_2$ in Eq. (1). Similarly,

$$\sigma^2_{Z(t_2)} = E[Z^2(t_2)] = 1$$

Therefore, the correlation coefficient of $Z(t_1)$ and $Z(t_2)$ is

$$\rho = \frac{\mathrm{Cov}[Z(t_1)Z(t_2)]}{\sigma_{Z(t_1)}\sigma_{Z(t_2)}}$$

$$= \cos[2\pi(t_1-t_2)]$$

Hence, the joint probability density function of $Z(t_1)$ and $Z(t_2)$

$$f_{Z(t_1),Z(t_2)}(z_1, z_2) = C \exp[-Q(z_1,z_2)]$$

where

$$C = \frac{1}{2\pi\sqrt{1-\cos^2[2\pi(t_1-t_2)]}}$$

$$= \frac{1}{2\pi \sin[2\pi(t_1-t_2)]}$$

$$Q(z_1,z_2) = \frac{1}{2\sin^2[2\pi(t_1-t_2)]}\{z_1^2 - 2\cos[2\pi(t_1-t_2)]z_1z_2 + z_2^2\}$$

(b) We note that the covariance of $Z(t_1)$ and $Z(t_2)$ depends only on the time difference t_1-t_2. The process Z(t) is therefore wide-sense stationary. Since it is Gaussian it is also strictly stationary.

Problem 1.5

(a) Let

$$X(t) = A + Y(t)$$

where A is a constant and Y(t) is a zero-mean random process. The autocorrelation function of X(t) is

$$\begin{aligned} R_X(\tau) &= E[X(t+\tau)\, X(t)] \\ &= E\{[A + Y(t+\tau)]\,[A + Y(t)]\} \\ &= E[A^2 + A\, Y(t+\tau) + A\, Y(t) + Y(t+\tau)\, Y(t)] \\ &= A^2 + R_Y(\tau) \end{aligned}$$

which shows that $R_X(\tau)$ contains a constant component equal to A^2.

(b) Let

$$X(t) = A_c \cos(2\pi f_c t + \theta) + Z(t)$$

where $A_c \cos(2\pi f_c t+\theta)$ represents the sinusoidal component of X(t) and θ is a random phase variable. The autocorrelation function of X(t) is

$$\begin{aligned} R_X(\tau) &= E[X(t+\tau)\, X(t)] \\ &= E\{[A_c \cos(2\pi f_c t + 2\pi f_c \tau + \theta) + Z(t+\tau)]\,[A_c \cos(2\pi f_c t + \theta) + Z(t)]\} \\ &= E[A_c^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta) \cos(2\pi f_c t + \theta)] \\ &\quad + E[Z(t+\tau)\, A_c \cos(2\pi f_c t + \theta)] \\ &\quad + E[A_c \cos(2\pi f_c t + 2\pi f_c \tau + \theta)\, Z(t)] \\ &\quad + E[Z(t+\tau)\, Z(t)] \\ &= (A_c^2/2) \cos(2\pi f_c \tau) + R_Z(\tau) \end{aligned}$$

which shows that $R_X(\tau)$ contains a sinusoidal component of the same frequency as X(t).

Problem 1.6

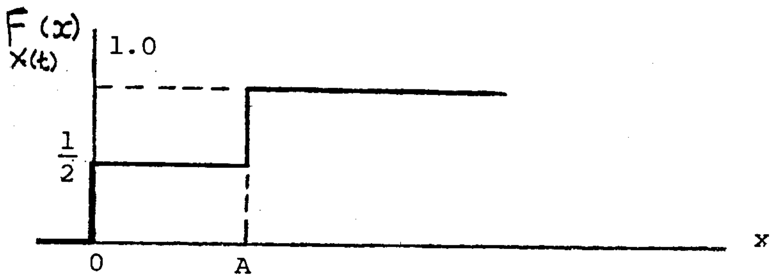
(a) We note that the distribution function of X(t) is

$$F_{X(t)}(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \le x \le A \\ 1, & A < x \end{cases}$$

and the corresponding probability density function is

$$f_{X(t)}(x) = \frac{1}{2}\,\delta(x) + \frac{1}{2}\,\delta(x - A)$$

which are illustrated below:



(b) By ensemble-averaging, we have

$$E[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx$$

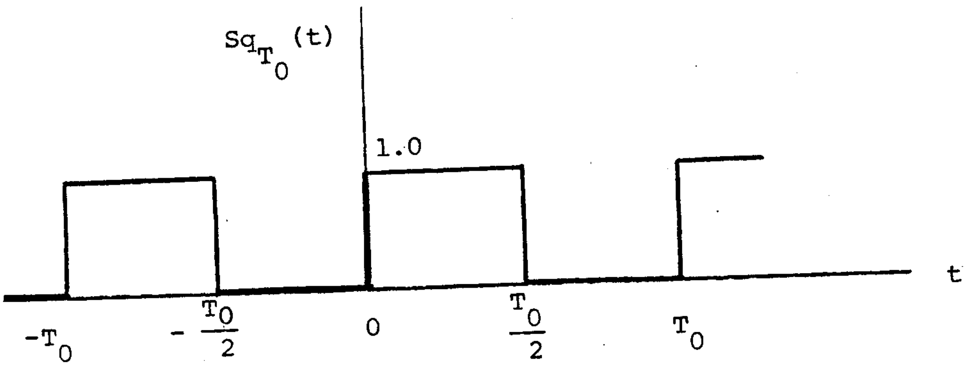
$$= \int_{-\infty}^{\infty} x\, [\tfrac{1}{2}\,\delta(x) + \tfrac{1}{2}\,\delta(x - A)]\, dx$$

$$= \frac{A}{2}$$

The autocorrelation function of X(t) is

$$R_X(\tau) = E[X(t+\tau)\, X(t)]$$

Define the square function $Sq_{T_0}(t)$ as the square-wave shown below:



Then, we may write

$$R_X(\tau) = E[A\, Sq_{T_0}(t - t_d + \tau) \cdot A\, Sq_{T_0}(t - t_d)]$$

$$= A^2 \int_{-\infty}^{\infty} Sq_{T_0}(t - t_d + \tau)\, Sq_{T_0}(t - t_d)\, f_{T_d}(t_d)\, dt_d$$

$$= A^2 \int_{-T_0/2}^{T_0/2} Sq_{T_0}(t - t_d + \tau)\, Sq_{T_0}(t - t_d) \cdot \frac{1}{T_0}\, dt_d$$

$$= \frac{A^2}{2}\left(1 - 2\frac{|\tau|}{T_0}\right), \quad |\tau| \leq \frac{T_0}{2}.$$

Since the wave is periodic with period T_0, $R_X(\tau)$ must also be periodic with period T_0.

(c) On a time-averaging basis, we note by inspection of Fig. P1.6 that the mean is

$$\langle x(t)\rangle = \frac{A}{2}$$

Next, the autocorrelation function

$$\langle x(t+\tau)x(t)\rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t+\tau)\, x(t)\, dt$$

has its maximum value of $A^2/2$ at $\tau = 0$, and decreases linearly to zero at $\tau = T_0/2$. Therefore,

$$\langle x(t+\tau)\, x(t)\rangle = \frac{A^2}{2}\left(1 - 2\frac{|\tau|}{T_0}\right), \quad |\tau| \leq \frac{T_0}{2}.$$

Again, the autocorrelation must be periodic with period T_0.

(d) We note that the ensemble-averaging and time-averaging procedures yield the same set of results for the mean and autocorrelation functions. Therefore, X(t) is ergodic in both the mean and the autocorrelation function. Since ergodicity implies wide-sense stationarity, it follows that X(t) must be wide-sense stationary.

Problem 1.7

(a) For $|\tau| > T$, the random variables $X(t)$ and $X(t+\tau)$ occur in different pulse intervals and are therefore independent. Thus,

$$E[X(t)\, X(t+\tau)] = E[X(t)]\, E[X(t+\tau)], \qquad |\tau| > T.$$

Since both amplitudes are equally likely, we have $E[X(t)] = E[x(t+\tau)] = A/2$. Therefore, for $|\tau| > T$,

$$\therefore R_X(\tau) = \frac{A^2}{4}.$$

For $|\tau| \leq T$, the random variables occur in the same pulse interval if $t_d < T - |\tau|$. If they do occur in the same pulse interval,

$$E[X(t)\, X(t+\tau)] = \frac{1}{2} A^2 + \frac{1}{2} 0^2 = \frac{A^2}{2}.$$

We thus have a conditional expectation:

$$\begin{aligned} E[X(t)\, X(t+\tau)] &= A^2/2, \qquad t_d < T - |\tau| \\ &= A^2/4, \text{ otherwise.} \end{aligned}$$

Averaging over t_d, we get

$$\begin{aligned} R_X(\tau) &= \int_0^{T-|\tau|} \frac{A^2}{2T}\, dt_d + \int_{T-|\tau|}^{T} \frac{A^2}{4T}\, dt_d \\ &= \frac{A^2}{4}\left(1 - \frac{|\tau|}{T}\right) + \frac{A^2}{4}, \qquad |\tau| \leq T \end{aligned}$$

(b) The power spectral density is the Fourier transform of the autocorrelation function. The Fourier transform of

$$\begin{aligned} g(\tau) &= 1 - \frac{|\tau|}{T}, \quad |\tau| \leq T \\ &= 0, \text{ otherwise,} \end{aligned}$$

is given by

$$G(f) = T\, \mathrm{sinc}^2(fT).$$

Therefore,

$$S_X(f) = \frac{A^2}{4}\delta(f) + \frac{A^2T}{4}\text{sinc}^2(fT) .$$

We next note that

$$\frac{A^2}{4}\int_{-\infty}^{\infty} \delta(f)\, df = \frac{A^2}{4} ,$$

$$\frac{A^2}{4}\int_{-\infty}^{\infty} T\text{sinc}^2(fT)\, df = \frac{A^2}{4} ,$$

$$\int_{-\infty}^{\infty} S_X(f)\, df \doteq R_X(0) = \frac{A^2}{2} .$$

It follows therefore that half the power is in the dc component.

Problem 1.8

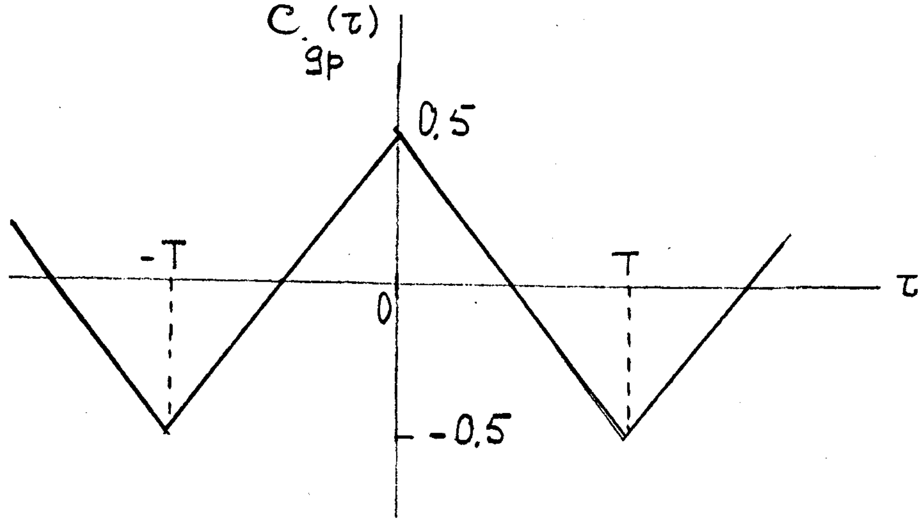
Since

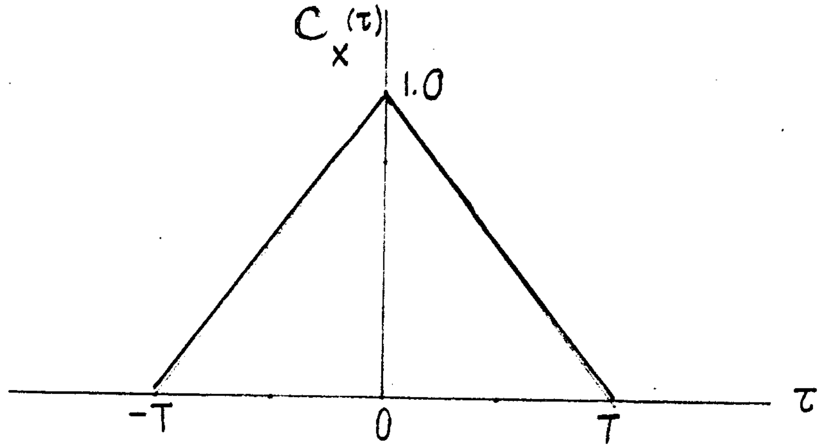
$$Y(t) = g_p(t) + X(t) + \sqrt{3/2}$$

and $g_p(t)$ and $X(t)$ are uncorrelated, then

$$\mathcal{C}_Y(\tau) = \mathcal{C}_{g_p}(\tau) + \mathcal{C}_X(\tau)$$

where $\mathcal{C}_{gp}(\tau)$ is the autocovariance of the periodic component and $\mathcal{C}_X(\tau)$ is the autocovariance of the random component. $\mathcal{C}_Y(\tau)$ is the plot in figure P1.8 shifted down by 3/2, removing the dc component. $\mathcal{C}_{gp}(\tau)$ and $\mathcal{C}_X(\tau)$ are plotted below:





Both $g_p(t)$ and $X(t)$ have zero mean.

(a) The average power of the periodic component $g_p(t)$ is therefore,

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p^2(t)\,dt = C_{g_p}(0) = \frac{1}{2}$$

(b) The average power of the random component $X(t)$ is

$$E[X^2(t)] = C_X(0) = 1$$

Problem 1.9

(a) $R_{XY}(\tau) = E[X(t+\tau)\,Y(t)]$

Replacing τ with $-\tau$:

$$R_{XY}(-\tau) = E[X(t-\tau)\,Y(t)]$$

Next, replacing t-τ with t, we get

$$R_{XY}(-\tau) = E[Y(t+\tau)\,X(t)]$$

$$= R_{YX}(\tau)$$

(b) Form the non-negative quantity

$$E[\{X(t+\tau) \pm Y(t)\}^2] = E[X^2(t+\tau) \pm 2X(t+\tau)\,Y(t) + Y^2(t)]$$

$$= E[X^2(t+\tau)] \pm 2E[X(t+\tau)\,Y(t)] + E[Y^2(t)]$$

$$= R_X(0) \pm 2R_{XY}(\tau) + R_Y(0)$$

Hence,

$$R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \geq 0$$

or

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

Problem 1.10

(a) The cascade connection of the two filters is equivalent to a filter with impulse response

$$h(t) = \int_{-\infty}^{\infty} h_1(u)\,h_2(t-u)\,du$$

The autocorrelation function of Y(t) is given by

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\,R_X(\tau - \tau_1 + \tau_2)\,d\tau_1\,d\tau_2$$

(b) The cross-correlation function of V(t) and Y(t) is

$$R_{VY}(\tau) = E[V(t+\tau)\,Y(t)]$$

The Y(t) and V(t+τ) are related by

$$Y(t) = \int_{-\infty}^{\infty} V(\lambda)\,h_2(t-\lambda)\,d\lambda$$

Therefore,

$$R_{VY}(\tau) = E[V(t+\tau)\int_{-\infty}^{\infty} V(\lambda)\,h_2(t-\lambda)\,d\lambda]$$

$$= \int_{-\infty}^{\infty} h_2(t-\lambda)\, E[V(t+\tau)\, V(\lambda)]\, d\lambda$$

$$= \int_{-\infty}^{\infty} h_2(t-\lambda)\, R_V(t+\tau-\lambda)\, d\lambda$$

Substituting λ for $t-\lambda$:

$$R_{VY}(\tau) = \int_{-\infty}^{\infty} h_2(\lambda)\, R_V(\tau+\lambda)\, d\lambda$$

The autocorrelation function $R_V(\tau)$ is related to the given $R_X(\tau)$ by

$$R_V(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\tau_1)\, h_1(\tau_2)\, R_X(\tau-\tau_1+\tau_2)\, d\tau_1 d\tau_2$$

Problem 1.11

(a) The cross-correlation function $R_{YX}(\tau)$ is

$$R_{YX}(\tau) = E[Y(t+\tau)\, X(t)]$$

The $Y(t)$ and $X(t)$ are related by

$$Y(t) = \int_{-\infty}^{\infty} X(u)\, h(t-u)\, du$$

Therefore,

$$R_{YX}(\tau) = E\left[\int_{-\infty}^{\infty} X(u)X(t)\, h(t+\tau-u)\, du\right]$$

$$= \int_{-\infty}^{\infty} h(t+\tau-u)\, E[X(u)X(t)]\, du$$

$$= \int_{-\infty}^{\infty} h(t+\tau-u)\, R_X(u-t)\, du$$

Replacing $t+\tau-u$ by u:

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} h(u)\, R_X(\tau-u)\, du$$

(b) Since $R_{XY}(\tau) = R_{YX}(-\tau)$, we have

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u)\, R_X(-\tau-u)\, du$$

Since $R_X(\tau)$ is an even function of τ:

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u)\, R_X(\tau+u)\, du$$

Replacing u by -u:

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(-u)\, R_X(\tau-u)\, du$$

(c) If X(t) is a white noise process with zero mean and power spectral density $N_0/2$, we may write

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

Therefore,

$$R_{YX}(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} h(u)\, \delta(\tau-u)\, du$$

Using the sifting property of the delta function:

$$R_{YX}(\tau) = \frac{N_0}{2} h(\tau)$$

That is,

$$h(\tau) = \frac{2}{N_0} R_{YX}(\tau)$$

This means that we may measure the impulse response of the filter by applying a white noise of power spectral density $N_0/2$ to the filter input, cross-correlating the filter output with the input, and then multiplying the result by $2/N_0$.

Problem 1.12

(a) The power spectral density consists of two components:

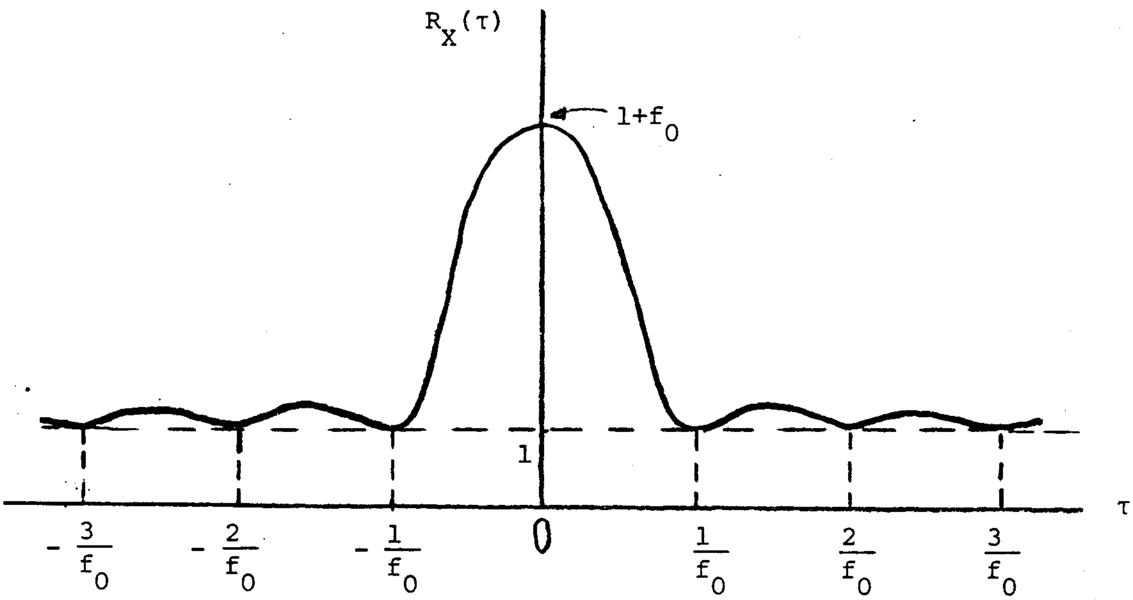
(1) A delta function $\delta(t)$ at the origin, whose inverse Fourier transform is one.

(2) A triangular component of unit amplitude and width $2f_0$, centered at the origin; the inverse Fourier transform of this component is $f_0\, \mathrm{sinc}^2(f_0\tau)$.

Therefore, the autocorrelation function of X(t) is

$$R_X(\tau) = 1 + f_0 \operatorname{sinc}^2(f_0\tau)$$

which is sketched below:



(b) Since $R_X(\tau)$ contains a constant component of amplitude 1, it follows that the dc power contained in X(t) is 1.

(c) The mean-square value of X(t) is given by

$$E[X^2(t)] = R_X(0)$$
$$= 1 + f_0$$

The ac power contained in X(f) is therefore equal to f_0.

(d) If the sampling rate is f_0/n, where n is an integer, the samples are uncorrelated. They are not, however, statistically independent. They would be statistically independent if X(t) were a Gaussian process.

Problem 1.13

The autocorrelation function of $n_2(t)$ is

$$R_{N_2}(t_1,t_2) = E[n_2(t_1)\, n_2(t_2)]$$

$$= E\{[n_1(t_1)\cos(2\pi f_c t_1 + \Theta) - n_1(t_1)\sin(2\pi f_c t_1+\Theta)]$$
$$\cdot [n_1(t_2)\cos(2\pi f_c t_2 + \Theta) - n_1(t_2)\sin(2\pi f_c t_2 + \Theta)]\}$$
$$= E[n_1(t_1)\, n_1(t_2)\cos(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta)$$
$$- n_1(t_1)\, n_1(t_2)\cos(2\pi f_c t_1 + \Theta)\sin(2\pi f_c t_2 + \Theta)$$
$$- n_1(t_1)\, n_1(t_2)\sin(2\pi f_c t_1 + \Theta)\cos(2\pi f_c t_2 + \Theta)$$

$$+ n_1(t_1)\, n_1(t_2) \sin(2\pi f_c t_1 + \Theta) \sin(2\pi f_c t_2 + \Theta)]$$

$$= E\{n_1(t_1)\, n_1(t_2) \cos[2\pi f_c(t_1 - t_2)]$$

$$- n_1(t_1)\, n_1(t_2) \sin[2\pi f_c(t_1 + t_2) + 2\Theta]\}$$

$$= E[n_1(t_1)\, n_1(t_2)] \cos[2\pi f_c(t_1 - t_2)]$$

$$- E[n_1(t_1)\, n_1(t_2)] \cdot E\{\sin[2\pi f_c(t_1 + t_2) + 2\Theta]\}$$

Since Θ is a uniformly distributed random variable, the second term is zero, giving

$$R_{N_2}(t_1, t_2) = R_{N_1}(t_1, t_2) \cos[2\pi f_c(t_1 - t_2)]$$

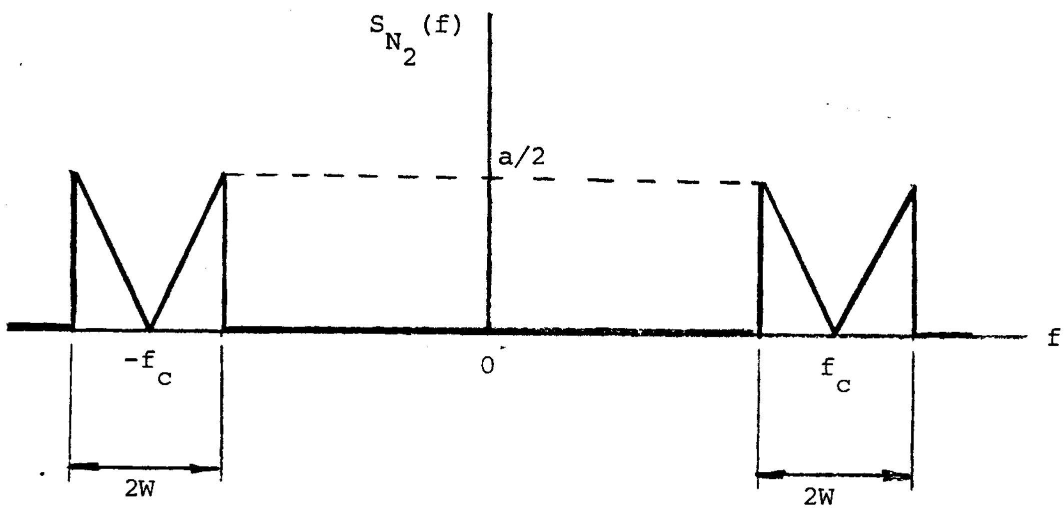
Since $n_1(t)$ is stationary, we find that in terms of $\tau = t_1 - t_2$:

$$R_{N_2}(\tau) = R_{N_1}(\tau) \cos(2\pi f_c \tau)$$

Taking the Fourier transforms of both sides of this relation:

$$S_{N_2}(f) = \frac{1}{2}\,[S_{N_1}(f + f_c) + S_{N_1}(f - f_c)]$$

With $S_{N_1}(f)$ as defined in Fig. P1.3, we find that $S_{N_2}(f)$ is as shown below:



Problem 1.14

The power spectral density of the random telegraph wave is

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau)\, d\tau$$

$$= \int_{-\infty}^{0} \exp(2\nu\tau) \exp(-j2\pi f\tau)\, d\tau$$

$$+ \int_{0}^{\infty} \exp(-2\nu\tau) \exp(-j2\pi f\tau)\, d\tau$$

$$= \frac{1}{2(\nu-j\pi f)} [\exp(2\nu\tau - j2\pi f\tau)]_{-\infty}^{0}$$

$$- \frac{1}{2(\nu+j\pi f)} [\exp(-2\nu\tau - j2\pi f\tau)]_{0}^{\infty}$$

$$= \frac{1}{2(\nu-j\pi f)} + \frac{1}{2(\nu+j\pi f)}$$

$$= \frac{\nu}{\nu^2+\pi^2 f^2}$$

The transfer function of the filter is

$$H(f) = \frac{1}{1 + j2\pi f RC}$$

Therefore, the power spectral density of the filter output is

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$= \frac{\nu}{[1 + (2\pi f RC)^2](\nu^2+\pi^2 f^2)}$$

To determine the autocorrelation function of the filter output, we first expand $S_Y(f)$ in partial fractions as follows

$$S_Y(f) = \frac{\nu}{1 - 4R^2C^2\nu^2} \left[- \frac{1}{(1/2RC)^2 + \pi^2 f^2} + \frac{1}{\nu^2 + \pi^2 f^2}\right]$$

Recognizing that

$$\exp(-2\nu|t|) \rightleftharpoons \frac{\nu}{\nu^2 + \pi^2 f^2}$$

$$\exp(-|t|/RC) \rightleftharpoons \frac{1/2RC}{(1/2RC)^2 + \pi^2 f^2}$$

we obtain the desired result:

$$R_Y(\tau) = \frac{\nu}{1-4R^2C^2\nu^2}\left[\frac{1}{\nu}\exp(-2\nu|\tau|) - 2RC\exp\left(-\frac{|\tau|}{RC}\right)\right]$$

Problem 1.15

We are given

$$y(t) = \int_{t-T}^{t} x(\tau)d\tau$$

For $x(t) = \delta(t)$, the impulse response of this running integrator is, by definition,

$$h(t) = \int_{t-T}^{t} \delta(\tau)d\tau$$

$$= 1 \text{ for } t - T \le 0 \le t \text{ or, equivalently, } 0 \le t \le T$$

Correspondingly, the frequency response of the running integrator is

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt$$

$$= \int_{0}^{T} \exp(-j2\pi ft)dt$$

$$= \frac{1}{j2\pi ft}[1 - \exp(-j2\pi fT)]$$

$$= T\,\text{sinc}\,(fT)\exp(-j\pi fT)$$

Hence the power spectral density $S_Y(f)$ is defined in terms of the power spectral density $S_X(f)$ as follows

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$= T^2 \text{sinc}^2(fT)S_X(f)$$

Problem 1.16

We are given a filter with the impulse response

$$h(t) = \begin{cases} a\exp(-at), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

The frequency response of the filter is therefore

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt$$

$$= \int_0^T a\exp(-at)\exp(-j2\pi ft)dt$$

$$= a\int_0^T \exp(-(a+j2\pi f)t)dt$$

$$= \frac{a}{a+j2\pi f}[-\exp(-(a+j2\pi f)t]_0^T$$

$$= \frac{a}{a+j2\pi f}[1-\exp(-(a+j2\pi f)T]$$

$$= \frac{a}{a+j2\pi f}[1-e^{-at}(\cos(2\pi fT)-j\sin(2\pi fT))$$

The squared magnitude response is

$$|H(f)|^2 = \left[\frac{a^2}{a^2+4\pi^2 f^2}(1-e^{-aT}\cos(2\pi fT))^2+(e^{-aT}\sin(2\pi fT))^2\right]$$

$$= \frac{a^2}{a^2+4\pi^2 f^2}[1-2e^{-aT}\cos(2\pi fT)+e^{-2aT}(\cos^2(2\pi fT)+\sin^2(2\pi fT))]$$

$$= \frac{a^2}{a^2+4\pi^2 f^2}[1-2e^{-aT}\cos(2\pi fT)+e^{-2aT}]$$

Correspondingly, we may write

$$S_Y(f) = \frac{a^2}{a^2+4\pi^2 f^2}[1-2e^{-aT}\cos(2\pi fT)+e^{-2aT}]S_X(f)$$

Problem 1.17

The autocorrelation function of X(t) is

$$R_X(\tau) = E[X(t+\tau)\,X(t)]$$

$$= A^2\,E[\cos(2\pi Ft + 2\pi F\tau - \Theta)\cos(2\pi Ft - \Theta)]$$

$$= \frac{A^2}{2}\,E[\cos(4\pi Ft + 2\pi F\tau - 2\Theta) + \cos(2\pi F\tau)]$$

Averaging over Θ, and noting that Θ is uniformly distributed over 2π radians, we get

$$R_X(\tau) = \frac{A^2}{2}\,E[\cos(2\pi F\tau)]$$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty} f_F(f)\cos(2\pi f\tau)\,df \qquad (1)$$

Next, we note that $R_X(\tau)$ is related to the power spectral density by

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\cos(2\pi f\tau)\,df \qquad (2)$$

Therefore, comparing Eqs. (1) and (2), we deduce that the power spectral density of X(t) is

$$S_X(f) = \frac{A^2}{2}\,f_F(f)$$

When the frequency assumes a constant value, f_c (say), we have

$$f_F(f) = \frac{1}{2}\delta(f-f_c) + \frac{1}{2}\delta(f+f_c)$$

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and, correspondingly,

$$S_X(f) = \frac{A^2}{4}\,\delta(f-f_c) + \frac{A^2}{4}\,\delta(f+f_c)$$

Problem 1.18

Let σ_X^2 denote the variance of the random variable X_k obtained by observing the random process X(t) at time t_k. The variance σ_X^2 is related to the mean-square value of X_k as follows

$$\sigma_X^2 = E[X_k^2] - \mu_X^2$$

where $\mu_X = E[X_k]$. Since the process X(t) has zero mean, it follows that

$$\sigma_X^2 = E[X_k^2]$$

Next we note that

$$E[X_k^2] = \int_{-\infty}^{\infty} S_X(f)df$$

We may therefore define the variance σ_X^2 as the total area under the power spectral density $S_X(f)$ as

$$\sigma_X^2 = \int_{-\infty}^{\infty} S_X(f)df \tag{1}$$

Thus with the mean $\mu_X = 0$ and the variance σ_X^2 defined by Eq. (1), we may express the probability density function of X_k as follows

$$f_{X_k}(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left(-\frac{x^2}{2\sigma_X^2}\right)$$

Problem 1.19

The input-output relation of a full-wave rectifier is defined by

$$Y(t_k) = |X(t_k)| = \begin{cases} X(t_k), & X(t_k) \geq 0 \\ -X(t_k), & X(t_k) \leq 0 \end{cases}.$$

The probability density function of the random variable $X(t_k)$, obtained by observing the input random process at time t_k, is defined by

$$f_{X(t_k)}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp(-\frac{x^2}{2\sigma^2})$$

To find the probability density function of the random variable $Y(t_k)$, obtained by observing the output random process, we need an expression for the inverse relation defining $X(t_k)$ in terms of $Y(t_k)$. We note that a given value of $Y(t_k)$ corresponds to 2 values of $X(t_k)$, of equal magnitude and opposite sign. We may therefore write

$$X(t_k) = -Y(t_k), \qquad X(t_k) < 0$$

$$X(t_k) = Y(t_k), \qquad X(t_k) > 0 .$$

In both cases, we have

$$\left| \frac{dX(t_k)}{dY(t_k)} \right| = 1 .$$

The probability density function of $Y(t_k)$ is therefore given by

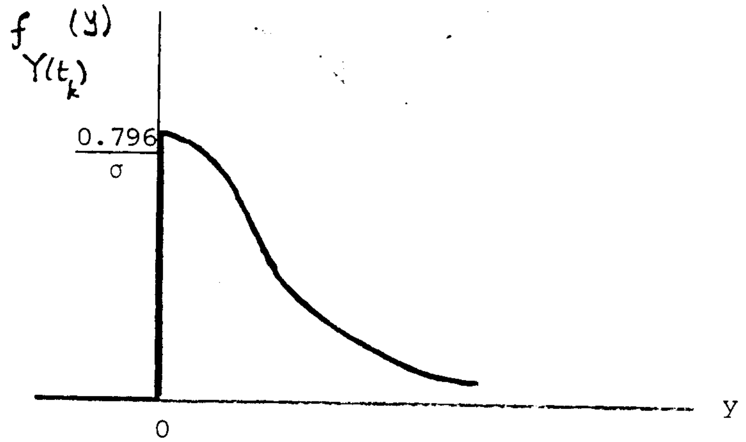
$$f_{Y(t_k)}(y) = f_{X(t_k)}(x = -y) \cdot \left| \frac{dX(t_k)}{dY(t_k)} \right| + f_{X(t_k)}(x = y) \cdot \left| \frac{dX(t_k)}{dY(t_k)} \right|$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \exp(-\frac{y^2}{2\sigma^2})$$

We may therefore write

$$f_{Y(t_k)}(y) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \exp(-\frac{y^2}{2\sigma^2}), & y \geq 0 \\ 0, & y < 0 . \end{cases}$$

which is illustrated below:



Problem 1.20

(a) The probability density function of the random variable $Y(t_k)$, obtained by observing the rectifier output $Y(t)$ at time t_k, is

$$f_{Y(t_k)}(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}\ \sigma_X} \exp(-\dfrac{y^2}{2\sigma_X}) , & y \geq 0 \\ 0 , & y < 0 \end{cases}$$

where $\sigma_X^2 = E[X^2(t_k)] - \{E[X(t_k)]\}^2$

$= E[X^2(t_k)]$

$= R_X(0)$

The mean value of $Y(t_k)$ is therefore

$$E[Y(t_k)] = \int_{-\infty}^{\infty} y\ f_{Y(t_k)}(y)\ dy$$

$$= \frac{1}{\sqrt{2\pi}\ \sigma_X} \int_0^{\infty} \sqrt{y}\ \exp(-\frac{y^2}{2\sigma_X^2})\ dy \qquad (1)$$

Put

$$\frac{y}{\sigma_X^2} = u^2$$

Then, we may rewrite Eq. (1) as

$$E[Y(t_k)] = \sqrt{\frac{2}{\pi}}\, \sigma_X^2 \int_0^\infty u^2 \exp(-\frac{u^2}{2})\, du$$

$$= \sigma_X^2$$

$$= R_X(0)$$

(b) The autocorrelation function of Y(t) is

$$R_Y(\tau) = E[Y(t+\tau)\, Y(t)]$$

Since $Y(t) = X^2(t)$, we have

$$R_Y(\tau) = E[X^2(t+\tau)\, X^2(t)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2\, x_2^2\, f_{X(t_k+\tau),X(t_k)}(x_1,x_2)\, dx_1\, dx_2 \qquad (2)$$

The $X(t_k+\tau)$ and $X(t_k)$ are jointly Gaussian with a joint probability density function defined by

$$f_{X(t_k+\tau),X(t_k)}(x_1, x_2) = \frac{1}{2\pi\, \sigma_X^2 \sqrt{1-\rho_X^2(\tau)}} \exp[-\frac{x_1^2 - 2\,\rho_X(\tau)\, x_1\, x_2 + x_2^2}{2\sigma_X^2\, (1-\rho_X^2(\tau))}]$$

where $\sigma_X^2 = R_X(0)$,

$$\rho_X(\tau) = \frac{Cov[X(t_k+\tau) X(t_k)]}{\sigma_X^2},$$

$$= \frac{R_X(\tau)}{R_X(0)}$$

Rewrite Eq. (2) in the form:

$$R_Y(\tau) = \frac{1}{2\pi\, \sigma_X^2 \sqrt{1-\rho_X^2(\tau)}} \int_{-\infty}^{\infty} x_2^2 \exp(-\frac{x_2^2}{2\sigma_X^2})\, g(x_2)\, dx_2 \qquad (3)$$

where

$$g(x_2) = \int_{-\infty}^{\infty} x_1^2 \exp\{-\frac{[x_1 - \rho_X(\tau)\, x_2]^2}{2\sigma_X^2[1 - \rho_X^2(\tau)]}\}\, dx_1$$

Let

$$u = \frac{x_1 - \rho_X(\tau)\ x_2}{\sigma_X \sqrt{1-\rho_X^2(\tau)}}$$

Then, we may express $g(x_2)$ in the form

$$g(x_2) = \sigma_X \sqrt{1-\rho_X^2(\tau)} \int_{-\infty}^{\infty} \exp(-\frac{u^2}{2})\ \{\rho_X^2(\tau)\ x_2^2 + \sigma_X^2[1 - \rho_X^2(\tau)]u^2 + 2\sigma_X\, \rho_X \sqrt{1-\rho_X^2(\tau)}\ ux_2\}\ du$$

However, we note that

$$\int_{-\infty}^{\infty} \exp(-\frac{u^2}{2})\ du = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} u \exp(-\frac{u^2}{2})\ du = 0$$

$$\int_{-\infty}^{\infty} u^2 \exp(-\frac{u^2}{2})\ du = \sqrt{2\pi}$$

Hence,

$$g(x_2) = \sigma_X \sqrt{2\pi[1-\rho_X^2(\tau)]}\ \{\rho_X^2(\tau)\ x_2^2 + \sigma_X^2\ [1 - \rho_X^2(\tau)]\}$$

Thus, from Eq. (3):

$$R_Y(\tau) = \frac{1}{\sqrt{2\pi}\ \sigma_X} \int_{-\infty}^{\infty} x_2^2 \exp(-\frac{x_2^2}{2\sigma_X^2})\ \{\rho_X^2(\tau)\ x_2^2 + \sigma_X^2\ [1-\rho_X^2(\tau)]\}\ dx_2$$

Using the results:

$$\int_{-\infty}^{\infty} x_2^2 \exp(-\frac{x_2^2}{2\sigma_X^2})\ dx_2 = \sqrt{2\pi}\ \sigma_X^3$$

$$\int_{-\infty}^{\infty} x_2^4 \exp(-\frac{x_2^2}{2\sigma_X^2})\ dx_2 = 3\sqrt{2\pi}\ \sigma_X^5$$

we obtain,

$$R_Y(\tau) = 3\sigma_X^4 \rho_X^2(\tau) + \sigma_X^4 [1 - \rho_X^2(\tau)]$$

$$= \sigma_X^4 [1 + 2\rho_X^2(\tau)]$$

Since $\sigma_X^2 = R_X(0)$

$$\rho_X(\tau) = \frac{R_X(\tau)}{R_X(0)}$$

we obtain

$$R_Y(\tau) = R_X^2(0) [1 + 2 \frac{R_X^2(\tau)}{R_X^2(0)}]$$

$$= R_X^2(0) + 2R_X^2(\tau)$$

The autocovariance function of Y(t) is therefore

$$C_Y(\tau) = R_Y(\tau) - \{E[Y(t_k)]\}^2$$
$$= R_X^2(0) + 2R_X^2(\tau) - R_X^2(0)$$
$$= 2R_X^2(\tau)$$

Problem 1.21

(a) The random variable $Y(t_1)$ obtained by observing the filter output of impulse response $h_1(t)$, at time t_1, is given by

$$Y(t_1) = \int_{-\infty}^{\infty} X(t_1-\tau)\, h_1(\tau)\, d\tau$$

The expected value of $Y(t_1)$ is

$$m_{Y_1} = E[Y(t_1)]$$

$$= H_1(0)\, m_X$$

where

$$H_1(0) = \int_{-\infty}^{\infty} h_1(\tau)\, d\tau$$

The random variable $Z(t_2)$ obtained by observing the filter output of impulse response $h_2(t)$, at time t_2, is given by

$$Z(t_2) = \int_{-\infty}^{\infty} X(t_2-u)\, h_2(u)\, du$$

The expected value of $Z(t_2)$ is

$$m_{Z_2} = E[z(t_2)]$$

$$= H_2(0)\, m_X$$

where

$$H_2(0) = \int_{-\infty}^{\infty} h_2(u)\, du$$

The covariance of $Y(t_1)$ and $Z(t_2)$ is

$$\mathrm{Cov}[Y(t_1)Z(t_2)] = E[(Y(t_1) - \mu_{Y_1})(Z(t_2) - \mu_{Z_2})]$$

$$= E[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (X(t_1-\tau) - \mu_X)\ (X(t_2-u) - \mu_X)\ h_1(\tau)\ h_2(u)\ d\tau\ du]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[(X(t_1-\tau) - \mu_X)\ (X(t_2-u) - \mu_X)]\ h_1(\tau)\ h_2(u)\ d\tau\ du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{C}_X(t_1-t_2-\tau+u)\ h_1(\tau)\ h_2(u)\ d\tau\ du$$

where $\mathcal{C}_X(\tau)$ is the autocovariance function of $X(t)$. Next, we note that the variance of $Y(t_1)$ is

$$\sigma^2_{Y_1} = E[(Y(t_1) - \mu_{Y_1})^2]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{C}_X(\tau-u)\ h_1(\tau)\ h_1(u)\ d\tau\ du$$

and the variance of $Z(t_2)$ is

$$\sigma^2_{Z_2} = E[(Z(t_2) - \mu_{Z_2})^2]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{C}_X(\tau-u)\ h_2(\tau)\ h_2(u)\ d\tau\ du$$

The correlation coefficient of $Y(t_1)$ and $Z(t_2)$ is

$$\rho = \frac{\mathrm{cov}[Y(t_1)Z(t_2)]}{\sigma_{Y_1}\ \sigma_{Z_2}}$$

Since $X(t)$ is a Gaussian process, it follows that $Y(t_1)$ and $Z(t_2)$ are jointly Gaussian with a probability density function given by

$$f_{Y(t_1),Z(t_2)}(y_1,z_2) = K \exp[-Q(y_1,z_2)]$$

where

$$K = \frac{1}{2\pi\sigma_{Y_1}\sigma_{z_2}\sqrt{1-\rho^2}}$$

$$Q(y_1,z_2) = \frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1-\mu_{Y1}}{\sigma_{Y_1}}\right)^2 - 2\rho\left(\frac{y_1-\mu_{Y1}}{\sigma_{Y_1}}\right)\left(\frac{z_2-\mu_{Z2}}{\sigma_{Z_2}}\right) + \left(\frac{z_2-\mu_{Z2}}{\sigma_{Z_2}}\right)^2\right]$$

(b) The random variables $Y(t_1)$ and $Z(t_2)$ are uncorrelated if and only if their covariance is zero. Since $Y(t)$ and $Z(t)$ are jointly Gaussian processes, it follows that $Y(t_1)$ and $Z(t_2)$ are statistically independent if $\mathrm{Cov}[Y(t_1)Z(t_2)]$ is zero. Therefore, the necessary and sufficient condition for $Y(t_1)$ and $Z(t_2)$ to be statistically independent is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_X(t_1-t_2-\tau+u)\ h_1(\tau)\ h_2(u)\ d\tau\ du = 0$$

for choices of t_1 and t_2.

Problem 1.22

(a) The filter output is

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

$$= \frac{1}{T} \int_{0}^{T} X(T-\tau)\, d\tau$$

Put $T-\tau=u$. Then, the sample value of $Y(t)$ at $t=T$ equals

$$Y = \frac{1}{T} \int_{0}^{T} X(u)\, du$$

The mean of Y is therefore

$$E[Y] = E[\frac{1}{T} \int_{0}^{T} X(u)\, du]$$

$$= \frac{1}{T} \int_{0}^{T} E[X(u)]\, du$$

$$= 0$$

The variance of Y is

$$\sigma_Y^2 = E[Y^2] - \{E[Y]\}^2$$

$$= R_Y(0)$$

$$= \int_{-\infty}^{\infty} S_Y(f)\, df$$

$$= \int_{-\infty}^{\infty} S_X(f)\, |H(f)|^2\, df$$

But

$$H(f) = \int_{-\infty}^{\infty} h(t) \exp(-j2\pi ft)\, dt$$

$$= \frac{1}{T} \int_0^T \exp(-j2\pi ft)\, dt$$

$$= \frac{1}{T} \left[\frac{\exp(-j2\pi ft)}{-j2\pi f}\right]_0^T$$

$$= \frac{1}{j2\pi fT} [1 - \exp(-j2\pi fT)]$$

$$= \text{sinc}(fT) \exp(-j\pi fT)$$

Therefore,

$$\sigma_Y^2 = \int_{-\infty}^{\infty} S_X(f)\, \text{sinc}^2(fT)\, df$$

(b) Since the filter input is Gaussian, it follows that Y is also Gaussian. Hence, the probability density function of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\, \sigma_Y} \exp\left(-\frac{y^2}{2\sigma_Y^2}\right)$$

where σ_Y^2 is defined above.

Problem 1.23

(a) The power spectral density of the noise at the filter output is given by

$$S_N(f) = \frac{N_0}{2} \left| \frac{j2\pi fL}{R+j2\pi fL} \right|^2$$

$$S_N(f) = \frac{N_0}{2} \frac{(2\pi f L/R)^2}{1+(2\pi f L/R)^2}$$

$$= \frac{N_0}{2} [1 - \frac{1}{1+(2\pi f L/R)^2}]$$

The autocorrelation function of the filter output is therefore

$$R_N(\tau) = \frac{N_0}{2} [\delta(\tau) - \frac{R}{2L} \exp(-\frac{R}{L} |\tau|)]$$

(b) The mean of the filter output is equal to H(0) times the mean of the filter input. The process at the filter input has zero mean. The value H(0) of the filter's transfer function H(f) is zero. It follows therefore that the filter output also has a zero mean.

The mean-square value of the filter output is equal to $R_N(0)$. With zero mean, it follows therefore that the variance of the filter output is

$$\sigma_N^2 = R_N(0)$$

Since $R_N(\tau)$ contains a delta function $\delta(\tau)$ centered on $\tau = 0$, we find that, in theory, σ_N^2 is infinitely large.

Problem 1.24

(a) The noise equivalent bandwidth is

$$W_N = \frac{1/2}{|H(0)|^2} \int_{-\infty}^{\infty} |H(f)|^2 \, df$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{df}{1 + (f/f_0)^{2n}}$$

$$= \int_{0}^{\infty} \frac{df}{1 + (f/f_0)^{2n}}$$

$$= \frac{\pi f_0}{2n \sin(\pi/2n)}$$

$$= \frac{f_0}{\mathrm{sinc}(1/2n)}$$

(b) When the filter order n approaches infinity, we have

$$W_N = f_0 \lim_{n \to \infty} \frac{1}{\mathrm{sinc}(1/2n)}$$

$$= f_0$$

Problem 1.25

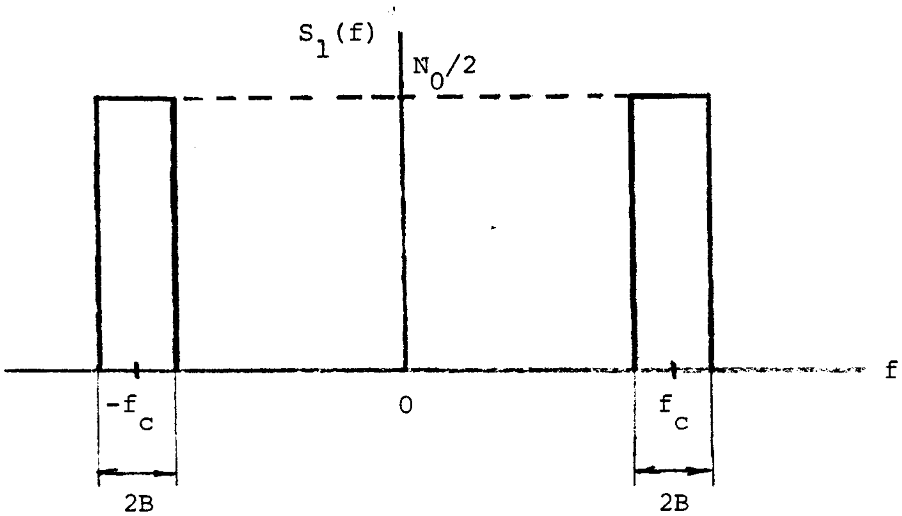
The process X(t) defined by

$$X(t) = \sum_{k=-\infty}^{\infty} h(t - \tau_k),$$

where $h(t - \tau_k)$ is a current pulse at time τ_k, is stationary for the following simple reason. There is no distinguishing origin of time.

Problem 1.26

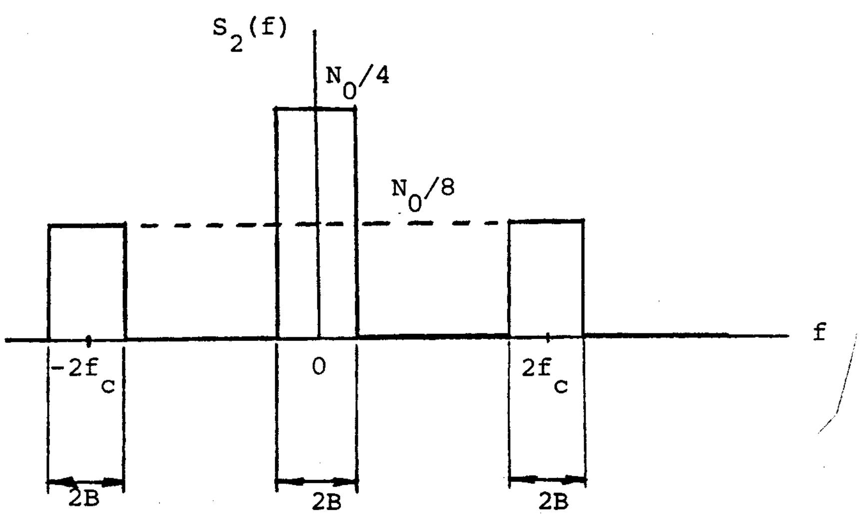
(a) Let $S_1(f)$ denote the power spectral density of the noise at the first filter output. The dependence of $S_1(f)$ on frequency is illustrated below:



Let $S_2(f)$ denote the power spectral density of the noise at the mixer output. Then, we may write

$$S_2(f) = \frac{1}{4}[S_1(f+f_c) + S_1(f-f_c)]$$

which is illustrated below:



The power spectral density of the noise n(t) at the second filter output is therefore defined by

$$S_0(f) = \begin{cases} \dfrac{N_0}{4}, & -B<f<B \\ 0, & \text{otherwise} \end{cases}$$

The autocorrelation function of the noise n(t) is

$$R_o(\tau) = \frac{N_0 B}{2} \operatorname{sinc}(2B\tau)$$

(b) The mean value of the noise at the system output is zero. Hence, the variance and mean-square value of this noise are the same. Now, the total area under $S_0(f)$ is equal to $(N_0/4)(2B) = N_0B/2$. The variance of the noise at the system output is therefore $N_0B/2$.

(c) The maximum rate at which n(t) can be sampled for the resulting samples to be uncorrelated is 2B samples per second.

Problem 1.27

(a) The autocorrelation function of the filter output is

$$R_X(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, R_W(\tau-\tau_1+\tau_2)\, d\tau_1\, d\tau_2$$

Since $R_W(\tau) = (N_0/2)\,\delta(\tau)$, we find that the impulse response h(t) of the filter must satisfy the condition:

$$R_X(\tau) = \frac{N_0}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, \delta(\tau-\tau_1+\tau_2)\, d\tau_1\, d\tau_2$$

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} h(\tau+\tau_2)\, h(\tau_2)\, d\tau_2$$

(b) For the filter output to have a power spectral density equal to $S_X(f)$, we have to choose the transfer function H(f) of the filter such that

$$S_X(f) = \frac{N_0}{2} |H(f)|^2$$

or

$$|H(f)| = \sqrt{\frac{2S_X(f)}{N_0}}$$

Problem 1.28

(a) Consider the part of the analyzer in Fig. 1.19 defining the in-phase component $n_I(t)$, reproduced here as Fig. 1:

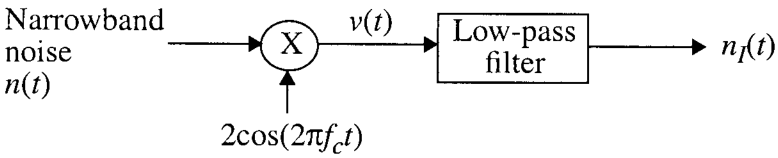


Figure 1

For the multiplier output, we have

$$v(t) = 2n(t)\cos(2\pi f_c t)$$

Applying Eq. (1.55) in the textbook, we therefore get

$$S_V(f) = [S_N(f - f_c) + S_N(f + f_c)]$$

Passing $v(t)$ through an ideal low-pass filter of bandwidth B, defined as one-half the bandwidth of the narrowband noise $n(t)$, we obtain

$$S_{N_I}(f) = \begin{cases} S_V(f) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} S_N(f - f_c) + S_N(f + f_c) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

For the quadrature component, we have the system shown in Fig. 2:

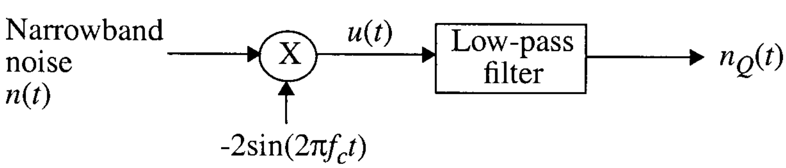


Fig. 2

The multiplier output $u(t)$ is given by

$$u(t) = -2n(t)\sin(2\pi f_c t)$$

Hence,

$$S_U(f) = [S_N(f - f_c) + S_N(f + f_c)]$$

and

$$S_{N_Q}(f) = \begin{cases} S_U(f) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} S_N(f - f_c) + S_N(f + f_c) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

Accordingly, from Eqs. (1) and (2) we have

$$S_{N_I}(f) = S_{N_Q}(f)$$

(b) Applying Eq. (1.78) of the textbook to Figs. 1 and 2, we obtain

$$S_{N_I N_Q}(f) = |H(f)|^2 S_{VU}(f) \qquad (3)$$

where

$$|H(f)| = \begin{cases} 1 & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

Applying Eq. (1.23) of the textbook to the problem at hand:

$$R_{VU}(\tau) = 2R_N(\tau)\sin(2\pi f_c \tau) = \frac{1}{j}R_N(\tau)(e^{j2\pi f_c \tau} - e^{-j2\pi f_c \tau})$$

Applying the Fourier transform to both sides of this relation:

$$S_{VU}(t) = \frac{1}{j}(S_N(f - f_c) - S_N(f + f_c)) \qquad (4)$$

Substituting Eq. (4) into (3):

$$S_{N_I N_Q}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)] & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

which is the desired result.

Problem 1.29

If the power spectral density $S_N(f)$ of narrowband noise $n(t)$ is symmetric about the midband frequency f_c we then have

$$S_N(f-f_c) = S_N(f+f_c) \quad \text{for} \quad -B \le f \le B$$

From part (b) of Problem 1.28, the cross-spectral densities between the in-phase noise component $n_I(t)$ and quadrature noise component $n_Q(t)$ are zero for all frequencies:

$$S_{N_I N_Q}(f) = 0 \quad \text{for all} \quad f$$

This, in turn, means that the cross-correlation functions $R_{N_I N_Q}(\tau)$ and $R_{N_Q N_I}(\tau)$ are both zero, that is,

$$E[N_I(t_k+\tau)N_Q(t_k)] = 0$$

which states that the random variables $N_I(t_k + \tau)$ and $N_Q(t_k)$, obtained by observing $n_I(t)$ at time $t_k + \tau$ and observing $n_Q(t)$ at time t_k, are orthogonal for all t.

If the narrow-band noise $n(t)$ is Gaussian, with zero mean (by virtue of the narrowband nature of $n(t)$), then it follows that both $N_I(t_k + \tau)$ and $N_Q(t_k)$ are also Gaussian with zero mean. We thus conclude the following:

- $N_I(t_k + \tau)$ and $N_Q(t_k)$ are both uncorrelated
- Being Gaussian and uncorrelated, $N_I(t_k + \tau)$ and $N_Q(t_k)$ are therefore statistically independent.

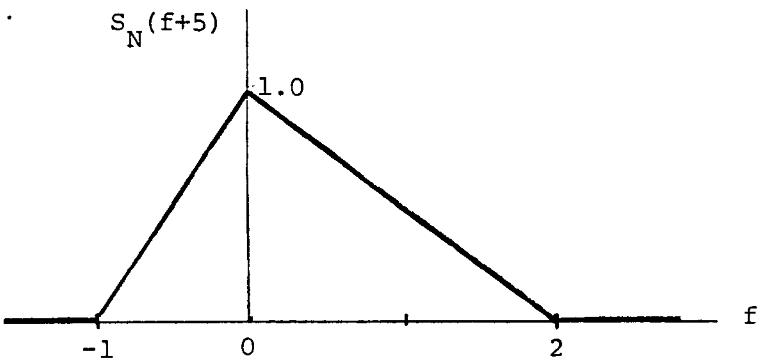
That is, the in-phase noise component $n_I(t)$ and quadrature noise component $n_Q(t)$ are statistically independent.

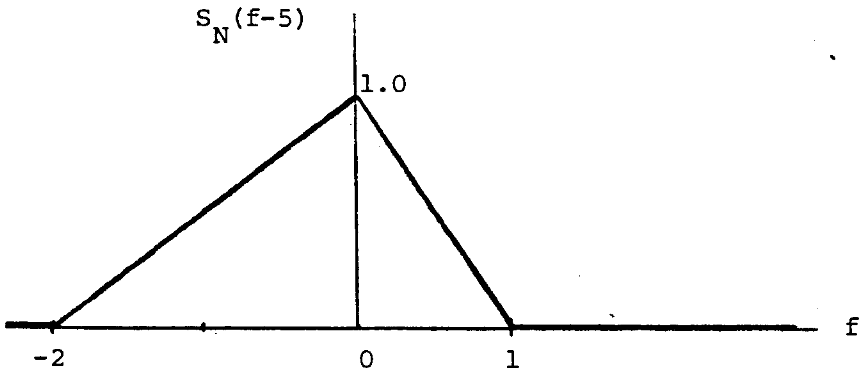
Problem 1.30

(a) The power spectral density of the in-phase component or quadrature component is defined by

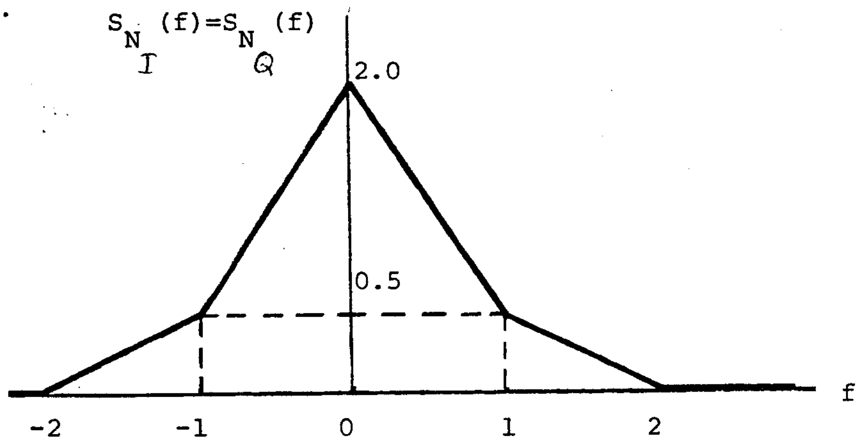
$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f+f_c) + S_N(f-f_c), & -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

We note that, for $-2 \le f \le 2$, the $S_N(f+5)$ and $S_N(f-5)$ are as shown below:





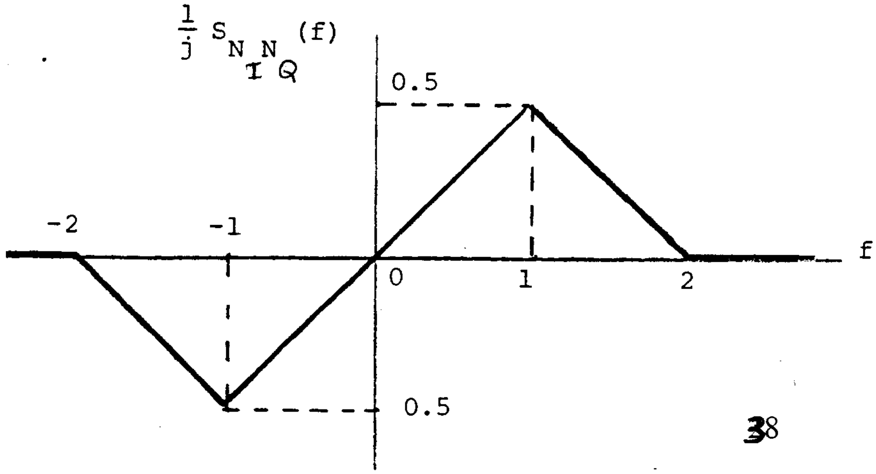
We thus find that $S_{N_I}(f)$ or $S_{N_Q}(f)$ is as shown below:



(b) The cross-spectral density $S_{N_I N_Q}(f)$ is defined by

$$S_{N_I N_Q}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)] \,, & -B \leq f \leq B \\ 0 \,, & \text{otherwise} \end{cases}$$

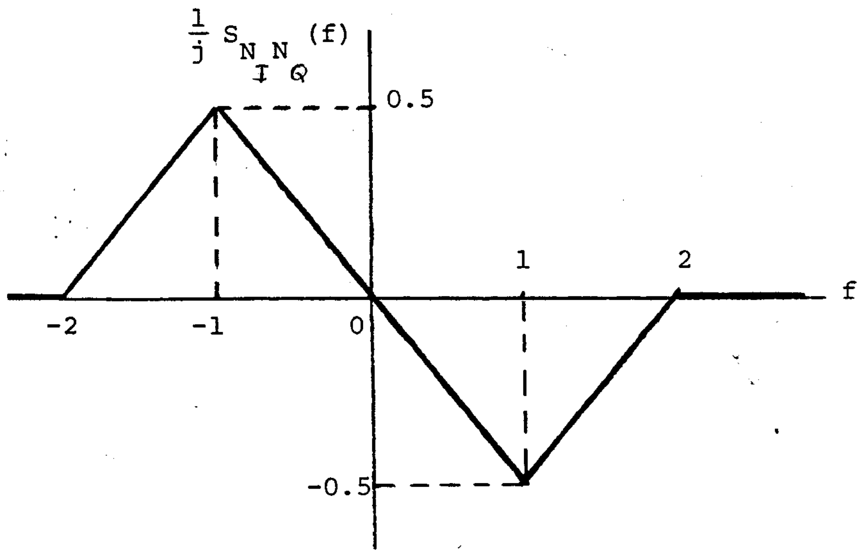
We therefore find that $S_{N_I N_Q}(f)/j$ is as shown below:



Next, we note that

$$S_{N_I N_Q}(f) = S^*_{N_I N_Q}(f)$$

We thus find that $S_{N_I N_Q}(f)$ is as shown below:



Problem 1.31

(a) Express the noise n(t) in terms of its in-phase and quadrature components as follows:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

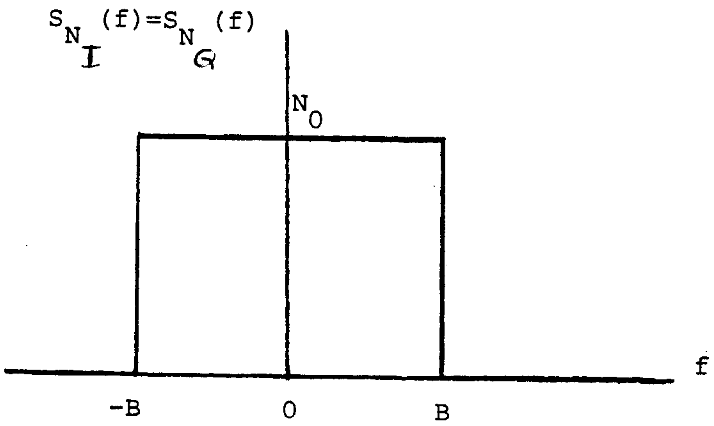
The envelope of n(t) is

$$r(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$$

which is Rayleigh-distributed. That is

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2}\exp(-\frac{r^2}{2\sigma^2}), & r \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

To evaluate the variance σ^2, we note that the power spectral density of $n_I(t)$ or $n_Q(t)$ is as follows



Since the mean of n(t) is zero, we find that

$$\sigma^2 = 2\,N_0 B$$

Therefore,

$$f_R(r) = \begin{cases} \frac{r}{2N_0B}\exp(-\frac{r^2}{4N_0B}), & r \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) The mean value of the envelope is equal to $\sqrt{\pi N_0 B}$, and its variance is equal to $0.858\ N_0 B$.

Problem 1.32

Autocorrelation of a Sinusoidal Wave Plus White Gaussian Noise

In this computer experiment, we study the statistical characterization of a random process $X(t)$ consisting of a sinusoidal wave component $A\cos(2\pi f_c t + \Theta)$ and a white Gaussian noise process $W(t)$ of zero mean and power spectral density $N_0/2$. That is, we have

$$X(t) = A\cos 2\pi f_c t + \Theta + W(t) \qquad (1)$$

where Θ is a uniformly distributed random variable over the interval$(-\pi,\pi)$. Clearly, the two components of the process $X(t)$ are independent. The autocorrelation function of $X(t)$ is therefore the sum of the individual autocorrelation functions of the signal (sinusoidal wave) component and the noise component, as shown by

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau) + \frac{N_0}{2}\delta(\tau) \qquad (2)$$

This equation shows that for $|\tau| > 0$, the autocorrelation function $R_X(\tau)$ has the same sinusoidal waveform as the signal component. We may generalize this result by stating that the presence of a periodic signal component corrupted by additive white noise can be detected by computing the autocorrelation function of the composite process $X(t)$.

The purpose of the experiment described here is to perform this computation using two different methods: (a) ensemble averaging, and (b) time averaging. The signal of interest consists of a sinusoidal signal of frequency $f_c = 0.002$ and phase $\theta = -\pi/2$, truncated to a finite duration T = 1000; the amplitude A of the sinusoidal signal is set to $\sqrt{2}$ to give unit average power. A particular realization $x(t)$ of the random process $X(t)$ consists of this sinusoidal signal and additive white Gaussian noise; the power spectral density of the noise for this realization is $(N_0/2) = 1000$. The original sinusoidal is barely recognizable in $x(t)$.

(a) For *ensemble-average computation* of the autocorrelation function, we may proceed as follows:

- Compute the product $x(t+\tau)x(t)$ for some fixed time t and specified time shift τ, where $x(t)$ is a particular realization of the random process $X(t)$.
- Repeat the computation of the product $x(t+\tau)x(t)$ for M independent realizations (i.e., sample functions) of the random process X(t).
- Compute the average of these computations over M.
- Repeat this sequence of computations for different values of τ.

The results of this computation are plotted in Fig. 1 for $M = 50$ realizations. The picture portrayed here is in perfect agreement with theory defined by Eq. (2). The important point to note here is that the ensemble-averaging process yields a clean estimate of the true autocorrelation function $R_X(\tau)$ of the random process $X(t)$. Moreover, the presence of the sinusoidal signal is clearly visible in the plot of $R_X(\tau)$ versus t.

(b) For the time-average estimation of the autocorrelation function of the process $X(t)$, we invoke ergodicity and use the formula

$$R_X(\tau) = \lim_{T \to \infty} R_x(\tau, T) \tag{3}$$

where $R_x(\tau,T)$ is the time-averaged autocorrelation function:

$$R_x(\tau, T) = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)dt \tag{4}$$

The $x(t)$ in Eq. (4) is a particular realization of the process $X(t)$, and $2T$ is the total observation interval. Define the *time-windowed function*

$$x_T(t) = \begin{cases} x(t), & -T \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

We may then rewrite Eq. (4) as

$$R_x(\tau, T) = \frac{1}{2T}\int_{-\infty}^{\infty} x_T(t+\tau)x_T(t)dt \tag{6}$$

For a specified time shift τ, we may compute $R_x(\tau,T)$ directly using Eq. (6). However, from a computational viewpoint, it is more efficient to use an *indirect* method based on Fourier transformation. First, we note From Eq. (6) that the time-averaged autocorrelation function $R_x(\tau,T)$ may be viewed as a scaled form of convolution in the τ-domain as follows:

$$R_x(\tau, T) = \frac{1}{2T}\int_{-\infty}^{\infty} x_T(\tau) \star\ x_T(-\tau) \tag{7}$$

where the star denotes convolution and $x_T(\tau)$ is simply the time-windowed function $x_T(t)$ with t replaced by τ. Let $X_T(f)$ denote the Fourier transform $x_T(\tau)$; note that $X_T(f)$ is the same as the Fourier transform $X(f,T)$. Since convolution in the τ-domain is transformed into multiplication in the frequency domain, we have the Fourier-transform pair:

$$R_x(\tau, T) = \frac{1}{2T}\int_{-\infty}^{\infty} |X_T(f)|^2 \tag{8}$$

The parameter $|X_T(f)|^2/2T$ is recognized as the periodogram of the process $X(t)$. Equation (8) is a mathematical description of the *correlation theorem*, which may be formally stated as follows: *The time-averaged autocorrelation function of a sample function pertaining to a random process and its periodogram, based on that sample function, constitute a Fourier-transform pair.*

We are now ready to describe the indirect method for computing the time-averaged autocorrelation function $R_x(\tau,T)$:

- Compute the Fourier transform $X_T(f)$ of time-windowed function $x_T(\tau)$.
- Compute the periodogram $|X_T(f)|^2/2T$.
- Compute the inverse Fourier transform of $|X_T(f)|^2/2T$.

To perform these calculations on a digital computer, the customary procedure is to use the fast Fourier transform (FFT) algorithm. With $x_T(\tau)$ uniformly sampled, the computational procedure described herein yields the desired values of $R_x(\tau,T)$ for $\tau = 0, \Delta, 2\Delta, \cdots, (N-1)\Delta$ where Δ is the sampling period and N is the total number of samples used in the computation. Figure 2 presents the results obtained in the time-averaging approach of "estimating" the autocorrelation function $R_X(\tau)$ using the indirect method for the same set of parameters as those used for the ensemble-averaged results of Fig. 1. The symbol $\hat{R}_X(\tau)$ is used to emphasize the fact that the computation described here results in an "estimate" of the autocorrelation function $R_X(\tau)$. The results presented in Fig. 2 are for a signal-to-noise ratio of + 10dB, which is defined by

$$\text{SNR} = \frac{A^2/2}{N_0/(2T)} = \frac{A^2 T}{N_0} \tag{9}$$

On the basis of the results presented in Figures 1 and 2 we may make the following observations:

- The ensemble-averaging and time-averaging approaches yield similar results for the autocorrelation function $R_X(\tau)$, signifying the fact that the random process $X(t)$ described herein is indeed ergodic.
- The indirect time-averaging approach, based on the FFT algorithm, provides an efficient method for the estimation of $R_X(\tau)$ using a digital computer.
- As the SNR is increased, the numerical accuracy of the estimation is improved, which is intuitively satisfying.

1 Problem 1.32

Matlab codes

```
% Problem 1.32a CS: Haykin
% Ensemble average autocorrelation
% M. Sellathurai


clear all
A=sqrt(2);
N=1000; M=1; SNRdb=0;
e_corrf_f=zeros(1,1000);
f_c=2/N;
t=0:1:N-1;

for trial=1:M

% signal
s=cos(2*pi*f_c*t);

%noise
snr = 10^(SNRdb/10);
wn = (randn(1,length(s)))/sqrt(snr)/sqrt(2);

%signal plus noise
s=s+wn;

% autocorrelation
[e_corrf]=en_corr(s,s, N);

%Ensemble-averaged autocorrelation
e_corrf_f=e_corrf_f+e_corrf;
end

%prints
plot(-500:500-1,e_corrf_f/M);
xlabel('(\tau)')
ylabel('R_X(\tau)')
```

```
% Problem 1.32b CS: Haykin
% time-averaged estimation of autocorrelation
% M. Sellathurai


clear all
A=sqrt(2);
N=1000; SNRdb=0;
f_c=2/N;
t=0:1:N-1;

% signal
s=cos(2*pi*f_c*t);%noise

%noise
snr = 10^(SNRdb/10);
wn = (randn(1,length(s)))/sqrt(snr)/sqrt(2);

%signal plus noise
s=s+wn;

% time -averaged autocorrelation
[e_corrf]=time_corr(s,N);

%prints
plot(-500:500-1,e_corrf);
xlabel('(\tau)')
ylabel('R_X(\tau)')
```

```
function [corrf]=en_corr(u, v, N)% funtion to compute the autocorreation/ cross-correlati
% ensemble average
% used in problem 1.32, CS: Haykin
% M. Sellathurai, 10 june 1999.

max_cross_corr=0;
tt=length(u);

for m=0:tt
shifted_u=[u(m+1:tt) u(1:m)];
corr(m+1)=(sum(v.*shifted_u))/(N/2);
if (abs(corr)>max_cross_corr)
max_cross_corr=abs(corr);
end
end

corr1=flipud(corr);
corrf=[corr1(501:tt) corr(1:500)];
```

```
function [corrf]=time_corr(s,N)
% funtion to compute the autocorreation/ cross-correlation
% time average
% used in problem 1.32, CS: Haykin
% M. Sellathurai, 10 june 1999.

X=fft(s);
X1=fftshift((abs(X).^2)/(N/2));
corrf=(fftshift(abs(ifft(X1))));
```

Answer to Problem 1.32

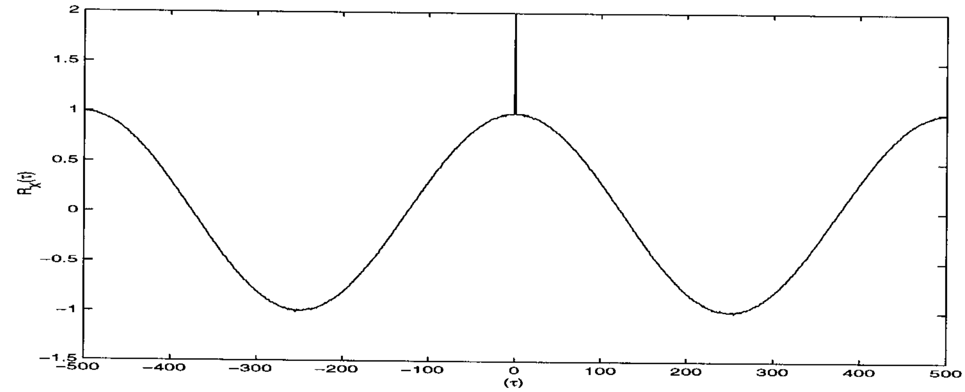


Figure 1: Ensemble averaging

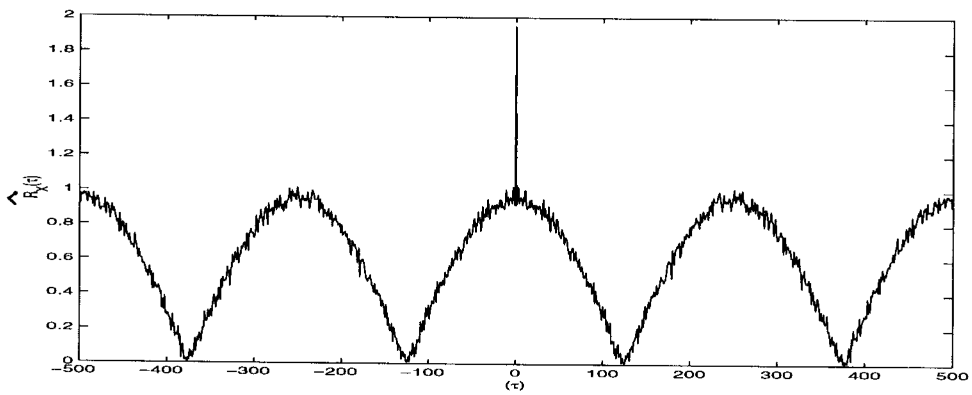


Figure 2: Time averaging

Problem 1.33

Matlab codes

```
% Problem 1.33 CS: Haykin
% multipath channel
% M. Sellathurai


clear all
Nf=0;Xf=0; % initializing counters

N=10000; % number of samples
M=2;  P=10;

a=1; % line of sight component component

for i=1:P

A=sqrt(randn(N,M).^2 + randn(N,M).^2);

xi=A.*cos(cos(rand(N,M)*2*pi) + rand(N,M)*2*pi); % inphase cpmponent
xq=A.*sin(cos(rand(N,M)*2*pi) + rand(N,M)*2*pi); % quadrature phase component

xi=(sum(xi'));
xq=(sum(xq'));

ra=sqrt((xi+a).^2+ xq.^2) ; % rayleigh, rician fading

[h X]=hist(ra,50);

Nf=Nf+h;
Xf=Xf+X;

end

Nf=Nf/(P);
Xf=Xf/(P);

% print
plot(Xf,Nf/(sum(Xf.*Nf)/20))
xlabel('v')
ylabel('f_v(v)')
```

Answer to Problem 1.33

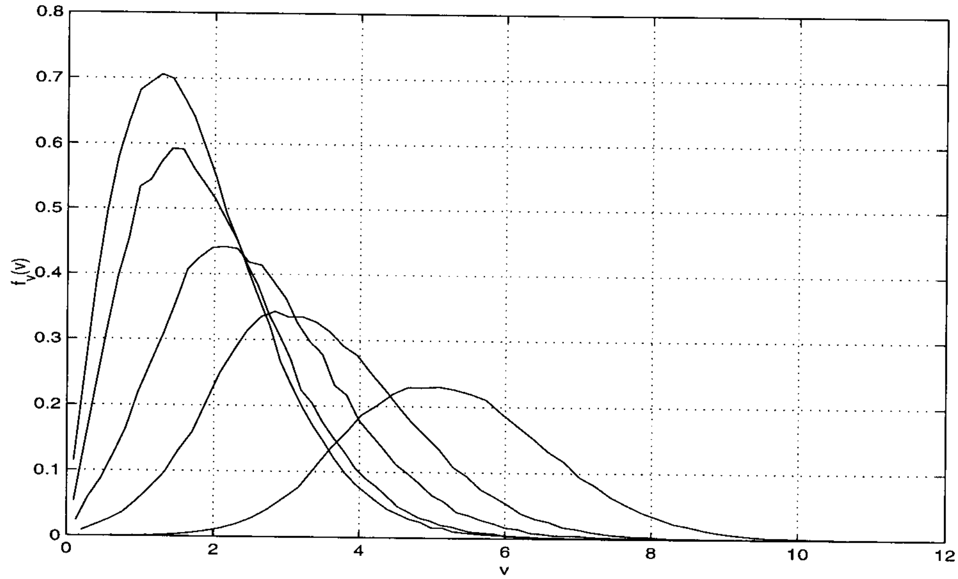


Figure 1 Rician distribution

CHAPTER 2

Continuous-wave Modulation

Problem 2.1

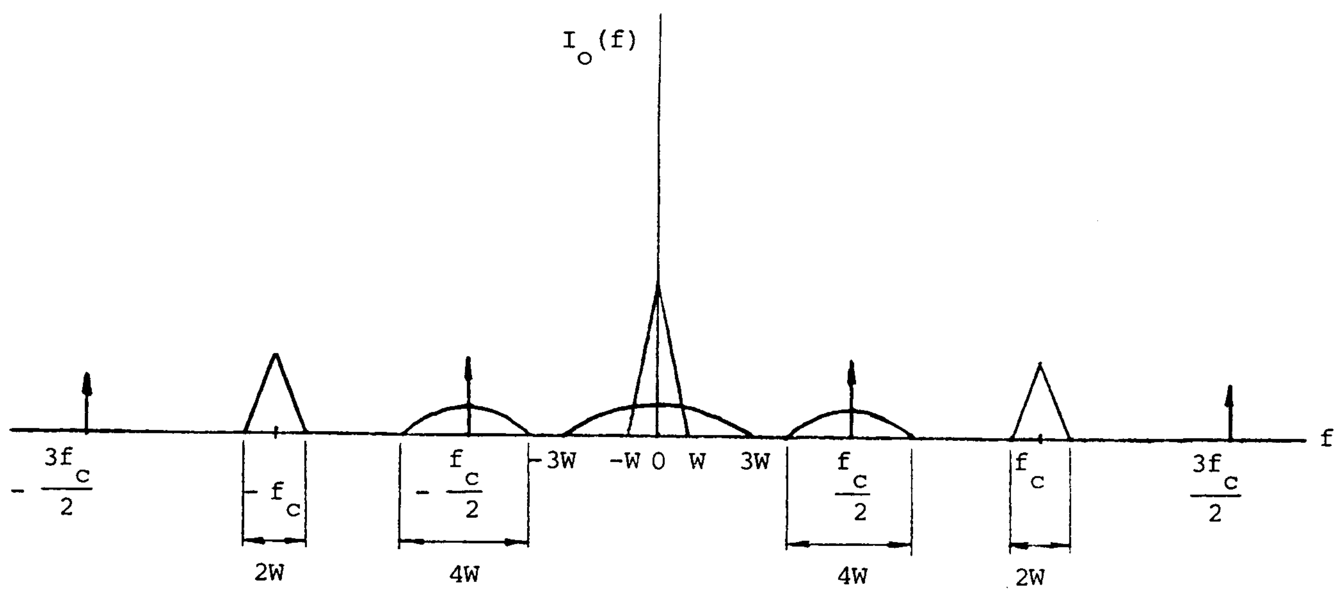
(a) Let the input voltage v_i consist of a sinusoidal wave of frequency $\frac{1}{2} f_c$ (i.e., half the desired carrier frequency) and the message signal m(t):

$$v_i = A_c \cos(\pi f_c t) + m(t)$$

Then, the output current i_o is

$$\begin{aligned} i_o &= a_1 v_i + a_3 v_i^3 \\ &= a_1[A_c\cos(\pi f_c t)+m(t)]+a_3[A_c\cos(\pi f_c t)+m(t)]^3 \\ &= a_1[A_c\cos(\pi f_c t)+m(t)] + \frac{1}{4}a_3A_c^3\,[\cos(3\pi f_c t)+3\cos(\pi f_c t)] \\ &\quad + \frac{3}{2}a_3A_c^2\, m(t)[1+\cos(2\pi f_c t)] + 3a_3A_c\cos(\pi f_c t)m^2(t) + a_3m^3(t) \end{aligned}$$

Assume that m(t) occupies the frequency interval $-W \le f \le W$. Then, the amplitude spectrum of the output current i_o is as shown below.

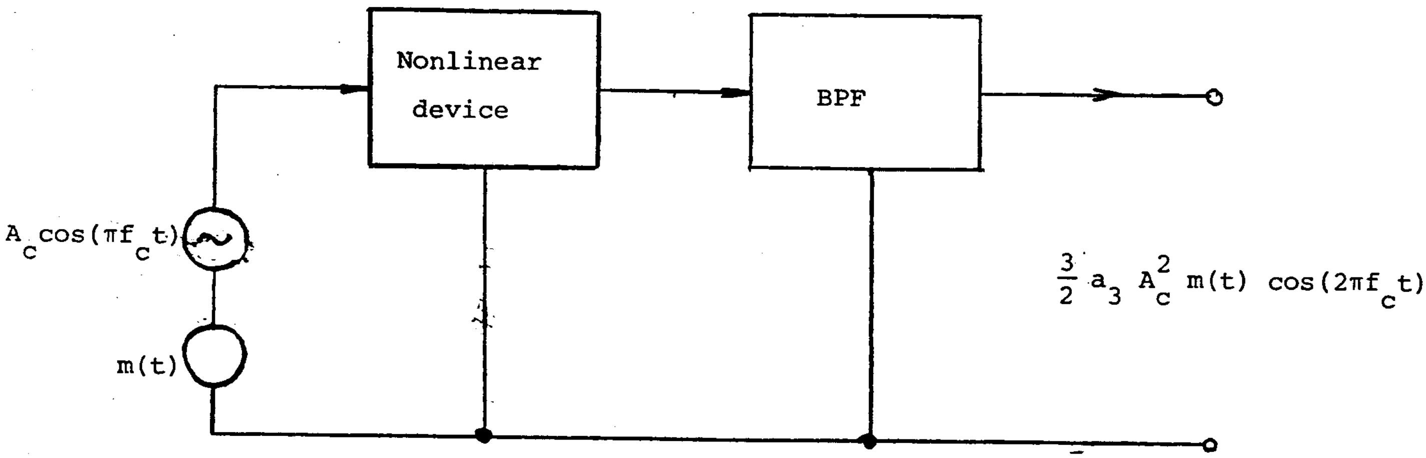


From this diagram we see that in order to extract a DSBSC wave, with carrier frequency f_c from i_o, we need a band-pass filter with mid-band frequency f_c and bandwidth 2W, which satisfy the requirement:

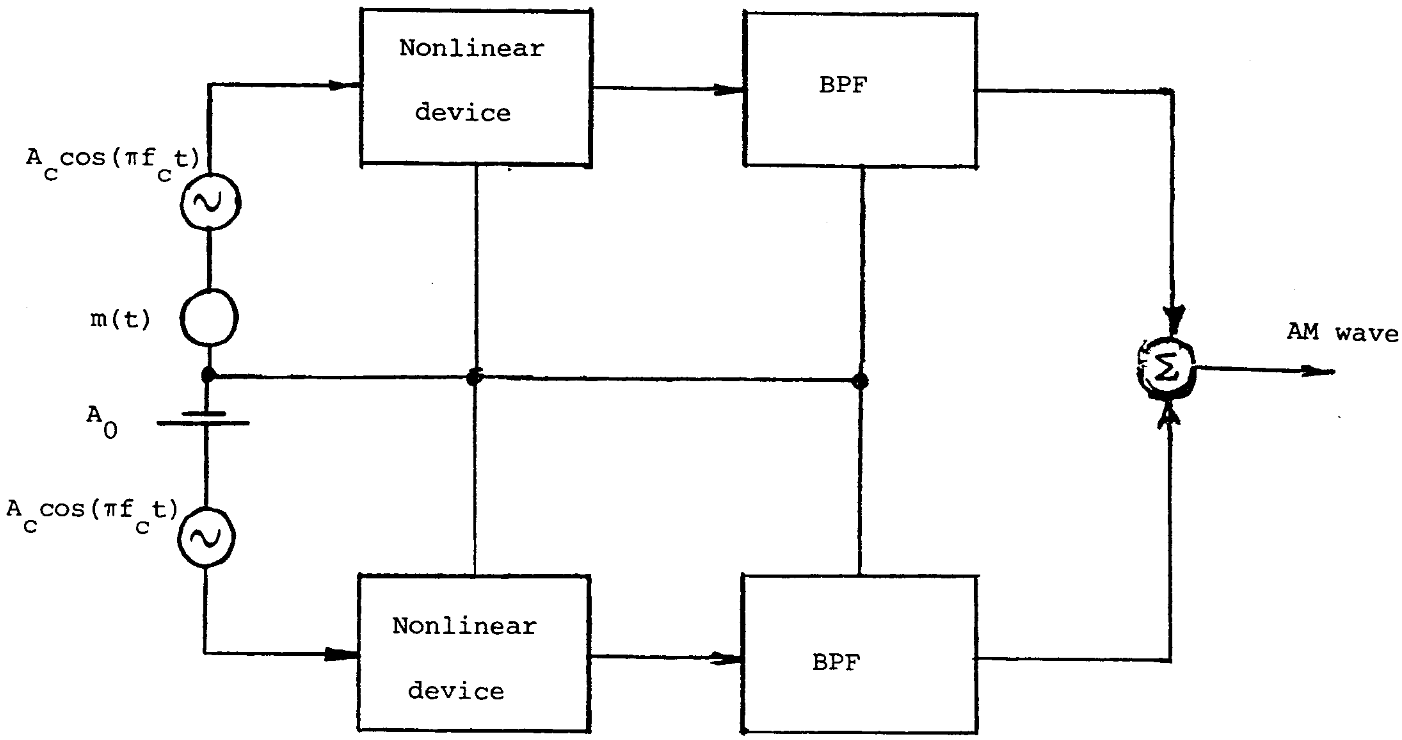
$$f_c - W > \frac{f_c}{2} + 2W$$

that is, $f_c > 6W$

Therefore, to use the given nonlinear device as a product mmodulator, we may use the following configuration:



(b) To generate an AM wave with carrier frequency f_c we require a sinusoidal component of frequency f_c to be added to the DSBSC generated in the manner described above. To achieve this requirement, we may use the following configuration involving a pair of the nonlinear devices and a pair of identical band-pass filters.



The resulting AM wave is therefore $\frac{3}{2}a_3 A_c^2[A_0+m(t)]\cos(2\pi f_c t)$. Thus, the choice of the dc level A_0 at the input of the lower branch controls the percentage modulation of the AM wave.

Problem 2.2

Consider the square-law characteristic:

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t) \tag{1}$$

where a_1 and a_2 are constants. Let

$$v_1(t) = A_c\cos(2\pi f_c t) + m(t) \tag{2}$$

Therefore substitutingEq. (2) into (1), and expanding terms:

$$v_2(t) = a_1 A_c\left[1 + \frac{2a_2}{A_1} m(t)\right]\cos(2\pi f_c t)$$
$$+ a_1 m(t) + a_2 m^2(t) + a_2 A_c^2 \cos^2(2\pi f_c t) \tag{3}$$

The first term in Eq. (3) is the desired AM signal with $k_a = 2a_2/a_1$. The remaining three terms are unwanted terms that are removed by filtering.

Let the modulating wave m(t) be limited to the band $-W \le f \le W$, as in Fig. 1(a). Then, from Eq. (3) we find that the amplitude spectrum $|V_2(f)|$ is as shown in Fig. 1(b). It follows therefore that the unwanted terms may be removed from $v_2(t)$ by designing the tuned filter at the modulator output of Fig. P2.2 to have a mid-band frequency f_c and bandwidth 2W, which satisfy the requirement that $f_c > 3W$.

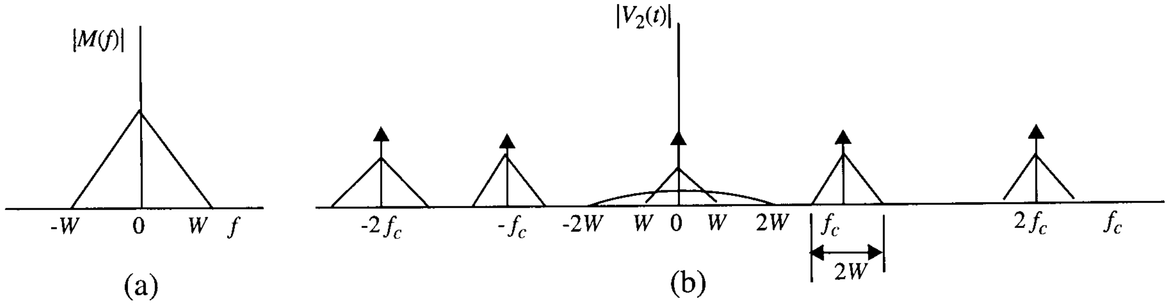


Figure 1

Problem 2.3

The generation of an AM wave may be accomplished using various devices; here we describe one such device called a *switching modulator.* Details of this modulator are shown in Fig. P2.3a, where it is assumed that the carrier wave $c(t)$ applied to the diode is large in amplitude, so that it swings right across the characteristic curve of the diode. We assume that the diode acts as an *ideal switch*, that is, it presents zero impedance when it is forward-biased [corresponding to $c(t) > 0$]. We may thus approximate the transfer characteristic of the diode-load resistor combination by a *piecewise-linear characteristic*, as shown in Fig. P2.3b. Accordingly, for an input voltage $v_1(t)$ consisting of the sum of the carrier and the message signal:

$$v_1(t) = A_c\cos(2\pi f_c t) + m(t) \tag{1}$$

where $|m(t)| << A_c$, the resulting load voltage $v_2(t)$ is

$$v_2(t) \approx \begin{cases} v_1(t), & c(t) > 0 \\ 0, & c(t) < 0 \end{cases} \tag{2}$$

That is, the load voltage $v_2(t)$ varies periodically between the values $v_1(t)$ and zero at a rate equal to the carrier frequency f_c. In this way, by assuming a modulating wave that is weak compared with the carrier wave, we have effectively replaced the nonlinear behavior of the diode by an approximately equivalent piecewise-linear time-varying operation.

We may express Eq. (2) mathematically as

$$v_2(t) \approx A_c \cos(2\pi f_c t) + m(t) g_{T_0}(t) \tag{3}$$

where $g_{T_0}(t)$ is a periodic pulse train of duty cycle equal to one-half, and period $T_0 = 1/f_c$, as in Fig. 1. Representing this $g_{T_0}(t)$ by its Fourier series, we have

$$g_{T_0}(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos[2\pi f_c t(2n-1)] \tag{4}$$

Therefore, substituting Eq. (4) in (3), we find that the load voltage $v_2(t)$ consists of the sum of two components:

1. The component

$$\frac{A_c}{2}\left[1 - \frac{4}{\pi A_c}m(t)\right]\cos(2\pi f_c t)$$

which is the desired AM wave with amplitude sensitivity $k_a = 4\pi A_c$. The switching modulator is therefore made more sensitive by reducing the carrier amplitude A_c; however, it must be maintained large enough to make the diode act like an ideal switch.

2. Unwanted components, the spectrum of which contains delta functions at 0, $\pm 2f_c$, $\pm 4f_c$, and so on, and which occupy frequency intervals of width $2W$ centered at 0, $\pm 3f_c$. $\pm 5f_c$, and so on, where W is the message bandwidth.

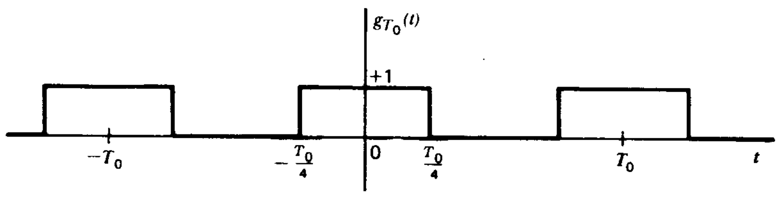


Fig. 1: Periodic pulse train

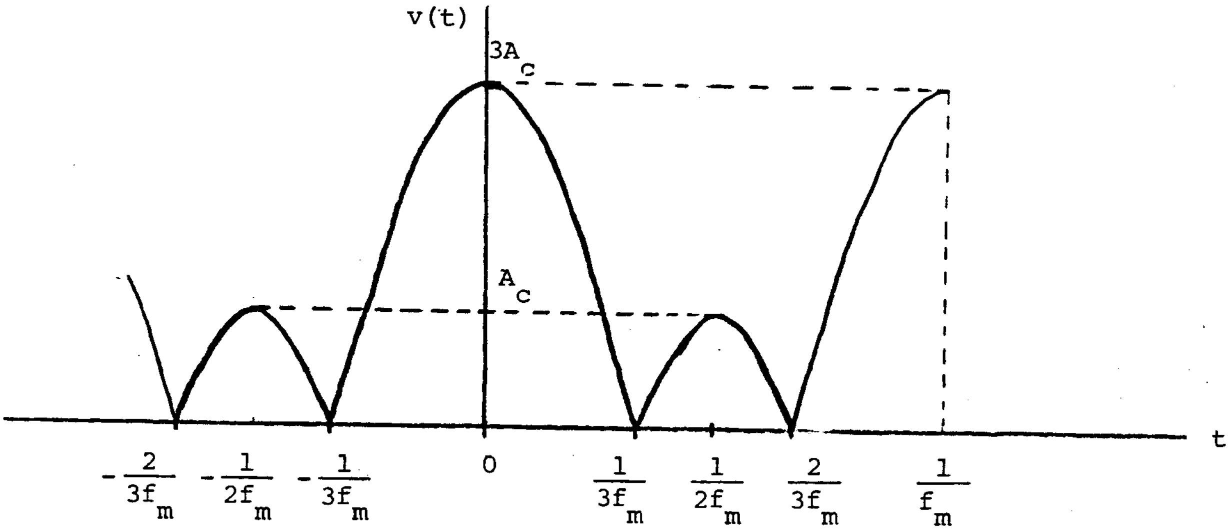
The unwanted terms are removed from the load voltage $v_2(t)$ by means of a band-pass filter with mid-band frequency f_c and bandwidth $2W$, provided that $f_c > 2W$. This latter condition ensures that the frequency separations between the desired AM wave the unwanted components are large enough for the band-pass filter to suppress the unwanted components.

Problem 2.4

(a) The envelope detector output is

$$v(t) = A_c|1+ \mu\cos(2\pi f_m t)|$$

which is illustrated below for the case when $\mu=2$.



We see that v(t) is periodic with a period equal to f_m, and an even function of t, and so we may express v(t) in the form:

$$v(t) = a_0 + 2 \sum_{n=1}^{\infty} a_n \cos(2n\pi f_m t)$$

where $$a_0 = 2f_m \int_0^{1/2f_m} v(t)dt$$

$$= 2A_c f_m \int_0^{1/3f_m} [1+2\cos(2\pi f_m t)]dt + 2A_c f_m \int_{1/3f_m}^{1/2f_m} [-1-2\cos(2\pi f_m t)]dt$$

$$= \frac{A_c}{3} + \frac{4A_c}{\pi} \sin(\frac{2\pi}{3}) \qquad (1)$$

$$a_n = 2f_m \int_0^{1/2f_m} v(t)\cos(2n\pi f_m t)\,dt$$

$$= 2A_c f_m \int_0^{1/3f_m} [1+2\cos(2\pi f_m t)]\cos(2n\pi f_m t)\,dt$$

$$+ 2A_c f_m \int_{1/3f_m}^{1/2f_m} [-1-2\cos(2\pi f_m t)]\cos(2n\pi f_m t)\,dt$$

$$= \frac{A_c}{n\pi}\left[2\sin\left(\frac{2n\pi}{3}\right) - \sin(n\pi)\right] + \frac{A_c}{(n+1)\pi}\left\{2\sin\left[\frac{2\pi}{3}(n+1)\right] - \sin[\pi(n+1)]\right\}$$

$$+ \frac{A_c}{(n-1)\pi}\left\{2\sin\left[\frac{2\pi}{3}(n-1)\right] - \sin[\pi(n-1)]\right\} \qquad (2)$$

For n=0, Eq. (2) reduces to that shown in Eq. (1).

(b) For n=1, Eq. (2) yields

$$a_1 = A_c\left(\frac{\sqrt{3}}{2\pi} + \frac{1}{3}\right)$$

For n=2, it yields

$$a_2 = \frac{A_c\sqrt{3}}{2\pi}$$

Therefore, the ratio of second-harmonic amplitude to fundamental amplitude in v(t) is

$$\frac{a_2}{a_1} = \frac{3\sqrt{3}}{2\pi+3\sqrt{3}} = 0.452$$

Problem 2.5

(a) The demodulation of an AM wave can be accomplished using various devices; here, we describe a simple and yet highly effective device known as the *envelope detector.* Some version of this demodulator is used in almost all commercial AM radio receivers. For it to function properly, however, the AM wave has to be narrow-band, which requires that the carrier frequency be large compared to the message bandwidth. Moreover, the percentage modulation must be less than 100 percent.

An envelope detector of the series type is shown in Fig. P2.5, which consists of a diode and a resistor-capacitor (RC) filter. The operation of this envelope detector is as follows. On a positive half-cycle of the input signal, the diode is forward-biased and the capacitor C charges up rapidly to the peak value of the input signal. When the input signal falls below this value, the diode becomes reverse-biased and the capacitor C discharges slowly through the load resistor R_l. The discharging process continues until the next positive half-cycle. When the input signal becomes greater than the voltage across the capacitor, the diode conducts again and the process is repeated. We assume that the diode is ideal, presenting resistance r_f to current flow in the forward-biased region and infinite resistance in the reverse-biased region. We further assume that the AM wave applied to the envelope detector is supplied by a voltage source of internal impedance R_s. The charging time constant $(r_f + R_s)\, C$ must be short compared with the carrier period $1/f_c$, that is

$$(r_f + R_s)C \ll \frac{1}{f_c} \tag{1}$$

so that the capacitor C charges rapidly and thereby follows the applied voltage up to the positive peak when the diode is conducting.

(b) The discharging time constant R_lC must be long enough to ensure that the capacitor discharges slowly through the load resistor R_l between positive peaks of the carrier wave, but not so long that the capacitor voltage will not discharge at the maximum rate of change of the modulating wave, that is

$$\frac{1}{f_c} \ll R_lC \ll \frac{1}{W} \tag{2}$$

where W is the message bandwidth. The result is that the capacitor voltage or detector output is nearly the same as the envelope of the AM wave.

Problem 2.6

Let

$$v_1(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

(a) Then the output of the square-law device is

$$\begin{aligned} v_2(t) &= a_1 v_1(t) + a_2 v_1^2(t) \\ &= a_1 A_c[1 + k_a m(t)]\cos(2\pi f_c t) \\ &+ \frac{1}{2} a_2 A_c^2 [1 + 2k_a m(t) + k_a^2 m^2(t)]\,[1 + \cos(4\pi f_c t)] \end{aligned}$$

(b) The desired signal, namely $a_2 A_c^2 k_a m(t)$, is due to the $a_2 v_1^2(t)$ - hence, the name "square-law detection". This component can be extracted by means of a low-pass filter. This is not the only contribution within the baseband spectrum, because the term $1/2\ a_2 A_c^2 k_a^2 m^2(t)$ will give rise to a plurality of similar frequency components. The ratio of wanted signal to distortion is $2/k_a m(t)$. To make this ratio large, the percentage modulation, that is, $|k_a m(t)|$ should be kept small compared with unity.

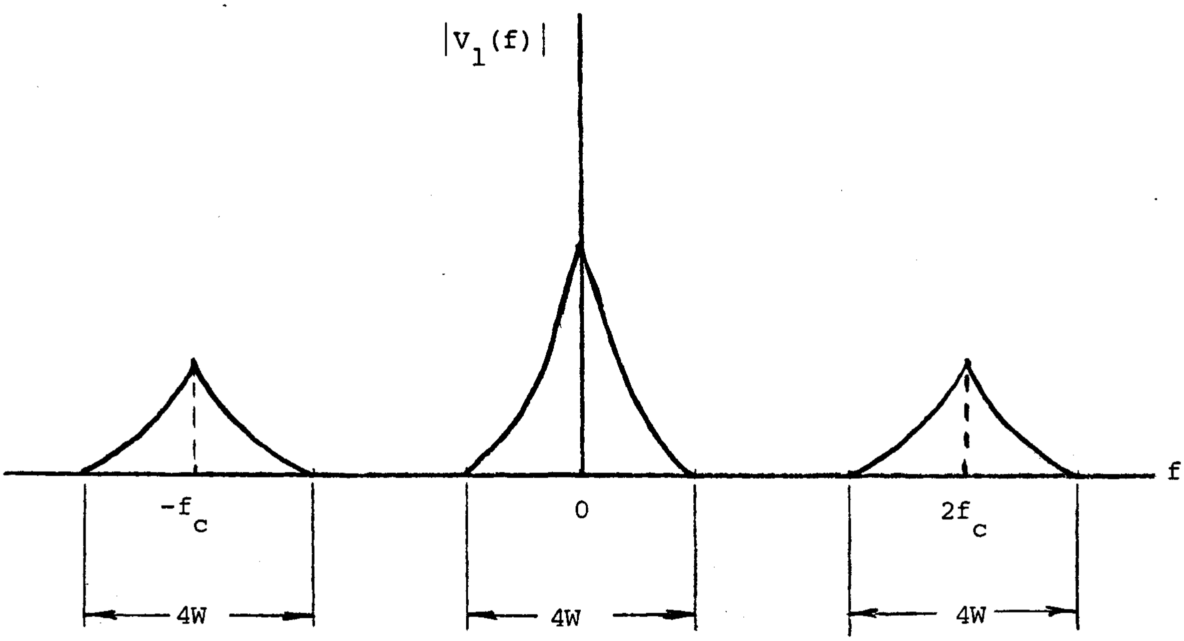
Problem 2.7

The squarer output is

$$v_1(t) = A_c^2\,[1+k_a m(t)]^2 \cos^2(2\pi f_c t)$$

$$= \frac{A_c^2}{2}\,[1+2k_a m(t)+m^2(t)][1+\cos(4\pi f_c t)]$$

The amplitude spectrum of $v_1(t)$ is therefore as follows, assuming that m(t) is limited to the interval $-W \leq f \leq W$:



Since $f_c > 2W$, we find that $2f_c-2W > 2W$. Therefore, by choosing the cutoff frequency of the low-pass filter greater than 2W, but less than $2f_c-2W$, we obtain the output

$$v_2(t) = \frac{A_c^2}{2}\,[1+k_a m(t)]^2$$

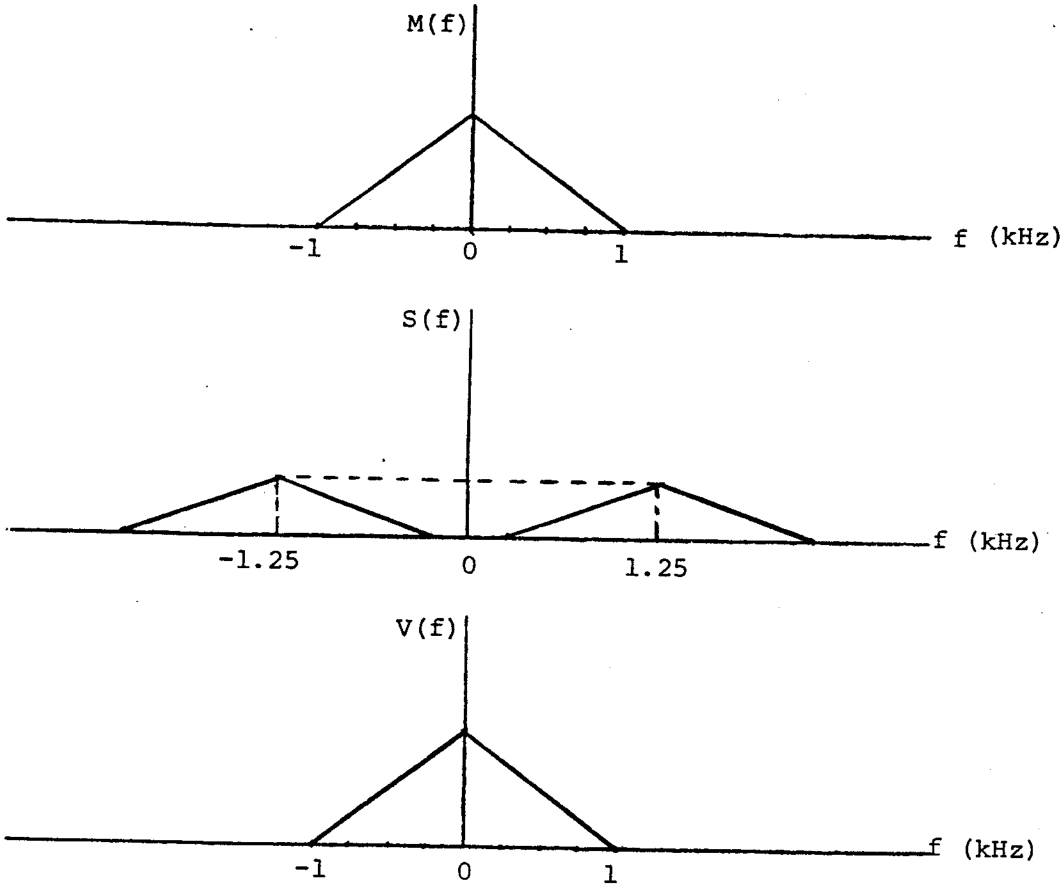
Hence, the square-rooter output is

$$v_3(t) = \frac{A_c}{\sqrt{2}}\,[1+k_a m(t)]$$

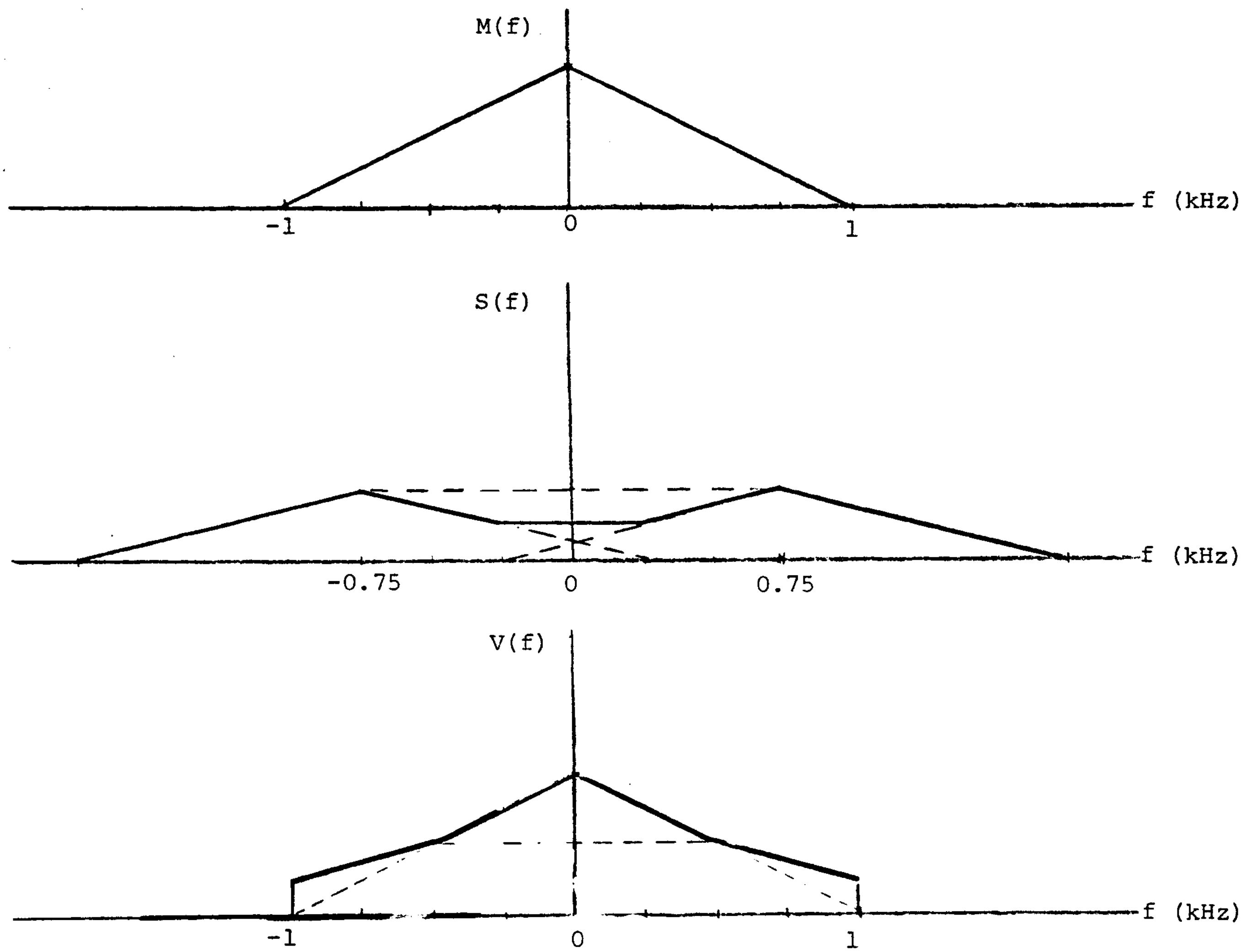
which, except for the dc component $\frac{A_c}{\sqrt{2}}$, is proportional to the message signal m(t).

Problem 2.8

(a) For f_c = 1.25 kHz, the spectra of the message signal m(t), the product modulator output s(t), and the coherent detector output v(t) are as follows, respectively:



(b) For the case when $f_c = 0.75$, the respective spectra are as follows:



To avoid sideband-overlap, the carrier frequency f_c must be greater than or equal to 1 kHz. The lowest carrier frequency is therefore 1 kHz for each sideband of the modulated wave s(t) to be uniquely determined by m(t).

Problem 2.9

The two AM modulator outputs are

$$s_1(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

$$s_2(t) = A_c[1 - k_a m(t)]\cos(2\pi f_c t)$$

Subtracting $s_2(t)$ from $s_1(t)$:

$$s(t) = s_2(t) - s_1(t)$$

$$= 2k_a A_c m(t)\cos(2\pi f_c t)$$

which represents a DSB-SC modulated wave.

Problem 2.10

(a) Multiplying the signal by the local oscillator gives:

$$s_1(t) = A_c m(t)\cos(2\pi f_c t)\cos[2\pi(f_c+\Delta f)t]$$

$$= \frac{A_c}{2} m(t)\{\cos(2\pi\Delta f t) + \cos[2\pi(2f_c+\Delta f)t]\}$$

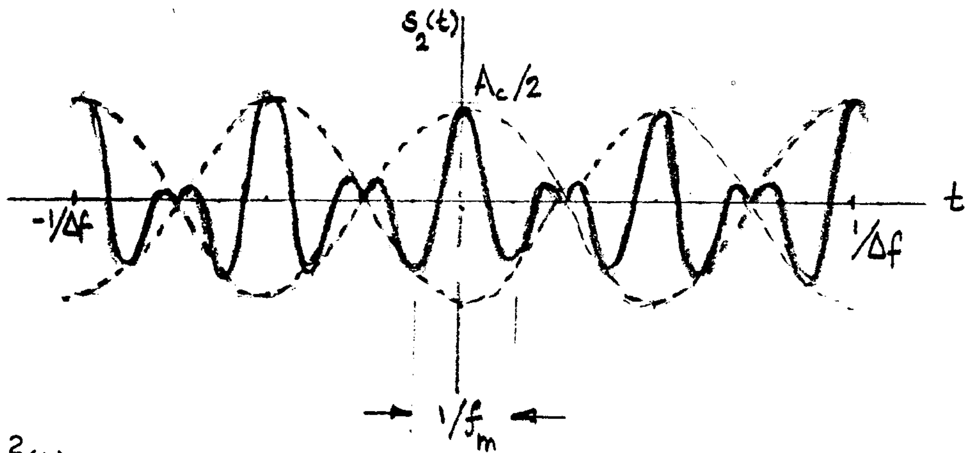
Low pass filtering leaves:

$$s_2(t) = \frac{A_c}{2} m(t)\cos(2\pi\Delta f t)$$

Thus the output signal is the message signal modulated by a sinusoid of frequency Δf.

(b) If $m(t) = \cos(2\pi f_m t)$,

then $s_2(t) = \frac{A_c}{2}\cos(2\pi f_m t)\cos(2\pi\Delta f t)$



Problem 2.11

(a) $y(t) = s^2(t)$

$$= A_c^2\cos^2(2\pi f_c t)m^2(t)$$

$$= \frac{A_c^2}{2}[1+\cos(4\pi f_c t)]m^2(t)$$

Therefore, the spectrum of the multiplier output is

$$Y(f) = \frac{A_c^2}{2}\int_{-\infty}^{\infty} M(\lambda)M(f-\lambda)d\lambda + \frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} M(\lambda)M(f-2f_c-\lambda)d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(f+2f_c-\lambda)d\lambda\right]$$

where $M(f) = F[m(t)]$.

(b) At $f=2f_c$, we have

$$Y(2f_c) = \frac{A_c^2}{2} \int_{-\infty}^{\infty} M(\lambda)M(2f_c-\lambda)d\lambda$$

$$+ \frac{A_c^2}{4} [\int_{-\infty}^{\infty} M(\lambda)M(-\lambda)d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(4f_c-\lambda)d\lambda]$$

Since $M(-\lambda) = M^*(\lambda)$, we may write

$$Y(2f_c) = \frac{A_c^2}{2} \int_{-\infty}^{\infty} M(\lambda)M(2f_c-\lambda)d\lambda$$

$$+ \frac{A_c^2}{4} [\int_{-\infty}^{\infty} |M(\lambda)|^2 d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(4f_c-\lambda)d\lambda] \quad (1)$$

With m(t) limited to $-W \leq f \leq W$ and $f_c > W$, we find that the first and third integrals reduce to zero, and so we may simplify Eq. (1) as follows

$$Y(2f_c) = \frac{A_c^2}{4} \int_{-\infty}^{\infty} |M(\lambda)|^2 d\lambda$$

$$= \frac{A_c^2 E}{4}$$

where E is the signal energy (by Rayleigh's energy theorem). Similarly, we find that

$$Y(-2f_c) = \frac{A_c^2}{4} E$$

The band-pass filter output, in the frequency domain, is therefore defined by

$$V(f) \simeq \frac{A_c^2}{4} E \ \Delta f[\delta(f-2f_c) + \delta(f+2f_c)]$$

Hence,

$$v(t) \simeq \frac{A_c^2}{2} E \ \Delta f \cos(4\pi f_c t)$$

Problem 2.12

The multiplexed signal is

$$s(t) = A_c\, m_1(t)\, \cos(2\pi f_c t) + A_c\, m_2(t)\, \sin(2\pi f_c t)$$

Therefore,

$$S(f) = \frac{A_c}{2} [M_1(f-f_c)+M_1(f+f_c)] + \frac{A_c}{2j} [M_2(f-f_c)-M_2(f+f_c)]$$

where $M_1(f) = F[m_1(t)]$ and $M_2(f) = F[m_2(t)]$. The spectrum of the received signal is therefore

$$R(f) = H(f)S(f)$$

$$= \frac{A_c}{2} H(f)\, [M_1(f-f_c)+M_1(f+f_c)+ \frac{1}{j}\, M_2(f-f_c)- \frac{1}{j} M_2(f+f_c)]$$

To recover $m_1(t)$, we multiply $r(t)$, the inverse Fourier transform of $R(f)$, by $\cos(2\pi f_c t)$ and then pass the resulting output through a low-pass filter, producing a signal with the following spectrum

$$F[r(t)\cos(2\pi f_c t)] = \frac{1}{2} [R(f-f_c)+R(f+f_c)]$$

$$= \frac{A_c}{4} H(f-f_c)[M_1(f-2f_c) + M_1(f) + \frac{1}{j} M_2(f-2f_c) - \frac{1}{j} M_2(f)]$$

$$+ \frac{A_c}{4} H(f+f_c)[M_1(f) + M_1(f+2f_c) + \frac{1}{j} M_2(f) - \frac{1}{j} M_2(f+2f_c)] \qquad (1)$$

The condition $H(f_c+f) = H^*(f_c-f)$ is equivalent to $H(f+f_c)=H(f-f_c)$; this follows from the fact that for a real-valued impulse response $h(t)$, we have $H(-f)=H^*(f)$. Hence, substituting this condition in Eq. (1), we get

$$F[r(t)\cos(2\pi f_c t)] = \frac{A_c}{2} H(f-f_c)M_1(f)$$

$$+ \frac{A_c}{4} H(f-f_c)[M_1(f-2f_c) + \frac{1}{j} M_2(f-2f_c)+M_1(f+2f_c) - \frac{1}{j} M_2(f+2f_c)]$$

The low-pass filter output, therefore, has a spectrum equal to $(A_c/2)\, H(f-f_c)M_1(f)$.

Similarly, to recover $m_2(t)$, we multiply $r(t)$ by $\sin(2\pi f_c t)$, and then pass the resulting signal through a low-pass filter. In this case, we get an output with a spectrum equal to $(A_c/2)\, H(f-f_c)M_2(f)$.

Problem 2.13

When the local carriers have a phase error ϕ, we may write

$$\cos(2\pi f_c t + \phi) = \cos(2\pi f_c t)\cos\phi - \sin(2\pi f_c t)\sin\phi$$

In this case, we find that by multiplying the received signal $r(t)$ by $\cos(2\pi f_c t+\phi)$, and passing the resulting output through a low-pass filter, the corresponding low-pass filter output in the receiver has a spectrum equal to $(A_c/2)\, H(f-f_c)\, [\cos\phi\, M_1(f) - \sin\phi\, M_2(f)]$. This indicates that there is cross-talk at the demodulator outputs.

Problem 2.14

The transmitted signal is given by

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

$$= A_c[V_0 + m_l(t) + m_r(t)]\cos(2\pi f_c t) + A_c[m_l(t) - m_r(t)]\sin(2\pi f_c t)$$

(a) The envelope detection of $s(t)$ yields

$$y_1(t) = A_c\sqrt{(V_0 + m_l(t) + m_r(t))^2 + (m_l(t) - m_r(t))^2}$$

$$= A_c(V_0 + m_l(t) + m_r(t))\sqrt{1 + \left(\frac{m_l(t) - m_r(t)}{V_0 + m_l(t) + m_r(t)}\right)^2}$$

To minimize the distortion in the envelope detector output due to the quadrature component, we choose the DC offset V_0 to be large. We may then approximate $y_1(t)$ as

$$y_1(t) \approx A(V_0 + m_l(t) + m_r(t))$$

which, except for the DC component $A_c V_0$, is proportional to the sum $m_l(t) + m_r(t)$.

(b) For coherent detection at the receiver, we need a replica of the carrier $A_c\cos(2\pi f_c t)$. This requirement can be satisfied by passing the received signal $s(t)$ through a narrow-band filter of mid-band frequency f_c. However, to extract the difference $m_l(t)$ - $m_r(t)$, we need $\sin(2\pi f_c t)$, which is obtained by passing the narrow-band filter output through a 90°-phase shifter. Then, multiplying $s(t)$ by $\sin(2\pi f_c t)$ andt low-pass filtering, we obtain a signal proportional to $m_l(t)$ - $m_r(t)$.

(c) To recover the original loudspeaker signals $m_l(t)$ and $m_r(t)$, we proceed as follows:

- Equalize the outputs of the envelope detector and coherent detector.
- Pass the equalized outputs through an audio demixer to produce $m_l(t)$ and $m_r(t)$.

Problem 2.15

(a) $s(t) = A_c(1 + k_a m(t))\cos(2\pi f_c t)$

$$= A_c\left(1 + \frac{k_a}{1+t^2}\right)\cos(2\pi f_c t)$$

To ensure 50 percent modulation, $k_a = 1$, in which case we get

$$s(t) = A_c\left(1 + \frac{1}{1+t^2}\right)\cos(2\pi f_c t)$$

(b) $s(t) = A_c m(t)\cos(2\pi f_c t)$

$$= \frac{A_c}{1+t^2}\cos(2\pi f_c t)$$

(c) $s(t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)]$

$$= \frac{A_c}{2}\left[\frac{1}{1+t^2}\cos(2\pi f_c t) - \frac{t}{1+t^2}\sin(2\pi f_c t)\right]$$

(d) $s(t) = \frac{A_c}{2}\left[\frac{1}{1+t^2}\cos(2\pi f_c t) + \frac{t}{1+t^2}\sin(2\pi f_c t)\right]$

As an aid to the sketching of the modulated signals in (c) and (d), the envelope of either SSB wave is

$$a(t) = \frac{1}{2}\sqrt{\frac{t^2+1}{(1+t^2)^2}} = \frac{1}{2}\sqrt{\frac{1}{1+t^2}}$$

Plots of the modulated signals in (a) to (d) are presented in Fig. 1 on the next page.

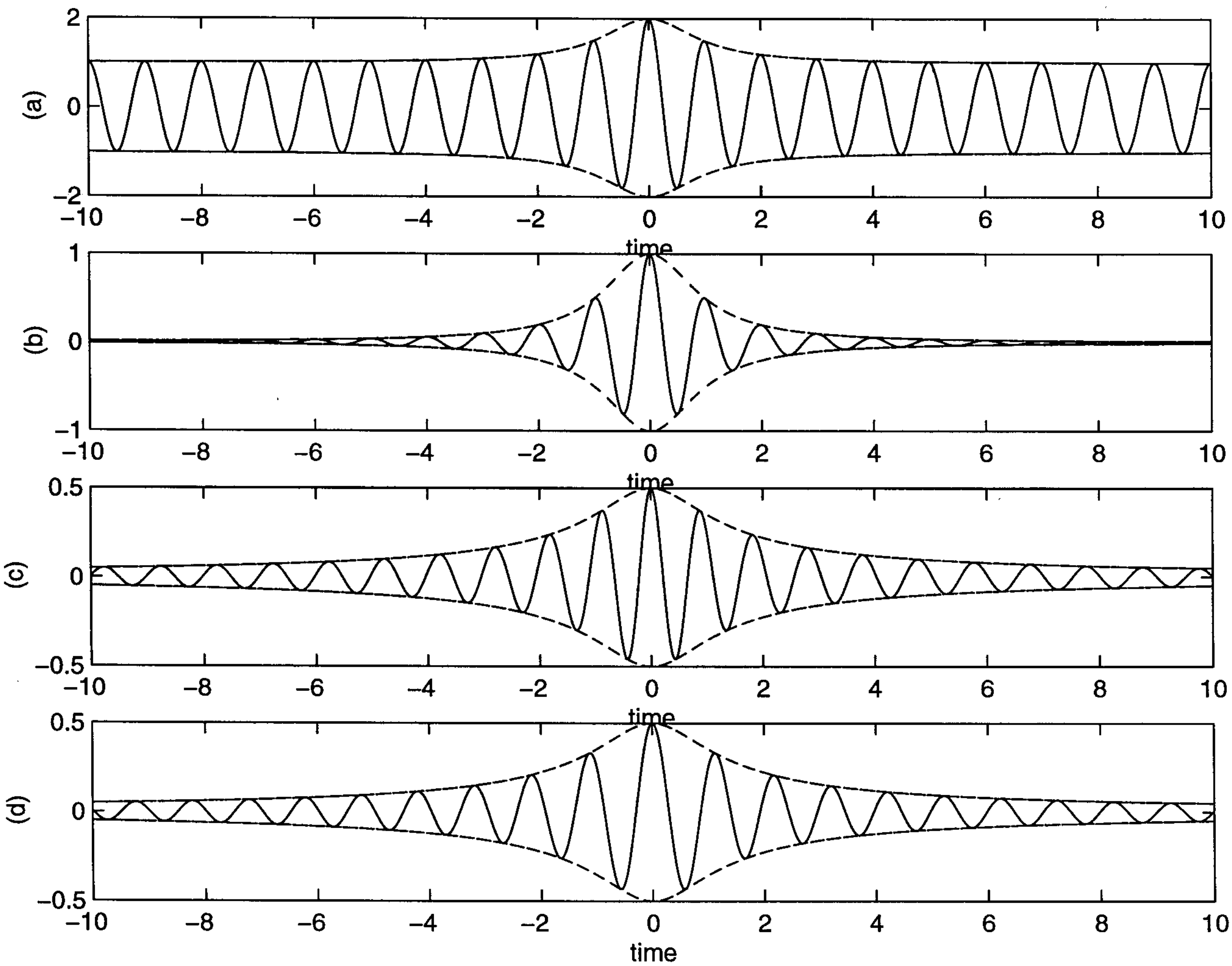


Figure 1

Problem 2.16

Consider first the modulated signal

$$s(t) = \frac{1}{2}m(t)\cos(2\pi f_c t) - \frac{1}{2}\hat{m}(t)\sin(2\pi f_c t) \qquad (1)$$

Let $S(f) = F[s(t)]$, $M(f) = F[m(t)]$, and $\hat{M}(f) = f[\hat{m}(t)]$ where $\hat{m}(t)$ is the Hilbert transform of the message signal $m(t)$. Then applying the Fourier transform to Eq. (1), we obtain

$$S(f) = \frac{1}{4}[M(f-f_c) + M(f+f_c)] - \frac{1}{4j}[\hat{M}(f-f_c) - \hat{M}(f+f_c)] \qquad (2)$$

From the definition of the Hilbert transform, we have

$$\hat{M}(f) = -j\,\mathrm{sgn}(f)M(f)$$

where sgn(f) is the signum function. Equivalently, we may write

$$-\frac{1}{j}\hat{M}(f-f_c) = \mathrm{sgn}(f-f_c)M(f-f_c)$$

$$-\frac{1}{j}\hat{M}(f+f_c) = \mathrm{sgn}(f+f_c)M(f+f_c)$$

(i) From the definition of the signum function, we note the following for $f > 0 =$ and $f > f_c$:

$$\mathrm{sgn}(f-f_c) = \mathrm{sgn}(f+f_c) = +1$$

Correspondingly, Eq. (2) reduces to

$$S(f) = \frac{1}{4}[M(f-f_c) + M(f+f_c)] + \frac{1}{4}[M(f-f_c) - M(f+f_c)]$$

$$= \frac{1}{2}M(f-f_c)$$

In words, we may thus state that, except for a scaling factor, the spectrum of the modulated signal $s(t)$ defined in Eq. (1) is the same as that of the DSB-SC modulated signal for $f > f_c$.

(ii) For $f > 0$ and $f < f_c$, we have

$$\mathrm{sgn}(f - f_c) = -1$$
$$\mathrm{sgn}(f + f_c) = +1$$

Correspondingly, Eq. (2) reduces to

$$S(f) = \frac{1}{4}[M(f - f_c) + M(f + f_c)] + \frac{1}{4}[-M(f - f_c) - M(f - f_c)]$$

$$= 0$$

In words, we may now state that for $f < f_c$, the modulated signal $s(t)$ defined in Eq. (1) is zero.

Combining the results for parts (i) and (ii), the modulated signal $s(t)$ of Eq. (1) represents a single sideband modulated signal containing only the upper sideband. This result was derived for $f > 0$. This result also holds for $f < 0$, the proof for which is left as an exercise for the reader.

Following a procedure similar to that described above, we may show that the modulated signal

$$s(t) = \frac{1}{2}m(t)\cos(2\pi f_c t) + \frac{1}{2}\hat{m}(t)\sin(2\pi f_c t) \tag{3}$$

represents a single sideband modulated signal containing only the lower sideband.

Problem 2.17

An error Δf in the frequency of the local oscillator in the demodulation of an SSB signal, measured with respect to the carrier frequency f_c, gives rise to distortion in the demodulated signal. Let the local oscillator output be denoted by $A_c' \cos(2\pi(f_c + \Delta f)t)$. The resulting demodulated signal is given by (for the case when the upper sideband only is transmitted)

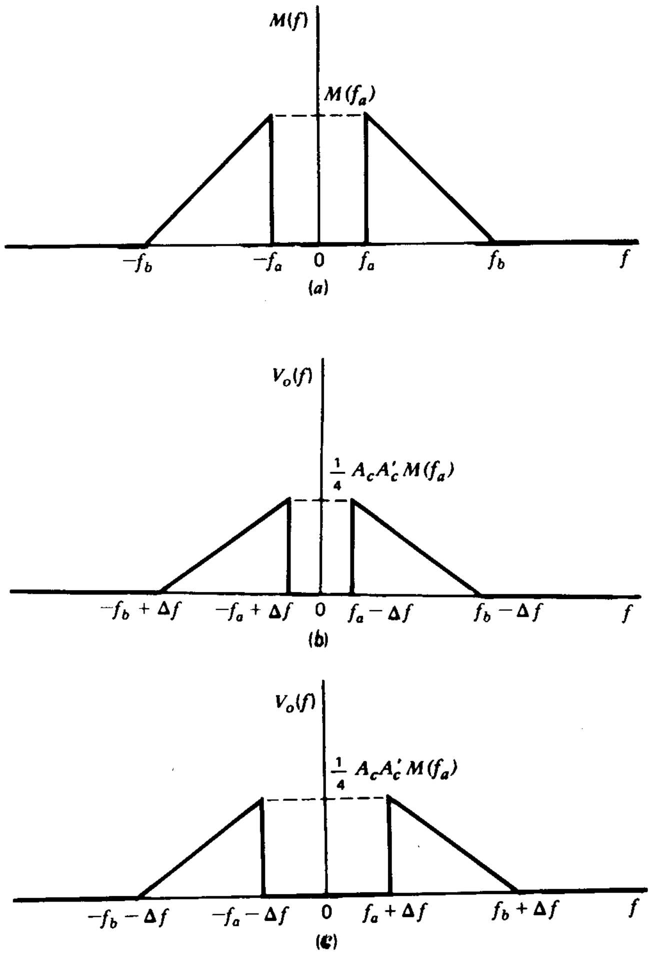
$$v_0(t) = \frac{1}{4} A_c A_c' \, [m(t)\cos(2\pi\Delta f t) + m(t)\sin(2\pi\Delta f t)]$$

This demodulated signal represents an SSB wave corresponding to a carrier frequency Δf.

The effect of frequency error Δf in the local oscillator may be interpreted as follows:

(a) If the SSB wave s(t) contains the upper sideband and the frequency error Δf is positive, or equivalently if s(t) contains the lower sideband and Δf is negative, then the frequency components of the demodulated signal $v_0(t)$ are shifted inward by the amount Δf compared with the baseband signal m(t), as illustrated in Fig. 1(b).

(b) If the incoming SSB wave s(t) contains the lower sideband and the frequency error Δf is positive, or equivalently if s(t) contains the upper sideband and Δf is negative, then the frequency components of the demodulated signal $v_0(t)$ are shifted outward by the amount Δf, compared with the baseband signal m(t). This is illustrated in Fig. 1c for the case of a baseband signal (e.g., voice signal) with an energy gap occupying the interval $-f_a \leq f \leq f_a$, in part (a) of the figure.

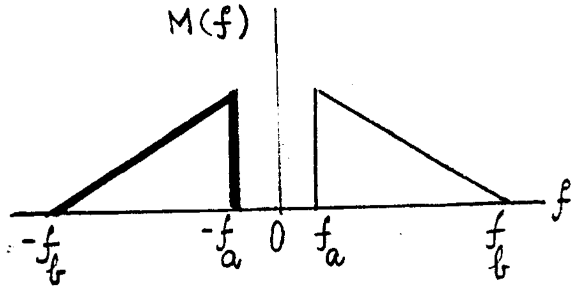


M(f)
M(f_a)
−f_b
−f_a
0
f_a
f_b
f
(a)
V_o(f)
1/4 A_c A'_c M(f_a)
−f_b + Δf
−f_a + Δf
0
f_a − Δf
f_b − Δf
f
(b)
V_o(f)
1/4 A_c A'_c M(f_a)
−f_b − Δf
−f_a − Δf
0
f_a + Δf
f_b + Δf
f

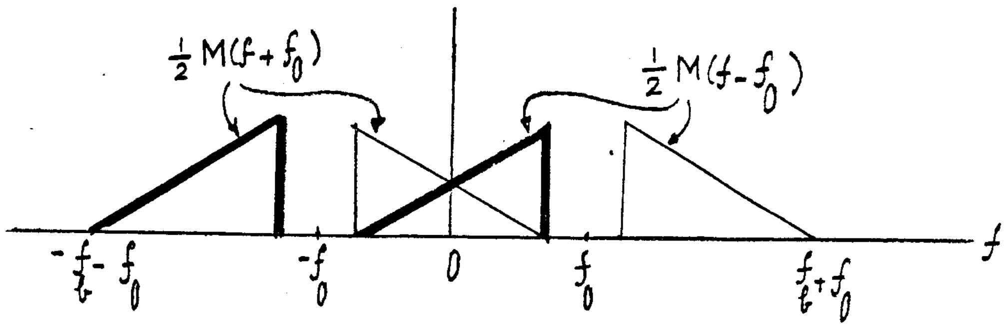
Fig. 1

Problem 2.18

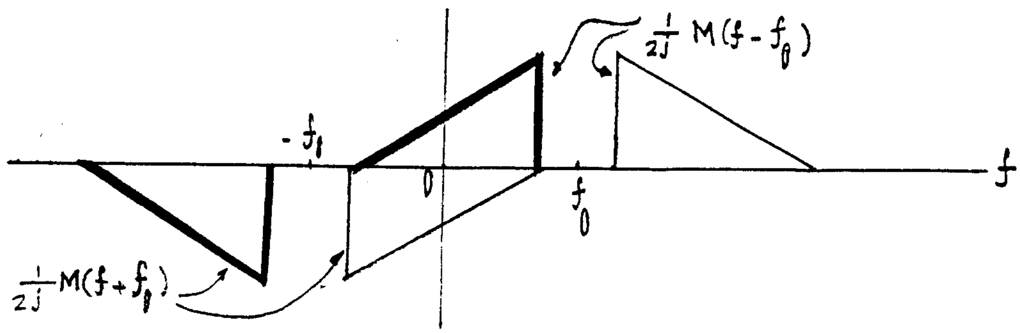
(a,b) The spectrum of the message signal is illustrated below:



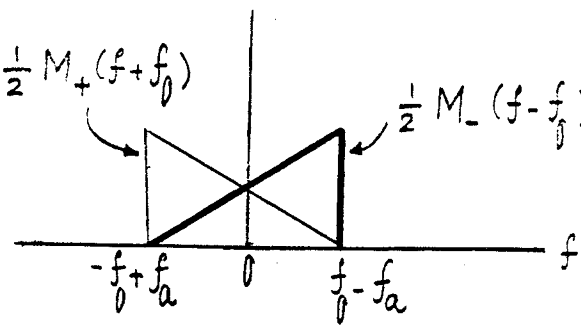
Correspondingly, the output of the upper first product modulator has the following spectrum:



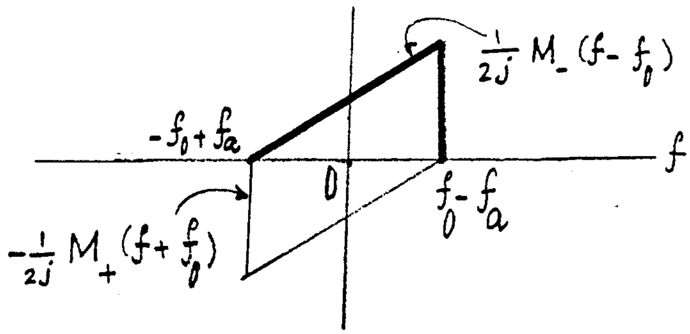
The output of the lower first product modulator has the spectrum:



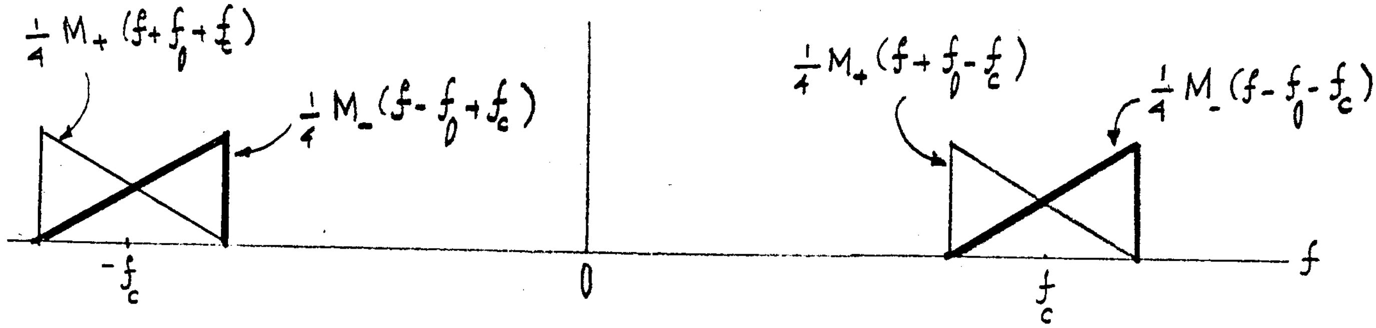
The output of the upper low pass filter has the spectrum:



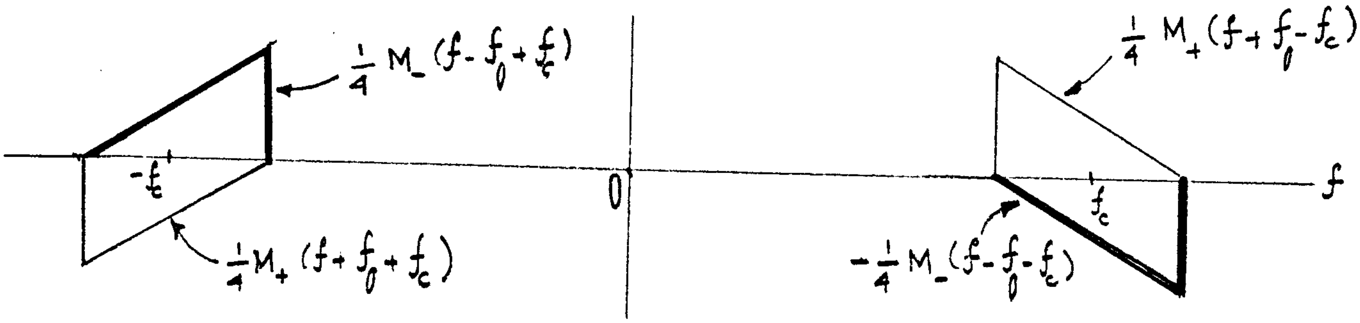
The output of the lower low pass filter has the spectrum:



The output of the upper second product modulator has the spectrum:



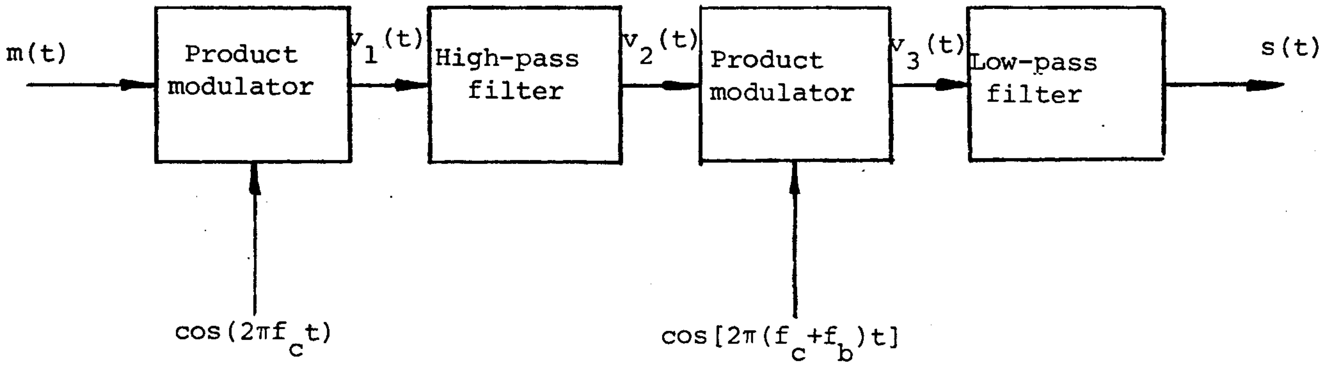
The output of the lower second product modulator has the spectrum:



Adding the two second product modulator outputs, their upper sidebands add constructively while their lower sidebands cancel each other.

(c) To modify the modulator to transmit only the lower sideband, a single sign change is required in one of the channels. For example, the lower first product modulator could multiply the message signal by $-\sin(2\pi f_0 t)$. Then, the upper sideband would be cancelled and the lower one transmitted.

Problem 2.19



(a) The first product modulator output is

$$v_1(t) = m(t)\cos(2\pi f_c t)$$

The second product modulator output is

$$v_3(t) = v_2(t)\cos[2\pi(f_c+f_b)t]$$

The amplitude spectra of m(t), $v_1(t)$, $v_2(t)$, $v_3(t)$ and s(t) are illustrated on the next page:

We may express the voice signal m(t) as

$$m(t) = \frac{1}{2}[m_+(t) + m_-(t)]$$

where $m_+(t)$ is the pre-envelope of m(t), and $m_-(t) = m_+^*(t)$ is its complex conjugate. The Fourier transforms of $m_+(t)$ and $m_-(t)$ are defined by (see Appendix 2)

$$M_+(f) = \begin{cases} 2M(f), & f > 0 \\ 0, & f < 0 \end{cases}$$

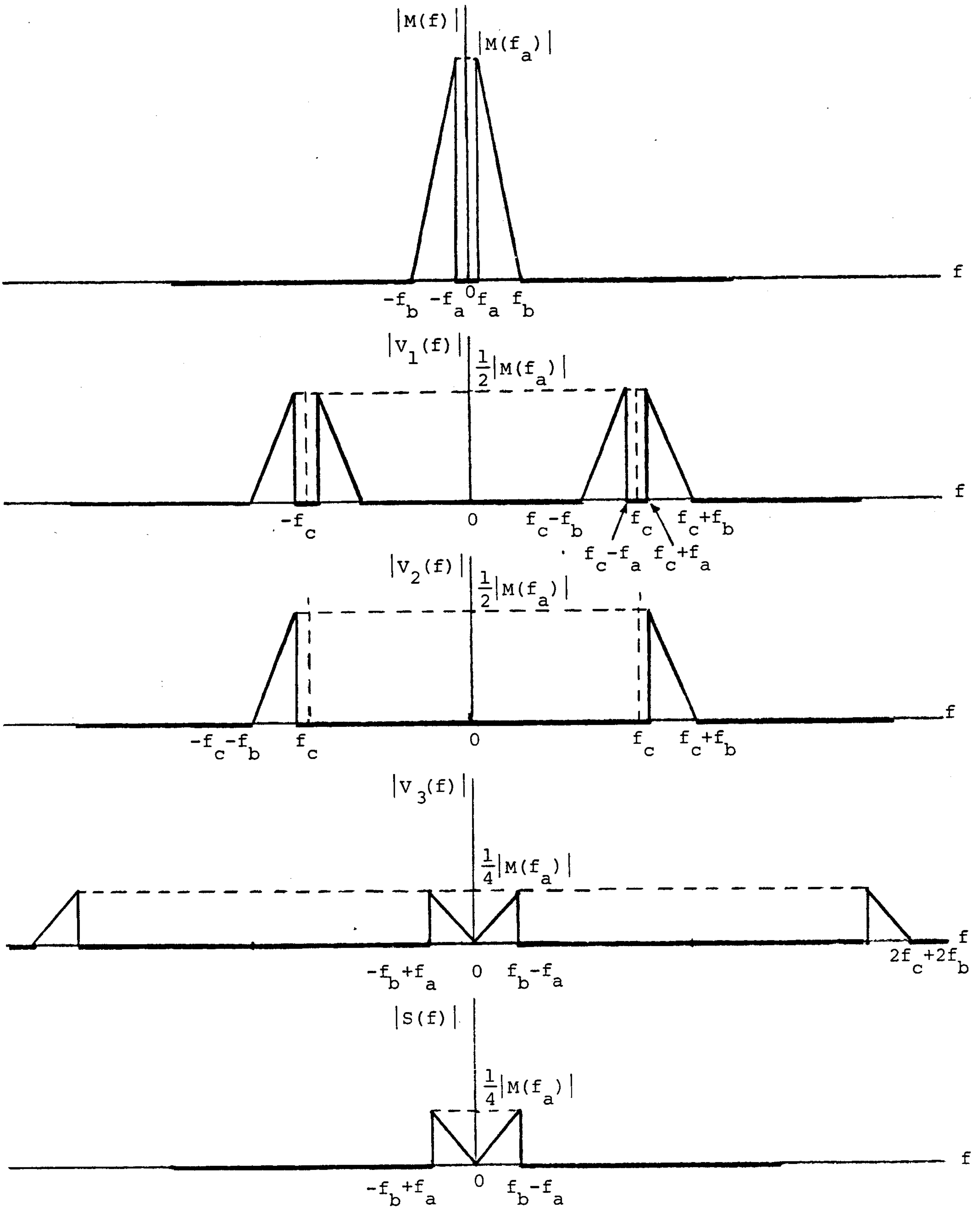
$$M_-(f) = \begin{cases} 0, & f > 0 \\ 2M(f), & f < 0 \end{cases}$$

Comparing the spectrum of s(t) with that of m(t), we see that s(t) may be expressed in terms of $m_+(t)$ and $m_-(t)$ as follows:

$$\begin{aligned} s(t) &= \frac{1}{8} m_+(t)\exp(-j2\pi f_b t) + \frac{1}{8} m_-(t)\exp(j2\pi f_b t) \\ &= \frac{1}{8}[m(t)+j\hat{m}(t)]\exp(-j2\pi f_b t) + \frac{1}{8}[m(t)-j\hat{m}(t)]\exp(j2\pi f_b t) \\ &= \frac{1}{4} m(t)\cos(2\pi f_b t) + \frac{1}{4}\hat{m}(t)\sin(2\pi f_b t) \end{aligned}$$

(b) With s(t) as input, the first product modulator output is

$$v_1(t) = s(t)\cos(2\pi f_c t)$$



|M(f)|
|M(f_a)|
f
-f_b -f_a 0 f_a f_b
|V_1(f)|
1/2|M(f_a)|
f
-f_c
0
f_c-f_b
f_c
f_c+f_b
f_c-f_a
f_c+f_a
|V_2(f)|
1/2|M(f_a)|
f
-f_c-f_b
f_c
0
f_c
f_c+f_b
|V_3(f)|
1/4|M(f_a)|
f
-f_b+f_a
0
f_b-f_a
2f_c+2f_b
|S(f)|
1/4|M(f_a)|
f
-f_b+f_a
0
f_b-f_a

C19

Problem 2.20

(a) Consider the system described in Fig. 1a, where $u(t)$ denotes the product modulator output, as shown by

$$u(t) = A_c m(t) \cos(2\pi f_c t)$$

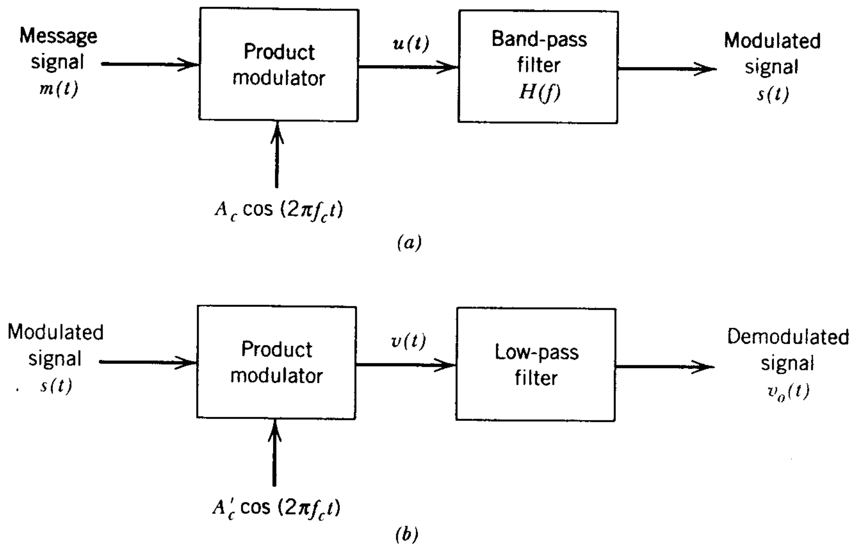


Figure 1: (a) Filtering scheme for processing sidebands. (b) Coherent detector for recovering the message signal.

Let $H(f)$ denote the transfer function of the filter following the product modulator. The spectrum of the modulated signal $s(t)$ produced by passing $u(t)$ through the filter is given by

$$S(f) = U(f)H(f)$$

$$= \frac{A_c}{2}[M(f-f_c) + M(f+f_c)]H(f) \tag{1}$$

where $M(f)$ is the Fourier transform of the message signal $m(t)$. The problem we wish to address is to determine the particular $H(f)$ required to produce a modulated signal $s(t)$ with desired spectral characteristics, such that the original message signal $m(t)$ may be recovered from $s(t)$ by coherent detection.

The first step in the coherent detection process involves multiplying the modulated signal $s(t)$ by a locally generated sinusoidal wave $A'_c \cos(2\pi f_c t)$, which is synchronous with the carrier wave $A_c \cos(2\pi f_c t)$, in both frequency and phase as in Fig. 1b. We may thus write

$$v(t) = A'_c \cos(2\pi f_c t) s(t)$$

Transforming this relation into the frequency domain gives the Fourier transform of $v(t)$ as

$$V(f) = \frac{A'_c}{2}[S(f-f_c) + S(f+f_c)] \tag{2}$$

Therefore, substitution of Eq. (1) in (2) yields

$$V(f) = \frac{A_c A'_c}{4} M(f)[H(f-f_c) + H(f+f_c)]$$

$$+ \frac{A_c A'_c}{4}[M(f-2f_c)H(f-f_c) + M(f+2f_c)H(f+f_c)] \tag{3}$$

(b) The high-frequency components of $v(t)$ represented by the second term in Eq. (3) are removed by the low-pass filter in Fig. 1b to produce an output $v_o(t)$, the spectrum of which is given by the remaining components:

$$V_o(f) = \frac{A_c A'_c}{2} M(f)[H(f-f_c) + H(f+f_c)] \tag{4}$$

For a distortionless reproduction of the original baseband signal $m(t)$ at the coherent detector output, we require $V_o(f)$ to be a scaled version of $M(f)$. This means, therefore, that the transfer function $H(f)$ must satisfy the condition

$$H(f-f_c) + H(f+f_c) = 2H(f_c) \tag{5}$$

where $H(f_c)$, the value of $H(f)$ at $f = f_c$, is a constant. When the message (baseband) spectrum $M(f)$ is zero outside the frequency range $-W \le f \le W$, we need only satisfy Eq. (5) for values of f in this interval. Also, to simplify the exposition, we set $H(f_c) = 1/2$. We thus require that $H(f)$ satisfies the condition:

$$H(f-f_c) + H(f+f_c) = 1, \qquad -W \le f \le W \tag{6}$$

Under the condition described in Eq. (6), we find from Eq. (4) that the coherent detector output in Fig. 1b is given by

$$v_o(t) = \frac{A_c A'_c}{2} m(t) \tag{7}$$

Equation (1) defines the spectrum of the modulated signal $s(t)$. Recognizing that $s(t)$ is a band-pass signal, we may formulate its time-domain description in terms of in-phase and quadrature components. In particular, $s(t)$ may be expressed in the canonical form

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t) \tag{8}$$

where $s_I(t)$ is the in-phase component of $s(t)$, and $s_Q(t)$ is its quadrature component. To determine $s_I(t)$, we note that its Fourier transform is related to the Fourier transform of $s(t)$ as follows:

$$S_I(f) = \begin{cases} S(f-f_c)+S(f+f_c), & -W \le f \le W \\ 0, & \text{elsewhere} \end{cases} \tag{9}$$

Hence, substituting Eq. (1) in (9), we find that the Fourier transform of $s_I(t)$ is given by

$$\begin{aligned} S_I(f) &= \frac{1}{2}A_c M(f)[H(f-f_c)+H(f+f_c)] \\ &= \frac{1}{2}A_c M(f); \qquad -W \le f \le W \end{aligned} \tag{10}$$

where, in the second line, we have made use of the condition in Eq. (6) imposed on $H(f)$. From Eq. (10) we readily see that the in-phase component of the modulated signal $s(t)$ is defined by

$$s_I(t) = \frac{1}{2}A_c m(t) \tag{11}$$

which, except for a scaling factor, is the same as the original message signal m(t).

To determine the quadrature component $s_Q(t)$ of the modulated signal $s(t)$, we recognize that its Fourier transform is defined in terms of the Fourier transform of $s(t)$ as follows:

$$S_Q(f) = \begin{cases} j[S(f-f_c)-S(f+f_c)] & -W \le f \le W \\ 0, & \text{elsewhere} \end{cases} \tag{12}$$

Therefore, substituting Eq. (11) in (12), we get

$$S_Q(f) = \frac{j}{2}A_c M(f)[H(f-f_c)-H(f+f_c)] \tag{13}$$

This equation suggests that we may generate $s_Q(t)$, except for a scaling factor, by passing the message signal $m(t)$ through a new filter whose transfer function is related to that of the filter in Fig. 1a as follows:

$$H_Q(f) = j[H(f-f_c) - H(f+f_c)], \qquad -W \le f \le W \tag{14}$$

Let $m'(t)$ denote the output of this filter produced in response to the input $m(t)$. Hence, we may express the quadrature component of the modulated signal $s(t)$ as

$$s_Q(t) = \frac{1}{2}A_c m'(t) \tag{15}$$

Accordingly, substituting Eqs. (11) and (15) in (8), we find that $s(t)$ may be written in the canonical form

$$m(t) = \frac{1}{2}A_c m(t)\cos(2\pi f_c t) - \frac{1}{2}A_c m'(t)\sin(2\pi f_c t) \tag{16}$$

There are two important points to note here:

1. The in-phase component $s_I(t)$ is completely independent of the transfer function $H(f)$ of the band-pass filter involved in the generation of the modulated wave $s(t)$ in Fig. 1a, so long as it satisfies the condition of Eq. (6).

2. The spectral modification attributed to the transfer function $H(f)$ is confined solely to the quadrature component $s_Q(t)$.

The role of the quadrature component is merely to interfere with the in-phase component, so as to reduce or eliminate power in one of the sidebands of the modulated signal $s(t)$, depending on the application of interest.

Problem 2.21

(a) Expanding s(t), we get

$$s(t) = \frac{1}{2} a\ A_m A_c \cos(2\pi f_c t) \cos(2\pi f_m t)$$
$$- \frac{1}{2} a\ A_m A_c \sin(2\pi f_c t) \sin(2\pi f_m t) + \frac{1}{2}(1-a)\ A_c A_m \cos(2\pi f_c t) \cos(2\pi f_m t)$$
$$+ \frac{1}{2}(1-a)\ A_m A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$$
$$= \frac{1}{2} A_m A_c \cos(2\pi f_c t) \cos(2\pi f_m t)$$
$$+ \frac{1}{2} A_m A_c\ (1-2a) \sin(2\pi f_c t) \sin(2\pi f_m t)$$

Therefore, the quadrature component is:

$$- \frac{1}{2} A_c A_m (1-2a) \sin(2\pi f_m t)$$

(b) After adding the carrier, the signal will be:

$$s(t) = A_c [1 + \frac{1}{2} A_m \cos(2\pi f_m t)] \cos(2\pi f_c t)$$
$$+ \frac{1}{2} A_c A_m (1-2a) \sin(2\pi f_m t) \sin(2\pi f_c t)$$

The envelope equals

$$a(t) = A_c \sqrt{[1 + \frac{1}{2} A_m \cos(2\pi f_m t)]^2 + [\frac{1}{2} A_m (1-2a) \sin(2\pi f_m t)]^2}$$

$$= A_c\ [1 + \frac{1}{2} A_m \cos(2\pi f_m t)] \sqrt{1 + \left[\frac{\frac{1}{2} A_m (1-2a) \sin(2\pi f_m t)}{1 + \frac{1}{2} A_m \cos(2\pi f_m t)}\right]^2}$$

$$= A_c\ [1 + \frac{1}{2} A_m \cos(2\pi f_m t)]\ d(t)$$

where d(t) is the distortion, defined by

$$d(t) = \sqrt{1 + \left[\frac{\frac{1}{2} A_m (1-2a) \sin(2\pi f_m t)}{1 + \frac{1}{2} A_m \cos(2\pi f_m t)}\right]^2}$$

(c) d(t) is greatest when a = 0.

Problem 2.22

Consider an incoming narrow-band signal of bandwidth 10 kHz, and mid-band frequency which may lie in the range 0.535–1.605 MHz. It is required to translate this signal to a fixed frequency band centered at 0.455 MHz. The problem is to determine the range of tuning that must be provided in the local oscillator.

Let f_c denote the mid-band frequency of the incoming signal, and f_l denote the local oscillator frequency. Then we may write

$$0.535 < f_c < 1.605$$

and

$$f_c - f_l = 0.455$$

where both f_c and f_l are expressed in MHz. That is,

$$f_l = f_c - 0.455$$

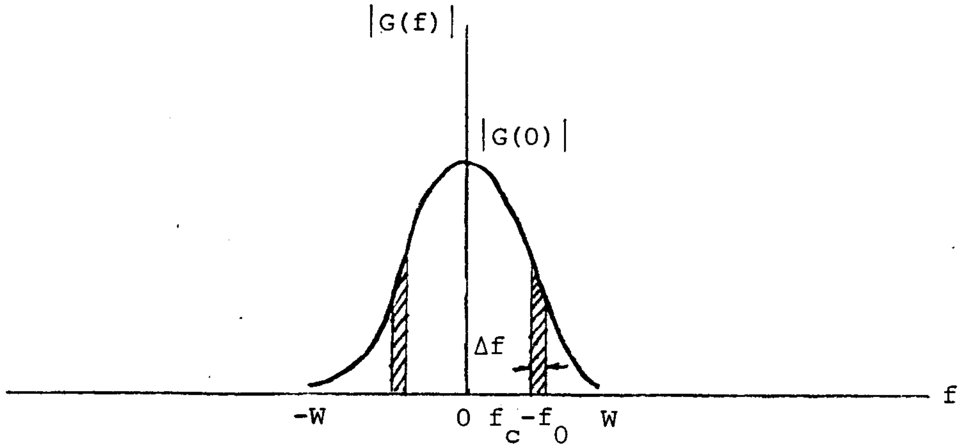
When $f_c = 0.535$ MHz, we get $f_l = 0.08$ MHz; and when $f_c = 1.605$ MHz, we get $f_l = 1.15$ MHz. Thus the required range of tuning of the local oscillator is 0.08–1.15 MHz.

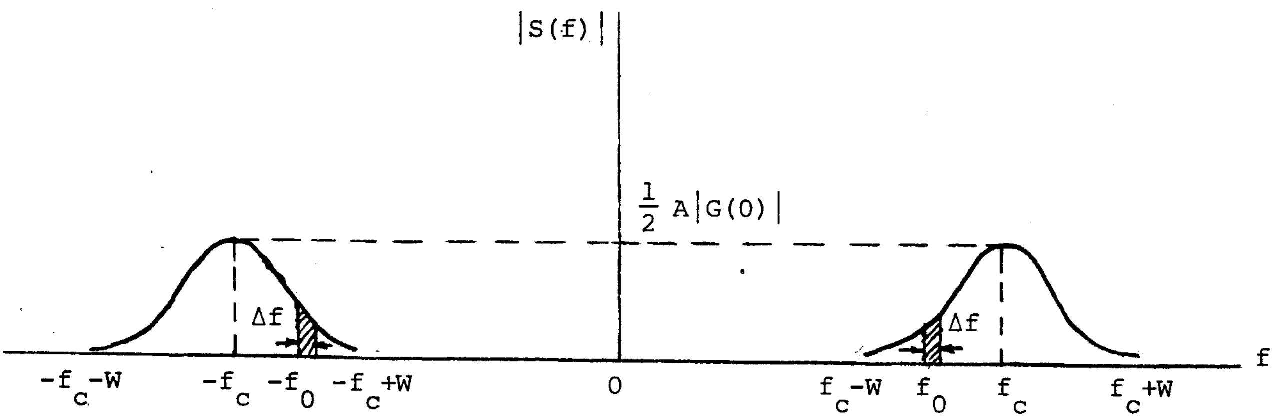
Problem 2.23

Let s(t) denote the multiplier output, as shown by

$$s(t) = A\ g(t) \cos(2\pi f_c t)$$

where f_c lies in the range f_0 to f_0+W. The amplitude spectra of s(t) and g(t) are related as follows:





With v(t) denoting the band-pass filter output, we thus find that the Fourier transform of v(t) is approximately given by

$$V(f) \simeq \frac{1}{2} A\, G(f_c - f_0) \quad , \quad f_0 - \frac{\Delta f}{2} \leq |f| \leq f_0 + \frac{\Delta f}{2}$$

The rms meter output is therefore (by using Rayleigh's energy theorem)

$$V_{rms} = [\int_{-\infty}^{\infty} v^2(t)dt]^{1/2}$$

$$= [\int_{-\infty}^{\infty} |V(f)|^2\, df]^{1/2} = [2(\frac{1}{4} A^2\, |G(f_c - f_o|^2)\, \Delta f]^{1/2}$$

$$= \frac{A}{\sqrt{2}} |G(f_c - f_o)|\, \sqrt{\Delta f}$$

Problem 2.24

For the PM case,

$$s(t) = A_c \cos[2\pi f_c t + k_p\, m(t)].$$

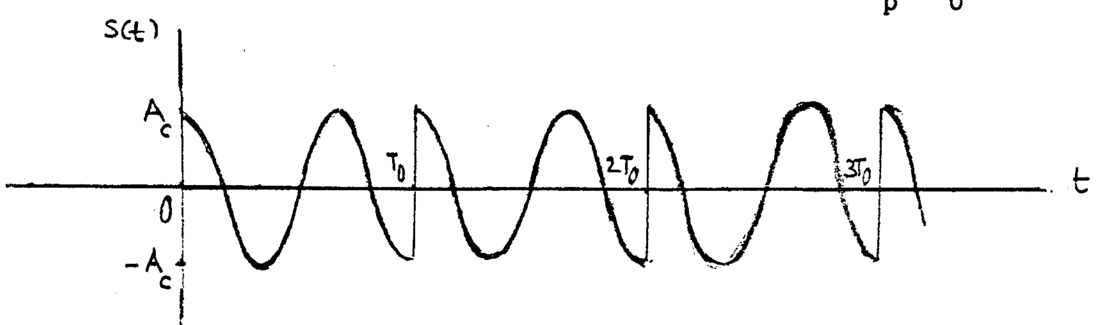
The angle equals

$$\theta_i(t) = 2\pi f_c t + k_p\, m(t).$$

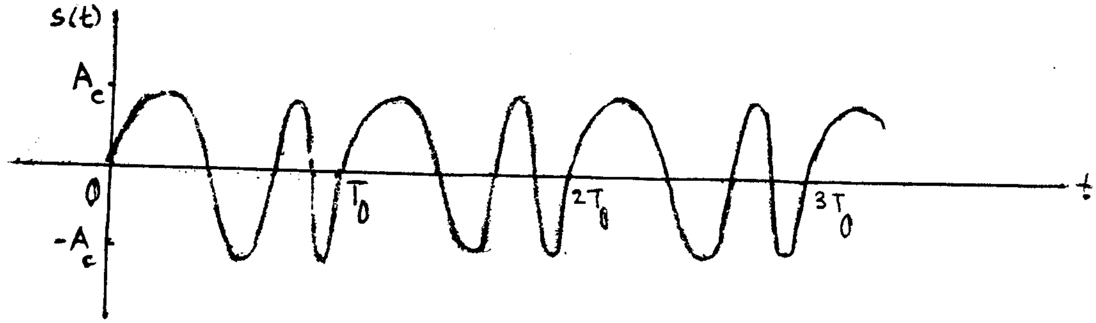
The instantaneous frequency,

$$f_i(t) = f_c + \frac{Ak_p}{2\pi T_0} - \sum_n \frac{Ak_p}{2\pi} \delta(t - nT_0),$$

is equal to $f_c + Ak_p/2\pi T_0$ except for the instants that the message signal has discontinuities. At these instants, the phase shifts by $-k_pA/T_0$ radians.

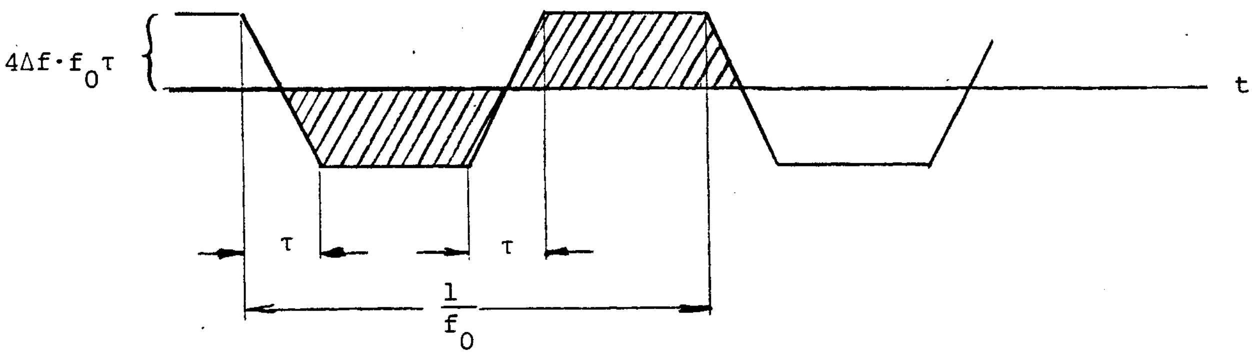


For the FM case, $f_i(t) = f_c + k_f\, m(t)$



Problem 2.25

The instantaneous frequency of the mixer output is as shown below:



The presence of negative frequency merely indicates that the phasor representing the difference frequency at the mixer output has reversed its direction of rotation.

Let N denote the number of beat cycles in one period. Then, noting that N is equal to the shaded area shown above, we deduce that

$$N = 2[4\Delta f \cdot f_0 \tau (\frac{1}{2f_0} - \tau) + 2\Delta f \cdot f_0 \tau^2]$$

$$= 4\Delta f \cdot \tau (1 - f_0 \tau)$$

Since $f_0 \tau << 1$, we have

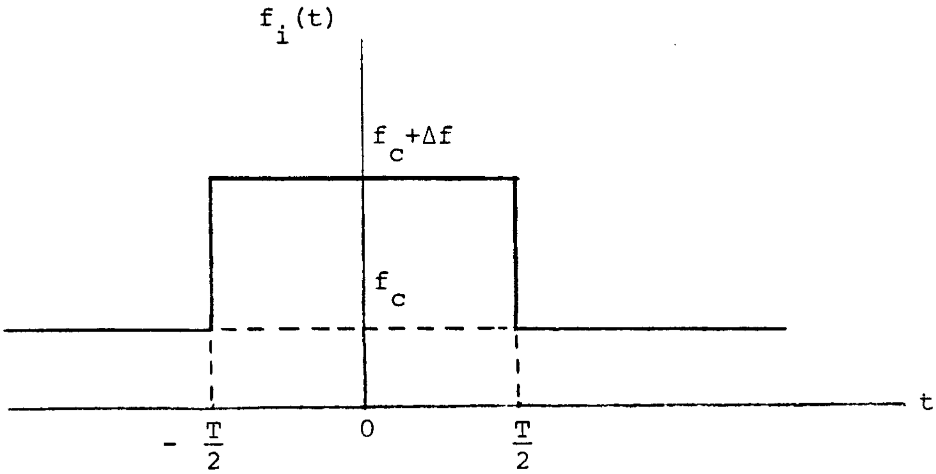
$$N \simeq 4\Delta f \cdot \tau$$

Therefore, the number of beat cycles counted over one second is equal to

$$\frac{N}{1/f_0} = 4\Delta f \cdot f_0 \tau.$$

Problem 2.26

The instantaneous frequency of the modulated wave s(t) is as shown below:



We may thus express s(t) as follows

$$s(t) = \begin{cases} \cos(2\pi f_c t), & t < -\frac{T}{2} \\ \cos[2\pi(f_c+\Delta f)t], & -\frac{T}{2} \le t \le \frac{T}{2} \\ \cos[2\pi f_c t), & \frac{T}{2} < t \end{cases}$$

The Fourier transform of s(t) is therefore

$$S(f) = \int_{-\infty}^{-T/2} \cos(2\pi f_c t) \exp(-j2\pi ft)\, dt$$

$$+ \int_{-T/2}^{T/2} \cos[2\pi(f_c+\Delta f)t] \exp(-j2\pi ft)\, dt$$

$$+ \int_{T/2}^{\infty} \cos(2\pi f_c t) \exp(-j2\pi ft)\, dt$$

$$= \int_{-\infty}^{\infty} \cos(2\pi f_c t) \exp(-j2\pi ft)\, dt$$

$$+ \int_{-T/2}^{T/2} \{\cos[2\pi(f_c+\Delta f)t - \cos(2\pi f_c t)\} \exp(-j2\pi ft)\, dt \qquad (1)$$

The second term of Eq. (1) is recognized as the difference between the Fourier transforms of two RF pulses of unit amplitude, one having a frequency equal to $f_c+\Delta f$ and the other having a frequency equal to f_c. Hence, assuming that $f_cT >> 1$, we may express S(f) as follows:

$$S(f) \approx \begin{cases} \frac{1}{2}\,\delta(f-f_c) + \frac{T}{2}\,\mathrm{sinc}[T(f-f_c-\Delta f)] - \frac{T}{2}\,\mathrm{sinc}[T(f-f_c)], & f > 0 \\ \\ \frac{1}{2}\,\delta(f+f_c) + \frac{T}{2}\,\mathrm{sinc}[T(f+f_c+\Delta f)] - \frac{T}{2}\,\mathrm{sinc}[T(f+f_c)], & f < 0 \end{cases}$$

Problem 2.27

For SSB modulation, the modulated wave is

$$s(t) = \frac{A_c}{2}\,[m(t)\,\cos(2\pi f_c t) \pm \hat{m}(t)\,\sin(2\pi f_c t)],$$

the minus sign applying when transmitting the upper sideband and the plus sign applying when transmitting the lower one.

Regardless of the sign, the envelope is

$$a(t) = \frac{A_c}{2}\sqrt{m^2(t) + \hat{m}^2(t)}\;.$$

(a) For upper sideband transmission, the angle,

$$\theta_i(t) = 2\pi f_c t + \tan^{-1}\left(\frac{\hat{m}(t)}{m(t)}\right)\;.$$

The instantaneous frequency is,

$$f_i(t) = \frac{1}{2\pi}\,\frac{d\theta_i(t)}{dt}$$

$$= f_c + \frac{m(t)\,\hat{m}'(t) - \hat{m}(t)\,m'(t)}{2\pi(m^2(t) + \hat{m}^2(t))}\;,$$

where ' denotes time derivative.

(b) For lower sideband transmission, we have

$$\theta_i(t) = 2\pi f_c t + \tan^{-1}\left(-\frac{\hat{m}(t)}{m(t)}\right)\;,$$

and

$$f_i(t) = f_c + \frac{\hat{m}(t)\,m'(t) - m(t)\,\hat{m}'(t)}{2\pi(m^2(t) + \hat{m}^2(t))}\;.$$

Problem 2.28

(a) The envelope of the FM wave s(t) is

$$a(t) = A_c \sqrt{1+\beta^2 \sin^2(2\pi f_m t)}$$

The maximum value of the envelope is

$$a_{max} = A_c \sqrt{1+\beta^2}$$

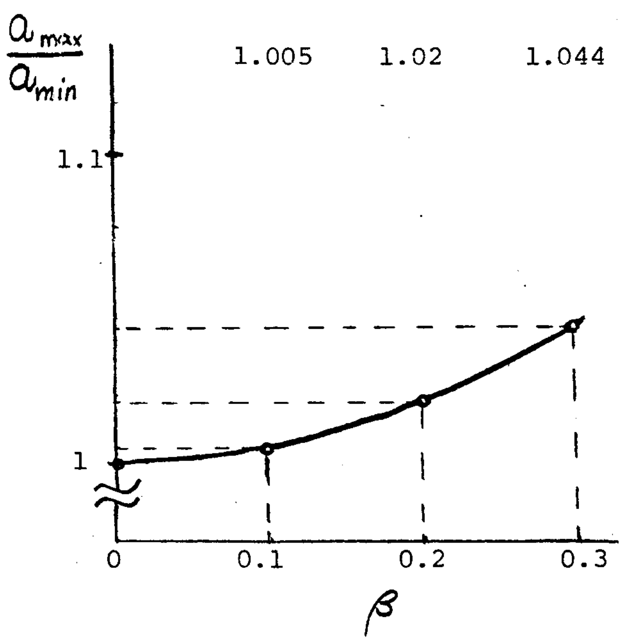
and its minimum value is

$$a_{min} = A_c$$

Therefore,

$$\frac{a_{max}}{a_{min}} = \sqrt{1+\beta^2}$$

This ratio is shown plotted below for $0 < \beta < 0.3$:



(b) Expressing s(t) in terms of its frequency components:

$$s(t) = A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c \cos[2\pi(f_c+f_m)t] - \frac{1}{2}\beta A_c \cos[2\pi(f_c-f_m)t]$$

The mean power of s(t) is therefore

$$P_1 = \frac{A_c^2}{2} + \frac{\beta^2 A_c^2}{8} + \frac{\beta^2 A_c^2}{8}$$

$$= \frac{A_c^2}{2}\left(1 + \frac{\beta^2}{2}\right)$$

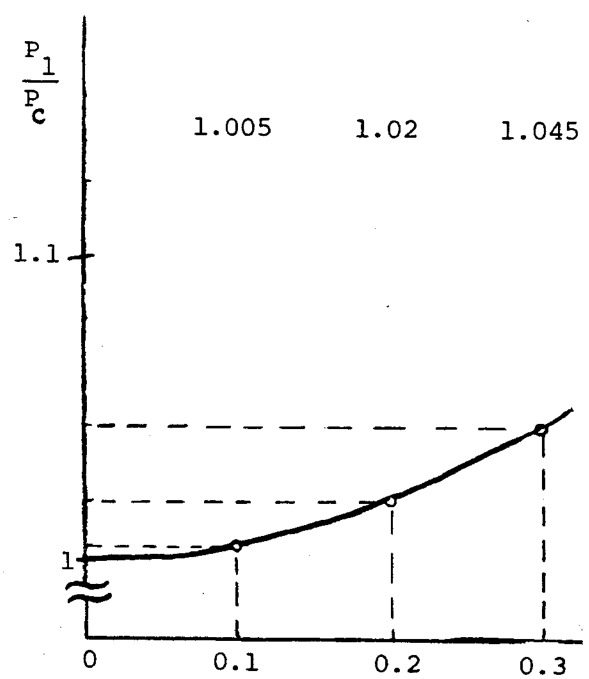
The mean power of the unmodulated carrier is

$$P_c = \frac{A_c^2}{2}$$

Therefore,

$$\frac{P_1}{P_c} = 1 + \frac{\beta^2}{2}$$

which is shown plotted below for $0 \leq \beta \leq 0.3$:



(c) The angle $\theta_i(t)$, expressed in terms of the in-phase component, $s_I(t)$, and the quadrature component, $s_Q(t)$, is:

$$\theta_i(t) = 2\pi f_c t + \tan^{-1}\left[\frac{s_I(t)}{s_Q(t)}\right]$$

$$= 2\pi f_c t + \tan^{-1}[\beta\sin(2\pi f_m t)]$$

Since $\tan^{-1}(x) \approx x - x^3/3 + \dots$,

$$\theta_i(t) \approx 2\pi f_c t + \beta\sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

The harmonic distortion is the power ratio of the third and first harmonics:

$$D_h = \left(\frac{\frac{1}{3}\beta^3}{\beta}\right)^2 = \frac{\beta^4}{9}$$

For $\beta = 0.3$, $D_h = 0.09\%$

Problem 2.29

(a) The phase-modulated wave is

$$s(t) = A_c\cos[2\pi f_c t + k_p A_m\cos(2\pi f_m t)]$$

$$= A_c\cos[2\pi f_c t + \beta_p\cos(2\pi f_m t)]$$

$$= A_c\cos(2\pi f_c t)\cos[\beta_p\cos(2\pi f_m t)] - A_c\sin(2\pi f_c t)\sin[\beta_p\cos(2\pi f_m t)] \qquad (1)$$

If $\beta_p \le 0.5$, then

$$\cos[\beta_p \cos(2\pi f_m t)] \simeq 1$$

$$\sin[\beta_p \cos(2\pi f_m t)] \simeq \beta_p \cos(2\pi f_m t)$$

Hence, we may rewrite Eq. (1) as

$$s(t) \simeq A_c \cos(2\pi f_c t) - \beta_p A_c \sin(2\pi f_c t) \cos(2\pi f_m t)$$

$$= A_c \cos(2\pi f_c t) - \frac{1}{2} \beta_p A_c \sin[2\pi(f_c+f_m)t]$$

$$- \frac{1}{2} \beta_p A_c \sin[2\pi(f_c-f_m)t] \tag{2}$$

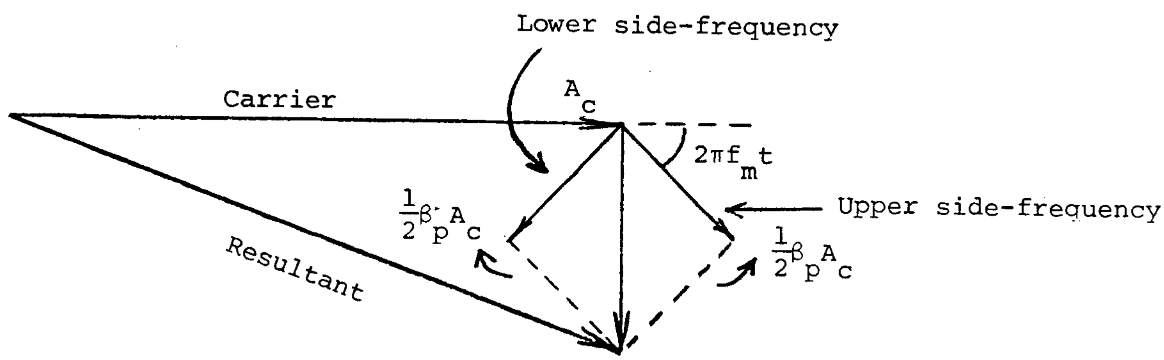
The spectrum of s(t) is therefore

$$S(f) \simeq \frac{1}{2} A_c[\delta(f-f_c) + \delta(f+f_c)]$$

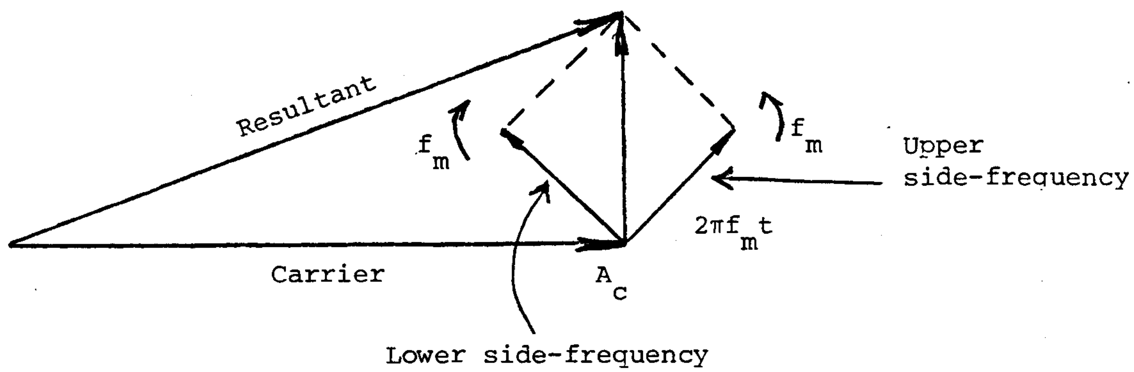
$$- \frac{1}{4j} \beta_p A_c[\delta(f-f_c-f_m) - \delta(f+f_c+f_m)]$$

$$- \frac{1}{4j} \beta_p A_c[\delta(f-f_c+f_m) - \delta(f+f_c-f_m)]$$

(b) The phasor diagram for s(t) is deduced from Eq. (2) to be as follows:



The corresponding phasor diagram for the narrow-band FM wave is as follows:



Comparing these two phasor diagrams, we see that, except for a phase difference, the narrow-band PM and FM waves are of exactly the same form.

Problem 2.30

The phase-modulated wave is

$$s(t) = A_c \cos[2\pi f_c t + \beta_p \cos(2\pi f_m t)]$$

The complex envelope of s(t) is

$$\tilde{s}(t) = A_c \exp[j\beta_p \cos(2\pi f_m t)]$$

Expressing $\tilde{s}(t)$ in the form of a complex Fourier series, we have

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_m t)$$

where

$$c_n = f_m \int_{-1/2f_m}^{1/2f_m} \tilde{s}(t) \exp(-j2\pi n f_m t)\, dt$$

$$= A_c f_m \int_{-1/2f_m}^{1/2f_m} \exp[j\beta_p \cos(2\pi f_m t) - j2\pi n f_m t]\, dt \qquad (1)$$

Let $2\pi f_m t = \pi/2 - \phi$.

Then, we may rewrite Eq. (1) as

$$c_n = -\frac{A_c}{2\pi} \exp\left(-\frac{jn\pi}{2}\right) \int_{3\pi/2}^{-\pi/2} \exp[j\beta_p \sin(\phi) + jn\phi]\, d\phi$$

The integrand is periodic with respect to ϕ with a period of 2π. Hence, we may rewrite this expression as

$$c_n = \frac{A_c}{2\pi} \exp(-\frac{jn\pi}{2}) \int_{-\pi}^{\pi} \exp[j\beta_p \sin(\phi) + jn\phi]\, d\phi$$

However, from the definition of the Bessel function of the first kind of order n, we have

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\, x \sin\phi - nj\phi)\, d\phi$$

Therefore,

$$c_n = A_c \exp(-\frac{jn\pi}{2}) J_{-n}(\beta_p)$$

We may thus express the PM wave s(t) as

$$s(t) = \mathrm{Re}[\tilde{s}(t) \exp(j2\pi f_c t)]$$

$$= A_c \mathrm{Re}[\sum_{n=-\infty}^{\infty} J_{-n}(\beta_p) \exp(-\frac{jn\pi}{2}) \exp(j2\pi n f_m t) \exp(j2\pi f_c t)]$$

$$= A_c \sum_{n=-\infty}^{\infty} J_{-n}(\beta_p) \cos[2\pi(f_c+nf_m)t - \frac{n\pi}{2}]$$

The band-pass filter only passes the carrier, the first upper side-frequency, and the first lower side-frequency, so that the resulting output is

$$s_o(t) = A_c J_0(\beta_p) \cos(2\pi f_c t) + A_c J_{-1}(\beta_p) \cos[2\pi(f_c+f_m)t - \frac{\pi}{2}]$$
$$+ A_c J_1(\beta_p) \cos[2\pi(f_c-f_m)t + \frac{\pi}{2}]$$
$$= A_c J_0(\beta_p) \cos(2\pi f_c t) + A_c J_{-1}(\beta_p) \sin[2\pi(f_c+f_m)t]$$
$$- A_c J_1(\beta_p) \sin[2\pi(f_c-f_m)t]$$

But

$$J_{-1}(\beta_p) = -J_1(\beta_p)$$

Therefore,

$$s_o(t) = A_c J_0(\beta_p) \cos(2\pi f_c t)$$
$$- A_c J_1(\beta_p) \{\sin[2\pi(f_c+f_m)t] + \sin[2\pi(f_c-f_m)t]\}$$
$$= A_c J_0(\beta_p) \cos(2\pi f_c t) - 2 A_c J_1(\beta_p) \cos(2\pi f_m t) \sin(2\pi f_c t)$$

The envelope of $s_o(t)$ equals

$$a(t) = A_c \sqrt{J_0^2(\beta_p) + 4J_1^2(\beta_p)\cos^2(2\pi f_m t)}$$

The phase of $s_o(t)$ is

$$\phi(t) = -\tan^{-1}\left[\frac{2\,J_1(\beta_p)}{J_0(\beta_p)}\cos(2\pi f_m t)\right]$$

The instantaneous frequency of $s_o(t)$ is

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d\phi(t)}{dt}$$

$$= f_c + \frac{2\,J_0(\beta_p)\,J_1(\beta_p)\sin(2\pi f_m t)}{J_0^2(\beta_p) + 4J_1^2(\beta_p)\cos^2(2\pi f_m t)}$$

Problem 2.31

(a) From Table A4.1, we find (by interpolation) that $J_0(\beta)$ is zero for

$\beta = 2.44,$

$\beta = 5.52,$

$\beta = 8.65,$

$\beta = 11.8,$

and so on.

(b) The modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f\,A_m}{f_m}$$

Therefore,

$$k_f = \frac{\beta f_m}{A_m}$$

Since $J_0(\beta) = 0$ for the first time when $\beta = 2.44$, we deduce that

$$k_f = \frac{2.44 \times 10^3}{2}$$

$$= 1.22 \times 10^3 \text{ hertz/volt}$$

Next, we note that $J_0(\beta) = 0$ for the second time when $\beta = 5.52$. Hence, the corresponding value of A_m for which the carrier component is reduced to zero is

$$A_m = \frac{\beta f_m}{k_f}$$

$$= \frac{5.52 \times 10^3}{1.22 \times 10^3}$$

$$= 4.52 \text{ volts}$$

Problem 2.32

For $\beta = 1$, we have

$J_0(1) = 0.765$

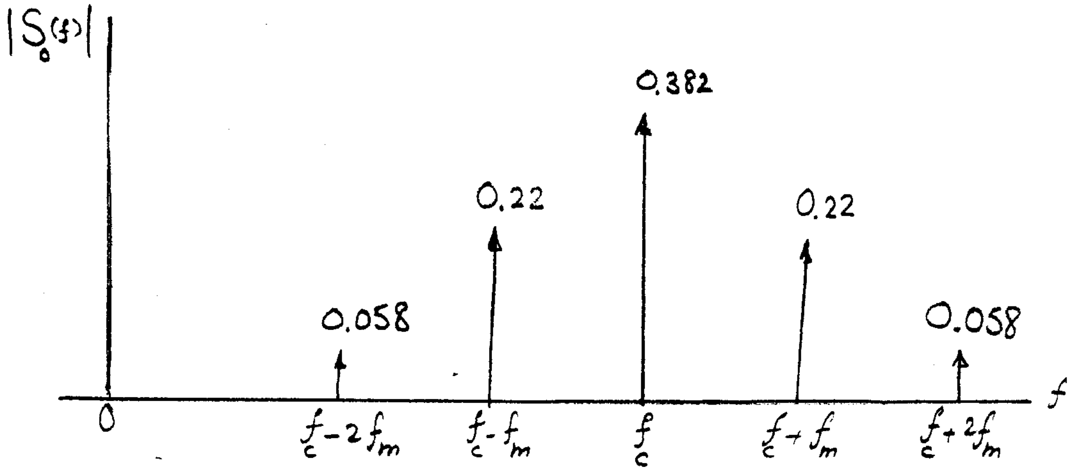
$J_1(1) = 0.44$

$J_2(1) = 0.115$

Therefore, the band-pass filter output is (assuming a carrier amplitude of 1 volt)

$$s_o(t) = 0.765 \cos(2\pi f_c t)$$
$$+ 0.44 \{\cos[2\pi(f_c+f_m)t] - \cos[2\pi(f_c-f_m)t]\}$$
$$+ 0.115 \{\cos[2\pi(f_c+2f_m)t] + \cos[2\pi(f_c-2f_m)t]\} ,$$

and the amplitude spectrum (for positive frequencies) is



Problem 2.33

(a) The frequency deviation is

$$\Delta f = k_f A_m = 25 \times 10^3 \times 20 = 5 \times 10^5 \text{ Hz}$$

The corresponding value of the modulation index is

$$\therefore \beta = \frac{\Delta f}{f_m} = \frac{5 \times 10^5}{10^5} = 5$$

The transmission bandwidth of the FM wave, using Carson's rule, is therefore

$$B_T = 2f_m(1+\beta) = 2\times100 \ (1+5) = 1200 \text{ kHz} = 1.2 \text{ MHz}.$$

(b) Using the universal curve of Fig. 3-36 we find that for $\beta=5$:

$$\frac{B_T}{\Delta f} = 3$$

Therefore,

$$B_T = 3\times500 = 1500 \text{ kHz} = 1.5 \text{ MHz}$$

(c) If the amplitude of the modulating wave is doubled, we find that

$$\Delta f = 1 \text{ MHz and } \beta = 10$$

Thus, using Carson's rule we obtain

$$B_T = 2\times100 \ (1+10) = 2200 \text{ kHz} = 2.2 \text{ MHz}$$

Using the universal curve of Fig. 3-36, we get

$$\frac{B_T}{\Delta f} = 2.75$$

and $B_T = 2.75$ MHz.

(d) If f_m is doubled, $\beta = 2.5$. Then, using Carson's rule, $B_T = 1.4$ MHz. Using the universal curve, $B_T/\Delta f = 4$, and

$$B_T = 4\Delta f = 2 \text{ MHz}.$$

Problem 2.34

(a) The angle of the PM wave is

$$\theta_i(t) = 2\pi f_c t + k_p\, m(t)$$

$$= 2\pi f_c t + k_p\, A_m \cos(2\pi f_m t)$$

$$= 2\pi f_c t + \beta_p \cos(2\pi f_m t)$$

where $\beta_p = k_p A_m$. The instantaneous frequency of the PM wave is therefore

$$f_i(t) = \frac{1}{2\pi} \frac{d\theta_i(t)}{dt}$$

$$= f_c - \beta_p f_m \sin(2\pi f_m t)$$

We see that the maximum frequency deviation in a PM wave varies linearly with the modulation frequency f_m.

Using Carson's rule, we find that the transmission bandwidth of the PM wave is approximately (for the case when $\beta_p \gg 1$)

$$B_T \simeq 2(f_m + \beta_p f_m) = 2f_m(1 + \beta_p) \simeq 2f_m \beta_p .$$

This shows that B_T varies linearly with f_m.

(b) In an FM wave, the transmission bandwidth B_T is approximately equal to $2\Delta f$, if the modulation index $\beta \gg 1$. Therefore, for an FM wave, B_T is effectively independent of the modulation frequency f_m.

Problem 2.35

The filter input is

$$v_1(t) = g(t)\, s(t)$$

$$= g(t) \cos(2\pi f_c t - \pi k t^2)$$

The complex envelope of $v_1(t)$ is

$$\tilde{v}_1(t) = g(t) \exp(-j\pi k t^2)$$

The impulse response h(t) of the filter is defined in terms of the complex impulse response $\tilde{h}(t)$ as follows

$$h(t) = \mathrm{Re}[\tilde{h}(t) \exp(j2\pi f_c t)]$$

With

$$h(t) = \cos(2\pi f_c t + \pi k t^2),$$

we have

$$\tilde{h}(t) = \exp(j\pi k t^2)$$

The complex envelope of the filter output is therefore (see Appendix 2)

$$\tilde{v}_0(t) = \frac{1}{2}\tilde{h}(t) \star \tilde{v}_i(t)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} g(\tau) \exp(-j\pi k\tau^2) \exp[j\pi k(t-\tau)]^2 d\tau$$

$$= \frac{1}{2} \exp(j\pi kt^2) \int_{-\infty}^{\infty} g(\tau) \exp(-j2\pi kt\tau)\ d\tau$$

$$= \frac{1}{2} \exp(j\pi kt^2)\ G(kt)$$

Hence,

$$|\tilde{v}_0(t)| = \frac{1}{2}\ |G(kt)|$$

This shows that the envelope of the filter output is, except for the scale factor of 1/2, equal to the magnitude of the Fourier transform of the input signal g(t), with kt playing the role of frequency f.

Problem 2.36

The overall frequency multiplication ratio is

$$n = 2x3 = 6$$

Assume that the instantaneous frequency of the FM wave at the input of the first frequency multiplier is

$$f_{i1}(t) = f_c + \Delta f \cos(2\pi f_m t)$$

The instantaneous frequency of the resulting FM wave at the output of the second frequency multiplier is therefore

$$f_{i2}(t) = nf_c + n\Delta f \cos(2\pi f_m t)$$

Thus, the frequency deviation of this FM wave is equal to

$$n\Delta f = 6x10 = 60 \text{ kHz}$$

and its modulation index is equal to

$$\frac{n\Delta f}{f_m} = \frac{60}{5} = 12$$

The frequency separation of the adjacent side-frequencies of this FM wave is unchanged at f_m = 5 kHz.

Problem 2.37

(a) Figure 1 shows the simplified block diagram of a typical FM transmitter (based on the indirect method) used to transmit audio signals containing frequencies in the range 100 Hz to 15 kHz. The narrow-band phase modulator is supplied with a carrier signal of frequency f_1 = 0.2 MHz by a crystal-controlled oscillator. The desired FM signal at the transmitter output is to have a carrier frequency f_c = 100 MHz and a minimum frequency deviation Δf = 75 kHz.

In order to limit the harmonic distortion produced by the narrow-band phase modulator, we restrict the modulation index β_1 of this modulator to a maximum value of 0.3 radians. Consider then the value β_1 = 0.2 radians, which certainly satisfies this requirement. The lowest modulation frequencies of 100 Hz produce a frequency deviation of Δf_1 = 20 Hz at the narrow-band phase modulator output, whereas the highest modulation frequencies of 15 kHz produce a frequency deviation of Δf_1 = 3 kHz. The lowest modulation frequencies are therefore of immediate concern, as they produce a much lower frequency deviation than the highest modulation frequencies. The requirement is therefore to ensure that the frequency deviation produced by the lowest modulation frequencies of 100 Hz is raised to 75 kHz.

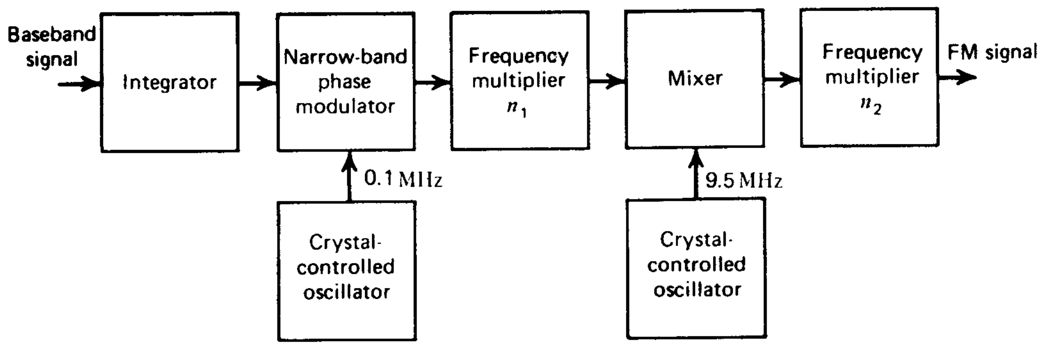


Figure 1

To produce a frequency deviation of Δf = 75 kHz at the FM transmitter output, the use of frequency multiplication is obviously required. Specifically, with Δf_1 = 20 Hz and Δf = 75 kHz, we require a total frequency multiplication ratio of 3750. However, using a straight frequency multiplication equal to this value would produce a much higher carrier frequency at the transmitter output than the desired value of 100 MHz. To generate an FM signal having both the desired frequency deviation and carrier frequency, we therefore need to use a *two-stage frequency multiplier* with an intermediate stage of frequency translation as illustrated in Fig. 1. Let n_1 and n_2 denote the respective frequency multiplication ratios, so that

$$n_1 n_2 = \frac{\Delta f}{\Delta f_1} = \frac{75000}{20} = 3750 \tag{1}$$

The carrier frequency $n_1 f_1$ at the first frequency multiplier output is translated downward to $(f_2 - n_1 f_1)$ by mixing it with a sinusoidal wave of frequency f_2 = 95 MHz, which is supplied by a second crystal-controlled oscillator. However, the carrier frequency at the input of the second frequency multiplier is required to equal f_c/n_2. Equating these two frequencies, we thus get

$$f_2 - n_1 f_1 = \frac{f_c}{n_2}$$

Hence, with f_1 = 0.1 MHz, f_2 = 9.5 MHz, and f_c = 100 MHz, we have

$$9.5 - 0.1 n_1 = \frac{100}{n_2} \qquad (2)$$

Solving Eqs. (1) and (2) for n_1 and n_2, we obtain

$n_1 = 75$
$n_2 = 50$

(b) Using these frequency multiplication ratios, we get the set of values indicated in the table below:

Table -Values of Carrier Frequency and Frequency Deviation at the Various Points in the Wide-band Frequency Modulator of Fig. 1

	At the Phase Modulator Output	At the First Frequency Multiplier Output	At the Mixer Output	At the Second Frequency Multiplier Output
Carrier frequency	0.1 MHz	7.5 MHz	2.0 MHz	100 MHz
Frequency deviation	20 Hz	1.5 kHz	1.5 kHz	75 kHz

Problem 2.38

(a) Let L denote the inductive component, C the capacitive component, and C_0 the capacitance of each varactor diode due to the bias voltage V_b acting alone. Then, we have

$$C_0 = 100 \ V_b^{-1/2} \ \text{pF}$$

and the corresponding frequency of oscillation is

$$f_0 = \frac{1}{2\pi\sqrt{L(C+C_0/2)}}$$

Therefore,

$$10^6 = \frac{1}{2\pi\sqrt{200 \times 10^{-6} \ (100 \times 10^{-12} + 50 \ V_b^{-1/2} \times 10^{-12})}}$$

Solving for V_b, we get

$$V_b = 3.52 \text{ volts}$$

(b) The frequency multiplication ratio is 64. Therefore, the modulation index of the FM wave at the frequency multiplier input is

$$\beta = \frac{5}{64} = 0.078$$

This indicates that the FM wave produced by the combination of L, C and the varactor diodes is a narrow-band one, which in turn means that the amplitude A_m of the modulating wave is small compared to V_b. We may thus express the instantaneous frequency of this FM wave as follows:

$$f_i(t) = \frac{1}{2\pi} \ [200 \times 10^{-6} \ \{100 \times 10^{-12} + 50 \times 10^{-12} \ [3.52 + A_m \sin(2\pi f_m t)]^{-1/2}\}]^{-1/2}$$

$$= \frac{10^7}{2\sqrt{2}\pi} \ \{1 + 0.266 \ [1 + \frac{A_m}{3.52} \sin(2\pi f_m t)]^{-1/2}\}^{-1/2}$$

$$\simeq \frac{10^7}{2\sqrt{2}\pi} \ \{1 + 0.266 \ [1 - \frac{A_m}{7.04} \sin(2\pi f_m t)]\}^{-1/2}$$

$$= 10^6 \ [1 - 0.03 \ A_m \sin(2\pi f_m t)]^{-1/2}$$

$$\simeq 10^6 \ [1 + 0.015 \ A_m \sin(2\pi f_m t)]$$

With a modulation index of 0.078, the corresponding value of the frequency deviation is

$$\Delta f = \beta f_m$$

$$= 0.078 \times 10^4 \text{ Hz}$$

Therefore,

$$0.015\, A_m \times 10^6 = 0.078 \times 10^4$$

where A_m is in volts. Solving for A_m, we get

$$A_m = 52 \times 10^{-3} \text{ volts.}$$

Problem 2.39

The transfer function of the RC filter is

$$H(f) = \frac{j2\pi fCR}{1+j2\pi fCR}$$

If $2\pi fCR \ll 1$ for all frequencies of interest, then we may approximate H(f) as

$$H(f) \simeq j2\pi fCR$$

However, multiplication by $j2\pi f$ in the frequency domain is equivalent to differentiation in the time domain. Therefore, denoting the RC filter output as $v_1(t)$, we may write

$$v_1(t) \simeq CR\,\frac{ds(t)}{dt}$$

$$= CR\,\frac{d}{dt}\{A_c \cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt]\}$$

$$= -CR\,A_c[2\pi f_c + 2\pi k_f m(t)] \sin[2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt]$$

The corresponding envelope detector output is

$$v_2(t) \simeq 2\pi f_c CR\,A_c \left|1 + \frac{k_f}{f_c} m(t)\right|$$

Since $k_f |m(t)| < f_c$ for all t, then

$$v_2(t) \simeq 2\pi f_c CR\,A_c [1 + \frac{k_f}{f_c} m(t)]$$

which shows that, except for a dc bias, the output is proportional to the modulating signal m(t).

Problem 2.40

The envelope detector input is

$$v(t) = s(t) - s(t-T)$$

$$= A_c \cos[2\pi f_c t + \phi(t)] - A_c \cos[2\pi f_c (t-T) + \phi(t-T)]$$

$$= -2A_c \sin[\frac{2\pi f_c(2t-T) + \phi(t) + \phi(t-T)}{2}] \sin[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}] \qquad (1)$$

where

$$\phi(t) = \beta \sin(2\pi f_m t)$$

The phase difference $\phi(t) - \phi(t-T)$ is

$$\phi(t) - \phi(t-T) = \beta \sin(2\pi f_m t) - \beta \sin[2\pi f_m (t-T)]$$

$$= \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t) \cos(2\pi f_m T) + \cos(2\pi f_m t) \sin(2\pi f_m T)]$$

$$\approx \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t) + 2\pi f_m T \cos(2\pi f_m t)]$$

$$= 2\pi \Delta f T \cos(2\pi f_m t)$$

where

$$\Delta f = \beta f_m.$$

Therefore, noting that $2\pi f_c T = \pi/2$, we may write

$$\sin[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}] \approx \sin[\pi f_c T + \pi \Delta f T \cos(2\pi f_m t)]$$

$$= \sin[\frac{\pi}{4} + \pi \Delta f T \cos(2\pi f_m t)]$$

$$= \sqrt{2} \cos[\pi \Delta f T \cos(2\pi f_m t)] + \sqrt{2} \sin[\pi \Delta f T \cos(2\pi f_m t)]$$

$$= \sqrt{2} + \sqrt{2}\, \pi \Delta f T \cos(2\pi f_m t)$$

where we have made use of the fact that $\pi \Delta f T \ll 1$. We may therefore rewrite Eq. (1) as

$$v(t) \approx -2\sqrt{2}\, A_c[1 + \pi \Delta f T \cos(2\pi f_m t)] \sin[\pi f_c(2t-T) + \frac{\phi(t) + \phi(t-T)}{2}]$$

Accordingly, the envelope detector output is

$$a(t) \approx 2\sqrt{2}\, A_c[1+\pi \Delta f T \cos(2\pi f_m t)]$$

which, except for a bias term, is proportional to the modulating wave.

Problem 2.41

(a) In the time interval $t-(T_1/2)$ to $t+(T_1/2)$, assume there are n zero crossings. The phase difference is $\theta_i(t+T_1/2) - \theta_i(t-T_1/2) = n\pi$. Also, the angle of an FM wave is

$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(t)\, dt.$$

Since m(t) is assumed constant, equal to m_1, $\theta_i(t) = 2\pi f_c t + 2\pi k_f m_1 t$. Therefore,

$$\theta_i(t+T_1/2) - \theta_i(t-T_1/2) = (2\pi f_c + 2\pi k_f m_1)\,[t+T_1/2 - (t-T_1/2)].$$
$$= (2\pi f_c + 2\pi k_f m_1)\, T_1.$$

But

$$f_i(t) = \frac{d\theta_i(t)}{dt} = 2\pi f_c + 2\pi k_f m_1 .$$

Thus,

$$\theta_i(t+T_1/2) - \theta_i(t-T_1/2) = f_i(t)\, T_1.$$

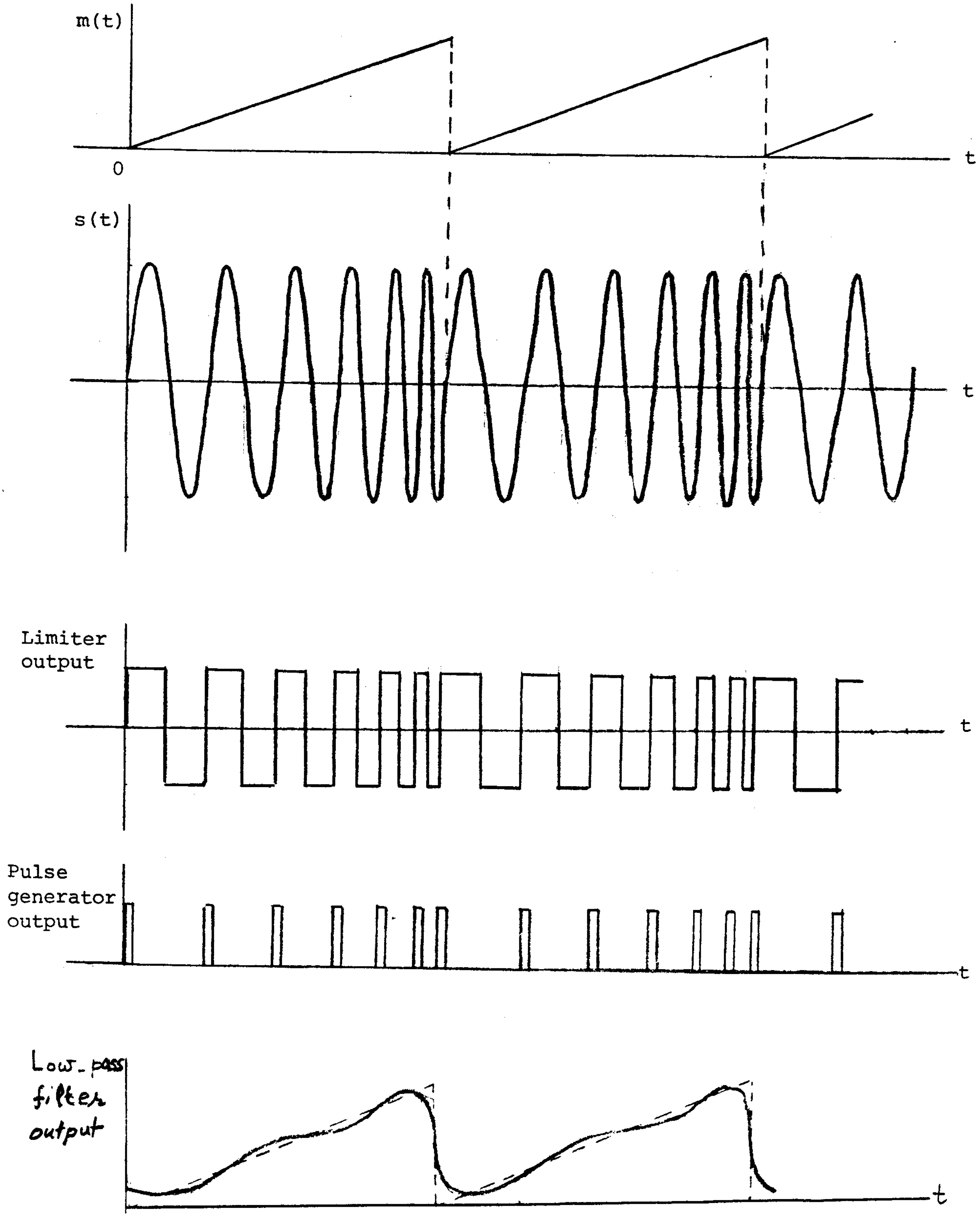
But this phase difference also equals $n\pi$. So,

$$f_i(t)\, T_1 = n\pi$$

and

$$f_i(t) = n\pi/T_1$$

(b) For a repetitive ramp as the modulating wave, we have the following set of waveforms


m(t)
0
t
s(t)
t
Limiter output
t
Pulse generator output
t
Low-pass filter output
t

Problem 2.42

The complex envelope of the modulated wave s(t) is

$$\tilde{s}(t) = a(t)\exp[j\phi(t)]$$

Since a(t) is slowly varying compared to $\exp[j\phi(t)]$, the complex envelope $\tilde{s}(t)$ is restricted effectively to the frequency band $-B_T/2 \le f \le B_T/2$. An ideal frequency discriminator consists of a differentiator followed by an envelope detector. The output of the differentiator, in response to $\tilde{s}(t)$, is

$$\tilde{v}_o(t) = \frac{d}{dt}\tilde{s}(t)$$

$$= \frac{d}{dt}\{a(t)\exp[j\phi(t)]\}$$

$$= \frac{da(t)}{dt}\exp[j\phi(t)] + j\frac{d\phi(t)}{dt}a(t)\exp[j\phi(t)]$$

$$= a(t)\exp[j\phi(t)]\left[\frac{1}{a(t)}\frac{da(t)}{dt} + j\frac{d\phi(t)}{dt}\right]$$

Since a(t) is slowly varying compared to $\phi(t)$, we have

$$\left|\frac{d\phi(t)}{dt}\right| \gg \left|\frac{1}{a(t)}\frac{da(t)}{dt}\right|$$

Accordingly, we may approximate $\tilde{v}_o(t)$ as

$$\tilde{v}_o(t) \simeq j\,a(t)\frac{d\phi(t)}{dt}\exp[j\phi(t)]$$

However, by definition

$$\phi(t) = 2\pi k_f \int_0^t m(t)\,dt$$

Therefore,

$$\tilde{v}_o(t) = j2\pi k_f\,a(t)\,m(t)\exp[j\phi(t)]$$

Hence, the envelope detector output is proportional to a(t) m(t) as shown by

$$|\tilde{v}_o(t)| \simeq 2\pi k_f\,a(t)\,m(t)$$

Problem 2.43

(a) The limiter output is

$$z(t) = \text{sgn}\{a(t)\cos[2\pi f_c t + \phi(t)]\}$$

Since a(t) is of positive amplitude, we have

$$z(t) = \mathrm{sgn}\{\cos[2\pi f_c t + \phi(t)]\}$$

Let

$$\psi(t) = 2\pi f_c t + \phi(t)$$

Then, we may write

$$\mathrm{sgn}[\cos\psi] = \sum_{n=-\infty}^{\infty} c_n \exp(jn\psi)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{sgn}[\cos\psi] \exp(-jn\psi)\, d\psi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} (-1) \exp(-jn\psi) d\psi + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} (+1) \exp(-jn\psi)\, d\psi \tag{1}$$

$$+ \frac{1}{2\pi}\int_{\pi/2}^{\pi} (-1) \exp(-jn\psi)\, d\psi$$

If $n \neq 0$, then

$$c_n = \frac{1}{2\pi(-jn)} [-\exp(\frac{jn\pi}{2})+\exp(jn\pi)+\exp(\frac{-jn\pi}{2})-\exp(\frac{jn\pi}{2})-\exp(-jn\pi)+\exp(\frac{-jn\pi}{2})]$$

$$= \frac{1}{\pi n}[2\sin(\frac{n\pi}{2})-\sin(n\pi)]$$

$$= \begin{cases} \frac{2}{\pi n}(-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

If $n=0$, we find from Eq. (1) that $c_n=0$. Therefore,

$$\mathrm{sgn}[\cos\psi] = \frac{2}{\pi}\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{n} (-1)^{(n-1)/2} \exp(jn\psi)$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos[\psi(2k+1)]$$

We may thus express the limiter output as

$$z(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos[2\pi f_c t(2k+1) + \phi(t)(2k+1)] \tag{2}$$

(b) Consider the term

$$\cos[2\pi f_c t(2k+1) + \phi(t)\ (2k+1)] = \mathrm{Re}\{\exp[j2\pi f_c t(2k+1)]\exp[j\phi(t)(2k+1)]\}$$
$$= \mathrm{Re}\{\exp[j2\pi f_c t(2k+1)][\exp(j\phi(t))]^{2k+1}\}$$

The function $\exp[j\phi(t)]$, representing the complex envelope of the FM wave with unit amplitude, is effectively low-pass in nature. Therefore, this term represents a band-pass signal centered about $\pm f_c(2k+1)$. Furthermore, the Fourier transform of $\{\exp[j\phi(t)]\}^{2k+1}$ is equal to that of $\exp[j\phi(t)]$ convolved with itself 2k times. Therefore, assuming that $\exp[j\phi(t)]$ is limited to the interval $-B_T/2 \le f \le B_T/2$, we find that $(\exp[j\phi(t)])^{2k+1}$ is limited to the interval $-(B_T/2)(2k+1) \le f \le (B_T/2)(2k+1)$.

Assuming that $f_c > B_T$, as is usually the case, we find that none of the terms corresponding to values of k greater than zero will overlap the spectrum of the term corresponding to k=0. Thus, if the limiter output is applied to a band-pass filter of bandwidth B_T and mid-band frequency f_c, all terms, except the term corresponding to k=0 in Eq. (2), are removed by the filter. The resulting filter output is therefore

$$y(t) = \frac{4}{\pi} \cos[2\pi f_c t + \phi(t)]$$

We thus see that by using the amplitude limiter followed by a band-pass filter, the effect of amplitude variation, represented by a(t) in the modulated wave s(t), is completely removed.

Problem 2.44

(a) Let the FM wave be defined by

$$s(t) = A_c \cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)\, dt]$$

Assuming that f_c is large compared to the bandwidth of s(t), we may express the complex envelope of s(t) as

$$\tilde{s}(t) = A_c \exp[j2\pi k_f \int_0^t m(t)\, dt]$$

But, by definition, the pre-envelope of s(t) is (see Appendix 2)

$$s_+(t) = \tilde{s}(t) \exp(j2\pi f_c t)$$

$$= s(t) + j\,\hat{s}(t)$$

where $\hat{s}(t)$ is the Hilbert transform of $s(t)$. Therefore,

$$s(t) + j\hat{s}(t) = A_c \exp[j2\pi k_f \int_0^t m(t)\,dt] \exp(j2\pi f_c t)$$

$$= A_c\{\cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt] + j \sin[2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt]\}$$

Equating real and imaginary parts, we deduce that

$$\hat{s}(t) = A_c \sin[2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt] \qquad (1)$$

(b) For the case of sinusoidal modulation, we have

$$m(t) = A_m \cos(2\pi f_m t)$$

The corresponding FM wave is

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

where

$$\beta = k_f A_m$$

Expanding s(t) in the form of a Fourier series, we get

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c+nf_m)t]$$

Noting that the Hilbert transform of $\cos[2\pi(f_c+nf_m)t]$ is equal to $\sin[2\pi(f_c+nf_m)t]$, and using the linearity property of the Hilbert transform, we find that the Hilbert transform of s(t) is

$$\hat{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \sin[2\pi(f_c+nf_m)\,t]$$

$$= A_c \sin[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

This is exactly the same result as that obtained by using Eq. (1). In the case of sinusoidal modulation, therefore, there is no error involved in using Eq. (1) to evaluate the Hilbert transform of the corresponding FM wave.

Problem 2.45

(a) The modulated wave s(t) is

$$s(t) = \exp[-\hat{\phi}(t)] \cos[2\pi f_c t + \phi(t)]$$

$$= \mathrm{Re}\{\exp[-\hat{\phi}(t)] \exp[j2\pi f_c t + j\phi(t)]\}$$
$$= \mathrm{Re}\{\exp[j2\pi f_c t + j(\phi(t) + j\hat{\phi}(t))]\}$$
$$= \mathrm{Re}\{\exp[j2\pi f_c t + j\phi_+(t)]\} \tag{1}$$

where $\phi_+(t)$ is the pre-envelope of the phase function $\phi(t)$, that is,

$$\phi_+(t) = \phi(t) + j\hat{\phi}(t)$$

Expanding the exponential function $\exp[j\phi_+(t)]$ in the form of an infinite series:

$$\exp[j\phi_+(t)] = \sum_{n=0}^{\infty} \frac{j^n}{n!} \phi_+^n(t) \tag{2}$$

Taking the Fourier transform of both sides of this relation, we may write

$$F\{\exp[j\phi_+(t)]\} = \sum_{n=0}^{\infty} \frac{j^n}{n!} F[\phi_+^n(t)]$$

For $n \geq 2$, we may express $\phi_+^n(t)$ as the product of $\phi_+(t)$ and $\phi_+^{n-1}(t)$. Hence,

$$F[\phi_+^n(t)] = \Phi_+(f) \star F[\phi_+^{n-1}(t)]$$

where $\phi_+(t) \rightleftharpoons \Phi_+(f)$, and $\star$ denotes convolution. Since $\Phi_+(f) = 0$ for $f < 0$, it follows that for all $n \geq 0$,

$$F[\Phi_+^n(t)] = 0, \qquad \text{for } f < 0$$

Hence,

$$F\{\exp[j\phi_+(t)]\} = 0 \qquad \text{for } f < 0$$

By using the frequency-shifting property of the Fourier transform, it follows that

$$F\{\exp[j\phi_+(t)] \exp(j2\pi f_c t)\} = 0 \quad \text{for } f < f_c \tag{3}$$

From Eq. (1),

$$s(t) = \frac{1}{2} \{\exp[j2\pi f_c t + j\phi_+(t)] + \exp[-j2\pi f_c t - j\phi_+^*(t)]\}$$

where $\phi_+^*(t)$ is the complex conjugate of $\phi_+(t)$. Therefore,

$$F[s(t)] = \frac{1}{2} F\{\exp[j2\pi f_c t + j\phi_+(t)]\} + \frac{1}{2} F\{\exp[-j2\pi f_c t - j\phi_+^*(t)\}$$

Applying the conjugate-function property of the Fourier transform to Eq. (3), we find that

$$F\{\exp[-j2\pi f_c t - j\phi_+^*(t)]\} = 0, \quad \text{for } f > -f_c.$$

Hence, it follows that the spectrum of s(t) is zero for $-f_c < f < f_c$. However, this spectrum is of infinite extent, because the expansion of s(t) contains an infinite number of terms, as in eq. (2).

(b) With

$$\phi(t) = \beta \sin(2\pi f_m t),$$

we find that

$$\hat{\phi}(t) = -\beta \cos(2\pi f_m t)$$

Therefore,

$$\begin{aligned}\phi_+(t) &= \beta \sin(2\pi f_m t) - j\beta \cos(2\pi f_m t)\\ &= -j\beta[\cos(2\pi f_m t) + j \sin(2\pi f_m t)]\\ &= -j\beta \exp(j2\pi f_m t)\end{aligned}$$

Hence,

$$\begin{aligned}\exp[j\phi_+(t)] &= \exp[\beta \exp(j2\pi f_m t)]\\ &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \exp(j2\pi n f_m t)\end{aligned}$$

The modulated wave s(t) is therefore

$$\begin{aligned}s(t) &= \mathrm{Re}\{\exp(j2\pi f_c t) \exp[j\phi_+(t)]\}\\ &= \mathrm{Re}[\exp(j2\pi f_c t) \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \exp(j2\pi n f_m t)]\\ &= \mathrm{Re}\{ \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \exp[j2\pi(f_c+nf_m)t]\}\\ &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \cos[2\pi(f_c+nf_m)t]\end{aligned}$$

Problem 2.46

After passing the received signal through a narrow-band filter of bandwidth 8kHz centered on $f_c = 200$kHz, we get

$$x(t) = A_c m(t)\cos(2\pi f_c t) + n'(t)$$

$$= A_c m(t)\cos(2\pi f_c t) + n'_I(t)\cos(2\pi f_c t) - n'_I(t)\sin(2\pi f_c t)$$

$$= (A_c m(t) + n_I(t))\cos(2\pi f_c t) - n'_Q(t)\sin(2\pi f_c t)$$

where $n'(t)$ is the narrow-band noise produced at the filter output, and $n'_I(t)$ and $n'_Q(t)$ are its in-phase and quadrature components. Coherent detection of $x(t)$ yields the output

$$y(t) = A_c m(t) + n'_I(t)$$

The average power of the modulated wave is

$$\frac{A_c^2 P}{4} = 10W$$

where P is the average power of $m(t)$. To calculate the average power of the in-phase noise component $n'_I(t)$, we refer to the spectra shown in Fig. 1:

- Part (a) of Fig. 1 shows the power spectral density of the noise $n(t)$, and a superposition of the frequency response of the narrow-band filter.
- Part (b) shows the power spectral density of the noise $n'_I(t)$ produced at the filter output.
- Part (c) shows the power spectral density of the in-phase component $n'_I(t)$ of $n'(t)$.

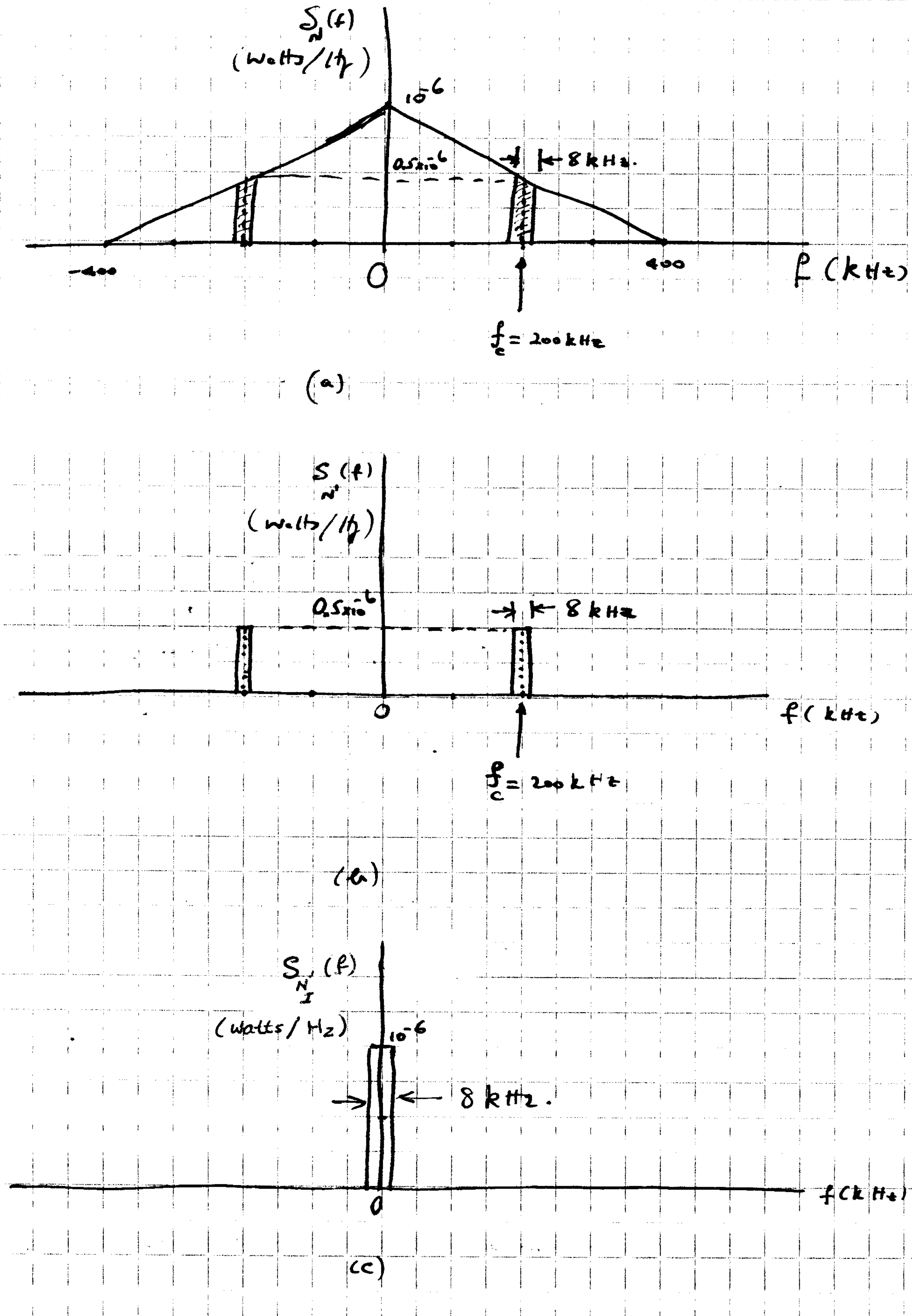
Note that since the bandwidth of the filter is small compared to the carrier frequency f_c, we have approximated the spectral characteristic of $n'(t)$ to be flat at the level of 0.5 x 10^{-6} watts/Hz. Hence, the average power of $n'_I(t)$ is (from Fig. 1c):

$$(10^{-6} \text{ watts/Hz})\ (8 \times 10^3) = 0.008 \text{ watts}$$

The output signal-to-noise ratio (SNR) is therefore

$$\frac{10}{0.008} = 1,250$$

Expressing this result in decibels, we have an output SNR of 31 dB.



S_N(f)
(Watts/Hz)
10^-6
0.5x10^-6
8 kHz.
-400
0
400
f (kHz)
f_c = 200 kHz
(a)
S_N'(f)
(Watts/Hz)
0.5x10^-6
8 kHz
0
f (kHz)
f_c = 200 kHz
(b)
S_N'_I (f)
(watts/Hz)
10^-6
8 kHz.
0
f (kHz)
(c)

Figure 1

Problem 2.47

From Problem 5.38, we note that the quadrature components of a narrow-band noise have autocorrelations:

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi f_c\tau) + \hat{R}_N(\tau)\sin(2\pi f_c\tau)$$

where $R_N(\tau)$ is the autocorrelation of the narrow-band noise, $\hat{R}_N(\tau)$ is the Hilbert transform of $R_N(\tau)$, and f_c is the band center. The cross-correlations of the quadrature components are

$$R_{N_IN_Q}(\tau) = -R_{N_IN_Q}(\tau) = R_N(\tau)\sin(2\pi f_c\tau) - \hat{R}_N(\tau)\cos(2\pi f_c\tau)$$

(a) For a DSBSC system,

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi f_c\tau) + \hat{R}_N(\tau)\sin(2\pi f_c\tau)$$

$$R_{N_IN_Q}(\tau) = -R_{N_QN_I}(\tau) = R_N(\tau)\sin(2\pi f_c t) - \hat{R}_N(\tau)\cos(2\pi f_c\tau)$$

where f_c is the carrier frequency, and $R_N(\tau)$ is the autocorrelation function of the narrow-band noise on the interval $f_c - W \leq f \leq f_c + W$.

(b) For an SSB system using the lower sideband,

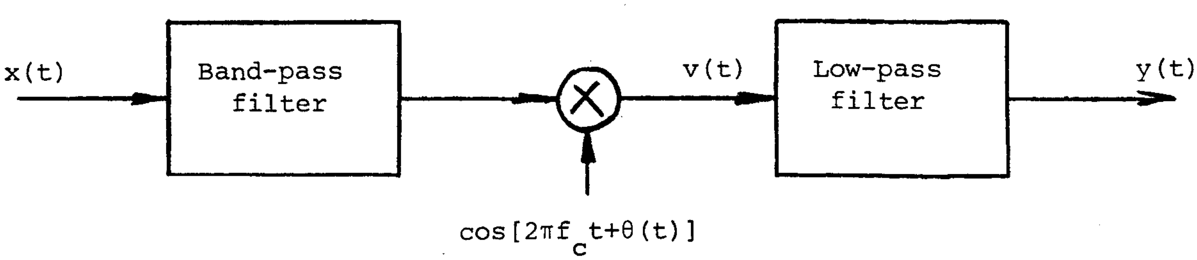
$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi(f_c - \tfrac{W}{2})\tau) + \hat{R}_N(\tau)\sin(2\pi(f_c - \tfrac{W}{2})\tau)$$

$$R_{N_IN_Q}(\tau) = -R_{N_QN_I}(\tau) = R_N(\tau)\sin(2\pi(f_c - \tfrac{W}{2})\tau) - \hat{R}_N(\tau)\cos(2\pi(f_c - \tfrac{W}{2})\tau)$$

where in this case, $R_N(\tau)$ is the autocorrelation of the narrow-band noise on the interval $f_c - W \leq f \leq f_c$.

(c) For an SSB system with only the upper sideband transmitted, the correlations are similar to (b) above, except that $(f_c - \frac{W}{2})$ is replaced by $(f_c + \frac{W}{2})$, and the narrow-band noise is on the interval $f_c \leq f \leq f_c + W$.

Problem 2.48



The signal at the mixer input is equal to $s(t) + n(t)$, where $s(t)$ is the modulated wave, and $n(t)$ is defined by

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

with

$$E[n_I^2(t)] = E[n_Q^2(t)] = N_0 B_T$$

The $s(t)$ is defined by for DSB-SC modulation

$$s(t) = A_c m(t)\cos(2\pi f_c t)$$

The mixer output is

$$v(t) = [s(t) + n(t)]\cos[2\pi f_c t + \theta(t)]$$

$$= \{[A_c m(t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)\}\cos 2\pi f_c t + \theta(t)]$$

$$= \frac{1}{2}[A_c m(t) + n_I(t)\{\cos[\theta(t)]\} + \cos[4\pi f_c t + \theta(t)]$$

$$+ \frac{1}{2}A_c n_Q(t)\{\sin[\theta(t)] - \sin[4\pi f_c t + \theta(t)]\}$$

The postdetection low-pass filter removes the high frequency components of $v(t)$, producing the output

$$y(t) = \frac{1}{2}[[A_c m(t) + n_I(t)]\cos[\theta(t)] + \frac{1}{2}A_c n_Q(t)\sin[\theta(t)] \tag{1}$$

When the phase error $\theta(t)$ is zero, we find that the message signal component of the receiver output is $\frac{1}{2}A_c m(t)$. The error at the receiver output is therefore

$$e(t) = y(t) - \frac{A_c}{2}m(t)$$

The mean-square value of this error is

$$\varepsilon = E[e^2(t)]$$

$$= E\left[\left(y(t) - \frac{A_c}{2}m(t)\right)^2\right] \tag{2}$$

Substituting Eq. (1) into (2), expanding the expectation, and noting that the processes $m(t)$, $\theta(t)$, $n_I(t)$ and $n_Q(t)$ are all independent of one another, we get

$$\varepsilon = \frac{A_c^2}{4}E[m^2(t)]E[(\cos^2\theta(t))] + \frac{1}{4}E[n_I^2(t)]E[\cos^2\theta(t)]$$

$$+ \frac{1}{4}E[n_Q^2(t)]E[\sin^2\theta(t)]$$

$$+ \frac{A_c^2}{4}E[m^2(t)] - \frac{A_c^2}{2}E[m^2(t)]E[\cos\theta(t)]$$

We now note that

$$E[n_I^2(t)] = E[n_Q^2(t)] = \sigma_N^2$$

$$E[n_I^2(t)]E[\cos^2\theta(t)] + E[n_Q^2(t)]E[\sin^2\theta(t)] = \sigma_N^2$$

Therefore,

$$\varepsilon = \frac{A_c^2}{2}E[m^2(t)]E\{[1 - \cos\theta(t)]^2\} + \frac{\sigma_N^2}{4}$$

$$= \frac{A_c^2 P}{4}E\{[1 + \cos\theta(t)]^2\} + \frac{\sigma_N^2}{4}$$

where $P = E[m^2(t)]$.

For small values of $\theta(t)$, we may use the approximation

$$1 - \cos\theta(t) \approx \frac{\sigma_N^2}{2}$$

Hence,

$$\varepsilon = \frac{A_c^2 P}{16} E[\theta^4(t)] + \frac{\sigma_N^2}{4}$$

Since $\theta(t)$ is Gaussian-distributed with zero mean and variance σ_θ^2, we have

$$E[\theta^4(t)] = 3\sigma_\theta^4$$

The mean-square error for the case of a DSBSC system is therefore

$$\varepsilon = \frac{3A_c^2 P \sigma_\theta^4}{16} + \frac{\sigma_N^2}{4}$$

Problem 2.49

Consider the case of a receiver using coherent detection, with an incoming single-sideband (SSB) modulated wave. We assume that only the lower sideband is transmitted, so that we can express the modulated wave as

$$s(t) = \frac{1}{2}CA_c\cos(2\pi f_c t)m(t) + \frac{1}{2}CA_c\sin(2\pi f_c t)\hat{m}(t) \tag{1}$$

where $\hat{m}(t)$ is the Hilbert transform of the message signal $m(t)$. The system-dependent scaling factor C is included to make the signal component $s(t)$ have the same units as the noise component $n(t)$. We may make the following observations concerning the in-phase and quadrature components of $s(t)$ in Eq. (1):

1. The two components $m(t)$ and $\hat{m}(t)$ are orthogonal to each other. Therefore, with the message signal $m(t)$ assumed to have zero mean, which is a reasonable assumption to make, it follows that $m(t)$ and $\hat{m}(t)$ are uncorrelated; hence, their power spectral densities are additive.
2. The Hilbert transform $\hat{m}(t)$ is obtained by passing $m(t)$ through a linear filter with a transfer function $-j\text{sgn}(f)$. The squared magnitude of this transfer function is equal to one for all f. Accordingly, we find that both $m(t)$ and $\hat{m}(t)$ have the same power spectral density.

Thus, using a procedure similar to that in Section 2.11, we find that the in-phase and quadrature components of the modulated signal $s(t)$ contribute an average power of $C^2A_c^2P/8$ each, where P is the average power of the message signal $m(t)$. The average power of $s(t)$ is therefore $C^2A_c^2P/4$. This result is half that in the DSB-SC receiver, which is intuitively satisfying.

The average noise power in the message bandwidth W is WN_0, as in the DSB-SC receiver. Thus the channel signal-to-noise ratio of a coherent receiver with SSB modulation is

$$(\text{SNR})_{C,\,\text{SSB}} = \frac{C^2A_c^2P}{4WN_0} \tag{2}$$

As illustrated in Fig. 1a, in an SSB system the transmission bandwidth B_T is W and the mid-band frequency of the power spectral density $S_N(f)$ of the narrow-band noise $n(t)$ is offset from the carrier frequency f_c by $W/2$. Therefore, we may express $n(t)$ as

$$n(t) = n_I(t)\cos\left[2\pi\left(f_c - \frac{W}{2}\right)t\right] - n_Q(t)\sin\left[2\pi\left(f_c - \frac{W}{2}\right)t\right] \tag{3}$$

The output of the coherent detector, due to the combined influence of the modulated signal $s(t)$ and noise $n(t)$, is thus given by

$$y(t) = \frac{1}{4}CA_c m(t) + \frac{1}{2}n_I(t)\cos(\pi Wt) + \frac{1}{2}n_Q(t)\sin(\pi Wt) \tag{4}$$

As expected, we see that the quadrature component $\hat{m}(t)$ of the modulated message signal $s(t)$ has been eliminated from the detector output, but unlike the case of DSB-SC modulation, the quadrature component of the narrow-band noise $n(t)$ now appears at the receiver output.

The message component in the receiver output is $CA_c m(t)/4$, and so we may express the average power of the recovered message signal as $C^2 A_c^2 P/16$. The noise component in the receiver output is $[n_I(t)\cos(\pi Wt) + n_Q(t)\sin(\pi Wt)]/2$. To determine the average power of the output noise, we note the following:

1. The power spectral density of both $n_I(t)$ and $n_Q(t)$ is as shown in Fig. 1b.
2. The sinusoidal wave $\cos(\pi Wt)$ is independent of both $n_I(t)$ and $n_Q(t)$. Hence, the power spectral density of $n'_I(t) = n_I(t)\cos(\pi Wt)$ is obtained by shifting $S_{N_I}(f)$ to the left by $W/2$, shifting it to the right by $W/2$, adding the shifted spectra, and dividing the result by 4. The power spectral density of $n'_Q(t) = n_Q(t)\sin(\pi Wt)$ is obtained in a similar way. The power spectral density of both $n'_I(t)$ and $n'_Q(t)$, obtained in this manner, is shown sketched in Fig. 1c.

From Fig. 1c we see that the average power of the noise component $n'_I(t)$ or $n'_Q(t)$ is $WN_0/2$. Therefore from Eq. (4), the average output noise power is $WN_0/4$. We thus find that the output signal-to-noise ratio of a system, using SSB modulation in the transmitter and coherent detection in the receiver, is given by

$$(\mathrm{SNR})_{O,\,\mathrm{SSB}} = \frac{C^2 A_c^2 P}{4WN_0} \tag{5}$$

Hence, from Eqs. (2) and (5), the figure of merit of such a system is

$$\left.\frac{(\mathrm{SNR})_O}{(\mathrm{SNR})_C}\right|_{\mathrm{SSB}} = 1 \tag{6}$$

where again we see that the factor C^2 cancels out.

Comparing Eqs. (5) and (6) with the corresponding results for DSB-SC modulation, we conclude that *for the same average transmitted (or modulated message) signal power and the same average noise power in the message bandwidth, an SSB receiver will have exactly the same output signal-to-noise ratio as a DSB-SC receiver, when both receivers use coherent detection for the recovery of the message signal.* Furthermore, in both cases, the noise performance of the receiver is the as that obtained by simply transmitting the message signal itself in the presence of the same noise. The only effect of the modulation process is to translate the message signal to a different frequency band to facilitate its transmission over a band-pass channel.

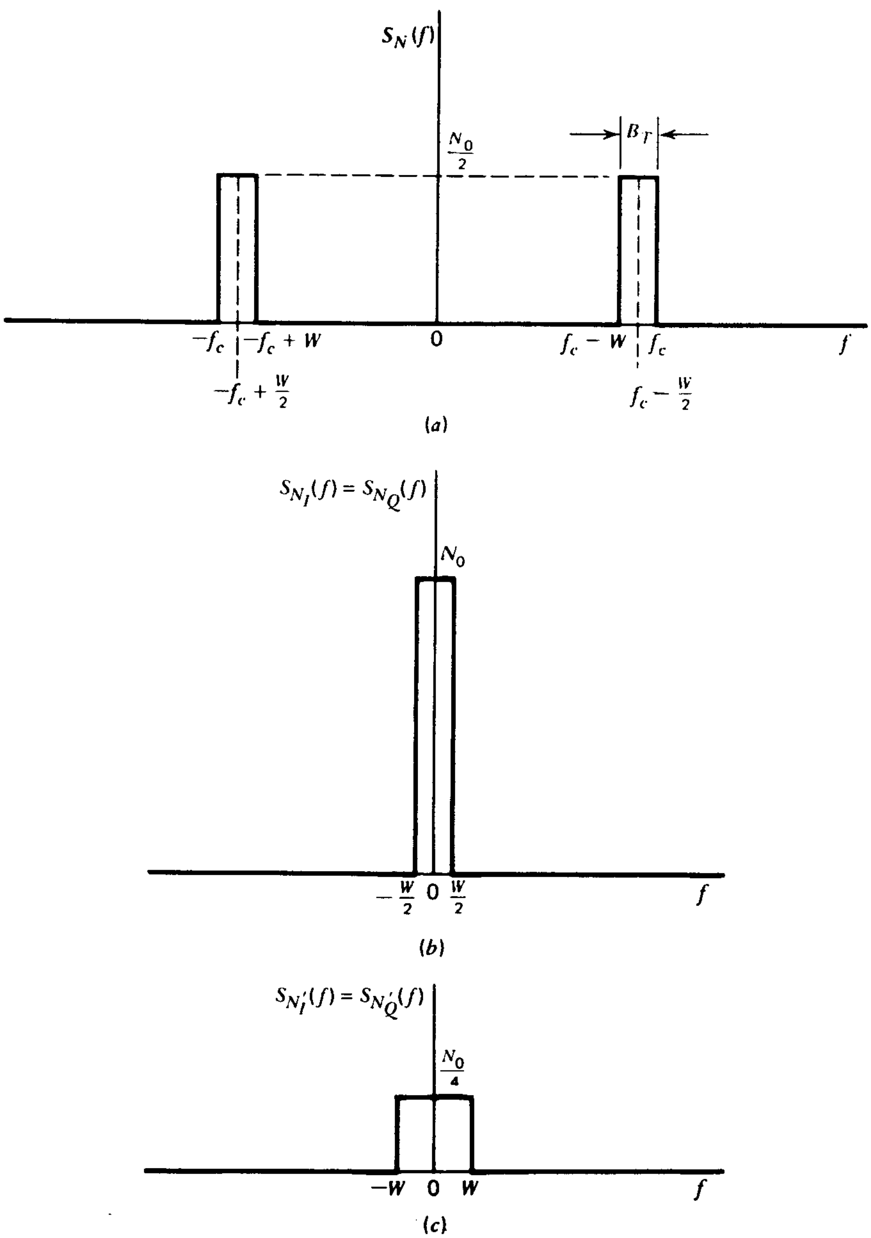
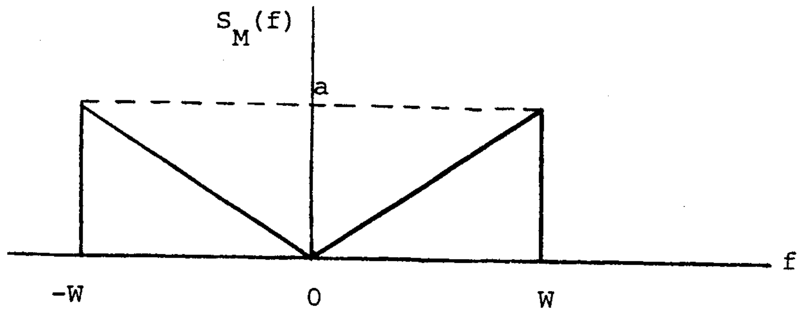


Figure 1

Problem 2.50

The power spectral density of the message signal m(t) is as follows



The average signal power is therefore

$$P = \int_{-\infty}^{\infty} S_M(f)\,df$$

$$= 2\int_0^W a\,\frac{f}{W}\,df$$

$$= aW$$

The corresponding value of the output signal-to-noise ratio of the SSB receiver is therefore, (using the solution to Problem 2.49)

$$(SNR)_O = \frac{A_c^2 P}{4WN_0}$$

$$= \frac{A_c^2 aW}{4WN_0}$$

$$= \frac{a A_c^2}{4N_0}$$

Problem 2.51

(a) If the probability

$$P(|n_s(t)| > \varepsilon A_c|1 + k_a m(t)|) \le \delta_1 ,$$

then, with a probability greater than $1 - \delta_1$, we may say that

$$y(t) \approx \{[A_c + A_c k_a m(t) + n_c(t)]^2\}^{1/2}$$

That is, the probability that the quadrature component $n_s(t)$ is negligibly small is greater than $1 - \delta_1$.

(b) Next, we note that if $k_a m(t) < -1$, then we get overmodulation, so that even in the absence of noise, the envelope detector output is badly distorted. Therefore, in order to avoid overmodulation, we assume that k_a is adjusted relative to the message signal m(t) such that the probability

$$P(A_c + A_c k_a m(t) + n_c(t) < 0) = \delta_2$$

Then, the probability of the event

$$y(t) \approx A_c[1 + k_a m(t)] + n_c(t)$$

for any value of t, is greater than $(1 - \delta_1)(1 - \delta_2)$.

(c) When δ_1 and δ_2 are both small compared with unity, we find that the probability of the event

$$y(t) \approx A_c[1 + k_a m(t)] + n_c(t)$$

for any value of t, is very close to unity. Then, the output of the envelope detector is approximately the same as the corresponding output of a coherent detector.

Problem 2.52

The received signal is

$$x(t) = A_c \cos(2\pi f_c t) + n(t)$$

$$= A_c \cos(2\pi f_c t) + n_c(t) \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

$$= [A_c + n_c(t)] \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

The envelope detector output is therefore

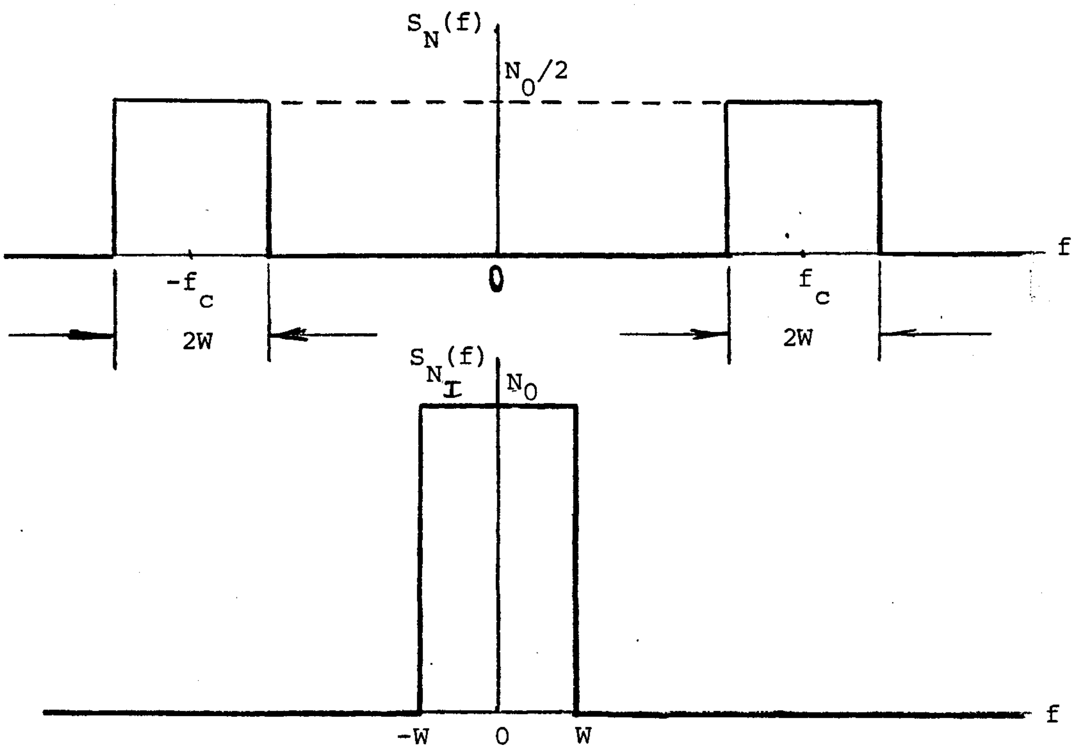
$$a(t) = \{[A_c + n_c(t)]^2 + n_s^2(t)\}^{1/2}$$

For the case when the carrier-to-noise ratio is high, we may approximate this result as

$$a(t) \simeq A_c + n_c(t)$$

The term A_c represents the useful signal component. The output signal power is thus A_c^2.

The power spectral densities of n(t) and $n_I(t)$ are as shown below:



The output noise power is $2N_0W$. The output signal-to-noise ratio is therefore

$$(\text{SNR})_0 = \frac{A_c^2}{2N_0W}$$

Problem 2.53

(a) From Section 1.12 of the textbook we recall that the envelope $r(t)$ of the narrow-band noise $n(t)$ is Rayleigh distributed; that is

$$f_R(r) = \frac{r}{\sigma_N^2}\exp\left(-\frac{r^2}{2\sigma_N^2}\right)$$

where σ_N^2 is the variance of the noise $n(t)$. For an AM system, the variance σ_N^2 is $2WN_0$. Therefore, the probability of the event that the envelope R of the narrow-band noise $n(t)$ is large compared to the carrier amplitude A_c is defined by

$$P(R \geq A_c) = \int_{A_c}^{\infty} f_R(r)dr$$

$$= \int_{A_c}^{\infty} \frac{r}{2WN_0}\exp\left(-\frac{r^2}{4WN_0}\right)dr$$

$$= \exp\left(-\frac{A_c^2}{4WN_0}\right) \tag{1}$$

Define the *carrier to noise ratio* as

$$\rho = \frac{\text{average carrier power}}{\text{average noise power in bandwidth of the modulated message signal}} \tag{2}$$

Since the bandwidth of the AM signal is $2W$, the average noise power in this bandwidth is $2WN_0$. The average power of the carrier is $A_c^2/2$. The carrier-to-noise ratio is therefore

$$\rho = \frac{A_c^2}{4WN_0} \tag{3}$$

(b) We may now use this definition to rewrite Eq. (1) in the compact form

$$P(R \geq A_c) = \exp(-\rho) \tag{4}$$

Solving $P(R \geq A_c) = 0.5$ for ρ, we get

$\rho = \log 2 = 0.69$

Similarly, for $P(R \geq A_c) = 0.01$, we get

$\rho = \log 100 = 4.6$

Thus with a carrier-to-noise ratio $10\log_{10}0.69 = -1.6$ dB, the envelope detector is expected to be well into the threshold region, whereas with a carrier-to-noise ratio $10\log_{10}4.6 = 6.6$ dB, the detector is expected to be operating satisfactorily. We ordinarily need a signal-to-noise ratio considerably greater than 6.6 dB for satisfactory intelligibility, and therefore threshold effects are seldom of great importance in AM receivers using envelope detection.

Problem 2.54

(a) Following a procedure similar to that described for the case of an FM system, we find that the input of the phase detector is

$$v(t) = A_c \cos[2\pi f_c t + \theta(t)]$$

where

$$\theta(t) = k_p\, m(t) + \frac{n_Q(t)}{A_c}$$

with $n_Q(t)$ denoting the quadrature noise component. The output of the phase discriminator is therefore,

$$y(t) = k_p\, m(t) + \frac{n_Q(t)}{A_c}$$

The message signal component of $y(t)$ is equal to $k_p\, m(t)$. Hence, the average output signal power is $k_p^2 P$, where P is the message signal power.

With the post detection low-pass filter following the phase detector restricted to the message bandwidth W, we find that the average output noise power is $2WN_0/A_c^2$.

Hence, the output signal-to-noise ratio of the PM system is

$$(SNR)_O = \frac{k_p^2 P A_c^2}{2WN_0}$$

(b) The channel signal-to-noise ratio of the PM system is the same as that of the corresponding FM system. That is,

$$(SNR)_O = \frac{A_c^2}{2WN_0}$$

The figure of merit of the PM system is therefore equal to k_p^2 P.

For the case of sinusoidal modulation, we have

$$m(t) = A_m \cos(2\pi f_m t)$$

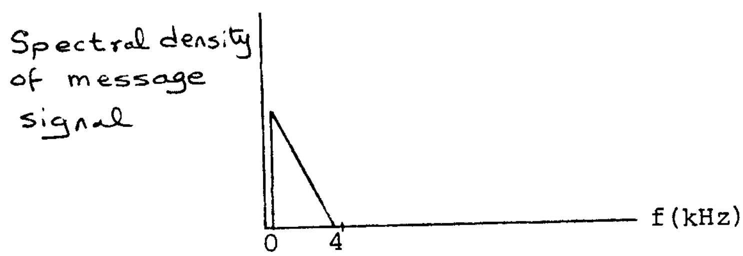
Hence,

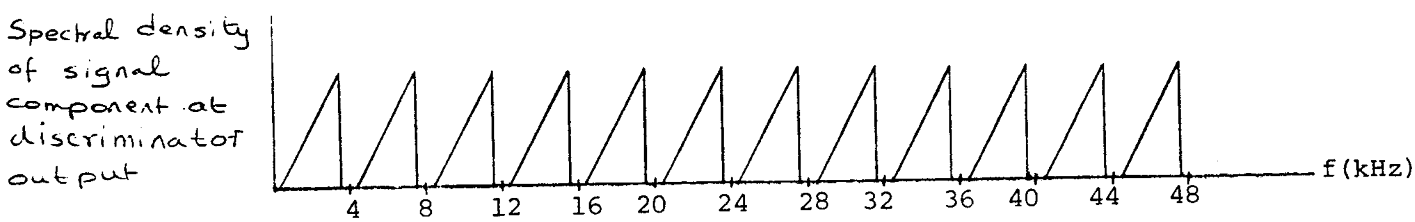
$$P = \frac{A_m^2}{2}$$

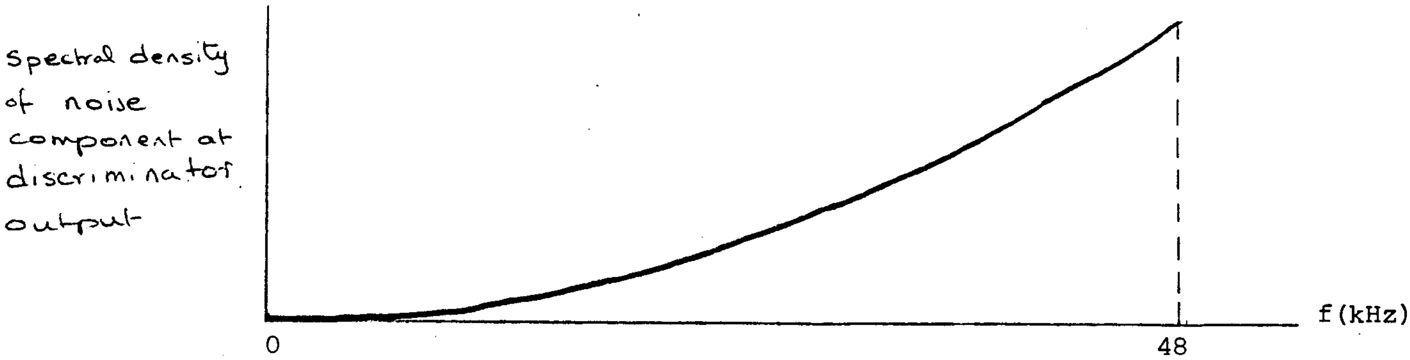
The corresponding value of the figure of merit for a PM system is thus equal to $\frac{1}{2}\beta_p^2$, where $\beta_p = k_p A_m$. On the other hand, the figure of merit for an FM system with sinusoidal modulation is equal to $\frac{3}{2}\beta^2$. We see therefore that for a specified phase deviation, the FM system is 3 times as good as the PM system.

Problem 2.55

(a) The power spectral densities of the original message signal, and the signal and noise components at the frequency discriminator output (for positive frequencies) are illustrated below:







(b) Each SSB modulated wave contains only the lower sideband. Let A_k and kf_0 denote the amplitude and frequency of the carrier used to generate the kth modulated wave, where f_0 = 4 kHz, and k = 1, 2, ..., 12. Then, we find that the kth modulated wave occupies the frequency interval $(k - 1)f_0 \le |f| \le kf_0$. We may define this modulated wave by

$$s_k(t) = \frac{A_k}{2} m(t) \cos(2\pi k f_0 t) + \frac{A_k}{2} \hat{m}(t) \sin(2\pi k f_0 t)$$

where m(t) is the original message signal, and $\hat{m}(t)$ is its Hilbert transform. Therefore, the average power of $s_k(t)$ is $A_k^2 P/4$, where P is the mean power of m(t).

We may express the output signal-to-noise ratio for the kth SSB modulated wave as follows:

$$(SNR)_0 = \frac{3A_c^2 k_f^2 (A_k^2 P/4)}{2N_0[k^3 f_0^3 - (k-1)^3 f_0^3]}$$

$$= \frac{3A_c^2 A_k^2 k_f^2 P}{8N_0 f_0^3 (3k^2 - 3k + 1)}$$

where A_c is the carrier amplitude of the FM wave. For equal signal-to-noise ratios, we must therefore choose the A_k so as to satisfy the condition

$$\frac{A_k^2}{3k^2 - 3k + 1} = \text{constant for } k = 1, 2, \ldots, 12.$$

Problem 2.56

The envelope $r(t)$ and phase $\psi(t)$ of the narrow-band noise $n(t)$ are defined by

$$r(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$$

$$\psi(t) = \tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right)$$

For a positive-going click to occur, we therefore require the following:

$n_I(t) \approx -A_c$

$n_Q(t)$ has a small positive value

$$\frac{d}{dt}\tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right) > 0$$

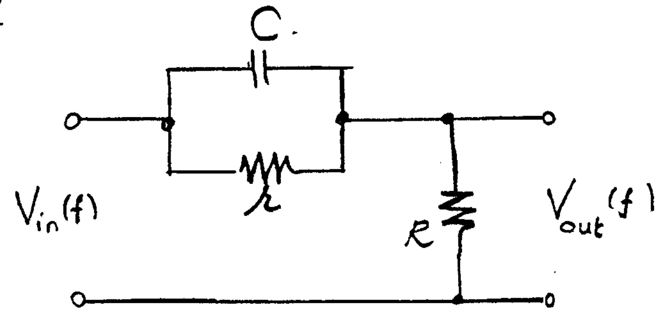
Correspondingly, for a negative-going click to occur, we require

$n_I(t) \approx A_c$

$n_Q(t)$ has a small negative value

$$\frac{d}{dt}\tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right) < 0$$

Problem 2.57



Let H(f) be $V_{out}(f)/V_{in}(f)$, or the transfer function of the filter. At low frequencies, the capacitor behaves as an open circuit. Then,

$$H(f) \simeq \frac{R}{r + R} \simeq \frac{R}{r}$$

Thus, the low frequencies of the input are frequency-modulated. At high frequencies, the capacitor behaves as a short circuit in relation to the resistor. Then,

$$H(f) \simeq \frac{R}{R + \frac{1}{j2\pi f C}} \simeq j2\pi f CR ,$$

and

$$v_{out}(t) \simeq RC \frac{d}{dt} v_{in}(t)$$

Frequency modulating the derivative of a waveform is equivalent to phase modulating the waveform. Thus, the high frequencies of the input are phase modulated.

Problem 2.58

(a) For the average power of the emphasized signal to be the same as the average power of the original message signal, we must choose the transfer function $H_{pe}(f)$ of the pre-emphasis filter so as to satisfy the relation

$$\int_{-\infty}^{\infty} S_M(f)df = \int_{-\infty}^{\infty} |H_{pe}|^2 S_M(f)df$$

With

$$S_M(f) = \begin{cases} \dfrac{S_0}{1 + (f/f_0)2}, & -W \leq f \leq W \\ 0, & \text{elsewhere.} \end{cases}$$

$$H_{pe}(f) = k(1 + \frac{jf}{f_0})$$

we have

$$\int_{-W}^{W} \frac{df}{1 + (f/f_0)^2} = k^2 \int_{-W}^{W} df$$

Solving for k, we get

$$k = [\frac{f_0}{W} \tan^{-1}(\frac{W}{f_0})]^{1/2} \tag{1}$$

(b) The improvement in output signal-to-noise ratio obtained by using pre-emphasis in the transmitter and de-emphasis in the receiver is defined by the ratio

$$D = \frac{2W^3}{3\int_{-W}^{W} f^2 |H_{de}(f)|^2 \, df}$$

$$= \frac{2W^3}{3\int_{-W}^{W} \frac{f^2}{k^2} \frac{df}{1 + (f/f_0)^2}}$$

$$= \frac{k^2 (W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]} \tag{2}$$

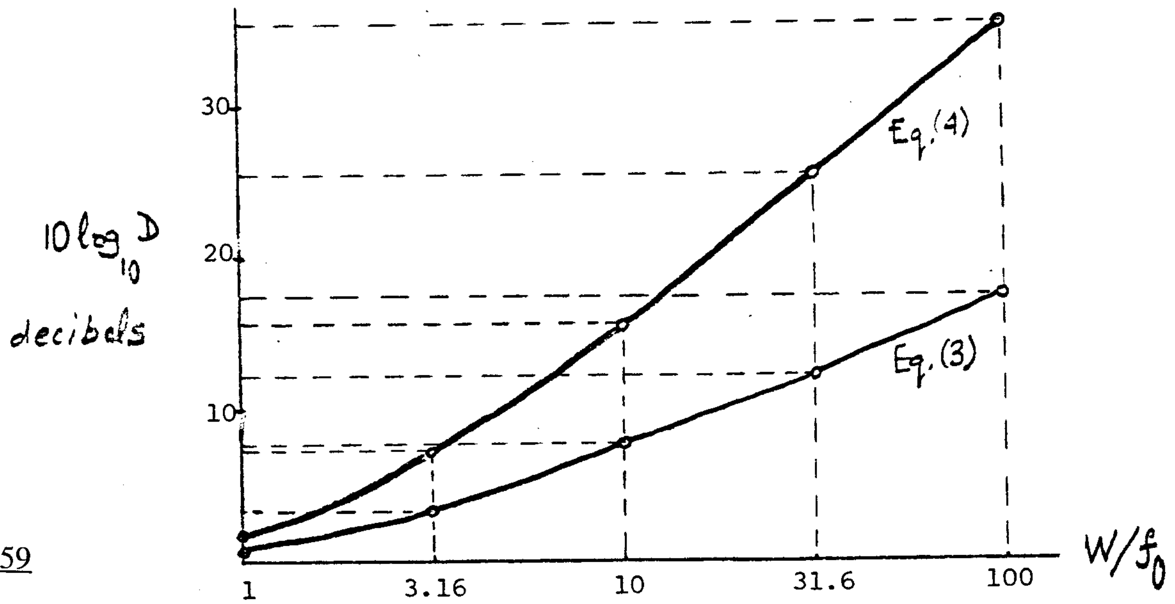
Substituting Eq. (1) in (2), we get

$$D = \frac{(W/f_0)^2 \tan^{-1}(W/f_0)}{3[(W/f_0) - \tan^{-1}(W/f_0)]} \tag{3}$$

This result applies to the case when the rms bandwidth of the FM system is maintained the same with or without pre-emphasis. When, however, there is no such constraint, we find from Example 4 of Chapter 6 that the corresponding value of D is

$$D = \frac{(W/f_0)^3}{3[(w/f_0) - \tan^{-1}(W/f_0)]} \tag{4}$$

In the diagram below, we have plotted the improvement D (expressed in decibels) versus the ratio W/f_0 for the two cases; when there is a transmission bandwidth constraint and when there is no such constraint:



Problem 2.59

In a PM system, the power spectral density of the noise at the phase discriminator output (in the absence of pre-emphasis and de-emphasis) is approximately constant. Therefore, the improvement in output signal-to-noise ratio obtained by using pre-emphasis in the transmitter and de-emphasis in the receiver of a PM system is given by

$$D = \frac{\int_0^W df}{\int_0^W |H_{de}(f)|^2 df}$$

With the transfer function $H_{de}(f)$ of the de-emphasis filter defined by

$$H_{de}(f) = \frac{1}{1 + (jf/f_0)},$$

we find that the corresponding value of D is

$$D = \frac{W}{\int_0^W \frac{df}{1 + (f/f_0)^2}}$$

$$= \frac{W/f_0}{\tan^{-1}(W/f_0)}$$

For the case when W = 15 kHz, f_0 = 2.1 kHz, we find that D = 5, or 7 dB. The corresponding value of the improvement ratio D for an FM system is equal to 13 dB (see Example 4 of Chapter 5). Therefore, the improvement obtained by using pre-emphasis and de-emphasis in a PM system is smaller by an amount equal to 6 dB.

Problem 2.60

Matlab codes

```
% Amplitude demodulation
%problem 2.60, CS: Haykin
% Mathini Sellathurai

clear all
Ac=1;
mue=0.5;
fc=20000;
fm=1000;
ts=1e-5;

% message signal
t=[0:250]*1e-5;
m=sin(2*pi*fm.*t);
plot(t, m)
xlabel('time (s)')
ylabel('Amplitude')
pause

% amplitude modulated signal
u=AM_mod(mue,m,ts,fc);
plot(t,u)
xlabel('time (s)')
ylabel('Amplitude')
pause

% demodulated signal
[t1, dem1]=AM_demod(mue,u,ts,fc);
plot(t1*ts, dem1)
xlabel('time (s)')
ylabel('Amplitude')


axis([0 2.5e-3 0 2])
```

```
function u=AM_mod(mue,m,ts,fc)
% Amplitude modulation
%used in problem 2.60, CS: Haykin
% Mathini Sellathurai
%

t=[0:length(m)-1]*ts;
c=cos(2*pi*fc.*t);
m_n=m/max(abs(m));
u=(1+mue*m_n).*c;
```

```
function [t, env]=AM_demod(mue,m,ts,fc)
% Amplitude demodulation
%used in problem 2.60, CS: Haykin
% Mathini Sellathurai
%

fs=1/ts;
fsofc=round(fs/fc);
n2=length(m);
v=zeros(1,round(n2/fsofc)); % initializing the envelope
R_L=1000; % load
C=0.01e-6; % capacitor


%demodulate the envelope
l=0; v(1)=m(1);
for k=1:fsofc:n2-fsofc
l=l+2;
v(l)=m(k)*exp(-ts/(R_L*C)/fsofc); % discharging
v(l+1)=m(k+fsofc); %charging
end

% envelope
t =0:fsofc/2:(length(v)-1)*fsofc/2;
env=v;
```

Answer to Problem 2.60

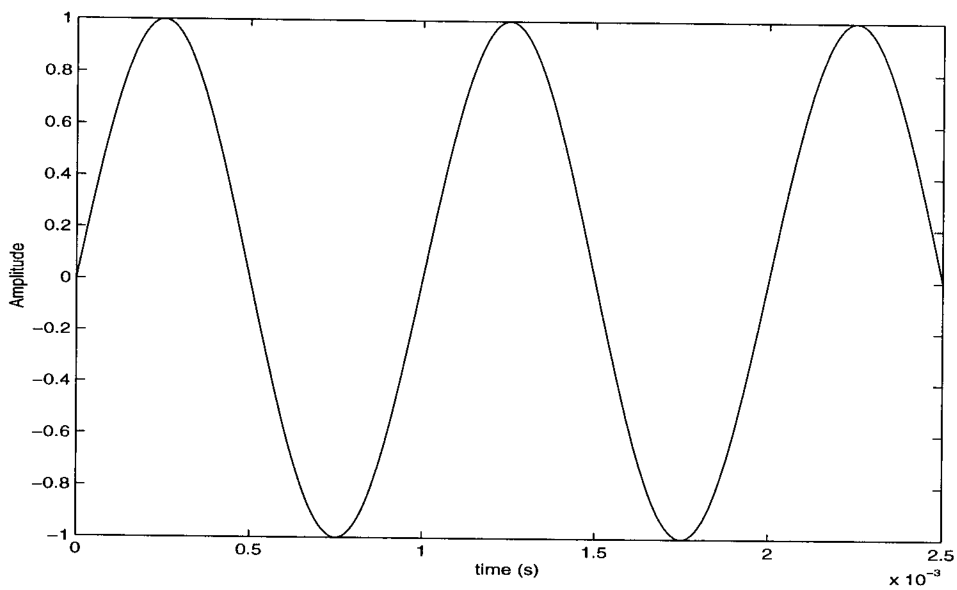


Figure 1: Message signal

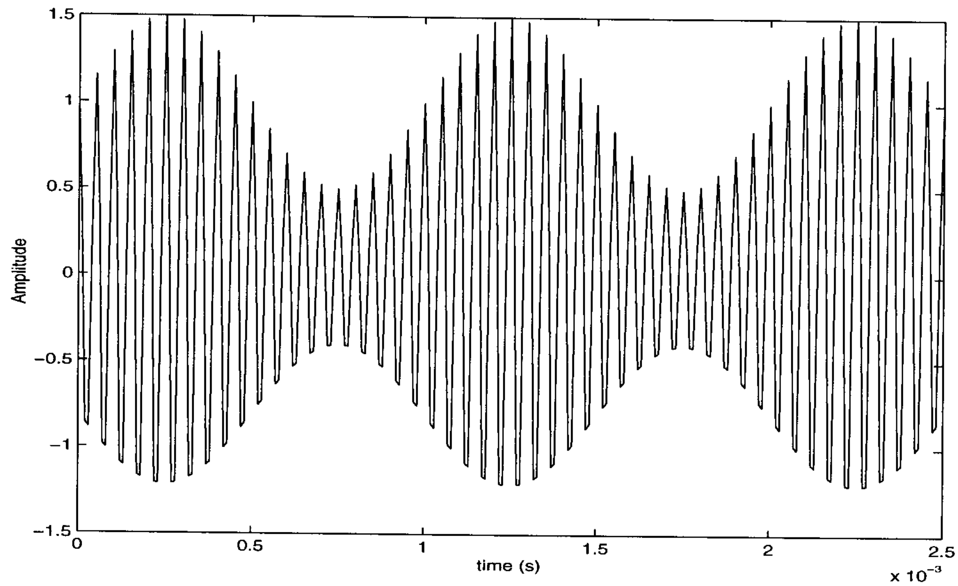


Figure 1: Amplitude modulated signal

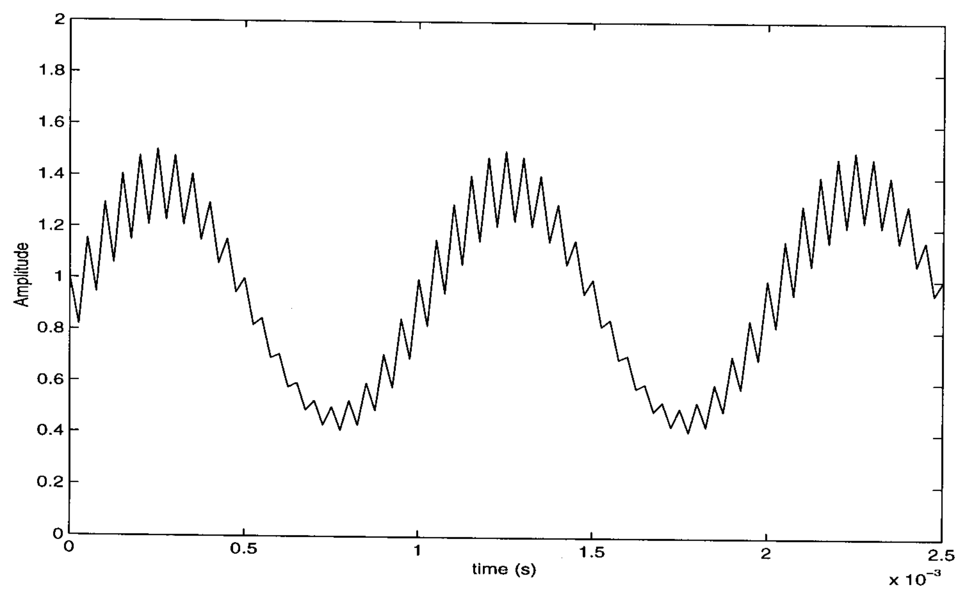


Figure 3: Demodulated signal

Problem 2.61

Matlab codes

```
% Problem 2.61 CS: Haykin
% phase lock loop and cycle slipping
% M. Sellathurai

% time interval
t0=0;tf=25;

% frequency step =0.125 Hz
delf=0.125;
u0=[0 -delf*2*pi];
[t,u]=ode23('lin',[t0 tf],u0); plot(t,u(:,2)/2/pi+delf);
xlabel('Time (s)')
ylabel('f_i (t), (Hz)')
pause

% frequency step =0.51 Hz
delf=0.5;
u0=[0 -delf*pi*2]';
[t,u]=ode23('lin',[t0 tf],u0); plot(t,u(:,2)/2/pi+delf);
xlabel('Time (s)');
ylabel('f_i (t), (Hz)');
pause;

% frequency step =7/12 Hz
delf=7/12;
u0=[0 -delf*pi*2]';
[t,u]=ode23('lin',[t0 tf],u0); plot(t,u(:,2)/2/pi+delf);
xlabel('Time (s)');
ylabel('f_i (t), (Hz)');
pause;

% frequency step =2/3 Hz
delf=2/3;
u0=[0 -delf*pi*2]';
[t,u]=ode23('lin',[t0 tf],u0); plot(t,u(:,2)/2/pi+delf);
xlabel('Time (s)');
ylabel('f_i (t), (Hz)');
```

```
function uprim =lin(t,u)
% used in Problem 2.61, CS: Haykin
% PLL
% Transfer function (1+as)/(1+bs),
% gain K=50/2/pi,
% natural frequency 1/2/pi
% damping 0.707
% Mathini Sellathurai

uprim(1)=u(2);
uprim(2)=-(1/50+1.2883*cos(u(1)))*u(2)-sin(u(1));
uprim=uprim';
```

Answer to Problem 2.61

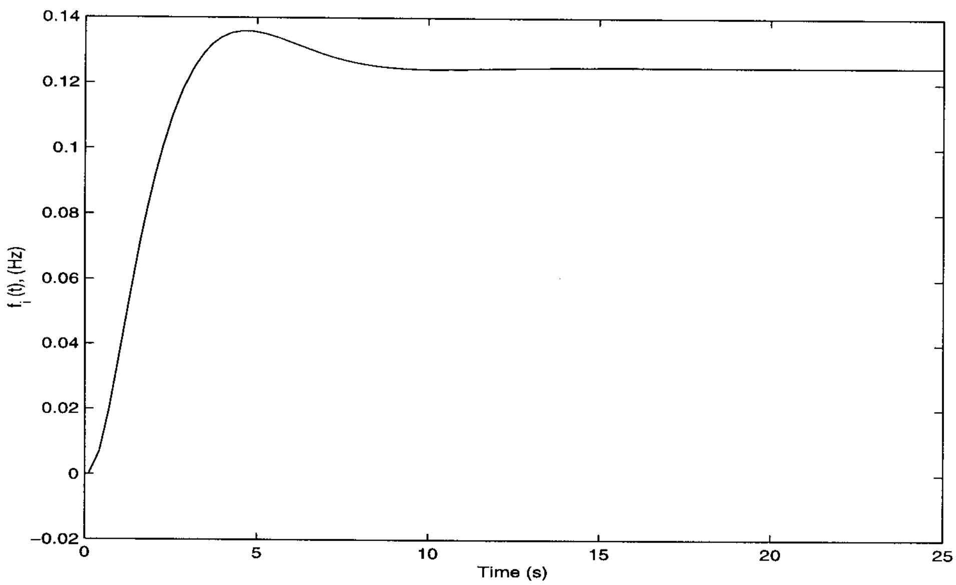


Figure 1: Variation in the instantaneous frequency of the PLL's voltage controlled oscillator for varying frequency step Δ f. (a) Δ f = 0.125 Hz

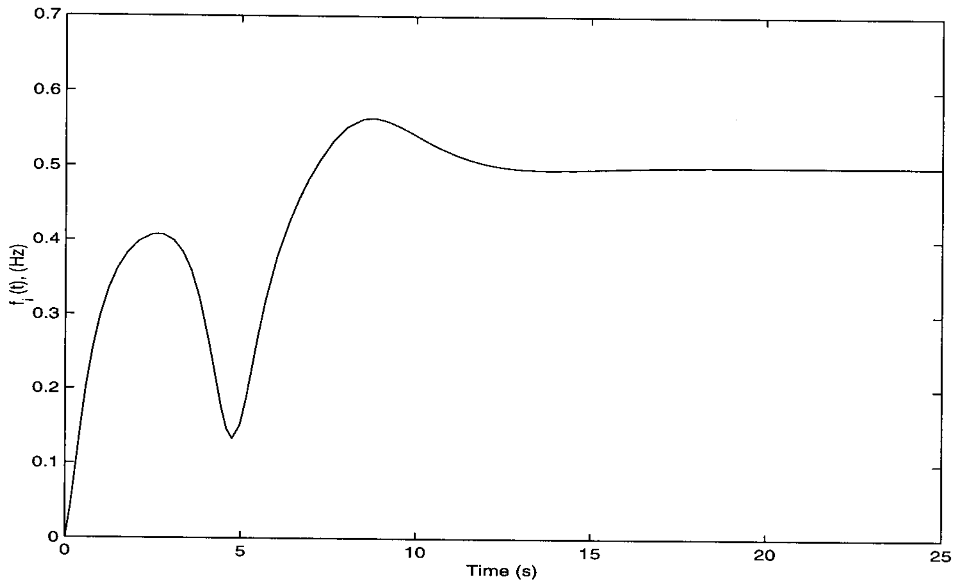


Figure 2: (b) Δ f = 0.5 Hz

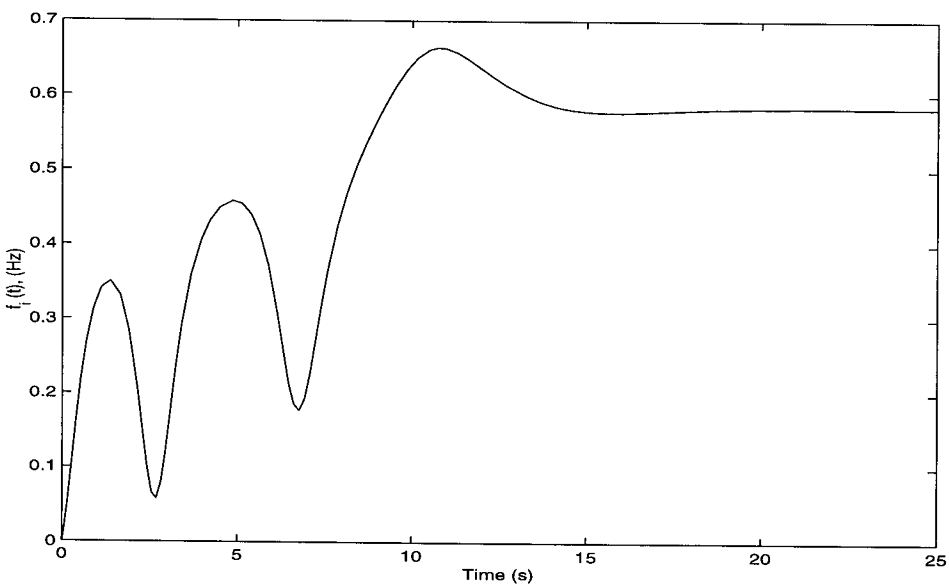


Figure 3: (b) Δ f = 7/12 Hz

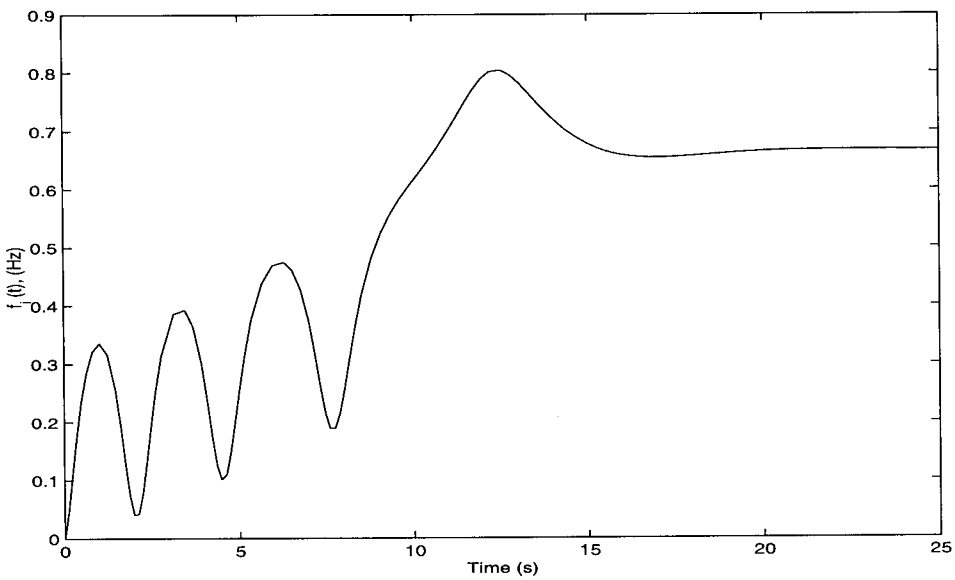


Figure 4: (b) Δ f = 2/3 Hz

CHAPTER 3

Pulse Modulation

Problem 3.1

Let $2W$ denote the bandwidth of a narrowband signal with carrier frequency f_c. The in-phase and quadrature components of this signal are both low-pass signals with a common bandwidth of W. According to the sampling theorem, there is no information loss if the in-phase and quadrature components are sampled at a rate higher than $2W$. For the problem at hand, we have

$f_c = 100$ kHz
$2W = 10$ kHz

Hence, $W = 5$ kHz, and the minimum rate at which it is permissible to sample the in-phase and quadrature components is 10 kHz.

From the sampling theorem, we also know that a physical waveform can be represented over the interval $-\infty < t < \infty$ by

$$g(t) = \sum_{n=-\infty}^{\infty} a_n \phi_n(t) \tag{1}$$

where $\{\phi_n(t)\}$ is a set of orthogonal functions defined as

$$\phi_n(t) = \frac{\sin\{\pi f_s(t - n/f_s)\}}{\pi f_s(t - n/f_s)}$$

where n is an integer and f_s is the sampling frequency. If $g(t)$ is a low-pass signal band-limited to W Hz, and $f_s \geq 2W$, then the coefficient a_n can be shown to equal $g(n/f_s)$. That is, for $f_s \geq 2W$, the orthogonal coefficients are simply the values of the waveform that are obtained when the waveform is sampled every $1/f_s$ second.

As already mentioned, the narrowband signal is two-dimensional, consisting of in-phase and quadrature components. In light of Eq. (1), we may represent them as follows, respectively:

$$g_I(t) = \sum_{n=-\infty}^{\infty} g_I(n/f_s)\phi_n(t)$$

$$g_Q(t) = \sum_{n=-\infty}^{\infty} g_Q(n/f_s)\phi_n(t)$$

Hence, given the in-phase samples $g_I\left(\frac{n}{f_s}\right)$ and quadrature samples $g_Q\left(\frac{n}{f_s}\right)$, we may reconstruct the narrowband signal $g(t)$ as follows:

$$g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$$

$$= \sum_{n=-\infty}^{\infty} \left[g_I\left(\frac{n}{f_s}\right)\cos(2\pi f_c t) - g_Q\left(\frac{n}{f_s}\right)\sin(2\pi f_c t) \right] \phi_n(t)$$

where f_c = 100 kHz and $f_s \geq$ 10 kHz, and where the same set of orthonormal basis functions is used for reconstructing both the in-phase and quadrature components.

Problem 3.2

(a) Consider a periodic train c(f) of rectangular pulses, each of duration T. The Fourier series expansion of c(t) (assuming that a pulse of the train is centered on the origin) is given by

$$c(t) = \sum_{n=-\infty}^{\infty} f_s \operatorname{sinc}(nf_s T)\exp(j2\pi n f_s t)$$

where f_s is the repetition frequency, and the amplitude of a rectangular pulse is assumed to be 1/T (i.e., each pulse has unit area). The assumption that $f_sT>>1$ means that the spectral lines (i.e., harmonics) of the periodic pulse train c(t) are well separated from each other.

Multiplying a message signal g(t) by c(t) yields

$$\begin{aligned} s(t) &= c(t)g(t) \\ &= \sum_{n=-\infty} f_s \operatorname{sinc}(nf_s T) g(t) \exp(j2\pi n f_s(t) \end{aligned} \tag{1}$$

Taking the Fourier transform of both sides of Eq. (1) and using the frequency-shifting property of the Fourier transform:

$$S(f) = \sum_{n=-\infty}^{\infty} f_s \operatorname{sinc}(nf_s T)\, G(f-nf_s) \tag{2}$$

where G(f) = F[g(t)]. Thus, the spectrum S(f) consists of frequency-shifted replicas of the original spectrum G(f), with the nthreplica being scaled in amplitude by the factor $f_s\operatorname{sinc}(nf_sT)$.

(b) In accordance with the sampling theorem, let it be assumed that

- The signal g(t) is band-limited with

$$G(f) = 0 \quad \text{for} \quad -W < f < W$$

- The sampling frequency f_s is defined by

$$f_s > 2W$$

Then, the different frequency-shifted replicas of G(f) involved in the construction of S(f) will <u>not</u> overlap. Under the conditions described herein, the original spectrum G(f), and therefore the signal g(t), can be recovered exactly (except for a trivial amplitude scaling) by passing s(t) through a low-pass filter of bandwidth W.

Problem 3.3

(a) $g(t) = \text{sinc}(200t)$

This sinc pulse corresponds to a bandwidth W = 100 Hz. Hence, the Nyquist rate is 200 Hz, and the Nyquist interval is 1/200 seconds.

(b) $g(t) = \text{sinc}^2(200t)$

This signal may be viewed as the product of the sinc pulse sinc(200t) with itself. Since multiplication in the time domain corresponds to convolution in the frequency domain, we find that the signal g(t) has a bandwidth equal to twice that of the sinc pulse sin(200t), that is, 200 Hz. The Nyquist rate of g(t) is therefore 400 Hz, and the Nyquist interval is 1/400 seconds.

(c) $g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$

The bandwidth of g(t) is determined by the highest frequency component of sinc(200t) or $\text{sinc}^2(200t)$, whichever one is the largest. With the bandwidth (i.e., highest frequency component of) the sinc pulse sinc(200 t) equal to 100 Hz and that of the squared sinc pulse $\text{sinc}^2(200t)$ equal to 200 Hz, it follows that the bandwidth of g(t) is 200 Hz. Correspondingly, the Nyquist rate of g(t) is 400 Hz, and its Nyquist interval is 1/400 seconds.

Problem 3.4

(a) The PAM wave is

$$s(t) = \sum_{n=-\infty}^{\infty} [1 + \mu m'(nT_s)]g(t-nT_s),$$

where g(t) is the pulse shape, and $m'(t) = m(t)/A_m = \cos(2\pi f_m t)$. The PAM wave is equivalent to the convolution of the instantaneously sampled $[1 + \mu m'(t)]$ and the pulse shape g(t):

$$s(t) = \left\{ \sum_{n=-\infty}^{\infty} [1+\mu m'(nT_s)]\,\delta(t-nT_s) \right\} \star g(t)$$

$$= \left\{ [1+\mu m'(t)] \sum_{n=-\infty}^{\infty} \delta(t-nT_s) \right\} \star g(t)$$

The spectrum of the PAM wave is,

$$S(f) = \left\{ [\delta(f) + \mu M'(f)] \star \frac{1}{T_s} \sum_{m=-\infty}^{\infty} \delta(f - \frac{m}{T_s}) \right\} G(f)$$

$$= \frac{1}{T_s} G(f) \sum_{m=-\infty}^{\infty} [\delta(f - \frac{m}{T_s}) + \mu M'(f - \frac{m}{T_s})]$$

For a rectangular pulse g(t) of duration T=0.45s, and with AT = 1, we have:

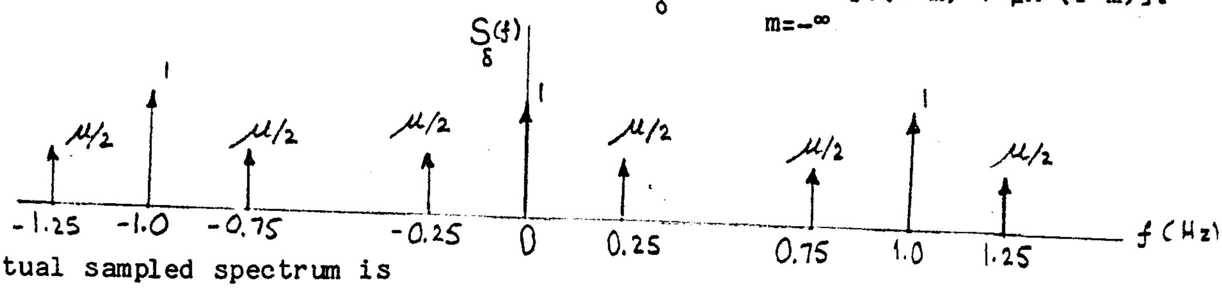
$$G(f) = AT\,\text{sinc}(fT)$$

$$= \text{sinc}(0.45f)$$

For $m'(t) = \cos(2\pi f_m t)$, and with $f_m = 0.25$ Hz, we have:

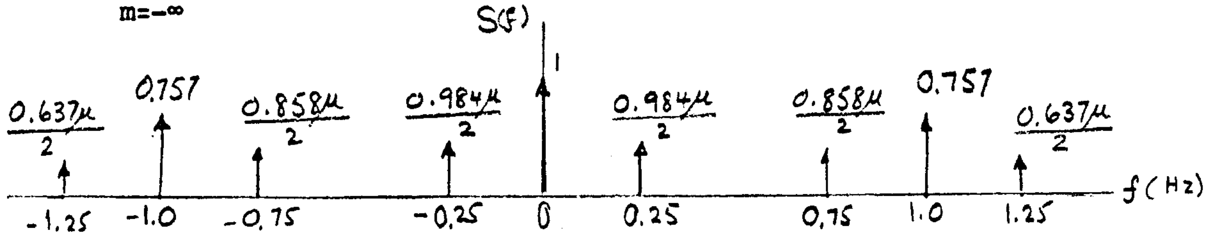
$$M'(f) = \frac{1}{2} [\delta(f-0.25) + \delta(f+0.25)]$$

For T_s = 1s, the ideally sampled spectrum is $S_\delta(f) = \sum_{m=-\infty}^{\infty} [\delta(f-m) + \mu M'(f-m)]$.

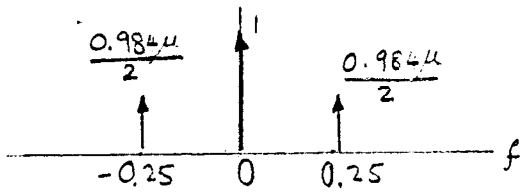


The actual sampled spectrum is

$$S(f) = \sum_{m=-\infty}^{\infty} \mathrm{sinc}(0.45f)[\delta(f-m) + \mu M'(f-m)]$$



(b) The ideal reconstruction filter would retain the centre 3 delta functions of S(f) or:



With no aperture effect, the two outer delta functions would have amplitude $\frac{\mu}{2}$. Aperture effect distorts the reconstructed signal by attenuating the high frequency portion of the message signal.

Problem 3.5

The spectrum of the flat-top pulses is given by

$$H(f) = T\,\mathrm{sinc}\,(fT)\exp(-j\pi fT)$$

$$= 10^{-4}\mathrm{sinc}\,(10^{-4}f)\exp(-j\pi f 10^{-4})$$

Let $s(t)$ denote the sequence of flat-top pulses:

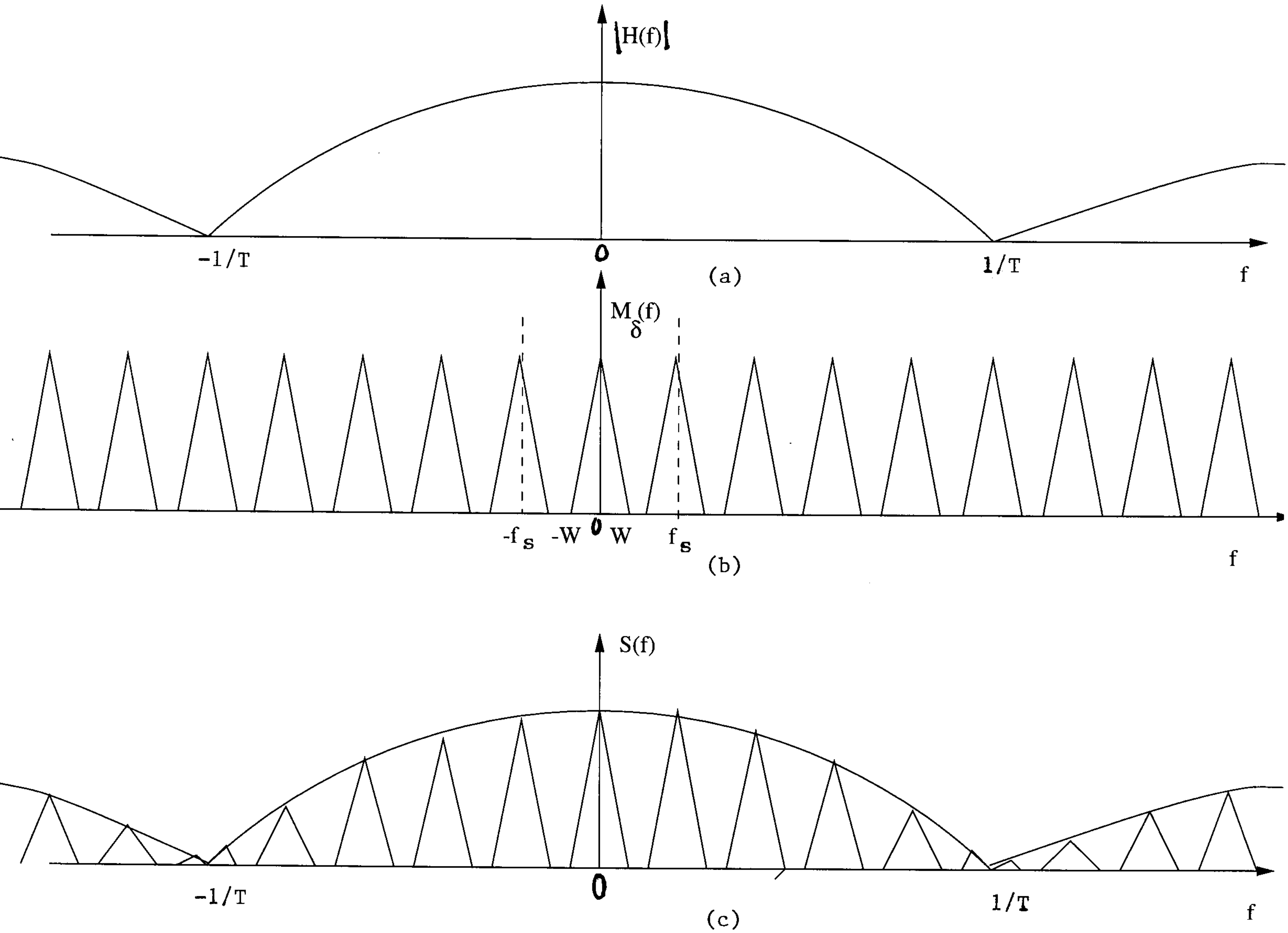
$$s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t-nT_s)$$

The spectrum $S(f) = F[s(t)]$ is as follows:

$$S(f) = f_s \sum_{k=-\infty}^{\infty} M(f-kf_s)H(f)$$

$$= f_s H(f) \sum_{k=-\infty}^{\infty} M(f-kf_s)$$

The magnitude spectrum $|S(f)|$ is thus as shown in Fig. 1c.



$1/T = 10{,}000$Hz
$f_s = 1{,}000$Hz
$W = 400$Hz

Figure 1

Problem 3.6

At $f = 1/2T_s$, which corresponds to the highest frequency component of the message signal for a sampling rate equal to the Nyquist rate, we find from Eq. (6-19) that the amplitude response

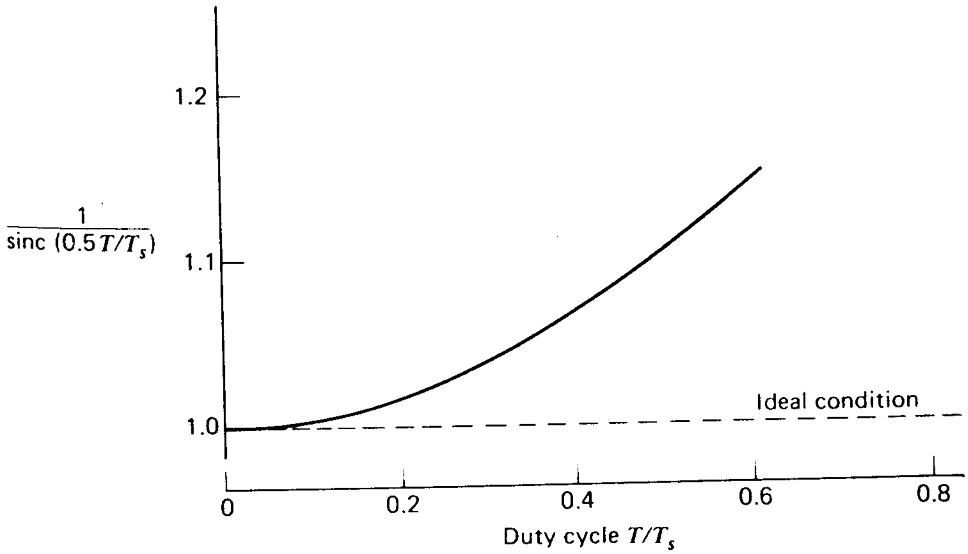


Figure 1

of the equalizer normalized to that at zero frequency is equal to

$$\frac{1}{\text{sinc}(0.5T/T_s)} = \frac{(\pi/2)(T/T_s)}{\sin[(\pi/2)(T/T_s)]}$$

where the ratio T/T_s is equal to the duty cycle of the sampling pulses. In Fig. 1, this result is plotted as a function of T/T_s. Ideally, it should be equal to one for all values of T/T_s. For a duty cycle of 10 percent, it is equal to 1.0041. It follows therefore that for duty cycles of less than 10 percent, the aperture effect becomes negligible, and the need for equalization may be omitted altogether.

Problem 3.7

Consider the full-load test tone $A\cos(2\pi f_m t)$. Denoting the kth sample amplitude of this signal by A_k, we find that the transmitted pulse is $A_k\, g(t)$, where g(t) is defined by the spectrum:

$$G(f) = \begin{cases} \frac{1}{2B_T} & |f| < B_T \\ 0, & \text{otherwise} \end{cases}$$

The mean value of the transmitted signal power is

$$P = E\{\lim_{L\to\infty} \frac{1}{2LT_s} \int_{-LT_s}^{LT_s} [\sum_{k=-L}^{L} A_k\, g(t)]^2 dt\}$$

$$= E[\lim_{L\to\infty} \frac{1}{2LT_s} \int_{-LT_s}^{LT_s} \sum_{k=-L}^{L} \sum_{n=-L}^{L} A_k A_n\, g^2(t)dt]$$

$$= \lim_{L\to\infty} \frac{1}{2LT_s} \sum_{k=-L}^{L} \sum_{n=-L}^{L} E[A_k A_n] \int_{-LT_s}^{LT_s} g^2(t)dt$$

where T_s is the sampling period. However,

$$E[A_k A_n] = \begin{cases} \frac{A^2}{2}, & k = n \\ 0, & \text{otherwsie} \end{cases}$$

Therefore,

$$P = \frac{A^2}{2T_s} \int_{-\infty}^{\infty} g^2(t)dt$$

Using Rayleigh's energy theorem, we may write

$$\int_{-\infty}^{\infty} g^2(t)dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

$$= \int_{-B_T}^{B_T} (\frac{1}{2B_T})^2 df$$

$$= \frac{1}{2B_T}$$

Therefore,

$$P = \frac{A^2}{4T_s B_T}$$

The average signal power at the receiver output is $A^2/2$. Hence, the output signal-to-noise ratio is given by

$$(SNR)_O = \frac{A^2/2}{B_T N_O}$$

$$= \frac{A^2}{2B_T N_O}$$

$$= \frac{2T_s P}{N_O}$$

By choosing $B_T=1/2T_s$, we get

$$(SNR)_O = \frac{P}{B_T N_O}$$

This shows that PAM and baseband signal transmission have the same signal-to-noise ratio for the same average transmitted power, with additive white Gaussian noise, and assuming the use of the minimum transmission bandwidth possible.

Problem 3.8

(a) The sampling interval is T_s = 125 μs. There are 24 channels and 1 sync pulse, so the time alloted to each channel is $T_c = T_s/25 = 5$ μs. The pulse duration is 1 μs, so the time between pulses is 4 μs.

(b) If sampled at the nyquist rate, 6.8 kHz, then T_s = 147 μs, T_c = 6.68 μs, and the time between pulses is 5.68 μs.

Problem 3.9

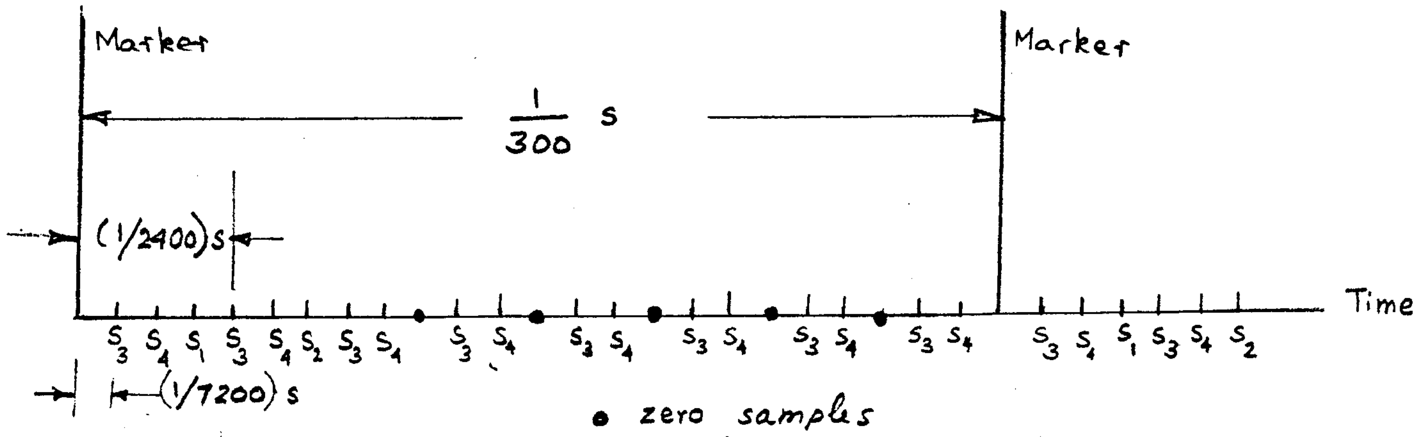
(a) The bandwidth required for each single sideband channel is 10 kHz. The total bandwidth for 12 channels is 120 kHz.

(b) The Nyquist rate for each signal is 20 kHz. For 12 TDM signals, the total data rate is 240 kHz. By using a sinc pulse whose amplitude varies in accordance with the modulation, and with zero crossings at multiples of (1/240) ms, we need a minimum bandwidth of 120 kHz.

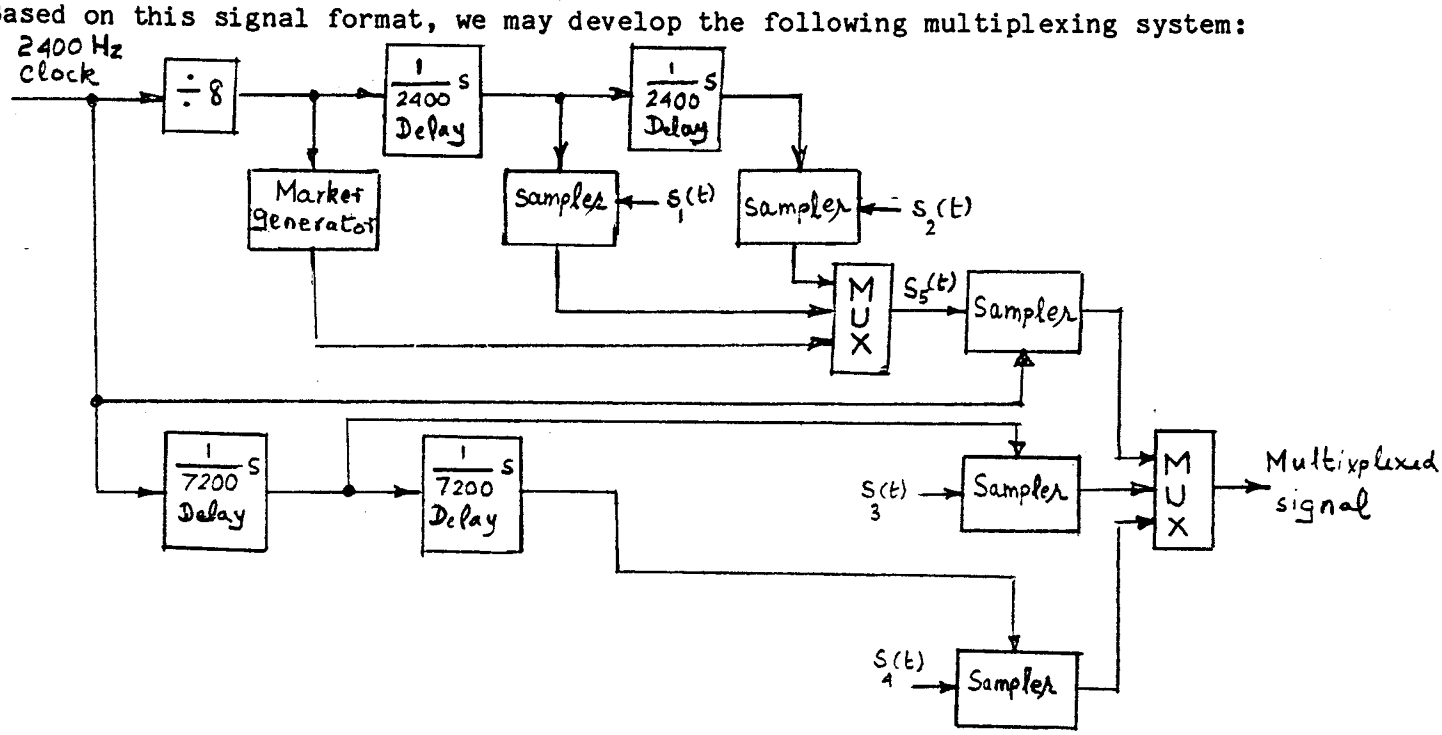
Problem 3.10

(a) The Nyquist rate for $s_1(t)$ and $s_2(t)$ is 160 Hz. Therefore, $\frac{2400}{2^R}$ must be greater than 160, and the maximum R is 3.

(b) With $R = 3$, we may use the following signal format to multiplex the signals $s_1(t)$ and $s_2(t)$ into a new signal, and then multiplex $s_3(t)$ and $s_4(t)$ and $s_5(t)$ including markers for synchronization:



Based on this signal format, we may develop the following multiplexing system:



Problem 3.11

In general, a line code can be represented as

$$s(t) = \sum_{n=-N}^{N} a_n g(t - nT_b)$$

Let $g(t) \rightleftharpoons G(f)$. We may then define the Fourier transform of $s(t)$ as

$$S(f) = \sum_{n=-N}^{N} a_n G(f) e^{-j\omega n T_b}$$

$$= G(f) \sum_{n=-N}^{N} a_n e^{-j\omega n T_b}$$

where $\omega = 2\pi f$. The power spectral density of $s(t)$ is

$$S_s(f) = \lim_{T \to \infty} \left[\frac{1}{T} |G(f)|^2 E \left| \sum_{n=-N}^{N} a_n e^{-j\omega n T_b} \right|^2 \right]$$

$$= |G(f)|^2 \lim_{T \to \infty} \left(\frac{1}{T} \sum_{n=-N}^{N} \sum_{m=-N}^{N} E[a_n a_m] e^{j(m-n)\omega T_b} \right)$$

where T is the duration of the binary data sequence, and E denotes the statistical expectation operator. Define the autocorrelation of the binary data sequence as

$$R(k) = E[a_n a_{n+k}]$$

By letting $m = n + k$ and $T = (2N + 1)T_b$, we may write

$$S_s(f) = |G(f)|^2 \lim_{N \to \infty} \left[\frac{1}{(2N+1)T_b} \sum_{n=-N}^{n=N} \sum_{k=-N-n}^{k=N-n} R(k) e^{jk\omega T_b} \right]$$

Replacing the outer sum over the index n by $2N$+1, we get

$$S_s(f) = \frac{|G(f)|^2}{T_b} \lim_{N \to \infty} \left[\frac{2N+1}{2N+1} \sum_{k=-N-n}^{k=N-n} R(k) e^{jk\omega T_b} \right]$$

$$= \frac{|G(f)|^2}{T_b} \sum_{k=-\infty}^{\infty} R(k) e^{jk\omega T_b} \tag{1}$$

where

$$R(k) = E[a_n a_{n+k}] = \sum_{i=1}^{I} (a_n a_{n+k}) p_i \tag{2}$$

where p_i is the probability of getting the product $(a_n\, a_{n+k})_i$ and there are I possible values for the $a_n\, a_{n+k}$ product. $G(f)$ is the spectrum of the pulse-shaping signal for representing a digital symbol. Eqs. (1) and (2) provide the basis for evaluating the spectra of the specified line codes.

(a) Unipolar NRZ signaling

For rectangular NRZ pulse shapes, the Fourier-transform pair is

$$g(t) = A\text{rect}\left(\frac{t}{T_b}\right) \rightleftharpoons G(f) = AT_b \text{sinc}(fT_b)$$

For unipolar NRZ signaling, the possible levels for a's are $+A$ and 0. For equiprobable symbols, we have the following autocorrelation values:

$$R(0) = \frac{1}{2}A^2 + \frac{1}{2} \times 0 = A^2/2$$

$$R(k) = \sum_{i=1}^{4} (a_n a_{n+k})_i p_i$$

$$= \frac{A^2}{4} + \frac{0}{4} + \frac{0}{4} + \frac{0}{4} = \frac{A^2}{4} \quad \text{for} \quad |k| > 0$$

Thus

$$R(k) = \begin{cases} A^2/2 & \text{for } k = 0 \\ A^2/4 & \text{for } k \neq 0 \end{cases} \tag{3}$$

Therefore, the power spectral density for unipolar NRZ signals, using formulas (1) and (3), is

$$S_s(f) = \frac{|AT_b \text{sinc}(fT_b)|^2}{T_b}\left[\frac{1}{4} + \frac{1}{4}\sum_{k=-\infty}^{\infty} e^{j2\pi kfT_b}\right]$$

$$= \frac{A^2T_b}{4}\text{sinc}^2(fT_b)\left[1 + \sum_{k=-\infty}^{\infty} e^{j2\pi kfT_b}\right]$$

But,

$$\sum_{k=-\infty}^{\infty} e^{j2\pi kfT_b} = \frac{1}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T_b}\right)$$

where $\delta(f)$ is a delta function in the frequency domain. Hence,

$$S_s(f) = \frac{A^2T_b}{4}\text{sinc}^2(fT_b)\left[1 + \frac{1}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T_b}\right)\right]$$

We also note that $\text{sinc}(fT_b) = 0$ at $f = \frac{n}{T_b}, n \neq 0$; we thus get

$$S_s(f) = \frac{A^2T_b}{4}\text{sinc}^2(fT_b)\left[1 + \frac{\delta(f)}{T_b}\right]$$

(b) Polar Non-return-to-zero Signaling

For polar NRZ signaling, the possible values for a's are $+A$ and $-A$. Assuming equiprobable symbols, we have

$$R(0) = \sum_{i=1}^{2}(a_n a_n)_i p_i$$

$$= \frac{A^2}{2} + \frac{(-A)^2}{2} = A^2$$

For $k \neq 0$, we have

$$R(k) = \sum_{i=1}^{4}(a_n a_{n+k})_i p_i$$

$$= \frac{A^2}{4} + 2\frac{(-A)(A)}{4} + 2\frac{(-A)(A)}{4} + \frac{(-A)^2}{4}$$

$$= 0$$

Thus,

$$R(k) = \begin{cases} A^2 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \tag{4}$$

The power spectral density for this case, using formulas (1) and (4), is

$$S(f) = A^2 T_b \mathrm{sinc}^2(f T_b)$$

(c) Return-to-zero Signaling

The pulse shape used for return-to-zero signaling is given by $g\left(\frac{t}{T_b/2}\right)$. We therefore have

$$G(f) = \frac{T_b}{2}\mathrm{sinc}(fT_b/2)$$

The autocorrelation for this case is the same as that for unipolar NRZ signaling. Therefore, the power spectral density of RZ signals is

$$S_s(f) = \frac{A^2 T_b}{16}\mathrm{sinc}^2(f T_b)\left[1 + \frac{t}{T_b}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T_b}\right)\right]$$

(d) Bipolar Signals

The permitted values of level a for bipolar signals are $+A$, $-A$, and 0, where binary symbol 1 is represented alternately by $+A$ and $-A$, and binary 0 is represented by level zero. We thus have the following autocorrelation function values:

$$R(0) = \frac{A^2}{2}$$

$$R(1) = \sum_{i=1}^{4} (a_n a_{n+1})_i p_i = -\frac{A^2}{4}$$

For $k > 1$,

$$R(k) = \sum_{i=1}^{5} (a_n a_{n+k})_i p_i = \frac{A^2}{8} - \frac{A^2}{8} = 0$$

Thus,

$$R(k) = \left\{ \begin{array}{ll} \frac{A^2}{2} & \text{for } k = 0 \\ -\frac{A^2}{4} & \text{for } |k| = 1 \\ 0 & \text{for } |k| > 1 \end{array} \right\} \tag{5}$$

The pulse duration for this case is equal to $T_b/2$. Hence,

$$G(f) = \frac{T_b}{2} \operatorname{sinc}\left(\frac{fT_b}{2}\right) \tag{6}$$

Using Equations (1), (5) and (6), the power spectral density of bipolar signals is

$$S_s(f) = \frac{\left|\frac{T_b}{2}\operatorname{sinc}\left(\frac{fT_b}{2}\right)\right|^2}{T_b} \left[\frac{A^2}{2} - \frac{A^2}{4}e^{j\omega T_b} - \frac{A^2}{4}e^{-j\omega T_b}\right]$$

$$= \frac{A^2 T_b}{8}\operatorname{sinc}^2\left(\frac{fT_b}{2}\right)\left[1 - \frac{1}{2}(e^{j\omega T_b} + e^{-j\omega T_b})\right]$$

$$= \frac{A^2 T_b}{8}\operatorname{sinc}^2\left(\frac{fT_b}{2}\right)[1 - \cos(2\pi f T_b)]$$

$$= \frac{A^2 T_b}{8} \text{sinc}^2\left(\frac{fT_b}{2}\right) \sin^2(\pi f T_b)$$

(e) Manchester Code

The permitted values of a's in the Manchester code are $+A$ and $-A$. Hence,

$$R(0) = \frac{1}{4}A^2 + \frac{1}{4}(-A)^2 + \frac{1}{4}(-A)^2 + \frac{1}{4}(A^2)$$

$$= A^2$$

For $k \neq 0$,

$$R(k) = \sum_{i=1}^{4} (a_n a_{n+k})_i p_i = \frac{A^2}{4} + \frac{(-A)(A)}{4} + \frac{A(-A)}{4} + \frac{(-A)^2}{4}$$

$$= 0$$

Thus,

$$R(k) = \begin{cases} A^2 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

The pulse shape of Manchester signaling is given by

$$g(t) = \text{rect}\left(\frac{t + T_b/4}{T_b/2}\right) - \text{rect}\left(\frac{t - T_b/4}{T_b/2}\right)$$

The pulse spectrum is therefore

$$G(f) = \frac{T_b}{2}\text{sinc}\left(\frac{fT_b}{2}\right) e^{j\omega T_b/4} - \frac{T_b}{2}\text{sinc}\left(\frac{fT_b}{2}\right) e^{-j\omega T_b/4}$$

$$= jT_b \text{sinc}\left(\frac{fT_b}{2}\right) \sin\left(\frac{2\pi f T_b}{4}\right)$$

Therefore, the power spectral density of Manchester NRZ has the form

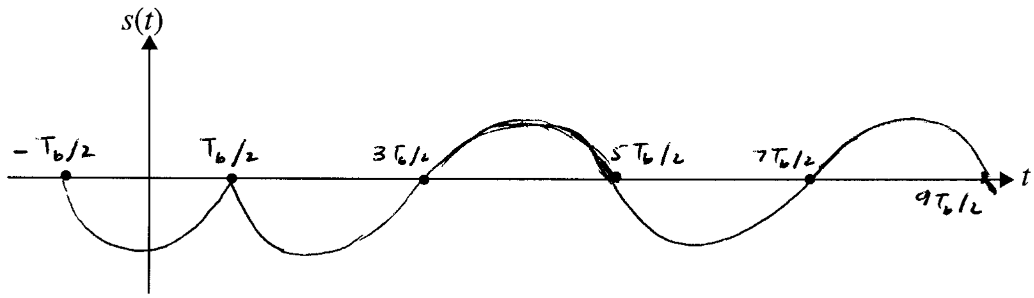
$$S_s(f) = A^2 T_b \operatorname{sinc}^2\left(\frac{fT_b}{2}\right)\sin^2\left(\frac{\pi f T_b}{2}\right)$$

Problem 3.12

Power spectral density of a binary data stream will not be affected by the use of differential encoding. The reason for this statement is that differential encoding uses the same pulse shaping functions as ordinary encoding methods. If the number of bits is high, then the probability of a symbol one and symbol zero are the same for both cases.

Problem 3.13

(a)



(b) $$g(t) = \begin{cases} \cos\left(\frac{\pi t}{T_b}\right), & \frac{-T_b}{2} < t \le \frac{T_b}{2} \\ 0, & \text{otherwise} \end{cases}$$

Equivalently, we may write

$$g(t) = \cos\left(\frac{\pi t}{T_b}\right) A \operatorname{rect}\left(\frac{t}{T_b}\right)$$

where rect(t) is a rectangular function of unit amplitude and unit duration. The Fourier transform of $g(t)$ is given by

$$G(f) = \frac{AT_b}{2}\left[\delta\left(f - \frac{2}{T_b}\right) + \delta\left(f + \frac{2}{T_b}\right)\right] \ast \operatorname{sinc}(fT_b)$$

where A denotes the pulse amplitude and $\ast$ denotes convolution in the frequency domain.

Using the replication property of the delta function $\delta(f)$, we get

$$G(f) = \frac{AT_b}{2}\left[\operatorname{sinc}\left(T_b\left(f-\frac{2}{T_b}\right)\right) + \operatorname{sinc}\left(T_b\left(f+\frac{2}{T_b}\right)\right)\right]$$

Using Eq. (1.52) of the textbook, the power spectral density of the binary data stream is

$$S(f) = \frac{|G(f)|^2}{T_b}$$

$$= \frac{A^2T_b}{4}\left[\operatorname{sinc}^2\left(T_b\left(f-\frac{2}{T_b}\right)\right) + \operatorname{sinc}^2\left(T_b\left(f+\frac{2}{T_b}\right)\right)\right.$$

$$\left. + 2\operatorname{sinc}\left(T_b\left(f-\frac{2}{T_b}\right)\right)\operatorname{sinc}\left(T_b\left(f+\frac{2}{T_b}\right)\right)\right] \qquad (1)$$

Note that the two spectral components $\operatorname{sinc}\left(T_b\left(f-\frac{2}{T_b}\right)\right)$ and $\operatorname{sinc}\left(T_b\left(f+\frac{2}{T_b}\right)\right)$ overlap in the frequency interval $-(1/T_b) \leq f \leq (1/T_b)$, hence the presence of cross-product terms in Eq. (1).

Figure 1 plots the normalized power spectral density $S(f)/(A^2T_b/4)$ versus the normalized frequency fT_b. The interesting point to note in this figure is the significant reduction in the power spectrum of the pulse-shaped data stream $x(t)$ in the interval $-1/T_b \leq f \leq 1/T_b$.

(c) The power spectral density of the standard form of polar NRZ signaling is

$$S(f) = A^2T_b\operatorname{sinc}^2(fT_b) \qquad (2)$$

Comparing this expression with that of Eq. (1), we observe the following differences:

	Polar NRZ signals using cosine pulses	Polar NRZ signals using rectangular pulses
$f = 0$	0	A^2T_b
$f = \pm 2/T_b$	$A^2T_b/4$	0

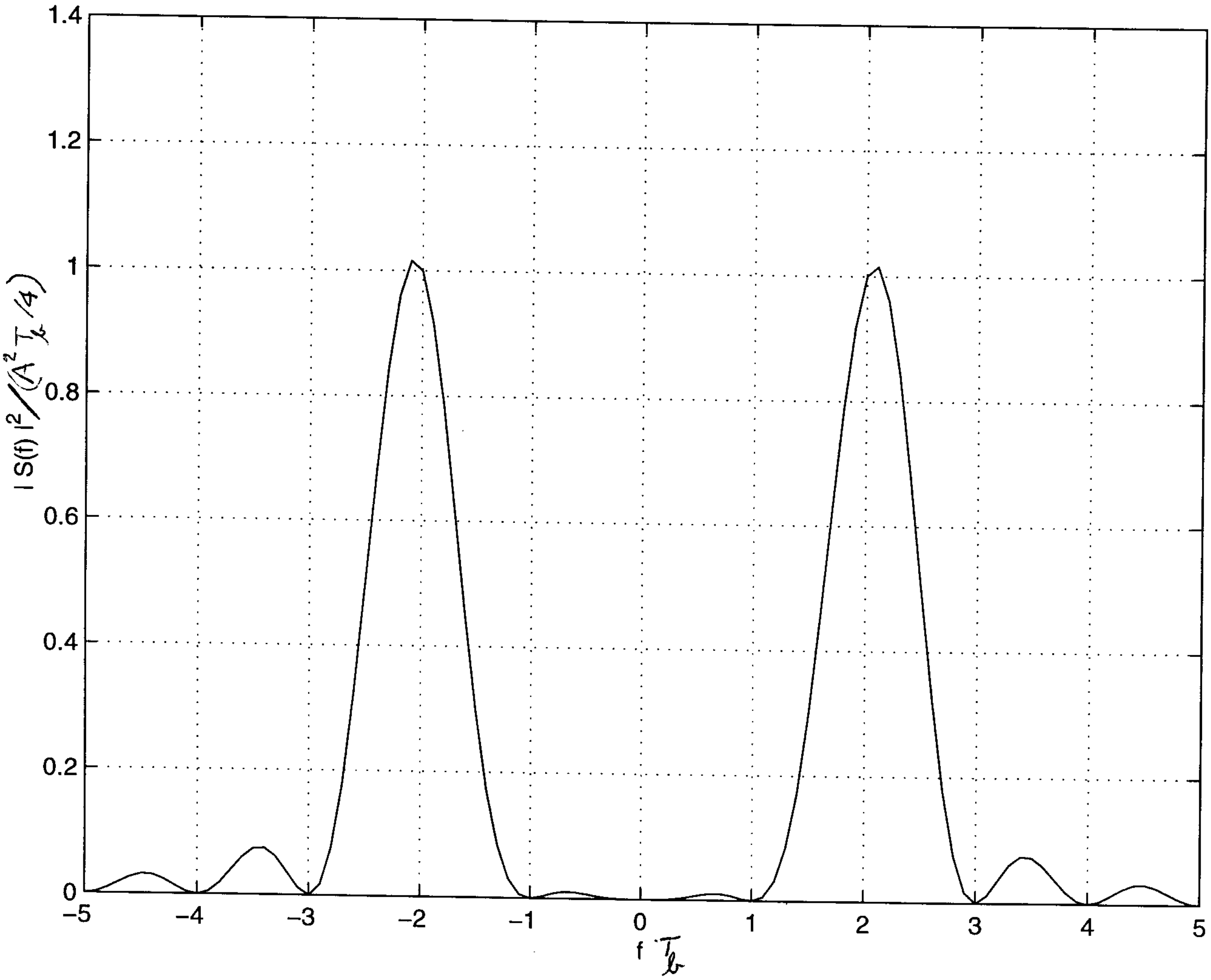
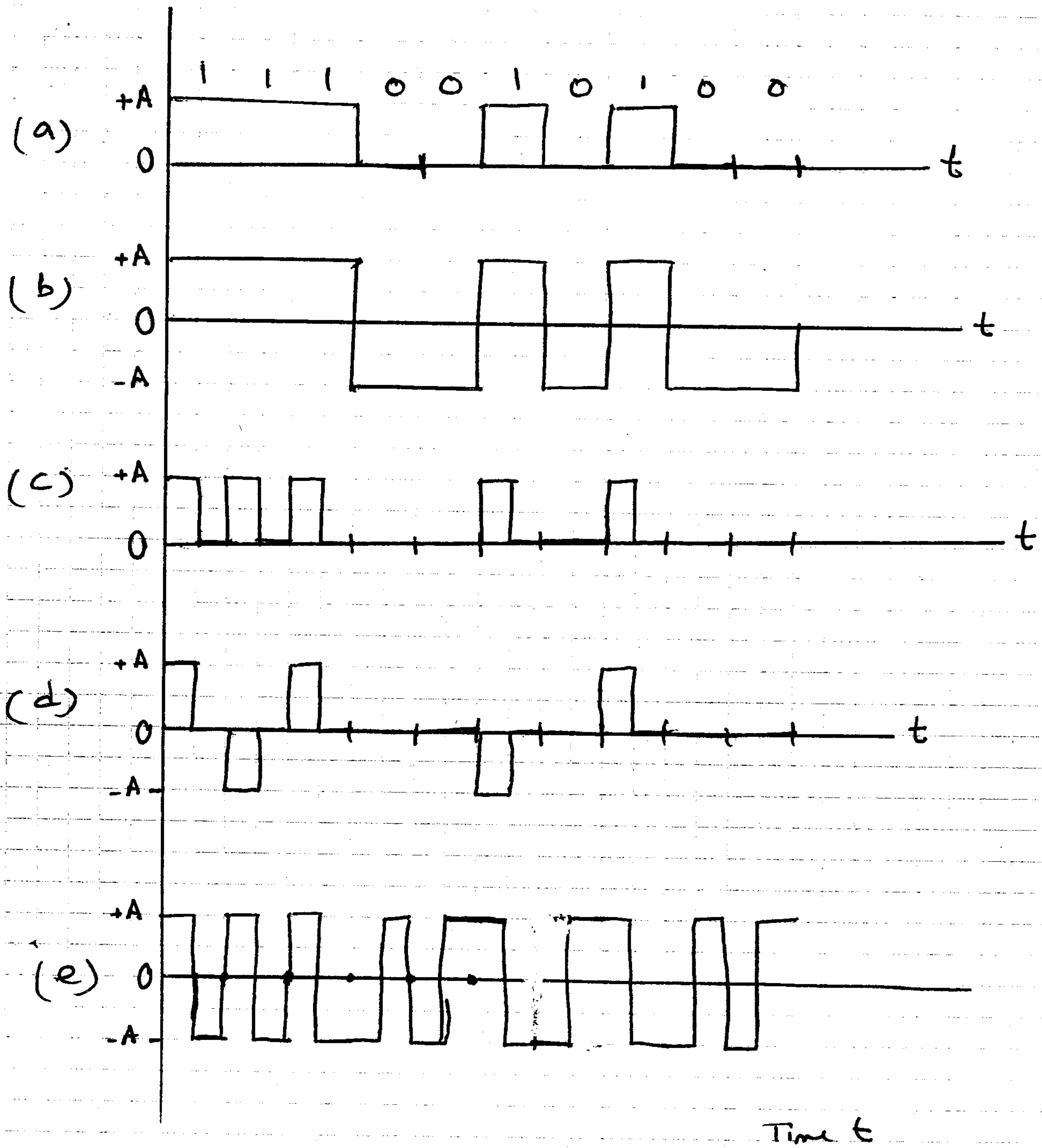
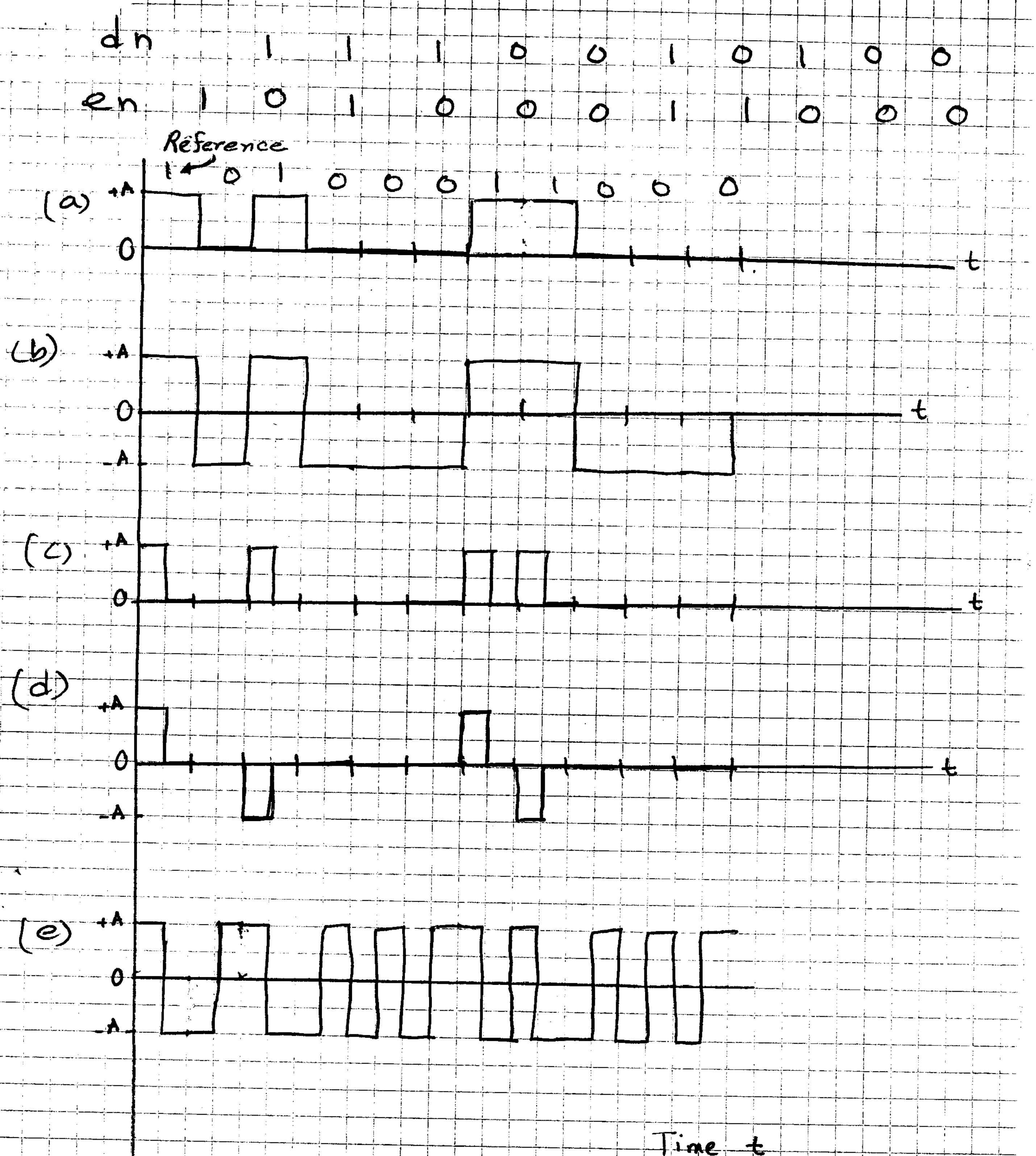


Figure 1

Problem 3.14



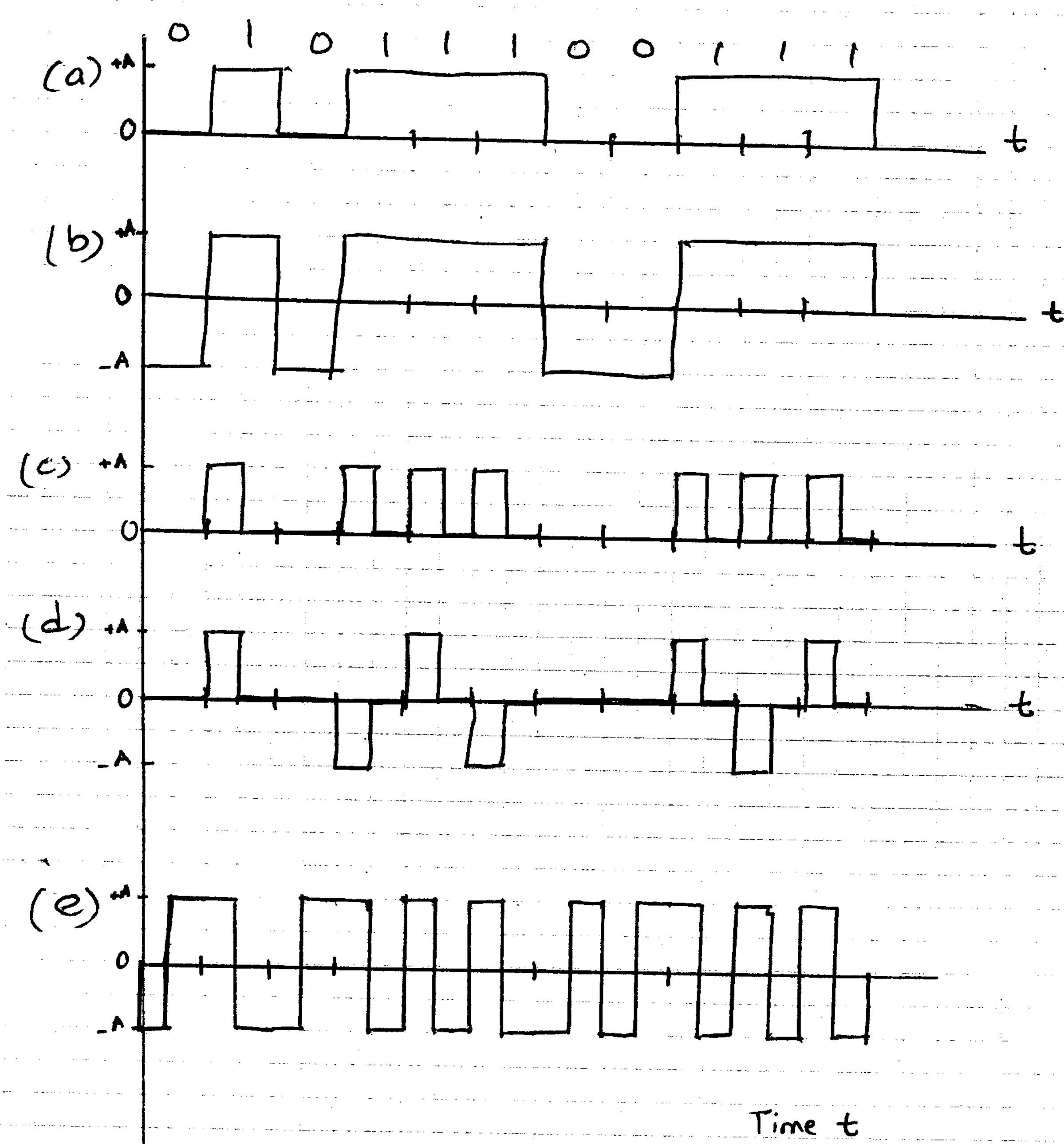
Problem 3.15(a)



Problem 3.15(b)

dn 1 1 1 0 0 1 0 1 0 0

en 0 1 0 1 1 1 0 0 1 1 1.



Problem 3.16

The minimum number of bits per sample is 7 for a signal-to-quantization noise ratio of 40 dB. Hence,

$$\begin{pmatrix}\text{The number of samples}\\ \text{in a duration of } 10s\end{pmatrix} = 8000 \times 10$$
$$= 8 \times 10^4 \text{ samples}$$

The minimum storage is therefore

$$= 7 \times 8 \times 10^4$$
$$= 5.6 \times 10^5$$
$$= 560 \text{ kbits}$$

Problem 3.17

Suppose that baseband signal m(t) is modeled as the sample function of a Gaussian random process of zero mean, and that the amplitude range of m(t) at the quantizer input extends from $-4A_{rms}$ to $4A_{rms}$. We find that samples of the signal m(t) will fall outside the amplitude range $8A_{rms}$ with a probability of overload that is less than 1 in 10^4. If we further assume the use of a binary code with each code word having a length n, so that the number of quantizing levels is 2", we find that the resulting quantizer step size is

$$\delta = \frac{8\,A_{rms}}{2^{\mathcal{R}}} \tag{1}$$

Substituting Eq. (1) to the formula for the output signal-to-quantization noise ratio, we get

$$(\text{SNR})_o = \frac{3}{16}(2^{2\mathcal{R}}) \tag{2}$$

Expressing the signal-to-noise ratio in decibels:

$$10\log_0(\text{SNR})_o = 6\mathcal{R} - 7.2 \tag{3}$$

This formula states that each bit in the code word of a PCM system contributes 6dB to the signal-to-noise ratio. It gives a good description of the noise performance of a PCM system, provided that the following conditions are satisfied:

1. The system operates with an average signal power above the error threshold, so that the effect of transmission noise is made negligible, and performance is thereby limited essentially by quantizing noise alone.
2. The quantizing error is uniformly distributed.
3. The quantization is fine enough (say $\mathcal{R} > 6$) to prevent signal-correlated patterns in the quantizing error waveform.
4. The quantizer is aligned with the amplitude range from $-4A_{rms}$ to $4A_{rms}$.

In general, conditions (1) through (3) are true of toll quality voice signals. However, when demands on voice quality are not severe, we may use a coarse quantizer corresponding to $\mathcal{R} \leq 6$. In such a case, degradation in system performance is reflected not only by a lower signal-to-noise ratio, but also by an undesirable presence of signal-dependent patterns in the waveform of quantizing error.

Problem 3.18

(a) Let the message bandwidth be W. Then, sampling the message signal at its Nyquist rate, and using an R-bit code to represent each sample of the message signal, we find that the bit duration is

$$T_b = \frac{T_s}{R} = \frac{1}{2WR}$$

The bit rate is

$$\frac{1}{T_b} = 2WR$$

The maximum value of message bandwidth is therefore

$$W_{max} = \frac{50 \times 10^6}{2 \times 7}$$

$$= 3.57 \times 10^6 \text{ Hz}$$

(b) The output signal-to-quantizing noise ratio is given by (see Example 2):

$$10 \log_{10}(SNR)_0 = 1.8 + 6R$$

$$= 1.8 + 6 \times 7$$

$$= 43.8 \text{ dB}$$

Problem 3.19

Let a signal amplitude lying in the range

$$x_i - \frac{1}{2}\delta_i \leq x \leq x_i + \frac{1}{2}\delta_i$$

be represented by the quantized amplitude x_i. The instantaneous square value of the error is $(x-x_i)^2$. Let the probability density function of the input signal be $f_X(x)$. If the step size δ_i is small in relation to the input signal excursion, then $f_X(x)$ varies little within the quantum step and may be approximated by $f_X(x_i)$. Then, the mean-square value of the error due to signals falling within this quantum is

$$E[Q_i^2] = \int_{x_i - \frac{1}{2}\delta_i}^{x_i + \frac{1}{2}\delta_i} (x-x_i)^2 f_X(x)\,dx$$

$$\simeq \int_{x_i - \frac{1}{2}\delta_i}^{x_i + \frac{1}{2}\delta_i} (x-x_i)^2 f_X(x_i)dx$$

$$= f_X(x_i) \int_{x_i - \frac{1}{2}\delta_i}^{x_i + \frac{1}{2}\delta_i} (x-x_i)^2 dx$$

$$= f_X(x_i) \int_{-\frac{1}{2}\delta_i}^{\frac{1}{2}\delta_i} x^2\, dx$$

$$= \frac{1}{12}\delta_i^3 f_X(x_i) \qquad (1)$$

The probability that the input signal amplitude lies within the ith interval is

$$p_i = \int_{x_i - \frac{1}{2}\delta_i}^{x_i + \frac{1}{2}\delta_i} f_X(x)dx \simeq f_X(x_i) \int_{x_i - \frac{1}{2}\delta_i}^{x_i + \frac{1}{2}\delta_i} dx = f_X(x_i)\delta_i \qquad (2)$$

Therefore, eliminating $f_X(x_i)$ between Eqs. (1) and (2), we get

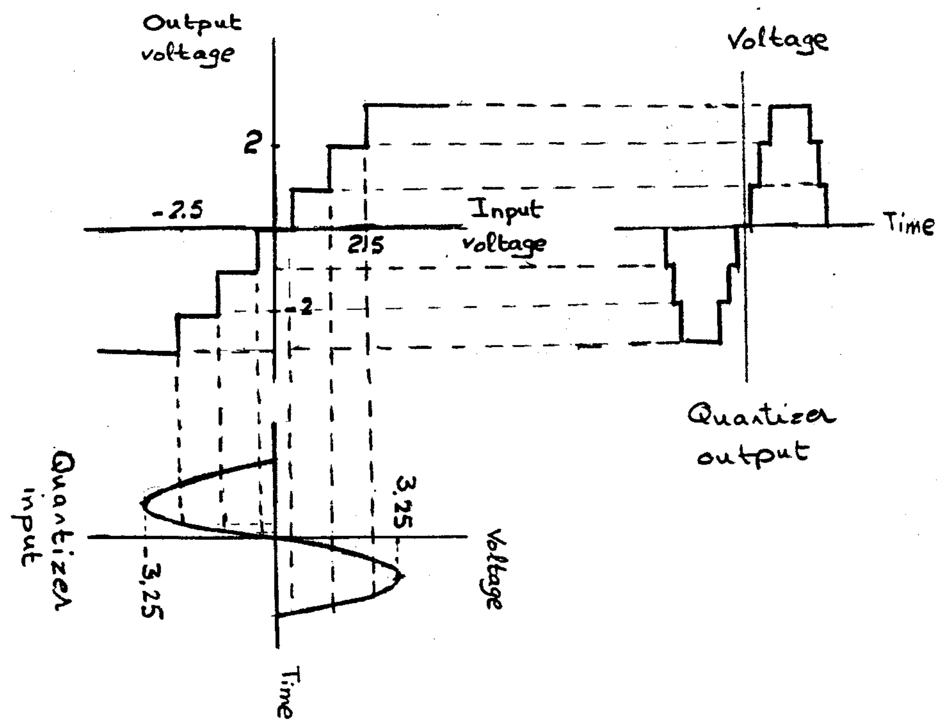
$$\therefore E[Q_i^2] = \frac{1}{12} p_i \delta_i^2$$

The total mean-square value of the quantizing error is the sum of that contributed by each of the several quanta. Hence,

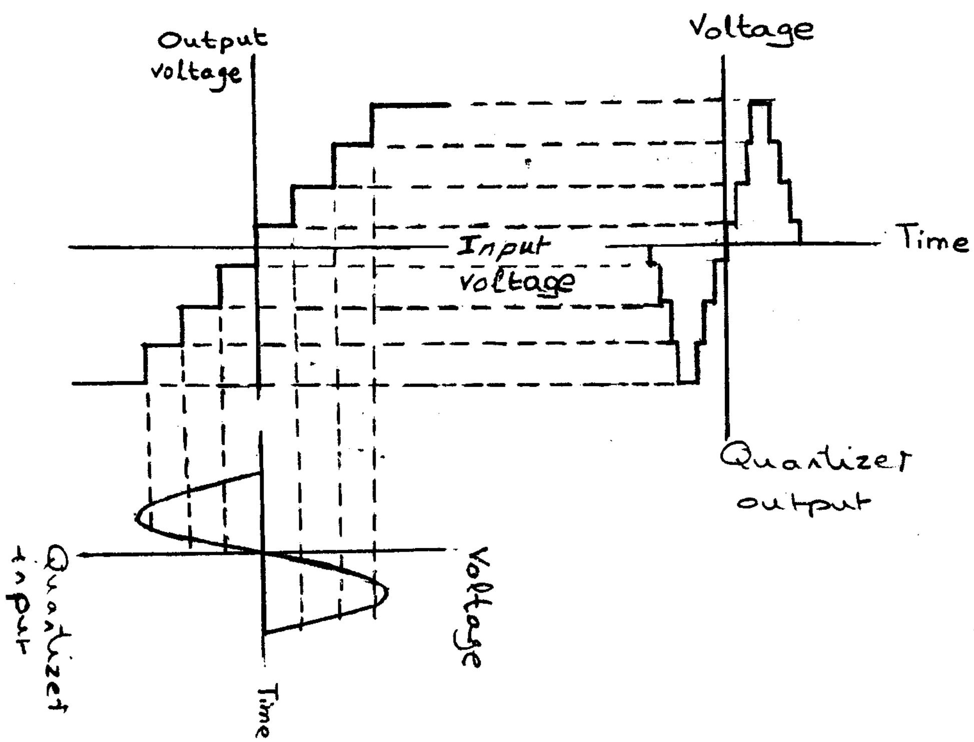
$$\sum_i E[Q_i^2] = \frac{1}{12} \sum_i p_i \delta_i^2$$

Problem 3.20

(a)

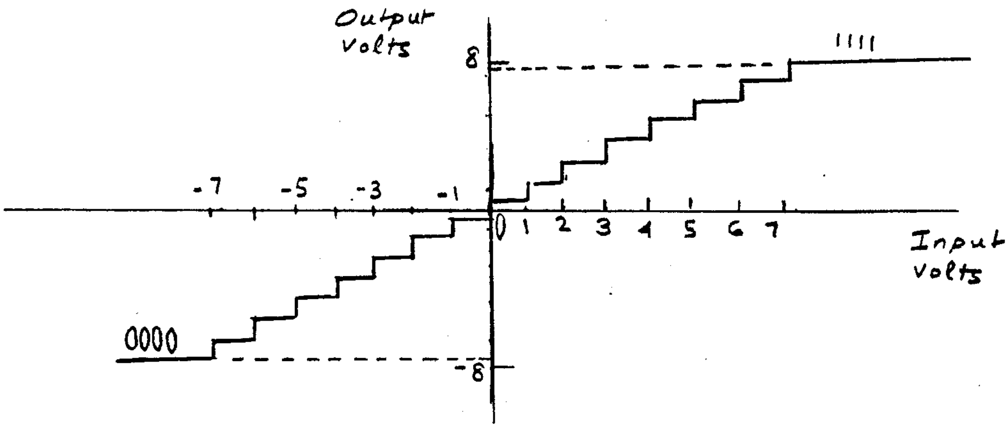


(b)



Problem 3.21

The quantizer has the following input-output curve:



At the sampling instants we have:

t	m(t)	code
-3/8	$-3\sqrt{2}$	0011
-1/8	$-3\sqrt{2}$	0011
+1/8	$3\sqrt{2}$	1100
+3/8	$3\sqrt{2}$	1100

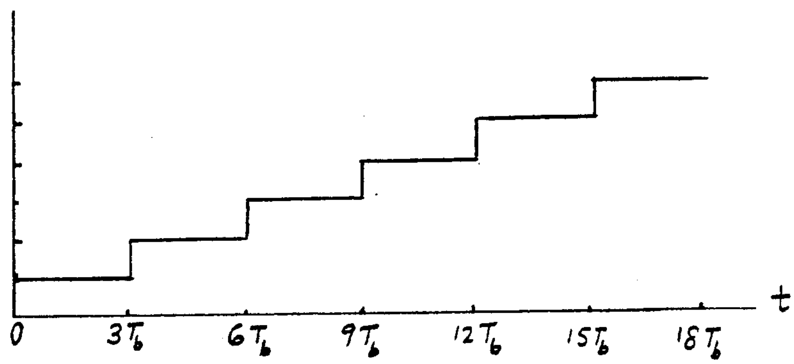
And the coded waveform is (assuming on-off signaling):

Problem 3.22

The transmitted code words are:

t/T_b	code
1	001
2	010
3	011
4	100
5	101
6	110

The sampled analog signal is



Problem 3.23

(a) The probability p_1 of any binary symbol being inverted by transmission through the system is usually quite small, so that the probability of error after n regenerations in the system is very nearly equal to n p_1. For very large n, the probability of more than one inversion must be taken into account. Let p_n denote the probability that a binary symbol is in error after transmission through the complete system. Then, p_n is also the probability of an odd number of errors, since an even number of errors restores the original value. Counting zero as an even number, the probability of an even number of errors is $1-p_n$. Hence

$$\begin{aligned} p_{n+1} &= p_n(1-p_1)+(1-p_n)p_1 \\ &= (1-2p_1)p_n+p_1 \end{aligned}$$

This is a linear difference equation of the first order. Its solution is

$$p_n = \frac{1}{2}\,[1-(1-2p_1)^n]$$

(b) If p_1 is very small and n is not too large, then

$$(1-2p_1)^n \approx 1-2p_1 n$$

and

$$\begin{aligned} p_n &\approx \frac{1}{2}[1-(1-2p_1 n)] \\ &= p_1 n \end{aligned}$$

Problem 3.24 - Regenerative repeater for PCM

Three basic functions are performed by regenerative repeaters: equalization, timing and decision-making.

Equalization: The equalizer shapes the incoming pulses so as to compensate for the effects of amplitude and phase distortion produced by the imperfect transmission characteristics of the channel.

Timing: The timing circuitry provides a periodic pulse train, derived from the received pulses, for sampling the equalized pulses at the instants of time where the signal-to-noise ratio is maximum.

Decision-making: The extracted samples are compared to a predetermined threshold to make decisions. In each bit interval, a decision is made whether the received symbol is 1 or 0 on the basis of whether the threshold is exceeded or not.

Problem 3.25

$m(t) = A\tanh(\beta t)$

To avoid slope overload, we require

$$\frac{\Delta}{T_s} \geq \max\left|\frac{dm(t)}{dt}\right| \tag{1}$$

$$\frac{dm(t)}{dt} = A\beta\,\mathrm{sech}^2(\beta t) \tag{2}$$

Hence, using Eq. (2) in (1):

$$\Delta \geq \max(A\beta\,\mathrm{sech}^2(\beta t)) \times T_s \tag{3}$$

$$\text{Since } \mathrm{sech}(\beta t) = \frac{1}{\cosh(\beta t)}$$

$$= \frac{2}{e^{+\beta t} + e^{-\beta t}}$$

it follows that the maximum value of sech(βt) is 1, which occurs at time $t = 0$. Hence, from Eq. (3) we find that $\Delta \geq A\beta T_s$.

Problem 3.26

The modulating wave is

$$m(t) = A_m \cos(2\pi f_m t)$$

The slope of m(t) is

$$\frac{dm(t)}{dt} = -2\pi f_m A_m \sin(2\pi f_m t)$$

The maximum slope of m(t) is equal to $2\pi f_m A_m$.

The maximum average slope of the approximating signal $m_a(t)$ produced by the delta modulator is δ/T_s, where δ is the step size and T_s is the sampling period. The limiting value of A_m is therefore given by

$$2\pi f_m A_m > \frac{\delta}{T_s}$$

or

$$A_m > \frac{\delta}{2\pi f_m T_s},$$

Assuming a load of 1 ohm, the transmitted power is $A_m^2/2$. Therefore, the maximum power that may be transmitted without slope-overload distortion is equal to $\delta^2/8\pi^2 f_m^2 T_s^2$.

Problem 3.27

$f_s = 10 f_{\text{Nyquist}}$

$f_{\text{Nyquist}} = 6.8$ kHz

$f_s = 10 \times 6.8 \times 10^3 = 6.8 \times 10^4$ Hz

$$\frac{\Delta}{T_s} \geq \max\left|\frac{dm(t)}{dt}\right|$$

For the sinusoidal signal $m(t) = A_m \sin(2\pi f_m t)$, we have

$$\frac{dm(t)}{dt} = 2\pi f_m A_m \cos(2\pi f_m t)$$

Hence,

$$\left|\frac{dm(t)}{dt}\right|_{\max} = |2\pi f_m A_m|_{\max}$$

or, equivalently,

$$\frac{\Delta}{T_s} \geq |2\pi f_m A_m|_{\max}$$

Therefore,

$$|A_m|_{\max} = \frac{\Delta}{T_s \times 2\pi \times f_m}$$

$$= \frac{\Delta f_s}{2\pi f_m}$$

$$= \frac{0.1 \times 6.8 \times 10^4}{2\pi \times 10^3}$$

$$= 1.08\text{V}$$

Problem 3.28

(a) From the solution to Problem 3.27, we have

$$A = \frac{\Delta f_s}{2\pi f_m} \quad \text{or} \quad \Delta = \frac{2\pi f_m A}{f_s} \tag{1}$$

The average signal power $= \dfrac{A^2}{2}$

$$= \frac{1}{2}\left(\frac{\Delta f_s}{2\pi f_m}\right)^2$$

With slope overload avoided, the only source of quantization of noise is granular noise. Replacing $\Delta/2$ for PCM with Δ for delta modulation, we find that the average quantization noise power is $\Delta^2/3$; for more details, see the solution to part (b) of Problem 3.30. The waveform of the reconstruction error (i.e., granular quantization noise) is a pattern of bipolar binary pulses characterized by (1) duration $= T_s = 1/f_s$, and (2) average power $= \Delta/3$. Hence, the autocorrelation function of the quantization noise is triangular in shape with a peak value of $\Delta^2/3$ and base $2T_s$, as shown in Fig. 1:

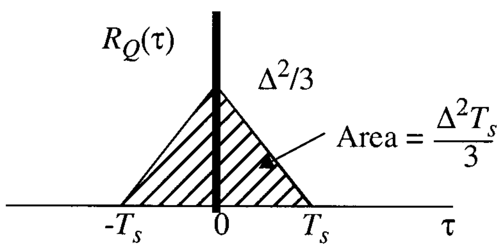


Fig. 1

From random process theory, we recall that

$$S_Q(f)\big|_{f=0} = \int_{-\infty}^{\infty} R_Q(\tau)\,d\tau$$

which, for the problem at hand, yields

$$S_Q(0) = \frac{\Delta^2 T_s}{3} = \frac{\Delta^2}{3f_s}$$

Typically, in delta modulation the sampling rate f_s is very large compared to the highest frequency component of the original message signal. We may therefore approximate the power spectral density of the granular quantization noise as

$$S_Q(f) \approx \begin{cases} \Delta^2/3f_s & -W \le f \le W \\ 0, & \text{otherwise} \end{cases}$$

where W is the bandwidth of the reconstruction filter at the demodulator output. Hence, the average quantization noise power is

$$N = \int_{-W}^{W} S_Q(f)df = \frac{2\Delta^2 W}{3f_s} \tag{2}$$

Substituting Eq. (2) into (1), we get

$$N = 2\left(\frac{2\pi f_m A}{f_s}\right)^2 \frac{W}{3f_s}$$

$$= \frac{8\pi^2 f_m^2 A^2 W}{3f_s^3}$$

(b) Correspondingly, output signal-to-noise ratio is

$$\text{SNR} = \frac{\left(\frac{1}{2}\right)A^2}{(8\pi^2 f_m^2 A^2 W)/3f_s^3}$$

$$= \frac{3f_s^3}{16\pi^2 f_m^2 W}$$

Problem 3.29

(a) $A \le \dfrac{\Delta f_s}{2\pi f_m}$

$$\Delta \ge \frac{2\pi f_m A}{f_s}$$

$$\Delta \geq \frac{2 \times \pi \times 10^3 \times 1}{50 \times 10^3}$$

$$= 0.126\text{V}$$

(b) $$(\text{SNR})_{\text{out}} = \frac{3}{8\pi^2} \frac{f_s^3}{f_m^2 W}$$

$$= \frac{3}{16\pi^2} \times \frac{(50 \times 10^3)^3}{10^6 \times 5 \times 10^3}$$

$$= 475$$

In decibels,

$$(\text{SNR})_{\text{out}} = 10\log_{10}475$$

$$= 26.8 \quad \text{dB}$$

Problem 3.30

(a) For linear delta modulation, the maximum amplitude of a sinusoidal test signal that can be used without slope-overload distortion is

$$A = \frac{\Delta f_s}{2\pi f_m}$$

$$= \frac{0.1 \times 60 \times 10^3}{2\pi \times 1 \times 10^3} \qquad f_s = 2 \times 3 \times 10^3$$

$$= 0.95\text{V}$$

(b) (i) Under the pre-filtered condition, it is reasonable to assume that the granular quantization noise is uniformly distributed between -Δ and +Δ. Hence, the variance of the quantization noise is

$$\sigma_Q^2 = \int_{-\Delta}^{\Delta} \frac{1}{2\Delta} q^2 dq$$

$$= \frac{1}{6\Delta}[q^3]_{-\Delta}^{\Delta}$$

$$= \frac{\Delta^2}{3}$$

The signal-to-noise ratio under the pre-filtered condition is therefore

$$(\mathrm{SNR})_{\mathrm{prefiltered}} = \frac{A^2/2}{\Delta^2/3}$$

$$= \frac{3A^2}{2\Delta^2}$$

$$= \frac{3 \times 0.95^2}{2 \times 0.1^2}$$

$$= 135$$
$$= 21.3 \text{ dB}$$

(ii)The signal-to-noise ratio under the post-filtered condition is

$$\left(\frac{S}{N}\right)_{\mathrm{postfiltered}} = \frac{3}{16\pi^2} \times \frac{f_s^3}{f_m^2 W}$$

$$= \frac{3}{16\pi^2} \times \frac{(60)^3}{(1)^2 \times 3}$$

$$= 1367$$

$$= 31.3 \text{ dB}$$

The filtering gain in signal-to-noise ratio due to the use of a reconstruction filter at the demodulator output is therefore 31.3 - 21.3 = 10 dB.

Problem 3.31

Let the sinusoidal signal $m(t) = A\sin\omega_0 t$, where $\omega_0 = 2\pi f_0$

The autocorrelation of the signal is

$$R_m(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$$

$$R_m(0) = \frac{A^2}{2}$$

$$R_m(1) = \frac{A^2}{2}\cos\left(\omega_0 \times \frac{1}{10\omega_0}\right)$$

$$= \frac{A^2}{2}\cos(0.1)$$

For this problem, we thus have

$$\mathbf{R}_m = [R_m(0)], \quad \mathbf{r}_m = [R_m(1)]$$

(a) The optimum solution is given by

$$\mathbf{w}_0 = \mathbf{R}_m^{-1}\mathbf{r}_m$$

$$= \frac{\frac{A^2}{2}\cos(0.1)}{\frac{A^2}{2}} = \cos(0.1)$$

$$= 0.995$$

(b) $J_{\min} = R_m(0) - \mathbf{r}_m^T\mathbf{R}_m^{-1}\mathbf{r}_m$

$$= \frac{A^2}{2} - \frac{A^2}{2}\cos(0.1) \times \frac{A^2}{2}\cos(0.1)/(A^2/2)$$

$$= \frac{A^2}{2}(1 - \cos^2(0.1))$$

$$= 0.005A^2$$

Problem 3.32

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0.8 & 0.6 \\ 0.8 & 1 & 0.8 \\ 0.6 & 0.8 & 1 \end{bmatrix}$$

$$\mathbf{r}_x = [0.8,\ 0.6,\ 0.4]^T$$

(a) $\mathbf{w}_0 = \mathbf{R}_x^{-1}\mathbf{r}_x$

$$= \begin{bmatrix} 1 & 0.8 & 0.6 \\ 0.8 & 1 & 0.8 \\ 0.6 & 0.8 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.8 \\ 0.6 \\ 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.875 \\ 0 \\ -0.125 \end{bmatrix}$$

(b) $J_{\min} = R_x(0) - \mathbf{r}_x^T\mathbf{R}_x^{-1}\mathbf{r}_x$

$$= R_x(0) - \mathbf{r}_x^T\mathbf{w}_0$$

$$= 1 - [0.8,\ 0.6,\ 0.4] \begin{bmatrix} 0.875 \\ 0 \\ -0.125 \end{bmatrix}$$

$$= 1 - (0.8 \times 0.875 - \ 0.4 \times \ 0.125)$$

$$= 1 - 0.7 + 0.05$$

$$= 0.35$$

Problem 3.33

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

$$\mathbf{r}_x = [0.8, \quad 0.6]^T$$

(a) $\mathbf{w}_0 = \mathbf{R}_x^{-1}\mathbf{r}_x$

$$= \begin{bmatrix} 0.8889 \\ -0.1111 \end{bmatrix}$$

(b) $J_{\min} = R_x(0) - \mathbf{r}_x^T\mathbf{R}_x^{-1}\mathbf{r}_x$

$$= 1 - 0.6444$$

$$= 0.3556$$

which is slightly worse than the result obtained with a linear predictor using three unit delays (i.e., three coefficients). This result is intuitively satisfying.

Problem 3.34

Input signal variance = $R_x(0)$

The normalized autocorrelation of the input signal for a lag of one sample interval is

$$\rho_x(1) = \frac{R_x(1)}{R_x(0)} = 0.75$$

Error variance $= R_x(0) - R_x(1)R_x^{-1}(0)R_x(1)$

$$= R_x(0)(1 - \rho_x^2(1))$$

$$\text{Processing gain} = \frac{R_x(0)}{R_x(0)(1 - \rho_x^2(1))}$$

$$= \frac{1}{1 - \rho_x^2(1)}$$

$$= \frac{1}{1 - (0.75)^2}$$

$$= 2.2857$$

Expressing the processing gain in dB, we have

$$10\log_{10}(2.2857) = 3.59 \text{ dB}$$

Problem 3.35

$$\text{Processing gain} = \frac{R_x(0)}{R_x(0)\left(1 - \dfrac{\mathbf{r}_x^T \mathbf{R}_x^{-1} \mathbf{r}_x}{R_x(0)}\right)}$$

(a) Three-tap predictor:
Processing gain = 2.8571
= 4.56 dB

(b) Two-tap predictor:
Processing gain = 2.8715
= 4.49 dB

Therefore, the use of a three-tap predictor in the DPCM system results an improvement of 4.56 - 4.49 = 0.07 dB over the corresponding system using a two-tap predictor.

Problem 3.36

(a) For DPCM, we have $10\log_{10}(\text{SNR})_0 = \alpha + 6n$ dB

For PCM, we have $10\log_{10}(\text{SNR})_0 = 4.77 + 6n - 20\log_{10}(\log(1 + \mu))$

where n is the number of quantization levels
SNR of DPCM

SNR = $\alpha + 6n$, where $-3 < \alpha < 15$

For n=8, the SNR is in the range of 45 to 63 dBs.

<u>SNR of PCM</u>

$$\begin{aligned}\text{SNR} &= 4.77 + 6n - 20\log_{10}(\log(2.56)) \\ &= 4.77 + 48 - 14.8783 \\ &= 38 \text{ dB}\end{aligned}$$

Therefore, the SNR improvement resulting from the use of DPCM is in the range of 7 to 25 dB.

(b) Let us assume that n_1 bits/sample are used for DPCM and n bits/sample for PCM

If $\alpha = 15$ dB, then we have

$$15 + 6n_1 = 6n - 10.0$$

Rearranging: $$(n - n_1) = \frac{10 + 15}{6}$$

$$= 4.18$$

which, in effect, represents a saving of about 4 bits/sample due to the use of DPCM.

If, on the other hand, we choose $\alpha = -3$ dB, we have

$$-3 + 6n_1 = 6n - 10$$

Rearranging: $$(n - n_1) = \frac{10 - 3}{6}$$

$$= \frac{7}{6}$$

$$= 1.01$$

which represents a saving of about 1 bit/sample due to the use of DPCM.

Problem 3.37

The transmitting prediction filter operates on exact samples of the signal, whereas the receiving prediction filter operates on quantized samples.

Problem 3.38

Matlab codes

```
% Problem 3.38, CS: Haykin
%flat-topped PAM signal
%and magnitude spectrum
% Mathini Sellathurai

%data
fs=8000; % sample frequency
ts=1.25e-4; %1/fs
pulse_duration=5e-5; %pulse duration

% sinusoidal sgnal;
td=1.25e-5; %sampling frequency of signal
fd=80000;
t=(0:td:100*td);
fm=10000;
s=sin(fm*t);

% PAM signal generation
pam_s=PAM(s,td,ts,pulse_duration);
figure(1);hold on
```

```
plot(t,s,'--');
plot(t(1:length(pam_s)),pam_s);
xlabel('time')
ylabel('magnitude')
legend('signal','PAM-signal');

% Computing magnitude spectrum S(f) of the signal
a=((abs(fft(pam_s)).^2));
a=a/max(a);
f=fs*(fs/fd:fs*(fs/fd):(length(a))*fs*(fs/fd);
figure(2)
plot(f,a);
xlabel('frequency');
ylabel('magnitude')

% finding the zeros
index=find(a<1e-5);

% finding the first zero
fprintf('Envelopes goes through zero for the first time = %6d\n', min(index)*fs*(fs/fd))
```

```
function pam_s=PAM(s,td,ts,pulse_duration)

% Problem 3.38, CS: Haykin
%flat-topped PAM signal
%used in   Problem 3.38, CS: Haykin
% Mathini Sellathurai

potd=pulse_duration/td;
tsotd=ts/td;

y=zeros(1,length(s));
tt=1:(tsotd):length(s);

for kk=1:length(tt);
y(tt(kk):tt(kk)+potd-1)=s(tt(kk)).*ones(1,potd);
end

pam_s=y(1:length(s)-potd);
```

Answers: 3.38

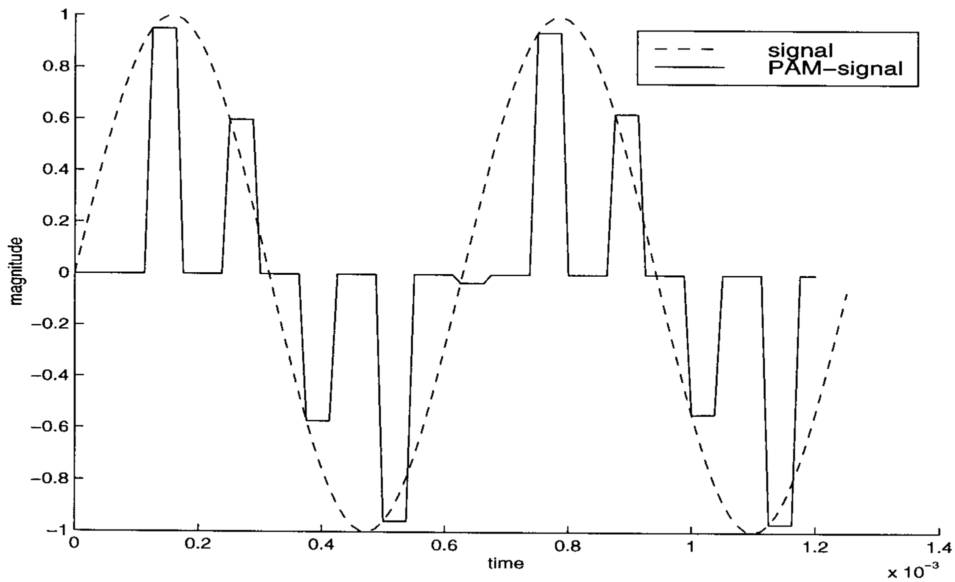


Figure 1: Flat-topped PAM signal

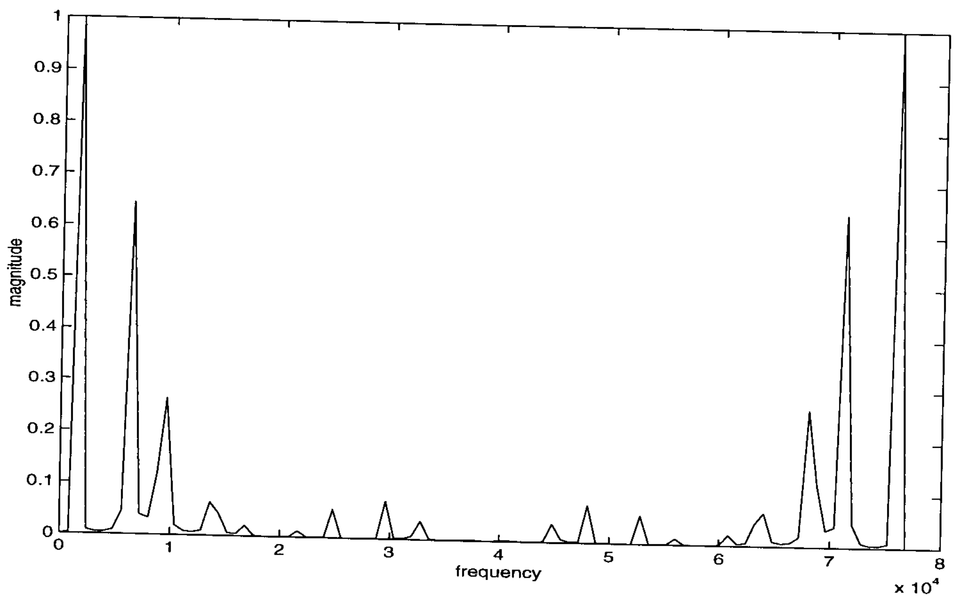


Figure .2: Magnitude spectrum of flat-topped PAM signal

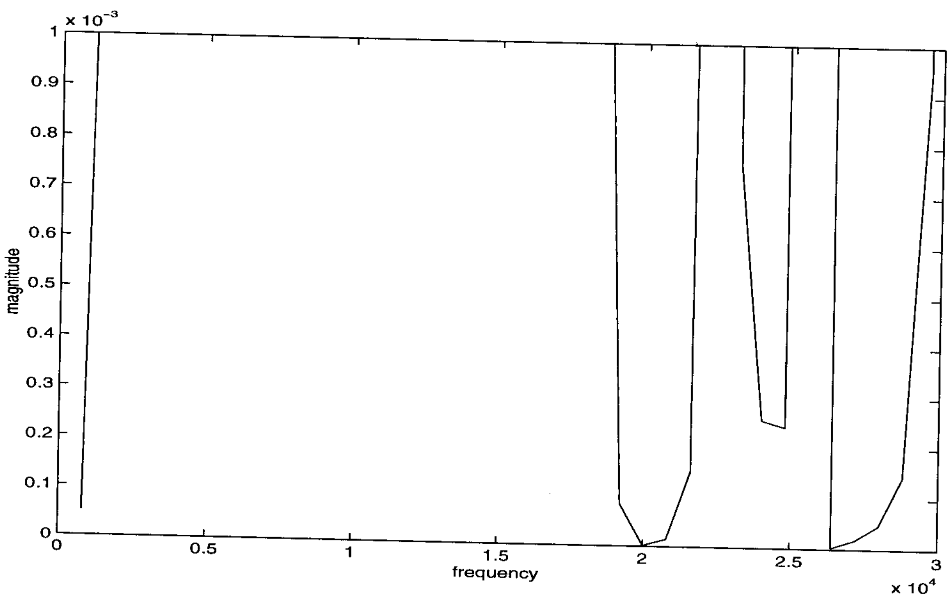


Figure .3: Zoomed magnitude spectrum of flat-topped PAM signal

Problem 3.39

Matlab codes

```
%problem 3.39, CS: Haykin
%mue-law pCM and uniform quantizing
%Mathini Sellathurai
clear all

%sinusoidal signal
t=[0:2*pi/100:2*pi];
a=sin(t);

% input signal to noise ratio in db
SNRdb=[-20 -15 -10 -5 0 5 10 15 20 25 ];

for nEN=1:10
  sqnrfm=0; sqnrfu=0;
for k=1:100
  snr = 10^(SNRdb(nEN)/10);
  wn= randn(1,length(a))/sqrt(snr); % noise
  a1=a+wn; %signal plus noise

   [a_quanu,codeu,sqnr_u]=u_pcm(a1,256); %call u-PCM
   [a_quanm,codem,sqnr_m]=mue_pcm(a1,256,255); %call mue-PCM

   sqnrfm=sqnrfm+sqnr_m;
   sqnrfu=sqnrfu+sqnr_u;
end
  SNROm(nEN)=sqnrfm/k; %bin-SNR-MUE-PCM
  SNROu(nEN)=sqnrfu/k;  %bin-SNR-U-PCM
end

%plots
figure;hold on;
plot(SNRdb,SNROu,'-+')
plot(SNRdb,SNROm,'-o')
xlabel('input signal-to-noise-ration in db')
 ylabel('output signal-to-noise-ration in db')
legend('uniform  PCM, 256 levels','mue-law PCM, mue=255')
```

```
function [a_q,snr]=u_pcm(a,n)
% function to generate uniform PCM for sinwave
%used in problem 3.39, CS: Haykin
%Mathini Sellathurai

n=length(a);
amax=max(abs(a));
a_q=a;
b_q=a_q;
d=2/n;
q=d.*[0:n-1];
q=q-((n-1)/2)*d;
for i=1:n
a_q(find((q(i)-d/2<= a_q) & (a_q <=q(i)+d/2)))=...
q(i).*ones(1,length(find((q(i)-d/2 <=a_q) & (a_q<=q(i)+d/2))));
b_q(find(a_q==q(i)))=(i-1).*ones(1,length(find(a_q==q(i))));
end
a_q =a_q*amax;

snr=20*log10(norm(a)/norm(a-a_q));
```

```
function [a_q,snr]=mue_pcm(s,n,mue)
% function to generate mue-law PCM for sinwave
%used in problem 3.39, CS: Haykin
%Mathini Sellathurai

a=max(abs(s));

% mue-law
y=(log(1+mue*abs(s/a))./log(1+mue)).*sign(s);
[y_q,code,sqn]=u_pcm(y,n);

%inverse mue-law
a_q=(((1+mue).^(abs(y_q))-1)./mue).*sign(y_q);
a_q=a_q*a;

%SNR
snr=20*log10(norm(s)/norm(s-a_quan));
```

Answer to Problem 3.39

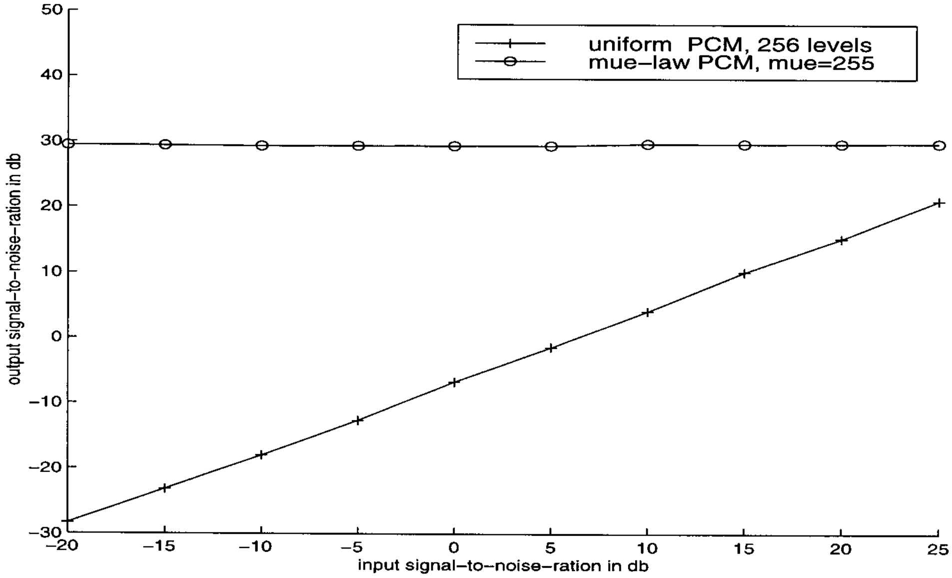


Figure 1 . input signal-to-noise ratio Vs. output signal-to-noise ratio for μ-law PAM and uniform PCM

Problem 3.40

Matlab codes

```
% Problem 3.40, CS: Haykin
%Normalized LMS- prediction
%of AR process/ speech signal
% Mathini Sellathurai

clear all
mue=0.05; % step size parameter, a value between 0 ans 2
p=2; % filter order
N=10; % size of data
M=1;% number of realizations

% initializing counters
err1=zeros(1,N-p);
xhat1=zeros(1,N-p);
x=zeros(1,N);

for m=1:M % 100 realizations

x(1:2)= [0.1 0.2];

%AR process
for k=3:N
x(k)=(0.8*x(k-1)-0.1*x(k-2))+0.1*rand(1);
end

% LMS prediction
[err, xhat]=LMS(x,mue,p);
err1=err1+err.^2;
xhat1=xhat1+xhat;
end

plot(err1/m,'-');
```

```
function [err, xhat]=LMS(xx,mue,p)
% function Normalized LMS
%p-order of the filter
%mue-step size parameter
%used in problem 3.40, CS: Haykin
%Mathini Sellathurai

% length of the data
N=length(xx);

% initializing weights and erros
w=zeros(p,N-p);
err=ones(1,N-p);
xhat=zeros(1,N-p);

%prediction
 l=1;
        for k=1:N-p
                h=xx(k:p+k-1);
                err(l)=(xx(k+p)-h*w(:,l));
                xhat(l)=h*w(:,l);
                xxx=xx(l+p-1)+xx(l+p-2);
                w(:,l+1)=w(:,l)+(mue/xxx)*h'*err(l);
              l=l+1;
        end
```

Answer to Problem 3.40

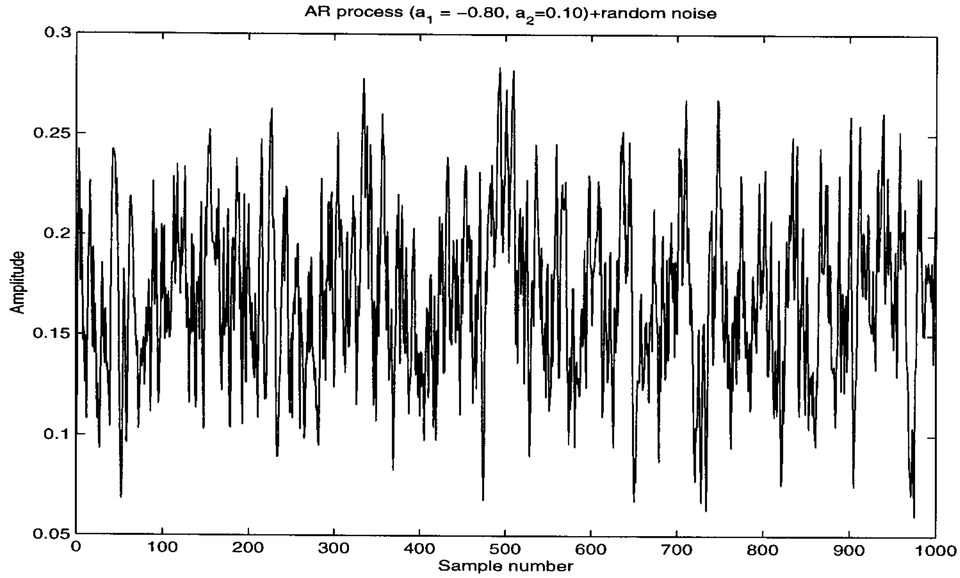


Figure 1 : Noisy-AR-process, $a_0 = -0.80$, $a_1 = 0.10$

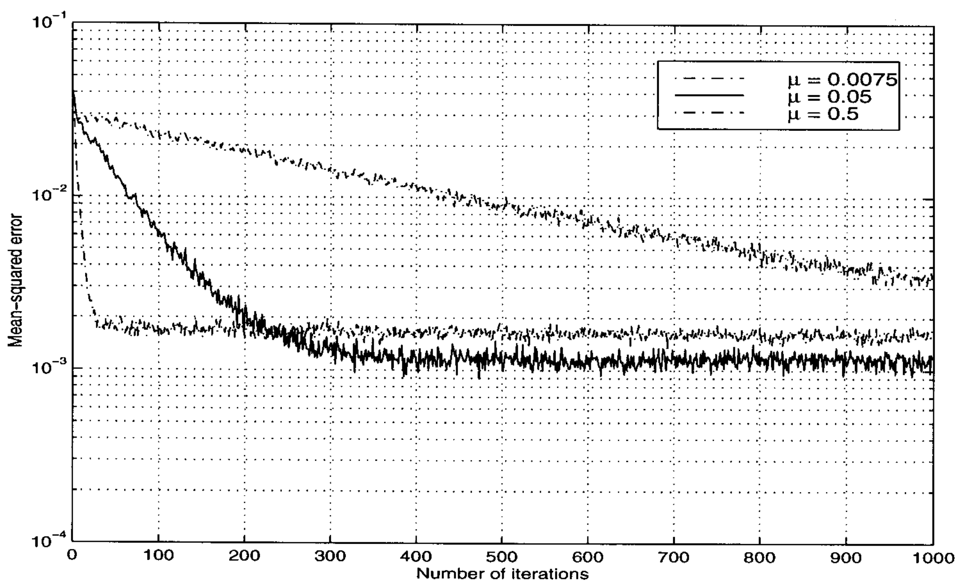


Figure 2 : Learning curves for $\mu = 0.0075, 0.05, 0.5$

CHAPTER 4

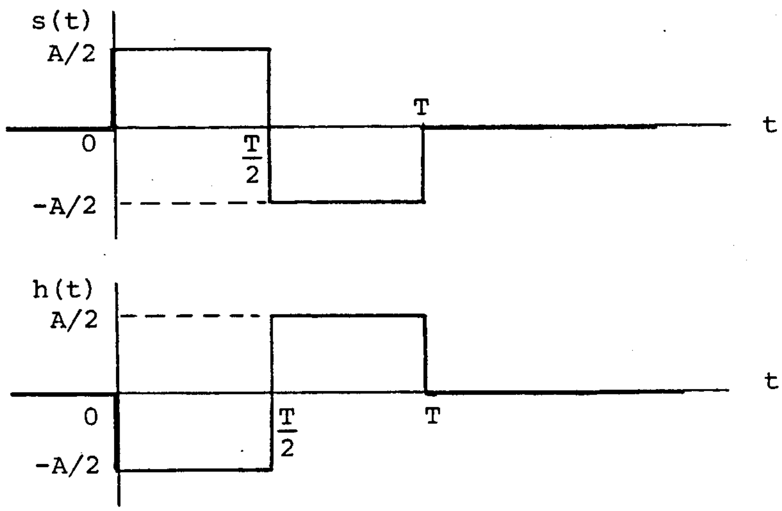
Baseband Pulse Transmission

Problem 4.1

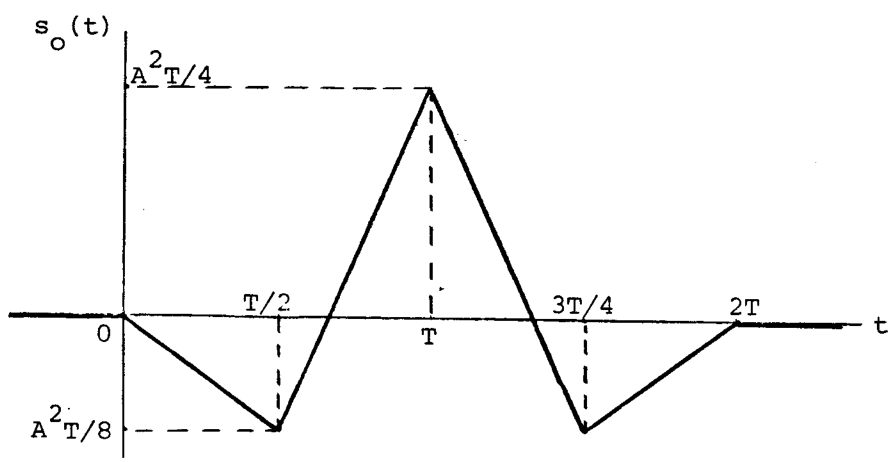
(a) The impulse response of the matched filter is

$h(t) = s(T-t)$

The $s(t)$ and $h(t)$ are shown below:



(b) The corresponding output of the matched filter is obtained by convolving $h(t)$ with $s(t)$. The result is shown below:



(c) The peak value of the filter output is equal to $A^2T/4$, occuring at $t=T$.

Problem 4.2

(a) The matched filter of impulse response $h_1(t)$ for pulse $s_1(t)$ is given in the solution to Problem 4.1. The matched filter of impulse response $h_2(t)$ for $s_2(t)$ is given by

$$h_2 = s_2(T-t)$$

which has the following waveform:

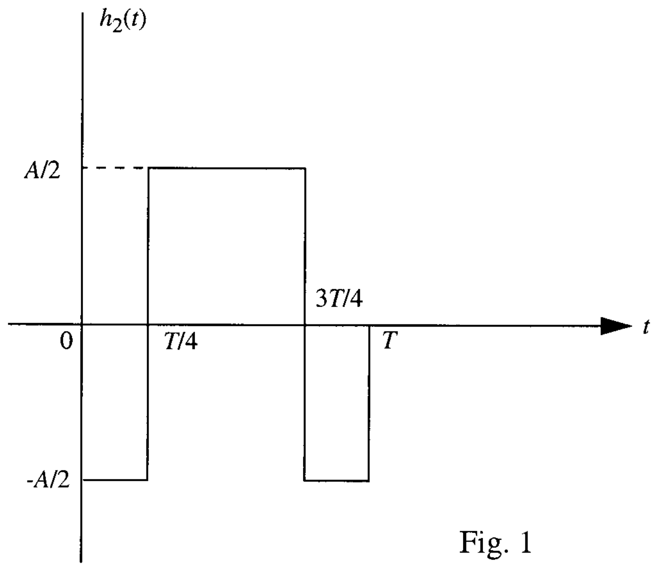


Fig. 1

(b) (i) The response of the matched filter, matched to $s_2(t)$ and due to $s_1(t)$ as input, is obtained by convolving $h_2(t)$ with $s_1(t)$, as shown by

$$y_{21}(t) = \int_0^T s_1(\tau)h_2(t-\tau)d\tau$$

The waveform of the output $y_{21}(t)$ so computed is plotted in Figure 2. This figure also includes the corresponding waveforms of input $s_1(t)$ and impulse response $h_2(t)$.

(ii) Next, the response of the matched filter, matched to $s_1(t)$ and due to $s_2(t)$ as input, is obtained by convolving $h_1(t)$ with $s_2(t)$, as shown by

$$y_{12}(t) = \int_0^T s_2(\tau)h_1(t-\tau)d\tau$$

Figure 3 shows the waveforms of input $s_2(t)$, impulse response $h_1(t)$, and response $y_{12}(t)$.

4.2(b)

(i)

Pulse $s_1(t)$

Filter $h_2(t)$

$h_2(t) = s_2(T-t)$

Filter response

$\int_0^T s_1(\tau)\, h_2(t-\tau)\, d\tau$

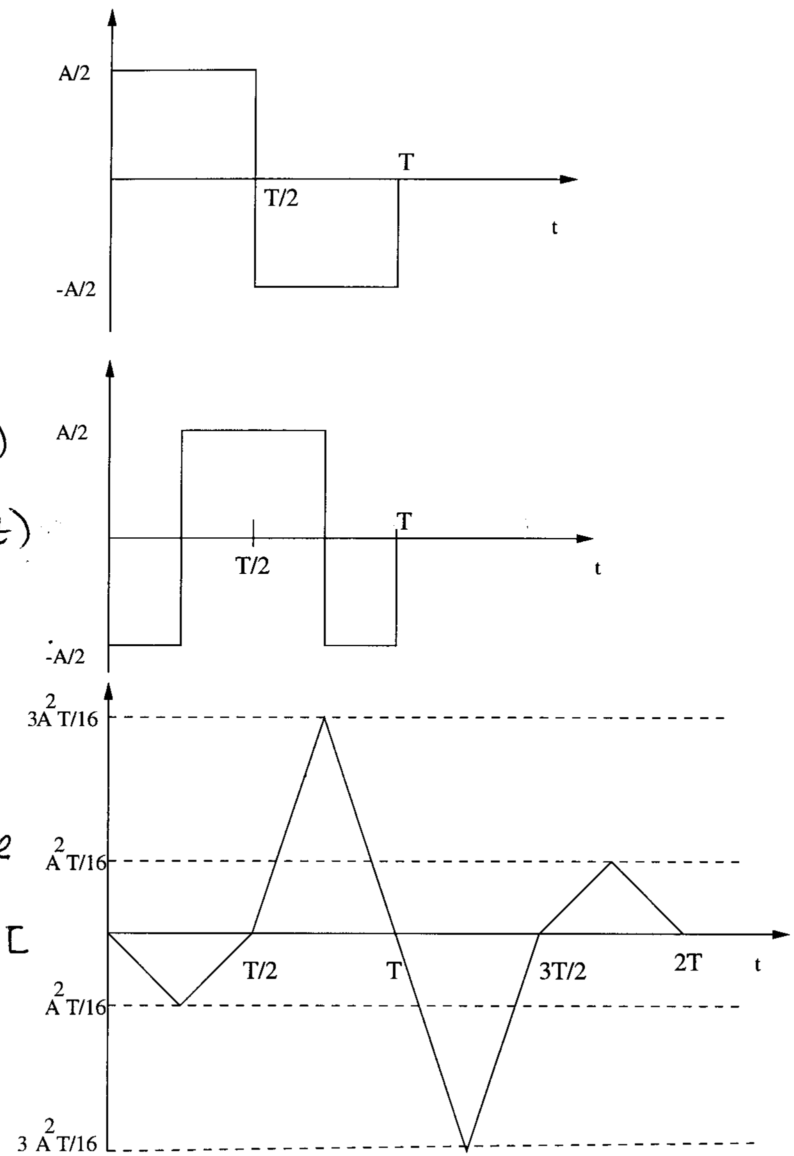


Fig. 3

4·2(b)

(ii)

Pulse $s_2(t)$

Filter $h_1(t)$

Filter matched to Pulse $s_1(t)$

Filter response $\int_0^T s_2(\tau)\, h_1(t-\tau)\, d\tau$

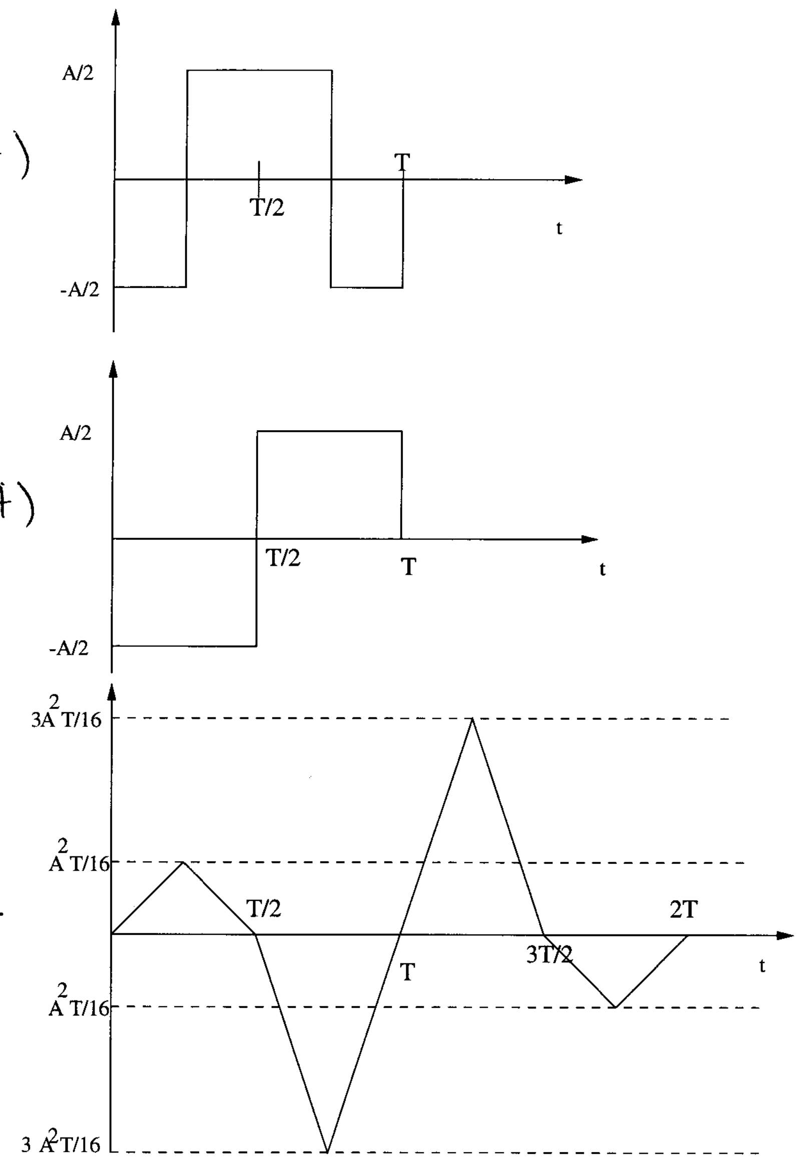


Fig. 3

Note that $y_{12}(t)$ is exactly the negative of $y_{21}(t)$. However, in both cases we find that at $t = T$, both outputs are equal to zero, as shown by

$$y_{21}(T) = y_{12}(T) = 0$$

For n pulses $s_1(t)$ $s_2(t),\ldots,s_n(t)$ that are orthogonal to each other over the interval [0,T], the n-dimensional matched filter has the following structure:

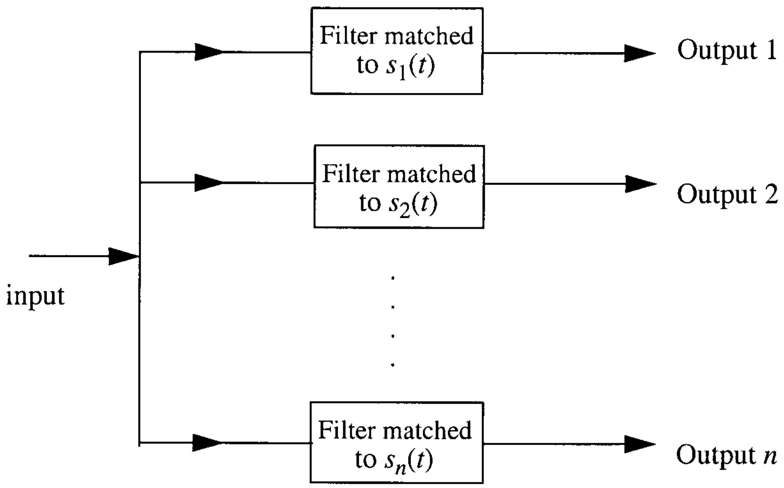


Fig. 4

Problem 4.3

Ideal low-pass filter with variable bandwidth. The transfer function of the matched filter for a rectangular pulse of duration τ and amplitude A is given by

$$H_{opt}(f) = \text{sinc}(fT)\exp(-j\pi fT) \qquad (1)$$

The amplitude response $|H_{opt}(f)|$ of the matched filter is plotted in Fig. 1(a). We wish to approximate this amplitude response with an ideal low-pass filter of bandwidth B. The amplitude response of this approximating filter is shown in Fig. 1(b). The requirement is to determine the particular value of bandwidth B that will provide the best approximation to the matched filter.

We recall that the maximum value of the output signal, produced by an ideal low-pass filter in response to the rectangular pulse occurs at $t = T/2$ for $BT \leq 1$. This maximum value, expressed in terms of the sine integral, is equal to $(2A/\pi)\text{Si}(\pi BT)$. The average noise power at the output of the ideal low-pass filter is equal to BN_0. The maximum output signal-to-noise ratio of the ideal low-pass filter is therefore

$$(\text{SNR})_0' = \frac{(2A/\pi)^2 \text{Si}^2(\pi BT)}{BN_0} \qquad (2)$$

Thus, using Eqs. (1) and (2), and assuming that $AT = 1$, we get

$$\frac{(\text{SNR})_0'}{(\text{SNR})_0} = \frac{2}{\pi^2 BT}\text{Si}^2(\pi BT)$$

This ratio is plotted in Fig. 2 as a function of the time-bandwidth product BT. The peak value on this curve occurs for $BT = 0.685$, for which we find that the maximum signal-to-noise ratio of the ideal low-pass filter is 0.84 dB below that of the true matched filter. Therefore, the " best" value for the bandwidth of the ideal low-pass filter characteristic of Fig. 1(b) is $B = 0.685/T$.

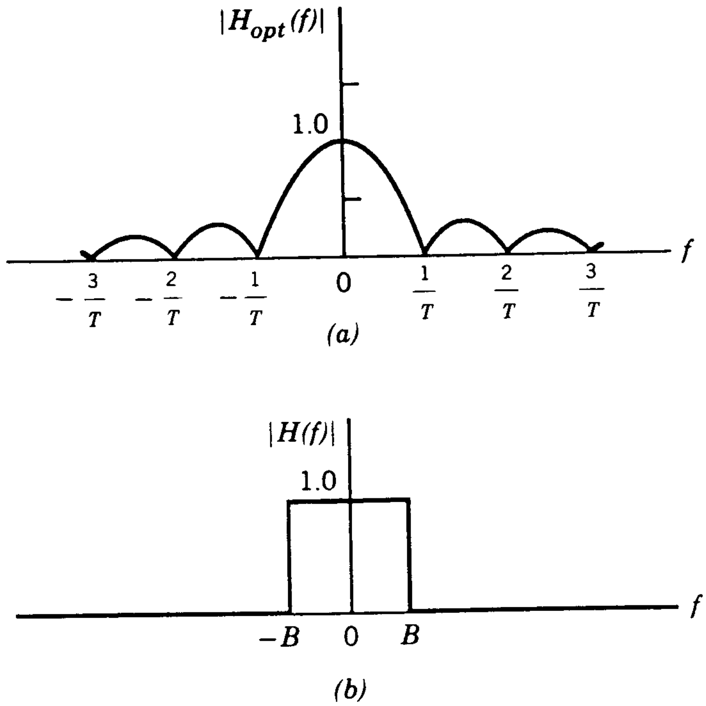


Figure 1

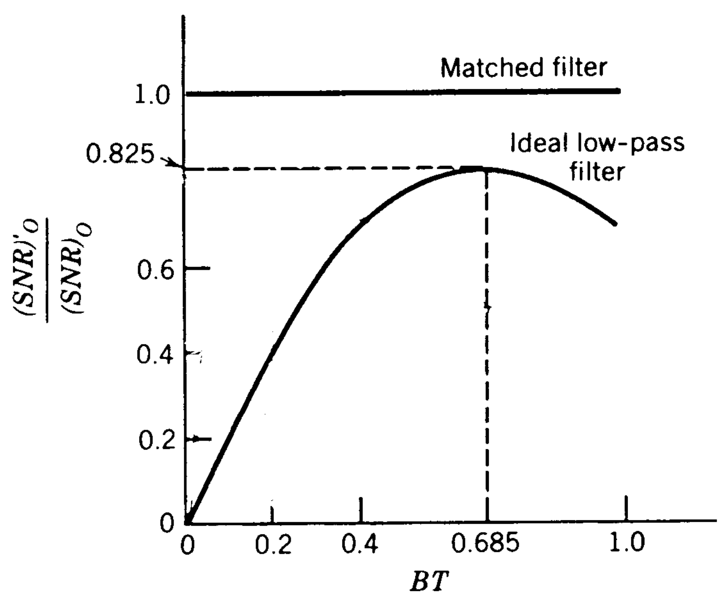
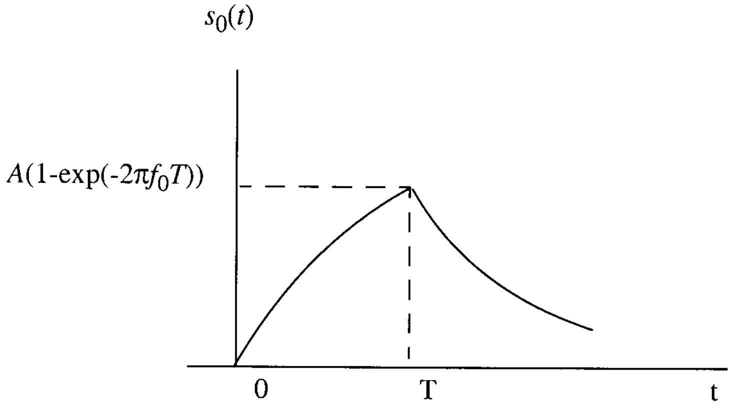


Figure 2

Problem 4.4

The output of the low-pass RC filter, produced by a rectangular pulse of amplitude A and duration T, is as shown below:



The peak value of the output pulse power is

$$P_{\text{out}} = A^2[1 - \exp(-2\pi f_0 T)]^2$$

where f_0 is the 3-dB cutoff frequency of the RC filter.

The average output noise power is

$$N_{\text{out}} = \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{df}{1 + (f/f_0)^2}$$

$$= \frac{N_0 \pi f_0}{2}$$

The corresponding value of the output signal-to-noise ratio is therefore

$$(\text{SNR})_{\text{out}} = \frac{2A^2}{N_0 \pi f_0}[1 - \exp(-2\pi f_0 T)]$$

Differentiating $(\text{SNR})_0$ with respect to $f_0 T$ and setting the result equal to zero, we find that $(\text{SNR})_{\text{out}}$ attains its maximum value at

$$f_0 = \frac{0.2}{T}$$

The corresponding maximum value of $(\text{SNR})_{\text{out}}$ is

$$(\mathrm{SNR})_{0,\max} = \frac{2A^2T}{0.2\pi N_0}[1 - \exp(-0.4\pi)]^2$$

$$= \frac{1.62A^2T}{N_0}$$

For a perfect matched filter, the output signal-to-noise ratio is

$$(\mathrm{SNR})_{0,\text{matched}} = \frac{2E}{N_0}$$

$$= \frac{2A^2T}{N_0}$$

Hence, we find that the transmitted energy must be increased by the ratio 2/1.62, that is, by 0.92 dB so that the low-pass RC filter with $f_0 = 0.2/T$ realizes the same performance as a perfectly matched filter.

Problem 4.5

(i) $p_0 > p_1$

The transmitted symbol is more likely to be 0. Hence, the average probability of symbol error is smaller when a 0 is transmitted than when a 1 is transmitted. In such a situation, the threshold λ in Figs. 4.5(a) and (b) in the textbook is moved to the right.

(ii) $p_1 > p_0$

The transmitted symbol is more likely to be 1. Hence, the average probability of symbol error is smaller when a 1 is transmitted than when a 0 is transmitted. In this second situation, the threshold λ in Figs. 4.5(a) and (b) in the textbook is moved to the left.

Problem 4.6

The average probability of error is

$$P_e = p_1 \int_{-\infty}^{\lambda} f_Y(y|1)dx + p_0 \int_{\lambda}^{\infty} f_Y(y|0)dx \tag{1}$$

An optimum choice of λ corresponds to minimum P_e. Differentiating Eq. (1) with respect to λ, we get:

$$\frac{\partial P_e}{\partial \lambda} = p_1 f_Y(\lambda|1) - p_0 f_Y(\lambda|0)$$

Setting $\dfrac{\partial P_e}{\partial \lambda} = 0$, we get the following condition for the optimum value of λ:

$$\frac{f_Y(\lambda_{opt}|1)}{f_Y(\lambda)_{opt}|0)} = \frac{p_0}{p_1}$$

which is the desired result.

Problem 4.7

In a binary PCM system, with NRZ signaling, the average probability of error is

$$P_e = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

The signal energy per bit is

$$E_b = A^2 T_b$$

where A is the pulse amplitude and T_b is the bit (pulse) duration. If the signaling rate is doubled, the bit duration T_b is reduced by half. Correspondingly, E_b is reduced by half.

Let $u = \sqrt{E_b/N_0}$. We may then set

$$P_e = 10^{-6} = \frac{1}{2}\,\text{erfc}(u)$$

Solving for u, we get

$$u = 3.3$$

When the signaling rate is doubled, the new value of P_e is

$$P_e' = \frac{1}{2}\,\text{erfc}\left(\frac{u}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\,\text{erfc}(2.33)$$

$$= 10^{-3}.$$

Problem 4.8

(a) The average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

where $E_b = A^2 T_b$. We may rewrite this formula as

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{A}{\sigma}\right) \tag{1}$$

where A is the pulse amplitude at $\sigma = \sqrt{N_0 T_b}$. We may view σ^2 as playing the role of noise variance at the decision device input. Let

$$u = \sqrt{\frac{E_b}{N_0}} = \frac{A}{\sigma}$$

We are given that

$$\sigma^2 = 10^{-2} \text{ volts}^2, \qquad \sigma = 0.1 \text{ volt}$$

$$P_e = 10^{-8}$$

Since P_e is quite small, we may approximate it as follows:

$$\mathrm{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}\,u}$$

We may thus rewrite Eq. (1) as (with $P_e = 10^{-8}$)

$$\frac{\exp(-u^2)}{2}\sqrt{\pi}u = 10^{-8}$$

Solving this equation for u, we get

$$u = 3.97$$

The corresponding value of the pulse amplitude is

$$A = \sigma u = 0.1 \times 3.97$$
$$= 0.397 \text{volts}$$

(b) Let σ^2_i denote the combined variance due to noise and interference; that is

$$\sigma_T^2 = \sigma^2 + \sigma_i^2$$

where σ^2 is due to noise and σ^2_i is due to the interference. The new value of the average probability of error is 10^{-6}. Hence

$$10^{-6} = \frac{1}{2}\,\mathrm{erfc}\left(\frac{A}{\sigma_T}\right) \tag{2}$$
$$= \frac{1}{2}\,\mathrm{erfc}(u_T)$$

where

$$u_T = \frac{A}{\sigma_T}$$

Equation (2) may be approximated as (with $P_e = 10^{-6}$)

$$\frac{\exp(-u_T^2)}{2\sqrt{\pi}\, u_T} \simeq 10^{-6}$$

Solving for u_T, we get

$$u_T = 3.37$$

The corresponding value of $\sigma^2{}_T$ is

$$\sigma_T^2 = \left(\frac{A}{u_T}\right)^2 = \left(\frac{0.397}{3.37}\right)^2 = 0.0138 \text{ volts}^2$$

The variance of the interference is therefore

$$\begin{aligned}\sigma_i^2 &= \sigma_T^2 - \sigma^2 \\ &= 0.0138 - 0.01 \\ &= 0.0038 \text{ volts}^2\end{aligned}$$

Problem 4.9

Consider the performance of a binary PCM system in the presence of channel noise; the receiver is depicted in Fig. 1. We do so by evaluating the average probability of error for such a system under the following assumptions:

1. The PCM system uses an on-off format, in which symbol 1 is represented by A volts and symbol 0 by zero volt.
2. The symbols 1 and 0 occur with equal probability.
3. The channel noise w(t) is white and Gaussian with zero mean and power spectral density $N_0/2$.

To determine the average probability of error, we consider the two possible kinds of error separately. We begin by considering the first kind of error that occurs when symbol 0 is sent and the receiver chooses symbol 1. In this case, the probability of error is just the probability that the correlator output in Fig. 1 will exceed the threshold λ owing to the presence of noise, so the transmitted symbol 0 is mistaken for symbol 1. Since the a priori probabilities of symbols 1 and 0 are equal, we have $p_0 = p_1$ Correspondingly, the expression for the threshold λ simplifies as follows:

$$\lambda = \frac{A^2 T_b}{2} \tag{1}$$

where T_b is the bit duration, and $A^2 T_b$ is the signal energy consumed in the transmission of symbol 1. Let y denote the correlator output:

$$y = \int_0^{T_b} s(t)x(t)dt \tag{2}$$

Under hypothesis H_0, corresponding to the transmission of symbol 0, the received signal x(t) equals the channel noise w(t). Under this hypothesis we may therefore describe the correlator output as

$$H_0 : y = A \int_0^{T_b} w(t)dt \tag{3}$$

Since the white noise w(t) has zero mean, the correlator output under hypothesis H_0 also has zero mean. In such a situation, we speak of a *conditional mean*, which (for the situation at hand) we describe by writing

$$\mu_0 = E[Y \mid H_0] = E\left[\int_0^{T_b} W(t)dt \right] = 0 \tag{4}$$

where the random variable Y represents the correlator output with y as its sample value and W(t) is a white-noise process with w(t) as its sample function. The subscript 0 in the conditional mean μ_0 refers to the condition that hypothesis H_0 is true. Correspondingly, let $\sigma^2{}_0$ denote the *conditional variance* of the correlator output, given that hypothesis H_0 is true. We may therefore write

$$\begin{aligned} \sigma_0^2 &= E[Y^2 \mid H_0] \\ &= E\left[\int_0^{T_b} \int_0^{T_b} W(t_1)W(t_2)dt_1\, dt_2 \right] \end{aligned} \tag{5}$$

The double integration in Eq. (5) accounts for the squaring of the correlator output. Interchanging the order of integration and expectation in Eq. (5), we may write

$$\begin{aligned} \sigma_0^2 &= \int_0^{T_b} \int_0^{T_b} E[W(t_1)W(t_2)]dt_1\, dt_2 \\ &= \int_0^{T_b} \int_0^{T_b} R_w(T_1 - t_2)dt_1\, dt_2 \end{aligned} \tag{6}$$

The parameter $R_w(t_1 - t_2)$ is the *ensemble-averaged autocorrelation function* of the white-noise process W(t). From random process theory, it is recognized that the autocorrelation function and power spectral density of a random process form a Fourier transform pair. Since the white-noise process W(t) is assumed to have a constant power spectral density of $N_0/2$, it follows that the autocorrelation function of such a process consists of a delta function weighted by $N_0/2$. Specifically, we may write

$$R_W(t_1 - t_2) = \frac{N_0}{2}\, \delta(\tau - t_1 + t_2) \tag{7}$$

Substituting Eq. (7) in (6), and using the property that the total area under the Dirac delta function $\delta(\tau - t_1 + t_2)$ is unity, we get

$$\sigma_0^2 = \frac{N_0 T_b A^2}{2} \tag{8}$$

The statistical characterization of the correlator output is computed by noting that it is Gaussian distributed, since the white noise at the correlator input is itself Gaussian (by assumption). In summary, we may state that under hypothesis H_0 the correlator output is a Gaussian random variable with zero mean and variance $N_0T_bA^2/2$, as shown by

$$f_0(y) = \frac{1}{\sqrt{\pi N_0 T_b}\ A} \exp\left(- \frac{y^2}{N_0 T_b A^2} \right) \tag{9}$$

where the subscript in $f_0(y)$ signifies the condition that symbol 0 was sent.

Figure 2(a) shows the bell-shaped curve for the probability density function of the correlator output, given that symbol 0 was transmitted. The probability of the receiver deciding in favor of symbol 1 is given by the area shown shaded in Fig. 2(a). The part of the y-axis covered by this area corresponds to the condition that the correlator output y is in excess of the threshold λ defined by Eq. (1). Let P_{e0} denote the *conditional probability of error, given that symbol 0 was sent.* Hence, we may write

$$\begin{aligned} p_{10} &= \int_{\lambda}^{\infty} f_0(y)\, dy \\ &= \frac{1}{\sqrt{\pi N_0 T_b}\ a} \int_{A^2T_b/2}^{\infty} \exp\left(- \frac{y^2}{N_0 T_b A^2} \right) dy \end{aligned} \tag{10}$$

Define

$$z = \frac{y}{\sqrt{N_0 T_b}\ A} \tag{11}$$

We may then rewrite Eq. (10) in terms of the new variable z as

$$p_{10} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{A^2T_b/2N_0}}^{\infty} \exp(-z^2)\, dz \tag{12}$$

which may be reformulated in terms of

complementary error function

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} \exp(-z^2)\, dz \tag{13}$$

Accordingly, we may redefine the conditional probability of error P_{e0} as

$$p_{10} = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{A^2 T_b}{4N_0}}\right) \tag{14}$$

Consider next the second kind of error that occurs when symbol 1 is sent and the receiver chooses symbol 0. Under this condition, corresponding to hypothesis H_1, the correlator input consists of a rectangular pulse of amplitude A and duration T_b plus the channel noise w(t). We may thus apply Eq. (2) to write

$$H_1 : y = A \int_0^{T_b} [A + w(t)]\, dt \tag{15}$$

The fixed quantity A in the integrand of Eq. (15) serves to shift the correlator output from a mean value of zero volt under hypothesis H_0 to a mean value of A^2T_b under hypothesis H_1. However, the conditional variance of the correlator output under hypothesis H_1 has the same value as that under hypothesis H_0. Moreover, the correlator output is Gaussian distributed as before. In summary, the correlator output under hypothesis H_1 is a Gaussian random variable with mean A^2T_b and variance $N_0T_b^2/2$, as depicted in Fig. 2(b), which corresponds to those values of the correlator output less than the threshold λ set at $A^2T_b/2$, From the symmetric nature of the Gaussian density function, it is clear that

$$p_{01} = p_{10} \tag{16}$$

Note that this statement is only true when the a priori probabilities of binary symbols 0 and 1 are equal; this assumption was made in calculating the threshold λ.

To determine the average probability of error of the PCM receiver, we note that the two possible kinds of error just considered are mutually exclusive events. Thus, with the a priori probability of transmitting a 0 equal to p_0, and the a priori probability of transmitting a 1 equal to p_1, we find that the *average probability of error*, P_e, is given by

$$P_e = p_0 P_{10} + p_1 P_{01} \tag{17}$$

Since $P_{01} = P_{10}$ and $p_0 + p_1 = 1$, Eq. (17) simplifies as

$$P_e = P_{10} = P_{01}$$

or

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2 T_b}{N_0}}\right) \tag{18}$$

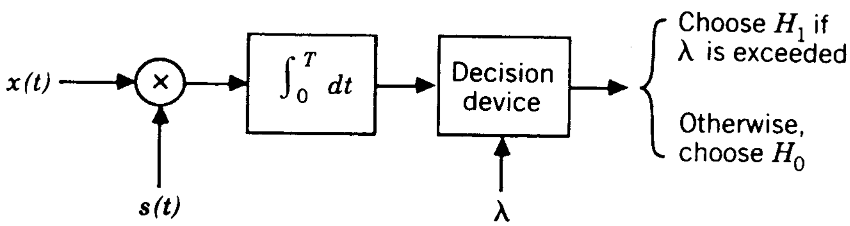


Figure 1

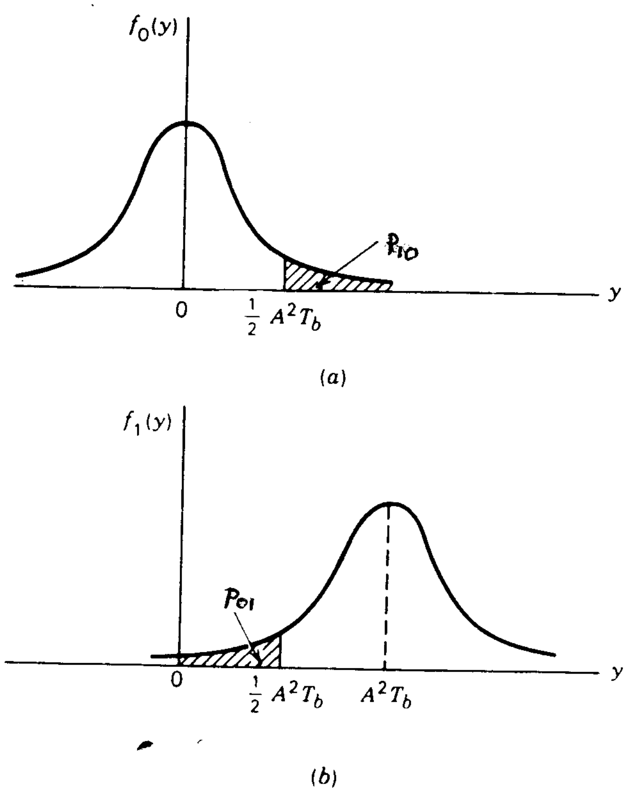


Figure 2

Problem 4.10

For unipolar RZ signaling, we have

Binary symbol 1: $s(t) = +A$ for $0 < t \le T/2$
and $s(t) = 0$ for $T/2 < t \le T$

Binary symbol 0: $s(t) = 0$ for $0 < t \le T$

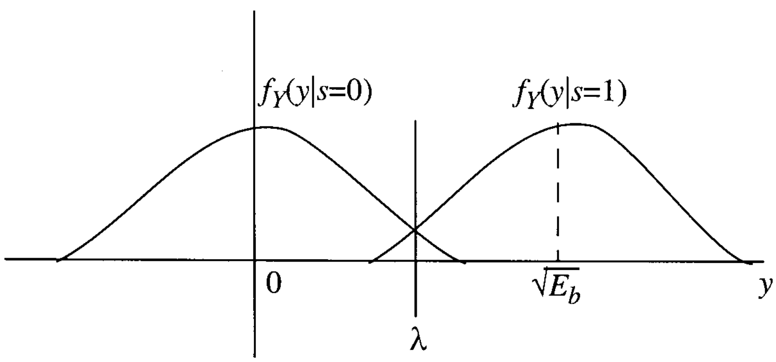
The a priori probabilities of symbols 1 and 0 are assumed to be equal, in which case we have $p_0 = p_1 = 1/2$.

To determine the average probability of error, we consider the two possible kinds of error separately. We begin by considering the first kind of error that occurs when symbol 0 is sent and the receiver chooses symbol 1. In this case, the probability of error is just the probability that the matched filter output will exceed the threshold λ owing to the presence of noise, so the transmitted symbol 0 is mistaken for symbol 1.

$$\text{Energy of symbol 1} = \frac{A^2 T_b}{2} = E_b$$

Energy of symbol 0 = 0

The conditional probability density function of the two signals is given below:



With symbols 1 and 0 assumed to be equiprobable, the optimum threshold is

$$\lambda = \frac{1}{2}\sqrt{E_b} = \frac{1}{2}\sqrt{\frac{A^2 T_b}{2}}$$

Given that symbol 0 was transmitted, the probability of error is simply the probability that $y > \lambda$, as shown by

$$P(\text{error}|0) = \int_{-\infty}^{\infty} f_Y(y|0)dy$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{N_0}\right) dy$$

Define a new variable z as

$$z = \frac{y}{\sqrt{N_0}}$$

We then have

$$P(\text{error}|0) = \frac{1}{\sqrt{\pi}} \int_{\lambda/\sqrt{N_0}}^{\infty} \exp(-z^2)dz$$

$$= \frac{1}{2}\text{erfc}\left(\frac{\lambda}{\sqrt{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2 T_b}{2N_0}}\right)$$

Similarly, $$P(\text{error}|1) = \int_{-\infty}^{\lambda} f_Y(y|(1)dy$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\lambda} \exp\left[-\frac{(y-\sqrt{E_b})^2}{N_0}\right] dy$$

Define $z = \dfrac{\sqrt{E_b} - y}{N_0}$, and so write

$$P(\text{error}|1) = \frac{1}{\sqrt{\pi}} \int_{\frac{\sqrt{E_b}-\lambda}{\sqrt{N_0}}}^{\infty} \exp(-z^2)dz$$

$$P(\text{error}|1) = \frac{1}{2}\text{erfc}\left(\frac{\sqrt{E_b}-\lambda}{\sqrt{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{\sqrt{E_b}}{2\sqrt{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2T_b}{2N_0}}\right)$$

The average probability of error is therefore

$$P_e = P(1)P(\text{error}|1) + P(\text{error}|0)P(0)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{E_b/N_0}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2T_b}{2N_0}}\right) \tag{1}$$

The average probability of error for on-off (i.e., unipolar NRZ) type of encoded signals is

$$\frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2T_b}{N_0}}\right)$$

Comparing this result with that of Eq. (1) for the unipolar RZ type of encoded signals, we immediately see that, for a prescribed noise spectral density N_0, the symbol energy in unipolar RZ signaling has to be doubled in order to achieve the same average probability of error as in unipolar NRZ signaling.

Problem 4.11

Probability of error for bipolar NRZ signal

Binary symbol 1 : $s(t) = \pm A$
Binary symbol 0: $s(t) = 0$
Energy of symbol 1 = $E_b = A^2T_b$

The absolute value of the threshold is $\lambda = \frac{1}{2}\sqrt{E_b} = \frac{1}{2}\sqrt{A^2 T_b}$.

Refering to Fig. 1 on the next page, we may write

$$P(\text{error}|\text{s=-A}) = \frac{1}{\sqrt{\pi N_0}}\int_{-\lambda}^{\lambda} \exp\left[\frac{-(y+\sqrt{E_b})^2}{N_0}\right] dy$$

Let $z = \dfrac{y+\sqrt{E_b}}{\sqrt{N_0}}$

Then,

$$P(\text{error}|s=-A) = \frac{1}{\sqrt{\pi}}\int_{-\frac{\lambda+\sqrt{E_b}}{\sqrt{N_0}}}^{\frac{\lambda+\sqrt{E_b}}{\sqrt{N_0}}} \exp(-z^2)dz$$

$$= \frac{1}{2}\left[\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right) - \operatorname{erfc}\left(\frac{3}{4}\sqrt{\frac{E_b}{N_0}}\right)\right]$$

Similarly, $P(\text{error}|s=+A) = P(\text{error}|s=-A)$

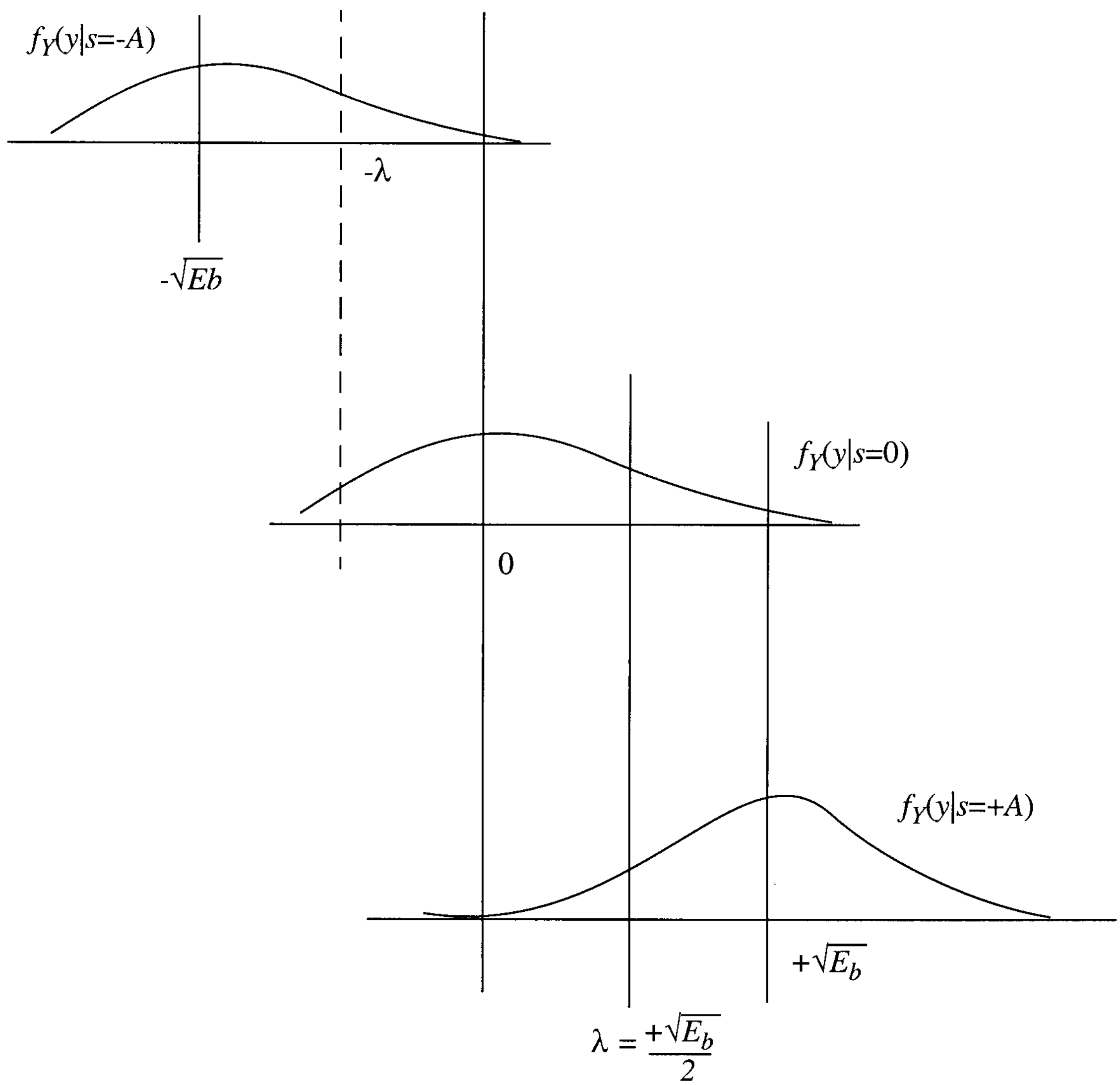
$$P(\text{error}|s=0) = \frac{2\times 1}{\sqrt{\pi N_0}}\int_{\lambda}^{\infty} \exp\left(\frac{-y^2}{N_0}\right) dy$$

$$= \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

The average probability of error is therefore

$P_e = P(s{=}\pm A)P(\text{error}|s{=}\pm A) + \quad P(s{=}0)P(\text{error}|s=0)$

The conditional probability density functions of symbols 1 and 0 are given in Fig. 1:



$f_Y(y|s=-A)$
$-\lambda$
$-\sqrt{Eb}$
$f_Y(y|s=0)$
0
$f_Y(y|s=+A)$
$+\sqrt{E_b}$
$\lambda = \frac{+\sqrt{E_b}}{2}$

Figure 1

$$P_e = \frac{1}{2}\times\frac{1}{2}\left[\text{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right) - \text{erfc}\left(\frac{3}{4}\sqrt{\frac{E_b}{N_0}}\right)\right] + \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

$$= \frac{3}{4}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right) - \frac{1}{4}\text{erfc}\left(\frac{3}{4}\sqrt{\frac{E_b}{N_0}}\right)$$

Problem 4.12

The rectangular pulse given in Fig. P4.12 is defined by

$g(t) = \text{rec}(t/T)$

The Fourier transform of $g(t)$ is given by

$$G(f) = \int_{-T/2}^{T/2} \exp(-j2\pi ft)dt$$

$$= T\,\text{sinc}\,(fT)$$

We thus have the Fourier-transform pair

$$\text{rec}(t/T) \rightleftharpoons T\,\text{sinc}\,(fT)$$

The magnitude spectrum $|G(f)|/T$ is plotted as the solid line in Fig. 1, shown on the next page.

Consider next a Nyquist pulse (raised cosine pulse with a rolloff factor of zero). The magnitude spectrum of this second pulse is a rectangular function of frequency, as shown by the dashed curve in Fig. 1.

Comparing the two spectral characteristics of Fig. 1, we may say that the rectangular pulse of Fig. P4.12 provides a crude approximation to the Nyquist pulse.

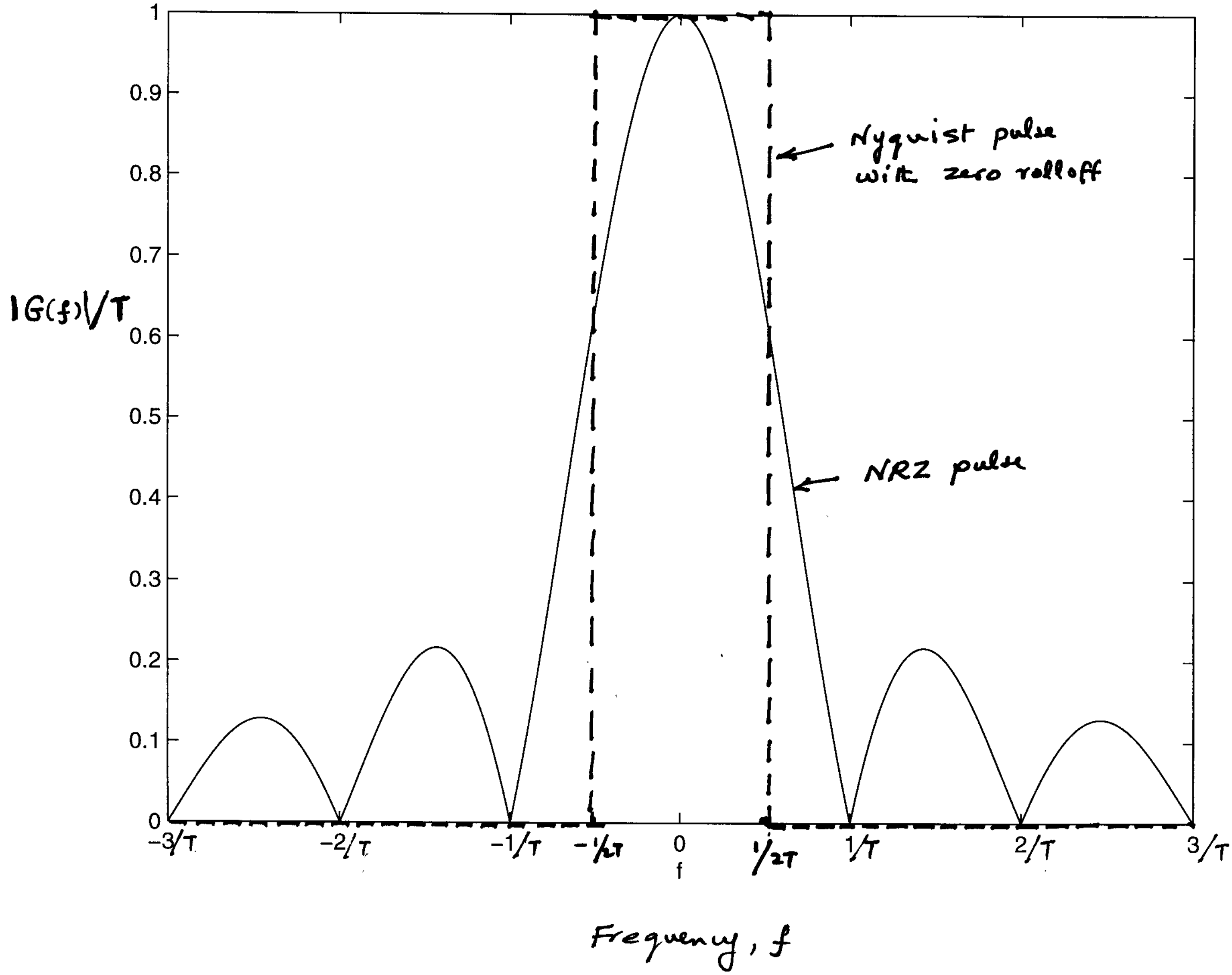


Figure 1 Spectral characteristics

Problem 4.13

Since P(f) is an even real function, its inverse Fourier transform equals

$$p(t) = 2\int_0^{\infty} P(f)\cos(2\pi ft)\,df \tag{1}$$

The P(f) is itself defined by Eq. (7.60) which is reproduced here in the form

$$P(f) = \begin{cases} \dfrac{1}{2W}, & 0 < |f| < f_1 \\ \dfrac{1}{4W}\left\{1 + \cos\left[\dfrac{\pi(|f| - f_1)}{2W - 2f_1}\right]\right\}, & f_1 < f < 2W - f_1 \\ 0, & |f| > 2W - f_1 \end{cases} \tag{2}$$

Hence, using Eq. (2) in (1):

$$p(t) = \frac{1}{W}\int_0^{f_1} \cos(2\pi ft)\,df + \frac{1}{2B}\int_{f_1}^{2W-f_1}\left[1 + \cos\left(\frac{\pi(f-f_1)}{2W\alpha}\right)\right]\cos(2\pi ft)\,df$$

$$= \left[\frac{(\sin(2\pi ft)}{2\pi Wt}\right] + \left[\frac{\sin(2\pi ft)}{4\pi Wt}\right]_{f_1}^{2W-f_1}$$

$$+ \frac{1}{4}W\left[\frac{\sin\left(2\pi ft + \dfrac{\pi(f-f_1)}{2W\alpha}\right)}{2\pi t + \pi/2W\alpha}\right]_{f_1}^{2W-f_1} + \frac{1}{4W}\left[\frac{\sin\left(2\pi ft - \dfrac{\pi(f-f_1)}{2W\alpha}\right)}{2\pi t - \pi/2W\alpha}\right]_{f_1}^{2W-f_1}$$

$$= \frac{\sin(2\pi f_1 t)}{4\pi Wt} + \frac{\sin[2\pi t(2W-f_1)]}{4\pi Wt}$$

$$- \frac{1}{4W}\frac{\sin(2\pi f_1 t) + \sin[2\pi t(2W-f_1)]}{2\pi t - \pi/2W\alpha} + \frac{\sin(2\pi f_1 t) + \sin[2\pi t(2W-f_1 t)]}{2\pi t - \pi/2W\alpha}$$

$$= \frac{1}{W}\left[\sin(2\pi f_1 t) + \sin[2\pi t(2W-f_1)]\right]\left[\frac{1}{4\pi t} - \frac{\pi t}{(2\pi t)^2 - (\pi/2W\alpha)^2}\right]$$

$$= \frac{1}{W}[\sin(2\pi Wt)\cos(2\pi\alpha W)]\left[\frac{-(\pi/2W\alpha)^2}{4\pi t\,[(2\pi t)^2 - (\pi/2W\alpha)^2}\right]$$

$$= \text{sinc}(2Wt)\cos(2\pi\alpha Wt)\left[\frac{1}{1 - 16\,\alpha^2 W^2 t^2}\right]$$

Problem 4.14

The minimum bandwidth, B_T, is equal to 1/2T, where T is the pulse duration. For 64 quantization levels, $\log_2 64 = 6$ bits are required.

Problem 4.15

The effect of a linear phase response in the channel is simply to introduce a constant delay τ into the pulse p(t). The delay τ is defined as $-1/(2\pi)$times the slope of the phase response; see Eq. 2.171.

H1

Problem 4.16

The Bandwidth B of a raised cosine pulse spectrum is $2W - f_1$, where $W = 1/2T_b$ and $f_1 = W(1-\alpha)$. Thus $B = W(1+\alpha)$. For a data rate of 56 kilobits per second, W = 28 kHz.

(a) For α = 0.25,

B = 28 kHz x 1.25

= 35 kHz

(b) B = 28 kHz x 1.5

= 42 kHz

(c) B = 49 kHz

(d) B = 56 kHz

Problem 4.17

The use of eight amplitude levels ensures that 3 bits can be transmitted per pulse. The symbol period can be increased by a factor of 3. All four bandwidths in problem 7.12 will be reduced to 1/3 of their binary PAM values.

Problem 4.18

(a) For a unity rolloff, raised cosine pulse spectrum, the bandwidth B equals 1/T, where T is the pulse length. Therefore, T in this case is 1/12kHz. Quarternary PAM ensures 2 bits per pulse, so the rate of information is

$$\frac{2 \text{ bits}}{T} = 24 \text{ kilobits per second.}$$

(b) For 128 quantizing levels, 7 bits are required to transmit an amplitude. The additional bit for synchronization makes each code word 8 bits. The signal is transmitted at 24 kilobits/s, so it must be sampled at

$$\frac{24 \text{ kbits/s}}{8 \text{ bits/sample}} = 3 \text{ kHz.}$$

The maximum possible value for the signal's highest frequency component is 1.5 kHz, in order to avoid aliasing.

Problem 4.19

The raised cosine pulse bandwidth $B = 2W - f_1$, where $W = 1/2T_b$. For this channel, $B = 75$ kHz. For the given bit duration, $W = 50$ kHz. Then,

$$f_1 = 2W - B$$
$$= 25 \text{ kHz}$$
$$\alpha = 1 - f_1/B_T$$
$$= 0.5$$

Problem 4.20

The duobinary technique has correlated digits, while the other two methods have independent digits.

Problem 4.21

(a)

binary sequence b_k		0	0	1	1	0	1	0	0	1
polar representation		-1	-1	1	1	-1	1	-1	-1	1
duobinary coder output c_k			-2	0	2	0	0	0	-2	0
receiver output $\hat{b}_k$		-1	-1	1	1	-1	1	-1	-1	1
output binary sequence		0	0	1	1	0	1	0	0	1

(b)

receiver input			0	0	2	0	0	0	-2	0
receiver output $\hat{b}_k$		-1	1	-1	1	-1	1	-1	-1	1
output binary sequence		0	1	0	1	0	1	0	0	1

We see that not only is the second digit in error, but also the error propagates.

Problem 4.22

(a)

binary sequence b_k		0	0	1	1	0	1	0	0	1
coded sequence d_k	1	1	1	0	1	1	0	0	0	1
polar representation	1	1	1	-1	1	1	-1	-1	-1	1
duobinary coder output c_k		2	2	0	0	2	0	-2	-2	0
receiver output		0	0	1	1	0	1	0	0	1

(b)

receiver input		2	0	0	0	2	0	-2	-2	0
receiver output		0	1	1	1	0	1	0	0	1

In this case we see that only the second digit is in error, and there is no error propagation.

Problem 4.23

(a) The correlative coder has output

$$z_n = y_n - y_{n-1}$$

Its impulse response is

$$h_k = \begin{cases} 1 & k = 0 \\ -1 & k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The frequency response is

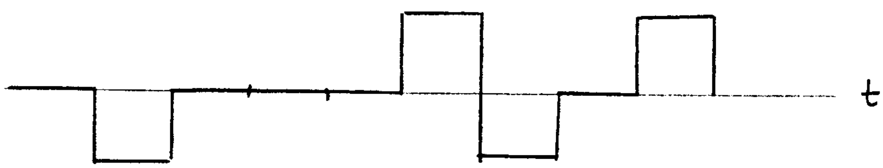
$$H(f) = \sum_{k=-\infty}^{\infty} h_k \exp(-j2\pi fkT_b)$$

$$= 1 - \exp(-j2\pi fT_b)$$

(b) Let the input to the differential encoder be x_n, the input to the correlative coder be y_n, and the output of the correlative coder be z_n. Then, for the sequence 010001101 in its on-off form, we have

x_n		0	1	0	0	0	1	1	0	1
y_n	1	1	0	0	0	0	1	0	0	1
z_n		0	-1	0	0	0	1	-1	0	1

Then z_n has the following waveform



The sequence z_n is a bipolar representation of the input sequence x_n.

Problem 4.24

(a) The output symbols of the modulo-2 adder are independent because:

1. the input sequence to the adder has independent symbols, and therefore

2. knowing the previous value of the adder does not improve prediction of the present value, i.e.

$$f(y_n | y_{n-1}) = f(y_n) ,$$

where y_n is the value of the adder output at time nT_b. The adder output sequence is another on-off binary wave with independent symbols. Such a wave has the power spectral density (from problem 4.10),

$$S_Y(f) = \frac{A^2}{4} \delta(f) + \frac{A^2 T_b}{4} \text{sinc}^2(fT_b) .$$

The correlative coder has the transfer function

$$H(f) = 1 - \exp(-j2\pi fT_b),$$

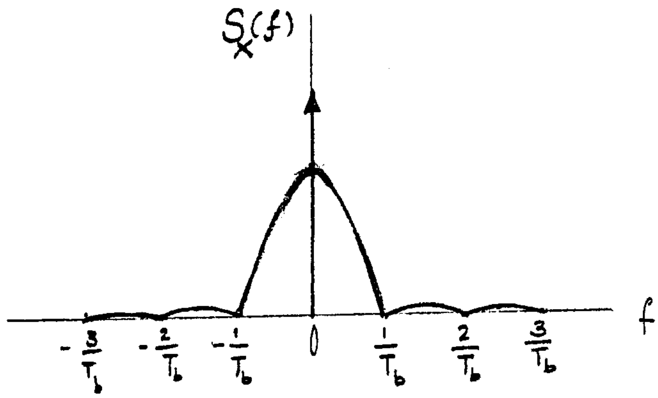
Hence, the output wave has the power spectral density

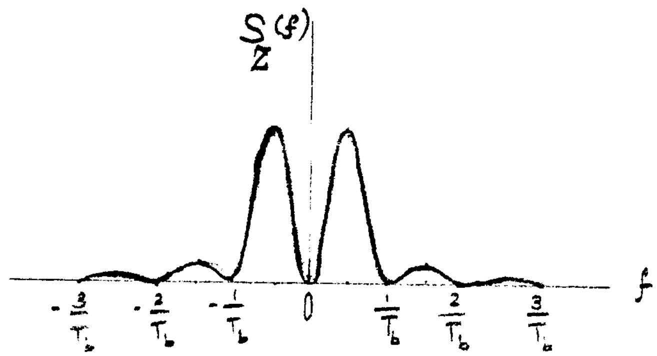
$$\begin{aligned} S_Z(f) &= |H(f)|^2 S_Y(f) \\ &= [1 - \exp(-j2\pi fT_b)]\,[1 - \exp(j2\pi fT_b)]\, S_Y(f) \\ &= [2 - 2\cos(2\pi fT_b)]\, S_Y(f) \\ &= 4\sin^2(\pi fT_b)\, S_Y(f) \\ &= 4\sin^2(\pi fT_b)\,[\frac{A^2}{4}\delta(f) + \frac{A^2T_b}{4}\operatorname{sinc}^2(fT_b)] \\ &= A^2T_b\sin^2(\pi fT_b)\operatorname{sinc}^2(fT_b) \end{aligned}$$

In the last line we have used the fact that

$$\sin(\pi fT_b) = 0 \text{ at } f = 0.$$

(b)





Note that the bipolar wave has no dc component.

(Note: The power spectral density of a bipolar signal derived in part (a) assumes the use of a pulse of full duration T_b. On the other hand, the result derived for a bipolar signal in part (d) of Problem 3.11 assumes the use of a pulse of half symbol duration T_b.)

Problem 4.25

The modified duobinary receiver estimate is $\hat{a}_k = c_k + \hat{a}_{k-2}$.

(a)	binary sequence a_k	0	1	1	1	0	0	1	0	1
	bipolar representation	−1	1	1	1	−1	−1	1	−1	1
	modified duobinary c_k			2	0	−2	−2	2	0	0
	receiver output $\hat{a}_k$	−1	1	1	1	−1	−1	1	−1	1
	output binary sequence	0	1	1	1	0	0	1	0	1
(b)	receiver input			0	0	−2	−2	2	0	0
	receiver output $\hat{a}_k$	−1	1	−1	1	−3	−1	−1	−1	−1
	output binary sequence	0	1	0	1	0	0	0	0	0

Here we see that not only is the third digit in error, but also the error propagates.

Problem 4.26

(a)	binary sequence b_k			0	1	1	1	0	0	1	0	1
	coded sequence a_k	0	0	0	1	1	0	1	0	0	0	1
	polar representation	−1	−1	−1	1	1	−1	1	−1	−1	−1	1
	modified duobinary c_k			0	2	2	−2	0	0	−2	0	2
	receiver output $\hat{b}_k = \lvert c_k \rvert$			0	2	2	2	0	0	2	0	2
	output binary sequence			0	1	1	1	0	0	1	0	1
(b)	receiver input			0	2	0	−2	0	0	−2	0	2
	receiver output			0	2	0	2	0	0	2	0	2
	output binary sequence			0	1	0	1	0	0	1	0	1

This time we find that only the third digit is in error, and there is no error propagation.

Problem 4.27

(a) Polar Signalling (M=2)

In this case, we have

$$m(t) = \sum_n A_n \operatorname{sinc}\left(\frac{t}{T} - n\right)$$

where $A_n = \pm A/2$. Digits 0 and 1 are thus represented by $-A/2$ and $+A/2$, respectively. The Fourier transform of m(t) is

$$M(f) = \sum_n A_n F\left[\operatorname{sinc}\left(\frac{t}{T} - n\right)\right]$$

$$= T \operatorname{rect}(fT) \sum_n A_n \exp(-j2\pi nfT)$$

Therefore, m(t) is passed through the ideal low-pass filter with no distortion.

The noise appearing at the low-pass filter output has a variance given by

$$\sigma^2 = \frac{N_0}{2T}$$

Suppose we transmit digit 1. Then, at the sampling instant, we obtain a random variable at the input of the decision device, defined by

$$X = \frac{A}{2} + N$$

where N denotes the contribution due to noise. The decision level is 0 volts. If $X > 0$, the decision device chooses symbol 1, which is a correct decision. If $X < 0$, it chooses symbol 0, which is in error. The probability of making an error is

$$P(X<0) = \int_{-\infty}^{0} f_X(x)\, dx$$

The expected value of X is A/2, and its variance is σ^2. Hence,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-\frac{(x - \frac{A}{2})^2}{2\sigma^2}]$$

$$P(X<0) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{0} \exp(-\frac{(x - \frac{A}{2})^2}{2\sigma^2})\, dx$$

$$= \frac{1}{2} \operatorname{erfc}(\frac{A}{2\sqrt{2}\sigma})$$

Similarly, if we transmit symbol 0, an error is made when X > 0, and the probability of this error is

$$P(X>0) = \frac{1}{2} \operatorname{erfc}(\frac{A}{2\sqrt{2}\sigma})$$

Since the symbols 1 and 0 are equally probable, we find that the average probability of error is

$$P_e = \frac{1}{2} P(X<0 \mid \text{transmit } 1) + \frac{1}{2} P(X>0 \mid \text{transmit } 0)$$

$$= \frac{1}{2} \operatorname{erfc}(\frac{A}{2\sqrt{2}\sigma})$$

(b) Polar ternary signaling

In this case we have

$$m(t) = \sum_n A_n \operatorname{sinc}(\frac{t}{T} - n)$$

where

$$A_n = 0, \pm A.$$

The 3 digits are defined as follows

Digit	Level
0	-A
1	0
2	+A

Suppose we transmit digit 2, which, at the input of the decision device, yields the random variable

$$X = A + N$$

The probability density function of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp(-\frac{(x-A)^2}{2\sigma^2})$$

The decision levels are set at -A/2 and A/2 volts. Hence, the probability of choosing digit 1 is

$$P(-\frac{A}{2} < X < \frac{A}{2}) = \int_{-A/2}^{A/2} \frac{1}{\sqrt{2\pi}\,\sigma} \exp[-\frac{(x-A)^2}{2\sigma^2}]\, dx$$

$$= \frac{1}{2}[\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma}) - \mathrm{erfc}(\frac{3A}{2\sqrt{2}\,\sigma})]$$

Next, the probability of choosing digit 0 is

$$P(X < -\frac{A}{2}) = \frac{1}{2}\,\mathrm{erfc}(\frac{3A}{2\sqrt{2}\,\sigma})$$

If we transmit digit 1, the random variable at the input of the decision device is

$$X = N$$

The probability density function of X is therefore

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp(-\frac{x^2}{2\sigma^2})$$

The probability of choosing digit 2 is

$$P(X > \frac{A}{2}) = \frac{1}{2}\,\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma})$$

The probability of choosing digit 0 is

$$P(X < -\frac{A}{2}) = \frac{1}{2}\,\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma})$$

Next, suppose we transmit digit 0. Then, the random variable at the input of the decision device is

$$X = -A + N$$

The probability density function of X is therefore

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp[-\frac{(x+A)^2}{2\sigma^2}]$$

The probability of choosing digit 1 is

$$P(-\frac{A}{2} < X < \frac{A}{2}) = \frac{1}{2}[\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma}) - \mathrm{erfc}(\frac{3A}{2\sqrt{2}\,\sigma})]$$

The probability of choosing digit 2 is

$$P(X > \frac{A}{2}) = \frac{1}{2}\mathrm{erfc}(\frac{3A}{2\sqrt{2}\sigma})$$

Assuming that digits 0, 1, and 2 are equally probable, the average probability of error is

$$\begin{aligned} P_e &= \frac{1}{3}[\frac{1}{2}\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma}) - \frac{1}{2}\mathrm{erfc}(\frac{3A}{2\sqrt{2}\,\sigma})] + \frac{1}{3}\cdot\frac{1}{2}[\mathrm{erfc}(\frac{3A}{2\sqrt{2}\,\sigma})] \\ &+ \frac{1}{3}\cdot\frac{1}{2}[\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma})] + \frac{1}{3}\cdot\frac{1}{2}[\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma})] \\ &+ \frac{1}{3}\cdot\frac{1}{2}[\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma}) - \mathrm{erfc}(\frac{3A}{2\sqrt{2}\,\sigma})] + \frac{1}{3}\cdot\frac{1}{2}\mathrm{erfc}(\frac{3A}{2\sqrt{2}\,\sigma}) \\ &= \frac{2}{3}\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma}) \end{aligned}$$

(c) Polar quaternary signaling

In this case, we have

$$A_n = \pm\frac{A}{2}, \pm\frac{3A}{2}$$

and the 4 digits are represented as follows

Digit	Level
0	$-\frac{3A}{2}$
1	$-\frac{A}{2}$
2	$+\frac{A}{2}$
3	$+\frac{3A}{2}$

Suppose we transmit digit 3, which, at the input of the decision device, yields the random variable:

$$X = \frac{3A}{2} + N.$$

The decision levels are 0, $\pm$ A. The probability of choosing digit 2 is

$$P(0 < X < A) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^A \exp[-\frac{(x - \frac{3A}{2})^2}{2\sigma^2}]\, dx$$

$$= \frac{1}{2}\,[\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma}) - \mathrm{erfc}(\frac{3A}{2\sqrt{2}\,\sigma})]$$

The probability of choosing digit 1 is

$$P(-A < X < 0) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-A}^0 \exp[-\frac{(x - \frac{3A}{2})^2}{2\sigma^2}]\, dx$$

$$= \frac{1}{2}\,[\mathrm{erfc}(\frac{3A}{2\sqrt{2}\,\sigma}) - \mathrm{erfc}(\frac{5A}{2\sqrt{2}\,\sigma})]$$

The probability of choosing digit 0 is

$$P(X < -A) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{-A} \exp[-\frac{(x - \frac{3A}{2})^2}{2\sigma^2}]\, dx$$

$$= \frac{1}{2}\,\mathrm{erfc}(\frac{5A}{2\sqrt{2}\,\sigma}).$$

Suppose next we transmit digit 2, obtaining

$$X = \frac{A}{2} + N.$$

The probability of choosing digit 3 is

$$P(X > A) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_A^\infty \exp[-\frac{(x - \frac{A}{2})^2}{2\sigma^2}]\, dx$$

$$= \frac{1}{2}\,\mathrm{erfc}(\frac{A}{2\sqrt{2}\,\sigma}).$$

The probability of choosing digit 1 is

$$P(-A < X < 0) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-A}^{0} \exp\left(-\frac{(x - \frac{A}{2})^2}{2\sigma^2}\right) dx$$

$$= \frac{1}{2}\left[\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\,\sigma}\right) - \operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right)\right]$$

The probability of choosing digit 0 is

$$P(X < -A) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{A} \exp\left[-\frac{(x - \frac{A}{2})^2}{2\sigma^2}\right] dx$$

$$= \frac{1}{2}\operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right).$$

Suppose next we transmit digit 1, obtaining

$$X = -\frac{A}{2} + N$$

The probability of choosing digit 0 is

$$P(X < -A) = \frac{1}{2}\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\,\sigma}\right)$$

The probability of choosing digit 2 is

$$P(0 < X < A) = \frac{1}{2}\left[\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\,\sigma}\right) - \operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right)\right]$$

The probability of choosing digit 3 is

$$P(X > A) = \frac{1}{2}\operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right).$$

Finally, suppose we transmit digit 0, obtaining

$$X = -\frac{3A}{2} + N$$

The probability of choosing digit 1 is

$$P(-A < X < 0) = \frac{1}{2}\left[\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\,\sigma}\right) - \mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right)\right]$$

The probability of choosing digit 2 is

$$P(0 < X < A) = \frac{1}{2}\left[\mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right) - \mathrm{erfc}\left(\frac{5A}{2\sqrt{2}\,\sigma}\right)\right]$$

The probability of choosing digit 3 is

$$P(X > A) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{5A}{2\sqrt{2}\,\sigma}\right)$$

Since all 4 digits are equally probable, with a probability of occurence equal to 1/4, we find that the average probability of error is

$$\begin{aligned} P_e = \frac{1}{4} \cdot 2 \cdot \frac{1}{2}\Big\{&\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\,\sigma}\right) - \mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right) \\ &+ \mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right) - \mathrm{erfc}\left(\frac{5A}{2\sqrt{2}\,\sigma}\right) \\ &+ \mathrm{erfc}\left(\frac{5A}{2\sqrt{2}\,\sigma}\right) \\ &+ \mathrm{erfc}\left(\frac{A}{2\sqrt{2}\,\sigma}\right) \\ &+ \mathrm{erfc}\left(\frac{A}{2\sqrt{2}\,\sigma}\right) - \mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right) \\ &+ \mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\,\sigma}\right)\Big\} \\ = \frac{3}{4}\,&\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\,\sigma}\right) . \end{aligned}$$

Problem 4.28

The average probability of error is (from the solution to Problem 7.23)

$$P_e = \left(1 - \frac{1}{M}\right)\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\,\sigma}\right) \tag{1}$$

The received signal-to-noise ratio is

$$(SNR)_R = \frac{A^2(M^2 - 1)}{12\ \sigma^2}$$

That is

$$\frac{A}{\sigma} = \sqrt{\frac{12(SNR)_R}{M^2 - 1}} \tag{2}$$

Substituting Eq. (2) in (1), we get

$$P_e = (1 - \frac{1}{M})\ \mathrm{erfc}(\sqrt{\frac{3(SNR)_R}{2(M^2-1)}})$$

With $P_e = 10^{-6}$, we may thus write

$$10^{-6} = (1 - \frac{1}{M})\ \mathrm{erfc}(u) \tag{3}$$

where

$$u^2 = \frac{3(SNR)_R}{2(M^2-1)}$$

For a specified value of M, we may solve Eq. (3) for the corresponding value of u. We may thus construct the following table:

M	u
2	3.37
4	3.42
8	3.45
16	3.46

We thus find that to a first degree of approximation, the minimum value of received signal-to-noise ratio required for $P_e < 10^{-6}$ is given by

$$\frac{3(SNR)_{R,min}}{2(M^2-1)} \simeq (3.42)^2$$

That is, $(SNR)_{R,\min} \approx 7.8\ \ (M^2 - 1)$

Problem 4.29

Typically, a cable contains many twisted pairs. Therefore, the received signal can be written as

$$r(n) = \sum_{i=1}^{N} v_i(n) + d(n), \qquad \text{large } N$$

where $d(n)$ is the desired signal and $\sum_{i=1}^{N} v_i(n)$ is due to cross-talk. Typically, the v_i are statistically independent and identically distributed. Hence, by using the central limit theorem, as N becomes infinitely large, the term $\sum_{i=1}^{N} v_i(n)$ is closely approximated by a Gaussian random variable for each time instant n.

Problem 4.30

(a) The power spectral density of the signal generated by the NRZ transmitter is given by

$$S(f) = \frac{\sigma^2}{T}|G(f)|^2 \tag{1}$$

where σ^2 is the symbol variance, T is the symbol duration, and

$$G(f) = \int_{-T/2}^{T/2} 1 \cdot e^{-j2\pi ft} dt = T\,\mathrm{sinc}\,(fT) = \frac{1}{R}\mathrm{sinc}\left(\frac{f}{R}\right) \tag{2}$$

is the Fourier transform of the generating function for NRZ symbols. Here, we have used the fact that the symbol rate $R = 1/T$. A 2B1Q code is a multi-level block code where each block has 2 bits and the bit rate $R = 2/T$ (i.e., m/T, where m is the number of bits in a block). Since the 2B1Q pulse has the shape of an NRZ pulse, the power spectral density of 2B1Q signals is given by

$$S_{2\mathrm{B1Q}} = \frac{\sigma^2}{T}|G_{2\mathrm{B1Q}}(f)|^2$$

where

$$G_{2\mathrm{B1Q}}(f) = \frac{\sin(2\pi(f/R))}{\sqrt{2}\pi f}$$

The factor $\sqrt{2}$ in the denominator is introduced to make the average power of the 2B1Q signal equal to the average power of the corresponding NRZ signal. Hence,

$$S_{2B1Q}(f) = \frac{\sigma^2}{T}\left(\frac{\sin(2\pi(f/R))}{\sqrt{2}\pi f}\right)^2$$

$$= \frac{2\sigma^2}{R}\,\mathrm{sinc}^2(2(f/R)) \tag{3}$$

(b) The transfer functions of pulse-shaping filters for the Manchester code, modified duobinary code, and bipolar return-to-zero code are as follows:

(i) Manchester code:

$$G(f) = \frac{j}{\pi f}\left[1 - \cos\left(\pi\frac{f}{R}\right)\right] \tag{4}$$

(ii)Modified duobinary code:

$$G(f) = \frac{1}{j\sqrt{2}\pi f}\left[\cos\left(3\pi\frac{f}{R}\right) - \cos\left(\pi\frac{f}{R}\right)\right] \tag{5}$$

(iii)Bipolar return-to-zero code:

$$G(f) = \frac{2}{\pi f}\left[\sin\left(\pi\frac{f}{2R}\right) \times \sin\left(\pi\frac{f}{R}\right)\right] \tag{6}$$

Hence, using Eqs. (4), (5), and (6) in the formula of Eq. (1) for the power spectral density of PAM line codes, we get the normalized spectral plots shown in Fig. 1. In this figure, the spectral density is normalized with respect to the symbol variance σ^2 and the frequency is normalized with respect to the data rate R.

From Fig. 1, we may make the following observations: Among the four line codes displayed here, the 2B1Q code has much of its power concentrated inside the frequency band $-R/2 \leq f \leq R/2$, which is much more compact than all the other three codes: Manchester code, modified duobinary code, and bipolar return-to-zero code.

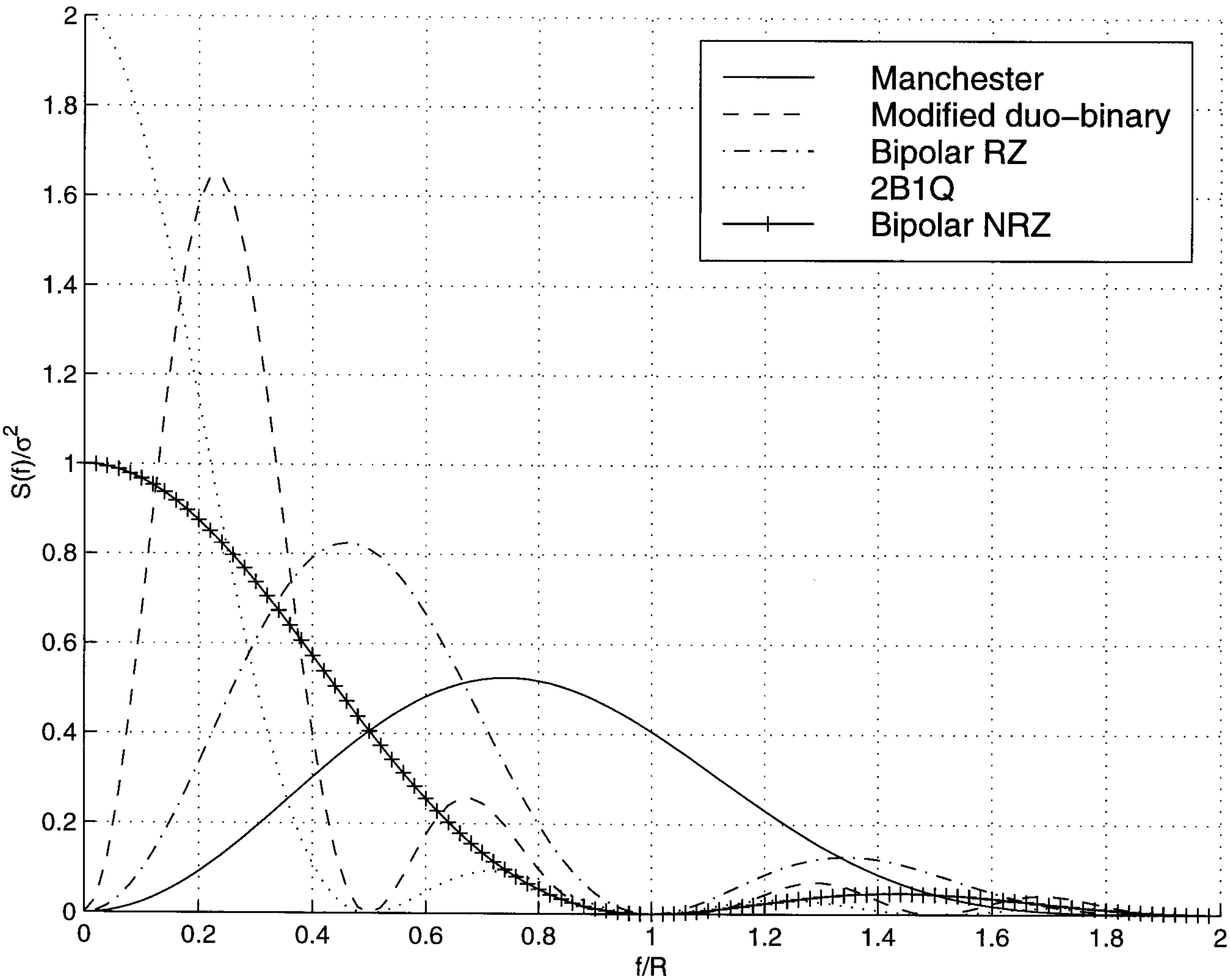
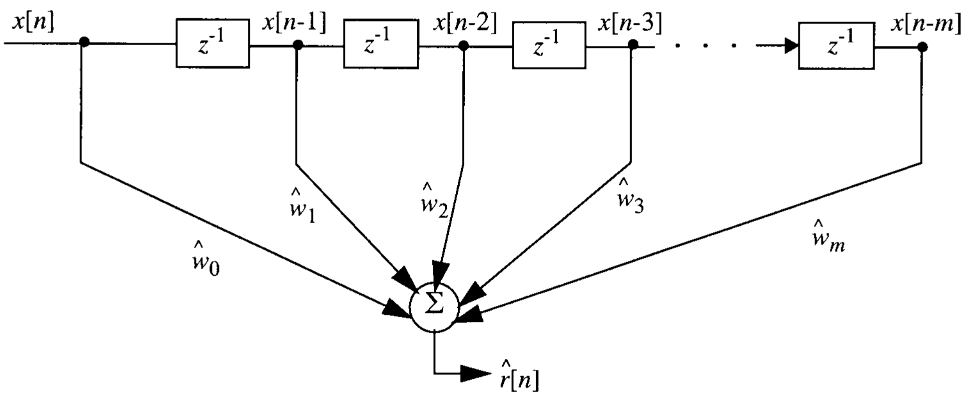

Manchester
Modified duo-binary
Bipolar RZ
2B1Q
Bipolar NRZ
S(f)/σ²
f/R
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Figure 1

Problem 4.31

The tapped-delay-line section of the adaptive filter is shown below:



$\hat{r}[n] = \mathbf{x}^T[n]\hat{\mathbf{w}}[n]$

$d[n] = x[n] + r[n]$

Error signal $e[n] = d[n] - \hat{r}[n]$

$\hat{\mathbf{w}}[n+1] = \hat{\mathbf{w}}[n] + \mu\mathbf{x}[n](d[n] - \mathbf{x}^T[n]\hat{\mathbf{w}}[n])$

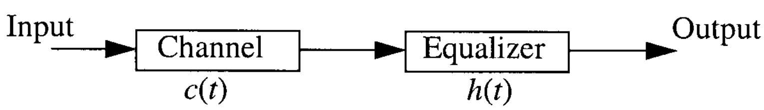
where $\hat{\mathbf{w}}[n] = [\hat{w}_0[n], \cdots, \hat{w}_m[n]^T]$

$\mathbf{x}[n] = [x[n], x[n-1], \cdots, x[n-m]^T]$

μ = learning parameter

Problem 4.32

(a)



The $h(t)$ is defined by

$$h(t) = \sum_{k=-N}^{N} w_k \delta(t - kT)$$

The impulse response of the cascaded system is given by the convolution sum

$$p_n = \sum_{j=-N}^{N} w_j c_{n-j}$$

where $p_n = p(nT)$. The kth sample of the output of the cascaded system due to the input sequence $\{I_n\}$ is defined by

$$\hat{I}_k = p_0 I_k + \sum_{n \neq k} I_n p_{k-n}$$

where $p_0 I_k$ is a scaled version of the desired symbol I_k. The summation term $\sum_{n \neq k} I_n p_{k-n}$ is the intersymbol interference.

The peak value of the interference is given by

$$D(N) = \sum_{\substack{n=-N \\ n \neq 0}}^{N} |p_n| = \sum_{\substack{n=-N \\ n \neq 0}}^{N} \left| \sum_{k=-N}^{N} w_k c_{n-k} \right|$$

To make the ISI equal to zero, we require

$$p_n = \sum_{k=-N}^{N} w_k c_{n-k} = \left\{ \begin{array}{ll} 1, & n = 0 \\ 0, & n \neq 0 \end{array} \right\}$$

(b) By taking the z-transform of the convolution sum

$$\sum_{k=-N}^{N} w_k c_{n-k}$$

and recalling that convolution in the discrete-time domain is transformed into multiplication in the z-domain, we get

$$P(z) = H(z)C(z)$$

For the zero-forcing condition, we require that $P(z) = 1$. Under this condition, we have

$$H(z) = 1/C(z)$$

which represents the transfer function of an inverse filter.

If the channel contains a spectral null at $f = 1/2T$ in its frequency response, the linear zero-forcing equalizer attempts to compensate for this null by introducing an infinite gain at frequency $f = 1/2T$. However, the channel distortion is compensated at the expense of enhancing additive noise: With $H(z) = 1/C(z)$, we find that when $C(z) = 0$,

$$H(z) = \infty$$

which results in noise enhancement.

Similarly, when the channel spectral response takes a smaller value, the equalizer will introduce a high gain at that frequency. Again, this tends to enhance the additive noise.

Problem 4.33

(a) Consider Eq. (4.108) of the textbook, which is rewritten as

$$\int_{-\infty}^{\infty}\left(R_q(t-\tau)+\frac{N_0}{2}\delta(t-\tau)\right)c(\tau)d\tau = q(-t)$$

Expanding the left-hand side:

$$\int_{-\infty}^{\infty}R_q(t-\tau)c(\tau)d\tau+\int_{-\infty}^{\infty}\frac{N_0}{2}\delta(t-\tau)c(\tau)d\tau = q(-t)$$

Applying the Fourier transform:

$$F\left\{\int_{-\infty}^{\infty}R_q(t-\tau)c(\tau)d\tau\right\} = F\{R_q(t-\tau)\} \times F\{(c(\tau))\}$$

$$= S_q(f)C(f)$$

$$F\left\{\int_{-\infty}^{\infty}\frac{N_0}{2}\delta(t-\tau)c(\tau)d\tau\right\} = \frac{N_0}{2}C(f)$$

$$F\{q(-t)\} = Q(-f) = Q^*(f)$$

In these three relations we have used the fact that convolution in the time domain corresponds to multiplication in the frequency domain.

Putting these results together, we get

$$S_q(f)C(f) + \frac{N_0}{2}C(f) = Q^*(f)$$

or

$$\left(S_q(f) + \frac{N_0}{2}\right)C(f) = Q^*(f)$$

which is the desired result.

(b) The autocorrelation function of the sequence is given by

$$R_q(\tau_1, \tau_2) = \sum_k q(kT_b - \tau_1)q(kT_b - \tau_2)$$

Using the fact that the autocorrelation function and power spectral density (PSD) form a Fourier transform pair, we may write

$$\begin{aligned}\text{PSD} &= F\{R_q(\tau_1, \tau_2)\} \\ &= F\left\{\sum_k q(kT_b - \tau_1)q(kT_b - \tau_2)\right\} \\ &= \frac{1}{T_b}\sum_k \left|Q\left(f + \frac{k}{T_b}\right)\right|^2\end{aligned}$$

where $F\{q(t)\} = Q(f)$

Problem 4.34

(a) The channel output is

$$x(t) = a_1 s(t-t_{01}) + a_2 s(t-t_{02})$$

Taking the Fourier transform of both sides:

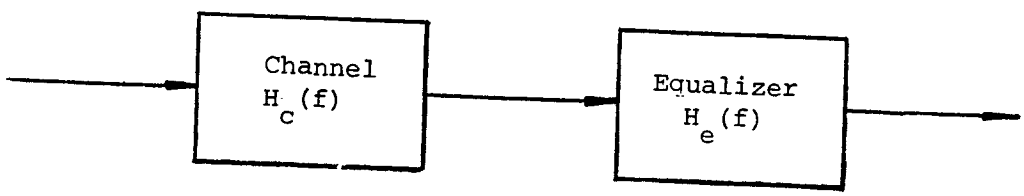
$$X(f) = [a_1 \exp(-j2\pi f t_{01}) + a_2 \exp(-j2\pi f t_{02})] S(f)$$

The transfer function of the channel is

$$H_c(f) = \frac{X(f)}{S(f)}$$

$$= a_1 \exp(-j2\pi f t_{01}) + a_2 \exp(-j2\pi f t_{02})$$

(b)



Ideally, the equalizer should be designed so that

$$H_c(f) H_e(f) = K_0 \exp(-j2\pi f t_0)$$

where K_0 is a constant gain and t_0 is the transmission delay. The transfer function of the equalizer is

$$H_e(f) = w_0 + w_1 \exp(-j2\pi fT) + w_2 \exp(-j4\pi fT)$$

$$= w_0 \left[1 + \frac{w_1}{w_0} \exp(-j2\pi fT) + \frac{w_2}{w_0} \exp(-j4\pi fT)\right] \qquad (1)$$

Therefore

$$H_e(f) = \frac{K_0 \exp(-j2\pi f t_0)}{H_c(f)}$$

$$= \frac{K_0 \exp(-j2\pi f t_0)}{a_1 \exp(-j2\pi f t_{01}) + a_2 \exp(-j2\pi f t_{02})}$$

$$= \frac{(K_0/a_1)\exp[-j2\pi f(t_0 - t_{01})]}{1 + \frac{a_2}{a_1}\exp[-j2\pi f(t_{02} - t_{01})]}$$

Since $a_2 \ll a_1$, we may approximate $H_e(f)$ as follows

$$H_e(f) = \frac{K_0}{a_1}\exp[-j2\pi f(t_0 - t_{01})]\ \{1 - \frac{a_2}{a_1}\exp[-j2\pi f(t_{02} - t_{01})]$$

$$+ (\frac{a_2}{a_1})^2 \exp[-j4\pi f(t_{02} - t_{01})]\} \qquad (2)$$

Comparing Eqs. (1) and (2), we deduce that

$$\frac{K_0}{a_1} \simeq w_0$$

$$t_0 - t_{01} \simeq 0$$

$$-\frac{a_2}{a_1} \simeq \frac{w_1}{w_0}$$

$$(\frac{a_2}{a_1})^2 \simeq \frac{w_2}{w_0}$$

$$T \simeq t_{02} - t_{01}$$

Choosing $K_0 = a_1$, we find that the tap weights of the equalizer are as follows

$$w_0 = 1$$

$$w_1 = -\frac{a_2}{a_1}$$

$$w_2 = (\frac{a_2}{a_1})^2$$

Problem 4.35

The Fourier transform of the tapped-delay-line equalizer output is defined by

$$Y_{out}(f) = H(f)\, X_{in}(f) \tag{1}$$

where H(f) is the equalizer's transfer function and $X_{in}(f)$ is the Fourier transform of the input signal. The input signal consists of a uniform sequence of samples, denoted by {x(nT)}. We may therefore write (see Eq. 6.2):

$$X_{in}(f) = \frac{1}{T}\sum_{k} X(f-\frac{k}{T}) \tag{2}$$

where T is the sampling period and s(t) is the signal from which the sequence of samples is derived. For perfect equalization, we require that

$$Y_{out}(f) = 1 \qquad \text{for all } f.$$

From Eqs. (1) and (2) we therefore find that

$$H(f) = \frac{T}{\sum_{k} X(f-k/T)} \tag{3}$$

Let the impulse response (sequence) of the equalizer be denoted by $\{w_n\}$. Assuming an infinite number of taps, we have

$$H(f) = \sum_{n=-\infty}^{\infty} w_n \exp(j2\pi fT)$$

We now immediately see that H(f) is in the form of a complex Fourier series with real coefficients defined by the tap weights of the equalizer. The tap-weights are themselves defined by

$$w_n = \frac{1}{T}\int_{-1/2T}^{1/2T} H(f)\exp(-j2\pi t/T), \qquad n=0, +1, +2,...$$

The transfer function H(f) is itself defined in terms of the input signal by Eq, (3). Accordingly, a tapped-delay-line equalizer of infinite length can approximate any function in the frequency interval (-1/2T, 1/2T).

Problem 4.36

(a) As an example, consider the following single-parameter model of a noisy system:

$$d[n] = w_0[n]x[n] + v[n]$$

where $x[n]$ is the input signal and $v[n]$ is additive noise. To track variations in the parameter $w_0[n]$, we may use the LMS algorithm, which is described by

$$\hat{w}[n+1] = \hat{w}[n] + \mu x[n]\left(\overbrace{(d[n] - \hat{w}[n]x[n])}^{\text{Error signal}}\right)$$

$$= (1 - \mu x^2[n])\hat{w}[n] + \mu x[n]d[n] \qquad (1)$$

To simplify matters, we assume that $\hat{w}[n]$ is independent of $x[n]$. Hence, taking the expectation of both sides of Eq. (1):

$$E[\hat{w}[n+1]] = (1 - \mu\sigma_x^2)E[\hat{w}[n]] + \mu r_{dx} \qquad (2)$$

where E is the statistical expectation operator, and

$$\sigma_x^2 = E[x^2[n]]$$

$$r_{dx} = E[d[n]x[n]]$$

Equation (2) represents a first-order difference equation in the mean value $E[\hat{w}[n]]$. For this difference equation to be convergent (i.e., for the system to be stable), we require that

$$|1 - \mu\sigma_x^2| < 1$$

or equivalently

(i) $1 - \mu\sigma_x^2 < 1$, i.e., $\mu > 0$

(ii) $-1 + \mu\sigma_x^2 < 1$, i.e., $\mu < \dfrac{2}{\sigma_x^2}$

Stated in yet another way, the LMS algorithm for the example considered herein is stable provided that the step-size parameter μ satisfies the following conditions:

$$0 < \mu < \frac{2}{\sigma_x^2}$$

where σ_x^2 is the variance of the input signal.

(b) When a small value is assigned to μ, the adaptation is slow, which is equivalent to the LMS algorithm having a long "memory". The excess mean-squared error after adaptation is small, on the average, because of the large amount of data used by the algorithm to estimate the gradient vector. On the other hand, when μ is large, the adaptation is relatively fast, but at the expense of an increase in the excess mean-squared error after adaptation. In this case, less data enter the estimation, hence a degraded estimation error performance. Thus, the reciprocal of the parameter μ may be viewed as the memory of the LMS algorithm.

Problem 4.37

A *decision-feedback equalizer* consists of a feedforward section, a feedback section, and a decision device connected together as shown in Fig. 1. The feed-forward section consists of a tapped-delay-line filter whose taps are spaced at the reciprocal of the signaling rate. The data sequence to be equalized is applied to this section. The feedback section consists of another tapped-delay-line filter whose taps are also spaced at the reciprocal of the signaling rate. The input applied to the feedback section consists of the decisions made on previously detected symbols of the input sequence. The function of the feedback section is to subtract out that portion of the intersymbol interference produced by previously detected symbols from the estimates of future samples.

Note that the inclusion of the decision device in the feedback loop makes the equalizer intrinsically *nonlinear* and therefore more difficult to analyze than an ordinary tapped-delay-line equalizer. Nevertheless, the mean-square error criterion can be used to obtain a mathematically tractable optimization of a decision-feedback equalizer. Indeed, the LMS algorithm can be used to jointly adapt both the feedforward tap-weights and the feedback tap-weights based on a *common* error signal. To be specific, let the augmented vector $\mathbf{c}_n$ denote the combination of the feedforward and feedback tap-weights, as shown by

$$\mathbf{c}_n = \begin{bmatrix} \hat{\mathbf{w}}_n^{(1)} \\ \hat{\mathbf{w}}_n^{(2)} \end{bmatrix} \qquad (1)$$

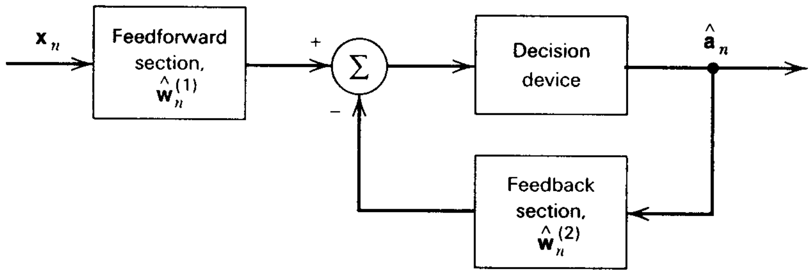


Figure 1

where the vector $\hat{\mathbf{w}}_n^{(1)}$ denotes the tap-weights of the feedforward section, and $\hat{\mathbf{w}}_n^{(2)}$ denotes the tap-weights of the feedback section. Let the augmented vector $\mathbf{v}_n$ denote the combination of input samples for both sections:

$$\mathbf{v}_n = \begin{bmatrix} \mathbf{x}_n \\ \hat{\mathbf{a}}_n \end{bmatrix} \tag{2}$$

where $\mathbf{x}_n$ is the vector of tap-inputs in the feedforward section, and $\hat{\mathbf{a}}_n$ is the vector of tap-inputs (i.e., present and past decisions) in the feedback section. The common error signal is defined by

$$e_n = a_n - \mathbf{c}_n^T \mathbf{v}_n \tag{3}$$

where the superscript T denotes matrix transposition and a_n is the polar representation of the nth transmitted binary symbol. The LMS algorithm for the decision-feedback equalizer is described by the update equations:

$$\hat{\mathbf{w}}_{n+1}^{(1)} = \hat{\mathbf{w}}_n^{(1)} + \mu_1 e_n \mathbf{x}_n$$

$$\hat{\mathbf{w}}_{n+1}^{(2)} = \hat{\mathbf{w}}_n^{(2)} + \mu_2 e_n \hat{\mathbf{a}}_n$$

where μ_1 and μ_2 are the step-size parameters for the feedforward and feedback sections, respectively.

Problem 4.38

Matlab codes

```
% Problem 4.38, CS: haykin
% Eyediagram
% baseband PAM transmission, M=4
%  Mathini Sellathurai
clear all

% Define the M-ary number, calculation sample frequency
M=4; Fs=20;

% Define the number of points in the calculation
Pd=500;

% Generate an integer message in range [0, M-1].
msg_d = exp_randint(Pd,1,M);

% Use square constellation PAM method for modulation
msg_a = exp_modmap(msg_d,Fs,M);

% nonlinear channel
alpha=0.0
```

```
msg_a=msg_a +alpha*msg_a.^2;

%raised cosine filtering
rcv_a=raisecos_n(msg_a,Fs);

% eye pattern
eyescat(rcv_a,0.5,Fs)
axis([-0.5 2.5 -1.5 1.5])
```

```
function y = exp_modmap(x, Fs,M);
% PAM modulation
% used in Problem 4.38
% Mathini Sellathurai

x=x-(M-1)/2;
x=2*x/(M-1)
y=zeros(length(x)*Fs,1);

p=0;
for k=1:Fs:length(y)
p=p+1;
y(k:(k+Fs-1))=x(p)*ones(Fs,1);
end
```

```
function out = exp_randint(p, q, r);
% random interger generator
%used for Problem 4.38
% Mathini Sellathurai

r = [0, r-1];
r = sort(r);
r(1) = ceil(r(1));
r(2) = floor(r(2));
if r(1) == r(2)
    out = ones(p, q) * r(1);
    return;
end;

d = r(2) - r(1);

r1 = rand(p, q);

out = ones(p,q)*r(1);

for i = 1:d
    index = find(r1 >= i/(d+1));
  out(index) = (r(1) + i) * index./index;
end;
```

Answer to Problem 4.38

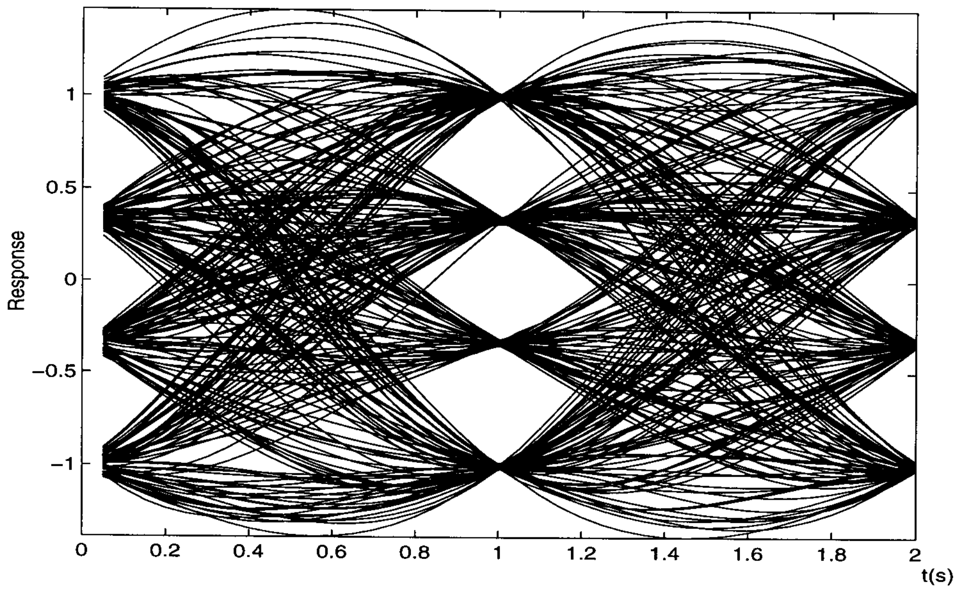


Figure 4 : Eye pattern for α=0

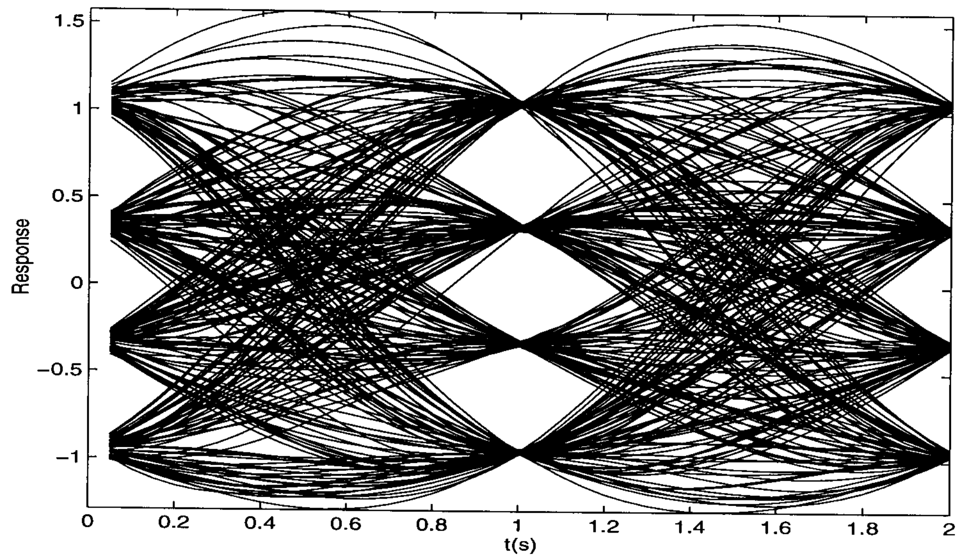


Figure 2: Eye pattern for α=0.05

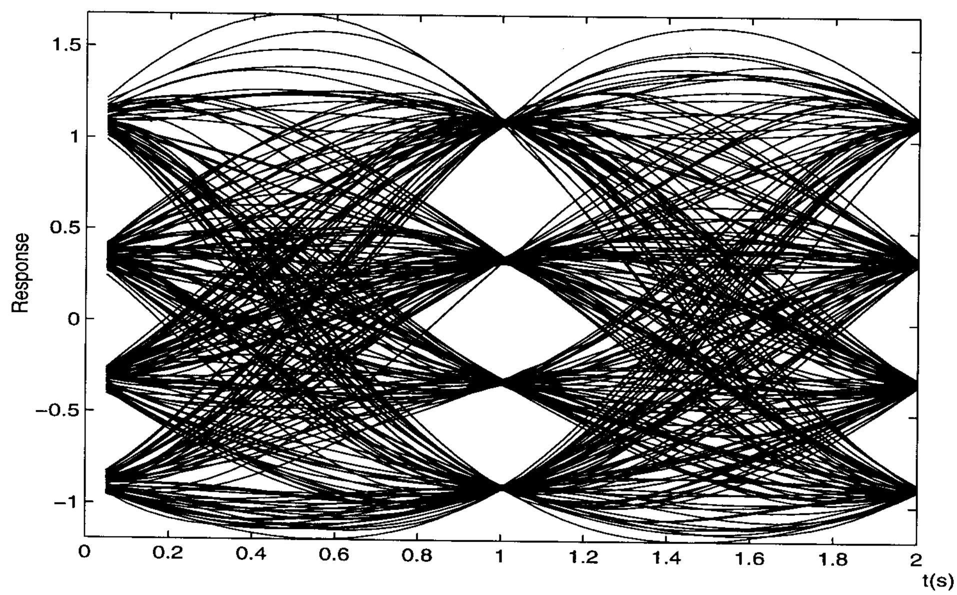


Figure 3: Eye pattern for α=0.1

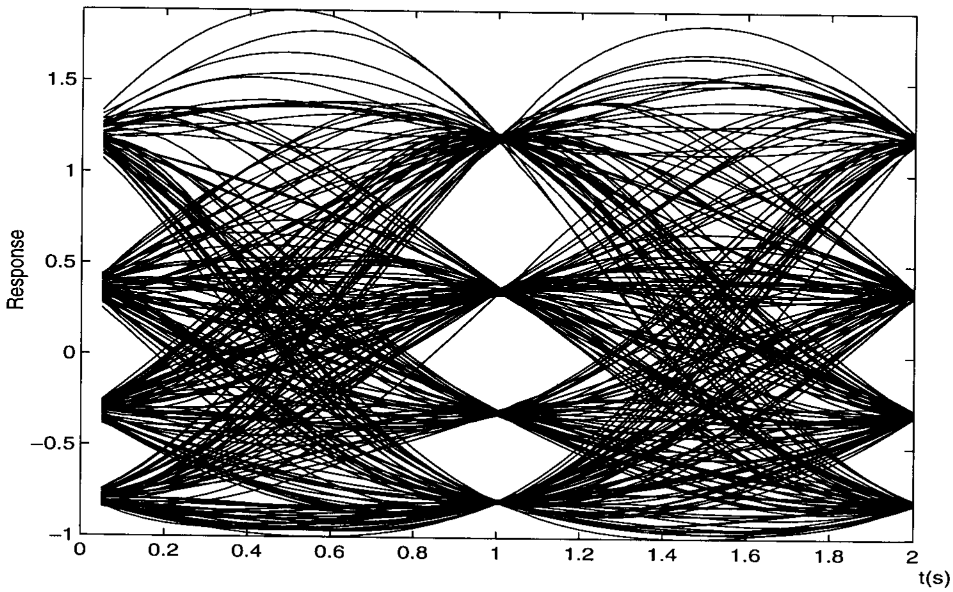


Figure 4 : Eye pattern for α=0.2

Problem 4.39

Matlab codes

```
% problem 4.39, CS: Haykin
%  root raised-cosine and raised cosine sequences
% M. Sellathurai

Data=[1 0 1 1 0 0]';
% sample frequency 20
sample_freq=20;

%generate antipodal signal
syms=PAM_mod(Data, sample_freq, 2);

% root raised cosine pulse
r_c_r = raisecos_sqrt(syms, sample_freq );

% normal raised cosine pulse
r_c_n= raisecos_n(syms, sample_freq );

% plots
t=length(r_c_r)-1;
figure; hold on
```

```
plot(0:1/20:t/20, r_c_r);
plot(0:1/20:t/20, r_c_n,'--');
xlabel('time')
legend('root raised-cosine','raised-cosine')
hold off
```

```
function osyms = raisecos_n(syms, sample_freq )
% function to generate raised-cosine sequence
% used in Problem 4.39, CS: Haykin
%M. Sellathurai

% size of data
[l_syms, w_syms] = size(syms);

% data
R=0.3;
W_T=[3, 3*3];

% Calculation of  Raised cosine pulse
W_T(1) = -abs(W_T(1));
time_T = [0 : 1/sample_freq : max(W_T(2), abs(W_T(1)))];
time_T_R = R * time_T;


    den = 1 - (2 * time_T_R).^2;
   index1 = find(den~= 0);
   index2 = find(den == 0);

    % when denominator not equal to zero
  b(index1) = sinc(time_T(index1)) .* cos(pi * time_T_R(index1)) ./ den(index1);

    % when denominator  equal to zero, (using  L'Hopital rule)
    if ~isempty(index2)
         b(index2) = 0;
  end;

b = [b(sample_freq * abs(W_T(1))+1 : -1 : 1), b(2 : sample_freq * W_T(2)+1)];
b=b(:)';
% filter parameters
order= floor(length(b)/2);
bb=[];
for i = 1: order
    bb = [bb; b(1+i:order+i)];
end;

[u, d, v] = svd(bb);
d = diag(d);

index = find(d/d(1) < 0.01);
    if isempty(index)
        o = length(bb);
    else
```

```
        o = index(1)-1;
    end;

a4 = bb(1);
u1 = u(1 : length(bb)-1, 1 : o);
v1 = v(1 :  length(bb)-1, 1 : o);
u2 = u(2 :  length(bb),   1 : o);

dd = sqrt(d(1:o));
vdd = 1 ./ dd;

uu = u1' * u2;
a1 = uu .* (vdd * dd');
a2 = dd .* v1(1, :)';
a3 = u1(1, :) .* dd';

[num, den] = ss2tf(a1, a2, a3, a4, 1);

 fsyms = zeros(l_syms+3*sample_freq, w_syms);
    for i = 1 : sample_freq : l_syms
        fsyms(i, :) = syms(i, :);
    end;

% filtering
for i = 1:w_syms
    fsyms(:, i) = filter(num, den, fsyms(:, i));
end;

 osyms = fsyms(( (3 - 1) * sample_freq  + 2):(size(fsyms, 1) - (sample_freq - 1)), :);
```

```
function osyms = raisecos_sqrt(syms, sample_freq )
% function to generate root raised-cosine sequence
% used in Problem 4.39, CS: Haykin
%M. Sellathurai

% size of data
[l_syms, w_syms] = size(syms);

% rolloff factor
R=0.3;
% window
W_T=[3, 3*3];

% Calculation of  Raised cosine pulse
W_T(1) = -abs(W_T(1));
time_T = [0 : 1/sample_freq : max(W_T(2), abs(W_T(1)))];

    den = 1 - (4 * time_T*R).^2;
   index1 = find(den ~= 0);
   index2 = find(den == 0);

    % when denominator not equal to zero
b(index1)=( cos((1 + R) * pi * time_T(index1))+...
(sinc((1-R)*time_T(index1))*(1-R)*pi/4/R))./den(index1)*4*R/ pi ;


    % when denominator  equal to zero t=\pm T/4/alpha
    if ~isempty(index2)
b(index2)=((1+2/pi)*sin(pi/4/R)+(1-2/pi)*cos(pi/4/R))*R/sqrt(2)
    end;

b(1)=1-R+4*R/pi; %t=0;

b = [b(sample_freq * abs(W_T(1))+1 : -1 : 1), b(2 : sample_freq * W_T(2)+1)];
b=b(:)';

% filter parameters
order= floor(length(b)/2);
bb=[];
for i = 1: order
    bb = [bb; b(1+i:order+i)];
end;

[u, d, v] = svd(bb);
d = diag(d);
```

```
index = find(d/d(1) < 0.01);
    if isempty(index)
        o = length(bb);
    else
        o = index(1)-1;
    end;

a4 = bb(1);
u1 = u(1 : length(bb)-1, 1 : o);
v1 = v(1 :  length(bb)-1, 1 : o);
u2 = u(2 :  length(bb),   1 : o);

dd = sqrt(d(1:o));
vdd = 1 ./ dd;

uu = u1' * u2;
a1 = uu .* (vdd * dd');
a2 = dd .* v1(1, :)';
a3 = u1(1, :) .* dd';

[num, den] = ss2tf(a1, a2, a3, a4, 1);

 fsyms = zeros(l_syms+3*sample_freq, w_syms);
    for i = 1 : sample_freq : l_syms
        fsyms(i, :) = syms(i, :);
    end;

% filtering
for i = 1:w_syms
    fsyms(:, i) = filter(num, den, fsyms(:, i));
end;

 osyms = fsyms(( (3 - 1) * sample_freq  + 2):(size(fsyms, 1) - (sample_freq - 1)), :);
```

Answer to Problem 4.39

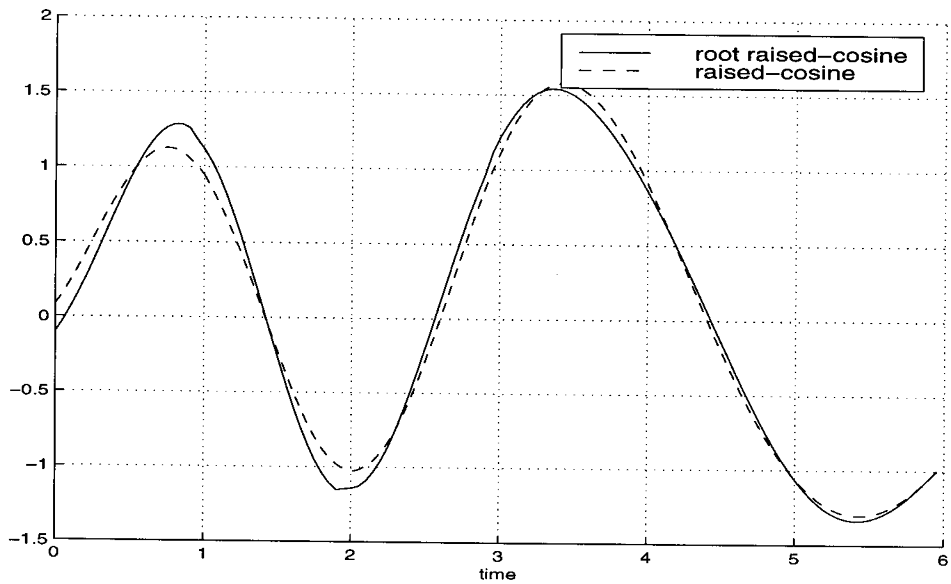


Figure 1: Raised-cosine and root raised-cosine pulse for sequence [101100]

CHAPTER 5

Problem 5.1

(a) Unipolar NRZ code.

The pair of signals $s_1(t)$ and $s_2(t)$ used to represent binary symbols 1 and 0, respectively are defined by

$$s_1(t) = \sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b$$

$$s_2(t) = 0, \qquad 0 \le t \le T_b$$

where E_b is the transmitted signal energy per bit and T_b is the bit duration. From the definitions of $s_1(t)$ and $s_2(t)$, it is clear that, in the case of unipolar NRZ signals, there is only one basis function of unit energy. The basis function is given by

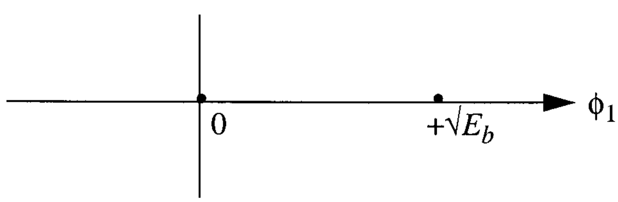
$$\phi_1(t) = \sqrt{\frac{1}{T_b}}, \qquad 0 \le t \le T_b$$

Then, we may expand the transmitted signals $s_1(t)$ and $s_2(t)$ in terms of $\phi_1(t)$ as follows:

$$s_1(t) = \sqrt{E_b}\phi_1(t), \qquad 0 \le t \le T_b$$

$$s_2(t) = 0, \qquad 0 \le t \le T_b$$

Hence, the signal-space diagram for unipolar NRZ code is $(+\sqrt{E_b}, 0)$, as shown



(b) Polar NRZ code.

In this code, binary symbols 1 and 0 are defined by

$$s_1(t) = +\sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b$$

$$s_2(t) = -\sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b$$

The basis function is given by

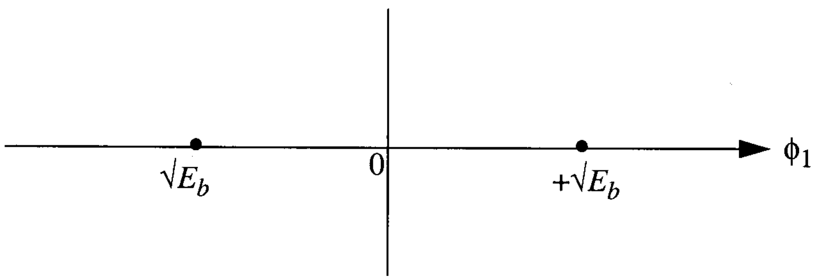
$$\phi_1(t) = \sqrt{\frac{1}{T_b}}, \qquad 0 \le t \le T_b$$

Then, the transmitted signals in terms of $\phi_1(t)$ are as follows:

$$s_1(t) = \sqrt{E_b}\phi_1(t) \qquad 0 \le t \le T_b$$

$$s_2(t) = -\sqrt{E_b}\phi_1(t) \qquad 0 \le t \le T_b$$

Hence, the signal-space diagram for the polar NRZ code is $(+\sqrt{E_b}, -\sqrt{E_b})$ as shown below:



(c) Unipolar return-to-zero code.

In this third code, binary symbols 1 and 0 are defined by

$$s_1(t) = +\sqrt{\frac{E_b}{T_b}}, \qquad 0 \le t \le T_b/2$$

$$= 0 \qquad T_b/2 \le t \le T_b$$

$$s_2(t) = 0 \qquad 0 \le t \le T_b$$

The energy of signal $s_1(t)$ is

$$E_1 = \int_0^{T_b/2} \left(\sqrt{\frac{E_b}{T_b}}\right)^2 dt + \int_{T_b/2}^{T_b} 0 dt$$

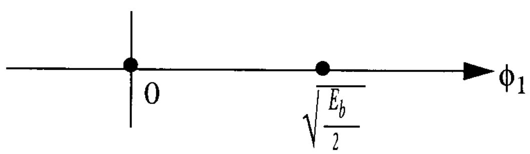
$$= \frac{E_b}{2}$$

The energy of signal $s_2(t)$ is zero.

The basis function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\frac{E_b}{2}}}$$

The signal-space diagram for the RZ code is as follows:



(d) Manchester code

Binary symbols 1 and 0 are defined by

$$s_1(t) = \begin{cases} \sqrt{\frac{E_b}{T_b}}, & 0 \le t \le T_b/2 \\ -\sqrt{\frac{E_b}{T_b}}, & T_b/2 \le t \le T_b \end{cases}$$

$$s_2(t) = \begin{cases} -\sqrt{\frac{E_b}{T_b}}, & 0 \le t \le T_b/2 \\ +\sqrt{\frac{E_b}{T_b}}, & T_b/2 \le t \le T_b \end{cases}$$

The energy of signal $s_1(t)$ is

$$E_1 = \int_0^{T_b/2} \left(\sqrt{\frac{E_b}{T_b}}\right)^2 dt + \int_{T_b/2}^{T_b} \left(\sqrt{\frac{E_b}{T_b}}\right)^2 dt$$

$$= E_b$$

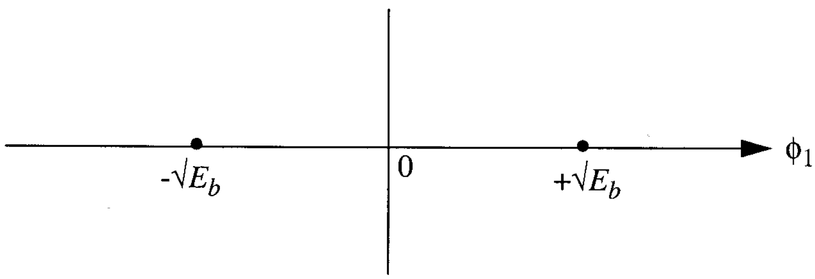
Similarly, the energy of symbol $s_2(t)$ is

$$E_2 = E_b$$

The basis function is given by

$$\phi_1(t) = = \frac{s_1(t)}{\sqrt{E_b}}$$

The signal-space diagram of the Manchester code is thus as follows:



Thus all the four line codes in this problem are one-dimensional.

Problem 5.2

The given 8-level PAM signal is defined by

$$s_i(t) = A_i \text{rect}\left(\frac{t}{T} - \frac{T}{2}\right)$$

The energy of signal $s_i(t)$ is given by

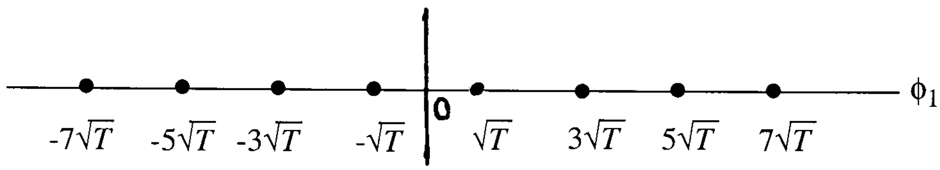
$$E_i = \int_0^T (A_i)^2 dt$$

$$= A_i^2 T, \qquad A_i = \pm 1, \pm 3, \pm 5, \pm 7$$

The basis function is given by

$$\phi_1(t) = \frac{s_i(t)}{\sqrt{E_i}} = \frac{s_i(t)}{A_i\sqrt{T}}$$

The signal-space diagram of the 8-level PAM signal is as follows:



Problem 5.3

Consider the signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ shown in Fig. 1a. We wish to use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for this set of signals.

Step 1 We note that the energy of signal $s_1(t)$ is

$$E_1 = \int_0^T s_1^2(t)dt$$

$$= \int_0^{T/3} (1)^2 dt$$

$$= \frac{T}{3}$$

The first basis function $\phi_1(t)$ is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$= \begin{Bmatrix} \sqrt{3/T}, & 0 \le t \le T/3 \\ 0, & \text{otherwise} \end{Bmatrix}$$

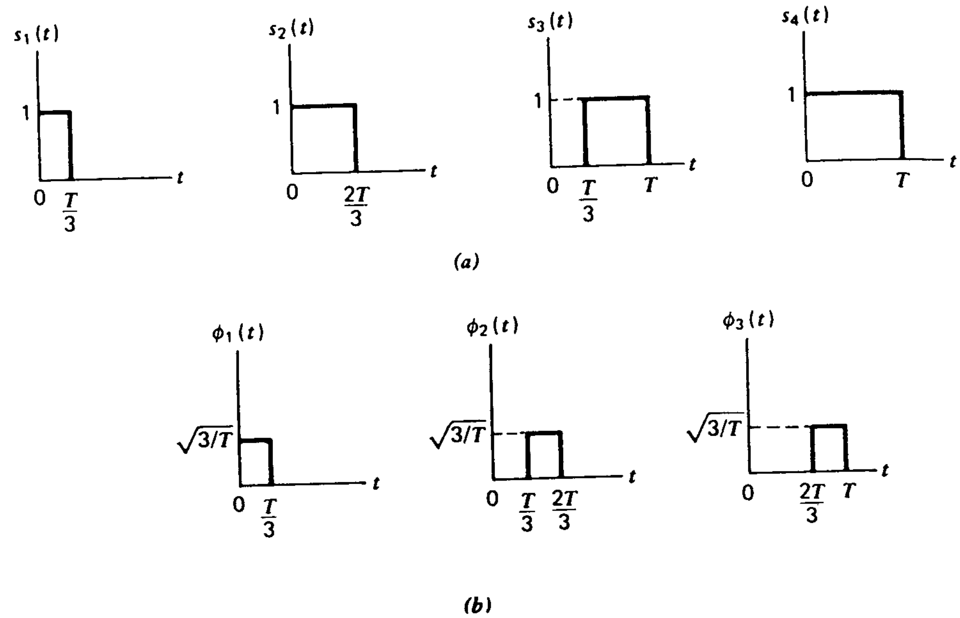


Figure 1

Step 2 Evaluating the projection of $s_2(t)$ onto $\phi_1(t)$, we find that

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt$$

$$= \int_0^{T/3} (1)\left(\sqrt{\frac{3}{T}}\right)dt$$

$$= \sqrt{\frac{3}{T}}$$

The energy of signal $s_2(t)$ is

$$E_2 = \int_0^T s_2^2(t)$$

$$= \int_0^{2T/3} (1)^2 dt$$

$$= \frac{2T}{3}$$

The second basis function $\phi_2(t)$ is therefore

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

$$= \begin{cases} \sqrt{3/T}, & T/3 \le 2T/3 \\ 0 & \text{otherwise} \end{cases}$$

Step 3 Evaluating the projection of $s_3(t)$ onto $\phi_1(t)$,

$$s_{31} = \int_0^T s_3(t)\phi_1(t)dt$$

$$= 0$$

and the coefficient s_{32} equals

$$s_{32} = \int_0^T s_3(t)\phi_1(t)dt$$

$$= \int_{T/3}^{2T/3} (1)\left(\sqrt{\frac{3}{T}}\right)dt$$

$$= \sqrt{\frac{3}{T}}$$

The corresponding value of the intermediate function $g_i(t)$, with $i = 3$, is therefore

$$g_3(t) = s_3(t) - s_{31}\phi_1(t) - s_{32}\phi_2(t)$$

$$= \begin{cases} 1, & 2T/3 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

Hence, the third basis function $\phi_3(t)$ is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}}$$

$$= \begin{cases} \sqrt{3/T}, & 2T/3 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

The orthogonalization process is now complete.

The three basis functions $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ form an orthonormal set, as shown in Fig. 1b. In this example, we thus have $M = 4$ and $N = 3$, which means that the four signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ described in Fig. 1a do not form a linearly independent set. This is readily confirmed by noting that $s_4(t) = s_1(t) + s_3(t)$. Moreover, we note that any of these four signals can be expressed as a linear combination of the three basis functions, which is the essence of the Gram-Schmidt orthogonalization procedure.

Problem 5.4

(a) We first observe that $s_1(t)$, $s_2(t)$ and $s_3(t)$ are linearly independent.

The energy of $s_1(t)$ is

$$E_1 = \int_0^1 (2)^2 dt = 4$$

The first basis function is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$= \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwsie} \end{cases}$$

Define

$$s_{21} = \int_0^T s_2(t)\ \phi_1(t)dt$$

$$= \int_0^1 (-4)(1)dt = -4$$

$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

$$= \begin{cases} -4, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the second basis function is

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}}$$

$$= \begin{cases} -1, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Define

$$s_{31} = \int_0^T s_3(t)\ \phi_1(t)dt$$

$$= \int_0^1 (3)(1)dt = 3$$

$$s_{32} = \int_T^{2T} s_3(t)\ \phi_2(t)dt$$

$$= \int_1^2 (3)(-1)dt = -3$$

$$g_3(t) = s_3(t) - s_{31}\ \phi_1(t) - s_{32}\ \phi_2(t)$$

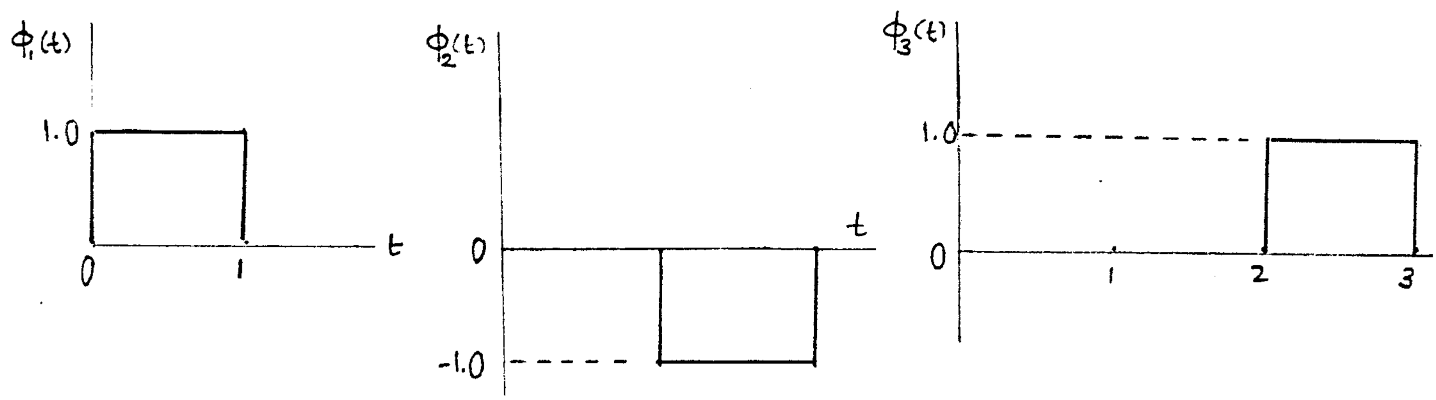
$$= \begin{cases} 3, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the thrid basis function is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}}$$

$$= \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

The three basis functions are as follows (graphically)



(b) $s_1(t) = 2\phi_1(t)$

$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$

$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$

Problem 5.5

Signals $s_1(t)$ and $s_2(t)$ are orthogonal to each other. The energy of $s_1(t)$ is

$$E_1 = \int_0^{T/2} 1^2 dt + \int_{T/2}^{T} (-1)^2 dt = T$$

The energy of $s_2(t)$ is

$$E_2 = \int_0^{T} 1^2 dt = T$$

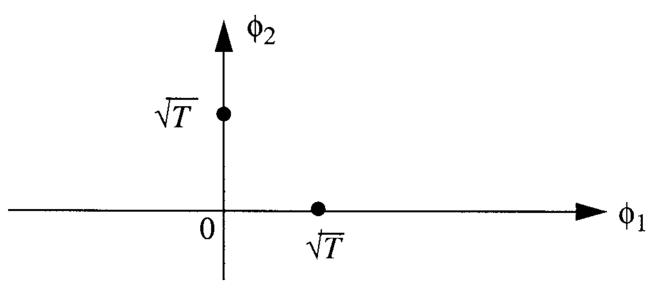
To represent the orthogonal signals $s_1(t)$ and $s_2(t)$, we need two basis functions. The first basis function is given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}$$

The second basis function is given by

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{E_2}} = \frac{s_2(t)}{\sqrt{T}}$$

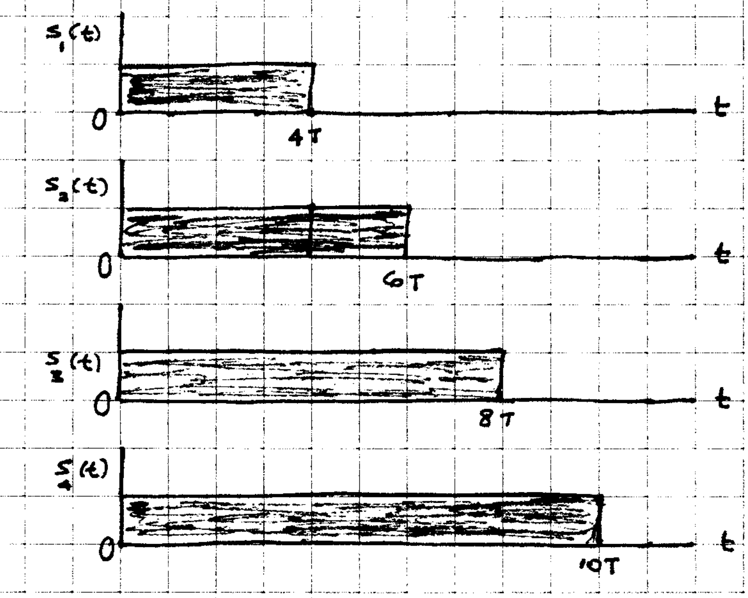
The signal-space diagram for $s_1(t)$ and $s_2(t)$ is as shown below:



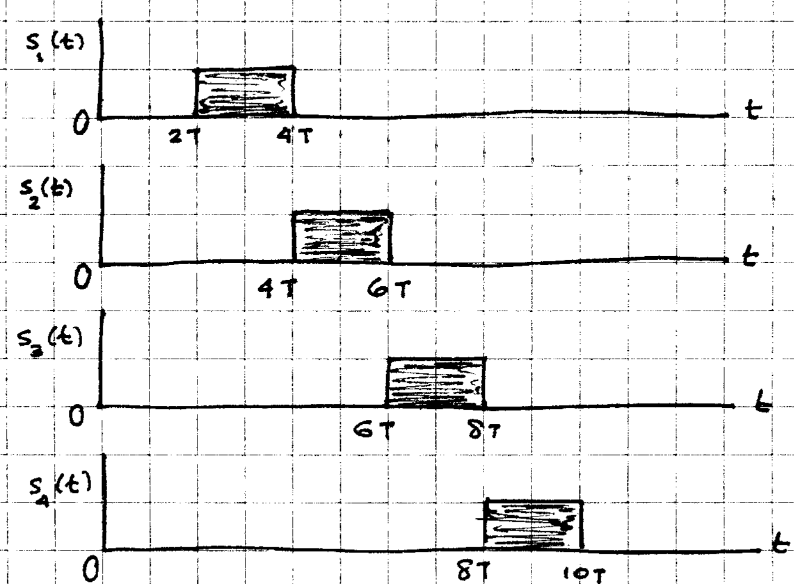
Problem 5.6

The common properties of PDM and PPM are as follows: In both cases a time parameter of the pulse is modulated and the pulses have a constant amplitude. In PDM, the samples of the message signals are used to vary the duration of the individual pulses, as illustrated in Fig. 1a for $M = 4$ on the next page. In PPM, the position of the pulse is varied in accordance with the message, while keeping the duration of the pulse constant, as illustrated in Fig. 1b for $M = 4$.

From these two illustrative figures, it is perfectly clear that the set of PDM signals is nonorthogonal, whereas the PDM signals form an orthogonal set.



(a) Pulse-duration modulation



(b) Pulse-position modulation

Figure 1

Problem 5.7

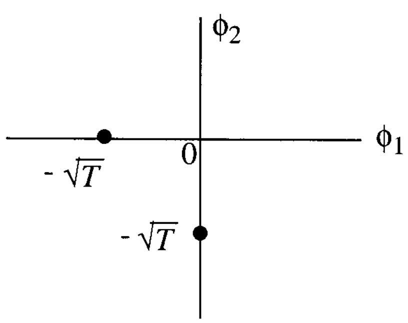
(a) The biorthogonal signals are defined as the negatives of orthogonal signals. Consider for example the two orthogonal signals $s_1(t)$ and $s_2(t)$ defined as follows:

$$s_1(t) = \sqrt{E}\phi_1(t)$$

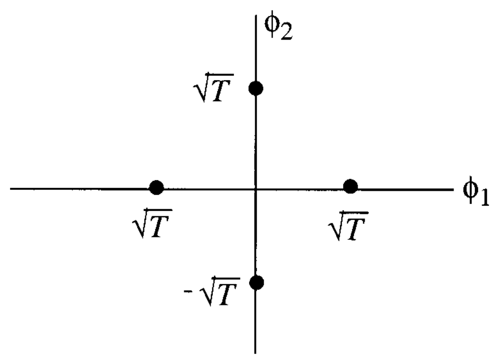
$$s_2(t) = \sqrt{E}\phi_2(t)$$

where $\phi_1(t)$ and $\phi_2(t)$ are orthonormal basis functions. The biorthogonal signals are given by $-s_1(t)$ and $-s_2(t)$, which are respectively expressed in terms of the basis functions as $\sqrt{E}\phi_1(t)$ and $-\sqrt{E}\phi_2(t)$. Hence, the inclusion of these two biorthogonal signals leaves the dimensionality of the signal-space diagram unchanged. This result holds for the general case of M orthogonal signals.

(b) The signal-space diagram for the biorthogonal signals corresponding to those shown in Fig. P5.5 is as shown in Fig. 1a. Incorporating this diagram with that of the solution to Problem 5.5, we get the 4-signal constellation shown in Fig. 1b.



(a)



(b)

Figure 1

Problem 5.8

(a) A pair of signals $s_i(t)$ and $s_k(t)$, belonging to an N-dimensional signal space, can be represented as linear combinations of N orthonormal basis functions. We thus write

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \quad \begin{array}{l} 0 \le t \le T \\ i = 1, 2 \end{array} \tag{1}$$

where the coefficients of the expansion are defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \qquad \begin{array}{l} i = 1, 2 \\ j = 1, 2 \end{array} \tag{2}$$

The real-valued basis functions $\phi_1(t)$ and $\phi_2(t)$ are orthonormal. Hence,

$$\int_0^T \phi_i(t)\phi_j(t) \quad = \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

The set of coefficients $\{s_{ij}\}_{j=1}^{N}$ may be viewed as an N-dimensional vector defined by

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \quad i = 1, 2, \ldots, M \tag{4}$$

where M is the number of signals in the setl, with $M \geq N$. The inner product of the pair of signal $s_i(t)$ and $s_k(t)$ is given by

$$\int_0^T s_i(t)s_k(t)dt \tag{5}$$

By substituting (1) in (5), we get the following result for the inner product:

$$\int_0^T \left[\sum_{j=1}^{N} s_{ij}\phi_j(t)\right] \left[\sum_{l=1}^{N} s_{kl}\phi_l(t)\right] dt$$

$$= \sum_{j=1}^{N} \sum_{l=1}^{N} s_{ij}s_{kl}\int_0^T \phi_j(t)\phi_l(t)dt \tag{6}$$

Since the $\phi_j(t)$ form an orthonormal set, then, in accordance with the two conditions of Eq. (3) and (4), the inner product of $s_i(t)$ and $s_k(t)$ reduces to

$$\int_0^T s_i(t)s_k(t)dt = \sum_{j=1}^{N} s_{ij}s_{kj}$$

$$= \mathbf{s}_i^T \mathbf{s}_k$$

(b) Consider next the squared Euclidean distance between $\mathbf{s}_i$ and $\mathbf{s}_k$, which can be expressed as follows:

$$\|\mathbf{s}_i - \mathbf{s}_k\|^2 = (\mathbf{s}_i - \mathbf{s}_k)^T(\mathbf{s}_i - \mathbf{s}_k)$$

$$= \mathbf{s}_i^T \mathbf{s}_i + \mathbf{s}_k^T \mathbf{s}_k - 2\mathbf{s}_i^T \mathbf{s}_k$$

$$= \int_0^T s_i^2(t)dt + \int_0^T s_k^2(t)dt - 2\int_0^T s_i(t)s_k dt$$

$$= \int_0^T (s_i(t) - s_k(t))^2 dt$$

Problem 5.9

Consider the pair of complex-valued signals $s_1(t)$ and $s_2(t)$, which are defined by

$$s_1(t) = a_{11}\phi_1(t) + a_{12}\phi_2(t) \tag{1}$$

$$s_2(t) = a_{21}\phi_1(t) + a_{22}\phi_2(t) \tag{2}$$

The basis functions $\phi_1(t)$ and $\phi_2(t)$ are real-valued and the coefficients a_{11}, a_{12}, a_{21} and a_{22} are complex-valued. We may denote the complex-valued coefficients as follows:

$a_{11} = \alpha_{11} + j\beta_{11}$
$a_{12} = \alpha_{12} + j\beta_{12}$
$a_{21} = \alpha_{21} + j\beta_{21}$
$a_{22} = \alpha_{22} + j\beta_{22}$

On this basis, we may represent the signals $s_1(t)$ and $s_2(t)$ by the following respective pair of vectors:

$$\mathbf{g}_1 = \begin{bmatrix} \alpha_{11} \\ \beta_{11} \\ \alpha_{12} \\ \beta_{12} \end{bmatrix}, \quad \mathbf{g}_2 = \begin{bmatrix} \alpha_{21} \\ \beta_{21} \\ \alpha_{22} \\ \beta_{22} \end{bmatrix}$$

The angle subtended between the vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ is

$$\cos\theta = \frac{\mathbf{g}_1^T \mathbf{g}_2}{\|\mathbf{g}_1\| \cdot \|\mathbf{g}_2\|}$$

$$= \frac{\alpha_{11}\alpha_{21} + \beta_{11}\beta_{21} + \alpha_{12}\alpha_{22} + \beta_{11}\beta_{22}}{\sqrt{\alpha_{11}^2\beta_{11}^2 + \alpha_{12}^2 + \beta_{12}^2} \cdot \sqrt{\alpha_{21}^2\beta_{21}^2 + \alpha_{22}^2 + \beta_{22}^2}}$$

$$= \frac{\mathbf{s}_1^H s_2}{\|\mathbf{s}_1\| \cdot \|\mathbf{s}_2\|}$$

where $\mathbf{s}_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$ and $\mathbf{s}_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$ are complex vectors. Recognizing that

$$\cos\theta = \frac{\mathbf{s}_2^H s_1}{\|\mathbf{s}_1\| \cdot \|\mathbf{s}_2\|} \le 1$$

we may go on to write

$$\frac{\int_{-\infty}^{\infty} s_1(t) s_2^*(t)\,dt}{\left(\int_{-\infty}^{\infty} |s_1(t)|^2 dt\right)^{1/2} \left(\int_{-\infty}^{\infty} |s_2(t)|^2 dt\right)^{1/2}} \le 1$$

The complex form of the Schwarz unequality becomes

$$\left| \int_{-\infty}^{\infty} s_1(t) s_2^*(t)\,dt \right|^2 \le \int_{-\infty}^{\infty} |s_1(t)|^2 dt \int_{-\infty}^{\infty} |s_2(t)|^2 dt$$

The equality holds when $s_1(t)$ and $s_2(t)$ are co-linear, that is, $s_1(t) = ks_2(t)$ where k is any real-valued constant.

Problem 5.10

$$E[X_j W'(t_k)] = E[(s_{ij} + W_j)W'(t_k)]$$

$$E[s_{ij}\ W'(t_k)] = s_{ij}\ E[W'(t_k)] = 0$$

We also note that

$$W'(t_k) = W(t_k) - \sum_{i=1}^{N} W_i\ \phi_i(t_k)$$

We therefore have

$$E[X_j W'(t_k)] = E[W_j\ W'(t_k)]$$

$$= E[W_j W(t_k)] - \sum_{i=1}^{N} \phi_i(t_k)\ E[W_j W_i]$$

But $E[W_j\ W(t_k)] = E[W(t_k) \int_0^T W(t)\ \phi_j(t)dt] = \int_0^T \phi_j(t)\ E[W(t_k)W(t)]dt$

$$= \int_0^T \phi_j(t) \cdot \frac{N_0}{2}\ \delta(t-t_k)dt = \frac{N_0}{2}\ \phi_j(t_k)$$

$$E[W_j W_i] = \begin{cases} \frac{N_0}{2}, & i=j \\ 0 & i \neq j \end{cases}$$

Hence, we get the final result

$$E[X_j\ W'(t_k)] = \frac{N_0}{2}\ \phi_j(t_k) - \frac{N_0}{2}\ \phi_j(t_k)$$

$$= 0.$$

Problem 5.11

For the noiseless case, the received signal $r(t) = s(t)$, $0 \le t \le T$.

(a) The correlator output is

$$y(T) = \int_0^T r(\tau)s(\tau)d\tau$$

$$y(T) = \int_0^T s^2(\tau)d\tau$$

$$= \int_0^T \sin^2\left(\frac{8\pi\tau}{T}\right)d\tau$$

$$= \int_0^T \frac{1}{2}\left[1 - \cos\left(\frac{16\pi\tau}{T}\right)\right]d\tau$$

$$= T/2$$

(b) The matched filter is defined by the impulse response

$$h(t) = s(T-t)$$

The matched filter output is therefore

$$y(t) = \int_{-\infty}^{\infty} r(\lambda)h(t-\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} s(\lambda)s(T-t+\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} \sin\left(\frac{8\pi\lambda}{T}\right)\sin\left(\frac{8\pi(T-t+\lambda)}{T}\right)d\lambda$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \cos\left[\frac{8\pi(T-t)}{T}\right]d\lambda - \frac{1}{2}\int_{-\infty}^{\infty} \cos\left[\frac{8\pi(T-t+\lambda)}{T}\right]d\lambda$$

Since $-T < \lambda \leq 0$, we have

$$y(t) = \frac{1}{2}\cos\left(\frac{8\pi(t-T)}{T}\right)\lambda\Bigg]_{\lambda=-T}^{\lambda=0}$$

$$- \frac{1}{2}\cdot\frac{T}{8\pi}\sin\left(\frac{8\pi(T-t+2\lambda)}{T}\right)\Bigg]_{\lambda=-T}^{\lambda=0}$$

$$= \frac{T}{2}\cos\left(\frac{8\pi(t-T)}{T}\right) - \frac{T}{16\pi}\sin\left(\frac{8\pi(t-T)}{T}\right) - \frac{T}{16\pi}\sin\left(\frac{8\pi(t+T)}{T}\right)$$

(c) When the matched filter output is sampled at $t = T$, we get

$$y(T) = T/2$$

which is exactly the same as the correlator output determined in part (a).

Problem 5.12

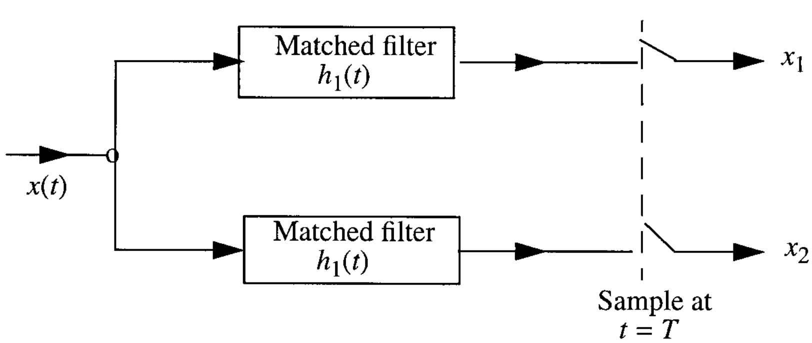
(a) The matched filter for signal $s_1(t)$ is defined by the impulse response

$$h_1(t) = s_1(T-t)$$

The matched filter for signal $s_2(t)$ is defined by the impulse response

$$h_2(t) = s_2(T-t)$$

The matched filter receiver is as follows



The receiver decides in favor of $s_2(t)$ if, for the noisy received signal,

$$x(t) = s_k(t) + w(t), \qquad \begin{array}{l} 0 \le t \le T \\ k = 1, 2 \end{array}$$

we find that $x_1 > x_2$. On the other hand, if $x_2 > x_1$, it decides in favor of $s_2(t)$. If $x_1 = x_2$, the decision is made by tossing a fair coin.

(b) Energy of signal $s_1(t)$ is given by

$$E_1 = \int_0^T (1)^2 dt + \int_T^{2T} (-1)^2 dt + \int_{2T}^{3T} (1)^2 dt$$

$$= 3T = E$$

Energy of signal $s_2(t)$ is

$$E_2 = \int_0^{T/2} (-1)^2 dt + \int_{T/2}^{3T/2} (1)^2 dt + \int_{3T/2}^{5T/2} (-1)^2 dt + \int_{5T/2}^{3T} (1)^2 dt$$

$$= 3T = E$$

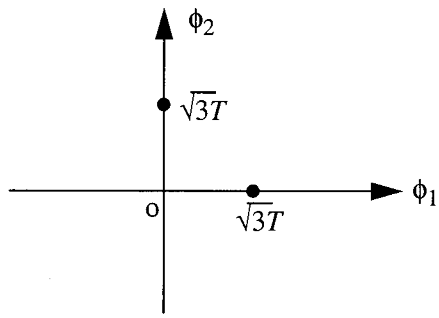
The orthornomal basis functions for the signal-space diagram of these two orthogonal signals are given by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{3T}}$$

and

$$\phi_2(t) = \frac{s_2(t)}{\sqrt{3T}}$$

The signal-space diagram of signals s_1 and s_2 is as follows:



The distance between the two signal points $s_1(t)$ and $s_2(t)$ is

$$d = \sqrt{2E} = \sqrt{6T}$$

The average probability of error is therefore

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{1}{2}\frac{d}{\sqrt{N_0}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{\frac{2E}{N_0}}\right)$$

For E/N_0, we therefore have

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{1}{2}\sqrt{2\times 4}\right)$$

$$= \frac{1}{2}\text{erfc}(\sqrt{2})$$

$$= 4\times 10^{-2}$$

Problem 5.13

Energy of binary symbol 1 represented by signal $s_1(t)$ is

$$E_1 = \int_0^{T/2}(+1)^2 dt + \int_{T/2}^{T}(-1)^2 dt = T$$

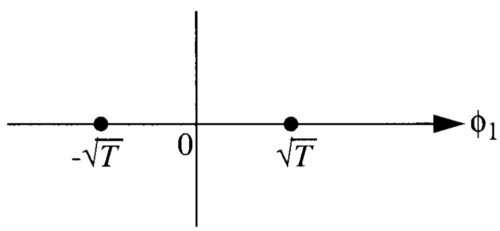
Energy of binary symbol 0 represented by signal $s_2(t)$ is the same as shown by

$$E_2 = \int_0^{T/2} (-1)^2 dt + \int_{T/2}^{T} (\ +1)^2 dt = T$$

The only basis function of the signal-space diagram is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{T}}$$

The signal-space diagram of the Manchester code using the doublet pulse is as follows:



Hence, the distance between the two signal points is $d = 2\sqrt{T}$. The average probability of error over an AWGN channel is given by

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{d}{2\sqrt{N_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{T}{N_0}}\right)$$

Problem 5.14

(a) Let Z denote the total observation space, which is divided into two parts Z_0 and Z_1. Whenever an observation falls in Z_0, we say H_0, and whenever an observation falls in Z_1, we say H_1. Thus, expressing the risk R in terms of the conditional probability density functions and the decision regions, we may write

$$R = C_{00}\, p_0 \int_{Z_0} f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x}$$

$$+ C_{10}\, p_0 \int_{Z_1} f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x}$$

$$+ C_{11}\, p_1 \int_{Z_1} f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x}$$

$$+ C_{01}\, p_1 \int_{Z_0} f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x} \tag{1}$$

For an N-dimensional observation space, the integrals in Eq. (1) are N-fold integrals.

To find the Bayes test, we must choose the decision regions Z_0 and Z_1 in such a manner that the risk R will be minimized. Because we require that a decision be made, this means that we must assign each point $\underline{x}$ in the observation space Z to Z_0 or Z_1; thus

$$Z = Z_0 + Z_1$$

Hence, we may rewrite Eq. (1) as

$$R = p_0C_{00} \int_{Z_0} f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x} + p_0C_{10} \int_{Z-Z_0} f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x}$$

$$+ p_1C_{11} \int_{Z-Z_0} f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x} + p_1C_{01} \int_{Z_0} f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x} \tag{2}$$

We observe that

$$\int_Z f_{\underline{X}|H_0}(\underline{x}|H_0)d\underline{x} = \int_Z f_{\underline{X}|H_1}(\underline{x}|H_1)d\underline{x} = 1$$

Hence, Eq. (2) reduces to

$$R = p_0C_{10} + p_1C_{11}$$
$$+ \int_{Z_0} \{-[p_0(C_{10}-C_{00})f_{\underline{X}|H_0}(\underline{x}|H_0)] + [p_1(C_{01}-C_{11})f_{\underline{X}|H_1}(\underline{x}|H_1)]\}d\underline{x} \tag{3}$$

The first two terms in Eq. (3) represent the fixed cost. The integral represents the cost controlled by those points $\underline{x}$ that we assign to Z_0. Since $C_{10} > C_{00}$ and $C_{01} > C_{11}$, we find that the two terms inside the square brackets are positive. Therefore, all values of $\underline{x}$ where the first term is larger than the second should be included in Z_0 because they contribute a negative amount to the integral. Similarly, all values of $\underline{x}$ where the second term is larger than the first should be excluded from Z_0 (i.e., assigned to Z_1) because they would contribute a positive amount to the integral. Values of $\underline{x}$ where the two terms are equal have no effect on the cost and may be assigned arbitrarily. Thus the decision regions are defined by the following statement: If

$$p_1(C_{01} - C_{11}) f_{\underline{X}|H_1}(\underline{x}|H_1) > p_0(C_{10}-C_{00}) f_{\underline{X}|H_0}(\underline{x}|H_0),$$

assign $\underline{x}$ to Z_1 and consequently say that H_1 is true. If the reverse is true, assign $\underline{x}$ to Z_0 and say H_0 is true.

Alternatively, we may write

$$\frac{f_{\underline{X}|H_1}(\underline{x}|H_1)}{f_{\underline{X}|H_0}(\underline{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0(C_{10}-C_{00})}{p_1(C_{01}-C_{11})}$$

The quantity on the left is the likelihood ratio:

$$\Lambda(\underline{x}) = \frac{f_{\underline{X}|H_1}(\underline{x}|H_1)}{f_{\underline{X}|H_0}(\underline{x}|H_0)}$$

Let

$$\lambda = \frac{p_0(C_{10}-C_{00})}{p_1(C_{01}-C_{11})}$$

Thus, Bayes criterion yields a likelihood ratio test described by

$$\Lambda(\underline{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

(b) For the minimum probability of error criterion, the likelihood ratio test is described by

$$\Lambda(\underline{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{p_0}{p_1}$$

Thus, we may view the minimum probability of error criterion as a special case of the Bayes criterion with the cost values defined as

$$C_{00} = C_{11} = 0$$

$$C_{10} = C_{01}$$

That is, the cost of a correct decision is zero, and the cost of an error of one kind is the same as the cost of an error of the other kind.

Problem 5.15

From the signal-space diagrams derived in the solution to Problem 5.1, we immediately observe the following:

1. Unipolar NRZ and unipolar RZ codes are non-minimum energy signals.
2. Polar NRZ and Manchester codes are minimum energy signals.

Problem 5.16

The orthonormal matrix that transforms the signal constellation shown in Fig. 5.11(a) of the textbook into the one shown in Fig. 5.11(b) is

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

To prove this statement, we note that the constellations of Fig. 5.11(a) is defined by the four points $\{(\alpha, \alpha), (-\alpha, \alpha), (-\alpha, -\alpha), (\alpha, -\alpha)\}$. The new constellation is defined by

$\mathbf{s}_{i,\text{ rotate}} = \mathbf{Q}\mathbf{s}_i$, which for $i = 1$ yields

$$\mathbf{s}_{1,\text{ rotate}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \qquad \text{for } \alpha = 1.$$

Similarly, $\mathbf{s}_{2,\text{ rotate}} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$

$$\mathbf{s}_{3,\text{ rotate}} = \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix}$$

$$\mathbf{s}_{4,\text{ rotate}} = \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}$$

Hence, the transformation from Fig. 5.11(a) to Fig. 5.11(b) is given by $\mathbf{Q}$, except for a scaling factor.

Problem 5.17

(a) The minimum distance between any two adjacent signal points in the constellation of Fig. P5.17a of the textbook is

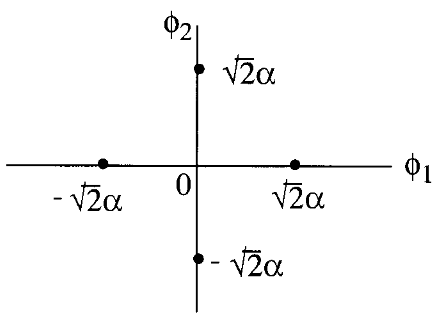
$$d_{\min}^{(a)} = 2\alpha$$

The minimum distance between any two adjacent signal points in the constellation of Fig. P5.17b of the textbook is

$$d_{\min}^{(b)} = \sqrt{(\sqrt{2}\alpha)^2 + (\sqrt{2}\alpha)^2} = 2\alpha$$

which is the same as $d_{\min}^{(a)}$. Hence, the average probability of symbol error using the constellation of Fig. P5.17a is the same as that of Fig. P5.17b.

(b) The constellation of Fig. P5.17a has minimum energy, whereas that of Fig. P5.17b is of non-minimum energy. Applying the minimum energy translate to the constellation of Fig. P5.17b, which involves translating it bodily to the left along the ϕ_1-axis by the amount $\sqrt{2}\alpha$, we get the corresponding minimum energy configuration:



Problem 5.18

Consider a set of three orthogonal signals denoted by $\{s_i(t)\}_{i=0}^{2}$, each with energy E_s. The average of these three signals is

$$a(t) = \frac{1}{3}\sum_{i=0}^{2} s_i(t)$$

Applying the minimum energy translate to the signal set $\{s_i(t)\}_{i=0}^{2}$, we get a new signal set defined by

$$s'_i(t) = s_i(t) - a(t), \qquad i = 0, 1, 2 \tag{1}$$

The signal energy of the new set is

$$E = \int_{-\infty}^{\infty} (s'_i(t))^2 dt$$

$$= \int_{-\infty}^{\infty} s_i^2(t)dt - 2\int_{-\infty}^{\infty} s_i(t)a(t)dt + \int_{-\infty}^{\infty} a(t)dt$$

$$= E_s - \frac{2}{3}E_s + \frac{1}{9}(3E_s)$$

$$= \frac{2}{3}E_s$$

The correlation coefficient ρ_{ij} between the signals $s'_i(t)$ and $s'_j(t)$ is given by

$$\rho_{ij} = \frac{E[s'_i(t)(s'_j(t))]}{E}$$

$$= \frac{3}{2E_s}\int_{-\infty}^{\infty} (s_i(t) - a(t))(s_j(t) - a(t))dt$$

$$= \frac{3}{2E_s}\left(\int_{-\infty}^{\infty} s_i(t)s_j(t)dt - \int_{-\infty}^{\infty} a(t)(s_i(t) + s_j(t))dt + \int_{-\infty}^{\infty} a^2(t)dt\right) \tag{2}$$

Since $s_i(t)$ and $s_j(t)$ are orthogonal by choice, Eq. (2) reduces to

$$\rho_{ij} = \frac{3}{2E_s}\left(0 - \frac{1}{3}E_s - \frac{1}{3}E_s + \frac{1}{9}(3E_s)\right)$$

$$= -\frac{1}{2} \quad \text{for } i \neq j$$

which is the maximum negative correlation that characterizes a simplex signal with $M = 3$. thus, the signal set $\{s'_i(t)\}_{i=0}^{2}$ defined in Eq. (1) is indeed a simplex signal.

To represent the signal set $\{s_i'(t)\}_{i=0}^{2}$ in geometric terms, we use the Gram-Schmidt orthogonalization procedure. Specifically, we first set

$$\phi_0(t) = \frac{s_0'(t)}{\sqrt{E}} \tag{3}$$

or equivalently

$$s_0'(t) = \sqrt{E}\phi_0(t)$$

The projection of $s_1'(t)$ unto $\phi_0(t)$ is

$$\begin{aligned} s_{10} &= \int_{-\infty}^{\infty} s_1'(t)\phi_0(t)dt \\ &= \frac{1}{\sqrt{E}}\int_{-\infty}^{\infty} s_1'(t)s_0'(t)dt \\ &= \sqrt{E}\left(\frac{1}{E}\int_{-\infty}^{\infty} s_1'(t)s_0'(t)dt\right) \\ &= \sqrt{E}\left(-\frac{1}{2}\right) \end{aligned}$$

The second basis function is therefore

$$\begin{aligned} \phi_1(t) &= \frac{s_1'(t) - s_{10}\phi_0(t)}{\sqrt{E - s_{10}^2}} \\ &= \frac{s_1'(t) + (\sqrt{E}/2)(\phi_0(t))}{\sqrt{E-(E/4)}} \\ &= \frac{2}{\sqrt{3E}}\left(s_1'(t) + \frac{\sqrt{E}}{2}\phi_0(t)\right) \end{aligned}$$

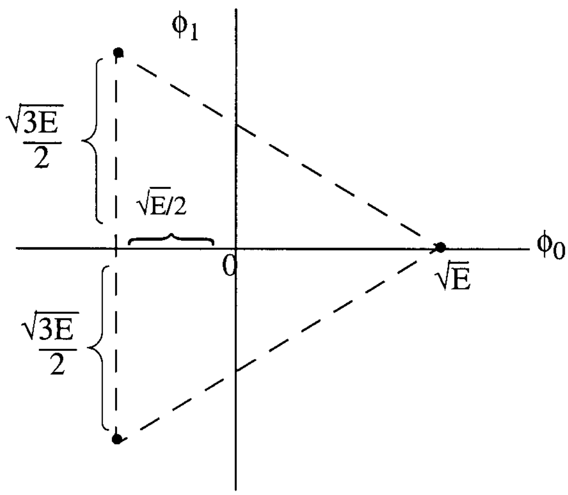
Accordingly, we may express $s_1'(t)$ in terms of the basis functions $\phi_0(t)$ and $\phi_1(t)$ as

$$s'_1(t) = -\frac{\sqrt{E}}{2}\phi_0(t) + \frac{\sqrt{3E}}{2}\phi_1(t) \qquad (4)$$

The remaining signal $s'_2(t)$ may be expressed in terms of $\phi_0(t)$ and $\phi_1(t)$ as

$$s'_2(t) = -\frac{\sqrt{E}}{2}\phi_0(t) - \frac{\sqrt{3E}}{2}\phi_1(t) \qquad (5)$$

Thus, using Eqs. (3) to (5), we may represent the simplex code by the following signal-space diagram:



Problem 5.19

(a) An upper bound on the complementary error function is given by

$$\mathrm{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}\ u}$$

Hence, we may bound the given P_e as follows:

$$P_e = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) < \frac{\exp\left(-\frac{E_b}{N_0}\right)}{2\sqrt{\pi E_b N_0}} = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right) \times \sqrt{\frac{N_0}{\pi E_b}} \qquad (1)$$

For large positive u, we may further simplify the upper bound on the complementary error function as shown here:

$$\text{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}}$$

Correspondingly, we may bound P_e as follows:

$$P_e < \frac{\exp(-E_b/N_0)}{2\sqrt{\pi}} \tag{2}$$

(b) For $E_b/N_0 = 9$, we get the following results:

(i) The exact calculation of P_e yields

$$P_e = \frac{1}{2}\text{erfc}(3)$$

$$= 1.0 \times 10^{-5}$$

(ii) Using the bound in (1), we have the approximate value:

$$P_e \approx \frac{\exp(-9)}{6\sqrt{\pi}}$$

$$= 1.16 \times 10^{-5}$$

(iii) Using the looser bound of (2), we have

$$P_e \approx \frac{\exp(-9)}{2\sqrt{\pi}}$$

$$= 3.48 \times 10^{-5}$$

As expected, the first bound is more accurate than the second bound for calculating P_e.

Problem 5.20

According to Eq. (5.91) of the textbook, the probability of error is over-bounded as follows:

$$P_e(m_i) \le \frac{1}{2} \sum_{\substack{k=1 \\ k \ne i}}^{M} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right), \quad i = 1, 2, \cdots, M \tag{1}$$

where d_{ik} is the distance between message points s_i and s_k. With the M transmitted messages assumed equally likely, the average probability of symbol error is overbounded as follows:

$$\begin{aligned} P_e &= \frac{1}{M}\sum_{i=1}^{M} P_e(m_i) \\ &\le \frac{1}{2M}\sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \ne i}}^{M} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \end{aligned} \tag{2}$$

The second line of Eq. (2) defines the union bound on the average probability of symbol error for any set of M equally likely signals in an AWGN channel. Equation (2) is particularly useful for the special case of a signal set that has a *symmetric geometry*, which is of common occurrence in practice. In such a case, the conditional error probability $P_e(m_i)$ is the same for all i, and so we may simplify Eq. (2) as

$$\begin{aligned} P_e &= P_e(m_i) \\ &\le \frac{1}{2} \sum_{\substack{k=1 \\ k \ne i}}^{M} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right), \quad \text{for all } i \end{aligned} \tag{3}$$

The complementary error function may be upper-bounded as follows:

$$\operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \le \frac{1}{\sqrt{\pi}} \exp\left(-\frac{d_{ik}^2}{2N_0}\right)$$

Hence, we may rewrite Eq. (3) as

$$P_e \le \frac{1}{2\sqrt{\pi}} \sum_{\substack{k=1 \\ k \ne i}}^{M} \exp\left(-\frac{d_{ik}^2}{2N_0}\right) \quad \text{for all } i \tag{4}$$

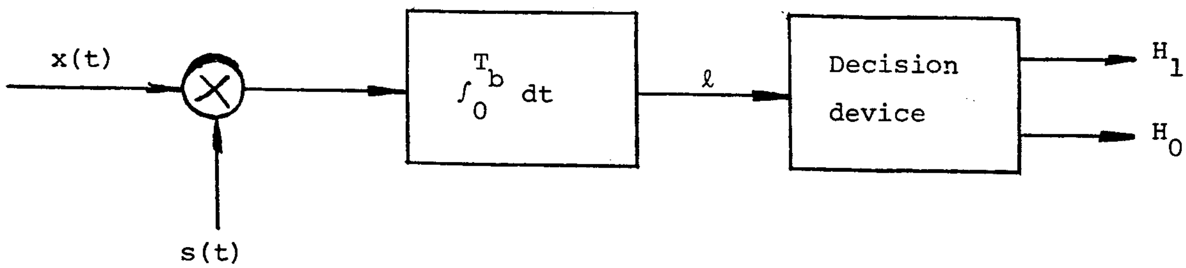
Provided that the transmitted signal energy is high enough compared to the noise spectral density N_0, the exponential term with the smallest distance d_{ik} will dominate the summation in Eq. (4). Accordingly, we may approximate the bound on P_e as

$$P_e \leq \frac{M_{\min}}{2\sqrt{\pi}} \exp\left[-\min_{\substack{i,k \\ i \neq k}}\left(\frac{d_{ik}^2}{2N_0}\right)\right] \tag{5}$$

where $M_{\min}$ is the number of transmitted signals that attain the minimum Euclidean distance for each m_i. Equation (5) describes a simplified form of the union bound for a symmetric signal set, which is easy to calculate.

Problem 6.1

(a) ASK with coherent reception



Denoting the presence of symbol 1 or symbol 0 by hypothesis H_1 or H_0, respectively, we may write

H_1: $x(t) = s(t) + w(t)$

H_0: $x(t) = w(t)$

where $s(t) = A_c\cos(2\pi f_c t)$, with $A_c = \sqrt{2E_b/T_b}$. Therefore,

$$\ell = \int_0^{T_b} x(t)\, s(t)\, dt$$

I4

If $\ell > E_b/2$, the receiver decides in favor of symbol 1. If $\ell < E_b/2$, it decides in favor of symbol 0.

The conditional probability density functions of the random variable L, whose value is denoted by ℓ, are defined by

$$f_{L|0}(\ell|0) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp(-\frac{\ell^2}{N_0 E_b})$$

$$f_{L|1}(\ell|1) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp[-\frac{(\ell - E_b)^2}{N_0 E_b}]$$

The average probability of error is therefore,

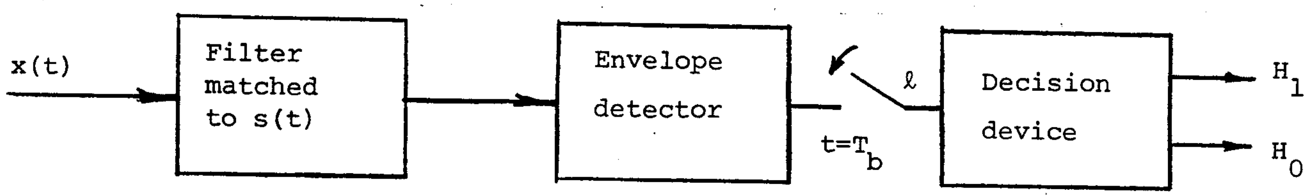
$$P_e = p_0 \int_{E_b/2}^{\infty} f_{L|0}(\ell|0)d\ell + p_1 \int_{-\infty}^{E_b/2} f_{L|1}(\ell|1)d\ell$$

$$= \frac{1}{2}\int_{E_b/2}^{\infty} \frac{1}{\sqrt{\pi N_0 E_b}} \exp(-\frac{\ell^2}{N_0 E_b})d\ell + \frac{1}{2}\int_{-\infty}^{E_b/2} \frac{1}{\sqrt{\pi N_0 E_b}} \exp[-\frac{(\ell - E_b)^2}{N_0 E_b}]d\ell$$

$$= \frac{1}{\sqrt{\pi N_0 E_b}} \int_{E_b/2}^{\infty} \exp(-\frac{\ell^2}{N_0 E_b})d\ell$$

$$= \frac{1}{2} \operatorname{erfc}(\frac{1}{2}\sqrt{E_b/N_0})$$

(b) ASK with noncoherent reception



In this case, the signal s(t) is defined by

$$s(t) = A_c \cos(2\pi f_c t + \theta)$$

where $A_c = \sqrt{2 E_b/T_b}$, and

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

For the case when symbol 0 is transmitted, that is, under hypothesis H_0, we find that the random variable L, at the input of the decision device, is Rayleigh-distributed:

$$f_{L|0}(\ell|0) = \frac{4\ell}{N_0 T_b} \exp(-\frac{2\ell^2}{N_0 T_b})$$

For the case when symbol 1 is transmitted, that is, under hypothesis H_1, we find that the random variable L is Rician-distributed:

$$f_{L|1}(\ell|1) = \frac{4\ell}{N_0T_b}\exp\left(-\frac{\ell^2 + A_c^2T_b^2/4}{N_0T_b/2}\right) I_0\left(\frac{2\ell A_c}{N_0}\right)$$

where $I_0(2\ell A_c/N_0)$ is the modified Bessel function of the first kind of zero order.

Before we can obtain a solution for the error performance of the receiver, we have to determine a value for the threshold. Since symbols 1 and 0 occur with equal probability, the minimum probability of error criterion yields:

$$\exp\left(-\frac{A_c^2T_b}{2N_0}\right) I_0\left(\frac{2\ell A_c}{N_0}\right) \underset{H_0}{\overset{H_1}{\gtrless}} 1 \qquad (1)$$

For large values of E_b/N_0, we may approximate $I_0(2\ell A_c/N_0)$ as follows:

$$I_0\left(\frac{2\ell A_c}{N_0}\right) \simeq \frac{\exp(2\ell A_c/N_0)}{\sqrt{4\pi\ell A_c/N_0}}$$

Using this approximation, we may rewrite Eq. (1) as follows:

$$\exp\left[\frac{A_c(4 - A_cT_b)}{2N_0}\right] \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\frac{4\pi\ell A_c}{N_0}}$$

Taking the logarithm of both sides of this relation, we get

$$\ell \underset{H_0}{\overset{H_1}{\gtrless}} \frac{A_cT_b}{4} + \frac{1}{2}\sqrt{\frac{\pi\ell N_0}{A_c}}$$

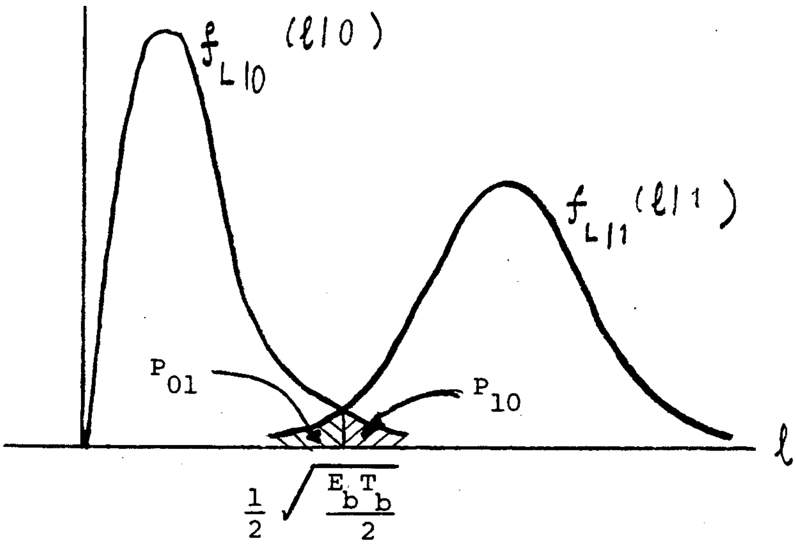
Neglecting the second term on the right hand side of this relation, and using the fact that

$$E_b = \frac{A_c^2T_b}{2}$$

we may write

$$\ell \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\sqrt{\frac{E_bT_b}{2}}$$

The threshold $\frac{1}{2}\sqrt{\frac{E_bT_b}{2}}$ is at the point corresponding to the crossover between the two probability density functions, as illustrated below.



The average probability of error is therefore

$$P_e = p_0 P_{10} + p_1 P_{01}$$

where

$$P_{10} = \int_{\sqrt{E_b T_b}/2\sqrt{2}}^{\infty} f_{L|0}(\ell|0)d\ell$$

$$= \int_{\sqrt{E_b T_b}/2\sqrt{2}}^{\infty} \frac{4\ell}{N_0 T_b} \exp(-\frac{2\ell^2}{N_0 T_b})d\ell$$

$$= \left[-\exp(-\frac{2\ell^2}{N_0 T_b})\right]_{\sqrt{E_b T_b}/2\sqrt{2}}^{\infty}$$

$$= \exp(-\frac{E_b}{4N_0})$$

$$P_{01} = \int_0^{\sqrt{E_b T_b}/2\sqrt{2}} f_{L|1}(\ell|1)d\ell$$

$$= \int_0^{\sqrt{E_b T_b}/2\sqrt{2}} \frac{4\ell}{N_0 T_b} \exp\left(-\frac{\ell^2 + A_c^2 T_b^2/4}{N_0 T_b/2}\right) I_0\left(\frac{2\ell A_c}{N_0}\right)d\ell$$

$$\simeq \int_0^{\sqrt{E_b T_b}/2\sqrt{2}} \frac{4\ell}{N_0 T_b} \exp\left(-\frac{\ell^2 + A_c^2 T_b^2/4}{N_0 T_b/2}\right) \cdot \frac{\exp(2\ell A_c/N_0)}{\sqrt{4\pi\ell A_c/N_0}} d\ell$$

$$= \int_0^{\sqrt{E_b T_b}/2\sqrt{2}} \sqrt{\frac{2\ell}{A_c T_b}} \sqrt{\frac{2}{\pi N_0 T_b}} \exp\left[-\frac{(\ell - A_c T_b/2)^2}{N_0 T_b/2}\right] d\ell \tag{2}$$

The integrand in Eq. (2) is the product of $\sqrt{2\ell/A_c T_b}$ and the probability density function of a Gaussian random variable of mean $A_c T_b/2$ and variance $N_0 T_b/4$. For high values of E_b/N_0, the standard deviation $\sqrt{N_0 T_b/4}$ is much less than the threshold $\sqrt{E_b T_b}/2\sqrt{2}$. Consequently, the area under the portion of the curve from 0 to $\sqrt{E_b T_b}/2\sqrt{2}$ is quite small, that is, $P_{01} \simeq 0$. Then, we may approximate the average probability of error as

$$P_e \simeq p_o P_{10}$$

$$= \frac{1}{2} \exp\left(-\frac{E_b}{4N_0}\right)$$

where it is assumed that symbols 0 and 1 occur with equal probability.

Problem 6.2

The transmitted binary PSK signal is defined by

$$s(t) = \begin{cases} \sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol 1} \\ -\sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol 0} \end{cases}$$

where the basis function $\phi(t)$ is defined by

$$\phi(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$$

The locally generated basis function in the receiver is

$$\phi_{\text{rec}}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \varphi)$$

$$= \sqrt{\frac{2}{T_b}}[\cos(2\pi f_c t)\cos\varphi - \sin(2\pi f_c t)\sin\varphi]$$

where φ is the phase error. The correlator output is given by

$$y = \int_0^{T_b} x(t)\varphi_{\text{rec}}(t)dt$$

where

$$x(t) = s_k(t) + w(t), \qquad k = 1, 2$$

Assuming that f_c is an integer multiple of $1/T_b$, and recognizing that $\sin(2\pi f_c t)$ is orthogonal to $\cos(2\pi f_c t)$ over the interval $0 \le t \le T_b$, we get

$$y = \pm\sqrt{E_b}\cos\varphi + W$$

when the plus sign corresponds to symbol 1 and the minus sign corresponds to symbol 0, and W is a zero-mean Gaussian variable of variance $N_0/2$. Accordingly, the average probability of error of the binary PSK system with phase error φ is given by

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b \cos\varphi}{N_0}}\right)$$

When $\varphi = 0$, this formula reduces to that for the standard PSK system equipped with perfect phase recovery. At the other extreme, when $\varphi = \pm 90°$, P_e attains its worst value of unity.

Problem 6.3

(a) The noiseless PSK signal is given by

$$s(t) = A_c\cos[2\pi f_c t + k_p m(t)]$$

$$= A_c\cos(2\pi f_c t)\cos[k_p m(t)] - A_c\sin(2\pi f_c t)\sin[k_p m(t)]$$

Since $m(t) = \pm 1$, it follows that

$$\cos[k_p m(t)] = \cos(\pm k_p) = \cos(k_p)$$

$$\sin[k_p m(t)] = \sin(\pm k_p) = \pm\sin(k_p) = m(t)\sin(k_p)$$

Therefore,

$$s(t) = A_c\cos(k_p)\cos(2\pi f_c t) - A_c m(t)\sin(k_p)\sin(2\pi f_c t) \tag{1}$$

The VCO output is

$$r(t) = A_v\sin[2\pi f_c t + \theta(t)]$$

The multiplier output is therefore

$$r(t)s(t) = \frac{1}{2}A_c A_v\cos(k_p)\{\sin[\theta(t)] + \sin[4\pi f_c t+\theta(t)]\}$$
$$- \frac{1}{2}A_c A_v m(t)\sin(k_p)\{\cos[\theta(t)] + \cos[4\pi f_c t+\theta(t)]\}$$

The loop filter removes the double-frequency components, producing the output

$$e(t) = \frac{1}{2}A_c A_v\cos(k_p)\sin[\theta(t)] - \frac{1}{2}A_c A_v m(t)\sin(k_p)\cos[\theta(t)]$$

Note that if $k_p = \pi/2$, (i.e., the carrier is fully deviated), there would be no carrier component for the PLL to track.

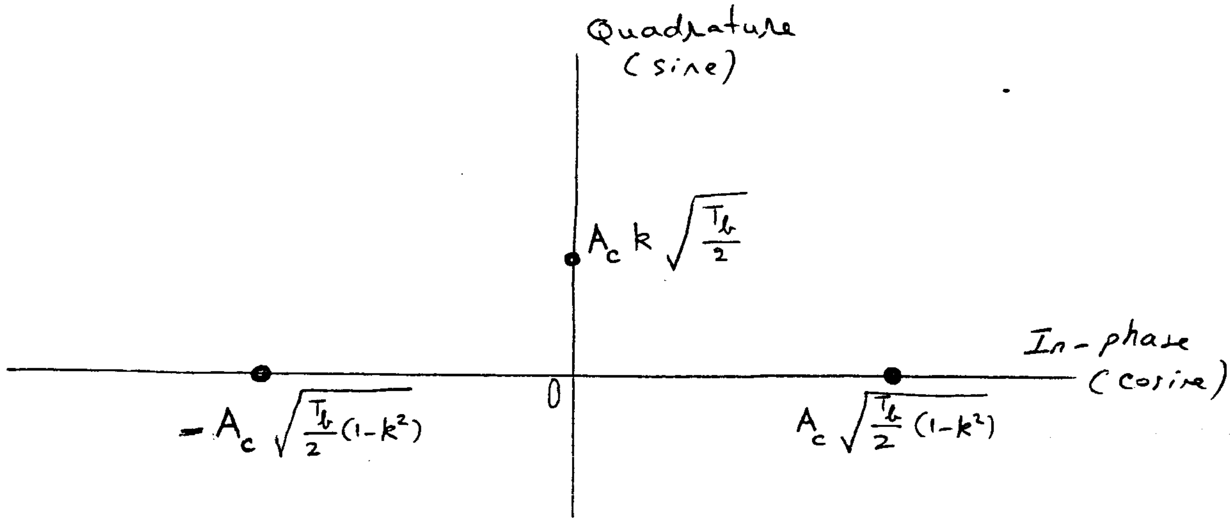
(b) Since the error signal tends to drive the loop into lock (i.e., $\theta(t)$ approaches zero), the loop filter output reduces to

$$e(t) = -\frac{1}{2}A_c A_v\sin(k_p)m(t)$$

which is proportional to the desired data signal m(t). Hence, the phase-locked loop may be used to recover m(t).

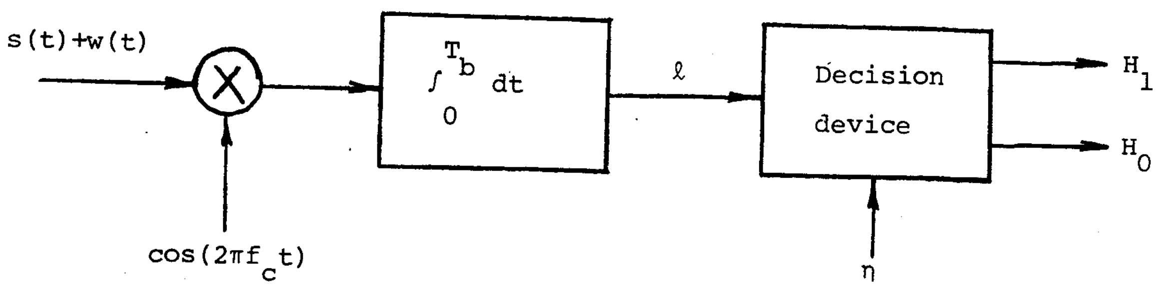
Problem 6.4

(a) The signal-space diagram of the scheme described in this problem is two-dimensional, as shown by



This signal-space diagram differs from that of the conventional PSK signaling scheme in that it is two-dimensional, with a new signal point on the quadrature axis at $A_c k\sqrt{T_b/2}$. If k is reduced to zero, the above diagram reduces to the same form as that shown in Fig. 8.14.

(b)



The signal at the decision device input is

$$\ell = \pm \frac{A_c}{2} \sqrt{1-k^2}\, T_b + \int_0^{T_b} w(t) \cos(2\pi f_c t)dt \tag{1}$$

Therefore, following a procedure similar to that used for evaluating the average probability of error for a conventional PSK system, we find that for the system defined by Eq. (1) the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b(1-k^2)/N_0}\right)$$

where $E_b = \frac{1}{2} A_c^2 T_b$.

(c) For the case when $P_e = 10^{-4}$ and $k^2 = 0.1$, we get

$$10^{-4} = \frac{1}{2}\,\mathrm{erfc}(u)$$

where $u^2 = \dfrac{0.9\, E_b}{N_0}$

Using the approximation

$$\mathrm{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}\, u}$$

we obtain

$$\exp(-u^2) - 2\sqrt{\pi} \times 10^{-4}\, u = 0$$

The solution to this equation is u = 2.64. The corresponding value of E_b/N_0 is

$$\frac{E_b}{N_0} = \frac{(2.64)^2}{0.9} = 7.74$$

Expressed in decibels, this value corresponds to 8.9 dB.

(d) For a conventional PSK system, we have

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{E_b/N_0}\right)$$

In this case, we find that

$$\frac{E_b}{N_0} = (2.64)^2 = 6.92$$

Expressed in decibels, this value corresponds to 8.4 dB. Thus, the conventional PSK system requires 0.5 dB less in E_b/N_0 then the modified scheme described herein.

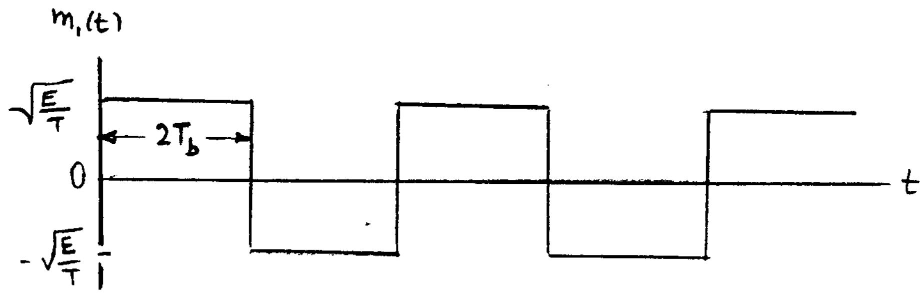
Problem 6.5

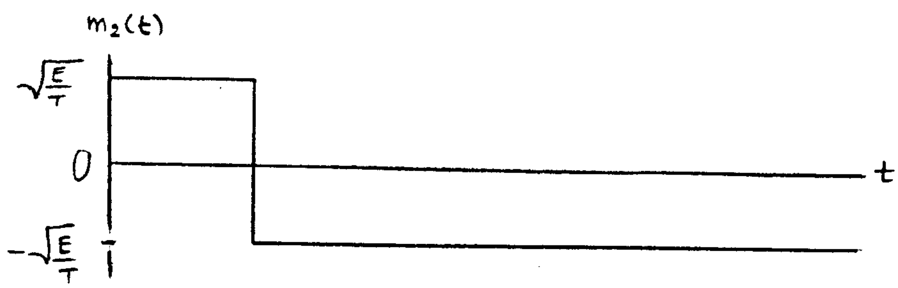
(a) The QPSK wave can be expressed as

$$s(t) = m_1(t)\cos(2\pi f_c t) + m_2(t)\sin(2\pi f_c t).$$

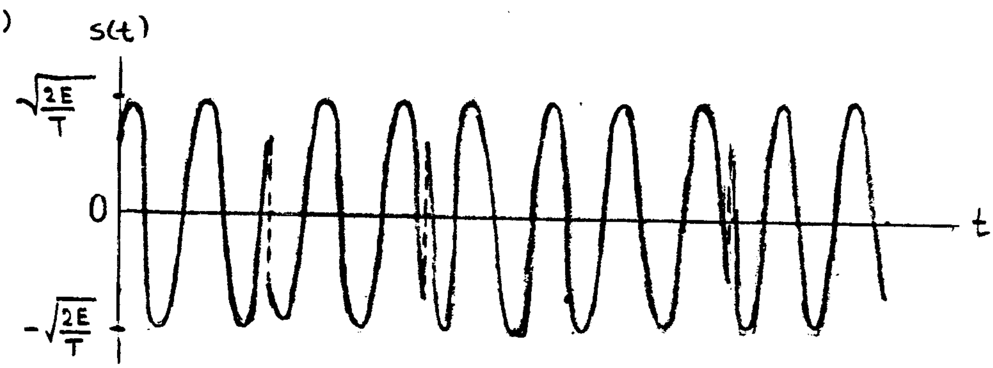
Dividing the binary wave into dibits and finding $m_1(t)$ and $m_2(t)$ for each dibit:

dibit	11	00	10	00	10
$m_1(t)$	$\sqrt{E/T}$	$-\sqrt{E/T}$	$\sqrt{E/T}$	$-\sqrt{E/T}$	$\sqrt{E/T}$
$m_2(t)$	$\sqrt{E/T}$	$-\sqrt{E/T}$	$-\sqrt{E/T}$	$-\sqrt{E/T}$	$-\sqrt{E/T}$





(b)



Problem 6.6

Let P_{eI} = average probability of symbol error in to the in-phase channel

P_{eQ} = average probability of symbol error in to the quadrature channel

Since the individual outputs of the in-phase and quadrature channels are statistically independent, the overall average probability of correct reception is

$$P_c = (1 - P_{eI})(1 - P_{eQ})$$
$$= 1 - P_{eI} - P_{eQ} + P_{eI} P_{eQ}$$

The overall average probability of error is therefore

$$P_e = 1 - P_c$$
$$= P_{eI} + P_{eQ} - P_{eI} P_{eQ}$$

Problem 6.7

Let **r** denote the received signal vector. Suppose that the signal corresponding to message point $\underset{\sim}{m}_1$ is transmitted. Then, referring to the signal-space diagram of Fig. 1, the conditional probability of error is given

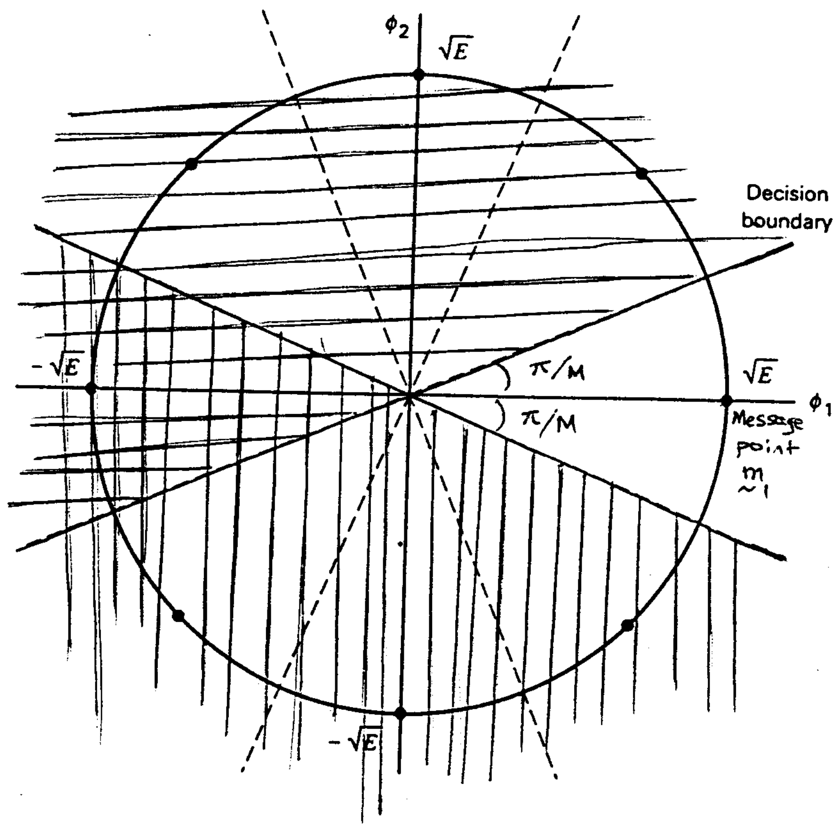
$$
\begin{aligned}
P_{e|\underset{\sim}{m}_1} &= P(r \text{ lies in shaded region}) \\
&= P(r \text{ lies in } \equiv) + P(R \text{ lies in } \|\|) \\
&\quad - P(r \text{ lies in } \|\|) \\
&= \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin\frac{\pi}{M}\right) + \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin\frac{\pi}{M}\right) \\
&\quad - P(r \text{ lies in } \|\|)
\end{aligned}
$$

Hence,

$$
P_{e|\underset{\sim}{m}_1} < \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin\frac{\pi}{M}\right)
$$

Assuming that all the message points are equally likely to be transmitted, we have $P_e = P_{e|\underset{\sim}{m}_1}$, and so

$$
P_e < \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin\frac{\pi}{M}\right)
$$

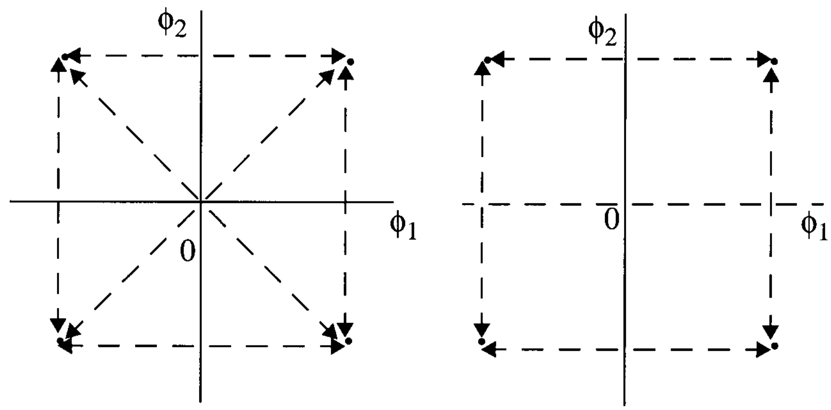


ϕ_2
$\sqrt{E}$
Decision
boundary
$-\sqrt{E}$
π/M
π/M
$\sqrt{E}$
ϕ_1
Message
point
$\underset{\sim}{m}_1$
$-\sqrt{E}$

Figure 1

Problem 6.8

Figures 6.10 and 6.10b of the textbook, reproduced here for convenience of presentation, depict the signal-space diagrams of QPSK and offset QPSK signals, respectively:



(a) QPSK (b) Offset QPSK

Figure 1

The two parts of this figure clearly show that the signal-space structure of the offset QPSK is basically the same as that of the standard QPSK. They only differ from each other in the way in which transition takes place from one signal point to another. Accordingly, they have the same power spectral density, as shown by

$$S(f) = E_b[\text{sinc}^2(2T_b(f-f_c)) + \text{sinc}^2(2T_b(f+f_c))]$$

where T_b is the bit duration and f_c is the carrier frequency.

Problem 6.9

(a) In vestigial-sideband (VSB) modulation, there are two basis functions:

- The double-bandwidth sinc function, defined by

$$\phi_1(t) = \sqrt{\frac{1}{T}}\text{sinc}\left(\frac{2t}{T}\right)\cos(2\pi f_c t) \tag{1}$$

where T is the symbol period and f_c is the carrier frequency.

- The Hilbert transform of $\phi_1(t)$, defined by

$$\phi_2(t) = \hat{\phi}_1(t)$$

$$= \sqrt{\frac{1}{T}}\operatorname{sinc}\left(\frac{2t}{T}\right)\sin(2\pi f_c t) \tag{2}$$

where it is assumed that $f_c > 2/T$.

(Here we have made use of the Hilbert-transform pair listed as entry 1 in Table A6.4, with the low-pass signal $m(t)$ set equal to $\sqrt{1/T}\operatorname{sinc} 2(t/T)$.)

The basis functions (1) and (2) imply the use of single-sideband modulation, which (as discussed in Chapter 2) is a special form of vestigial sideband modulation. We have chosen these definitions merely to simplify the discussion. The use of VSB substitutes a realizable function for the sinc function that is unrealizable in practice.

Based on the definitions of the basis functions $\phi_1(t)$ and $\phi_2(t)$ given in Eqs. (1) and (2), it may be tempting to choose $2/T$ as the symbol rate for successive transmission of binary symbols using binary VSB. However, such a choice of signaling destroys the orthonormality of $\phi_1(t)$ and $\phi_2(t)$; that is,

$$\int_0^T \phi_i(t)\phi_j\left(t-\frac{T}{2}\right)dt \neq \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$$

To maintain orthogonality of $\phi_1(t)$ and $\phi_2(t)$, successive translations of these basis functions must be integer multiples of $1/T$, as shown by

$$\int_0^T \phi_i(t)\phi_j(t-kT)dt = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$$

for any integer k.

Suppose, however, we restrict k to assume only odd integer values, and choose the carrier frequency f_c to be an odd integer multiple of $1/2T$, that is,

$$f_c = \frac{l}{2T}, \qquad l = \text{odd integer} \tag{3}$$

We then have the following two properties:

(i) $$\int_0^T \phi_1(t)\phi_2\left(t - \frac{kT}{2}\right)dt = 0 \text{ for all odd integer } k \tag{4}$$

(ii) $$\sin\left(2\pi f_c\left(t - \frac{kT}{2}\right)\right) = \sin(2\pi f_c t)\cos(kl\pi/2) - \cos(2\pi f_c t)\sin(kl\pi/2)$$

$$= \cos(2\pi f_c t) \qquad \text{for} \quad \begin{array}{l} k = \text{odd integer} \\ l = \text{odd integer} \end{array}$$

With such a choice, the implementation of the digital VSB transmission system is equivalent to a time-varying one-dimensional data transmission system, which operates at the rate of $2/T$ dimensions per second.

(b) The optimum receiver for the digital VSB transmission system just described consists of a pair of matched filters, that are matched to the two basis functions $\phi_1(t)$ and $\phi_2(t)$ as defined in Eqs. (1) and (2). However, in order to conform to the design choices imposed on integer k and carrier frequency f_c as described in Eqs. (4) and (3), the instants of time at which the two matched filter outputs are sampled are staggered by $T/2$ with respect to each other. The two sequences of samples so obtained are subsequently interleaved so as to produce a single one-dimensional data stream as the overall receiver output. The delay by $T/2$ is identical to what is actually done in the offset QPSK, thereby establishing the equivalence of the digital VSB system to the offset QPSK.

Problem 6.10

Assuming that modulator initially resides in a phase state of zero, we may construct the following sequence of events in response to the input sequence 01101000.

Step k	Phase θ_{k-1} (radians)	Input dibit	Phase change $\Delta\theta_k$ (radians)	Transmitted phase θ_k (radians)
1	0	01	$3\pi/4$	$3\pi/4$
2	$3\pi/4$	10	$-\pi/4$	$\pi/2$
3	$\pi/2$	10	$-\pi/4$	$\pi/4$
4	$\pi/4$	00	$\pi/4$	$\pi/2$

Problem 6.11

The output of a π/4-shifted QPSK modulator may be expressed in terms of its in-phase and quadrature components as

$$s(t) = \sqrt{\frac{2E}{T}}\cos(i\pi/4)\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin(i\pi/4)\sin(2\pi f_c t) \qquad i = 0, 1, 2, \ldots, 7$$

The different values of interger i correspond to the eight possible phase states in which the modulator can reside. But, unlike the 8-PSK modulator, the phase states of the π/4-shifted QPSK modulator are divided into two QPSK groups that are shifted by π/4 relative to each other.

Therefore, $s_I(t) = \sqrt{\frac{2E}{T}}\cos(i\pi/4)$

$$s_Q(t) = \sqrt{\frac{2E}{T}}\sin(i\pi/4)$$

The orthonormal-basis functions for π/4-shifted QPSK may be defined as

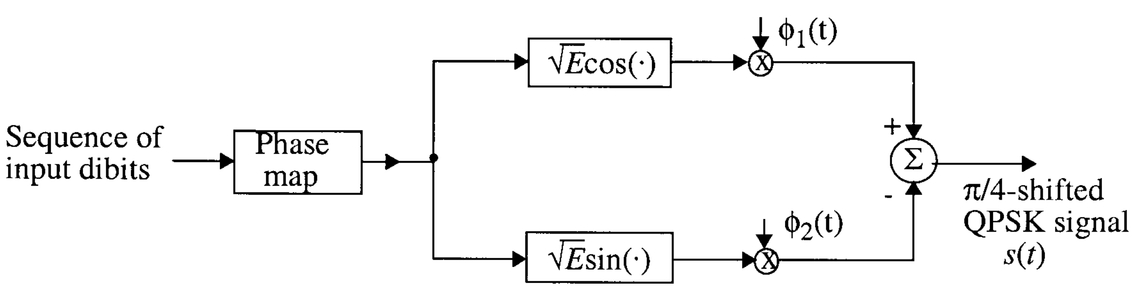
$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

Then the π/4-shifted QPSK signal is defined in terms of these two basis functions as

$$s(t) = \sqrt{E}\cos(i\pi/4)\phi_1(t) - \sqrt{E}\sin(i\pi/4)\phi_2(t)$$

On the basis of this representation, we may thus set up the following scheme for generating π/4-shifted QPSK signals:



Problem 6.12

A $\pi/4$-shifted DQPSK signal can be expressed as follows:

$$s(t) = \sqrt{\frac{2E}{T}}\cos(\phi_{k-1}+\Delta\phi_k)\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin(\phi_{k-1}+\Delta\phi_k)\sin(2\pi f_c t)$$

$$= \sqrt{\frac{2E}{T}}\cos(2\pi f_c t + \phi_{k-1} + \Delta\phi_k)$$

where $\phi_{k-1} + \Delta\phi_k = \phi_k$ and ϕ_{k-1} is the absolute angle of symbol k-1, and $\Delta\phi_k$ is the differentially encoded phase change. In the demodulation process, the change in phase ϕ_k occurring over one symbol interval needs to be determined.

If we demodulate the $\pi/4$-shifted DQPSK signal using a FM discriminator, the output of the FM discriminator is given by

$$v_{\text{out}}(t) = K\frac{d[2\pi f_c t + \phi_k]}{dt}$$

$$= K\left[2\pi f_c + \frac{d\phi_k}{dt}\right]$$

$$= K[2\pi f_c + \Delta\phi_k]$$

where K is a constant. In a balanced FM discriminator, the DC offset $2\pi f_c K$ will not appear at the output. Hence, the output of the FM discriminator is $K\Delta\phi_k$.

Problem 6.13

The output of a $\pi/4$-shifted DQPSK modulator may be expressed as

$$s(t) = \sqrt{\frac{2E}{T}}\cos(\theta_{k-1}+\Delta\theta_k)\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin(\theta_{k-1}+\Delta\theta_k)\sin(2\pi f_c t)$$

$$= \sqrt{\frac{2E}{T}}[\cos\theta_{k-1}\cos\Delta\theta_k - \sin\theta_{k-1}\sin\Delta\theta_k]\cos(2\pi f_c t)$$

$$-\sqrt{\frac{2E}{T}}[\sin\theta_{k-1}\cos\Delta\theta_k + \cos\theta_{k-1}\sin\Delta\theta_k]\sin(2\pi f_c t)$$

Let $I_k = \cos(\theta_{k-1} + \Delta\theta_k)$ and $Q_k = \sin(\theta_{k-1} + \Delta\theta_k)$. We may then write

$$I_k = \cos\theta_{k-1}\cos\Delta\theta_k - \sin\theta_{k-1}\cos\Delta\theta_k$$

$$= \cos(\theta_{k-2} + \Delta\theta_{k-1})\cos\Delta\theta_k - \sin(\theta_{k-2} + \Delta\theta_{k-1})\cos\Delta\theta_k$$

from which we readily deduce the recursion

$$I_k = I_{k-1}\cos\Delta\theta_k - Q_{k-1}\sin\Delta\theta_k$$

Similarly, we may show that

$$Q_k = \sin\theta_{k-1}\cos\Delta\theta_k + \cos\theta_{k-1}\sin\Delta\theta_k$$

$$= Q_{k-1}\cos\Delta\theta_k + I_{k-1}\sin\Delta\theta_k$$

From the definition of I_k and Q_k, we immediately see that I_k and Q_k may also be expressed as

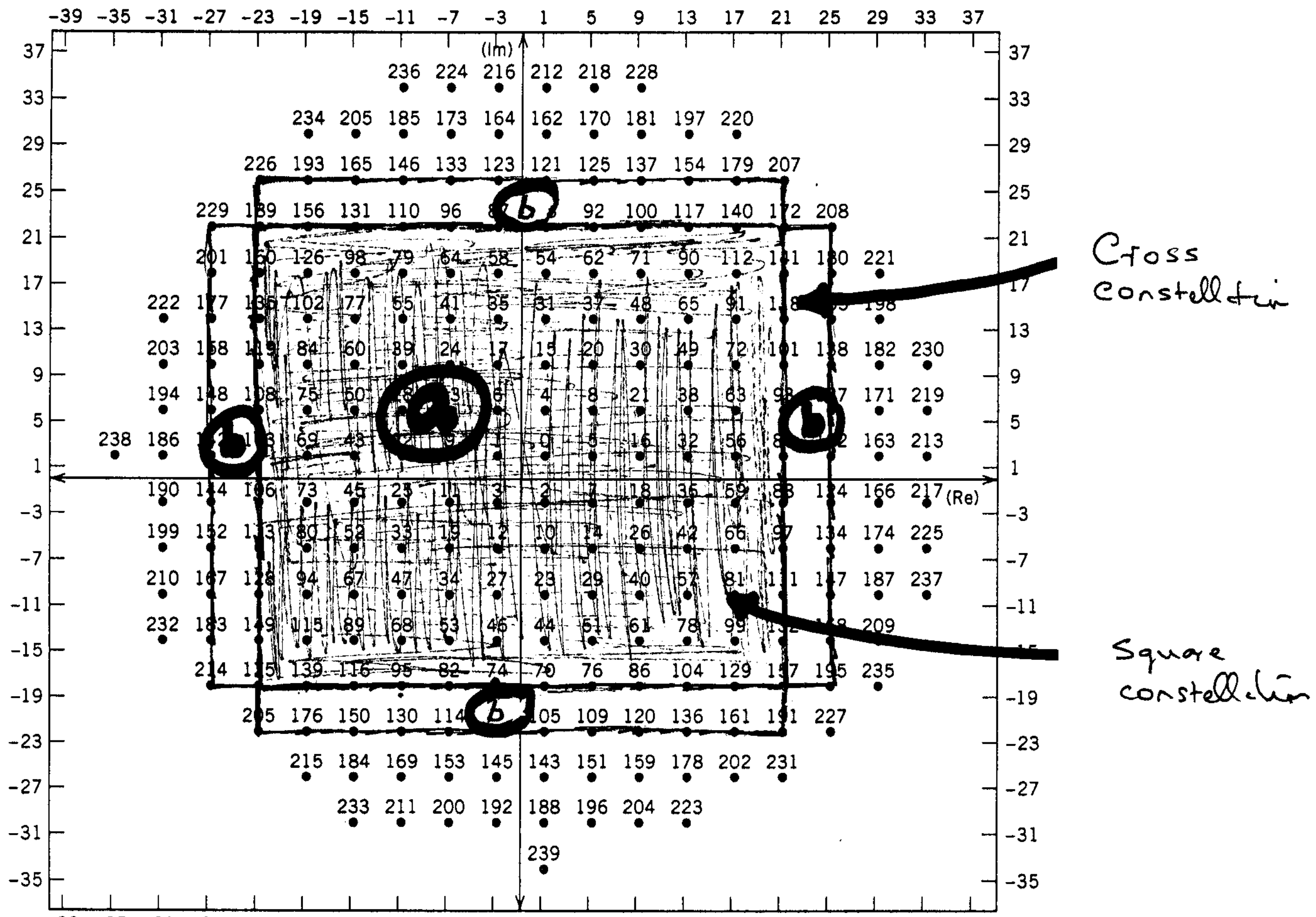
$$I_k = \cos\theta_k$$

and

$$Q_k = \sin\theta_k$$

which are the desired results.

Problem 6.14



Quarter - superconstellation of V.34 modem

Problem 6.15

The transmission bandwidth of 256-QAM signal is

$$B = \frac{2R_b}{\log_2 M}$$

where R_b is the bit rate given by $1/T_b$ and $M = 256$. Thus

$$B_{256} = \frac{2(1/T_b)}{\log_2 256} = \frac{2}{16T_b} = \frac{1}{8T_b}$$

The transmission bandwidth of 64-QAM is

$$B_{64} = \frac{2(1/T_b)}{\log_2 64} = \frac{2}{8T_b} = \frac{1}{4T_b}$$

Hence, the bandwidth advantage of 256-QAM over 64-QAM is

$$\frac{1}{4T_b} - \frac{1}{8T_b} = \frac{1}{8T_b}$$

The average energy of 256-QAM signal is

$$E_{256} = \frac{2(M-1)E_0}{3} = \frac{2(256-1)E_0}{3}$$

$$= 170E_0$$

where E_0 is the energy of the signal with the lowest amplitude. For the 64-QAM signal, we have

$$E_{64} = \frac{2(63)}{3}E_0 = 42E_0$$

Therefore, the increase in average signal energy resulting from the use of 256-QAM over 64-QAM, expressed in dBs, is

$$10\log_{10}\left(\frac{170E_0}{42E_0}\right) \approx 10\log_{10}(4)$$

$$= 6 \text{ dB}$$

Problem 6.16

The probability of symbol error for 16-QAM is given by

$$P_e = 2\left(1 - \frac{1}{\sqrt{M}}\right)\mathrm{erfc}\left(\sqrt{\frac{3E_{\mathrm{av}}}{2(M-1)N_0}}\right)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = 2\left(1 - \frac{1}{4}\right)\mathrm{erfc}\left(\sqrt{\frac{3E_{\mathrm{av}}}{30N_0}}\right)$$

Solving this equation for E_{av}/N_0,

$$\begin{aligned}\frac{E_{\mathrm{av}}}{N_0} &= 58 \\ &= 17.6\mathrm{dB}\end{aligned}$$

The probability of symbol error for 16-PSK is given by

$$P_e = \mathrm{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/M)\right)$$

Setting $P_e = 10^{-3}$, we get

$$10^{-3} = \mathrm{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/16)\right)$$

Solving this equation for E/N_0, we get

$$\frac{E}{N_0} = 142 = 21.5\mathrm{dB}$$

Hence, on the average, the 16-PSK demands 21.5 - 17.6 = 3.9 dB more symbol energy than the 16-QAM for $P_e = 10^{-3}$.

Thus the 16-QAM requires about 4 dB less in signal energy than the 16-PSK for a fixed N_0 and $P_e = 10^{-3}$, However, for this advantage of the 16-QAM over the 16-PSK to be realized, the channel must be linear.

Problem 6.17

(a) An M-ary QAM signal is defined by

$$s_k(t) = \sqrt{\frac{2E}{T}}a_k\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}a_k\sin(2\pi f_c t) \tag{1}$$

We can redefine the M-ary QAM signal in terms of a general pulse-shaping function $g(t)$ as

$$s_k(t) = a_k g(t-kT)\cos(2\pi f_c t) - b_k g(t-kT)\sin(2\pi f_c t) \tag{2}$$

The M-ary QAM signal $s(t)$ for an infinite succession of input symbols can be expressed as

$$s(t) = \sum_{k=-\infty}^{\infty} s_k(t)$$

$$= \sum_{k=-\infty}^{\infty} \{a_k g(t-kT)\cos(2\pi f_c t) - b_k g(t-kT)\sin(2\pi f_c t)\}$$

$$= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} (a_k + jb_k)g(t-kT)e^{j2\pi f_c t}\right\}$$

$$= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} A_k g(t-kT)e^{j2\pi f_c t}\right\} \tag{3}$$

where A_k is a complex number defined by

$$A_k = a_k + jb_k$$

By multiplying Eq. (3) by $\exp(-j2\pi f_c kT) \times \exp(j2\pi f_c kT)$, we get

$$s(t) = R_e\left\{\sum_{k=-\infty}^{\infty} A_k g(t-kT)\exp(-j2\pi f_c kT)\exp(j2\pi f_c t)\right\}$$

$$= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} \tilde{A}_k \tilde{g}(t-kT)\right\} \tag{4}$$

where $\tilde{A}_k = A_k \exp(j2\pi f_c kT)$

$$\tilde{g}(t) = g(t)\exp(j2\pi f_c t)$$

The scalar $\tilde{A}_k$ is a rotated version of the complex representation of the kth transmitted signal.

Equation (4), representing a QAM signal, appears to be carrierless, therefore, it is equivalent to a CAP system.

(b) A CAP signal is defined as

$$s(t) = \mathrm{Re}\left\{ \sum_{k=-\infty}^{\infty} \tilde{A}_k \tilde{g}(t-kT) \right\}$$

$$= \mathrm{Re}\left\{ \sum_{k=-\infty}^{\infty} \tilde{A}_k \exp(j2\pi f_c kT) g(t-kT) \exp(j2\pi f_c (t-kT)) \right\}$$

$$= \mathrm{Re}\left\{ A_k \sum_{k=-\infty}^{\infty} g(t-kT) \exp(j2\pi f_c t) \right\}$$

$$= \sum_{k=-\infty}^{\infty} a_k g(t-kT)\cos(2\pi f_c t) - \sum_{k=-\infty}^{\infty} b_k g(t-kT)\sin(2\pi f_c t) \tag{5}$$

Now the pulse shaping functions of CAP signal, $g(t - kT)$, may be replaced by $\sqrt{\frac{2E}{T}}$ for $0 \leq t \leq T$, and the formulaton in Eq. (5) can be rewritten as

$$s(t) = \sum_{k=-\infty}^{\infty} \sqrt{\frac{2E}{T}} a_k \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} b_k \sin(2\pi f_c t) \tag{6}$$

The kth signal of the signal $s(t)$ defined in Eq. (6) is given by

$$s_k(t) = \sqrt{\frac{2E}{T}} a_k \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} b_k \sin(2\pi f_c t), \quad 0 \leq t \leq T \tag{7}$$

The signal formulation given in Eq. (7) is recognized as the M-ary QAM signal of Eq. (1).

Problem 6.18

The CAP signal can be expressed as

$$s(t) = \sum_{n=-\infty}^{\infty} a_n p(t-nT) - \sum_{n=-\infty}^{\infty} b_n \hat{p}(t-nT)$$

where $\hat{p}(t)$ is the Hilbert transform of the pulse $p(t)$, and $A_n = a_n + jb_n$. The CAP signal $s(t)$ can be written in the equivalent form:

$$s(t) = \left[\sum_{n=-\infty}^{\infty} a_n \delta(t-nT)\right] \star p(t)$$

$$-\left[\sum_{n=-\infty}^{\infty} b_n \delta(t-nT)\right] \star \hat{p}(t)$$

where $\delta(t)$ is the delta function, and the star denotes convolution in the time domain. Hence, the power spectral density of $s(t)$ is

$$S_s(f) = \frac{\sigma_a^2}{T}|P(f)|^2 + \frac{\sigma_b^2}{T}|\hat{P}(f)|^2$$

where σ_a^2 and σ_b^2 are the variances of symbol a_k and b_k, respectively, where $p(t) \rightleftharpoons P(f)$ and $\hat{p}(t) \rightleftharpoons \hat{P}(f)$. Noting that $|\hat{P}(f)| = P(f)$, we thus have

$$S_s(f) = \frac{\sigma_a^2 + \sigma_b^2}{T}|P(f)|^2$$

Next, noting that

$$\sigma_a^2 + \sigma_b^2 = \sigma_A^2 = \frac{1}{L}\sum_{i=1}^{L}(a_i^2 + b_i^2)$$

we finally get

$$S_s(f) = \sigma_A^2|P(f)|^2$$

Problem 6.19

From the defining equations (6.74) and (6.75) of the textbook, we have

$$p(t) = g(t)\cos(2\pi f_c t) \tag{1}$$

and

$$\hat{p}(t) = g(t)\sin(2\pi f_c t) \tag{2}$$

Applying the Fourier transform to Eqs. (1) and (2), we get

$$P(f) = \frac{1}{2}[G(f-f_c)+G(f+f_c)] \tag{3}$$

and

$$\hat{P}(f) = \frac{1}{2j}[G(f-f_c)-G(f+f_c)] \tag{4}$$

Accordingly, we may determine $p(t)$ and $\hat{p}(t)$ by proceeding as follows:

- Given $G(f)$, use Eqs. (3) and (4) to evaluate $P(f)$ and $\hat{P}(f)$.
- Using the inverse Fourier transform, compute $p(t) = F^{-1}[P(f)]$ and $\hat{p}(t) = F^{-1}[\hat{P}(f)]$.

Problem 6.20

For binary FSK, the two signal vectors are

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}$$

$$\mathbf{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

where E_b is the signal energy per bit. The inner products of these two signal vectors with the observation vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

are as follows, respectively,

$$(\mathbf{x},\mathbf{s}_1) = \sqrt{E_b}\, x_1$$

$$(\mathbf{x},\mathbf{s}_2) = \sqrt{E_b}\, x_2$$

where $(\mathbf{x}, \mathbf{s}_i) = \mathbf{x}^T\mathbf{s}_i$ for i=1,2. The condition

$$\mathbf{x}^T\mathbf{s}_1 > \mathbf{x}^T\mathbf{s}_2$$

is therefore equivalent to

$$\sqrt{E_b}\, x_1 > \sqrt{E_b}\, x_2$$

Cancelling the common factor $\sqrt{E_b}$, we get

$$x_1 > x_2$$

which is the desired condition for making a decision in favor of symbol 1.

Problem 6.21

The bit duration is

$$T_b = \frac{1}{2.5\text{x}10^6\text{Hz}} = 0.4\ \mu s$$

The signal energy per bit is

$$E_b = \frac{1}{2} A_c^2 T_b$$
$$= \frac{1}{2}(10^{-6})^2 \text{ x } 0.4 \text{ x } 10^{-6} = 2 \text{ x } 10^{-19} \text{ joules}$$

(a) Coherent Binary FSK

The average probability of error is

$$P_e = \frac{1}{2}\,\text{erfc}(\sqrt{E_b/2N_0})$$

$$= \frac{1}{2}\,\text{erfc}(\sqrt{2\text{x}10^{-19}/4\text{x}10^{-20}})$$

$$= \frac{1}{2}\,\text{erfc}(\sqrt{5})$$

Using the approximation

$$\text{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}\,u}$$

we obtain the result

$$P_e = \frac{1}{2}\frac{\exp(-5)}{\sqrt{5\pi}} = 0.85 \text{ x } 10^{-3}$$

(b) MSK

$$P_e = \text{erfc}(\sqrt{E_b/N_0})$$

$$= \text{erfc}(\sqrt{10})$$

$$\simeq \frac{\exp(-10)}{\sqrt{10\pi}}$$

$$= 0.81 \times 10^{-5}$$

(c) Noncoherent Binary FSK .

$$P_e = \frac{1}{2}\exp(-\frac{E_b}{2N_0})$$

$$= \frac{1}{2}\exp(-5)$$

$$= 3.37 \times 10^{-3}$$

Problem 6.22

(a) The correlation coefficient of the signals $s_0(t)$ and $s_1(t)$ is

$$\rho = \frac{\int_0^{T_b} s_0(t)s_1(t)dt}{[\int_0^{T_b} s_0^2(t)dt]^{1/2}\ [\int_0^{T_b} s_1^2(t)dt]^{1/2}}$$

$$= \frac{A_c^2 \int_0^{T_b} \cos[2\pi(f_c+\frac{1}{2}\Delta f)t]\cos[2\pi(f_c-\frac{1}{2}\Delta f)t]\ dt}{[\frac{1}{2}A_c^2T_b]^{1/2}\ [\frac{1}{2}A_c^2T_b]^{1/2}}$$

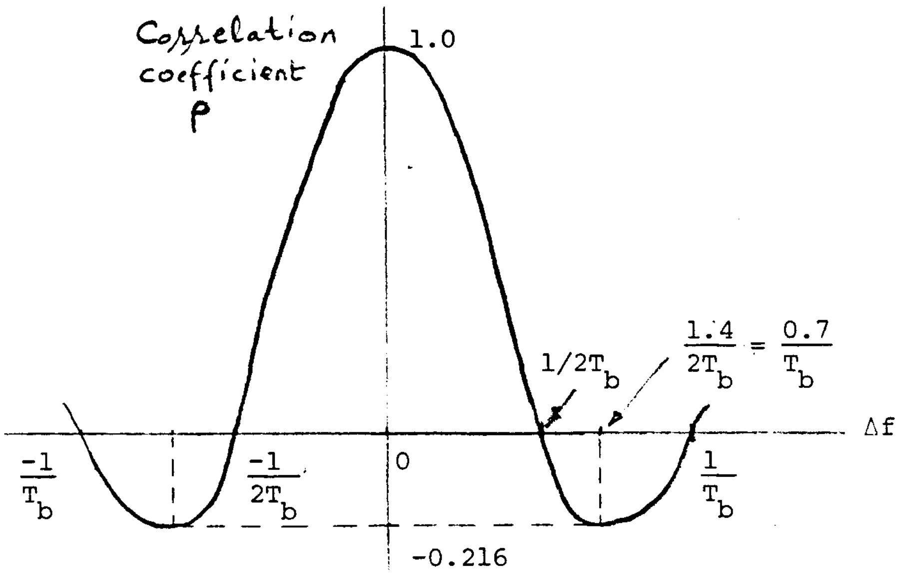
$$= \frac{1}{T_b}\int_0^{T_b} [\cos(2\pi\Delta f t) + \cos(4\pi f_c t)]\ dt$$

$$= \frac{1}{2\pi T_b}\left[\frac{\sin(2\pi\Delta f T_b)}{\Delta f} + \frac{\sin(4\pi f_c T_b)}{2f_c}\right] \tag{1}$$

Since $f_c \gg \Delta f$, then we may ignore the second term in Eq. (1), obtaining

$$\rho \simeq \frac{\sin(2\pi\Delta f T_b)}{2\pi T_b\ \Delta f} = \mathrm{sinc}(2\Delta f T_b)$$

(b) The dependence of ρ on Δf is as shown in Fig. 1.



Correlation coefficient ρ
1.0
1/2T_b
1.4/2T_b = 0.7/T_b
Δf
-1/T_b
-1/2T_b
0
1/T_b
-0.216

Fig. 1

$s_0(t)$ and $s_1(t)$ are othogonal when $\rho = 0$. Therefore, the minimum value of Δf for which they are orthogonal, is $1/2T_b$.

(c) The average probability of error is given by

$$E_b = \frac{1}{2}\,\mathrm{erfc}(\sqrt{E_b(1-\rho)/2N_0})$$

The most negative value of ρ is -0.216, occuring at $\Delta f = 0.7/T_b$. The minimum value of P_e is therefore

$$P_{e,min} = \frac{1}{2}\,\mathrm{erfc}(\sqrt{0.608E_b/N_0})$$

(d) For a coherent binary PSK system, the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}(\sqrt{E_b/N_0})$$

Therefore, the E_b/N_0 of this coherent binary FSK system must be increased by the factor 1/0.608 = 1.645 (or 2.16 dB) so as to realize the same average probability of error as a coherent binary PSK system.

Problem 6.23

(a) Since the two oscillators used to represent symbols 1 and 0 are independent, we may view the resulting binary FSK wave as the sum of two on-off keying (OOK) signals. One OOK signal operates with the oscillator of frequency f_1. The second OOK signal operates with the oscillator of frequency f_2.

The power spectral density of a random binary wave $X_1(t)$, in which symbol 1 is represented by A volts and symbol 0 by zero volts, is given by (see Problem 4.10)

$$S_{X_1}(f) = \frac{A^2}{4}\,\delta(f) + \frac{A^2T_b}{4}\,\mathrm{sinc}^2(fT_b)$$

where T_b is the bit duration. When this binary wave is multiplied by a sinusoidal wave of unit amplitude and frequency $f_c + \Delta f/2$, we get the first OOK signal with

$$A = \sqrt{2E_b/T_b}$$

The power spectral density of this OOK signal equals

$$S_1(f) = \frac{1}{4}\,[S_{X_1}(f - f_c - \frac{\Delta f}{2}) + S_{X_1}(f + f_c + \frac{\Delta f}{2})]$$

The power spectral density of the random binary wave $X_2(t) = \overline{X_1(t)}$, in which symbol 1 is represented by zero volts and symbol 0 by A volts, is given by

$$S_{X_2}(f) = S_{X_1}(f)$$

When $X_2(t)$ is multiplied by the second sinusoidal wave of unit amplitude and frequency $f_c - \Delta f/2$, we get the second OOK signal whose power spectral density equals

$$S_2(f) = \frac{1}{4}\,[S_{X_2}(f - f_c + \frac{\Delta f}{2}) + S_{X_2}(f + f_c - \frac{\Delta f}{2})]$$

The power spectral density of the FSK signal equals:

$$S_{FSK}(f) = S_1(f) + S_2(f)$$

$$= \frac{E_b}{8T_b}\,[\delta(f - f_c - \frac{\Delta f}{2}) + \delta(f + f_c + \frac{\Delta f}{2}) + \delta(f - f_c + \frac{\Delta f}{2}) + \delta(f + f_c - \frac{\Delta f}{2})]$$

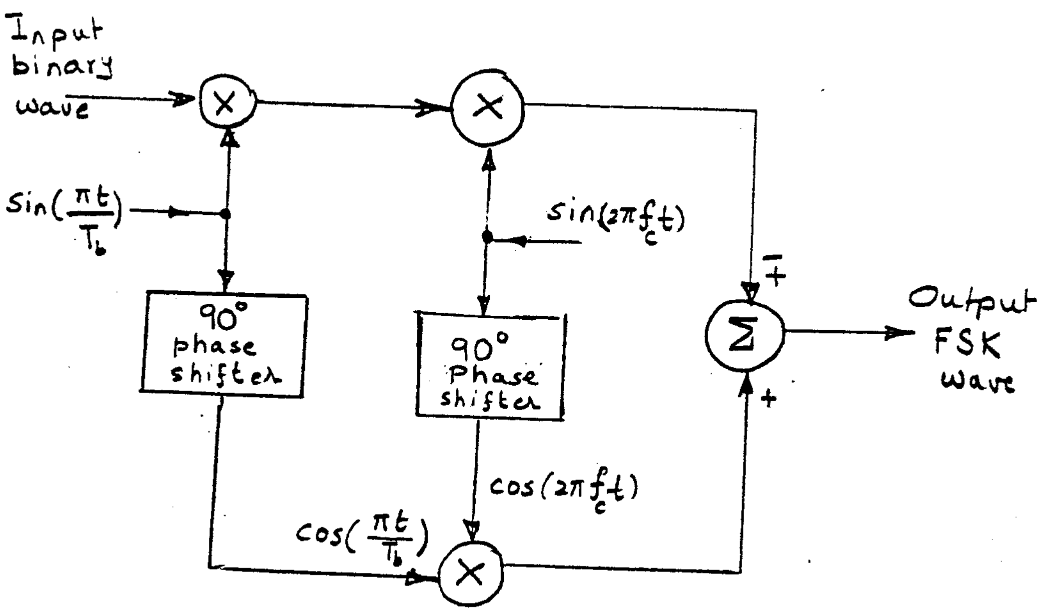
$$+ \frac{E_b}{8}\,\{\mathrm{sinc}^2[T_b(f - f_c - \frac{\Delta f}{2})] + \mathrm{sinc}^2[T_b(f + f_c + \frac{\Delta f}{2})]$$

$$+ \mathrm{sinc}^2[T_b(f - f_c + \frac{\Delta f}{2})] + \mathrm{sinc}^2[T_b(f + f_c - \frac{\Delta f}{2})]\}$$

This result shows that the power spectrum of this binary FSK wave contains delta functions at $f = f_c \pm \Delta f/2$.

(b) At high values of x, the function sinc(x) falls off as 1/x. Hence, at high frequencies, S_{FSK} falls off as $1/f^2$.

Problem 6.24



Problem 6.25

The similarities between offset QPSK and MSK are that both have a half-symobl delay between the in-phase and quadrature components of each data symbol, and both have the same probability of error.

The differences between the two techniques are: (1) the basis functions for offset QPSK are sinusoids multiplied by a rectangle function, while the basis functions for MSK are sinusoids multiplied by half a cosine pulse, and (2) offset QPSK is a form of phase modulation while MSK is a form of frequency modulation.

Problem 6.26

For coherent MSK, the probability of error is

$$P_e = \text{erfc}(\sqrt{E_b/N_0}) ,$$

while for noncoherent MSK, (i.e., noncoherent binary FSK)

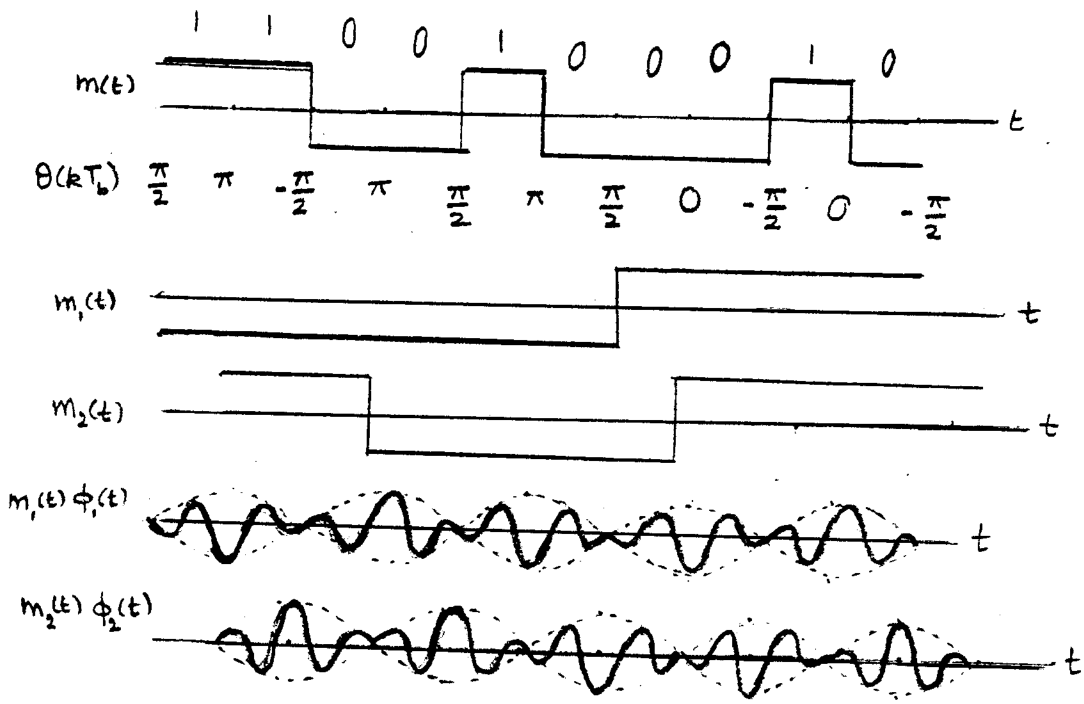
$$P_e = \frac{1}{2} \exp\left(- \frac{E_b}{2N_0}\right) .$$

To maintain $P_e = 10^{-5}$ for coherent MSK, $\frac{E_b}{N_0} = 9.8$. To maintain the same probability of symbol error for noncoherent MSK,

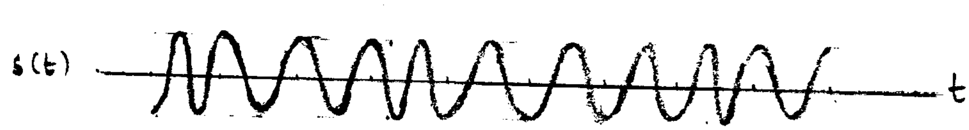
$$\frac{E_b}{N_0} = 21.6, \text{ which is an increase of 3.4 dB.}$$

Problem 6.27

(a)



(b)



Problem 6.28

(a) The Fourier transform of $h(t)$ is given by (using entry 5 of the Fourier-transform pairs Table A6.3)

$$H(f) = \frac{\sqrt{\pi}}{\alpha} \cdot \frac{1}{\sqrt{\pi/\alpha}} \exp\left(-\pi \times \frac{f^2}{\pi/\alpha^2}\right)$$

$$= \exp-(f^2\alpha^2) \tag{1}$$

Substituting $\alpha = (\sqrt{\log 2/2}/W)$ into (1), we get

$$H(f) = \exp\left(-f^2 \frac{\log 2}{2} \times \frac{1}{W^2}\right)$$

$$= \exp\left(-\frac{\log 2}{2}\left(\frac{f}{W}\right)^2\right) \tag{2}$$

Let f_0 denote the 3-dB cut-off frequency of the GMSK signal. Then, by definition,

$$|H(f_0)| = \frac{1}{\sqrt{2}}|H(0)|$$

$$= \frac{1}{\sqrt{2}}$$

Hence, from Eq. (2) it follows that

$$\exp\left(-\frac{\log 2}{2}\left(\frac{f_0}{W}\right)^2\right) = \frac{1}{\sqrt{2}}$$

or

$$\exp\left(\log 2\left(\frac{f_0}{W}\right)^2\right) = 2$$

Taking the logarithm of both sides, we readily find that

$f_0 = W$

The 3-dB bandwidth (cut-off frequency) of the filter used to shape GMSK signals is therefore W.

(b) The response of the filter to a rectangular pulse of unit amplitude and duration T centered on the origin is given by

$$g(t) = \int_{-T/2}^{T/2} h(t-\tau)d\tau$$

$$= \int_{-T/2}^{T/2} \frac{\sqrt{\pi}}{\alpha}\exp\left[\frac{-\pi^2(t-\tau)^2}{\alpha^2}\right]d\tau \qquad (3)$$

Let $k = \dfrac{\pi(t-\tau)}{\alpha}$, and

$$dk = -\frac{\pi}{\alpha}d\tau$$

Hence, we may reqrite Eq. (3) as

$$g(t) = -\int_{k_1}^{k_2} \frac{\sqrt{\pi}}{\alpha}\exp(-k^2)\frac{\alpha}{\pi}dk \qquad (4)$$

where $k_1 = \dfrac{\pi(t+T/2)}{\alpha}$ and

$$k_2 = \frac{\pi(t-T/2)}{\alpha}$$

Equation (4) is finally rewritten as

$$g(t) = -\frac{1}{2}\left\{\frac{2}{\sqrt{\pi}}\int_0^{k_2} \exp(-k^2)dk + \frac{2}{\sqrt{\pi}}\int_{k_1}^{0} \exp(-k^2)dk\right\}$$

$$= -\frac{1}{2}\mathrm{erf}(k_2) + \frac{1}{2}\mathrm{erf}(k_1)$$

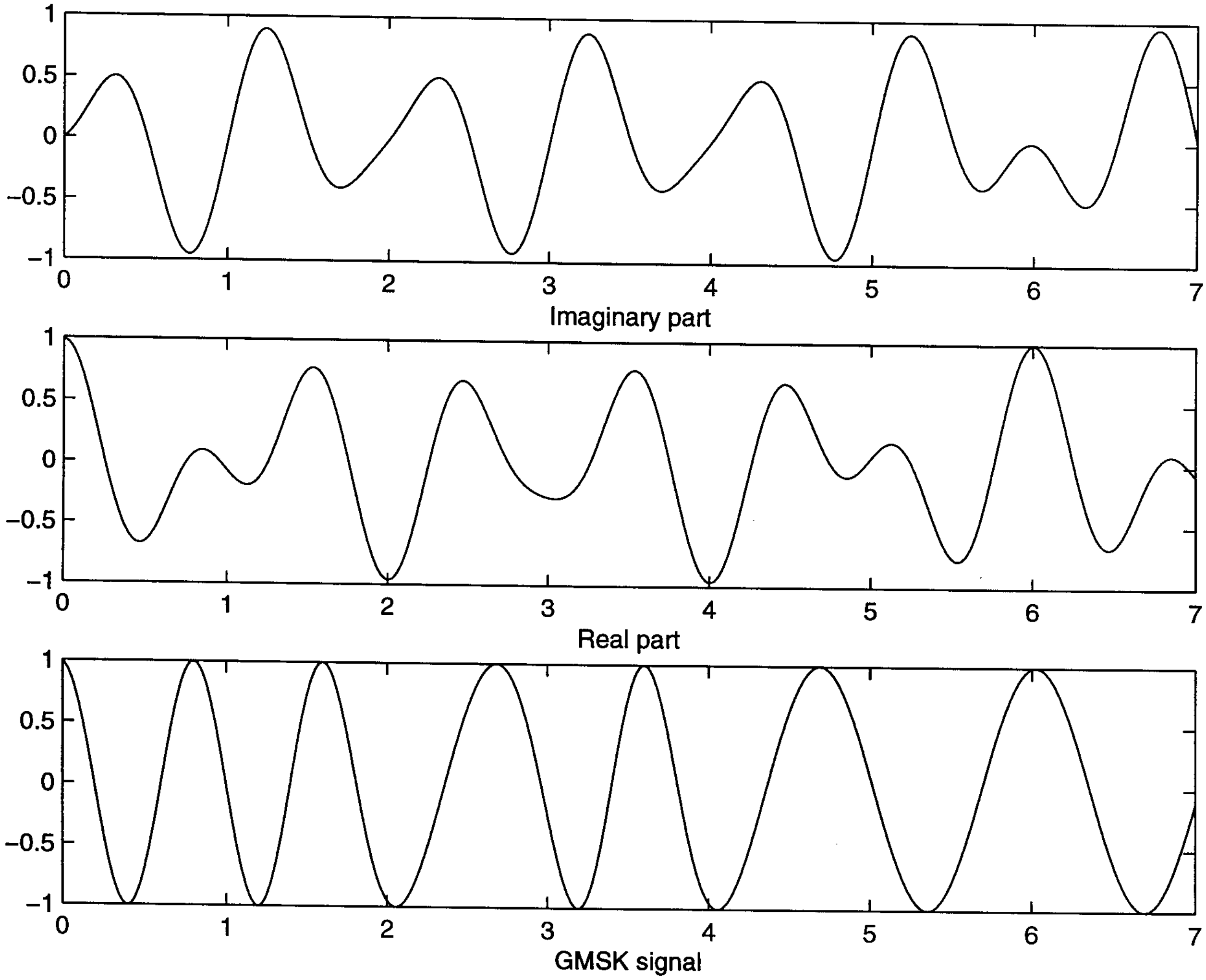
$$= -\frac{1}{2}[1 - \mathrm{erfc}(k_2)] + \frac{1}{2}[1 - \mathrm{erfc}(k_1)]$$

$$= \frac{1}{2}\mathrm{erfc}(k_2) - \frac{1}{2}\mathrm{erfc}(k_1)$$

$$= \frac{1}{2}\left\{\mathrm{erfc}\left[\frac{\pi(t - T/2)}{\alpha}\right] - \mathrm{erfc}\left[\frac{\pi(t - T/2)}{\alpha}\right]\right\}$$

$$= \frac{1}{2}\left\{\mathrm{erfc}\left[\pi\sqrt{\frac{2}{\log 2}}WT\left(\frac{t}{T} - \frac{1}{2}\right)\right] - \mathrm{erfc}\left[\pi\sqrt{\frac{2}{\log 2}}WT\left(\frac{t}{T} + \frac{1}{2}\right)\right]\right\}$$

Problem 6.29



The GMSK signal, displayed in the bottom waveform, is very similar to that of the MSK signal in Fig. 6.30, both of which are produced by the input sequence 1101000. This objective is indeed the idea behind the GMSK signal.

Problem 6.30

Comparing the standard MSK and Gaussian-filtered GMS signals, we note the following:

(a) Similarities

- For a given input sequence, the waveforms produced by the MSK and GMSK modulators are very similar, as illustrated by comparing the GMSK signal displayed in the solution to Problem 6.29 and the corresponding MSK signal displayed in Fig. 6.30 of the textbook for the input sequence 1101000.
- They both have a constant envelope.

(b) Differences

The use of GMSK results in a slight degradation in performance compared to the standard MSK for a time-bandwidth product $WT_b = 0.3$. However, the GMSK makes up for this loss in performance by providing a more compact power-spectral characteristic.

Problem 6.31

In the binary FSK case, the transmitted signal is defined by

$$s_i(t) = \begin{cases} \sqrt{2\frac{E_b}{T_b}} \cos(2\pi f_i t), & 0 \le r \le T_b \\ 0, & \text{elsewhere} \end{cases} \tag{1}$$

where the carrier frequency f_i equals one of two possible values f_1 and f_2. The transmission of frequency f_1 represents symbol 1, and the transmission of frequency f_2 represents symbol 0. For the noncoherent detection of this frequency-modulated wave, the receiver consists of a pair of matched filters followed by envelope detectors, as in Fig. 1. The filter in the upper channel of the receiver is matched to $\sqrt{2/T_b}\cos(2\pi f_1 t)$ and the filter in the lower channel is matched to $\sqrt{2/T_B}\cos(2\pi f_2 t), 0 \le t \le T_b$. The resulting envelope detector outputs are sampled at $t = T_b$, and their values are compared. Let l_1 and l_2 denote the envelope samples of the upper and lower channels, respectively, Then, if $l_1 > l_2$, the receiver decides in favor of symbol 1, and if $l_1 < l_2$ it decides in favor of symbol 0.

Suppose symbol 1 or frequency f_1 is transmitted. Then a correct decision will be made by the receiver if $l_1 > l_2$. If, however, the noise is such that $l_1 < l_2$, the receiver decides in favor of symbol 0, and an erroneous decision will have been made. To calculate the probability of error, we must have the probability density functions of the random variables L_1 and L_2 whose sample values are denoted by l_1 and l_2, respectively.

When frequency f_1 is transmitted, and there is no synchronism between the receiver and transmitter, the received signal x(t) is of the form

$$\begin{aligned} x(t) &= \sqrt{\frac{2E_b}{T_B}} \cos(2\pi f_1 t + \theta) + w(t) \\ &= \sqrt{\frac{2E_b}{T_b}} \cos\theta \cos(2\pi f_1 t) - \sqrt{\frac{2E_b}{T_b}} \sin\theta \sin(2\pi f_1 t) + w(t), \qquad 0 \le t \le T_b \end{aligned} \tag{2}$$

Let

$$x_{ci} = \int_0^{T_b} x(t) \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t)dt, \qquad i = 1,2 \tag{3}$$

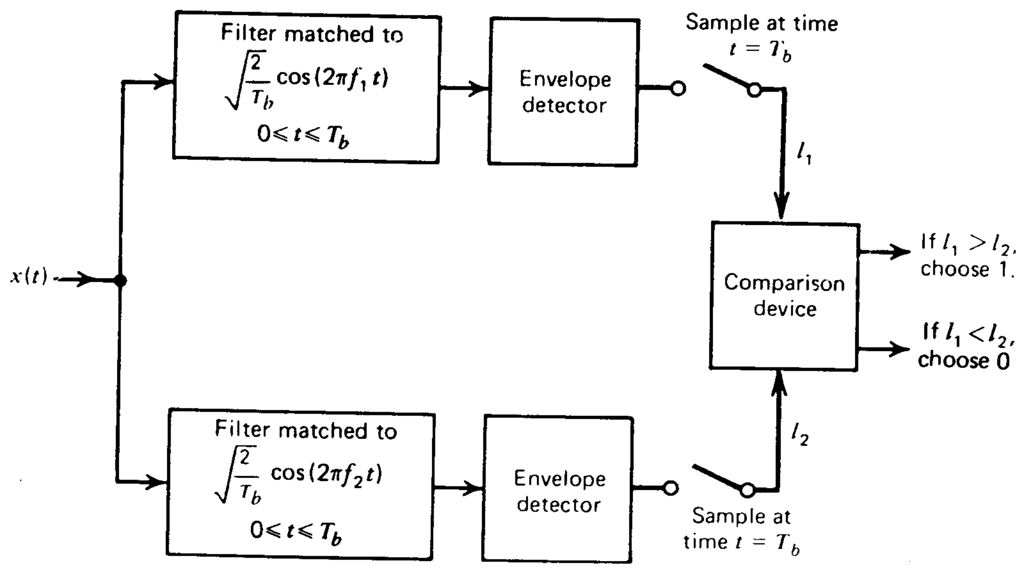


Figure 1

and

$$x_{si} = \int_0^{T_b} x(t) \sqrt{\frac{2}{T_b}} \sin(2\pi f_i t)dt, \qquad i = 1,2, \tag{4}$$

The x_{ci} and x_{si} i=1,2, define the coordinates of the received signal point. Note that, although each transmitted signal $s_i(t)$, i=1,2, is represented by a point in a two-dimensional space, the presence of the unknown phase θ makes it necessary to use four orthonormal basis functions in order to resolve the received signal x(t). With the received signal x(t) having the form shown in Eq. (1), we find that the output of the upper channel in the receiver of Fig. 1 equals

$$l_1 = \sqrt{x_{c1}^2 + x_{s1}^2} \tag{5}$$

where

$$x_{c1} = \sqrt{E_b} \cos \theta + w_{c1} \tag{6}$$

and

$$x_{s1} = -\sqrt{E_b} \sin \theta + w_{s1} \tag{7}$$

On the other hand, the corresponding value of the lower channel output is

$$l_2 = \sqrt{x_{c2}^2 + x_{s2}^2} \tag{8}$$

where

$$x_{c2} = w_{c2} \tag{9}$$

and

$$x_{s2} = w_{s2} \tag{10}$$

The w_{ci} and w_{si}, i=1,2, are related to the noise w(t) as follows:

$$w_{ci} = \int_0^{T_b} w(t) \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t) dt, \qquad i = 1,2 \tag{11}$$

and

$$w_{si} = \int_0^{T_b} w(t) \sqrt{\frac{2}{T_b}} \sin(2\pi f_i t) dt, \qquad i = 1,2, \tag{12}$$

Accordingly, w_{ci} and w_{si}, i=1,2, are sample values of independent Gaussian random variables of zero mean and variance $N_0/2$.

When symbol 1 or frequency f_1 is transmitted, we see from Eqs. (9) and (10) that x_{c2} and x_{s2} are sample values of two Gaussian and statistically independent random variables, X_{c2} and X_{s2}, with zero mean and variance $N_0/2$. Accordingly, the lower channel output l_2, related to x_{c2} and x_{s2} by Eq. (8), is the sample value of a Rayleigh-distributed random variable L_2. We may thus express the conditional probability density function of L_2, given that symbol 1 was transmitted, as follows:

$$f_{L2|1}(l_2 \mid 1) = \frac{2l_2}{N_0} \exp\left(-\frac{l_2^2}{N_0} \right), \qquad l_2 \geq 0 \tag{13}$$

Again under the condition that symbol 1 or frequency f_1 is transmitted, we see from Eqs. (6) and (7) that x_{c1} and x_{s1} are sample values of two Gaussian and statistically independent random variables, X_{c1} and X_{s1}, with mean values equal to $\sqrt{E_b}\cos\theta$ and $\sqrt{E_b}\sin\theta$, respectively, and variance $N_0/2$. Therefore, the joint probability density function of X_{c1} and X_{s1}, given that symbol 1 was transmitted and that the random phase $\Theta = \theta$, may be expressed as follows

$$f_{X_{c1},X_{s1}|1,\Theta}(x_{c1}, x_{s1} \mid \theta) = \frac{1}{\pi N_0} \exp\left\{ -\frac{1}{N_0} \left[(x_{c1} - \sqrt{E_b}\cos\theta)^2 + (x_{s1} + \sqrt{E_b}\sin\theta)^2 \right] \right\} \tag{14}$$

Define the transformations

$$x_{c1} = l_1 \cos\psi_1 \qquad)15)$$

and

$$x_{s1} = l_1 \sin\psi_1 \tag{16}$$

where $\psi_1 = \tan^{-1}(x_{s1}/x_{c1})$, with $0 \leq \psi_1 \leq 2\pi$. Then, applying this transformation and following a procedure similar to that described in Section 5.12, we find that the upper channel output l_1 is the sample value of a Rician-distributed random variable L_1. Hence, the conditional probability density function of L_1, given that symbol 1 was transmitted and that the random phase $\Theta = \theta$, is given by the Rician distribution

$$\begin{aligned} f_{L1|\Theta}(l_1 \mid 1, \theta) &= \int_0^{2\pi} f_{L1,\Psi|1,\Theta}(l_1, \psi \mid 1, \theta)\, d\psi \\ &= \frac{2l_1}{N_0} \exp\left(-\frac{l_1^2 + E_b}{N_0} \right) I_0\left(\frac{2l_1\sqrt{E_b}}{N_0} \right), \qquad l_1 \geq 0 \end{aligned} \tag{17}$$

where $I_0(2l_1\sqrt{E_b}/N_0)$ is the modified Bessel function of the first kind of zero order. Since Eq. (17) does not depend on θ, which is to be expected, it follows that the conditional probability density function of L_1, given that symbol 1 was transmitted, is

$$f_{L1|1}(l_1 \mid 1) = \frac{2l_1}{N_0} \exp\left(-\frac{l_1^2 + E_b}{N_0} \right) I_0\left(\frac{2l_1 \sqrt{E_b}}{N_0} \right), \qquad l_1 \geq 0 \tag{18}$$

Note that by putting $E_b = 0$ and recognizing that $I_0(0) = 1$, Eq. (18) reduces to a Rayleigh distribution.

When symbol 1 is transmitted, the receiver makes an error whenever the envelope sample l_2 obtained from the lower channel (due to noise alone) exceeds the envelope sample l_1 obtained from the upper channel (due to signal plus noise), for all possible values of l_1. Consequently, the probability of this error is obtained by integrating $f_{L2|1}(l_2 \mid 1)$ with respect to l_2 from l_1 to infinity, and then averaging over all possible values of l_1. That is to say,

$$\begin{aligned} p_{01} &= P(l_2 > l_1 \mid \text{symbol 1 was sent}) \\ &= \int_0^\infty dl_1 f_{L_1|1} \int_0^\infty dl_2 f_{L_2|1}(l_2 \mid 1) \end{aligned} \tag{19}$$

where the inner integral is the conditional probability of error for a *fixed* value of l_1, given that symbol 1 was transmitted, and the outer integral is the average of this conditional probability for all possible values of l_1. Since the random variable L_2 is Rayleigh-distributed when symbol 1 is transmitted, the inner integral in Eq. (19) is equal to $\exp(-l_1/2N_0)$. Thus, using Eq. (18) in (19), we get

$$p_{01} = \int_0^\infty \frac{2l_1}{N_0} \exp\left(-\frac{2l_1^2 + E_b}{N_0} \right) I_0\left(\frac{2l_1 \sqrt{E_b}}{N_0} \right) dl_1 \tag{20}$$

Define a new variable v related to l_1 by

$$v = \frac{2l_1}{\sqrt{N_0}} \tag{21}$$

Then, changing the variable of integration from l_1 to v, we may rewrite Eq. (20) in the form

$$p_{01} = \frac{1}{2} \exp\left(-\frac{E_b}{2N_0} \right) \int_0^\infty v \exp\left(-\frac{v^2 + a^2}{2} \right) I_0(av)dv \tag{22}$$

where $a = \sqrt{E_b/N_0}$. The integral in Eq. (22) represents the total area under the normalized form of the Rician distribution. Since this integral must be equal to one, we may simplify Eq. (22) as

$$p_{01} = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right) \tag{23}$$

Similarly, when symbol 0 or frequency f_2 is transmitted, we may show that p_{10}, the probability that $l_1 > l_2$ and therefore the probability that the receiver makes an error by deciding in favor of symbol 1, has the same value as in Eq. (23). Thus, averaging p_{10} and p_{01}, we find that the average probability of symbol error for the noncoherent binary FSK equals

$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right) \tag{24}$$

which is exactly the same as that in Eq. (6.163) in the textbook.

Comparing the effort involved in the derivation of Eq. (24) presented in this problem with that in deriving Eq. (6.163), we clearly see the elegance of the approach adopted in the textbook.

Problem 6.32

Let

$$x(t) = A_c \cos(2\pi f_c t + \theta)$$

$$= A_c \cos(2\pi f_c t) \cos\theta - A_c \sin(2\pi f_c t) \sin\theta$$

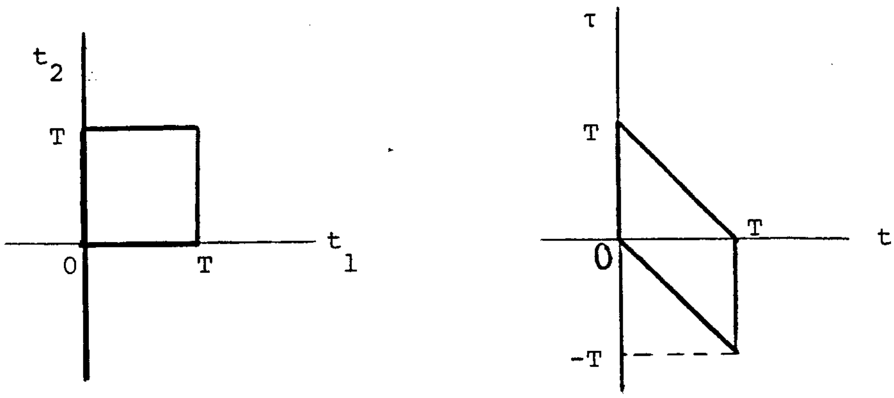
The output of the square-law envelope detector in Fig. P8.2, sampled at time t=T, is given by

$$y(T) = \left[\int_0^T x(t) \cos(2\pi f_c t)\, dt\right]^2 + \left[\int_0^T x(t) \sin(2\pi f_c t)\, dt\right]^2$$

This may be written as

$$y(T) = \int_0^T \int_0^T x(t_1)x(t_2)[\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) + \sin(2\pi f_c t_1)\sin(2\pi f_c t_2)]dt_1 dt_2 \qquad (1)$$

Put $t_1 = t$, and $t_2 = t+\tau$. This transformation is illustrated below:



Then, we may rewrite Eq. (1) as follows

$$y(T) = \int_0^T \int_{-t}^{T-t} x(t)x(t+\tau)\, [\cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi f_c \tau)$$

$$+ \sin(2\pi f_c t)\sin(2\pi f_c t + 2\pi f_c \tau)]\, dt\, d\tau \qquad (2)$$

However,

$$\cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi f_c \tau) + \sin(2\pi f_c t)\sin(2\pi f_c t + 2\pi f_c \tau) = \cos(2\pi f_c \tau)$$

Therefore, we may simplify Eq. (2) as follows

$$y(T) = \int_0^T \int_{-t}^{T-t} x(t)\, x(t+\tau) \cos(2\pi f_c \tau)\, d\tau\, dt$$

$$= 2\int_0^T \int_0^{T-t} x(t)\, x(t+\tau) \cos(2\pi f_c \tau)\, d\tau\, dt, \quad 0 \le \tau \le T \qquad (3)$$

Define

$$R_X(\tau) = \int_0^{T-t} x(t)\, x(t+\tau)\, dt \qquad 0 \leq \tau \leq T$$

Then, we may rewrite Eq. (3) in terms of $R_X(\tau)$ as follows

$$y(T) = 2 \int_0^{T} R_X(\tau) \cos(2\pi f_c \tau)\, d\tau$$

$$= 2\, S_X(f_c) \qquad (3)$$

where

$$S_X(f) = \int_0^{T} R_X(\tau) \cos(2\pi f_c \tau)\, d\tau$$

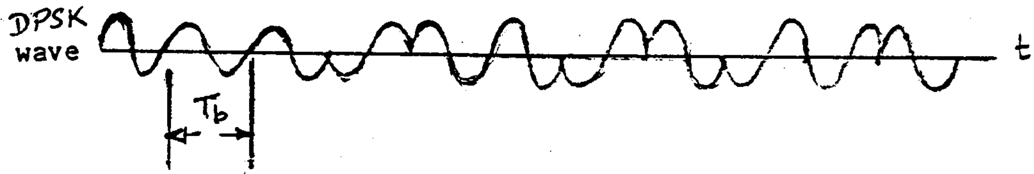
Equation (3) is the desired result.

Problem 6.33

(a)

b_k		1	1	0	0	1	0	0	0	1	0
d_{k-1}		1	1	1	0	1	1	0	1	0	0
d_k	1	1	1	0	1	1	0	1	0	0	1
Transmitted phase	0	0	0	π	0	0	π	0	π	π	0

The waveform of the DPSK signal is thus as follows:



(b) Let

x_I = output of the integrator in the in-phase channel

x_Q = output of the integrator in the quadrature channel

x_I' = one-bit delayed version of x_I

x_Q' = one-bit delayed version of x_Q

l_I = in-phase channel output

$= x_I x_I'$

l_Q = quadrature channel output

$= x_Q x_Q'$

$y = l_I + l_Q$

Transmitted phase (radians)	0	0	0	π	0	0	π	0	π	π	0
Polarity of x_I	+	+	+	-	+	+	-	+	-	-	+
Polarity of x_I'		+	+	+	-	+	+	-	+	-	-
Polarity of l_I		+	+	-	-	+	-	-	-	+	-
Polarity of x_Q	-	-	-	+	-	-	+	-	+	+	-
Polarity of x_Q'		-	-	-	+	-	-	+	-	+	+
Polarity of l_Q		+	+	-	-	+	-	-	-	+	-
Polarity of y		+	+	-	-	+	-	-	-	+	-
Reconstructed data stream		1	1	0	0	1	0	0	0	1	0

Problem 6.34

Coherent M-ary PSK requires exact knowledge of the carrier frequency and phase for the receiver to be accurately synchronized to the transmitter. When carrier recovery at the receiver is impractical, we may use differential encoding based on the phase difference between successive symbols at the cost of some degradation in performance. If the incoming data are encoded by a phase-shift rather than by absolute phase, the receiver performs detection by comparing the phase of one symbol with that of the previous symbol, and the need for a coherent reference is thereby eliminated. This procedure is the same as that described for binary DPSK. The exact calculation of probability of symbol error for the differential detection of differential M-ary PSK (commonly referred to as M-ary DPSK) is much too complicated for $M > 2$. However, for large values of E/N_0 and $M \geq 4$, the probability of symbol error is approximately given by

$$P_e \approx \operatorname{erfc}\left(\sqrt{\frac{2E}{N_0}}\sin\left(\frac{\pi}{2M}\right)\right), \qquad M \geq 4 \tag{1}$$

For coherent M-ary PSK, the corresponding formula for the average probability of symbol error is approximately given by

$$P_e \approx \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}}\sin\left(\frac{\pi}{M}\right)\right) \tag{2}$$

(a) Comparing the approximate formulas of Eqs. (1) and (2), we see that for $M \geq 4$ an M-ary DPSK system attains the same probability of symbol error as the corresponding coherent M-ary PSK system provided that the transmitted energy per symbol is increased by the following factor:

$$k(M) = \frac{\sin^2\left(\frac{\pi}{M}\right)}{2\sin^2\left(\frac{\pi}{2M}\right)}, \qquad M \geq 4$$

(b) For example, $k(4) = 1.7$. That is, differential QPSK (which is noncoherent) is approximately 2.3 dB poorer in performance than coherent QPSK.

Problem 6.35

(a) For coherent binary PSK,

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{E_b}{N_0}\right) .$$

For P_e to equal 10^{-4}, $\sqrt{E_b/N_0} = 2.64$. This yields $E_b/N_0 = 7.0$. Hence $E_b = 3.5 \times 10^{-10}$. The required average carrier power is 0.35 mW.

(b) For DPSK,

$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right) .$$

For P_e to equal 10^{-4}, we have $\frac{E_b}{N_0} = 8.5$. Hence $E_b = 4.3 \times 10^{-10}$. The required average power is 0.43 mW.

Problem 6.36

(a) For a coherent PSK system, the average probability of error is

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left[\sqrt{(E_b/N_0)_1}\right]$$

$$\simeq \frac{1}{2}\,\frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\,\sqrt{(E_b/N_0)_1}} \qquad (1)$$

For a DPSK system, we have

$$P_e = \frac{1}{2}\exp[-(E_b/N_0)_2] \qquad (2)$$

Let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then, we may use Eqs. (1) and (2) to obtain

$$\sqrt{\pi}\ \sqrt{(E_b/N_0)_1} = \exp\ \delta$$

We are given that

$$\left(\frac{E_b}{N_0}\right)_1 = 7.2$$

Hence,

$$\delta = \ln[\sqrt{7.2\pi}]$$
$$= 1.56$$

Therefore,

$$10\ \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10\ \log_{10}\ 7.2 = 8.57\ \text{dB}$$

$$10\ \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10\ \log_{10}(7.2 + 1.56)$$
$$= 9.42\ \text{dB}$$

The separation between the two (E_b/N_0) ratios is therefore $9.42 - 8.57 = 0.85$ dB.

(b) For a coherent PSK system, we have

$$P_e = \frac{1}{2}\ \text{erfc}[\sqrt{(E_b/N_0)_1}]$$
$$\simeq \frac{1}{2}\ \frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\ \sqrt{(E_b/N_0)_1}} \qquad (3)$$

For a QPSK system, we have

$$P_e = \text{erfc}[\sqrt{(E_b/N_0)_2}]$$
$$\simeq \frac{\exp[-(E_b/N_0)_2]}{\sqrt{\pi}\ \sqrt{(E_b/N_0)_2}} \qquad (4)$$

Here again, let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then we may use Eqs. (3) and (4) to obtain

$$\frac{1}{2} = \frac{\exp(-\delta)}{\sqrt{1 + \delta/(E_b/N_0)_1}} \tag{5}$$

Taking logarithms of both sides:

$$-\ln 2 = -\delta - 0.5 \ln[1 + \delta/(E_b/N_0)_1]$$

$$\simeq -\delta - 0.5 \frac{\delta}{(E_b/N_0)_1}$$

Solving for δ:

$$\delta \simeq \frac{\ln 2}{1 + 0.5/(E_b/N_0)_1}$$

$$= 0.65$$

Therefore,

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(7.2) = 8.57 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(7.2 + .65)$$

$$= 8.95 \text{ dB.}$$

The separation between the two (E_b/N_0) ratios is 8.95 - 8.57 = 0.38 dB.

(c) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2} \operatorname{erfc}[\sqrt{(E_b/2N_0)_1}]$$

$$= \frac{1}{2} \frac{\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_1\right)}{\sqrt{\pi}\sqrt{(E_b/2N_0)_1}} \tag{6}$$

For a noncoherent binary FSK system, we have

$$P_e = \frac{1}{2} \exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_2\right) \tag{7}$$

Hence,

$$\sqrt{\frac{\pi}{2}\left(\frac{E_b}{N_0}\right)_1} = \exp\left(\frac{\delta}{2}\right) \tag{8}$$

We are given that $(E_b/N_0)_1 = 13.5$. Therefore,

$$\delta = \ln\left(\frac{13.5\ \pi}{2}\right)$$

$$= 3.055$$

We thus find that

$$10\ \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10\ \log_{10}(13.5)$$

$$= 11.3\ \text{dB}$$

$$10\ \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10\ \log_{10}(13.5 + 3.055)$$

$$= 12.2\ \text{dB}$$

Hence, the separation between the two (E_b/N_0) ratios is 12.2 - 11.3 = 0.9 dB.

(d) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2}\ \text{erfc}[\sqrt{(E_b/2N_0)_1}]$$

$$= \frac{1}{2}\ \frac{\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_1\right)}{\sqrt{\pi}\ \sqrt{(E_b/2N_0)_1}} \tag{9}$$

For a MSK system, we have

$$P_e = \frac{1}{2}\ \text{erfc}[\sqrt{(E_b/2N_0)_2}] \tag{10}$$

$$\simeq \frac{\exp[-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_2]}{\sqrt{\pi}\ \sqrt{(E_b/2N_0)_2}} \tag{10}$$

Hence, using Eqs. (9) and (10), we

$$\ln 2 - \frac{1}{2}\ \ln[1 + \frac{\delta}{(E_b/N_0)_1}] = \frac{1}{2}\ \delta \tag{11}$$

Noting that

$$\frac{\delta}{(E_b/N_0)_1} \ll 1$$

we may approximate Eq. (11) to obtain

$$\ln 2 - \frac{1}{2}\ [\frac{\delta}{(E_b/N_0)_1}] \simeq \frac{1}{2}\ \delta \tag{11}$$

Solving for δ, we obtain

$$\delta = \frac{2 \ln 2}{1 + \frac{1}{(E_b/N_0)_1}}$$

$$= \frac{2 \times 0.693}{1 + \frac{1}{13.5}}$$

$$= 1.29$$

We thus find that

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10 \log_{10}(13.5) = 10 \times 1.13 = 11.3 \text{ dB}$$

$$10 \log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10 \log_{10}(13.5 + 1.29) = 11.7 \text{ dB}$$

Therefore, the separation between the two (E_b/N_0) ratios is 11.7 - 11.3 = 0.4 dB.

Problem 6.37

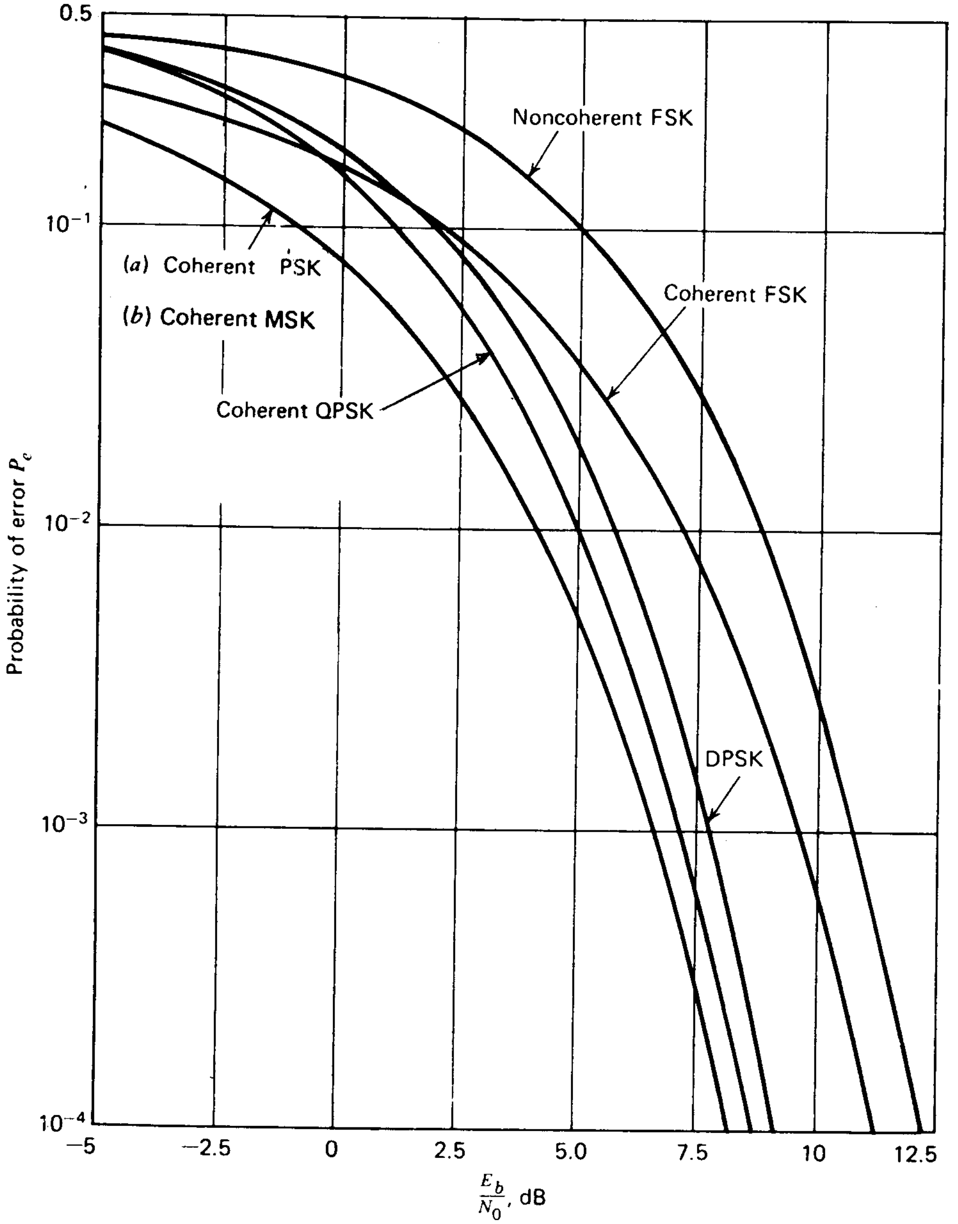


Figure 1 Comparison of the noise performances of different PSK and FSK systems.

The important point to note here, in comparison to the results plotted in Fig. 1 is that the error performance of the coherent QPSK is slightly degraded with respect to that of coherent PSK and coherent MSK. Otherwise, the observations made in Section 8.18 still hold here.

Problem 6.38

The average power for any modulation scheme is

$$P = \frac{E_b}{T_b} .$$

This can be demonstrated for the three types given by integrating their power spectral densities from $-\infty$ to ∞,

$$P = \int_{-\infty}^{\infty} S(f)\, df$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} [S_B(f - f_c) + S_B(f + f_c)]df$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} S_B(f)df .$$

The baseband power spectral densities for each of the modulation techniques are:

	PSK	QPSK	MSK
$S_B(f)$	$2E_b \operatorname{sinc}^2(fT_b)$	$4E_b \operatorname{sinc}^2(2fT_b)$	$\frac{32E_b}{\pi^2}\left[\frac{\cos(2\pi fT_b)}{16f^2T_b^2 - 1}\right]^2$

Since $\int_{-\infty}^{\infty} a \operatorname{sinc}^2(ax)dx = 1$, $P = \frac{E_b}{T_b}$ is easily derived for PSK and QPSK. For MSK we have

$$P = \frac{16E_b}{\pi^2} \int_{-\infty}^{\infty} \left[\frac{\cos(2\pi f T_b)}{16f^2T_b^2 - 1}\right]^2 df$$

$$= \frac{16E_b}{\pi^2 T_b} \int_{-\infty}^{\infty} \frac{\cos^2(2\pi x)}{(16x^2 - 1)^2} dx$$

$$= \frac{8E_b}{\pi^2 T_b} \int_{-\infty}^{\infty} \frac{1 + \cos(4\pi x)}{16x^2(x^2 - \frac{1}{16})} dx$$

$$= \frac{E_b}{16\pi^2 T_b} \int_{-\infty}^{\infty} \frac{\cos 0 + \cos(4\pi x)}{(x^2 - \frac{1}{16})^2} dx$$

From integral tables, (see Appendix AII-6)

$$\int_0^x \frac{\cos(ax)dx}{(b^2 - x^2)^2} = \frac{\pi}{4b^3}[\sin(ab) - ab\cos(ab)]$$

For $a = 0$, the integral is 0.

For $a = 4\pi$, $b = \frac{1}{4}$, we have

$$P = \frac{E_b}{16\pi^2 T_b} \int_{-\infty}^{\infty} \frac{\cos(ax)}{(b^2 - x^2)^2} dx = \frac{E_b}{T_b}$$

For the three schemes, the values of $S(f_c)$ are as follows:

	PSK	QPSK	MSK
$S(f_c)$	$\frac{E_b}{2}$	E_b	$\frac{8E_b}{\pi^2}$

Hence, the noise equivalent bandwidth for each technique is as follows:

	PSK	QPSK	MSK
Bandwidth	$\frac{1}{T_b}$	$\frac{1}{2T_b}$	$\frac{0.62}{T_b}$

Problem 6.39

(a) Table 1, presented below, describes the differential quadrant coding for the V.32 modem of Fig. 6.48a in the textbook, which may operate with nonredundant coding at 9,600 b/s. The entries in the table correspond to the following:

Present inputs: $Q_{1,n}Q_{2,n}$
Previous outputs: $I_{1,n\text{-}1}I_{2,n\text{-}1}$
Present outputs: $I_{1,n}\ I_{2,n}$

Table 1

Input dibit		Previous output dibit		Present output dibit	
$Q_{1,n}$	$Q_{2,n}$	$I_{1,n\text{-}1}$	$I_{2,n\text{-}1}$	$I_{1,n}$	$I_{2,n}$
0	1	0	0	0	0
0	1	0	1	0	1
0	1	1	0	1	0
0	1	1	1	1	1
0	0	0	0	0	1
0	0	0	1	1	1
0	0	1	0	0	0
0	0	1	1	1	0
1	0	0	0	1	1
1	0	0	1	1	0
1	0	1	0	0	1
1	0	1	1	0	0
1	1	0	0	1	0
1	1	0	1	0	0
1	1	1	0	1	1
1	1	1	1	0	1

(b) Table 2, presented below, describes the mapping from the four bits $I_{1,n-1}I_{2,n-1}$, $Q_{3,n}Q_{4,n}$ to the output coordinates of the V.32 modem.

Table 2

Present output dibit		Present input dibit		Output coordinates		
$I_{1,n}$	$I_{2,n}$	$Q_{3,n}$	$Q_{4,n}$	ϕ_1	ϕ_2	
0	1	0	0	1	-1	4th quadrant
0	1	0	1	1	-3	
0	1	1	0	3	-1	
0	1	1	1	3	-3	
0	0	0	0	-1	-1	3rd quadrant
0	0	0	1	-3	-1	
0	0	1	0	-1	-3	
0	0	1	1	-3	-3	
1	0	0	0	-1	1	2nd quadrant
1	0	0	1	-1	3	
1	0	1	0	-3	1	
1	0	1	1	-3	3	
1	1	0	0	1	1	1st quadrant
1	1	0	1	3	1	
1	1	1	0	1	3	
1	1	1	1	3	3	

(b) We are given the current input quadbit:

$$Q_{1,n}Q_{2,n}Q_{3,n}Q_{4,n} = 0001$$

and the previous output dibit:

$$I_{1,n-1}I_{2,n-1} = 01$$

From Table 1, we find that the resulting present output dibit is

$$I_{1,n}I_{2,n} = 11$$

Hence, using this result, together with the given input dibit $Q_{3,n}Q_{4,n} = 01$ in Table 2, we find that the coordinates of the modem output are as follows:

$\phi_1 = 3, \text{ and } \phi_2 = 1$

We may check this result by consulting Table 6.10 and Fig. 6.49 of the textbook. With $Q_{1,n}Q_{2,n} = 00$ we find from Table 6.10 that the modem experiences a phase change of 90°. With $I_{1,n-1}\, I_{2,n-1} = 01$, we find from Fig. 6.49 that the modem was previously residing in the fourth quadrant. Hence, with a rotation of 90° in the counterclockwise direction, the modem moves into the first quadrant. With $Q_{3,n}Q_{4,n} = 01$, we readily find from Fig. 6.49 that

$$\phi_1 = 3, \text{ and } \phi_2 = 1$$

which is exactly the same as the result deduced from Tables 1 and 2 of the solutions manual.

For another example, suppose we are given

$$Q_{1,n}Q_{2,n}Q_{3,n}Q_{4,n} = 1011$$

and

$$I_{1,n-1}I_{2,n-1} = 11$$

Then, from Table 1, we find that

$$I_{1,n}I_{2,n} = 00$$

Next, from Table 2, we find that the output coordinates are $\phi_1 = -3$ and $\phi_2 = -3$. Confirmation that these results are in perfect accord with the calculations based on Table 6.10 and Figure 6.49 is left as an exercise for the reader.

Problem 6.40

(a) The average signal-to-noise ratio is defined by

$$(\text{SNR})_{\text{av}} = \frac{P_{\text{av}}}{\sigma^2} \tag{1}$$

where P_{av} is the average transmitted power, and σ^2 is the channel noise variance. The transmitted signal is defined by

$$s_k(t) = a_k\cos(2\pi f_c t) - b_k\sin(2\pi f_c t), \qquad 0 \le t \le T$$

where (a_k, b_k) is the kth symbol of the QAM signal, and T is the symbol duration. The power spectrum of $s_k(t)$ has the following graphical form:

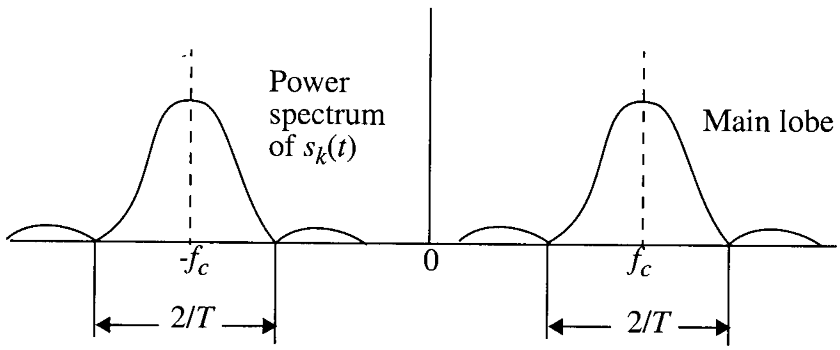


Fig. 1

On the basis of this diagram, we may use the null-to-null bandwidth of the power spectrum in Fig. 1 as the channel bandwidth:

$$B = \frac{2}{T} \quad \text{or} \quad T = \frac{2}{B}$$

The average transmitted power is

$$P_{\text{av}} = \frac{1}{T}E_{\text{av}} = \frac{BE_{\text{av}}}{2} \tag{2}$$

where E_{av} is the average signal energy per symbol.

To calculate the noise variance σ^2, refer to the following figure:

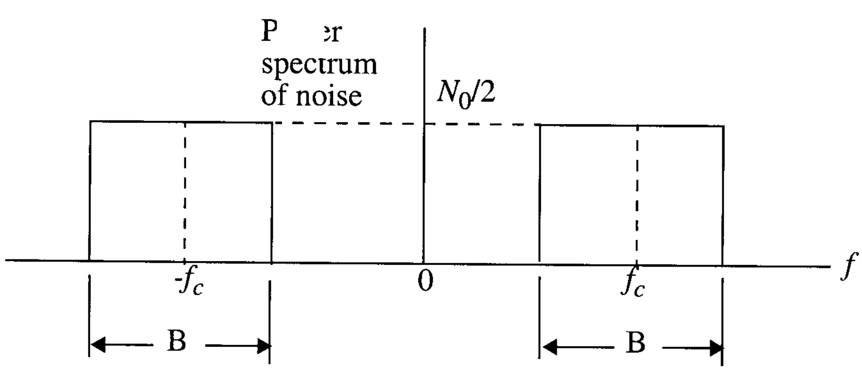


Fig. 2

The noise variance is therefore

$$\sigma^2 = N_0 B \tag{3}$$

Hence, substituting Eqs. (2) and (3) into (1):

$$(\mathrm{SNR})_{av} = \frac{BE_{av}/2}{N_0 B}$$

$$= \frac{1}{2}\left(\frac{E_{av}}{N_0}\right)$$

Expressing the SNR in decibels, we may thus write

$$10\log_{10}(\mathrm{SNR})_{av} = -3 + 10\log_{10}\left(\frac{E_{av}}{N_0}\right), \text{ dB}$$

Given the value $10\log_{10}$, $(E_{av}/N_0) = 20$ dB or $E_{av}/N_0 = 100$, we thus have

$$10\log_{10}(\mathrm{SNR})_{av} = 17 \text{ dB}$$

(b) With M = 16, the average probability of symbol error is

$$P_e = 2\left(1 - \frac{1}{\sqrt{M}}\right)\mathrm{erfc}\left(\sqrt{\frac{3E_{av}}{2(M-1)N_0}}\right)$$

$$= 2\left(1 - \frac{1}{4}\right)\mathrm{erfc}\left(\sqrt{\frac{100}{10}}\right)$$

$$= 1.16 \times 10^{-5}$$

Problem 6.41

We are given the following set of passband basis functions:

$$\{\phi(t)\cos(2\pi f_n t), \phi(t)\sin(2\pi f_n t)\}_{n=1}^{N}$$

where $f_n = \frac{n}{T}$, $\qquad n = 1, 2, \cdots, N$

and $\phi(t) = \sqrt{\frac{2}{T}}\,\mathrm{sinc}\left(\frac{t}{T}\right), \qquad -\infty < t < \infty$

Property 1

$$\int_{-\infty}^{\infty} (\phi(t)\cos(2\pi f_n t))(\phi(t)\sin(2\pi f_n t))dt = 0 \quad \text{for all } n \tag{1}$$

To prove this property, we use the following relation from Fourier transform theory:

$$\int_{-\infty}^{\infty} (\phi(t)\cos(2\pi f_n t))(\phi(t)\sin(2\pi f_n t))dt = 0 \quad \text{for all } n \tag{2}$$

where $g_i(t) \rightleftharpoons G_i(f)$ for $i = 1,2$, and the asterisk denotes complex conjugation. For the problem at hand, we have

$$g_1(t) = \sqrt{\frac{2}{T}}\,\mathrm{sinc}\left(\left(\frac{t}{T}\right)\cos(2\pi f_n t)\right)$$

$$g_2(t) = \sqrt{\frac{2}{T}}\,\mathrm{sinc}\left(\left(\frac{t}{T}\right)\sin(2\pi f_n t)\right)$$

The Fourier transform of the sinc function is

$$F\left[\mathrm{sin}c\left(\frac{t}{T}\right)\right] = T\mathrm{rect}(fT)$$

where

$$\mathrm{rect}(fT) = \left\{\begin{array}{ll} 1 \text{ for } -\frac{1}{2T} \le f \le \frac{1}{2T} & \\ 0 \qquad \text{otherwise} & \end{array}\right\}$$

Hence,

$$G_1(f) = \sqrt{\frac{T}{2}}[\mathrm{rect}((f - f_n)T) + \mathrm{rect}((f + f_n)T)]$$

$$G_2(f) = \frac{1}{j}\sqrt{\frac{T}{2}}[\mathrm{rect}((f - f_n)T) - \mathrm{rect}((f + f_n)T)]$$

Let I_1 denote the integral on the left-hand side of Eq. (1). We may then use Eq. (2) to write

$$I_1 = j\left(\frac{T}{2}\right)\int_{-\infty}^{\infty}[\mathrm{rect}^2((f - f_n)T) - \mathrm{rect}^2((f + f_n)T)]df \tag{3}$$

where the integrand is depicted as follows:

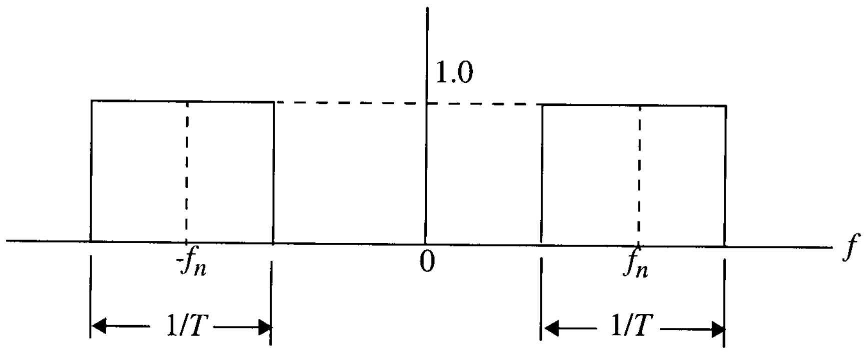


Fig. 1

From Fig. 1 we immediately see that the areas under the two rectangular functions are exactly equal. Hence, Eq. (3) is zero, thereby proving Property 1 for any n.

<u>Property 2</u>

$$\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2}}\phi(t)e^{j2\pi f_n t}\right)\left(\frac{1}{\sqrt{2}}\phi(t)e^{j2\pi f_n t}\right)^* dt = \begin{cases} 1 \text{ for } k = n \\ 2 \text{ for } k \neq n \end{cases} \tag{4}$$

Let I_2 denote the integral in Eq. (4). When $k = n$, we have

$$I_2 = \left(\frac{T}{2}\right)\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2}}\phi(t)e^{j2\pi f_n t}\right)\left(\frac{1}{\sqrt{2}}\phi(t)e^{(-j)2\pi f_n t}\right)dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\phi^2(t)dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{T}}\mathrm{sinc}\left(\frac{t}{T}\right)\right)^2 dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} \text{sinc}^2\left(\frac{t}{T}\right) dt$$

Using Rayleigh's energy theorem, namely,

$$\int_{-\infty}^{\infty} g^2(t)dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

we may write

$$\frac{1}{T}\int_{-\infty}^{\infty} \text{sinc}^2\left(\frac{t}{T}\right) dt = \text{sinc}^2(\lambda)d\lambda, \quad \lambda = t/T$$

$$= \int_{-\infty}^{\infty} \text{rect}^2(f)df$$

$$= 1$$

which proves Property 2 for $k = n$.

To prove Property 2 for $k \neq n$, let

$$g_1(t) = \phi(t)e^{j2\pi f_n t}$$

$$= \sqrt{\frac{2}{T}}\text{sinc}\left(\frac{t}{T}\right)e^{j2\pi f_n t}$$

$$g_2(t) = \phi(t)e^{j2\pi f_k t}$$

$$= \sqrt{\frac{2}{T}}\text{sinc}\left(\frac{t}{T}\right)e^{j2\pi f_k t}, \quad f_k \neq f_n$$

Then applying the following relation from Fourier transform theory,

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)\ dt = \int_{-\infty}^{\infty} G_1(f)G_2^*(f)\ df$$

we may rewrite the integral I_2 of Eq. (4) as

$$I_2 = \frac{2}{T}\int_{-\infty}^{\infty} F\left[\operatorname{sinc}\left(\frac{t}{T}\right)e^{j2\pi f_n t}\right]F\left[\operatorname{sinc}\left(\frac{t}{T}\right)e^{j2\pi f_n t}\right]df$$

Since $f_n = n/T$ by definition, we may depict the two Fourier transforms constituting the integrand of I_2 as shown in Fig. 2 for the worst possible case of $k = n+1$:

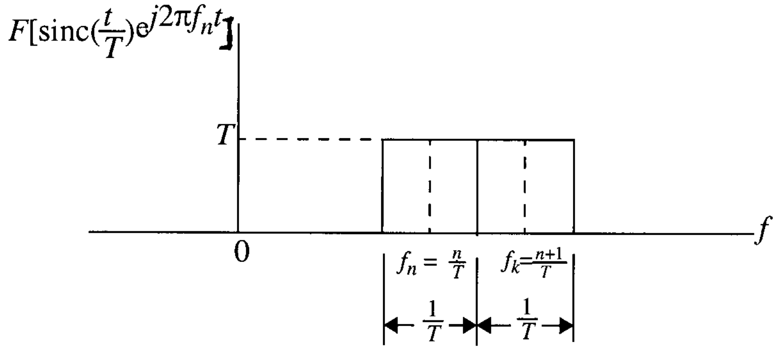


Fig. 2

From Fig. 2 we immediately see that the Fourier transforms of $\phi(t)e^{j2\pi f_n t}$ and $\phi(t)e^{j2\pi f_k t}$ will never overlap for $k \neq n$. Hence, the integral I_2 is zero, proving the rest of Property 2.

Property 3

$$\int_{-\infty}^{\infty} (\phi(t) * h(t))e^{j2\pi f_n t}(\phi(t) * h(t)e^{j2\pi f_k t})^* \, dt = 0 \quad \text{for} \quad k \neq n$$

where the star denotes convolution. From the convolution theorem, we have

$$F[\phi(t) * h(t)] = \Phi(f)H(f)$$

where $\Phi(f) = F[\phi(t)]$ and $\Phi(f) = F[h(t)]$. For $k \neq n$, the picture portrayed in Fig. 2 remains equally valid except for the fact that the basic rectangular spectrum is now replaced by that rectangular spectrum multiplied by the frequency response $H(f)$. This multiplication does not affect the nonoverlapping nature of the two spectra representing $(\phi(t) * h(t))e^{j2\pi f_n t}$ and $(\phi(t) * h(t)e^{j2\pi f_k t})$ for $k \neq n$, hence, proving Property 3.

Problem 6.42

Step 1 - Set $k = 0$ and the initial noise-to-signal ratio $NSR(k) = 0$. Sort the subchannels used in ascending order (i.e., from the smallest to largest ones).

Step 2 - Update the number of subchannels used by setting $k = k+1$.

Step 3 - Compute $NSR(k+1) = NSR(k) + \frac{\sigma_k^2}{g_k^2}$

Step 4 - Set $\lambda(k) = \frac{1}{k}[P_k + \Gamma NSR(k)]$

Step 5 - If $P_k = \lambda(k) - \Gamma\left(\frac{\sigma_k^2}{g_k^2}\right) < 0$

then compute $P_l = \lambda(k-1) - P\left(\frac{\sigma_l^2}{P_l^2}\right)$

and

$$B_l = \log_2\left(1 + \frac{P_l g_l^2}{\Gamma\sigma_l^2}\right)$$

for $l = 1, 2, \cdots, k-1$

Otherwise, go to step 2.

For notations, refer to Section 6.12.

(The algorithm presented here is adapted from T. Starr, J.M. Cioffi, and P.J. Silverman (1999);' see the bibliography.)

Problem 6.43

(a) $P_1 + P_2 + P_3 = P$ (1)

$$P_1 - K = -\Gamma\frac{\sigma^2}{g_1^2} = -\Gamma\sigma^2 \tag{2}$$

$$P_2 - K = -\Gamma\frac{\sigma^2}{g_2^2} = -\Gamma\frac{\sigma^2}{l_1} \tag{3}$$

$$P_3 - K = -\Gamma\frac{\sigma^2}{g_3^2} = -\Gamma\frac{\sigma^2}{l_2} \tag{4}$$

Adding Eqs. (2), (3) and (4), and then using Eq. (1):

$$3K = P + \Gamma\sigma^2\left(1 + \frac{1}{l_1} + \frac{1}{l_2}\right)$$

Solving for K, we thus have

$$K = \frac{P}{3} + \frac{\Gamma\sigma^2}{3}\left(1 + \frac{1}{l_1} + \frac{1}{l_2}\right)$$

With this value of K at hand, we next solve for P_1, P_2, and P_3, obtaining

$$P_1 = \frac{P}{3} + \frac{\Gamma\sigma^2}{3}\left(\frac{1}{l_1} + \frac{1}{l_2} - 2\right)$$

$$P_2 = \frac{P}{3} + \frac{\Gamma\sigma^2}{3}\left(1 + \frac{1}{l_2} - \frac{1}{l_1}\right)$$

$$P_3 = \frac{P}{3} + \frac{\Gamma\sigma^2}{3}\left(1 + \frac{1}{l_1} - \frac{2}{l_2}\right)$$

(b) $$P_1 = \frac{10}{3} + \frac{1}{3}\left(\frac{3}{2} + 3 - 2\right)$$

$$= \frac{1}{3}(10 + 2.5)$$

$$= \frac{12.5}{3}$$

$$P_2 = \frac{10}{3} + \frac{1}{3}(1 + 3 - 3)$$
$$= \frac{11}{3}$$

$$P_3 = \frac{10}{3} + \frac{1}{3}\left(1 + \frac{3}{2} - 6\right)$$

$\mathbf{W} = \mathbf{U}^{\dagger}\mathbf{w}$

Each element of the vector $\mathbf{X}$ in Eq. (2) represents an independent channel, and so we write

$$X_n = \lambda_n A_n + W_n, \qquad n = 1, 2, \cdots, N$$

(b) In the multichannel transmission model, the channel capacity of the entire system in bits per transmission is given by

$$R = \frac{1}{N}\sum_{n=1}^{N+\nu} R_n$$

$$= \frac{1}{2N}\sum_{n=1}^{N+\nu} \log_2\left(1 + \frac{P_n}{\Gamma\sigma^2}\right)$$

$$= \frac{1}{2}\log_2\left[\prod_{n=1}^{N}\left(1 + \frac{P_n}{\Gamma\sigma_n^2}\right)\right]^{1/(n+\nu)} \tag{3}$$

where ν is the length of the channel impulse response. We may also express the R as follows:

$$R = \frac{1}{2}\log_2\left(1 + \frac{1}{\Gamma}(\mathrm{SNR})_{\text{vector coding}}\right) \tag{4}$$

Hence, combining (3) and (4):

$$\Gamma + (\mathrm{SNR})_{\text{vector coding}} = \Gamma\left(\prod_{n=1}^{N}\left(1 + \frac{P_n}{\Gamma\sigma_n^2}\right)\right)^{\frac{1}{(N+\nu)}}$$

$$(\mathrm{SNR})_{\text{vector coding}} = \Gamma\left(\prod_{n=1}^{N}\left(1 + \frac{P_n}{\Gamma\sigma_n^2}\right)\right)^{\frac{1}{(N+\nu)}} - \Gamma$$

(c) As the block length goes to infinity, we may ignore ν, in which case the channel matrix $\mathbf{H}$ becomes nearly an N x N matrix. Therefore, $\mathbf{H}$ may be decomposed as

$$\mathbf{H} = \mathbf{Q}^H \Lambda \mathbf{Q}$$

where $\mathbf{Q}$ is an orthonormal matrix, and Λ is a diagonal matrix of eigenvalues (i.e., singular values). Correspondingly, the singular values approach the magnitude of the Fourier transform.

Even though a vector coding receiver and discrete multitone receiver converge to the same performance, they are not the same:

- The subchannel gains are the same in both cases, but in vector modulation all subchannels have zero phase, while in DMT the subchannels have arbitrary phase angles.
- Unlike DMT, a vector-coding system does not require the use of a cyclic prefix.
- The computational complexity of the vector-coding multichannel system is much greater

$$s_i(t) = \sqrt{\frac{2E}{T}}\cos\left(\frac{\pi}{T}(n_c + i)t\right), \qquad 0 \le t \le T \text{ and } i = 1, 2, \cdots, M \tag{1}$$

where E is the symbol energy, T is the symbol period, and the carrier frequency $f_c = n_c/2T$ for some fixed integer n_c. The signals $s_i(t)$ for $i = 1, 2, \cdots, M$ constitute an orthogonal set over the interval $0 \le t \le T$, as shown by

$$\int_0^T s_i(t)s_j(t)dt = 0 \text{ for } i \ne j \tag{2}$$

Each frequency in Eq. (1) (i.e., specified value of integer i) is modulated with binary data. The net result is a set of parallel carriers, each of which contains a certain portion of the incoming user's data. What we have just described is a form of orthogonal frequency-division multiplexing (OFDM).

Problem 6.47

(a) The M-ary PSK signal is given by

$$y(t) = \sqrt{\frac{2E}{T}}\cos\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right), \qquad i = 1, 2, \cdots, M \tag{1}$$

The output of the Mth power-law device is the Mth power of the input signal $y(t)$:

$$z(t) = \left(\frac{2E}{T}\right)^{\frac{M}{2}} \cos^M\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right) \tag{2}$$

The signal $z(t)$ generates a frequency component at Mf_c, which can be used to drive a phase-locked loop tuned to Mf_c. Specifically, expanding Eq. (2), we get

$$z(t) = \left(\frac{2E}{T}\right)^{\frac{M}{2}}\left\{\binom{M}{\frac{M}{2}}\frac{1}{2^M} + \frac{1}{2^{M-1}}\sum_{k=1}^{(M/2)}\binom{M}{\frac{M}{2}-k}\cos\left[2\pi(2k)f_c t + (2k)\frac{2\pi}{M}(i-1)\right]\right\}$$

$$= \left(\frac{2E}{T}\right)^{\frac{M}{2}}\left\{\frac{1}{2^{M-1}}\binom{M}{0}\cos[2\pi M f_c t + 2\pi(i-1)] \qquad \right\}$$

$$+\frac{1}{2^{M-1}}\binom{M}{1}\cos\left[2\pi(M-1)f_c t+\frac{(M-1)2\pi(i-1)}{M}\right]$$

$$+\ldots+\frac{1}{2^{M-1}}\binom{M}{(M/2)-1}\cos\left(4\pi f_c t+\frac{4\pi}{M}(i-1)\right)$$

$$\left.+\binom{M}{M/2}\frac{1}{2^M}\right\}$$

Therefore according to the first term of this series expansion, the output of the Mth power law device contains a tone of frequency Mf_c, where f_c is the original carrier frequency.

(b) The phase-locked loop is set to a frequency equal to Mf_c. The phase-locked loop acts as a narrow-band filter, thereby passing the sinusoidal component of frequency Mf_c and rejecting the other components.

(c) Consider, for example, the simple case of binary PSK. Since a squaring loop contains a squaring device at its input end, it is clear that changing the sign of the input signal leaves the sign of the recovered carrier unaltered. In other words, the squaring loop with M=2 exhibits a 180° phase ambiguity. Generalizing this result, we may say that Mth power loop for M-ary PSK exhibits M phase ambiguities in the interval $[0,2\pi]$.

One method of resolving the phase ambiguity problem is to exploit differential encoding. Specifically, the incoming data sequence is first differentially encoded before modulation, resulting in a small degradation in noise performance. This method is called the coherent detection of differentially encoded M-ary PSK. As such, this method of modulation is different from the M-ary DPSK considered in Problem 6.34. For the special case of coherent detection of differentially encoded binary PSK, the average probability of symbol error is given by

$$P_e = \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)-\frac{1}{2}\operatorname{erfc}^2\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{3}$$

In the region where $(E_b/N_0) >> 1$, the second term on the right-hand side of Eq. (3) has a negligible effect; hence, this modulation scheme has an average probability of symbol error practically the same as that for coherent QPSK or MSK. For the coherent detection of differentially encoded QPSK, the average probability of symbol error is given by

$$P_e = 2\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)-2\operatorname{erfc}^2\left(\sqrt{\frac{E_b}{N_0}}\right)+\operatorname{erfc}^3\left(\sqrt{\frac{E_b}{N_0}}\right)-\frac{1}{4}\operatorname{erfc}^4\left(\sqrt{\frac{E_b}{N_0}}\right)$$

For large E_b/N_0, this average probability of symbol error is approximately twice that of coherent QPSK.

Problem 6.48

(a) Assuming that the input data sequence $x[n]$ is measured in volts, and recognizing that the symbol a_n is dimensionless, then from Eq. (6.271) we find that the error signal $e[n]$ is also measured in volts. Then, with the phase estimate $\hat{\theta}[n]$ measured in radians, it follows that the step-size parameter γ for carrier recovery in Eq. (6.272) is measured in radians/volts.

(b) From Eq. (6.282) defining the error signal in terms of the input data sequence, we see that the error signal is measured in volts squared. Hence, with $c[n]$ responsible for timing recovery, measured in seconds, it follows that the step-size parameter γ for timing recovery in Eq. (6.286) is measured in seconds/volts^2.

Problem 6.49

(a) The complex envelope of the received waveform is given by

$$\tilde{r}(t) = \tilde{s}(t) + \tilde{w}(t) \tag{1}$$

$$\text{where } \tilde{s}(t) = e^{j(2\pi\nu t + \theta)} \sum_{k=0}^{L_0-1} a_k g(t - kT - \tau)$$

and $w(t)$ is the channel noise. The parameter ν represents the frequency offset, θ is the carrier phase we want to estimate, τ is the timing error, $\{a_k\}$ is the sequence of information symbols, T is the symbol period, and $g(t)$ is the signaling pulse shape.

The likelihood function $L(r|\tilde{\theta})$ is given by

$$L(r|\tilde{\theta}) = \exp\left(\frac{1}{N_0}\int_0^{T_0} Re\{\tilde{r}(t)\tilde{s}(t)\}dt \;--\; \frac{1}{2N_0}\int_0^{T_0} |\tilde{s}(t)|^2 dt\right) \tag{2}$$

$$\text{where } \tilde{s}(t) = e^{j(2\pi\nu t + \theta)} \sum_{k=0}^{L_0-1} a_k g(t - kT - \tau)$$

Since $|\tilde{s}(t)|$ is independent of the carrier phase θ, the log-likelihood function of θ is given by

$$l(\theta)=\log(L(r|\tilde{\theta})) = Re\left\{\int_0^{T_0} \tilde{r}(t)\tilde{s}^*(t)dt\right\} \tag{3}$$

where $\int_0^{T_0} \tilde{r}(t)\tilde{s}^*(t)dt = e^{-j\tilde{\theta}}\sum_{k=0}^{L_0-1} a_k^* x(k)$

where $x(k)$ represents the sample taken at time $t = kT + \tau$ in the formula for convolution:

$$x(t)=[r(t)e^{-j2\pi\nu t}] * g(-t)$$

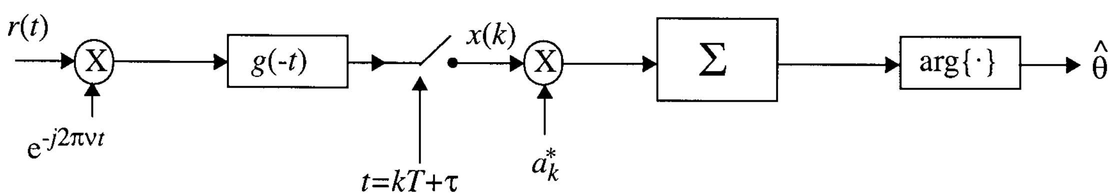
Therefore,

$$l(\theta) = Re\left\{e^{-j\tilde{\theta}}\sum_{k=0}^{L_0-1} a_k^* x(k)\right\}$$

The maximum of $l(\theta)$, i.e., maximum likelihood estimation of θ, is achieved for

$$\hat{\theta} = \arg\left\{\sum_{k=0}^{L_0-1} a_k^* x(k)\right\} \tag{4}$$

(b) Hence, from Eq. (4) we deduce the following system for estimating the phase θ :



Problem 6.51

Matlab codes

```
% Problem 6.51 a. CS: Haykin
%effect of a dispersive channel on BPSK signals
% M. Sellathurai


% number of bits and number of samples per bit
no_of_syms =10;
no_of_bits=no_of_syms*1;
samples_per_bit=16;


% generating bits
Bits=[1 1 0 1 1 0 1 0 0 1]';


% generating QPSK signals
[syms]=BPSK_mod(no_of_bits, Bits);

ts=1e-3/16;
l=length(syms);

% baseband signal
s=zeros(samples_per_bit*(l),1);
for k=1:l-1
for kk=0:(samples_per_bit-1)
s((k-1)*samples_per_bit+kk+1,1)=syms(1,k);
end
end

t=0:ts:(length(s)-1)*ts;

% channel bandwidths 2B=12, filter order 2N=10
B=6; N=5;
H12=butter_channel(2*B,N);
TT12=conv(H12, s);

% channel bandwidths  2B=16 filter order 2N=10
B=8;
H16=butter_channel(2*B,N);
TT16=conv(H16, s);
```

```
% channel bandwidth 2*B=20 filter order 2N=10
B=10;
H20=butter_channel(2*B,N);
TT20=conv(H20, s);


%channel bandwidth  2*B=24 filter order 2N=10
B=12;
H24=butter_channel(2*B,N);
TT24=conv(H24, s);

%  channel bandwidth 2*B=30 filter order 2N=10
B=15;
H30=butter_channel(2*B,N);
TT30=conv(H30, s);

% prints

subplot(2,3,1)
hold on
for k=1:10
plot(k, [syms(k)]','o')
 line([k, k], [0 syms(k)])
end
xlabel('Bit period');
title('Transmitted bits');
hold off


subplot(2,3,2)
[ m start]=max(real(H12));
hold on
plot(t,s,'--');
plot(t,real(TT12(start:160+start-1)));
xlabel('time (s)');
title('Baseband BPSK, BW=12kHz');
hold off

subplot(2,3,3)
[m start]=max(real(H16));
hold on
plot(t,s,'--');
plot(t,real(TT16(start:160+start-1)));
xlabel('time (s)');
title(' Baseband BPSK, BW=16kHz');
hold off
```

```
subplot(2,3,4) .
[m start]=max(real(H20));
hold on
plot(t,s,'--');
plot(t,real(TT20(start:160+start-1)));
xlabel('time (s)');
title('Baseband BPSK, BW=20kHz');
hold off

subplot(2,3,5)
[m start]=max(real(H24));
hold on
plot(t,s,'--');
plot(t,real(TT24(start:160+start-1)));
xlabel('time (s)');
title('Baseband BPSK, BW=24kHz');
hold off

subplot(2,3,6)
[m start]=max(real(H30));
hold on
plot(t,s,'--');
plot(t,real(TT30(start:160+start-1)));
xlabel('time (s)');
title('Baseband BPSK, BW=30kHz');
hold off
```

```
% Problem 6.51 b. CS: Haykin
%effect of a dispersive channel on QPSK signals
% M. Sellathurai


% number of bits and number of samples per bit
no_of_syms =5;
no_of_bits=no_of_syms*2;
samples_per_bit=16;

% generating bits
%Bits=round(rand(no_of_bits,1));
Bits=[1 1 0 1 1 0 1 0 0 1]';

% generating QPSK signals
[syms]=QPSK_mod(no_of_bits, Bits);

l=length(syms);

% baseband signal
s=zeros(samples_per_bit*(l-1),1);
for k=1:l-1
for kk=0:(samples_per_bit-1)
s((k-1)*samples_per_bit+kk+1,1)=syms(1,k);
end
end

t=0:ts:(length(s)-1)*ts;

% channel bandwidths 2B=12, filter order 2N=10
B=6; N=5;
H12=butter_channel(2*B,N);
TT12=conv(H12, s);

% channel bandwidths  2B=16 filter order 2N=10
B=8;
H16=butter_channel(2*B,N);
TT16=conv(H16, s);


% channel bandwidth 2*B=20 filter order 2N=10
B=10;
H20=butter_channel(2*B,N);
TT20=conv(H20, s);
```

```
%channel bandwidth  2*B=24 filter order 2N=10
B=12;
H24=butter_channel(2*B,N);
TT24=conv(H24, s);

%  channel bandwidth 2*B=30 filter order 2N=10
B=15;
H30=butter_channel(2*B,N);
TT30=conv(H30, s);


% prints
subplot(2,3,1)
hold on
for k=1:10
plot(k, [2*Bits(k)-1]','o')
line([k, k], [0 (2*Bits(k)-1)])
end
xlabel('Bit period');
title('Transmitted bits');
hold off


subplot(2,3,2)
[ m start]=max(real(H12));
hold on
plot(t,s,'--');
plot(t,real(TT12(start:64+start-1)));
xlabel('time (s)');
title('Baseband QPSK, BW=12kHz');
hold off

subplot(2,3,3)
[m start]=max(real(H16));
hold on
plot(t,s,'--');
plot(t,real(TT16(start:64+start-1)));
xlabel('time (s)');
title(' Baseband QPSK, BW=16kHz');
hold off

subplot(2,3,4)
[m start]=max(real(H20));
hold on
plot(t,s,'--');
plot(t,real(TT20(start:64+start-1)));
```

```
xlabel('time (s)');
title('Baseband QPSK, BW=20kHz');
hold off

subplot(2,3,5)
[m start]=max(real(H24));
hold on
plot(t,s,'--');
plot(t,real(TT24(start:64+start-1)));
xlabel('time (s)');
title('Baseband QPSK, BW=24kHz');
hold off

subplot(2,3,6)
[m start]=max(real(H30));
hold on
plot(t,s,'--');
plot(t,real(TT30(start:64+start-1)));
xlabel('time (s)');
title('Baseband QPSK, BW=30kHz');
hold off
```

```
% Problem 6.51 c. CS: Haykin
%effect of a dispersive channel on MSK signals
% M. Sellathurai


% number of bits and number of samples per bit
no_of_syms =5;
no_of_bits=no_of_syms*2;
samples_per_bit=16;

% generating bits
Bits=[1 1 0 1 1 0 1 0 0 0]';

% generating QPSK signals
[s,phase]=MSK_mod(no_of_bits,samples_per_bit,Bits);

% channel bandwidths 2B=12, filter order 2N=10
B=6; N=5;
H12=butter_channel(2*B,N);
TT12=conv(H12, s);

% channel bandwidths  2B=16 filter order 2N=10
B=8;
H16=butter_channel(2*B,N);
TT16=conv(H16, s);


% channel bandwidth 2*B=20 filter order 2N=10
B=10;
H20=butter_channel(2*B,N);
TT20=conv(H20, s);


%channel bandwidth  2*B=24 filter order 2N=10
B=12;
H24=butter_channel(2*B,N);
TT24=conv(H24, s);

%  channel bandwidth 2*B=30 filter order 2N=10
B=15;
H30=butter_channel(2*B,N);
TT30=conv(H30, s);
ts=1e-3/16;
t=0:ts:(length(s)-1)*ts

% prints
```

```
subplot(2,3,1)
hold on
for k=1:10
plot(k, [2*Bits(k)-1]','o')
line([k, k], [0 (2*Bits(k)-1)])
end
xlabel('Bit period');
title('Transmitted bits');
hold off


subplot(2,3,2)
[ m start]=max(real(H12));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT12(start+5:165+start-1)));
xlabel('time (s)');
title('MSK (envelope), BW=12kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,3)
[m start]=max(real(H16));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT16(start+5:165+start-1)));
xlabel('time (s)');
title(' MSK  (envelope), BW=16kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,4)
[m start]=max(real(H20));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT20(start+5:165+start-1)));
xlabel('time (s)');
title('MSK  (envelope), BW=20kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,5)
[m start]=max(real(H24));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT24(start+5:165+start-1)));
```

```
xlabel('time (s)');
title('MSK  (envelope), BW=24kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,6)
[m start]=max(real(H30));
hold on
plot(t,abs(s),'--');
plot(t(1:155),abs(TT30(start+5:160+start-1)));
xlabel('time (s)');
title('MSK  (envelope), BW=30kHz');
hold off
axis([0, 0.01,0.9,1.1 ])
```

```
% Problem 6.51 d. CS: Haykin
%effect of a dispersive channel on GMSK signals
% M. Sellathurai


% number of bits and number of samples per bit
no_of_syms =5;
no_of_bits= no_of_syms*2;
samples_per_bit=16;

% generating bits
Bits=[1 1 0 1 1 0 1 0 0 0]';

% generating GMSK signals, WTb=0.3
[s, phase]=GMSK_mod(no_of_bits,samples_per_bit,Bits);

% channel bandwidths 2B=12, filter order 2N=10
B=6; N=5;
H12=butter_channel(2*B,N);
TT12=conv(H12, s);

% channel bandwidths  2B=16 filter order 2N=10
B=8;
H16=butter_channel(2*B,N);
TT16=conv(H16, s);


% channel bandwidth 2*B=20 filter order 2N=10
B=10;
H20=butter_channel(2*B,N);
TT20=conv(H20, s);


%channel bandwidth  2*B=24 filter order 2N=10
B=12;
H24=butter_channel(2*B,N);
TT24=conv(H24, s);

%  channel bandwidth 2*B=30 filter order 2N=10
B=15;
H30=butter_channel(2*B,N);
TT30=conv(H30, s);

ts=1e-3/16;
t=0:ts:(length(s)-1)*ts
```

```
% prints
subplot(2,3,1)
hold on
for k=1:10
plot(k, [2*Bits(k)-1]','o')
line([k, k], [0 (2*Bits(k)-1)])
end
xlabel('Bit period');
title('Transmitted bits');
hold off
subplot(2,3,2)

[ m start]=max(real(H12));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT12(start:160+start-1)));
xlabel('time (s)');
title('GMSK (envelope), BW=12kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,3)
[m start]=max(real(H16));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT16(start:160+start-1)));
xlabel('time (s)');
title(' GMSK  (envelope), BW=16kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,4)
[m start]=max(real(H20));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT20(start:160+start-1)));
xlabel('time (s)');
title('GMSK  (envelope), BW=20kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,5)
[m start]=max(real(H24));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT24(start:160+start-1)));
```

```
xlabel('time (s)');
title('GMSK  (envelope), BW=24kHz');
hold off
axis([0, 0.01,0.9,1.1 ])

subplot(2,3,6)
[m start]=max(real(H30));
hold on
plot(t,abs(s),'--');
plot(t,abs(TT30(start:160+start-1)));
xlabel('time (s)');
title('GMSK  (envelope), BW=30kHz');
hold off
axis([0, 0.01,0.9,1.1 ])
```

```
function [amp]=BPSK_mod(no_of_bits, b)
% used in problem 6.51(a), CS: Haykin
% BPSK modulation
% Mathini Sellathurai
amp=[];

l=1;
m=size(b,1);

for k=1:1:m
if (b(k)==0 )
amp(l)= (-1);
elseif (b(k)==1 )
amp(l)= 1;
end
l=l+1;
end
```

```
function [amp]=QPSK_mod(no_of_bits, b)
% used in problem 6.51(b), CS: Haykin
% QPSK modulation
% Mathini Sellathurai
amp=[];

l=1;
m=size(b,1);
for k=1:2:m

if (b(k)==0 & b(k+1) == 0)
amp(l)= (-1+i*-1)/sqrt(2);
elseif (b(k)==1 & b(k+1) == 0)
amp(l)= (1-i*1)/sqrt(2);
elseif (b(k)==1 & b(k+1) == 1)
amp(l)= (1+i*1)/sqrt(2);
else (b(k)==0& b(k+1) == 1)
amp(l)= (-1+i*1)/sqrt(2);

end
l=l+1
end
```

```
function [amp,phase]=MSK_mod(no_of_bits, samples_per_bit, b)
% used in problem 6.51(c), CS: Haykin
% MSK signal generator
% Mathini Sellathurai

amp=[];

ini_phase=0;


for k=1:no_of_bits

ee=b(k);

for kk=0:samples_per_bit-1

% NRZ signal generator
if ee==0
      ee=-1;
elseif ee==1
      ee=1;
end

      phase((k-1)*samples_per_bit+kk+1)=ini_phase +ee*(pi/(2*samples_per_bit));
      ini_phase=phase((k-1)*samples_per_bit+kk+1);

end
end

phase=rem(phase,2*pi);
in=cos(phase);
quad=sin(phase);
amp=in+i*quad;
```

```
function [amp, phase1]=GMSK_mod(no_of_bits, samples_per_bit, b)
% used in problem 6.51(d), CS: Haykin
% GMSK signal generator
% Mathini Sellathurai

amp=[];

for k =1:no_of_bits

 %Generating NRZ sequence
  if  b(k,1)==0
im_bits(k,1)= -1;
  else
im_bits(k,1) =1;
  end
end

impulse_bits=im_bits;
Bits_to_transmit=max(size(impulse_bits));
BT=0.6;
inphase=0;
data(1,4)=0;
t=0;
for i=0:3

for k=0:(samples_per_bit -1)
       co =GMSK_co(i-2,k+8,samples_per_bit,BT);
       qmskcoef(1,i*samples_per_bit+k+1)=co;
end
end

for bitcount=1:Bits_to_transmit
       ini_phase=inphase;
       ini_phase=rem(ini_phase+data(1,4)*pi/2,2*pi);

data(1,1)=impulse_bits(bitcount,1);
for i =4:-1:2
data(1,i)=data(1,i-1);
end

inphase=ini_phase;

for pha_loop=1:samples_per_bit
      phase=inphase;
for i=0:3
phase=phase+pi/2*data(1,i+1)*qmskcoef(1,samples_per_bit*i+pha_loop);
```

```
end
samples_store(1,t+1)=t;
t=t+1;
phase=rem(phase,2*pi);
phase1(1, pha_loop+samples_per_bit*(bitcount-1))=phase;
rephase(1,pha_loop+samples_per_bit*(bitcount-1))=cos(phase);
quphase(1,pha_loop+samples_per_bit*(bitcount-1))=sin(phase);
end
end

amp=rephase+j*quphase;
```

```
function [co] = GMSK_co(a, b, samples_per_bit,bt)
% used in GMSK signal generation Problem 651d, CS: Haykin
% Mathini Sellathurai

alpha=bt*5.336446225;
T=a+b/samples_per_bit;
co=T*erf(T*alpha)+exp(-alpha*alpha*T*T)/(alpha *1.772453855);
co=co-(T-1)*erf((T-1)*alpha)-exp(-alpha*alpha*(T-1)*(T-1))/(alpha*sqrt(pi));
co=0.5+0.5*co;
```

```
function  hb=butter_channel(f,N)
% Used in Problem 6.51
% Butterworth filter of order 2N=10;
% M. Sellathurai

[B, A]=butter(N, f/64);
[H,w]=freqz(B,A,128,'whole');
hb=ifft(H);
```

Answer to Problem 6.51

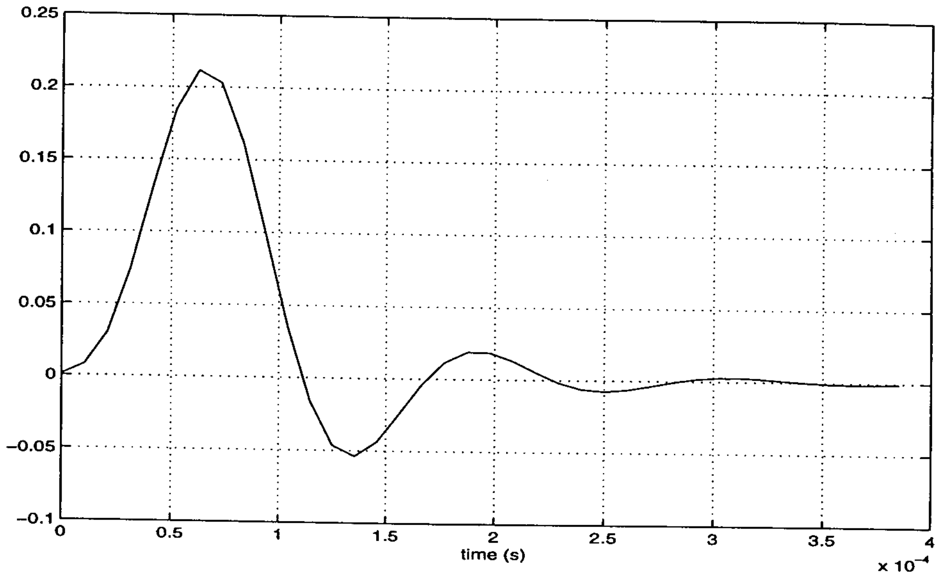
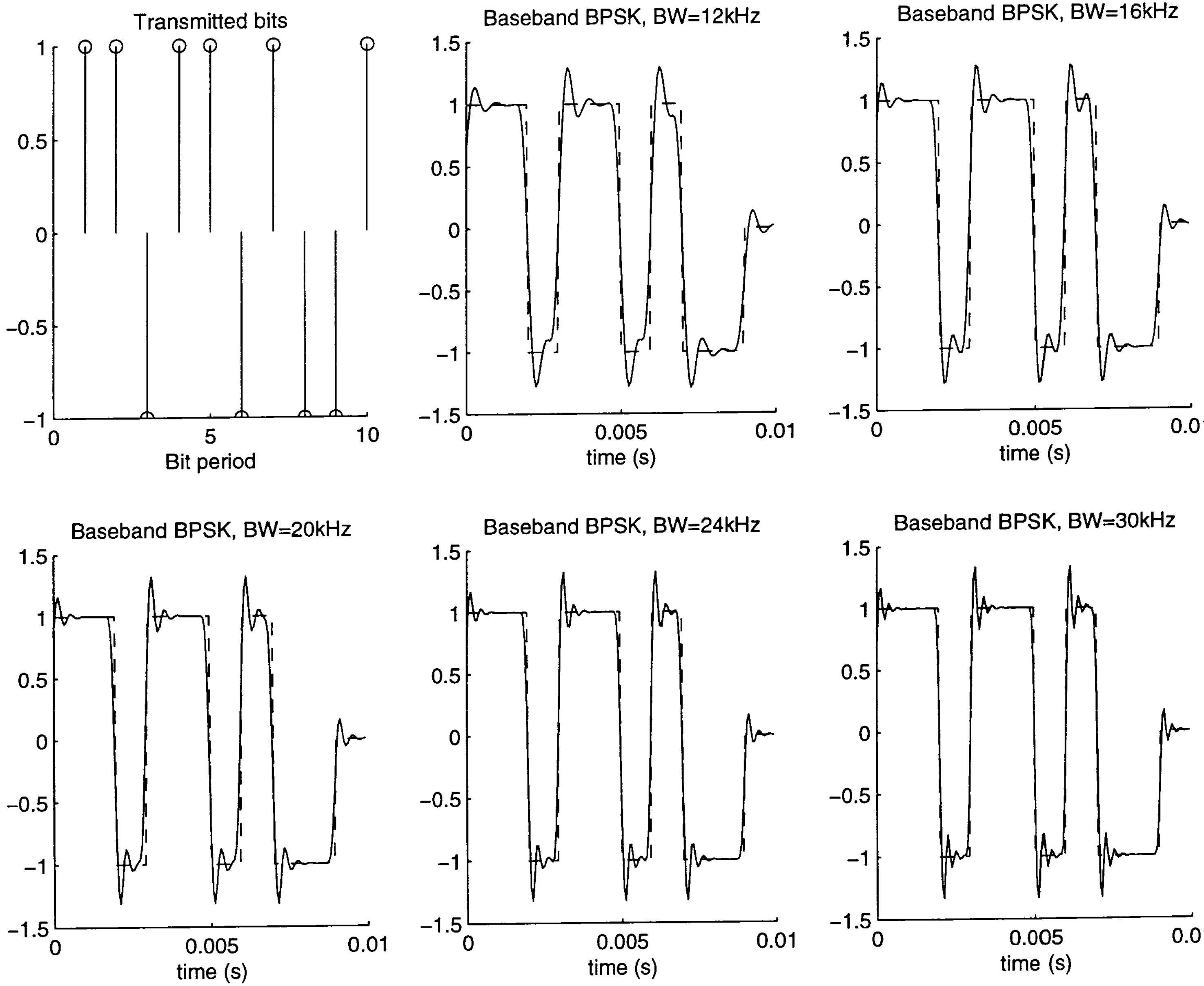
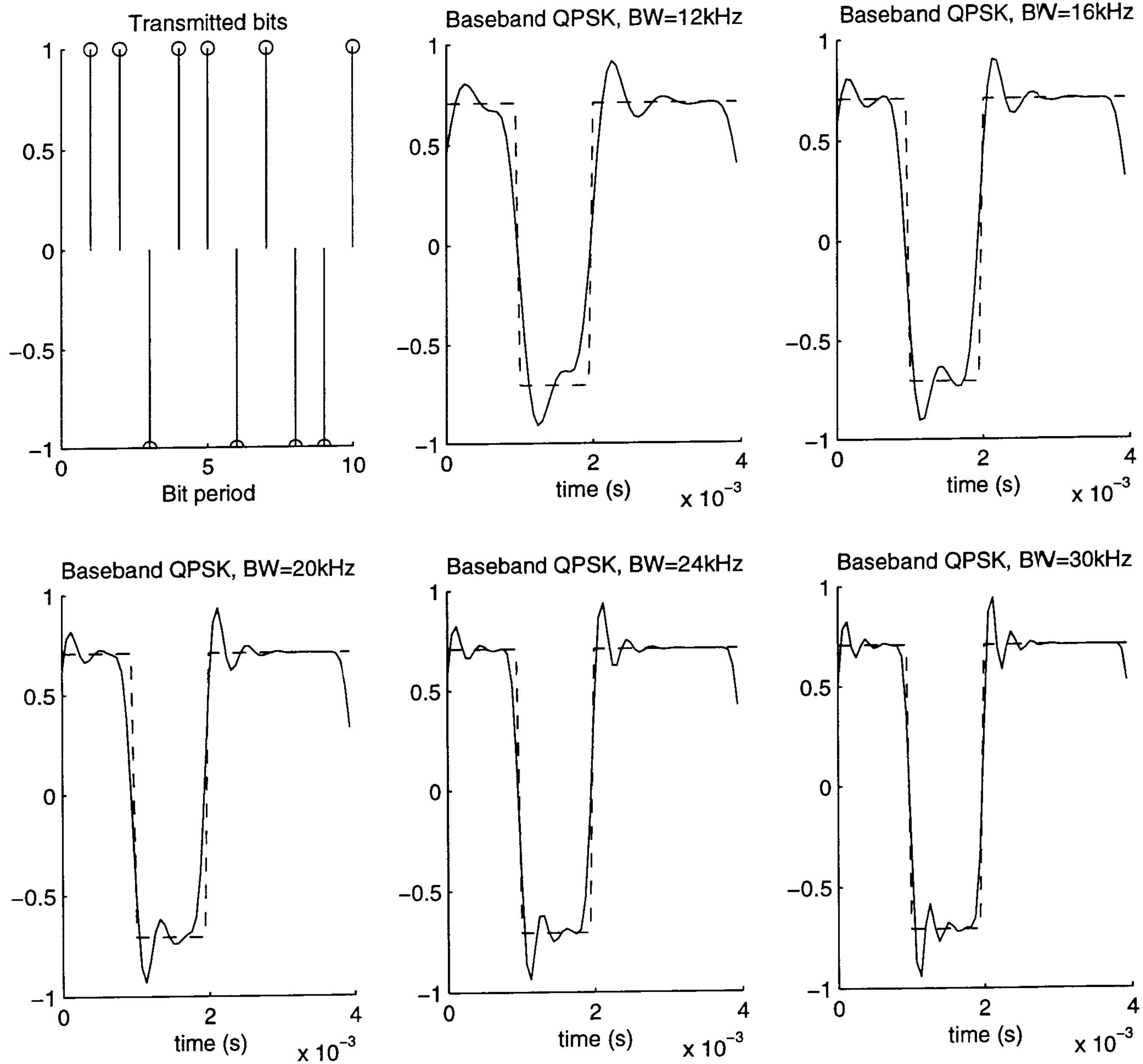
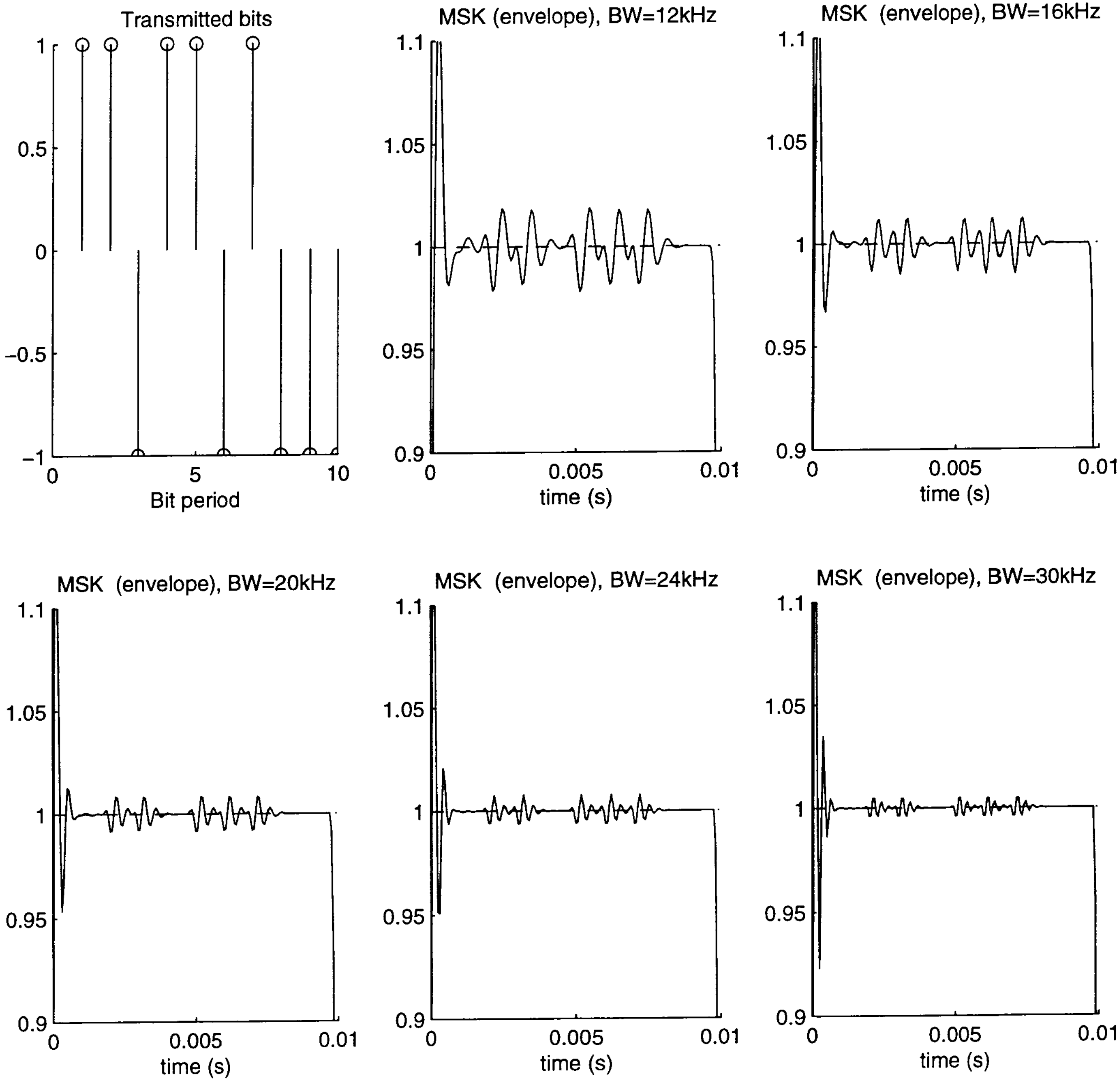


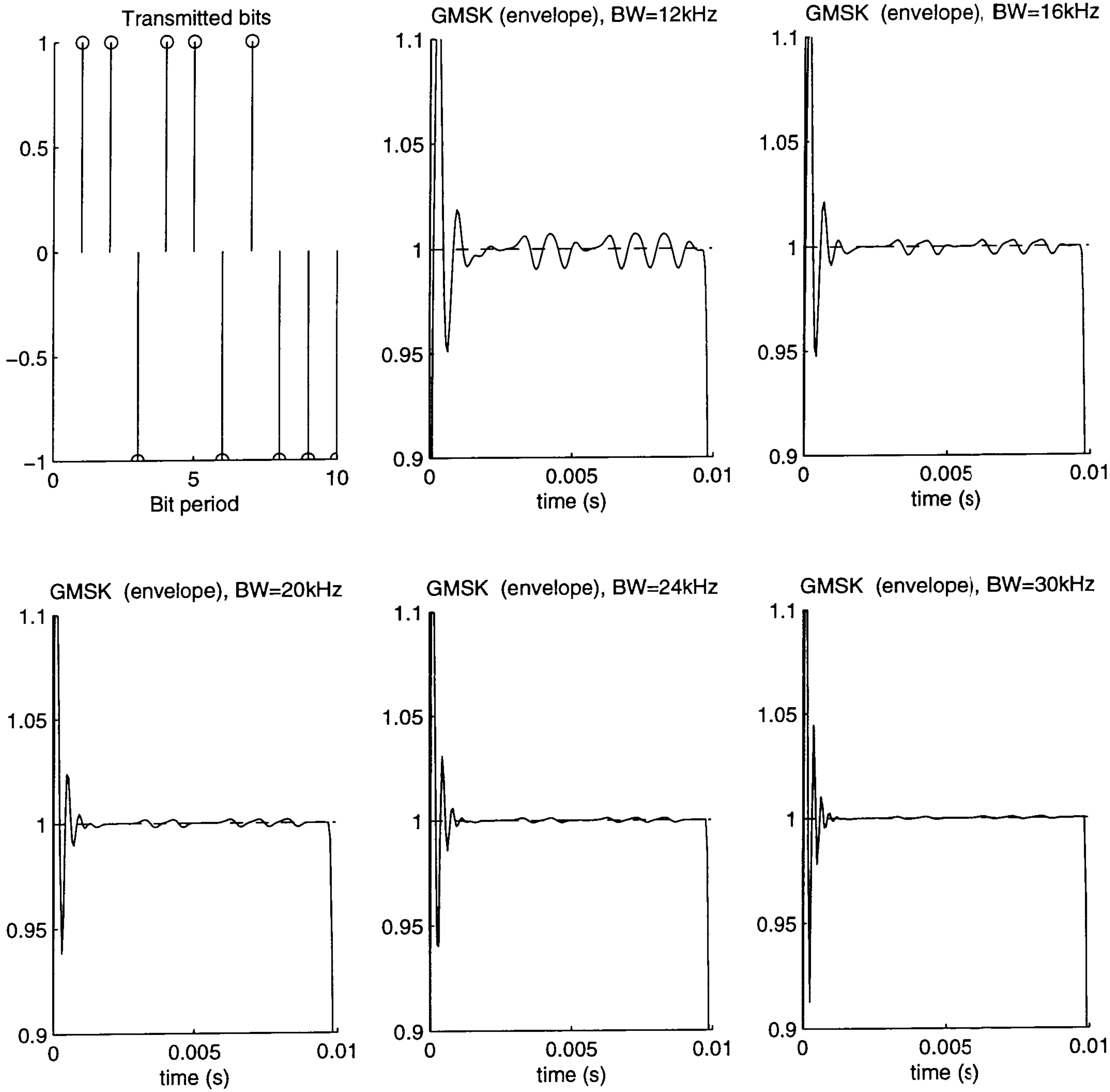
Figure 1: Butterworth baseband filter of order N=5



Transmitted bits
Bit period
Baseband BPSK, BW=12kHz
Baseband BPSK, BW=16kHz
Baseband BPSK, BW=20kHz
Baseband BPSK, BW=24kHz
Baseband BPSK, BW=30kHz
time (s)


Transmitted bits
Bit period
Baseband QPSK, BW=12kHz
Baseband QPSK, BW=16kHz
Baseband QPSK, BW=20kHz
Baseband QPSK, BW=24kHz
Baseband QPSK, BW=30kHz
time (s)
x 10^{-3}


Transmitted bits
Bit period
MSK (envelope), BW=12kHz
MSK (envelope), BW=16kHz
MSK (envelope), BW=20kHz
MSK (envelope), BW=24kHz
MSK (envelope), BW=30kHz
time (s)


Transmitted bits
Bit period
GMSK (envelope), BW=12kHz
GMSK (envelope), BW=16kHz
GMSK (envelope), BW=20kHz
GMSK (envelope), BW=24kHz
GMSK (envelope), BW=30kHz
time (s)
1.1
1.05
1
0.95
0.9
0
0.005
0.01
0.5
0
−0.5
−1
5
10

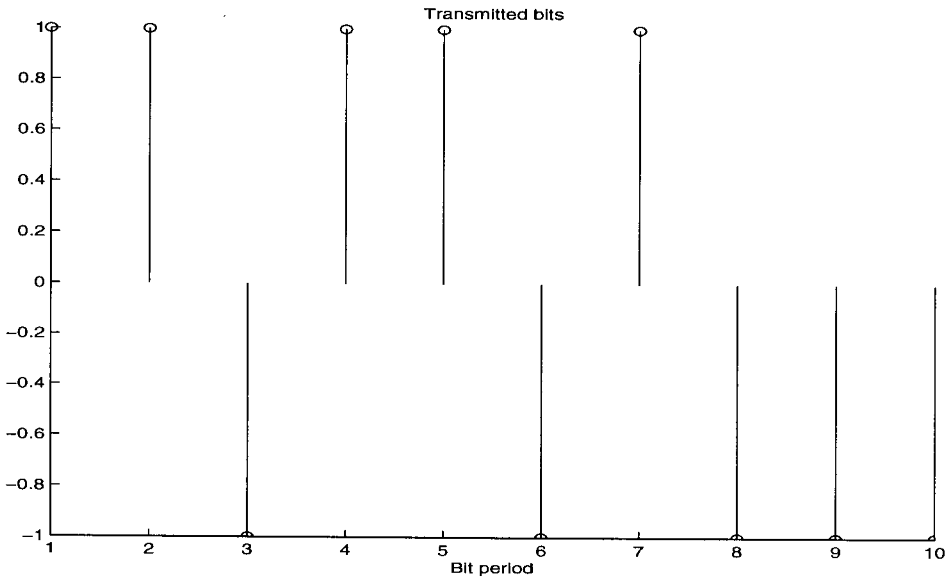


Figure 2 Transmitted bits

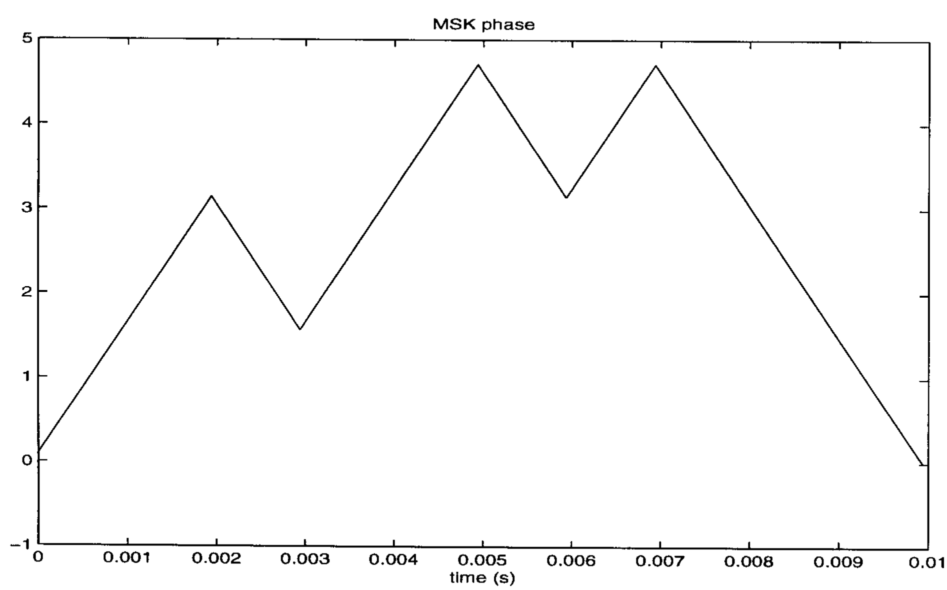


Figure 3 Phase of baseband MSK signal

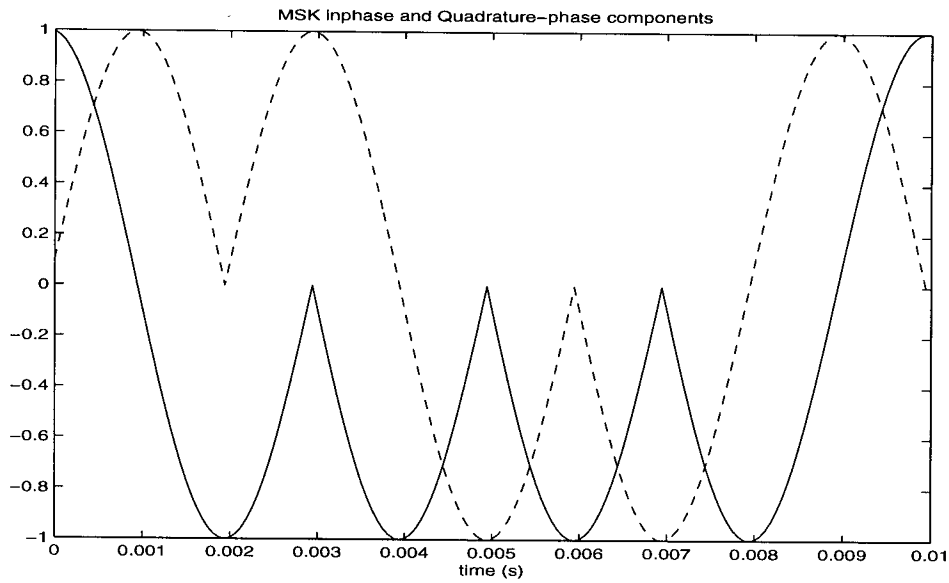


Figure 4 I and Q components of baseband MSK signal

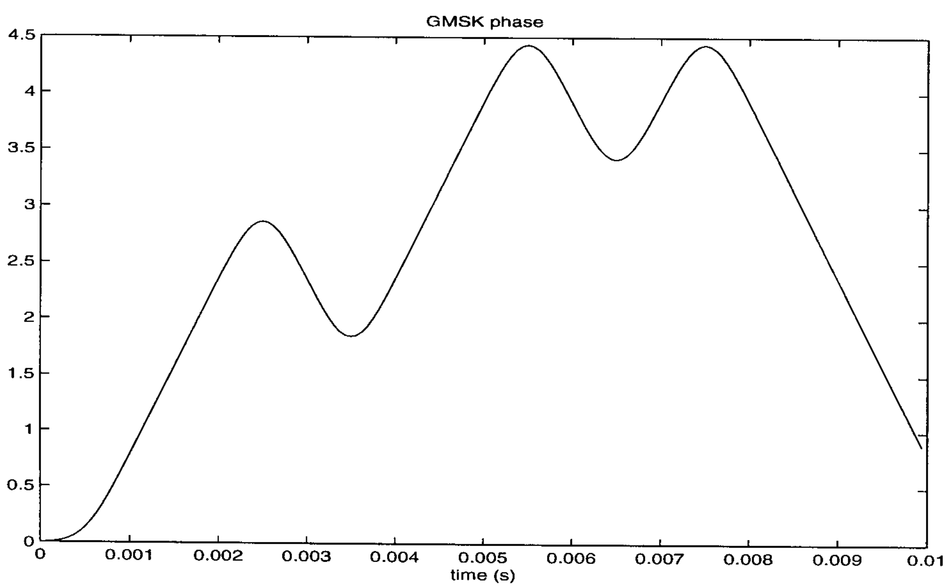


Figure 5 Phase of baseband GMSK signal

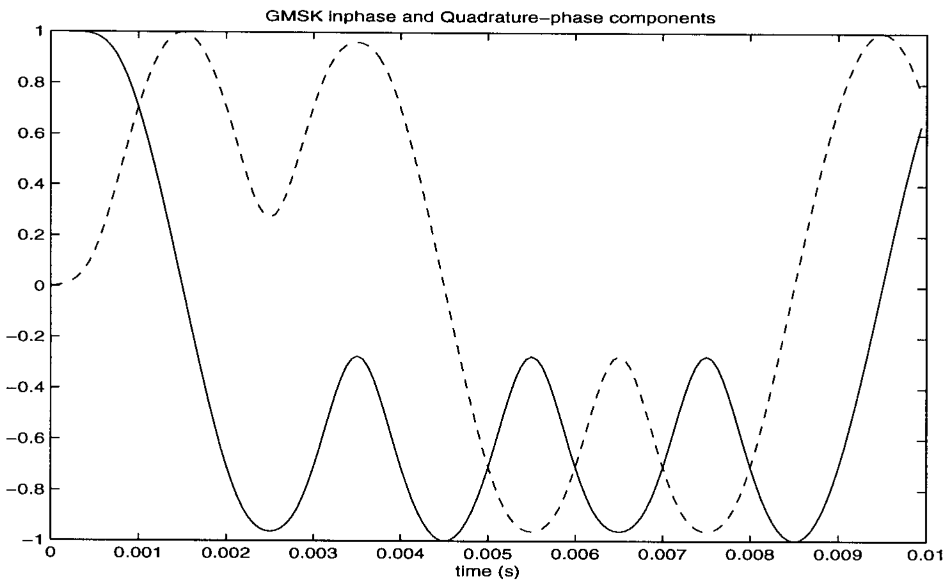


Figure 6: I and Q components of baseband GMSK signal

CHAPTER 7

Spread-spectrum Modulation

Problem 7.1

(a) The PN sequence length is

$$N = 2^m - 1 = 2^4 - 1 = 15$$

(b) The chip duration is

$$T_c = \frac{1}{10^7} s = 0.1 \ \mu s$$

(c) The period of the PN sequence is

$$T = NT_c$$
$$= 15 \times 0.1 = 1.5 \ \mu s$$

Problem 7.2

Shift number	Shift-register contents	Modulo-2 adder output	Shift-register output
0	1000		
1	0100	0 + 0 = 0	0
2	0010	0 + 0 = 0	0
3	1001	1 + 0 = 1	0
4	1100	0 + 1 = 1	1
5	0110	0 + 0 = 0	0
6	1011	1 + 0 = 1	0
7	0101	1 + 1 = 0	1
8	1010	0 + 1 = 1	1

9	1101	1 + 0 = 1	0
10	1110	0 + 1 = 1	1
11	1111	1 + 0 = 1	0
12	0111	1 + 1 = 0	1
13	0011	1 + 1 = 0	1
14	0001	1 + 1 = 0	1
15	1000	0 + 1 = 1	1

The output sequence is therefore

$$11\,\underbrace{000100110101111}_{\text{one period}}\,0001$$

Problem 7.3

(a) From both Table 7.2a and Table 7.2b we note the following:

Balance property:

Number of 1s in one period = 16
Number of 0s in one period = 15

Hence, the number of 1s exceeds the number of 0s by one.

(b) Run property:

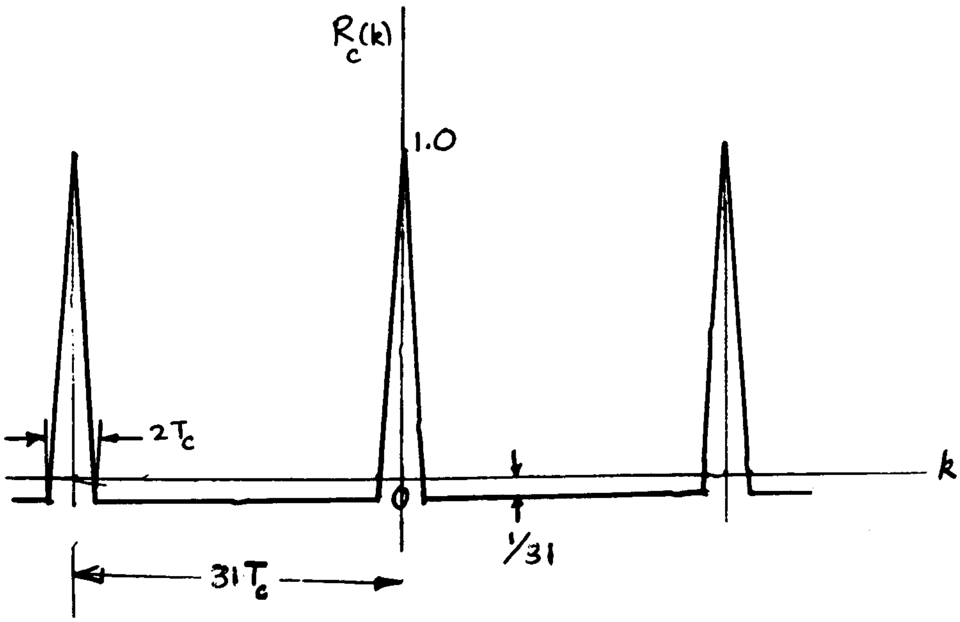
In both Tables 7.2a and 7 2b, we count a total of 8 runs of 1s and a total of 8 runs of 0s. Moreover, we note the following:

Runs of length 1 : 4
Runs of length 2 : 2
Runs of length 3 : 1

(c) Autocorrelation function:

$$R_c(k) = \begin{cases} 1, & k = lN \\ -\dfrac{1}{N}, & k \neq lN \end{cases}$$

Hence, we have (not to scale)



Problem 7.4

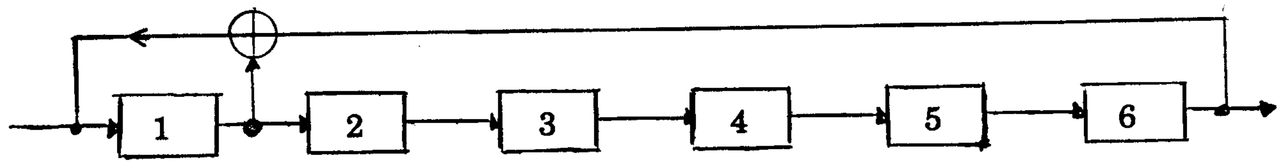


Table 1

Feedback symbol	State of Feedback-shift register	Output symbol
	1 0 0 0 0 0	
1	1 1 0 0 0 0	0
1	1 1 1 0 0 0	0
1	1 1 1 1 0 1	0
1	1 1 1 1 1 0	0
1	1 1 1 1 1 1	0
0	0 1 1 1 1 1	1
1	1 0 1 1 1 1	1
0	0 1 0 1 1 1	1
1	1 0 1 0 1 1	1
0	0 1 0 1 0 1	1
1	1 0 1 0 1 0	1
1	1 1 0 1 0 1	0
0	0 1 1 0 1 0	1
0	0 0 1 1 0 1	0
1	1 0 0 1 1 0	1
1	1 1 0 0 1 1	0
0	0 1 1 0 0 1	1
1	1 0 1 1 0 0	1
1	1 1 0 1 1 0	0
1	1 1 1 0 1 1	0
0	0 1 1 1 0 1	1
1	1 0 1 1 1 0	1

Table 1 continued

Feedback symbol	State of feedback-shift register	Output symbol
1	1 1 0 1 1 1	0
0	0 1 1 0 1 1	1
1	1 0 1 1 0 1	1
0	0 1 0 1 1 0	1
0	0 0 1 0 1 1	0
1	1 0 0 1 0 1	1
0	0 1 0 0 1 0	1
0	0 0 1 0 0 1	0
1	1 0 0 1 0 0	1
1	1 1 0 0 1 0	0
1	1 1 1 0 0 1	0
0	0 1 1 1 0 0	1
0	0 0 1 1 1 0	0
0	0 0 0 1 1 1	0
1	1 0 0 0 1 1	1
0	0 1 0 0 0 1	1
1	1 0 1 0 0 0	1
1	1 1 0 1 0 0	0
1	1 1 1 0 1 0	0
1	1 1 1 1 0 1	0
0	0 1 1 1 1 0	1
0	0 0 1 1 1 1	0
1	1 0 0 1 1 1	1
0	0 1 0 0 1 1	1
1	1 0 1 0 0 1	1
0	0 1 0 1 0 0	1
0	0 0 1 0 1 0	1

Table 1 continued

Feedback symbol	State of feedback-shift register	Output symbol
0	0 0 0 1 0 1	0
1	1 0 0 0 1 0	1
1	1 1 0 0 0 1	0
0	0 1 1 0 0 0	1
0	0 0 1 1 0 0	0
0	0 0 0 1 1 0	0
0	0 0 0 0 1 1	0
1	1 0 0 0 0 1	0
0	0 1 0 0 0 0	1
0	0 0 1 0 0 0	0
0	0 0 0 1 0 0	0
0	0 0 0 0 1 0	0
0	0 0 0 0 0 1	0
1	1 0 0 0 0 0	1

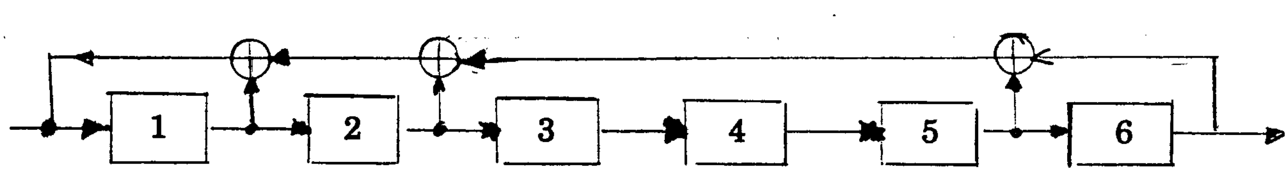


Table 2

Feedback symbol	State of feedback-shift register 1 0 0 0 0 0	Output symbol
1	1 1 0 0 0 0	0
0	0 1 1 0 0 0	0
1	1 0 1 1 0 0	0
1	1 1 0 1 1 0	0
1	1 1 1 0 1 1	0
0	0 1 1 1 0 1	1
0	0 0 1 1 1 0	1
1	1 0 0 1 1 1	0
1	1 1 0 0 1 1	1
0	0 1 1 0 0 1	1
0	0 0 1 1 0 0	1
0	0 0 0 1 1 0	0
1	1 0 0 0 1 1	0
1	1 1 0 0 0 1	1
1	1 1 1 0 0 0	1
0	0 1 1 1 0 0	0
1	1 0 1 1 1 0	0
0	0 1 0 1 1 1	0
1	1 0 1 0 1 1	1
1	1 1 0 1 0 1	1
1	1 1 1 0 1 0	1
1	1 1 1 1 0 1	0
1	1 1 1 1 1 0	1
1	1 1 1 1 1 1	0
0	0 1 1 1 1 1	1

Table 2 continued

Feedback symbol	State of feedback-shift register	Output symbol
1	1 0 1 1 1 1	1
1	1 1 0 1 1 1	1
0	0 1 1 0 1 1	1
1	1 0 1 1 0 1	1
0	0 1 0 1 1 0	1
0	0 0 1 0 1 1	0
0	0 0 0 1 0 1	1
1	1 0 0 0 1 0	1
0	0 1 0 0 0 1	0
0	0 0 1 0 0 0	1
0	0 0 0 1 0 0	0
0	0 0 0 0 1 0	0
1	1 0 0 0 0 1	0
0	0 1 0 0 0 0	1
1	1 0 1 0 0 0	0
1	1 1 0 1 0 0	0
0	0 1 1 0 1 0	0
0	0 0 1 1 0 1	0
1	1 0 0 1 1 0	1
0	0 1 0 0 1 1	0
1	1 0 1 0 0 1	1
0	0 1 0 1 0 0	1
1	1 0 1 0 1 0	0
0	0 1 0 1 0 1	0
0	0 0 1 0 1 0	1

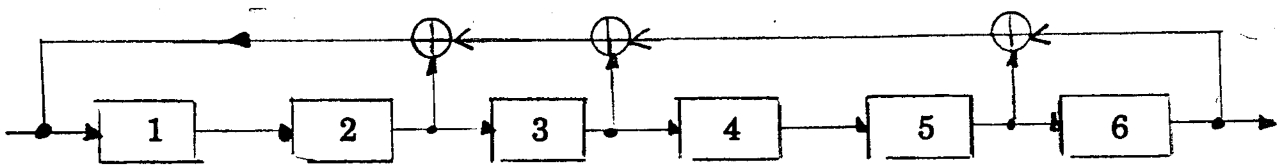


Table 3

Feedback symbol	State of feed-back shift register 1 0 0 0 0 0	Output symbol
0	0 1 0 0 0 0	0
1	1 0 1 0 0 0	0
1	1 1 0 1 0 0	0
1	1 1 1 0 1 0	0
1	1 1 1 1 0 1	0
1	1 1 1 1 1 0	1
1	1 1 1 1 1 1	0
0	0 1 1 1 1 1	1
0	0 0 1 1 1 1	1
1	1 0 0 1 1 1	1
0	0 1 0 0 1 1	1
1	1 0 1 0 0 1	1
0	0 1 0 1 0 0	1
1	1 0 1 0 1 0	0
0	0 1 0 1 0 1	0
0	0 0 1 0 1 0	1
0	0 0 0 1 0 1	0
1	1 0 0 0 1 0	1
1	1 1 0 0 0 1	0
0	0 1 1 0 0 0	1
0	0 0 1 1 0 0	0

Table 3 continued

Feedback symbol	State of feedback-shift register	Output symbol
1	1 0 0 1 1 0	0
1	1 1 0 0 1 1	0
1	1 1 1 0 0 1	1
1	1 1 1 1 0 0	1
0	0 1 1 1 1 0	0
1	1 0 1 1 1 1	0
1	1 1 0 1 1 1	1
1	1 1 1 0 1 1	1
0	0 1 1 1 0 1	1
1	1 0 1 1 1 0	1
0	0 1 0 1 1 1	0
1	1 0 1 0 1 1	1
1	1 1 0 1 0 1	1
0	0 1 1 0 1 0	1
1	1 0 1 1 0 1	0
0	0 1 0 1 1 0	1
0	0 0 1 0 1 1	0
1	1 0 0 1 0 1	1
1	1 1 0 0 1 0	1
0	0 1 1 0 0 1	0
1	1 0 1 1 0 0	1
1	1 1 0 1 1 0	0
0	0 1 1 0 1 1	0
0	0 0 1 1 0 1	1
0	0 0 0 1 1 0	1

Table 3 continued

Feedback symbol	State of feedback-shift register	Output symbol
1	1 0 0 0 1 1	0
0	0 1 0 0 0 1	1
0	0 0 1 0 0 0	1
1	1 0 0 1 0 0	0
0	0 1 0 0 1 0	0
0	0 0 1 0 0 1	0
0	0 0 0 1 0 0	1
0	0 0 0 0 1 0	0
1	1 0 0 0 0 1	0
1	1 1 0 0 0 0	1
1	1 1 1 0 0 0	0
0	0 1 1 1 0 0	0
0	0 0 1 1 1 0	0
0	0 0 0 1 1 1	0
0	0 0 0 0 1 1	1
0	0 0 0 0 0 1	1
1	1 0 0 0 0 0	1

Table 3 continued

Feedback symbol	State of feedback-shift register	Output symbol
1	1 0 0 1 0 1	0
0	0 1 0 0 1 0	1
0	0 0 1 0 0 1	0
1	1 0 0 1 0 0	1
1	1 1 0 0 1 0	0
1	1 1 1 0 0 1	0
1	1 1 1 1 0 0	1
0	0 1 1 1 1 0	0
0	0 0 1 1 1 1	0
0	0 0 0 1 1 1	1
0	0 0 0 0 1 1	1
0	0 0 0 0 0 1	1
1	1 0 0 0 0 0	1

Problem 7.5

	State of feedback-shift register	Output symbol
Initial state	1 0 0 0 0	
	0 1 0 0 0	0
	0 0 1 0 0	0
	0 0 0 1 0	0
	0 0 0 0 1	0
	1 1 1 0 1	1
	1 0 0 1 1	1
	1 0 1 0 0	1
	0 1 0 1 0	0
	0 0 1 0 1	0
	1 1 1 1 1	1
	1 0 0 1 0	1
	0 1 0 0 1	0
	1 1 0 0 1	1
	1 0 0 0 1	1
	1 0 1 0 1	1
	1 0 1 1 1	1
	1 0 1 1 0	1
	0 1 0 1 1	0
	1 1 0 0 0	1
	0 1 1 0 0	0
	0 0 1 1 0	0
	0 0 0 1 1	0
	1 1 1 0 0	1
	0 1 1 1 0	0
	0 0 1 1 1	0
	1 1 1 1 0	1
	0 1 1 1 1	0
	1 1 0 1 0	1
	0 1 1 0 1	0
	1 1 0 1 1	1
	1 0 0 0 0	1

The 31-element code generated by the scheme shown in Fig. P9.2 is exactly the same as that described in Table 9.2b. Note, however, the code described in Tab.e 9.2b appears in reversed order to that described in the above table; this reversal is clearly of a trivial nature.

Problem 7.6

(a) The modulo-2 sum of b(t) and c(t), on a pulse-by-pulse basis, is as follows

		b(t) = 0	b(t) = 1
c(t)	0	0	1
	1	1	0

(b) If symbol 0 is represented by a sinusoid of zero phase shift, and symbol 1 is represented by a sinusoid of 180° phase shift, the output of the modulo-2 adder takes on the same form as that described in Table 7.3 of the text.

Problem 7.7

$$j(t) = \sqrt{2J}\cos(2\pi f_c t + \theta)$$

The basis functions are

$$\phi_k(t) = \begin{cases} \sqrt{\dfrac{2}{T_c}}\cos(2\pi f_c t), & kT_c \le t \le (k+1)T_c \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{\phi}_k(t) = \begin{cases} \sqrt{\dfrac{2}{T_c}}\sin(2\pi f_c t), & kT_c \le t \le (k+1)T_c \\ 0, & \text{otherwise} \end{cases}$$

Hence, we may express the jamming signal j(t) as

$$j(t) = \sqrt{JT_c}\ \cos\theta \sum_{k=0}^{N-1} \phi_k(t)$$

$$- \sqrt{JT_c}\ \sin\theta \sum_{k=0}^{N-1} \tilde{\phi}_k(t)$$

Problem 7.8

The processing gain is

$$\frac{T_b}{T_c} = \frac{1/T_c}{1/T_b}$$

The spread bandwidth of the transmitted signal is proportional to $1/T_c$. The despread bandwidth of the received signal is proportional to $1/T_b$. Hence,

$$\text{Processing gain} = \frac{\text{spread bandwidth of transmitted signal}}{\text{despread bandwidth of received signal}}$$

Problem 7.9

$$m = 19$$

$$N = 2^m - 1 = 2^{19} - 1 \simeq 2^{19}$$

The processing gain is

$$10\log_{10}N \simeq 10\log_{10}2^{19}$$

$$= 190 \times 0.3$$

$$= 57 \text{ dB}$$

Problem 7.10

(a) Processing gain = $10\log_{10}(2^m-1) = 10\log_{10}(2^{19}-1) = 57$ dB

(b) Antijam margin = (Processing gain) - $10\log_{10}\left(\frac{E_b}{N_0}\right)$

The probability of error is

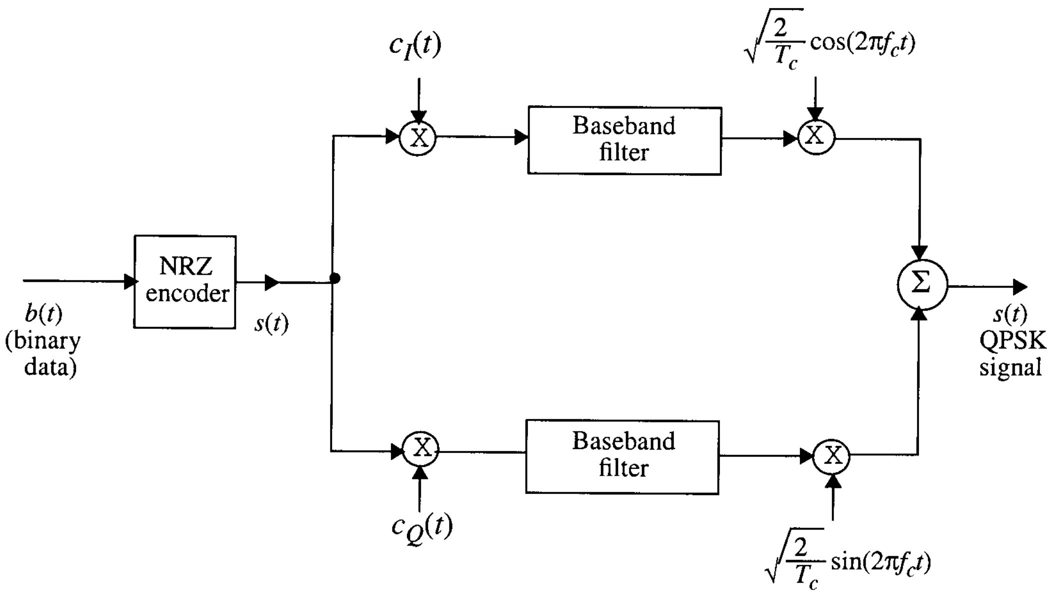
$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

With $P_e = 10^{-5}$, we have $E_b/N_0 = 9$. Hence,

Antijam margin $= 57 - 10\log_{10}9 = 57 - 9.5$
$= 47.5$ dB

Problem 7.11

The DS/QPSK signal modulator is given below:



The DS/QPSK modulated signal is

$$x(t) = \pm\sqrt{\frac{E}{T}}c_I(t)\cos(2\pi f_c t) \pm \sqrt{\frac{E}{T}}c_Q(t)\sin(2\pi f_c t)$$

where $\quad c_I(t) = \{c_{0,I}(t), c_{1,I}(t), \cdots, c_{N;-1,I}(t)\}$ and

$$c_Q(t) = \{c_{0,Q}(t), c_{1,Q}(t), \cdots, c_{N-1,Q}(t)\}$$

denote the spreading sequences for $0 \le t \le T_s$, which are applied to the in-phase and quadrature channels of the modulator.

Consider the following set of orthonormal basis functions:

$$\phi_{c_{I,k}}(t) = \begin{cases} \sqrt{\frac{2}{T_c}}\cos(2\pi f_c t), & kT_c \le t \le (k+1)T_c \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_{c_{Q,k}}(t) = \begin{cases} \sqrt{\dfrac{2}{T_c}}\sin((2\pi f_c t), \;) & kT_c \le t \le (k+1)T_c \\ 0, & \text{otherwise} \end{cases}$$

where T_c is the chip duration; $k = 0, 1, 2, \cdots, N\text{-}1$, and $N = T/T_c$, that is, N is the number of chips per bit.

The DS/QPSK modulated signal can be written as follows (using the set of basis functions):

$$s(t) = \pm\sqrt{\frac{T_c E}{2T}} \cdot \sqrt{\frac{2}{T_c}}\cos(2\pi f_c t)c_I(t) \pm \sqrt{\frac{T_c E}{2T}} \cdot \sqrt{\frac{2}{T_c}}\sin(2\pi f_c t)c_Q(t)$$

$$= \pm\sqrt{\frac{E_b}{2N}}\sum_{k=0}^{N\text{-}1} c_{I,k}\phi_{C_{I,k}}(t) \pm\sqrt{\frac{E_b}{2N}}\sum_{k=0}^{N\text{-}1} C_{Q,k}(t)\phi_{c_{Q,k}}(t)$$

The channel output at the receiving end of the system has the following form

$$x(t) = s(t) + j(t)$$

where $j(t)$ denotes the interference signal. We may express the interference signal using the $2N$-dimensional basis functions as follows:

$$j(t) = \sum_{k=0}^{N\text{-}1} c_{I,k}(t)\phi_{c_{I,k}}(t) + \sum_{k=0}^{N\text{-}1} j_{cQ,k}(t)\phi_{c_{Q,k}}(t)$$

where

$$j_{c_{I,k}} = \int_{kT_b}^{(k+1)T_b} j(t)\phi_{c_{I,k}}(t)dt$$

$$j_{c_{Q,k}} = \int_{kT_b}^{(k+1)T_b} j(t)\phi_{c_{Q,k}}(t)dt$$

$$k = 0, 1, \cdots, N\text{-}1$$

The average power of the interferer is given by

$$J = \frac{1}{T_b}\sum_{k=0}^{N-1} j_{c_{I,k}}^2 + \frac{1}{T_b}\sum_{k=0}^{N-1} j_{c_{Q_{I,k}}}^2$$

Assuming that the power is equally distributed between the in-phase and quadrature components:

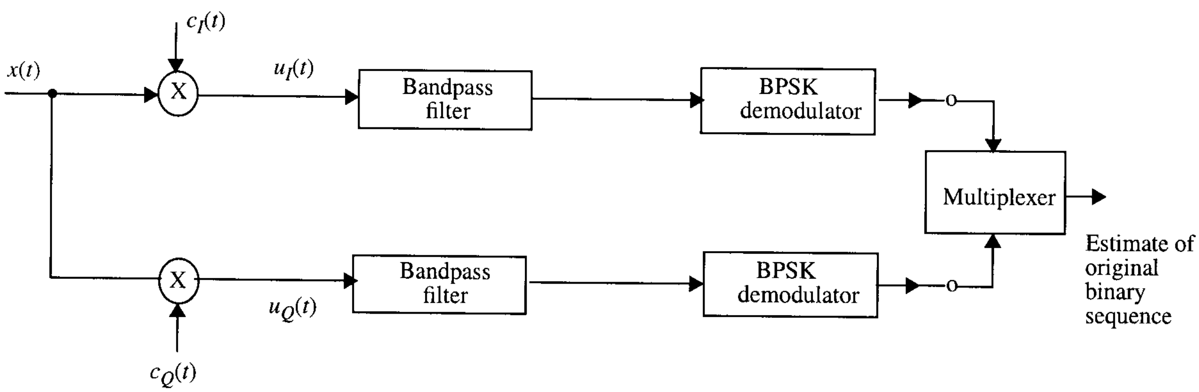
$$J = \frac{1}{T_b}\sum_{k=0}^{2(N-1)} j_{c_{I,k}}^2$$

The mean of the interference signal is zero. The variance of the interference signal is therefore

$$\sigma_{jam}^2 = \frac{1}{2N}\sum_{k=0}^{2(N-1)} j_{c_{I,k}}^2$$

$$= \frac{JT_c}{2}$$

Demodulation



There are two stages of demodulation. First, the received signal $x(t)$ is multiplied by the despreading sequences $c_I(t)$ and $c_Q(t)$, yielding

$$u_I(t) = \pm\sqrt{\frac{E}{T}}\cos(2\pi f_c t) \pm c_I(t)c_Q(t)\sqrt{\frac{E}{T}}\sin(2\pi f_c t) + c_I(t)j(t)$$

$$u_Q(t) = \pm\sqrt{\frac{E}{T}}\sin(2\pi f_c t) \pm c_Q(t)c_I(t)\sqrt{\frac{E}{T}}\cos(2\pi f_c t) + c_Q(t)j(t)$$

The second terms in the right-hand side of $u_I(t)$ and $u_Q(t)$ are filtered by the bandpass filters, and the BPSK demodulators recover estimates of their respective binary sequences. Finally, the multiplexer reconstructs the original binary data stream.

Processing gain

The signal-to-noise ratio at the output of the receiver is

$$(\text{SNR})_0 = \frac{\text{Instantaneous peak signal power}}{\sigma_{\text{jam}}^2}$$

$$= \frac{E}{JT_c/2} = \frac{2E}{JT_c}$$

The signal-to-noise ratio at the input of the coherent receiver is

$$(\text{SNR})_I = \frac{\text{average input-signal power}}{\text{average interferer power}}$$

$$= \frac{E/T}{J} = \frac{E}{JT}$$

We may therefore write

$$10\log_{10}\left[\frac{(\text{SNR})_0}{(\text{SNR})_I}\right] = 10\log_{10}\left(\frac{2T}{T_c}\right) = 3 + 10\log_{10}\left(\frac{T}{T_c}\right)$$

The QPSK processing gain= T/T_c

$$= \frac{2T_b}{T_c}$$

That is,

$$PG_{\text{QPSK}} = 2[PG_{\text{BPSK}}]$$

Solving for the antenna aperture:

Problem 7.12

The processing gain (PG) is

$$PG = \frac{\text{FH bandwidth}}{\text{symbol rate}}$$

$$= \frac{W_c}{R_s}$$

$$= 5 \times 4 = 20$$

Hence, expressed in decibels,

$$PG = 10\log_{10}20$$

$$= 26 \text{ db}$$

Problem 7.13

The processing gain is

$$PG = 4 \times 4$$

$$= 16$$

Hence, in decibels,

$$PG = 10\log_{10}16$$

$$= 12 \text{ dB}$$

Problem 7.13

Matlab codes

```
% Problem 7.13(a), CS: Haykin
% Generating 63-chip PN sequences
% polynomial1(x)  = x^6 + x + 1
% polynomial2(x) = x^6 + x^5 + x^2 + x + 1
% Mathini Sellathurai, 10.05.1999

% polynomials
pol1=[1 0 0 0 0 1 1];
pol2=[1 1 0 0 1 1 1];

% chip size
N=63;

% generating the PN sequence
pnseq1 = PNseq(pol1);
pnseq2 = PNseq(pol2);

% mapping  antipodal signals (0-->-1,  1-->1)
u=2*pnseq1-1;
v=2*pnseq2-1;
```

```
% autocorrelation of pnseq1
[corrf]=pn_corr(u, u, N)

% prints
plot(-61:62,corrf(2:125)); axis([-62, 62,-10, 80])
xlabel(' Delay \tau')
ylabel(' Autocorrelation function R_{c}(\tau)')

pause

%autocorrelation of pnseq2
[corrf]=pn_corr(v, v, N)


% prints
plot(-61:62,corrf(2:125)); axis([-62, 62,-10, 80])
xlabel(' Delay \tau')
ylabel(' Autocorrelation function R_{c}(\tau)')

pause

% cross correlation of pnseq1, pnseq2
[c_corr]=pn_corr(u, v, N)

% prints
plot(-61:62,c_corr(2:125)); axis([-62, 62,-20, 20])
xlabel(' Delay \tau')
ylabel(' Cross-correlation function R_{ji}(\tau)')
```

```
% Problem 7.13 (b), CS: Haykin
% Generating 63-chip PN sequences
% polynomial1(x)  = x^6 + x + 1
% polynomial2(x) = x^6 + x^5 + x^2 + x + 1
% Mathini Sellathurai, 10.05.1999

% polynomials
pol1=[1 1 1 0 0 1 1];
pol2=[1 1 0 0 1 1 1];

% chip size
N=63;

% generating the PN sequence
pnseq1 = PNseq(pol1);
pnseq2 = PNseq(pol2);

% mapping  antipodal signals (0-->-1,  1-->1)
u=2*pnseq1-1;
v=2*pnseq2-1;

% autocorrelation of pnseq1
[corrf]=pn_corr(u, u, N)

% prints
plot(-61:62,corrf(2:125)); axis([-62, 62,-10, 80])
xlabel(' Delay \tau')
ylabel(' Autocorrelation function R_{c}(\tau)')

pause

%autocorrelation of pnseq2
[corrf]=pn_corr(v, v, N)


% prints
plot(-61:62,corrf(2:125)); axis([-62, 62,-10, 80])
xlabel(' Delay \tau')
ylabel(' Autocorrelation function R_{c}(\tau)')

pause

% cross correlation of pnseq1, pnseq2
[c_corr]=pn_corr(u, v, N)

% prints
```

```
plot(-61:62,c_corr(2:125)); axis([-62, 62,-20, 20])
xlabel(' Delay \tau')
ylabel(' Cross-correlation function R_{ji}(\tau)')
```

```
function x =  PNseq(p)
% Linear shift register for generating PN  sequence of polynomial p
% used for problems 7.13, 7.14 of CS: Haykin
% Mathini Sellathurai, 10.05.1999

N = length(p) - 1; % order of the polynomial
p = fliplr(p);
X = [1 zeros(1, N-1)];
n = 1;

for i = 1 : n*(2^N - 1)
  x(i) = X(1);
  X = [X(2:N) p(N+1) * rem(sum(p(1:N) .* X(1:N)), 2)];
end
```

0

```
function [corrf]=pn_corr(u, v, N)

% funtion to compute the autocorreation/ cross-correlation
%  function of two PN sequences
% used in problem 7.13, 7.14, CS: Haykin
% Mathini Sellathurai, 10 june 1999.

max_cross_corr=0;

for m=0:N
shifted_u=[u(m+1:N) u(1:m)];
corr(m+1)=(sum(v.*shifted_u));
if (abs(corr)>max_cross_corr)
max_cross_corr=abs(corr);
end
end

corr1=flipud(corr);
corrf=[corr1(2:N) corr];
```

Answer to Problem 7.13

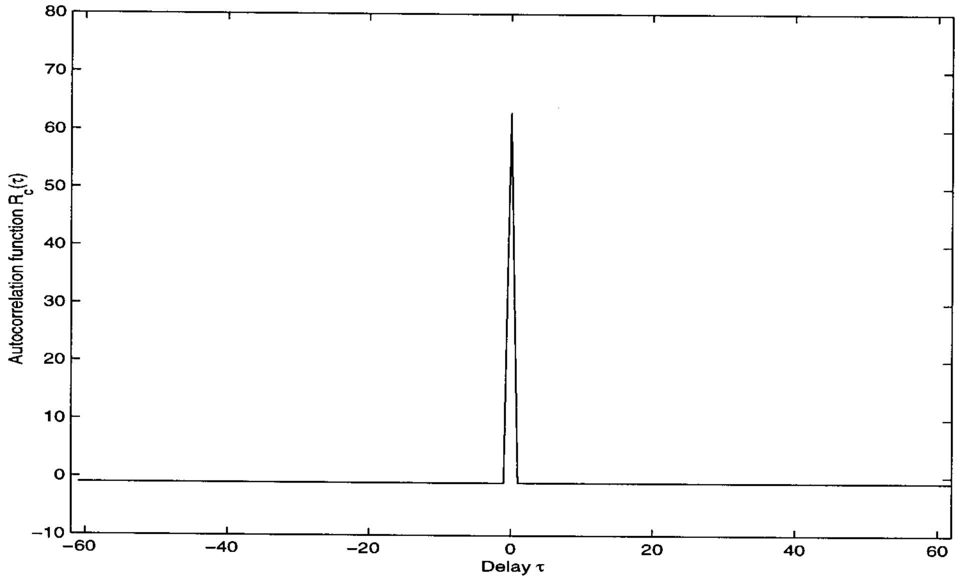


Figure 1: Autocorrelation function of [6,5,2,1],[6,1]

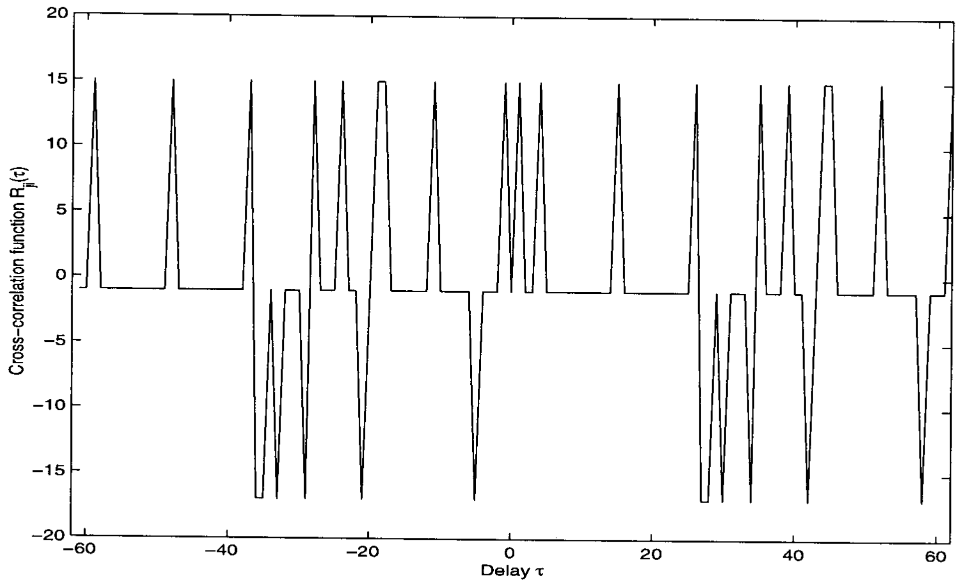


Figure 2: Cross-correlation function of [6,5,2,1],[6,1]

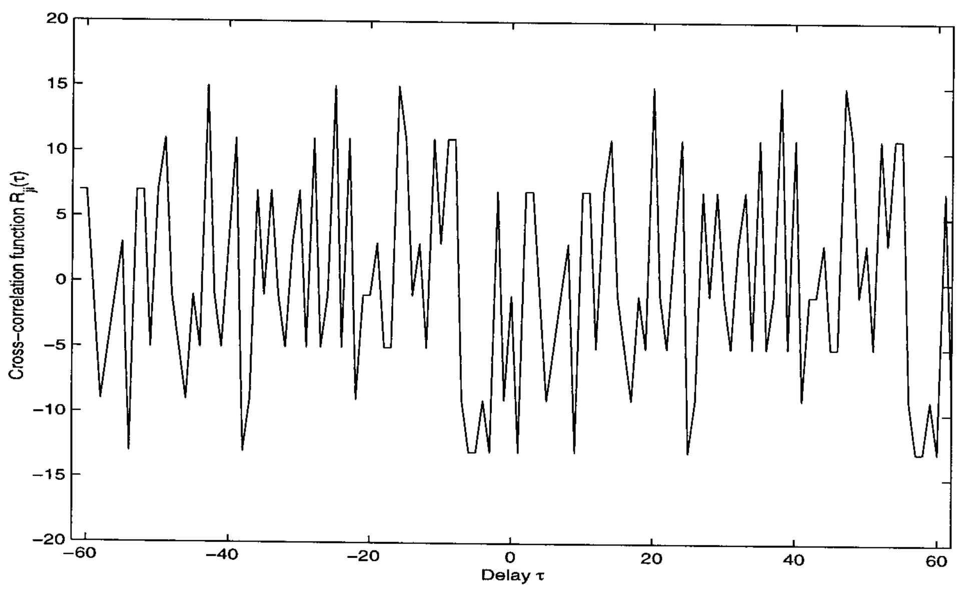


Figure 3: Cross-correlation function of [6,5,2,1],[6,5,4,1]

Problem 7.14

Matlab codes

```
% Problem 7.14 (a), CS: Haykin
% Generating 31-chip PN sequences
% polynomial1(x)  = x^5 + x^2 + 1
% polynomial2(x) = x^5 + x^3  + 1
% Mathini Sellathurai, 10.05.1999

% polynomials
pol1=[1 0 0 1 0 1];
pol2=[1 0 1 0 0 1];

% chip size
N=31;

% generating the PN sequence
pnseq1 = PNseq(pol1);
pnseq2 = PNseq(pol2);

% mapping  antipodal signals (0-->-1,  1-->1)
u=2*pnseq1-1;
v=2*pnseq2-1;

% cross correlation of pnseq1, pnseq2
[c_corr]=pn_corr(u, v, N)

% prints
plot(-30:31,c_corr); axis([-30, 31,-15, 15])
xlabel(' Delay \tau')
ylabel(' Cross-correlation function R_{ji}(\tau)')
```

```
% Problem 7.14 (b), CS: Haykin
% Generating 63-chip PN sequences
% polynomial1(x)  = x^5 + x^3 + 1
% polynomial2(x) = x^5 + x^4 + x^2 + x + 1
% Mathini Sellathurai, 10.05.1999

% polynomials
pol1=[1 0 1 0 0 1];
pol2=[1 1 0 1 1 1];

% chip size
N=31;

% generating the PN sequence
pnseq1 = PNseq(pol1);
pnseq2 = PNseq(pol2);

% mapping  antipodal signals (0-->-1,  1-->1)
u=2*pnseq1-1;
v=2*pnseq2-1;

% cross correlation of pnseq1, pnseq2
[c_corr]=pn_corr(u, v, N)

% prints
plot(-30:31,c_corr); axis([-30, 31,-10, 10])
xlabel(' Delay \tau')
ylabel(' Cross-correlation function R_{ji}(\tau)')
```

```
% Problem 7.14 (c), CS: Haykin
% Generating 63-chip PN sequences
% polynomial1(x)  = x^5 + x^4 + x^3+1
% polynomial2(x) = x^5 + x^4 + x^2 + x + 1
% Mathini Sellathurai, 10.05.1999

% polynomials
pol1=[1 1 1 1 0 1];
pol2=[1 1 0 1 1 1];

% chip size
N=31;

% generating the PN sequence
pnseq1 = PNseq(pol1);
pnseq2 = PNseq(pol2);

% mapping  antipodal signals (0-->-1,  1-->1)
u=2*pnseq1-1;
v=2*pnseq2-1;

% cross correlation of pnseq1, pnseq2
[c_corr]=pn_corr(u, v, N)

% prints
plot(-30:31,c_corr); axis([-30, 30,-10, 10])
xlabel(' Delay \tau')
ylabel(' Cross-correlation function R_{ji}(\tau)')
```

Answer to Problem 7.14

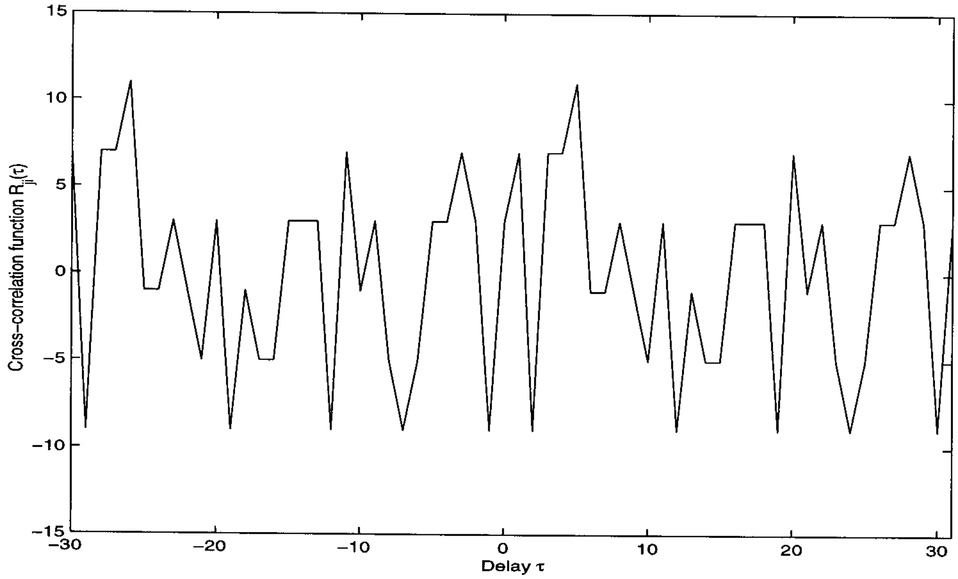


Figure 1: Cross-correlation function of [5,3],[5,2]

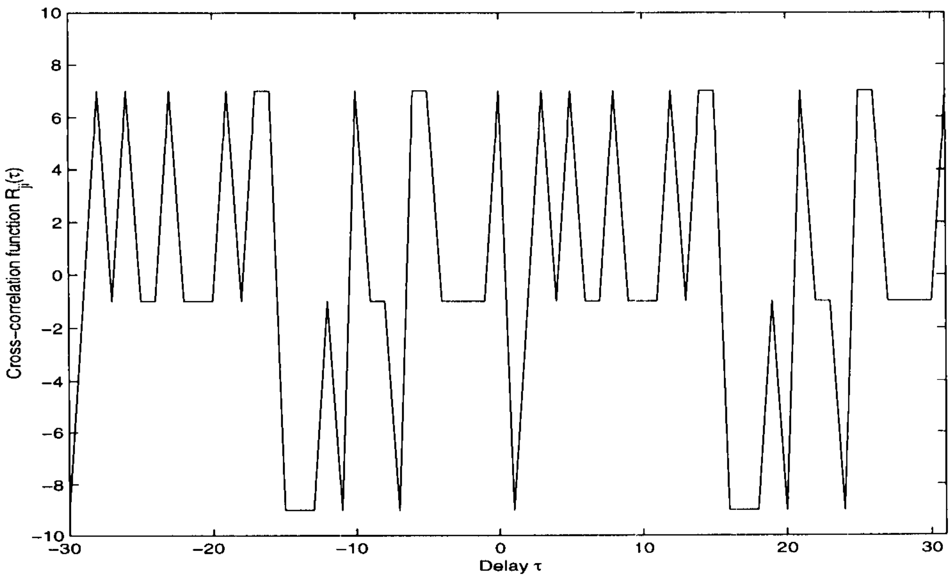


Figure 2: Cross-correlation function of [6,5,2,1],[6,1]

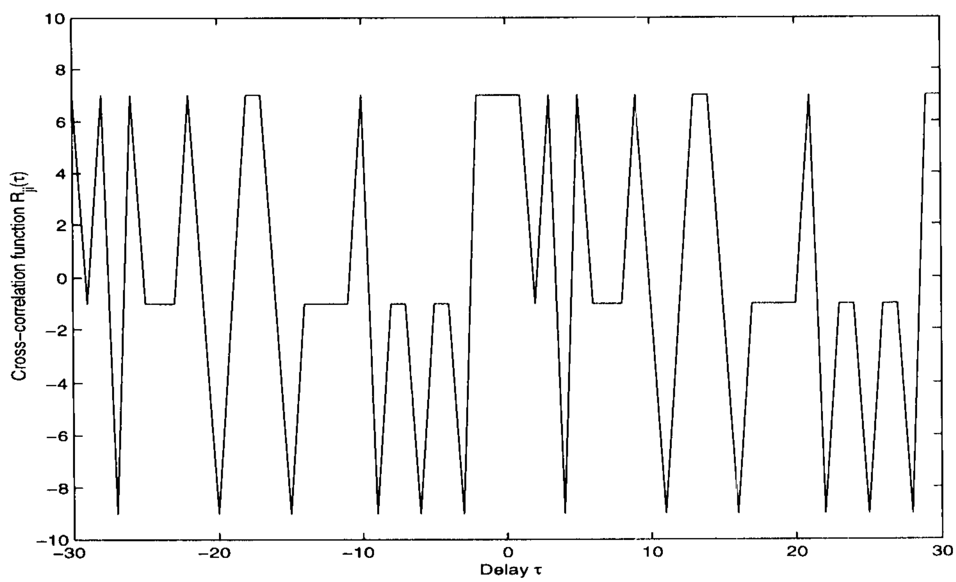


Figure 3: Cross-correlation function of [6,5,2,1],[6,5,4,1]

Problem 8.1

(a) Free space loss $= 10\log_{10}\left(\frac{4\pi d}{\lambda}\right)^2$

$$= 20\log_{10}\left(\frac{4\times\pi\times 150}{3\times 10^8/4\times 10^9}\right)\text{dB}$$

$$= 88 \text{ dB}$$

(b) The power gain of each antenna is

$$10\log_{10}G_r = 10\log_{10}G_t = 10\log_{10}\left(\frac{4\times\pi\times A}{\lambda^2}\right)$$

$$= 10\log_{10}\left(\frac{4\times\pi\times\pi\times 0.6}{(3/40)^2}\right)$$

$$= 36.24 \text{ dB}$$

(c) Received Power= Transmitted power $+G_r$ - Free space loss
= 1 + 36.24 - 88
= -50.76 dBW

Problem 8.2

The antenna gain and free-space loss at 12 GHz can be calculated by simply adding $20\log_{10}(12/4)$ for the values calculated in Problem 8.1 for downlink frequency 4 GHz. Specifically, we have:

(a) Free-space loss= $88 + 20\log_{10}(3)$
= 97.54 dB

(b) Power gain of each antenna
= $36.24 + 20\log_{10}(3)$
= 45.78 dB

(c) Received power = -50.76 dBW

The important points to note from the solutions to Problems 8.1 and 8.3 are:
1. Increasing the operating frequency produces a corresponding increase in free-space loss, and an equal increase in the power gain of each antenna.
2. The net result is that, in theory, the received power remains unchanged.

Problem 8.3

The Friis free-space equation is given by

$$P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi d}\right)^2$$

(a) Using the relationship

$$A_r = \frac{\lambda^2}{4\pi} G_r, \quad \text{and} \quad A_t = \frac{\lambda^2}{4\pi} G_t \text{, we may write}$$

$$P_r = P_t \left[\frac{4\pi A_t}{\lambda^2}\right] \left[\frac{4\pi A_r}{\lambda^2}\right] \left[\frac{\lambda}{4\pi d}\right]^2$$

$$= \frac{P_t A_t A_r}{\lambda^2 d^2} \tag{1}$$

(b) $$P_r = P_t \left[\frac{4\pi A_t}{\lambda^2}\right] G_r \left(\frac{\lambda}{4\pi d}\right)^2$$

$$= \frac{P_t A_t G_r}{4\pi d^2} \tag{2}$$

In both Eqs. (1) and (2) the dependent variable is the received signal power, but the independent variables are different.

(c) Equation (1) is the appropriate choice for calculating P_r performance when the dimensions of both the transmitting and receiving antennas are already fixed. Equation (1) states that for fixed size antennas, the received power increases as the wavelength is decreased.

Equation (2) is the appropriate choice when both A_t and G_r are fixed and the requirement is to determine the required value of the average transmitted power P_t in order to realize a specified P_r.

Problem 8.4

The free space loss is given by

$$L_{\text{free space}} = \left(\frac{4\pi d}{\lambda}\right)^2$$

According to the above formulation for free space loss, free space loss is frequency dependent. Path loss, as characterized in this formulation, is a definition based on the use of an isotropic receiving antenna ($G_r = 1$).

The power density, $\rho(d)$, is a function of distance and is equal to

$$\rho(d) = \frac{\text{EIRP}}{4\pi d^2}$$

The received power of an isotropic antenna is equal to

$$P_r = \rho(d) \times \frac{\lambda^2}{4\pi}$$

$$= \frac{\text{EIRP}}{4\pi d^2} \times \frac{\lambda^2}{4\pi}$$

$$= \frac{\text{EIRP}}{\left(\frac{4\pi d}{\lambda}\right)^2}$$

$$= \text{EIRP}/L_{\text{free-space}} \tag{1}$$

Equation (1) states the power received by an isotropic antenna is equal to the effective transmitted power EIRP, reduced only by the path loss. However, when the receiving antenna is not isotropic, the received power is modified by the receiving antenna gain G_r, that is, Eq. (1) is multiplied by G_r

Problem 8.5

In a satellite communication system, satellite power is limited by the permissible antenna size. Accordingly, a sensible design strategy is to have the path loss on the downlink smaller than the pass loss on the uplink. Recognizing the inverse dependence of path loss on the wavelength λ, it follows that we should have

$$\lambda_{\text{uplink}} < \lambda_{\text{downlink}}$$

or, equivalently,

$$f_{\text{uplink}} > f_{\text{downlink}}$$

Problem 8.6

Received power in dBW is defined by

$$Pr = \text{EIRP} + G_r - \text{Free-space loss} \tag{1}$$

For these three components, we have

(1) $\text{EIRP} = 10\log_{10}(P_t G_t)$

$$= 10\log_{10}P_t + 10\log_{10}(G_t)$$

$$= 10\log_{10}(0.1) + 10\log_{10}(G_t) \tag{2}$$

Transmit antenna gain (in dB):

$$10\log_{10}G_t = 10\log_{10}\left(\frac{4\times\pi\times 0.7\times\pi/4}{(3/40)^2}\right)$$

$$= 30.89 \text{ dB} \tag{3}$$

(2) Receive antenna gain:

$$10\log_{10}G_r = 10\log_{10}\left(\frac{4\times\pi\times 0.55\times\pi\times 5^2}{(3/40)^2}\right)$$

$$= 49.84 \text{ dB} \tag{4}$$

(3) Free-space loss:

$$L_p = 20\log_{10}\left(\frac{4\times\pi\times R}{\lambda}\right)$$

$$= 20\log_{10}\left(\frac{4\times\pi\times 4\times 10^7}{3/40}\right)$$

$$= 196.25 \text{ dB} \tag{5}$$

Hence, using Eqs. (1) to (5), we find that

$$P_r = 10\log_{10}(0.1) + 30.89+49.84 - 196.52$$

$$= -206.52 + 8.073$$

$$= -125 \text{ dBW}$$

Problem 8.7

(a) RMS value of thermal noise = $\sqrt{E[v^2]} = \sqrt{4kTR\Delta f}$ volts, where k is Boltzmann's constant equal to 1.38 x 10^{-23}, T is the absolute temperature in degrees Kelvin, and R is the resistance in ohmns. Hence,

$$\text{RMS value} = \sqrt{4 \times 1.38 \times 10^{-23} \times 290 \times 75 \times 1 \times 10^6}$$

$$= \sqrt{4 \times 1.38 \times 290 \times 75 \times 10^{-17}}$$

$$= 1.096 \text{ x } 10^{-6} \text{ volts}$$

(b) The maximum available noise power delivered to a matched load is

$$kT\Delta f = 1.38 \times 10^{-23} \times 290 \times 10^6$$

$$= 4.0 \text{ x } 10^{-15} \text{ watts}$$

Problem 8.8

The wavegude loss is 1 dB; that is,

$$G_{\text{waveguide}} = 0.78$$

The noise temperature at the input to the LNA due to the combined presence of antenna and waveguide is

$$T_e = G_{\text{waveguide}} \times T_{\text{antenna}} + (1 - G_{\text{waveguide}})T_{\text{waveguide}}$$

$$= 0.78 \times 50 + 290(1 - 0.78)$$

$$= 102.8K$$

The overall noise temperature of the system is

$$T_{\text{system}} = T_e + 50 + \frac{500}{200} + \frac{1000}{200}$$

$$= 160.3K$$

The system noise temperature referred to the antenna terminal is

$$160.3/0.78 = 205.5K$$

<u>Problem 8.9</u>

In this problem, we are given the noise figures (F) and the available power gains (G) of the devices. By using the following relationship, we can estimate the equivalent noise temperature of each device:

$$F = \frac{T + T_e}{T}$$

$$T_e = T(F\text{-}1)$$

where T is room temperature (290K) and T_e is the equivalent noise temperature.

(a) The equivalent noise temperatures of the given four components are

<u>Waveguide</u>

$$T_{\text{waveguide}} = 290(2-1)$$
$$= 290K$$

<u>Mixer</u>

$$T_{\text{mixer}} = 290(3-1)$$
$$= 580K$$

<u>Low-noise RF amplifier</u>

$$T_{RF} = 290(1.7\text{-}1)$$
$$= 203K$$

<u>IF amplifier</u>

$$T_{IF} = 290(5-1)$$
$$= 1160K$$

(b) The effective noise temperature at the input to the LNA due to the antenna and waveguide is

$$T_e = G_{\text{waveguide}} \times T_{\text{antenna}} + (1 - G_{\text{waveguide}}) \times T_{\text{waveguide}}$$

$$= 0.2 \times 50 + 290(1 - 0.2)$$
$$= 242K$$

The effective noise temperature of the system is

$$T_{\text{system}} = T_e + T_{\text{RF}} + \frac{T_{\text{mixer}}}{G_{\text{RF}}} + \frac{T_{\text{IF}}}{G_{\text{RF}} \times G_{\text{mixer}}}$$

$$= 242 + 203 + \frac{580}{10} + \frac{1160}{10 \times 5}$$

$$= 526.2K$$

Problem 8.10

(a) For the uplink power budget, the ratio $\frac{C}{N}$ is given by

$$\left.\frac{C}{N}\right|_{\text{uplink}} = \phi_s - G_I - BO_I + \frac{G}{T} - k - L_r$$

where

ϕ_s = Power density at saturation
G_I = Gain of 1m^2
BO_I = Power back-off
$\frac{G}{T}$ = Figure of Merit
k = Boltzmann constant in dBK
L_r = Losses due to rain

For the given satellite system, we have

$$\left.\frac{C}{N}\right|_{\text{uplink}} = -81 - 44.5 - 0.0 + 1.9 + 228.6 + 0.0$$

$$= 105.0 \text{ dB-Hz}$$

where we have used the following gain of 1m^2 antenna:

$$G_l = 10\log_{10}\left(\frac{4 \times \pi \times 1}{(3 \times 10^8 / 14 \times 10^9)^2}\right)$$

$$= 44.5 \text{ dB}$$

Boltzmann constant k = -228.6 dB

(b) Given the data rate in the uplink = 33.9 Mb/s and link margin of 6 dB, the required $\frac{E_b}{N_0}$ is

$$\left(\frac{E_b}{N_0}\right)_{\text{required}} = \left(\frac{C}{N_0}\right)_{\text{uplink}} - (10\log_{10}M + 10\log_{10}R)$$

$$= 105 - 6 - 10\log_{10}(33.9 \times 10^6)$$

$$= 105 - 6 - 75.3$$

$$= 23.7 \text{ dB}$$

Equivalently, we have

$$\frac{E_b}{N_0} = 234$$

Given the use of 8-PSK, the symbol error rate is defined by

$$P_e = \text{erfc}\left(\sqrt{\frac{E}{N_0}}\sin(\pi/8)\right)$$

For 8-PSK

$$\frac{E}{N_0} = \frac{3E_b}{N_0} = 234 \times 3 = 702$$

Hence,

$$P_e = \text{erfc}(\sqrt{702} \times \sin(\pi/8))$$
$$\approx 0$$

This result further confirms the statement we made in Example 8.2 in that the satellite communication system is essentially downlink-limited. Recognizing that we have more powerful resources available at an earth station then at a satellite, it would seem reasonable that the BER at the satellite can be made practically zero by transmitting enough signal power along the uplink.

Problem 8.11

For the downlink, the relationship between

$\left(\frac{C}{N_0}\right)$ and $\left(\frac{E_b}{N_0}\right)_{\text{req}}$, expressed in decibels, is described by

$$\left(\frac{C}{N_0}\right)_{\text{downlink}} = \left(\frac{E_b}{N_0}\right)_{\text{req}} + 10\log M + 10\log R \qquad (1)$$

where M is the margin and R is the bit rate in bits/second.

Solving Eq. (1) for the the link margin in dB and evaluating it for the problem at hand, we get

$$10\log_{10}M = 85 - 10 - 10\log_{10}(10^6)$$

$$= 5\text{dB}$$

For the downlink budget, the equation for $\left(\frac{C}{N_0}\right)$, expressed in decibels, is as follows:

$$\left(\frac{C}{N_0}\right)_{\text{downlink}} = \text{EIRP} + \left(\frac{G_r}{T}\right)_{\text{dB}} - L_{\text{freespace}} - 10\log_{10}k$$

where k is Boltzmann's constant.

For a satisfactory reception at any situation, we consider additional losses due to rain etc. up to the calculated link margin of 5 dB. Hence, we may write

$$\left(\frac{C}{N_0}\right)_{\text{downlink}} = \text{EIRP} + \left(\frac{G_r}{T}\right)_{\text{dB}} - L_{\text{freespace}} - 10\log_{10}k - 10\log_{10}M(\text{dB}) \qquad (2)$$

where

$$\text{EIRP} = 57 \text{ dBW}$$
$$L_{\text{freespace}} = \text{free-space loss}$$

$$= 92.4 + 20\log_{10}(12.5) + 20\log_{10}(40,000)$$
$$= 206 \text{ dB}$$
$$10\log_{10}k = 228.6 \text{ dBK}$$
$$10\log_{10}M = 5 \text{ dB}$$

Using these values in Eq. (2) and solving for G_r/T, we get

$$\left(\frac{G_r}{T}\right)_{\text{dB}} = 85 - 57 + 206 - 228.6 + 5$$

$$= 10.4 \text{ dB}$$

With $T = 310\text{K}$, we thus find

$$G_r = 10.4 + 10\log_{10}(310)$$

$$= 35.31 \text{ dB}$$

The receiving antenna gain in is given by

$$10\log_{10}G_r = 10\log_{10}\left(\frac{4\pi A\eta}{\lambda^2}\right)$$

For a dish antenna (circular) with diameter D, the area A equals $\pi D^2/4$. Thus,

$$10\log_{10}G_r = 20\log_{10}D + 20\log_{10}f + 10\log_{10}(\eta) + 20.4(\text{dB})$$

where D is measured in meters and f is measured in GHz. Solving for the antenna diameter for the given system, we finally get

$D_{\min} = 0.6$ meters

Problem 8.12

(a) Similarities between satellite and wireless communications:

- They are both bandwidth-limited.
- They both rely on multiple-access techniques for their operation.
- They both have uplink and downlink data transmissions.
- The performance of both systems is influenced by intersymbol interference and external interference signals.

(b) Major differences between satellite and wireless communications:

- Multipath fading and user mobility are characteristic features of wireless communications, which have no counterparts in satellite communications.
- The carrier frequency for satellite communications is in the gigahertz range (Ku-band), whereas in satellite communications it is in the megahertz range.
- Satellite communication systems provide broad area coverage, whereas wireless communications provide local coverage with provision for mobility in a cellular type of layout.

Problem 8.13

In a wireless communication system, transmit power is limited at the mobile unit, whereas no such limitation exists at the base station. A sensible design strategy is to make the path loss (i.e., free-space loss) on the downlink as small as possible, which, in turn, suggests that we make

$(\text{Path loss})_{\text{uplink}} < (\text{Path loss})_{\text{downlink}}$

Recognizing that path loss is inversely proportional to wavelength, it follows that

$\lambda_{\text{uplink}} > \lambda_{\text{downlink}}$

or, equivalently,

$f_{\text{uplink}} < f_{\text{downlink}}$

Problem 8.14

The phase difference between the direct and reflected waves can be expressed as

$$\phi = \frac{2\pi d}{\lambda}\left[\sqrt{\left(\frac{h_b + h_m}{d}\right)^2 + 1} - \sqrt{\left(\frac{h_b - h_m}{d}\right)^2 + 1}\right] \tag{1}$$

where λ is the wave length. For large d, Eq. (1) may be approximated as

$$\phi \approx \frac{4\pi(h_b h_m)}{\lambda d} \text{ radians}$$

With perfect reflection (i.e., reflected coefficient of the ground is -1) and assuming small ϕ (i.e., large d), the received power P_r is defined by

$$P_r = P_o\left|1 - e^{j\phi}\right|^2 \approx P_o \sin^2\left(\frac{4\pi(h_b h_m)}{\lambda d}\right)$$

$$\approx P_o\left(\frac{4\pi(h_b h_m)}{\lambda d}\right)^2 \qquad (2)$$

$$\text{where } P_o = P_t G_b G_m\left(\frac{\lambda}{4\pi d}\right)^2 \qquad (3)$$

Using Eq. (3) in (2):

$$P_r = P_t G_b G_m\left(\frac{4\pi(h_b h_m)}{\lambda d}\right)^2\left(\frac{\lambda}{4\pi d}\right)^2$$

$$= P_t G_b G_m\left(\frac{h_b^2 h_m^2}{d^4}\right)$$

which shows that the received power is inversely proportional to the fourth power of distance d between the two antennas.

Problem 8.15

The complex (baseband) impulse response of a wireless channel may be described by

$$\tilde{h}(t) = a_1 e^{-j\phi_1}\delta(t-\tau) + a_2 e^{-j\phi_2}\delta(t-\tau) \qquad (1)$$

where the amplitudes a_1 and a_2 are Rayleigh distributed, and the phase angles ϕ_1 and ϕ_2 are uniformly distributed. This model assumes (2) the presence of two different clusters with each one consisting of a large number of scatterers, and (2) the absence of line-of-sight paths in the wireless environment. Define

$$h(t) = \tilde{h}(t)e^{j\phi_1}$$

$$\theta = \phi_2 - \phi_1$$

We may then rewrite Eq. (1) in the form

$$h(t) = a_1\delta(t-\tau) + a_2 e^{-j\theta}\delta(t-\tau)$$

as stated in the problem.

(a) (i) The transfer function of the model is

$$H(f) = F[h(t)]$$

$$= a_1 e^{-j2\pi f\tau_1} + a_2 e^{-j(2\pi f\tau_2 + \theta)}$$

(ii) The power-delay profile of the model is

$$P_h = E[|h(t)|^2]$$

$$= E[a_1\delta(t-\tau_1) + a_2 e^{-j\theta}\delta(t-\tau_2)(a_1\delta(t-\tau)) + a_2 e^{j\theta}\delta(t-\tau_2)]$$

$$= E[a_1^2\delta^2(t-\tau_1) + a_2^2\delta^2(t-\tau_2) + a_1 a_2 \cos\theta\delta(t-\tau_1)\delta(t-\tau_2)]$$

$$= E[a_1^2]\delta^2(t-\tau_1) + E[a_2^2]\delta^2(t-\tau_2) \qquad (1)$$

(b) The magnitude response of the model is

$$|H(f)| = \left|a_1 e^{-j2\pi f\tau_1} + a_2 e^{-j2\pi f(\tau_2 + \theta)}\right|$$

$$= \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(2\pi f(\tau_2 - \tau_1) + \theta)}$$

which exhibits frequency selectivity due to two factors: (1) variations in the coefficients a_1 and a_2, and (2) variations inthe delay difference τ_2-τ_1.

Problem 8.16

The multipath influence on a communication system is usually described in terms of two effects - selective fading and intersymbol interference. In a Rake receiver, selective fading is mitigated by detecting the echo signals individually, using a correlation method, and adding them algebraically (with the same sign) rather than vectorially, and intersymbol interference is dealt with by reinserting different delays into the various detected echoes so that they fall into step again.

Making each correlator perform at its assigned value of delay can be done by inserting the right amount of delay in either the reference (called the *delayed-reference*) or received signals (called the *delayed-signal*). Independent of the form of the reference signals employed, the output SNR from the integrating filters is substantially the same for both configurations, under the assumption that the length of the delay T_d is significantly smaller than the symbol duration T. Each integrating filter responds to signals only within about $\pm 1/T$ of the frequency f. Therefore, the noises adding shorter than T, regardless of the form of reference signal. The only difference between the tap circuit contributions of the delayed-signal scheme and those of the delayed-reference scheme is that the latter are staggered in time by various fractions of T_d, and since such staggering is small compared to the significant fluctuation period of the contributions, we conclude that the noise outputs of the two configurations are equivalent.

However, there are three practical advantages of the delayed-signal scheme over the delayed-reference scheme. First, one delay line instead of two is required. Second, in the latter configuration, corresponding taps in the mark (symbol 0) and space (symbol 1) lines would have to be adjusted and be kept in phase coincidence. Third, coherent intersymbol interference (eliminated in the delayed-signal scheme) is still present in the latter scheme, (Price and Green 1958), see the Bibliography.

Problem 8.17

(a) The output of the linear combiner is given by

$$x(t) = \sum_{j=1}^{N} \alpha_j x_j(t)$$

$$= \sum_{j=1}^{N} \alpha_j (z_j m(t) + n_j(t))$$

$$= \underbrace{\sum_{j=1}^{N} \alpha_j z_j m(t)}_{\text{signal}} + \underbrace{\sum_{j=1}^{N} \alpha_j n_j(t)}_{\text{noise}}$$

The output signal-to-noise ratio is therefore

$$(\text{SNR})_0 = \frac{\text{Average signal power}}{\text{Average noise power}}$$

$$= \frac{E\left[\sum_{j=1}^{N} \alpha_j (z_j m(t))\right]^2}{E\left[\sum_{j=1}^{N} \alpha_j n_j(t)\right]^2}$$

$$= \frac{E\left[\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\alpha_k z_j z_k m^2(t)\right]}{E\left[\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\alpha_k n_j(t) n_k(t)\right]}$$

$$= \frac{\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\alpha_k z_j z_k E[m^2(t)]}{\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\alpha_k \; E[n_j(t) n_k(t)]} \tag{1}$$

Using the following expectations

$E[m^2(t)] = 1$ for all t (i.e., unit message power)

$$E[n_j(t)n_k(t)] = \begin{cases} \sigma_j^2 & \text{for } k = j \\ 0 & \text{for } k \neq j \end{cases}$$

we find that Eq. (1) simplifies to

$$(\mathrm{SNR})_0 = \frac{\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\alpha_k z_j z_k}{\sum_{j=1}^{N}\alpha_j^2\sigma_j^2}$$

$$= \frac{\left(\sum_{j=1}^{N}\alpha_j z_j\right)^2}{\sum_{j=1}^{N}\alpha_j^2\sigma_j^2} \tag{2}$$

(b) Equation (2) can be rewritten in the equivalent form

$$(\text{SNR})_0 = \frac{\left[\sum_{j=1}^{N} \alpha_j \sigma_j (z_j/\sigma_j)\right]^2}{\sum_{j=1}^{N} \alpha_j^2 \sigma_j^2}$$

$$= \frac{\left[\sum_{j=1}^{N} u_j (\text{SNR})_j^{1/2}\right]^2}{\sum_{j=1}^{N} u_j^2} \tag{3}$$

where $u_j = \alpha_j \sigma_j$ and $(\text{SNR})_j = z_j^2/\sigma_j^2$. We now invoke the Schwarz inequality, which, in discrete form for the problem at hand, is stated as follows

$$\left(\sum_{j=1}^{N} u_j (\text{SNR})_j^{1/2}\right)^2 \le \left(\sum_{j=1}^{N} u_j^2\right)\left(\sum_{j=1}^{N} ((\text{SNR})_j^{1/2})^2\right) \tag{4}$$

Hence, inserting this inequality into the right-hand side of Eq. (3), we may write

$$(\text{SNR})_0 \le \sum_{j=1}^{N} (\text{SNR})_j$$

which proves the formula under subpart (i).

To prove subpart (ii), we recall that the Schwarz inequality of Eq. (4) is satisfied with the equality sign if (except for a scaling factor)

$$(\text{SNR})_j^{1/2} = u_j$$

or, equivalently,

$$\frac{z_j}{\sigma_j} = \alpha_j \sigma_j$$

That is,

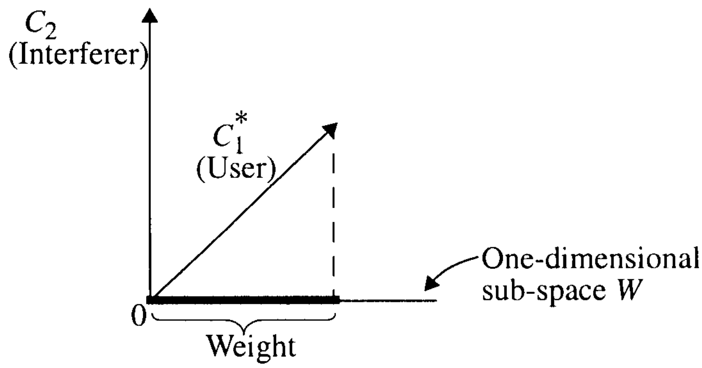
$$\alpha_j = \frac{z_j}{\sigma_j^2}$$

Problem 8.18

For the problem at hand, we have $M = N = 2$. Therefore,

$$M - N + 1 = 1$$

and so the weight subspace W is one-dimensional. We thus have the following representation for the action of the antenna array:



Problem 8.19

(a) The cost function is

$$J = \frac{1}{2}|e[n]|^2 = \frac{1}{2}e[n]e^*[n]$$

where the error signal is

$$e[n] = d[n] - \sum_{k=1}^{M} w_k^*[n]x_k[n]$$

Let

$$w_k[n] = a_k[n] + jb_k[n]$$

Hence

$$\frac{\partial J}{\partial a_k} = \frac{1}{2}e[n]\frac{\partial e^*[n]}{\partial a_k} + \frac{1}{2}e^*[n]\frac{\partial e[n]}{\partial a_k}$$

$$= -\frac{1}{2}e[n]x_k^*[n] - \frac{1}{2}e^*[n]x_k[n]$$

$$= -\frac{1}{2}\mathrm{Re}\{x_k[n]e^*[n]\} \tag{1}$$

$$\frac{\partial J}{\partial b_k} = \frac{1}{2}e[n]\frac{\partial e^*[n]}{\partial b_k} + \frac{1}{2}e^*[n]\frac{\partial e[n]}{\partial b_k}$$

$$= -\frac{j}{2}e[n]x_k^*[n] + \frac{j}{2}e^*[n]x_k[n]$$

$$= -\mathrm{Im}\{x_k[n](e^*[n])\} \tag{2}$$

The adjustment applied to the kth weight is therefore

$$\Delta w_k[n] = \Delta a_k[n] + j\Delta b_k[n]$$

$$= -\mu\frac{\partial J}{\partial a_k} - j\mu\frac{\partial J}{\partial b_k} \tag{3}$$

where μ is the step-size parameter. Substituting Eqs. (1) and (2) into (3),

$$\Delta w_k[n] = \mu\mathrm{Re}\{x_k[n]e^*[n]\} + \mu\mathrm{Im}\{x_k[n]e^*[n]\}$$

$$= \mu x_k[n]e^*[n]$$

(b) The complex LMS algorithm is described by the following pair of relations:

$$w_k[n+1] = w_k[n] + \Delta w_k[n]$$

$$= w_k[n] + \mu x_k[n]e^*[n], \qquad k = 1, 2, \cdots, M$$

$$e(n) = d[n] - \sum_{k=1}^{M} w_k^* x_k[n]$$

Problem 8.20

(a) We are told that the speed of response of the weights in the LMS algorithm is proportional to the average signal power at the antenna array input. Conversely, we may say that the average signal power at the array input is proportional to the speed of response of the weights in the LMS algorithm. Moreover, the maximum speed of response of the LMS weights is proportional to R_b/f_{max}, where R_b is the bit rate and f_{max} is the maximum fade rate in Hz. It follows therefore that the dynamic range of the average signal power at the antenna array input is proportional to R_b/f_{max}, as shown by

$$P_{max} = \alpha R_b / f_{max} \text{ watts} \tag{1}$$

where α is the proportionality constant.

(b) For $\alpha = 0.2$, $R_b = 32 \times 10^3$ b/s, and $f_{max} = 70$ Hz, the use of Eq. (1) yields

$$P_{max} = 0.2 \times 32 \times 10^3 / 70$$

$$= 640/7$$

$$= 91 \text{ watts}$$

which is somewhat limited in value.

Problem 8.21

(a) According to the Wiener filter, derived for the case of complex data, the optimum weight vector is defined by

$$\mathbf{R}_x \mathbf{w}_o = \mathbf{r}_{xd} \tag{1}$$

where

$\mathbf{R}_x$ = correlation matrix of the input signal vector $\mathbf{x}[n]$

$$= E[\mathbf{x}[n]\mathbf{x}^H[n]] \tag{2}$$

$\mathbf{r}_{xd}$ = cross-correlation vector between $\mathbf{x}[n]$ and desired response $d[n]$

$$= E[\mathbf{x}[n]\mathbf{d}^*[n]] \tag{3}$$

$\mathbf{w}_o$ = optimum weight vector.

Note that the formulation of Eq. (1) is based on the premise that the array output is defined as the inner product $\mathbf{w}^H\mathbf{x}[n]$. The Wiener filter for real data is a special case of Eq. (1), where the Hermitian transpose H in Eq. (2) is replaced by ordinary transposition and the complex conjugation in Eq. (3) is omitted. Assuming that the input $\mathbf{x}[n]$ and desired response $d[n]$ are jointly ergodic, we may use the following estimates for $\mathbf{R}_x$ and $\mathbf{r}_{xd}$:

$$\hat{\mathbf{R}}_x = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}[n]\mathbf{x}^H[n] \tag{4}$$

$$\hat{\mathbf{r}}_{\mathbf{x}d} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}[n]d^*[n] \tag{5}$$

where K is the total number of snapshots used to train the antenna array. Correspondingly, the estimate of the optimum weight vector $\mathbf{w}_o$ is computed as

$$\hat{\mathbf{w}} = \hat{\mathbf{R}}_x^{-1}\hat{\mathbf{r}}_{\mathbf{x}d} \tag{6}$$

where $\mathbf{R}_x^{-1}$ is the inverse of $\mathbf{R}_x$.

(b) The DMI algorithm for computing the estimate $\hat{\mathbf{w}}$ may now proceed as follows:

1. Collect K snapshots of data denoted by

   $$\{\mathbf{x}[k], d[k]\}_{k=1}^{K}$$

   where K is sufficiently large for $\hat{\mathbf{w}}$ to approach $\mathbf{w}_o$ and yet small enough to ensure stationarity of the data.

2. Use Eqs. (4) and (5) to compute the correlation estimates $\hat{\mathbf{R}}_x$ and $\hat{\mathbf{r}}_{\mathbf{x}d}$.

3. Invert the correlation matrix $\hat{\mathbf{R}}_x$ and then use Eq. (6) to compute the weight estimate $\hat{\mathbf{w}}$.

For an antenna array consisting of M elements, the matrix $\hat{\mathbf{R}}_x$ is an M-by-M matrix and $\hat{\mathbf{r}}_{\mathbf{x}d}$ is an M-by-1 vector. Therefore, the inversion of $\hat{\mathbf{R}}_x$ and its multiplication by $\hat{\mathbf{r}}_{\mathbf{x}d}$ requires multiplications and additions on the order of M^3.

Chapter 9

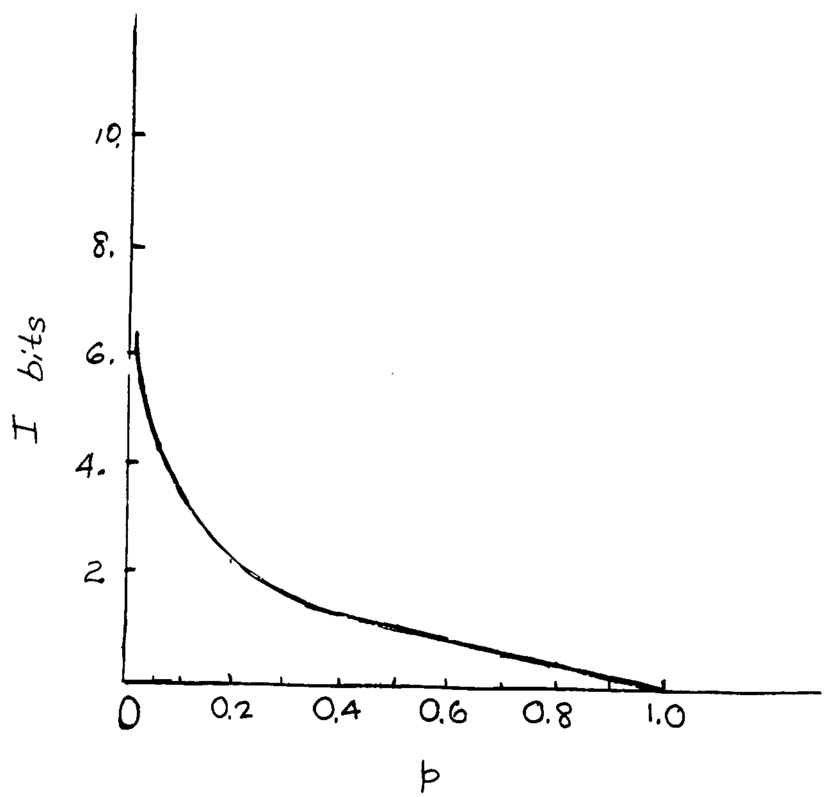
Fundamental limits in Information Theory

Problem 9.1

Amount of information gained by the occurrence of an event of probability p is

$$I = \log_2\left(\frac{1}{p}\right) \text{ bits}$$

I varies with p as shown below:



Problem 9.2

Let the event $S=s_k$ denote the emission of symbol s_k by the source, Hence,

$$I(s_k) = \log_2\left(\frac{1}{p}\right) \text{ bits}$$

s_k	s_0	s_1	s_2	s_3
p_k	0.4	0.3	0.2	0.1
$I(s_k)$ bits	1.322	1.737	2.322	3.322

Problem 9.3

Entropy of the source is

$$\begin{aligned} H(S) &= p_0\log_2\left(\frac{1}{p_0}\right) + p_1\log_2\left(\frac{1}{p_1}\right) + p_2\log_2\left(\frac{1}{p_2}\right) + p_3\log_2\left(\frac{1}{p_3}\right) \\ &= \frac{1}{3}\log_2(3) + \frac{1}{6}\log_2(6) + \frac{1}{4}\log_2(4) + \frac{1}{4}\log_2(4) \\ &= 0.528 + 0.431 + 0.5 + 0.5 \\ &= 1.959 \text{ bits} \end{aligned}$$

Problem 9.4

Let X denote the number showing on a single roll of a die. With a die having six faces, we note that p_X is 1/6. Hence, the entropy of X is

$$\begin{aligned} H(X) &= p_X \log_2\left(\frac{1}{p_X}\right) \\ &= \frac{1}{6}\log_2(6) = 0.431 \text{ bits} \end{aligned}$$

Problem 9.5

The entropy of the quantizer output is

$$H = -\sum_{k=1}^{4} P(X_k)\log_2 P(X_k)$$

where X_k denotes a representation level of the quantizer. Since the quantizer input is Gaussian with zero mean, and a Gaussian density is symmetric about its mean, we find that

$$P(X_1) = P(X_4)$$

$$P(X_2) = P(X_3)$$

The representation level $X_1 = 1.5$ corresponds to a quantizer input $+1 \le Y < \infty$. Hence,

$$\begin{aligned} P(X_1) &= \int_1^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\ &= \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{4}{\sqrt{2}}\right) \\ &= 0.1611 \end{aligned}$$

The representation level $X_2 = 0.5$ corresponds to the quantizer input $0 \le Y < 1$. Hence,

$$\begin{aligned} P(X_2) &= \int_0^{1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \\ &= \frac{1}{2}\,\mathrm{erf}\left(\frac{4}{\sqrt{2}}\right) \\ &= 0.3389 \end{aligned}$$

Accordingly, the entropy of the quantizer output is

$$\begin{aligned} H &= -2\left[0.1611\log_2\left(\frac{1}{0.1611}\right) + 0.3389\log_2(0.3389)\right] \\ &= 1.91 \text{ bits} \end{aligned}$$

Problem 9.6

(a) For a discrete memoryless source:

$$P(\sigma_i) = P(s_{i_1})\, P(s_{i_1}) \dots P(s_{i_n})$$

Noting that $M = K^n$, we may therefore write

$$\begin{aligned}
\sum_{i=0}^{M-1} P(\sigma_i) &= \sum_{i=0}^{M-1} P(s_{i_1})\, P(s_{i_2}) \dots P(s_{i_n}) \\
&= \sum_{i_1=0}^{K-1} \sum_{i_2=0}^{K-1} \cdots \sum_{i_n=0}^{K-1} P(s_{i_1})\, P(s_{i_2}) \dots P(s_{i_n}) \\
&= \sum_{i_1=0}^{K-1} P(s_{i_1}) \sum_{i_2=0}^{K-1} P(s_{i_2}) \cdots \sum_{i_n=0}^{K-1} P(s_{i_n}) \\
&= 1
\end{aligned}$$

(b) For $k = 1,2,\dots,n$, we have

$$\sum_{i=0}^{M-1} P(\sigma_i) \log_2\left(\frac{1}{p_{i_k}}\right) = \sum_{i=0}^{M-1} P(s_{i_1})\, P(s_{i_2}) \cdots P(s_{i_n}) \log_2\left(\frac{1}{p_{i_k}}\right)$$

For k=1, say, we may thus write

$$\begin{aligned}
\sum_{i=0}^{M-1} P(\sigma_i) \log_2\left(\frac{1}{p_{i_1}}\right) &= \sum_{i=0}^{K-1} P(s_{i_1}) \log_2\left(\frac{1}{p_{i_1}}\right) \sum_{i=0}^{K-1} P(s_{i_2}) \cdots \sum_{i=0}^{K-1} P(s_{i_n}) \\
&= \sum_{i=0}^{K-1} p_{i_1} \log_2\left(\frac{1}{p_{i_1}}\right) \\
&= H(S)
\end{aligned}$$

Clearly, this result holds not only for k=1 but also k=2,...,n.

(c)

$$H(S^n) = \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(\sigma_i)}$$

$$= \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(s_{i_1})\, P(s_{i_2}) \cdots P(s_{i_n})}$$

$$= \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(s_{i_1})} + \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(s_{i_2})}$$

$$+ \cdots + \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \frac{1}{P(s_{i_n})}$$

Using the result of part (b), we thus get

$$H(S^n) = H(S) + H(S) + \cdots + H(S)$$
$$= n\, H(S)$$

Problem 9.7

(a) The entropy of the source is

$$H(S) = 0.7 \log_2 \frac{1}{0.7} + 0.15 \log_2 \frac{1}{0.15} + 0.15 \log_2 \frac{1}{0.15}$$
$$= 0.258 + 0.4105 + 0.4105$$
$$= 1.079 \text{ bits}$$

(b) The entropy of second-order extension of the source is

$$H(S^2) = 2 \times 1.079 = 2.158 \text{ bits}$$

Problem 9.8

The entropy of text is defined by the smallest number of bits that are required, on the average, to represent each letter.

According to Lucky[†], English text may be represented by less than 3 bits per character, because of the redundancy built into the English language. However, the spoken equivalent of English text has much less redundancy; its entropy is correspondingly much higher than 3 bits. It follows therefore from the source coding theorem that the number of bits required to represent (store) text is much smaller than the number of bits required to represent (store) its spoken equivalent.

Problem 9.9

(a) With K equiprobable symbols, the probability of symbol s_k is

$$p_k = P(s_k) = \frac{1}{K}$$

The average code-word length is

$$\bar{L} = \sum_{k=0}^{K-1} p_k l_k$$

$$= \frac{1}{K} \sum_{k=0}^{K-1} l_k$$

The choice of a fixed code-word length $l_k=l_0$ for all k yields the value $\bar{L}=l_0$. With K symbols in the code, any other choice for l_k yields a value for $\bar{L}$ no less than l_0.

(b) Entropy of the source is

[†]R.W. Lucky, "Silicon Dreams", p.111 (St. Martin's Press, 1989).

$$H(S) = \sum_{k=0}^{K-1} p_k \log_2\left(\frac{1}{p_k}\right)$$

$$= \sum_{k=0}^{K-1} \frac{1}{K} \log_2 K = \log_2 K$$

The coding efficiency is therefore

$$\eta = \frac{H(S)}{\bar{L}} = \frac{\log_2 K}{l_0}$$

For η=1, we have

$$l_0 = \log_2 K$$

To satisfy this requirement, we choose

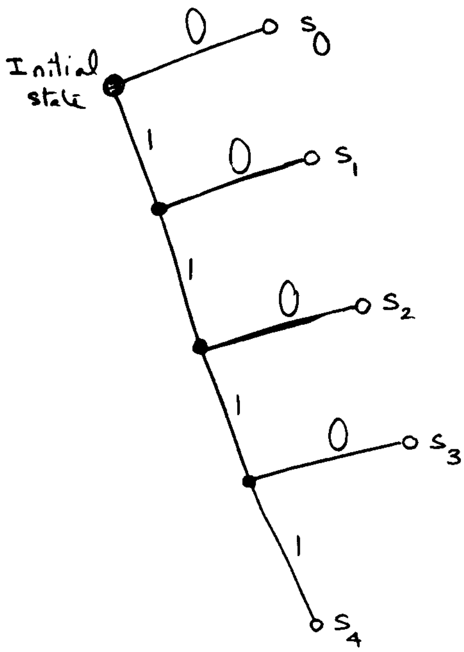
$$K = 2^{l_0}$$

where l_0 is an integer.

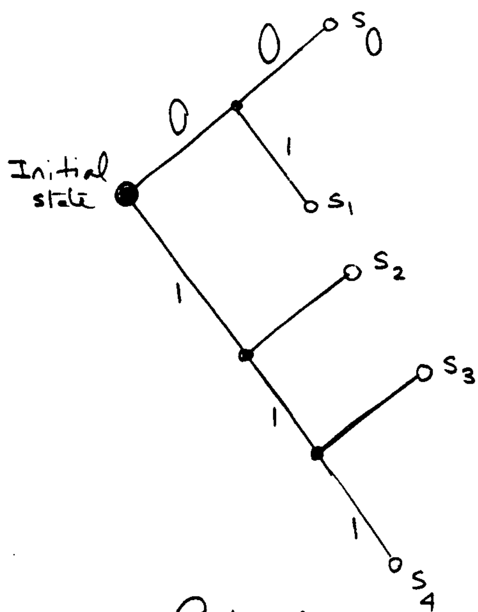
Problem 9.10

A prefix code is defined as a code in which no code work is the prefix of any other code word. By inspection, we see therefore that codes I and IV are prefix codes, whereas codes II and III are not.

To draw the decision tree for a prefix code, we simply begin from some starting node, and extend branches forward until each symbol of the code is represented. We thus have:



Code I

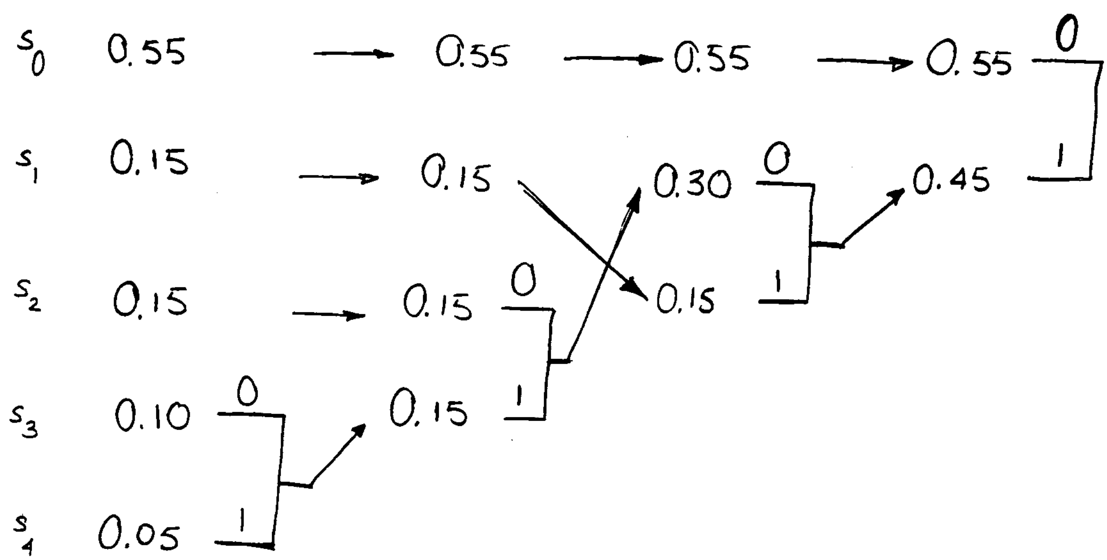


Code IV

Problem 9.11

We may construct two different Huffman codes by choosing to place a combined symbol as low or as high as possible when its probability is equal to that of another symbol.

We begin with the Huffman code generated by placing a combined symbol as low as possible:



The source code is therefore

s_0	0
s_1	1 1
s_2	1 0 0
s_3	1 0 1 0
s_4	1 0 1 1

The average code-word length is therefore

$$\bar{L} = \sum_{k=0}^{4} p_k l_k$$

$$= 0.55(1) + 0.15(2) + 0.15(3) + 0.1(4) + 0.05(4)$$

$$=1.9$$

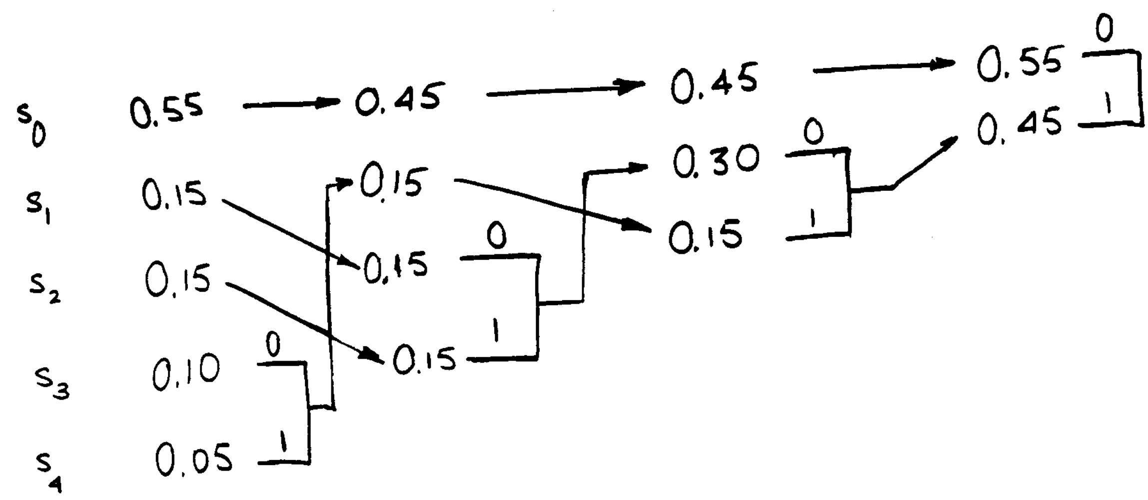
The variance of $\bar{L}$ is

$$\sigma^2 = \sum_{k=0}^{4} p_k(l_k - \bar{L})^2$$

$$= 0.55(-0.9)^2 + 0.15(0.1)^2 + 0.15(1.1)^2 + 0.1(2.1)^2 + 0.05(2.1)^2$$

$$= 1.29$$

Next placing a combined symbol as high as possible, we obtain the second Huffman code:



Correspondingly, the Huffman code is

s_0	0
s_1	1 0 0
s_2	1 0 1
s_3	1 1 0
s_4	1 1 1

The average code-word length is

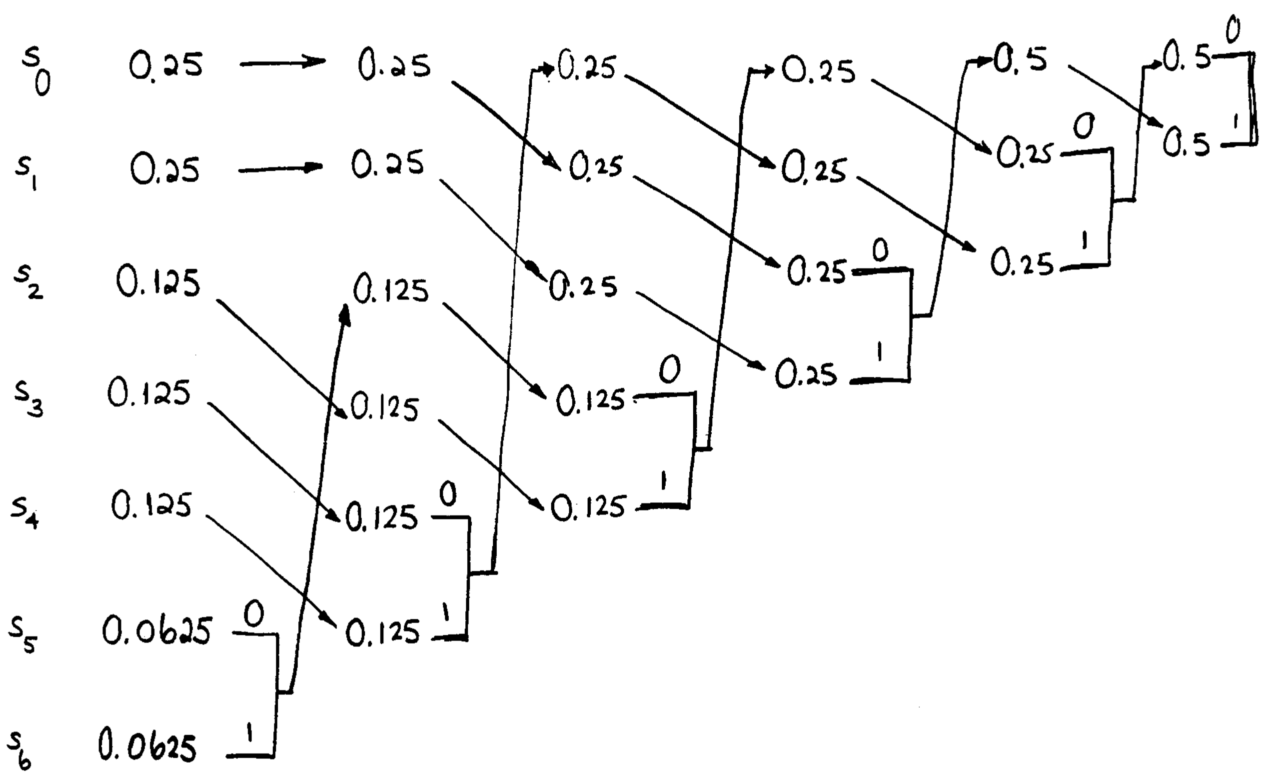
$$\bar{L} = 0.55(1) + (0.15 + 0.15 + 0.1 + 0.05)(3)$$
$$= 1.9$$

The variance of $\bar{L}$ is

$$\sigma^2 = 0.55(-0.9)^2 + (0.15 + 0.15 + 0.1 + 0.05)(1.1)^2$$
$$= 0.99$$

The two Huffman codes described herein have the same average code-word length but different variances.

Problem 9.12



The Huffman code is therefore

s_0	1 0
s_1	1 1
s_2	0 0 1
s_3	0 1 0
s_4	0 1 1
s_5	0 0 0 0
s_6	0 0 0 1

The average code-word length is

$$\bar{L} = \sum_{k=0}^{6} p_k l_k$$

$$= 0.25(2)(2) + 0.125(3)(3) + 0.0625(4)(2)$$

$$= 2.625$$

The entropy of the source is

$$H(S) = \sum_{k=0}^{6} p_k \log_2\left(\frac{1}{p_k}\right)$$

$$= 0.25(2) \log_2\left(\frac{1}{0.25}\right) + 0.125(3) \log_2\left(\frac{1}{0.125}\right)$$

$$+ 0.0625(2) \log_2\left(\frac{1}{0.0625}\right)$$

$$= 2.625$$

The efficiency of the code is therefore

$$\eta = \frac{H(S)}{\bar{L}} = \frac{2.625}{2.625} = 1$$

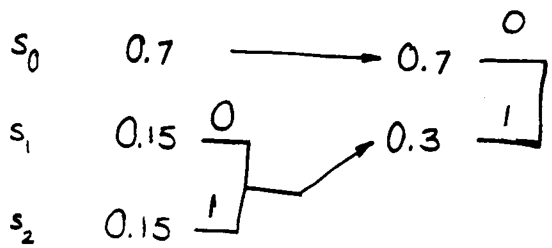
We could have shown that the efficiency of the code is 100% by inspection since

$$\eta = \frac{\sum_{k=0}^{6} p_k \log_2(1/p_k)}{\sum_{k=0}^{6} p_k l_k}$$

where $l_k = \log_2(1/p_k)$.

Problem 9.13

(a)



The Huffman code is therefore

s_0 0
s_1 1 0
s_2 1 1

The average code-word length is

$$\bar{L} = 0.7(1) + 0.15(2) + 0.12(2)$$
$$= 1.3$$

(b) For the extended source we have

Symbol	s_0s_0	s_0s_1	s_0s_2	s_1s_0	s_2s_0	s_1s_1	s_1s_2	s_2s_1	s_2s_2
Probability	0.49	0.105	0.105	0.105	0.105	0.0225	0.0225	0.0225	0.0225

Applying the Huffman algorithm to the extended source, we obtain the following source code:

s_0s_0	1
s_0s_1	0 0 1
s_0s_2	0 1 0
s_1s_0	0 1 1
s_2s_0	0 0 0 0
s_1s_1	0 0 0 1 0 0
s_1s_2	0 0 0 1 0 1
s_2s_1	0 0 0 1 1 0
s_2s_2	0 0 0 1 1 1

The corresponding value of the average code-word length is

$$\bar{L}_2 = 0.49(1) + 0.105(3)(3) + 0.105(4) + 0.0225(4)(4)$$

$$= 2.395 \text{ bits/extended symbol}$$

$$\frac{\bar{L}_2}{2} = 1.1975 \text{ bits/symbol}$$

(c) The original source has entropy

$$H(S) = 0.7 \log_2\left(\frac{1}{0.7}\right) + 0.15(2) \log_2\left(\frac{1}{0.15}\right)$$

$$= 1.18$$

According to Eq. (10.28),

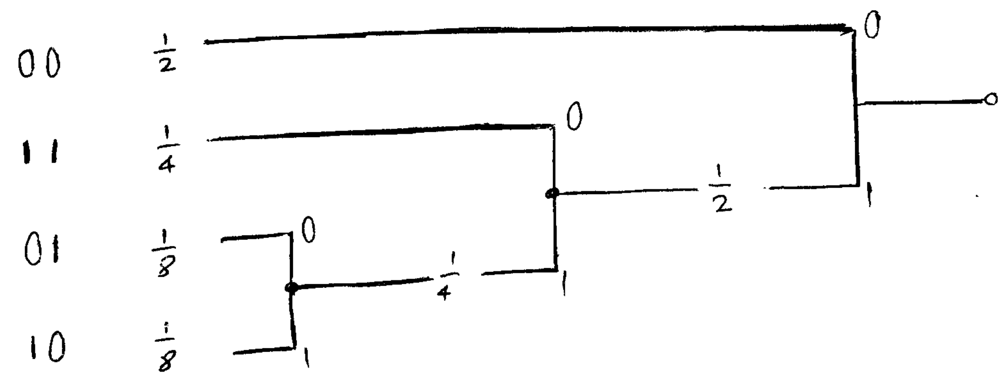
$$H(S) \leq \frac{\bar{L}_n}{n} \leq H(S) + \frac{1}{n}$$

This is a condition which the extended code satisfies.

Problem 9.14

Symbol	Huffman Code	Code-word length
A	1	1
B	0 1 1	3
C	0 1 0	3
D	0 0 1	3
E	0 0 1 1	4
F	0 0 0 0 1	5
G	0 0 0 0 0	5

Problem 9.15



Computer code	Probability	Huffman Code
0 0	$\frac{1}{2}$	0
1 1	$\frac{1}{4}$	1 0
0 1	$\frac{1}{8}$	1 1 0
1 0	$\frac{1}{8}$	1 1 1

The number of bits used for the instructions based on the computer code, in a probabilistic sense, is equal to

$$2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}\right) = 2 \text{ bits}$$

On the other hand, the number of bits used for instructions based on the Huffman code, is equal to

$$1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 3 \times \frac{1}{8} = \frac{7}{4}$$

The percentage reduction in the number of bits used for instruction, realized by adopting the Huffman code, is therefore

$$100 \times \frac{1/4}{2} = 12.5\%$$

Problem 9.16

Initial step

Subsequences stored: 0

Data to be parsed: 1 1 1 0 1 0 0 1 1 0 0 0 1 0 1 1 0 1 0 0 ...

Step 1

Subsequences stored: 0, 1, 11

Data to be parsed: 1 0 1 0 0 1 1 0 0 0 1 0 1 1 0 1 0 0 ..

Step 2

Subsequences stored: 0, 1, 11, 10

Data to be parsed: 1 0 0 1 1 0 0 0 1 0 1 1 0 1 0 0 ....

Step 3

Subsequences stored: 0, 1, 11, 10, 100

Data to be parsed: 1 1 0 0 0 1 0 1 1 0 1 0 0 ...

Step 4

Subsequences stored: 0, 1, 11, 10, 100, 110

Data to be parsed: 0 0 1 0 1 1 0 1 0 0 ...

<u>Step 5</u>

Subhsequences stored: 0, 1, 11, 10, 100, 110, 00

Data to be parsed: 1 0 1 1 0 1 0 0 ....

<u>Step 6</u>

Subsequences stored: 0, 1, 11, 10, 100, 110, 00, 101

Data to be parsed: 1 0 1 0 0 ...

<u>Step 7</u>

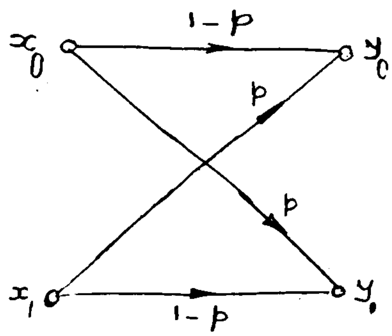
Subsequences stored: 0, 1, 11, 10, 100, 110, 00, 101, 1010

Data to be parsed: 0 ....

Now that we have gone as far as we can go with data parsing for the given sequence, we write

Numerical positions	1	2	3	4	5	6	7	8	9
Subsequences	0,	1,	11,	10,	100,	110,	00,	101,	1010
Numerical representations			22,	21,	41,	31,	11,	42,	81
Binary encoded blocks			0101,	0100,	0100,	0110,	0010,	1001,	10000

Problem 9.17



$$p(x_0) = p(x_1) = \frac{1}{2}$$

$$\begin{aligned} p(y_0) &= (1-p)p(x_0) + p\,p(x_1) \\ &= (1-p)\left(\frac{1}{2}\right) + p\left(\frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} p(y_1) &= p\,p(x_0) + (1-p)\,p(x_1) \\ &= p\left(\frac{1}{2}\right) + (1-p)\left(\frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

Problem 9.18

$$p(x_0) = \frac{1}{4}$$

$$p(x_1) = \frac{3}{4}$$

$$p(y_0) = (1 - p)\,(\frac{1}{4}) + p(\frac{3}{4})$$

$$= \frac{1}{4} + \frac{p}{2}$$

$$p(y_i) = p(\frac{1}{4}) + (1 - p)\,(\frac{3}{4})$$

$$= \frac{3}{4} - \frac{p}{2}$$

Problem 9.19

From Eq.(9.52) we may express the mutual information as

$$I(X,Y) = \sum_{j=0}^{1} \sum_{k=0}^{1} p(x_j,y_k) \log_2\left(\frac{p(x_j,y_k)}{p(x_j)\,p(y_k)}\right)$$

The joint probability $p(x_j,y_k)$ has the following four possible values:

$j = k = 0$: $p(x_0,y_k) = p_0(1-p) = (1 - p_1)(1 - p)$

$j = 0, k = 1$: $p(x_0,y_1) = p_0 p = (1 - p_1)p$

$j = 1, k = 0$: $p(x_1,y_0) = p_1 p$

$j = k = 1$: $p(x_1,y_1) = p(1 - p_1)$

where $p_0 = p(x_0)$ and $p_1 = p(x_1)$

The mutual information is therefore

$$I(X;Y) = (1-p_1)(1-p)\log_2\left(\frac{(1-p_1)(1-p)}{(1-p_1)((1-p_1)(1-p) + p_1 p)}\right)$$
$$+ (1-p_1)\, p \log_2\left(\frac{(1-p_1)\, p}{(1-p_1)((1-p_1)\, p + p_1(1-p))}\right)$$
$$+ p_1 p \log_2\left(\frac{p_1 p}{p_1((1-p_1)(1-p) + p_1 p)}\right)$$
$$+ p_1(1-p)\log_2\left(\frac{p_1(1-p)}{p_1((1-p_1)p + p_1(1-p)}\right)$$

Rearranging terms and expanding algorithms:

$$I(X;Y) = p \log_2 p + (1-p)\log_2(1-p)$$
$$- \left[p_1(1-p) + (1-p_1)p\right]\log_2\left[p_1(1-p) + (1-p_1)p\right]$$
$$- \left[p_1 p + (1-p_1)(1-p)\right]\log_2\left[p_1 p + (1-p_1)(1-p)\right]$$

Treating the transition probability p of the channel as a running parameter, we may carry out the following numerical evaluations:

<u>p=0</u>:

$I(X;Y) = - p_1 \log_2 p_1 - (1 - p_1) \log_2 (1 - p_1)$
$p_1 = 0.5$, $I(X;Y) = 1.0$

<u>p=01</u>:

$I(X;Y) = - 0.469 - (0.1 + 0.8\, p_1) \log_2 (0l1 + 0.8\, p_1)$
$\qquad - (0.9 - 0.8\, p_1) \log_2 (0.9 - 0.9\, p_1)$
$p_1 = 0.5$, $I(X;Y) = 0.531$

<u>p=0.2</u>:

$I(X;Y) = - 0.7219 - (0.2 + 0.6\, p_1) \log_2 (0.2 + 0.6\, p_1)$
$\qquad - (0.8 - 0.6\, p_1) \log_2 (0.8 - 0.6\, p_1)$
$p_1 = 0.5$, $I(X;Y) = 0.278$

p=0.3:

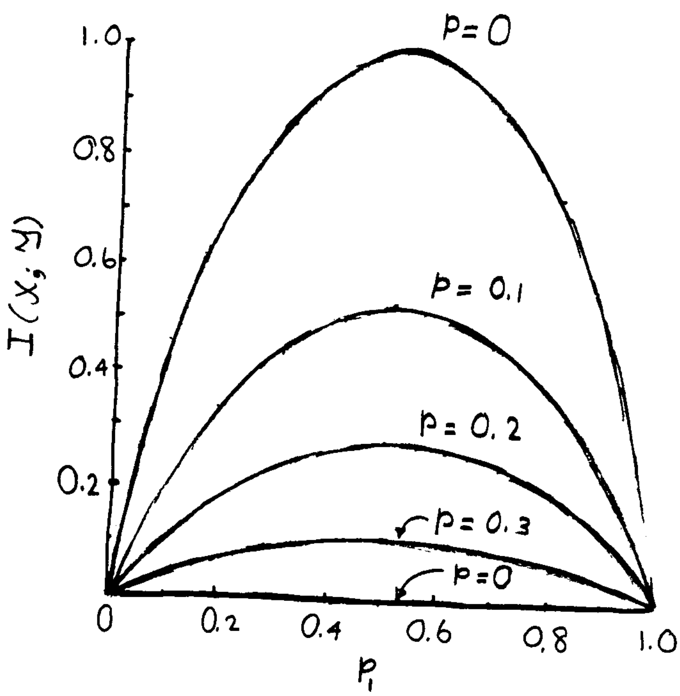
$$I(X;Y) = -0.88129 - (0.3 + 0.4\,p_1)\log_2(0.3 + 0.4\,p_1) - (0.7 - 0.4\,p_1)\log_2(0.7 - 0.4\,p_1)$$

$p_1 = 0.5, \quad I(X;Y) = 0.1187$

p=0.5:

$I(X;Y) = 0$

Thus, treating the a priori probability p_1 as a variable and the transition probability p as a running parameter, we get the following plots:



Problem 9.20

From the plots of I(X;Y) versus p_1 for p as a running parameter, that were presented in the solution to Problem 10.19 we observe that I(X;Y) attains its maximum value at p_1=0.5 for any p. Hence, evaluating the mutual information I(X;Y) at p_1=0.5, we get the channel capacity:

$$\begin{aligned} C &= 1 + p\log_2 p + (1-p)\log_2(1-p) \\ &= 1 - H(p) \end{aligned}$$

where H(p) is the entropy function of p.

Problem 9.21

(a) Let

$$z = p_1(1 - p) + (1 - p_1)\, p = (1 - p_0)\,(1 - p) + p_0 p$$

Hence,

$$p_1 p + (1 - p_1) = 1 - p_1\,(1 - p) - (1 - p_1)\, p$$
$$= 1 - z$$

Correspondingly, we may rewrite the expression for the mutual information I(X;Y) as

$$I(X;Y) = H(z) - H(p)$$

where

$$H(z) = - z \log_2 z - (1 - z) \log_2 (1 - z)$$
$$H([) = - p \log_2 p - (1 - p) \log_2 (1 - p)$$

(b) Differentiating I(X;Y) with respect to p_0 (or p_1) and setting the result equal to zero, we find that I(X;Y) attains its maximum value at $p_0 = p_1 = 1/2$.

(c) Setting $p_0 = p_1 - 1/2$ in the expression for the variable z, we get:

$$z = 1 - z = 1/2$$

Correspondingly, we have

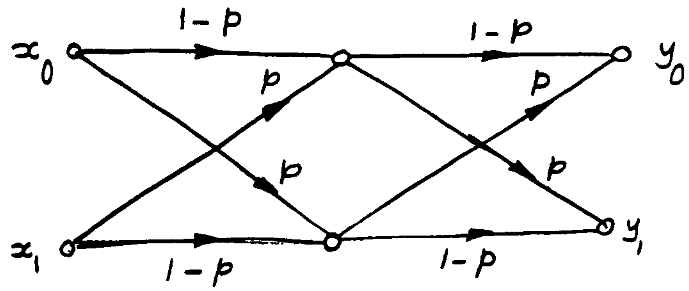
$$H(z) = 1$$

We thus get the following value for the channel capacity:

$$C = I(X;Y)\Big|_{p_0 = p_1 = 1/2}$$
$$= 1 - H(p)$$

where H(p) is the entropy function of the channel transition probability p.

Problem 9.22

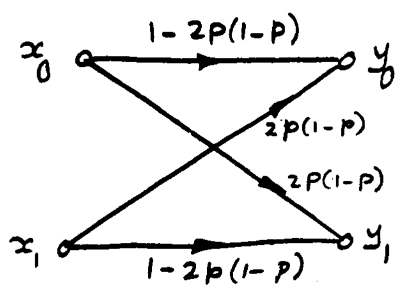


From this diagram, we obtain (by inspection)

$$P(y_0 \mid x_0) = (1 - p)^2 + p^2 = 1 - 2p(1 - p)$$

$$P(y_0 \mid x_1) = p(1 - p) + (1 - p)p = 2p(1 - p)$$

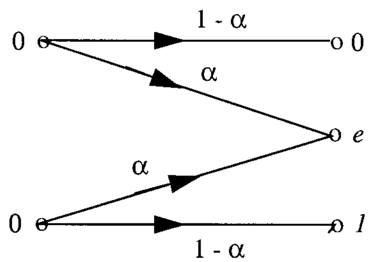
Hence, the cascade of two binary symmetric channels with a transition probability p is equivalent to a single binary symmetric channel with a transition probability equal to 2p(1 - p), as shown below:



Correspondingly, the channel capacity of the cascade is

$$\begin{aligned} C &= 1 - H(2p(1 - p)) \\ &= 1 - 2p(1 - p) \log_2 [2p(1 - p)] - (1 - 2p + 2p^2) \log_2 (1 - 2p + 2p^2) \end{aligned}$$

Problem 9.23



The mutual information for the erasure channel is

$$\mathbf{I(X;Y)} = \sum_{j=0}^{1}\sum_{k=0}^{2} p(x_j, y_k)\log_2\left(\frac{p(x_j, y_k)}{p(x_j)p(y_k)}\right) \tag{1}$$

The joint probabilities for the channel are

$p(x_0, y_0) = (1-\alpha)p_0$ $\qquad$ $p(x_1, y_0) = 0$ $\qquad$ $p(x_0, y_2) = p_0\alpha$

$p(x_0, y_1) = 0$ $\qquad$ $p(x_1, y_1) = (1-\alpha)p_1$ $\qquad$ $p(x_1, y_2) = p_1\alpha$

where $p_0 + p_1 = 1$. Substituting these values in (1), we get

$$\mathbf{I(X;Y)} = (1-\alpha)\left[p_0\log_2\left(\frac{1}{p_0}\right) + (1-p_0)\log_2\left(\frac{1}{1-p_0}\right)\right]$$

Since the transition probability $p = (1 - \alpha)$ is fixed, the mutual information **I(X;Y)** is maximized by choosing the a priori probability p_0 to maximize $H(p_0)$. This maximization occurs at $p_0 = 1/2$,. for which $H(p_0) = 1$. Hence, the channel capacity C of the erasure channel is $1 - \alpha$.

Problem 9.24

(a) When each symbol is repeated three times, we have

Messages	Unused signals	Channel outputs
000	001	000
	010	001
	011	010
	100	100
	101	101
	110	110
111		111

We note the following:

1. The probability that no errors occur in the transmission of three 0s or three 1s is $(1 - p)^3$.
2. The probability of just one error occurring is $3p(1 - p)^2$.
3. The probability of two errors occurring is $3p^2(1 - p)$.
4. The probability of receiving all three bits in error is p^3.

With the decision-making based on a majority vote, it is clear that contributions 3 and 4 lead to the probability of error

$$P_3 = 3p^2(1-p) + p^3$$

(b) When each symbol is transmitted five times, we have

Messages	Unused signals	Channel outputs
00000		00000
	00001	00001
	00010	00010
	00011	00011
	⋮	⋮
	11110	11110
11111		11111

The probability of zero, one, two, three, four, or five bit errors in transmission is as follows, respectively:

$$(1-p)^5$$
$$5p\,(1-p)^4$$
$$10p^2(1-p)^3$$
$$10p^3(1-p)^2$$
$$5p^4(1-p)$$
$$p^5$$

The last three contributions constitute the probability of error

$$P_e = p^5 + 5p^4(1-p) + 10p^3(1-p)^2$$

(a) For the general case of n=2m + 1, we note that the decision-making process (based on a majority vote) makes an error when m+1 bits or more out of the n bits of a message are received in error. The probability of i message bits being received in error is $\binom{n}{i} p^i (1-p)^{n-i}$.
Hence, the probability of error is (in general)

$$P_e = \sum_{i=m+1}^{n} \binom{n}{i} p_i (1-p)^{n-i}$$

The results derived in parts (a) and (b) for m=1 and m=2 are special cases of this general formula.

Problem 9.25

The differential entropy of a random variable is independent of its mean. To evaluate the differential entropy of a Gaussian vector **X**, consisting of the components $X_1, X_2, \ldots, X_n$, we may simplify our task by setting the mean of **X** equal to zero. Under this condition, we may express the joint probability density function of the Gaussian vector **X** as

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(\sqrt{2\pi})^n \, \sigma_1\sigma_2\ldots\sigma_n} \exp\left(-\frac{x_1^2}{2\sigma_1^2} - \frac{x_2^2}{2\sigma_2^2} - \ldots - \frac{x_n^2}{2\sigma_n^2} \right)$$

The logarithm of $f_{\mathbf{X}}(\mathbf{x})$ is

$$\log_2 f_{\mathbf{x}}(\mathbf{x}) = -\log_2\left((2\pi)^{n/2}\,\sigma_1\sigma_2\ldots\sigma_n\right) - \left(\frac{x_1^2}{2\sigma_1^2} + \frac{x_2^2}{2\sigma_2^2} + \ldots + \frac{x_n^2}{2\sigma_n^2} \right) \log_2 e$$

Hence, the differential entropy of $\mathbf{X}$ is

$$\begin{aligned} h(\mathbf{X}) &= -\int\int\ldots\int f_{\mathbf{X}}(\mathbf{x}) \log_2\left(f_{\mathbf{X}}(\mathbf{x})\right) d\mathbf{x} \\ &= \log_2\left((2\pi)^{n/2}\sigma_1\sigma_2\ldots\sigma_n\right) \int\int\ldots\int f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x} \\ &+ \log_2 e \int\int\ldots\int \left(\frac{x_1^2}{2\sigma_1^2} + \frac{x_2^2}{2\sigma_2^2} + \ldots + \frac{x_n^2}{2\sigma_n^2} \right) f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x} \end{aligned}$$

We next note that

$$\int\int\ldots\int f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x} = 1$$

$$\int\int\ldots\int x_i^2 \, f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x} = \sigma_i^2 \qquad i = 1,2,\ldots,n$$

Hence, we may simplify (1) as

$$h(\mathbf{X}) = \log_2\left[(2\pi)^{n/2}\sigma_1\sigma_2 \dots \sigma_n\right] + \frac{n}{2}\log_2 e\,,$$

$$= \log_2\left[2\pi(\sigma_1^2\sigma_2^2 \dots \sigma_n^2)^{1/n}\right]^{n/2} + \frac{n}{2}\log_2 e$$

$$= \frac{n}{2}\left[2\pi(\sigma_1^2\sigma_2^2 \dots \sigma_n^2)^{1/n}\right] + \frac{n}{2}\log_2 e$$

$$= \frac{n}{2}\log_2\left[2\pi e(\sigma_1^2\sigma_2^2 \dots \sigma_n^2)^{1/n}\right]$$

When the individual variances are equal:

$$\sigma_1^2 = \sigma_2^2 = \dots \sigma_n^2 = \sigma^2$$

$$(\sigma_1^2\sigma_2^2 \dots \sigma_n^2)^{1/n} = \sigma^2$$

Correspondingly, the differential entropy of **X** is

$$h(\mathbf{X}) = \frac{n}{2}\log_2(2\pi e\sigma^2)$$

Problem 9.26

(a) The differential entropy of a random variable X is

$$h(X) = -\int_{-\infty}^{\infty} f_X(x)\log_2 f_X(x)dx$$

The constraint on the value x of the random variable X is

$$|x| \leq M$$

Using the method of Lagrange multipliers, we find that h(X), subject to this constraint, is maximized when

$$\int_{-M}^{M} \left[-f_X(x) \log_2 f_X(x) + \lambda \, f_X(x) \right] dx$$

is stationary. Differentiating $-f_X(x)\log_2 f_X(x) + \lambda f_X(x)$ with respect to $f_X(x)$, and then setting the result equal to zero, we get

$$-\log_2 e + \lambda = \log_2 f_x(x)$$

This shows that $f_X(x)$ is independent of x. Hence, for the differential entropy h(X) under the constraints $|x| \leq M$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$ to be maximum, the random variable X must be uniformly distributed:

$$f_X(x) = \begin{cases} 1/2M, & -M \leq x < M \\ 0, & \text{otherwise} \end{cases}$$

(b) The maximum differential entropy of the uniformly distributed random variable X is

$$\begin{aligned} h(X) &= \int_{-M}^{M} \frac{1}{2M} \log_2(2M)dx \\ &= \frac{1}{2M} \log_2(2M) \int_{-M}^{M} dx \\ &= \log_2(2M) \end{aligned}$$

Problem 9.27

(a)

$$I(X;Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\log_2\left[\frac{f_X(x|y)}{f_X(x)}\right]dx\,dy \qquad (1)$$

$$I(Y;X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\log_2\left[\frac{f_Y(y|x)}{f_Y(y)}\right]dx\,dy \qquad (2)$$

From Bayes' rule, we have

$$\frac{f_X(x|y)}{f_X(x)} = \frac{f_Y(y|x)}{f_Y(y)}$$

Hence, $I(X;Y) = I(Y;X)$.

(b) We note that

$$f_{X,Y}(x,y) = f_X(x|y)f_Y(y)$$

Therefore, we may rewrite (1) as

$$I(X;Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\log_2\left[\frac{f_{X,Y}(x,y)}{f_X(x)f_Y(y)}\right]dx\,dy$$

From the continuous version of the fundamental inequality, we have

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\log_2\left(\frac{f_{X,Y}(x,y)}{f_x(x)f_y(y)}\right)dx\,dy \geq 0$$

which is satisfied with equality if and only if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

Hence,

$$I(X;Y) \geq 0$$

(c) By definition, we have

$$h(X|Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \log_2\left[\frac{1}{f_X(x|y)}\right] dx\, dy \tag{3}$$

By definition, we also have

$$h(X) = \int_{-\infty}^{\infty} f_X(x) \log_2\left[\frac{1}{f_X(x)}\right] dx$$

Since

$$\int_{-\infty}^{\infty} f_Y(y|x)\, dy = 1,$$

we may rewrite the expression for h(X) as

$$h(X) = \int_{-\infty}^{\infty} f_X \log_2\left[\frac{1}{f_X(x)}\right] dx \int_{-\infty}^{\infty} f_Y(y|x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_Y(y|x)\, f_X(x) \log_2\left[\frac{1}{f_X(x)}\right] dx\, dy$$

But

$$f_Y(y|x)\, f_X(x) = f_{X,Y}(x,y)$$

Hence,

$$h(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\log_2\left[\frac{1}{f_X(x)}\right]dx \tag{4}$$

Next, we subtract (3) from (4), obtaining

$$\begin{aligned} h(X) - h(X|Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\left[\log_2\frac{1}{f_X(x)} - \log_2\frac{1}{f_X(x|y)}\right]dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\log_2\left[\frac{f_X(x|y)}{f_X(x)}\right]dx\,dy \\ &= I(X;Y) \end{aligned}$$

(d) Using the symmetric property derived in part (a), we may also write

$$I(Y;X) = h(Y) - h(Y|X)$$

Problem 9.28

By definition, we have

$$h(Y|X) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\log_2\left[\frac{1}{f_Y(y|x)}\right]dx\,dy$$

Since

$$f_{X,Y}(x,y) = f_Y(y|x)f_X(x)$$

we may also write

$$h(Y|X) = \int_{-\infty}^{\infty} f_X(x)\,dx \int_{-\infty}^{\infty} f_Y(y|x)\log_2\left[\frac{1}{f_Y(y|x)}\right]dy \tag{1}$$

The variable Y is related to X as

$$Y = X + N$$

Hence, the conditional probability density function $f_Y(y \mid x)$ is identical to that of N except for a translation of x, the given value of X. Let $f_N(n)$ denote the probability density function of N. Then

$$f_Y(y|x) = f_N(y-x)$$

Correspondingly, we may write

$$\begin{aligned}\int_{-\infty}^{\infty} f_Y(y|x)\log_2\left[\frac{1}{f_Y(y|x)}\right]dy &= \int_{-\infty}^{\infty} f_N(y-x)\log_2\left[\frac{1}{f_N(y-x)}\right]dy \\ &= \int_{-\infty}^{\infty} f_N(n)\log_2\left[\frac{1}{f_N(n)}\right]dn \\ &= h(N)\end{aligned} \tag{2}$$

where, in the second line, we have used n = y-x and dn = dy for a given x. Substituting Eq. (2) in (1):

$$\begin{aligned}h(Y|X) &= h(N)\int_{-\infty}^{\infty} f_X(x)\,dx \\ &= h(N)\end{aligned}$$

Problem 9.29

(a) Channel bandwidth B = 3.4 kHz
Received signal-to-noise ratio SNR = $10^3 \equiv 30$ dB

Hence, the channel capacity is

$$C = B \log_2(1 + \text{SNR})$$
$$= 3.4 \times 10^3 \log_2(1 + 10^3)$$
$$= 33.9 \times 10^3 \text{ bits/second}$$

(b) $4800 = 3.4 \times 10^3 \log_2 (1 + \text{SNR})$

Solving for the unknown SNR, we get

$$\text{SNR} = 1.66 \equiv 2.2 \text{ dB}$$

Problem 9.30

With 10 distinct brightness levels with equal probability, the information in each level is $\log_2 10$ bits. With each picture frame containing 3×10^5 elements, the information content of each picture frame is $3 \times 10^5 \log_2 10$ bits. Thus, a rate of information transmission of 30 frames per second corresponds to

$$30 \times 3 \times 10^5 \log_2 10 = 9 \times 10^6 \log_2 10 \text{ bits/second}$$

That is, the channel capacity is

$$C = 9 \times 10^6 \log_2 10 \text{ bits/second}$$

From the information capacity theorem:

$$C = B \log_2 (1 + SNR)$$

With a signal-to-noise ratio SNR = 10^3 = 30 dB, the channel bandwidth is therefore

$$B = \frac{C}{\log_2(1 + SNR)}$$

$$= \frac{9 \times 10^6 \log_2 10}{\log_2 1001}$$

$$= 3 \times 10^3 \text{ Hz}$$

Problem 9.31

The information in each element is $\log_2 10$ bits.

The information in each picture is $[3 \log_2 (10)] \times 10^5$ bits.

The transmitted information rate is $[9 \log_2 (10)] \times 10^6$ bits/second.

The channel must have this capacity. From the information capacity theorem,

$$c = B \log_2(1 + SNR).$$

Therefore,

$$[9 \log_2(10)] \times 10^6 \text{ bits/second} = B\log_2(1 + 1000).$$

Solving for B, we get

$$B = 9 \times 10^6 \text{Hz} \bullet \left(\frac{\log_2(10)}{\log_2(1001)}\right) = 3 \times 10^6 \text{Hz}$$

Problem 9.32

Figure 1 shows the 64-QAM constellation. Under the condition that the transmitted signal energy per symbol is maintained the same as that in Fig. 1, we get the tightly packed constellation of Fig. 2. We now find that the Euclidean distance between adjacent signal points in the tightly packed constellation of Fig. 2 is larger than that of the 64-QAM constellation in Fig. 1. From Eq. (5.95) of the textbook, we recall that an increase in the minimum Euclidean distance d_{min} results in a corresponding reduction in the average probability of symbol error. It follows therefore that, with the signal energy per symbol being maintained the same in the two constellations of Figs. 1 and 2, a digital communication systems using the tightly packed constellation of Fig. 2 produces a smaller probability of error than the corresponding 64-QAM system represented by Fig. 1.

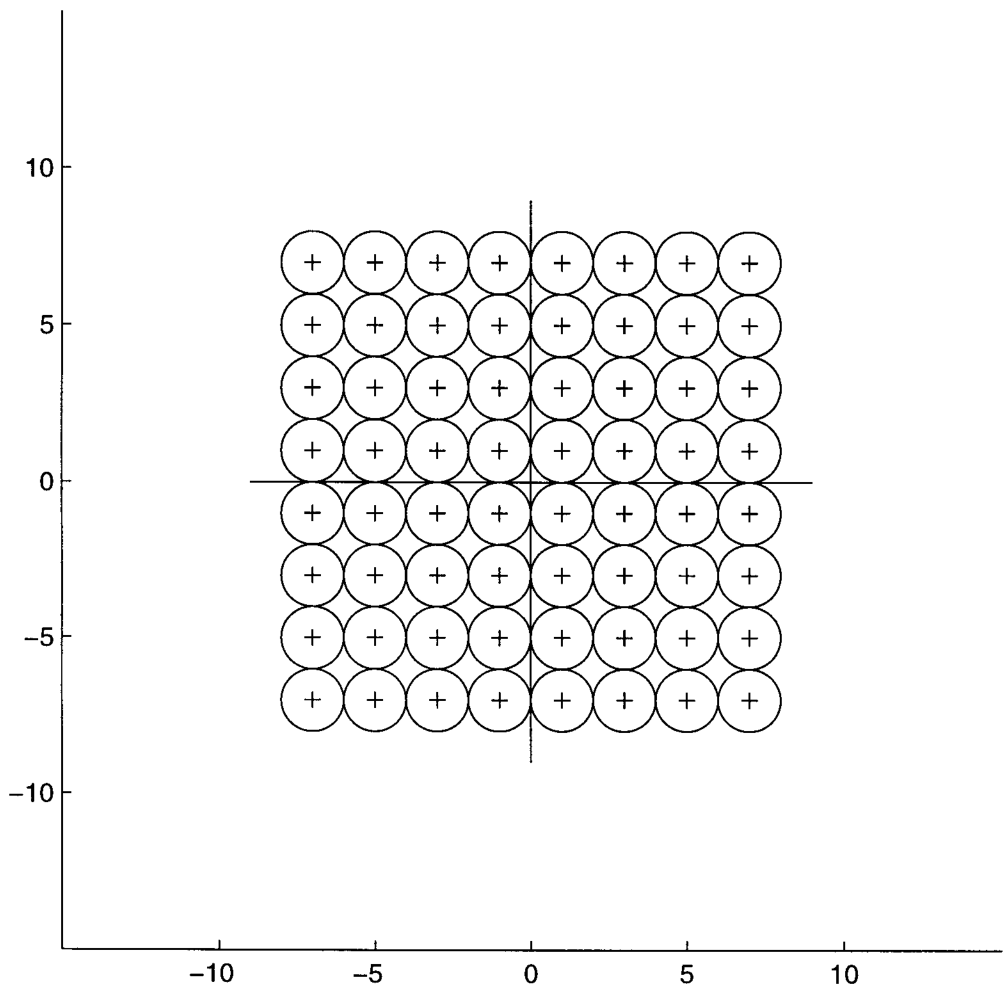


Figure 1

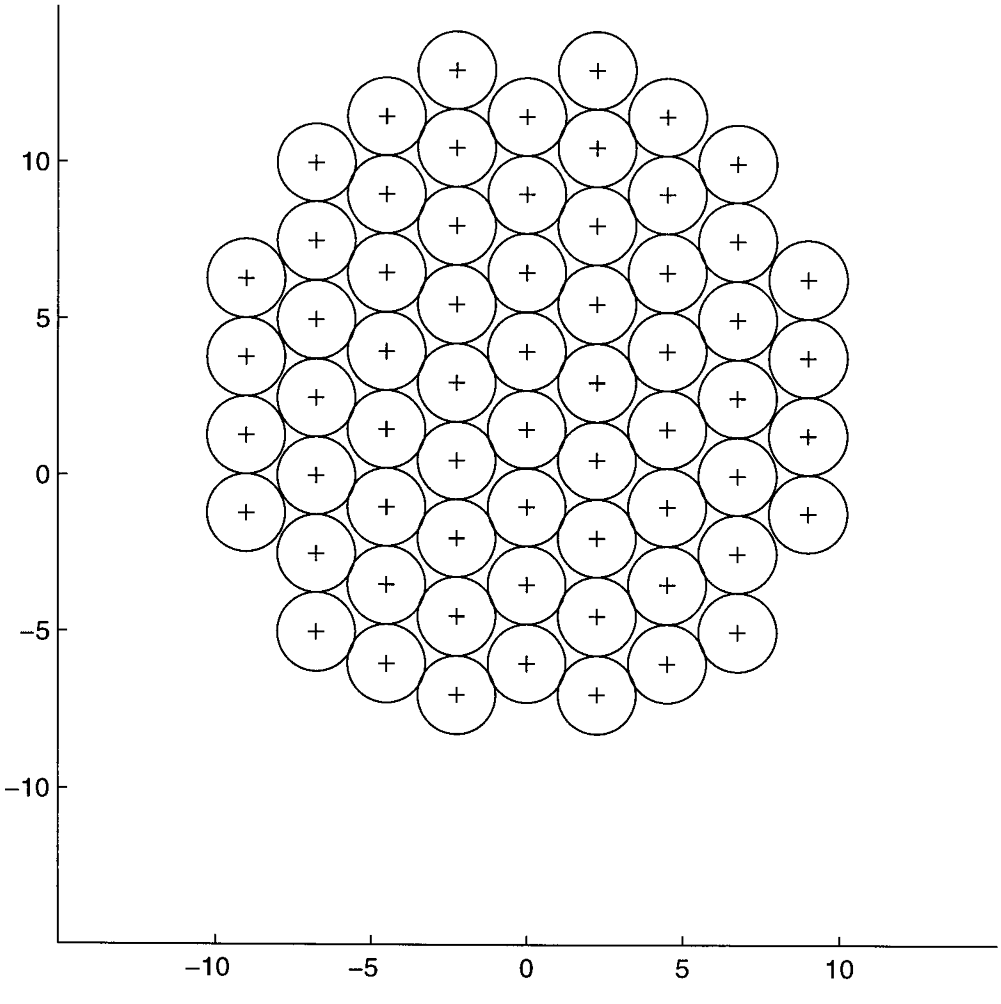


Figure 2

Problem 9.33

In the textbook, the capacity of the NEXT-dominated channel is derived as

$$C = \frac{1}{2}\int_{F_a} \log_2\left(1 + \frac{|H_{\text{NEXT}}(f)|^2}{|H(f)|^2}\right) df$$

where F_a is the set of positive and negative frequencies for which $S_x(f) > 0$, where $S_x(f)$ is the power spectral density of the transmitted signal.

For the NEXT-dominated channel described in the question, the capacity is

$$C = \frac{1}{2}\int_{F_a} \log_2\left(1 + \frac{\beta f^{3/2}}{\exp(-\alpha\sqrt{f})}\right) df$$

$$= \frac{1}{2}\int_{F_a} \log_2\left(1 + \frac{\beta f^{3/2}}{\exp\left(-\frac{klf^{1/2}}{l_o}\right)}\right) df$$

where β, k, l and f_o are all constants pertaining to the transmission medium. This formula for capacity can only be evaluated numerically for prescribed values of these constants.

Problem 9.34

For k=1, Eq. (9.138) reduces to

$$10\ \log_2(\mathrm{SNR}) = 6\ \log_2 N + C_1\ \mathrm{dB} \tag{1}$$

Expressing Eq. (3.33) in decibels, we have

$$10\ \log_2(\mathrm{SNR}) = 6R + 10\ \log_{10}\left(\frac{3P}{m_{max}^2}\right) \tag{2}$$

The number of bits per sample R, is defined by

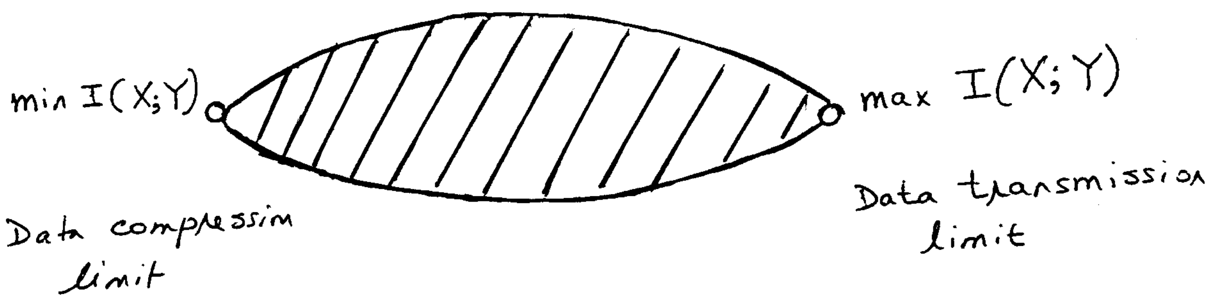
$$R = \log_2 N$$

We thus see that Eqs. (1) and (2) are equivalent, with

$$C_1 = 10\ \log_{10}\left(\frac{3P}{m_{max}^2}\right)$$

Problem 9.35

The rate distortion function and channel capacity theorem may be summed up diagrammatically as follows:



According to the rate distortion theory, the data compression limit set by minimizing the mutual information I(X;Y) lies at the extreme left of this representation; here, the symbol Y represents the data compressed form of the source symbol X. On the other hand, according to the channel capacity theorem the data transmission limit is defined by maximizing the mutual information I(X;Y) between the channel input X and channel output Y. This latter limit lies on the extreme right of the representation shown above.

1 Problem 9. 36

Matlab codes

```
% Computer Problem in Chapter 9
% Figure:  The minimum achievable BER as a function of
% Eb/NO for several different code rates using binary signaling.
% This program calculates the Minimum required Eb/NO
% for BPSK signalling at unit power over AWGN channel
% given a rate and an allowed BER.
% Code is based on Brandon's C code.
% Ref: Brendan J. Frey,  Graphical models for machine
% learning and digital communications, The MIT Press.
% Mathini Sellathurai

EbNo=double( [7.85168, 7.42122, 6.99319, 6.56785, 6.14714, 5.7329, 5.32711, ...
4.92926, 4.54106, 4.16568, 3.80312, 3.45317, 3.11902, 2.7981, 2.49337, 2.20617, ...
1.93251, 1.67587, 1.43313, 1.20671, 0.994633, 0.794801, 0.608808, 0.434862, ...
0.273476, 0.123322, -0.0148204, -0.144486, -0.266247, -0.374365, -0.474747, -0.5708, ...
-0.659038, -0.736526, -0.812523, -0.878333, -0.944802, -0.996262, -1.04468, ...
-1.10054, -1.14925, -1.18536, -1.22298, -1.24746, -1.27394, -1.31061, -1.34588, ...
-1.37178, -1.37904, -1.40388, -1.42553, -1.45221, -1.43447, -1.44392, -1.46129, ...
-1.45001, -1.50485, -1.50654, -1.50192, -1.45507, -1.60577, -1.52716, -1.54448, ...
-1.51713, -1.54378, -1.5684]);

rate= double([9.989372e-01, 9.980567e-01, 9.966180e-01, 9.945634e-01, ...
9.914587e-01, 9.868898e-01, 9.804353e-01, 9.722413e-01, 9.619767e-01, 9.490156e-01, ...
9.334680e-01, 9.155144e-01, 8.946454e-01, 8.715918e-01, 8.459731e-01, 8.178003e-01, ...
7.881055e-01, 7.565174e-01, 7.238745e-01, 6.900430e-01, 6.556226e-01, ...
6.211661e-01, 5.866480e-01, 5.525132e-01, 5.188620e-01, 4.860017e-01, 4.539652e-01, ...
4.232136e-01, 3.938277e-01, 3.653328e-01, 3.382965e-01, 3.129488e-01, 2.889799e-01, ...
2.661871e-01, 2.451079e-01, 2.251691e-01, 2.068837e-01, 1.894274e-01, ...
1.733225e-01, 1.588591e-01, 1.453627e-01, 1.326278e-01, 1.210507e-01, 1.101504e-01, ...
1.002778e-01, 9.150450e-02, 8.347174e-02, 7.598009e-02, 6.886473e-02, 6.266875e-02, ...
5.698847e-02, 5.188306e-02, 4.675437e-02, 4.239723e-02, 3.851637e-02, 3.476062e-02,...
 3.185243e-02, 2.883246e-02, 2.606097e-02, 2.332790e-02, 2.185325e-02, ...
1.941896e-02, 1.764122e-02, 1.586221e-02, 1.444108e-02, 1.314112e-02]);
N=66;


b=double([1e-5]);  % Allowed BER
% Rate R (bits per channel usage)
r=double([1/32, 1/16,0.1,0.2,0.3,0.4,0.5, 0.6, 0.7, 0.8,0.85,0.95]);
```

```
le=zeros(1,length(r)); % initialize buffer for Eb/NO
  for p=1:length(r)
   c = double(r(p)*(1.0+b*log(b)+(1.0-b)*log(1.0-b)/log(2.0)));
    i=N-1;
  % Minimum Eb/NO calculations
             while ( (i>=0) & (c>rate(i)) )
                   i=i-1;
             end
     i=i+1;

if  ( (i>0) |  (i<N ) )
                   e =double( EbNo(i)+(EbNo(i-1)-EbNo(i))*(c-rate(i))/(rate(i-1)-rate(i))
                   le(p)=10*log10((10^(e/10))*c/r(p));
                  display(le)
             else
                 display('values out of range')
             end
  end
plot(10*log10(r),le,'-')
xlabel('Rate (dB)')
ylabel('Minimum E_b/N_0 (dB)')
axis([10*log10(1/32), 0, -2 4])
```

```
% Computer Experiment in Chapter 9
% Program to create the figure for the minimum
% Eb/N0 needed for error-free communication
% with a rate R code, over an AWGN channel
% using binary signaling
% This program calculates the Minimum required Eb/N0
% for BPSK signalling at unit power over AWGN channel
% given a rate and an allowed BER.
% Code is based on Brandon's C code.
% Ref: Brendan J. Frey,  Graphical models for machine
% learning and digital communications, The MIT Press.
% Mathini Sellathurai

EbNo= double([7.85168, 7.42122, 6.99319, 6.56785, 6.14714, 5.7329, 5.32711, ...
4.92926, 4.54106, 4.16568, 3.80312, 3.45317, 3.11902, 2.7981, 2.49337, 2.20617, ...
1.93251, 1.67587, 1.43313, 1.20671, 0.994633, 0.794801, 0.608808, 0.434862, ...
0.273476, 0.123322, -0.0148204, -0.144486, -0.266247, -0.374365, -0.474747, -0.5708, ...
-0.659038, -0.736526, -0.812523, -0.878333, -0.944802, -0.996262, -1.04468, ...
-1.10054, -1.14925, -1.18536, -1.22298, -1.24746, -1.27394, -1.31061, -1.34588, ...
-1.37178, -1.37904, -1.40388, -1.42553, -1.45221, -1.43447, -1.44392, -1.46129, ...
-1.45001, -1.50485, -1.50654, -1.50192, -1.45507, -1.60577, -1.52716, -1.54448, ...
-1.51713, -1.54378, -1.5684]);

rate=double( [9.989372e-01, 9.980567e-01, 9.966180e-01, 9.945634e-01, ...
9.914587e-01, 9.868898e-01, 9.804353e-01, 9.722413e-01, 9.619767e-01, 9.490156e-01, ...
9.334680e-01, 9.155144e-01, 8.946454e-01, 8.715918e-01, 8.459731e-01, 8.178003e-01, ...
7.881055e-01, 7.565174e-01, 7.238745e-01, 6.900430e-01, 6.556226e-01, ...
6.211661e-01, 5.866480e-01, 5.525132e-01, 5.188620e-01, 4.860017e-01, 4.539652e-01, ...
4.232136e-01, 3.938277e-01, 3.653328e-01, 3.382965e-01, 3.129488e-01, 2.889799e-01, ...
2.661871e-01, 2.451079e-01, 2.251691e-01, 2.068837e-01, 1.894274e-01, ...
1.733225e-01, 1.588591e-01, 1.453627e-01, 1.326278e-01, 1.210507e-01, 1.101504e-01, ...
1.002778e-01, 9.150450e-02, 8.347174e-02, 7.598009e-02, 6.886473e-02, 6.266875e-02, ...
5.698847e-02, 5.188306e-02, 4.675437e-02, 4.239723e-02, 3.851637e-02, 3.476062e-02,...
 3.185243e-02, 2.883246e-02, 2.606097e-02, 2.332790e-02, 2.185325e-02, ...
1.941896e-02, 1.764122e-02, 1.586221e-02, 1.444108e-02, 1.314112e-02]);
N=66;


b=double(0.5:-1e-5:1e-5); % Allowed BER
rrr=double([0.99,1/2,1/3,1/4,1/5,1/8]);  % Rate R(bits/channel usage)
le=zeros(1,length(b));

for rr=1:length(rrr)
  r=rrr(rr);
    for p=1:length(b)
      c = double(r*(1.0+b(p)*log(b(p))+(1.0-b(p))*log(1.0-b(p))/log(2.0)));
```

```
        i=N-1;

              while ( (i>=0) & (c>rate(i)) )
                    i=i-1;
              end
      i=i+1;

if  ( (i>0) |  (i<N ) )
                    e = double(EbNo(i)+(EbNo(i-1)-EbNo(i))*(c-rate(i))/(rate(i-1)-rate(i))
                    le(p)=10*log10((10^(e/10))*c/r);

              else
                  display('values out of range')
              end
       end
   plot(le,10*log10(b),'-')
end
xlabel('E_b/N_0  (dB)')
ylabel('Minimum BER')
axis([-2 1 -50 -10])
```

Answer to Problem 9.36

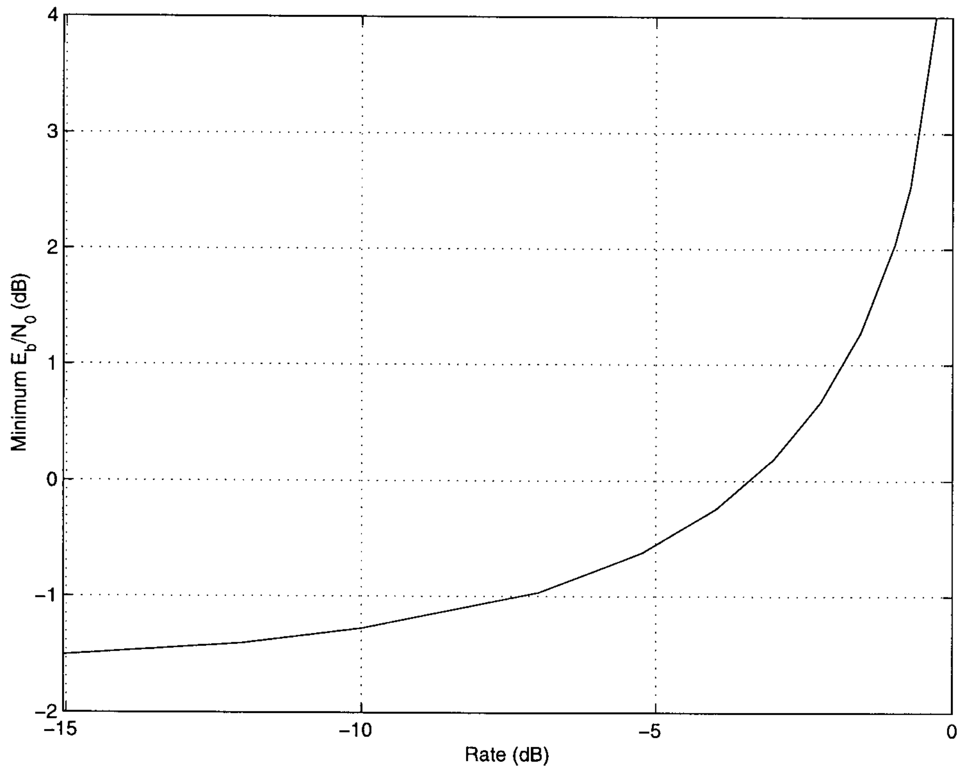


Figure 1: The minimum Eb/N0 needed for error-free communication with a rate R code, over an AWGN channel using binary signaling

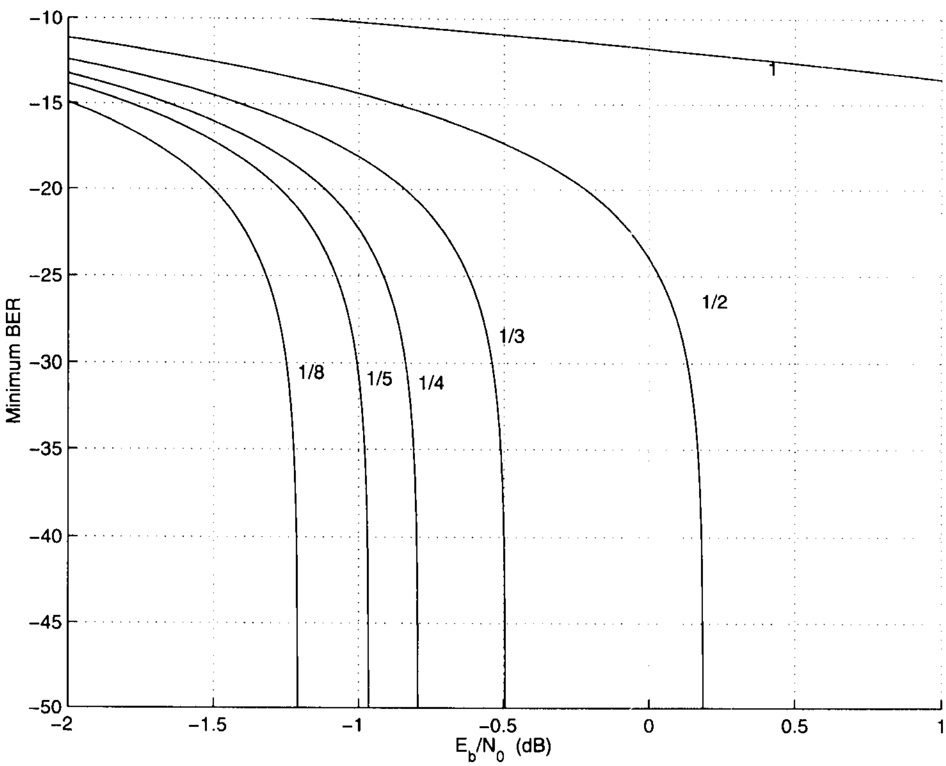


Figure 2: The minimum achievable BER as a function of Eb/N0 for several different code rates using binary signaling

Chapter 10

Error-Control Coding

Problem 10.1

The matrix of transition probabilities of a discrete memoryless channel with 2 inputs and Q outputs may be written as

$$P = \begin{bmatrix} p(0|0) & p(1|0) & p(2|0) & \cdots & p(Q-1|0) \\ p(0|1) & p(1|1) & p(2|1) & \cdots & p(Q-1|1) \end{bmatrix}$$

For a symmetric channel,

$$p(j|0) = p(Q-1-j|1), \qquad j = 0,1,\ldots,Q-1$$

Moreover, each row of the matrix **P** contains the same set of numbers, and each column of the matrix **P** contains the same set of numbers. For example, for Q=4, we may write

$$P = \begin{bmatrix} a & a & b & b \\ b & b & a & a \end{bmatrix}$$

The sum of the elements of each row of matrix **P** must add up to one. Hence, for this example,

$$2a + 2b = 1$$

The probability of receiving symbol j is

$$p(j) = p(j|0)\,p(0) + p(j|1)p(1)$$

For equally likely input symbols:

$$p(0) = p(1) = \frac{1}{2}$$

Hence,

$$p(j) = \frac{1}{2}\left[p(j|0) + p(Q-1-j|0)\right]$$

For the example of Q=4, we have

$$p(j) = \frac{1}{2}(a+b)$$

$$= \frac{1}{4}, \qquad j = 0,1,2,3$$

In general, we may write

$$p(j) = \frac{1}{Q}, \qquad j = 0,1,\ldots,Q-1$$

Problem 10.2

For a binary PSK channel, the probability density function of the correlator output in the receiver is

$$f_X(x|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}\left(x+\sqrt{E_b}\right)^2\right]$$

$$f_X(x|1) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}\left(x-\sqrt{E_b}\right)^2\right]$$

Let

$$y = \sqrt{\frac{2}{N_0}}\ x$$

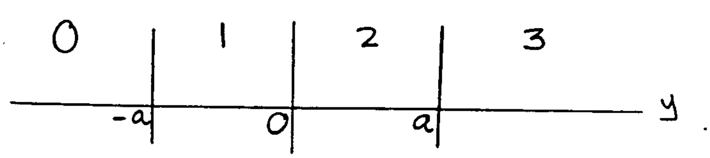
$$dy = \sqrt{\frac{2}{N_0}}\ dx$$

y pertains to a Gaussian variable of mean $\pm\sqrt{\frac{2E_b}{N_0}}$ and unit variance. We may therefore express the channel transition probability as

$$p(y|0) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(y + \sqrt{\frac{2E_b}{N_0}}\right)^2\right]$$

$$p(y|1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(y - \sqrt{\frac{2E_b}{N_0}}\right)^2\right]$$

where $-\infty < y < \infty$.



$$p(0|0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(y + \sqrt{\frac{2E_b}{N_0}}\right)^2\right] dy$$

$$= \frac{1}{2} \operatorname{erfc}\left(\frac{a}{\sqrt{2}} - \sqrt{\frac{E_b}{N_0}}\right)$$

$$p(1|0) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{0} \exp\left[-\frac{1}{2}\left(y + \sqrt{\frac{2E_b}{N_0}}\right)^2\right] dy$$

$$= \frac{1}{2}\left[\operatorname{erfc}\left(-\sqrt{\frac{E_b}{N_0}}\right) - \operatorname{erfc}\left(\frac{a}{\sqrt{2}} - \sqrt{\frac{E_b}{N_0}}\right)\right]$$

$$p(2|0) = \frac{1}{2\pi} \int_{0}^{a} \exp\left[-\frac{1}{2}\left(y + \sqrt{\frac{2E_b}{N_0}}\right)^2\right] dy$$

$$= \frac{1}{2}\left[\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) - \operatorname{erfc}\left(\frac{a}{\sqrt{2}} + \sqrt{\frac{E_b}{N_0}}\right)\right]$$

$$p(3|0) = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} \exp\left[-\frac{1}{2}\left(y + \sqrt{\frac{2E_b}{N_0}}\right)\right]^2 dy$$

$$= \frac{1}{2} \operatorname{erfc}\left(\frac{a}{\sqrt{2}} + \sqrt{\frac{E_b}{N_0}}\right)$$

We also note that

$p(3|0) = p(0|1)$
$p(2|0) = p(1|1)$
$p(1|0) = p(2|1)$
$p(0|0) = p(3|1)$

Hence, the channel is symmetric.

Problem 10.3

From the solution to Problem 10.2, we readily note the following:

$$p(y|0) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(y + \sqrt{\frac{2E}{N_0}}\right)^2\right], \qquad -\infty < y < \infty$$

$$p(y|1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(y - \sqrt{2\frac{E}{N_0}}\right)^2\right], \qquad -\infty < y < \infty$$

where E is the code symbol energy.

Problem 10.4

Message Sequence	Single-parity-check code
0 0 0	0 0 0 0
0 0 1	0 0 1 1
0 1 0	0 1 0 1
0 1 1	0 1 1 0
1 0 0	1 0 0 1
1 0 1	1 0 1 0
1 1 0	1 1 0 0
1 1 1	1 1 1 1

Problem 10.5

For the (4,1) repetition code, the parity check matrix is

$$H = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

For a (7,4) Hamming code, we have

$$H = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & \vdots & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 1 & 1 & 1 \end{bmatrix}$$

For the Hamming code, the parity check matrix **H** is more structured than that for the repetition code. Indeed, the matrix **H** for the Hamming code includes that for the repetition code as a submatrix.

Problem 10.6

The generator matrix for the (7.4) Hamming code is

$$G = \begin{bmatrix} 1 & 1 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \vdots & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & \vdots & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

The parity-check matrix is

$$H = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & \vdots & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 1 & 1 & 1 \end{bmatrix}$$

Hence,

$$HG^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{mod-2}$$

Problem 10.7

(a) Viewing the matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & \vdots & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 0 & 1 & 1 & 1 \end{bmatrix}$$

as a generator matrix, we may define the code vector $\underset{\sim}{c}$ in terms of the message vector **m** as

$$\underset{\sim}{c} = \mathbf{m}\, H$$

The message word length is

$$n - k = 7-4 = 3$$

Hence, we may construct the following table

Message word	Code word	Hamming weight
0 0 0	0 0 0 0 0 0 0	0
0 0 1	0 0 1 0 1 1 1	4
0 1 0	0 1 0 1 1 1 0	4
0 1 1	0 1 1 1 0 0 1	4
1 0 0	1 0 0 1 0 1 1	4
1 0 1	1 0 1 1 1 0 0	4
1 1 0	1 1 0 0 1 0 1	4
1 1 1	1 1 1 0 0 1 0	5

(b) The minimum value of the Hamming weight defines the Hamming distance of the dual code as

$$d_{min} = 4$$

Problem 10.8

(a) For a (5,1) repetition code:

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \vdots & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

$$H^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The syndrome is

$$s = e\,H^T$$

where **e** is the error pattern. For a single error, we thus have

Error pattern	Syndrome
0 0 0 0 1	1 1 1 1
0 0 0 1 0	0 0 0 1
0 0 1 0 0	0 0 1 0
0 1 0 0 0	0 1 0 0
1 0 0 0 0	1 0 0 0

(b) For two errors in the received word, we have

Error pattern	Syndrome
0 0 0 1 1	1 1 1 0
0 0 1 0 1	1 1 0 1
0 1 0 0 1	1 0 1 1
1 0 0 0 1	0 1 1 1
0 0 1 1 0	0 0 1 1
0 1 0 1 0	0 1 0 1
1 0 0 1 0	1 0 0 1
0 1 1 0 0	0 1 1 0
1 0 1 0 0	1 0 1 0
1 1 0 0 0	1 1 0 0

We note that the syndromes for all single-error and double-error patterns are distinct. This is intuitively satisfying since a (5,1) repetition code is capable of correcting up to two errors in the received vector

$$y = e + c$$

Problem 10.9

$$g(X) = 1 + X + X^3$$

$$c(X) = m(X)\,g(X)$$

Hence, we may construct the following table:

Message word	m(X)	c(X)	Code word
0 0 0 0	0	0	0 0 0 0 0 0 0
0 0 0 1	X^3	$X^3 + X^4 + X^6$	0 0 0 1 1 0 1
0 0 1 0	X^2	$X^2 + X^3 + X^5$	0 0 1 1 0 1 0
0 1 0 0	X	$X + X^2 + X^4$	0 1 1 0 1 0 0
1 0 0 0	1	$1 + X + X^3$	1 1 0 1 0 0 0
0 0 1 1	$X^2 + X^3$	$X^2 + X^4 + X^5 + X^6$	0 0 1 0 1 1 1
0 1 1 0	$X + X^2$	$X + X^3 + X^4 + X^5$	0 1 0 1 1 1 0
1 1 0 0	$1 + X$	$1 + X^2 + X^3 + X^4$	1 0 1 1 1 0 0
0 1 0 1	$X + X^3$	$X + X^2 + X^3 + X^6$	0 1 1 1 0 0 1
1 0 1 0	$1 + X^2$	$1 + X + X^2 + X^5$	1 1 1 0 0 1 0
1 0 0 1	$1 + X^3$	$1 + X + X^4 + X^6$	1 1 0 0 1 0 1
0 1 1 1	$X + X^2 + X^3$	$X + X^5 + X^6$	0 1 0 0 0 1 1
1 1 1 0	$1 + X + X^2$	$1 + X^5 + X^6$	1 0 0 0 1 1 0
1 0 1 1	$1 + X^2 + X^3$	$1+X+X^2+X^3+X^4+X^5+X^6$	1 1 1 1 1 1 1
1 1 0 1	$1 + X + X^3$	$1 + X^2 + X^6$	1 0 1 0 0 0 1
1 1 1 1	$1 + X + X^2 + X^3$	$1+ X^3 + X^5 + X^6$	1 0 0 1 0 1 1

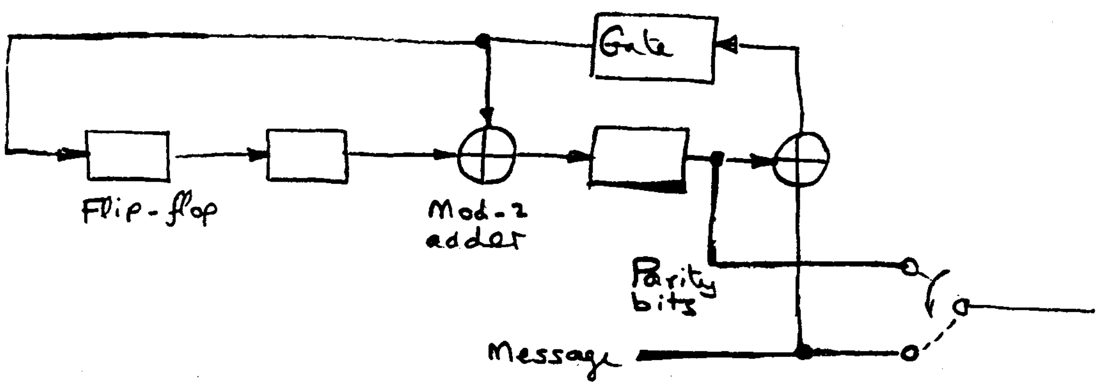
Comparing the code word to the message word, we see that the cyclic code generated by multiplying g(X) and c(X) is not a systematic code.

Problem 10.10

Consider the generator polynomial

$$g(X) = 1 + X^2 + X^3$$

The encoder corresponding to this g(X) is as follows:



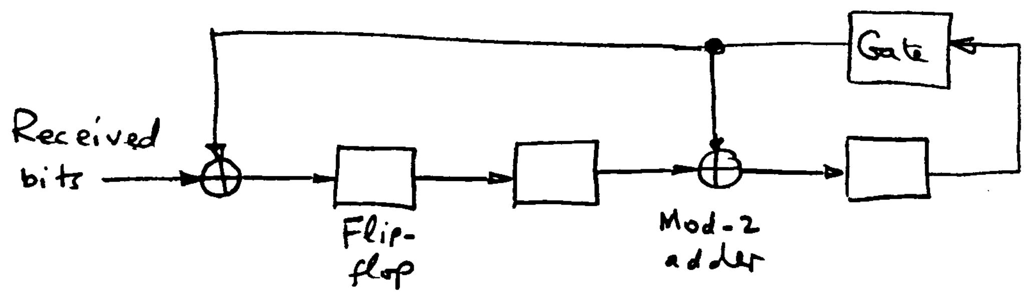
The generator matrix **G** associated with this encoder is

$$G + \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

To reduce this matrix to a systematic form, we add row 1 to 2, add rows 1 and 2 to row 3, and add rows 2 and 3 to row 4:

$$
G = \begin{bmatrix} 1 & 0 & 1 & \vdots & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & \vdots & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & \vdots & 0 & 0 & 1 & 0 \\ \underbrace{0 \quad 1 \quad 1}_{P} & & & \vdots & \underbrace{0 \quad 0 \quad 0 \quad 1}_{I_4} \end{bmatrix}
$$

For the syndrome calculator, we have



Given that

$$G = [P \;\vdots\; I_4]$$

$$H = [I_3 \;\vdots\; P^T]$$

we find that the parity-check matrix is

$$
H = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & \vdots & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \vdots & 1 & 1 & 0 & 1 \end{bmatrix}
$$

In Example 3 of the text, the message sequence (1001) was applied to the encoder and the output code word was 0111001. For the above encoder, the parity bits are 110 and the code word is then 1101001. In particular, we have

Shift	Input	Register contents
		0 0 0
1	1	1 0 1
2	0	1 1 1
3	0	1 1 0
4	1	1 1 0

If we were to make an error in the middle bit and receive 1100001, then circulating it through the syndrome calculator, we have

Shift	Input	Register contents
		0 0 0
1	1	1 0 0
2	0	0 1 0
3	0	0 0 1
4	0	1 0 1
5	0	1 1 1
6	1	0 1 0
7	1	1 0 1

From the parity check matrix we see that the syndrome calculator output 101 corresponds to the error pattern 0001000. The corrected code word is therefore 1101001.

Problem 10.11

The error polynomial is

$$e(X) = r(X) + c(X)$$

We are given

$$c(X) = X + X^2 + X^3 + X^6$$

$$r(X) = X + X^3 + X^6$$

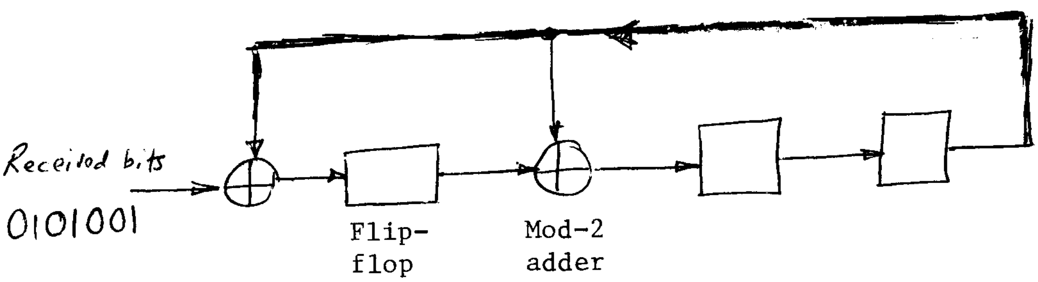
The error polynomial is therefore

$$e(X) = X^2$$

Consider next the syndrome polynomial s(X). The syndrome calculator for the generator polynomial

$$g(X) = 1 + X + X^3$$

is shown in Fig. 10.11; this calculator is reproduced here for convenience of presentation:



Circulating the received bits through the syndrome calculator, we may construct the following table:

Initial state	0 0 0
	1 0 0
	0 1 0
	0 0 1
	0 1 0
	0 0 1
	0 1 0
	0 0 1

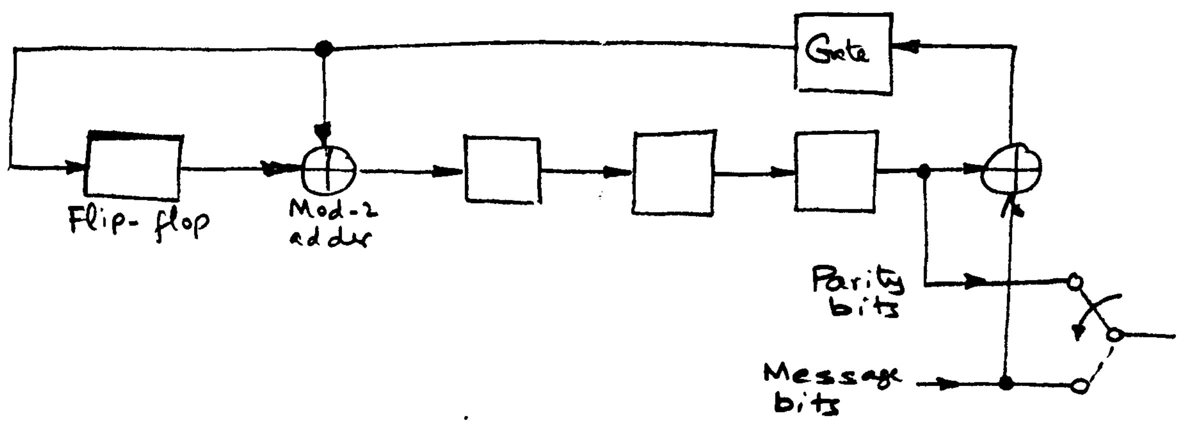
Here, the syndrome polynomial is

$$s(X) = X^2$$

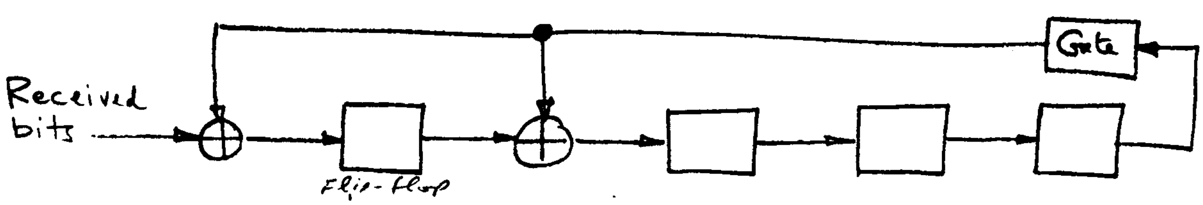
which, for the problem at hand, is the same as the error polynomial. This result demonstrates the property of the syndrome polynomial, stating that it is the same as the error polynomial when the transmission errors are confined to the parity-check bits. In Problem 11.11 the third parity-check bit is received in error.

Problem 10.12

The encoder structure is



The syndrome calculator is

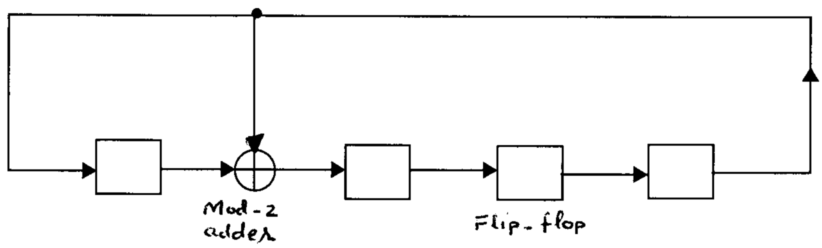


Problem 10.13

(a) A maximal-length code is the dual of the corresponding Hamming code. The generator polynomial of a (15,11) Hamming code is given as $1 + X + X^4$. We may therefore define the feedback connections of the corresponding (15,4) maximal-length code by choosing the primitive polynomial

$$h(X) = 1 + X + X^4$$

The feedback connections are therefore [4,1], which agrees with entry 3 of Table 7.1. Specifically, the encoder of the (15,4) maximal-length encoder is as follows:



(b) The generator polynomial of the (15,4) maximal-length code is

$$g(X) = \frac{1 + X^{15}}{h(X)} = \frac{1 + X^{15}}{1 + X + X^4}$$

Performing this division modulo-2, we obtain

$$g(X) = 1 + X + X^2 + X^3 + X^5 + X^7 + X^8 + X^{11}$$

(This computation is left as an exercise for the reader.) Assuming that the initial state of the encoder is 0001, we find that the output sequence is (111101011001000). Here we recognize that the length of the coded sequence is $2^4 - 1 = 15$. The output sequence repeats itself periodically every 15 bits.

Problem 10.14

(a) $n = 2^m - 1 = 31$ symbols

Hence, the number of bits per symbol in the code is

$m = 5$ bits

(b) Block length = 31 x 5 = 155 bits

(c) Minimum distance of the code is

$$\begin{aligned} d_{min} &= 2t + 1 \\ &= n-k + 1 \\ &= 31 - 15 + 1 \\ &= 17 \text{ symbols} \end{aligned}$$

(d) Number of correctable symbols is

$$\begin{aligned} t &= \frac{1}{2}(n - k) \\ &= 8 \text{ symbols} \end{aligned}$$

Problem 10.15

The encoder is realized by inspection:

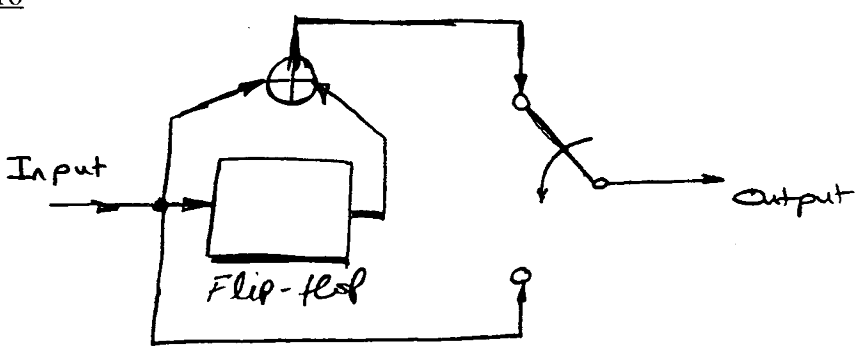
$g^{(1)} = (1,0,1)$

$g^{(2)} = (1,1,0)$

$g^{(3)} = (1,1,1)$

For the Hamming code, the parity check matrix **H** is more structured than that for the repetition code. Indeed, the matrix **H** for the Hamming code includes that for the repetition code as a submatrix.

Problem 10.16



Using this encoder, we may construct the following table by inspection:

Message	1	0	1	1	1	1	...
Output	11	10	11	01	01	01	...

Original message

The code is in fact systematic.

Problem 10.17

The generator polynomials are

$$g^{(1)}(X) = 1 + X + X^2 + X^3$$
$$g^{(2)}(X) = 1 + X + \quad X^3$$

The message polynomial is

$$m(X) = 1 + X^2 + X^3 + X^4 + \dots$$

Hence,

$$\begin{aligned} c^{(1)}(X) &= g^{(1)}(X)\, m(X) \\ &= 1 + X + X^3 + X^4 + X^5 + \dots . \end{aligned}$$

$$\begin{aligned} c^{(2)}(X) &= g^{(2)}(X)\, m(X) \\ &= 1 + X + X^2 + X^3 + \qquad X^6 + X^7 + \dots \end{aligned}$$

Hence,

$$\{c^{(1)}\} = 1,1,0,1,1,1,...$$

$$\{c^{(2)}\} = 1,1,1,1,0,0,...$$

The encoder output is therefore 11, 11, 01, 11, 10, 10.

Problem 10.18

The encoder of Fig. 10.13 (b) has three generator sequences for each of the two input paths; they are as follows (from top to bottom)

$$g_1^{(1)} = (1,1) \qquad g_1^{(2)} = (1,0), \qquad g_1^{(3)} = (1,1)$$

$$g_2^{(1)} = (0,1), \qquad g_2^{(2)} = (1,1), \qquad g_2^{(3)} = (0,0)$$

Hence,

$$g_1^{(1)}(X) = 1 + X, \qquad g_1^{(2)}(X) = 1, \qquad g_1^{(3)}(X) = 1 + X$$

$$g_2^{(1)}(X) = X, \qquad g_2^{(2)}(X) = 1 + X, \qquad g_2^{(3)}(X) = 0$$

The incoming message sequence 10111... enters the encoder two bits at a time; hence

$m^{(1)} = 1\ 1 \ldots.$

$m^{(2)} = 0\ 1 \ldots.$

The message polynomials are therefore

$m_1(X) = 1 + X + \ldots$

$m_2(X) = X + \ldots$

Hence, the output polynomials are

$$c^{(1)}(X) = g_1^{(1)}(X)\, m_1(X) + g_2^{(1)}(X)\, m_2(X)$$

$$= (1 + X)(1 + X + \ldots) + X(X + \ldots)$$

$$= 1 + \ldots.$$

$$c^{(2)}(X) = g_1^{(2)}(X)\, m_1(X) + g_2^{(2)}(X)\, m_2(X)$$

$$= (1)\,(1 + X + \ldots) + (1 + X)\,(X + \ldots)$$

$$= 1 + X + \ldots \quad + X + X^2 + \ldots$$

$$= 1 + X^2 + \ldots$$

$$c^{(3)}(X) = g_1^{(3)}(X)\, m_1(X) + g_2^{(3)}(X)\, m_2(X)$$

$$= (1 + X)\,(1 + X + \ldots) + (0)\,(X + \ldots)$$

$$= 1 + X^2 + \ldots$$

The output sequences are correspondingly as follows:

$c^{(1)} = 1, 0, \ldots$

$c^{(2)} = 1, 0, \ldots$

$c^{(3)} = 1, 0, \ldots$

The encoder output is therefore (1,1,1), (0,0,0), ...

Problem 10.19

1. If the input sequence is 00, the encoder output is 00, 00, 00, 00.

2. If the input sequence is 11, the message polynomial is

$$m(X) = 1 + X$$

The two generator polynomials are

$$g^{(1)}(X) = 1 + X + X^2$$

$$g^{(2)}(X) = 1 + X^2$$

Hence,

$$c^{(1)}(X) = (1 + X)(1 + X + X^2)$$

$$= 1 + X^3$$

$$c^{(2)}(X) = (1 + X)(1 + X^2)$$

$$= 1 + X + X^2 + X^3$$

The encoder output is 11, 01, 01, 11

3. If the input sequence is 01, the message polynomial is

$$m(X) = X$$

Hence,

$$c^{(1)}(X) = X(1 + X + X^2)$$

$$= X + X^2 + X^3$$

$$c^{(2)}(X) = X(1 + X^2)$$

$$= X + X^3$$

The encoder output is 00, 11, 10, 11

4. Finally, if the input sequence is 10, the message polynomial is

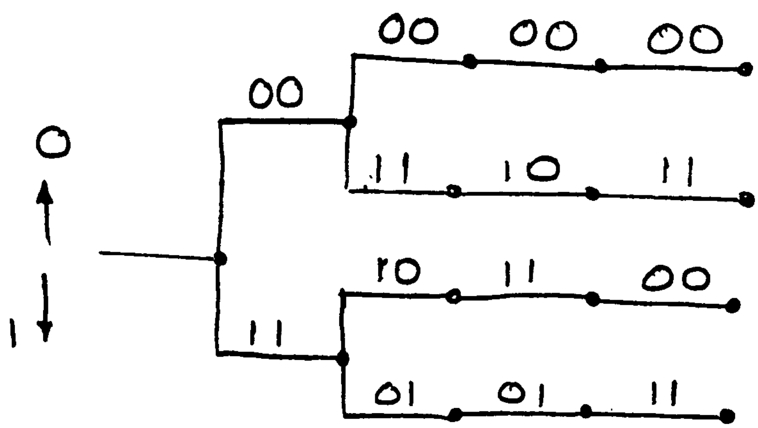
$$m(X) = 1$$

Correspondingly,

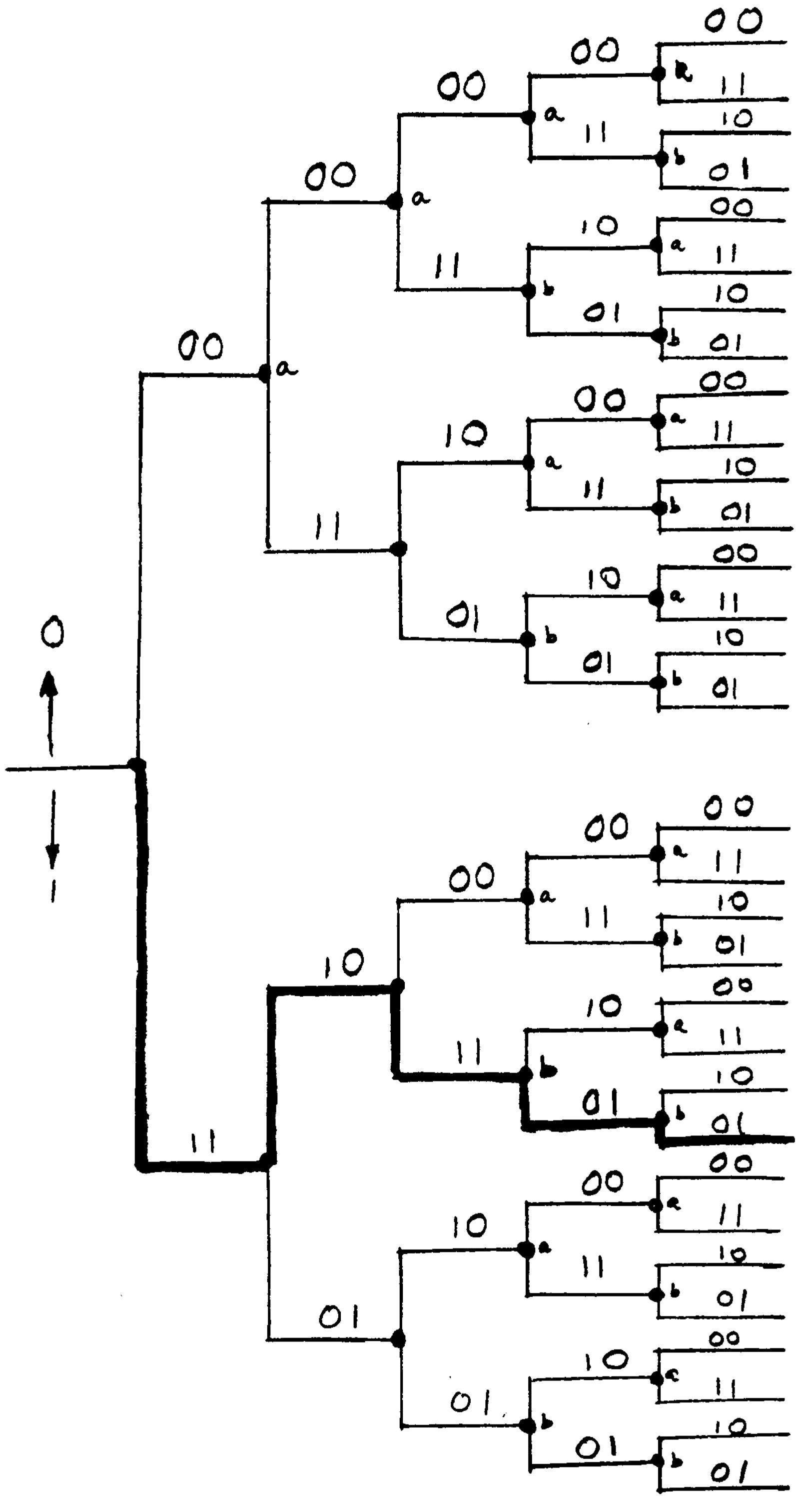
$$c^{(1)}(X) = g^{(1)}(X)$$
$$= 1 + X + X^2$$
$$c^{(2)}(X) = g^{(2)}(X)$$
$$= 1 + X^2$$

Hence, the encoder output is 11, 10, 11, 00.

The encoder outputs calculated above are in perfect accord with the entries of the code tree:

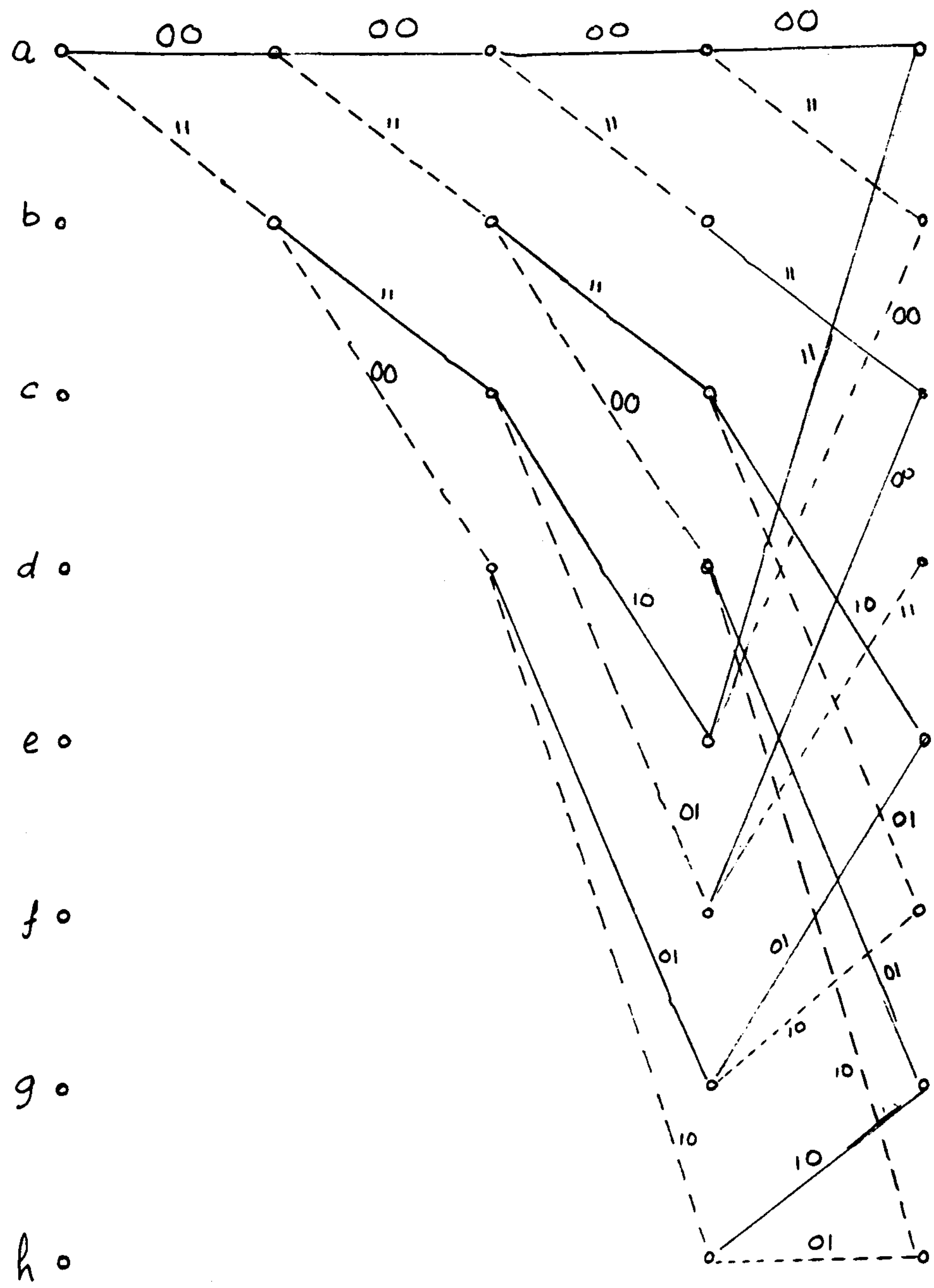


Problem 10.20



The output sequence is 11, 10, 11, 01, 01

Problem 10.21

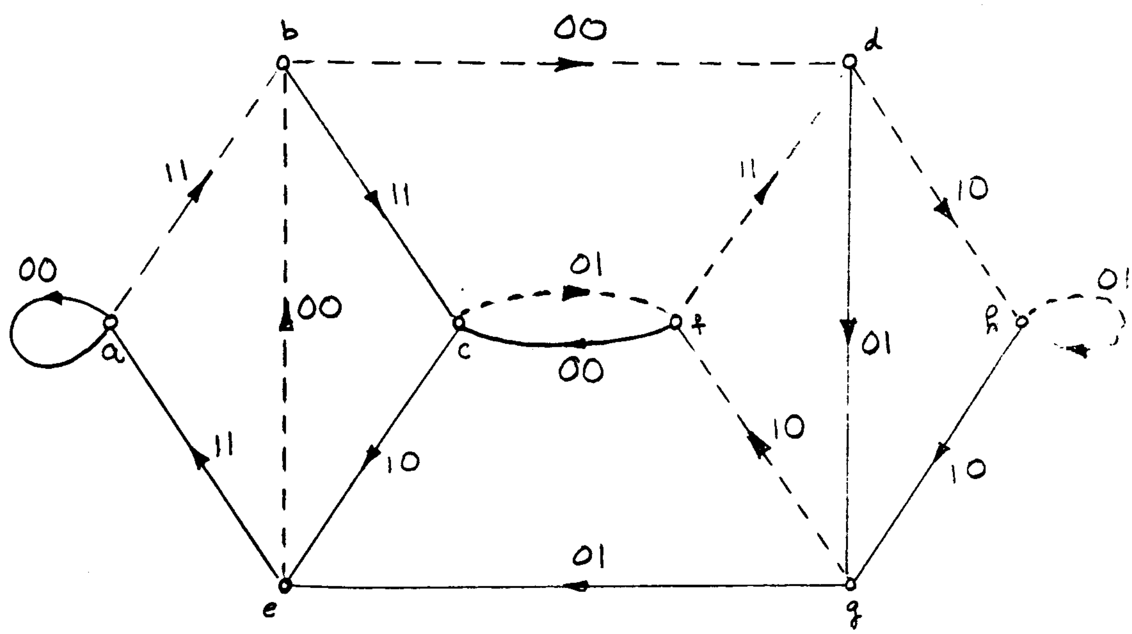


Problem 10.22

The encoder of Fig. P10-17 has eight states:

State	Register contents
a	0 0 0
b	1 0 0
c	0 1 0
d	1 1 0
e	0 0 1
f	1 0 1
g	0 1 1
h	1 1 1

The state diagram of the encoder is as follows:



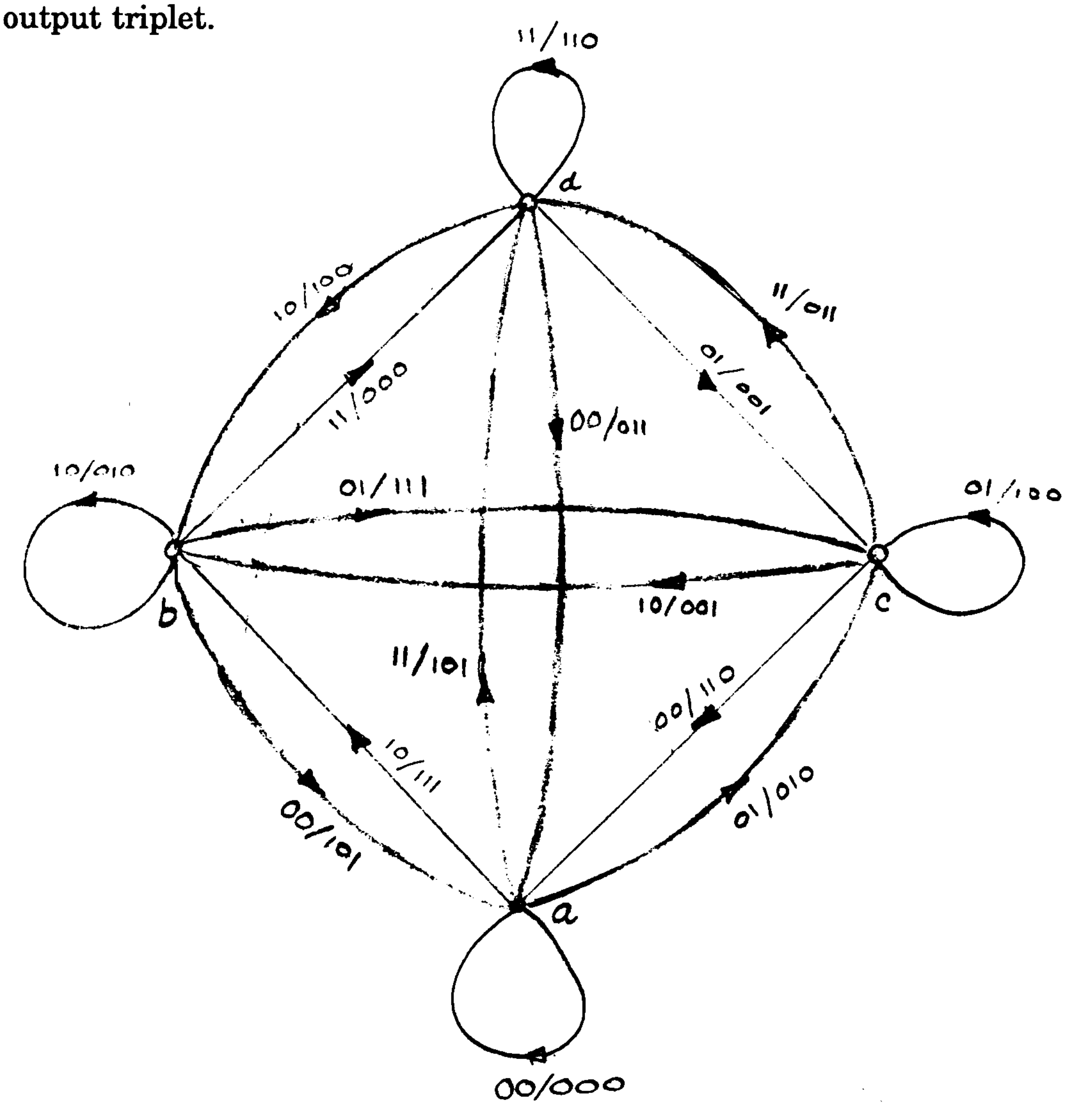
In this diagram, a solid line corresponds to an input of 0 and a dashed line corresponds to an input of 1.

Problem 10.23

(a) The encoder of Fig. 10-13b has four states:

State	Register contents
a	0,0
b	1,0
c	0,1
d	1,1

In the state diagram shown below, each branch is labeled with the input dibit followed by the output triplet.



(b) Starting from the all-zero state a, the incoming sequence 10111... produces the path a b d ... Equivalently, we have the decoded (output) sequence (111), (000), ..., which is exactly the same result calculated in Problem 10.18.

Problem 10.24

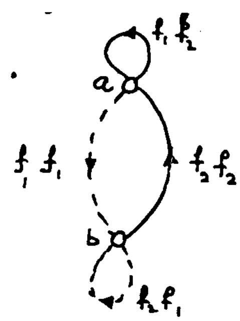
An MSK system has two distinct phase states

State	Phase,radians
a	0
b	π

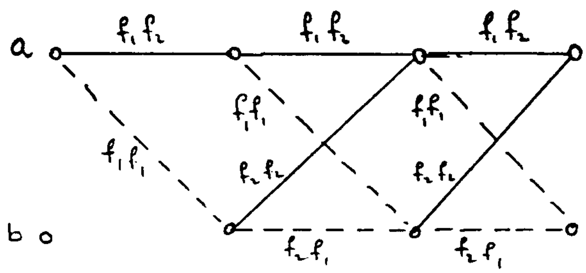
The transmission of a 1 increases the phase by $\pi/2$, whereas the transmission of a 0 decreases the phase by $\pi/2$. Correspondingly, the transmission of dibit 10 or 01 leaves the state of MSK unchanged, whereas the transmission of dibit 00 or 11 movesthe system from one state to the other. For the output, we have

Input dibit	Output frequencies
1 1	$f_1\ f_1$
0 1	$f_2\ f_1$
1 0	$f_1\ f_2$
0 0	$f_2\ f_2$

We may thus construct the following state diagram for MSK:

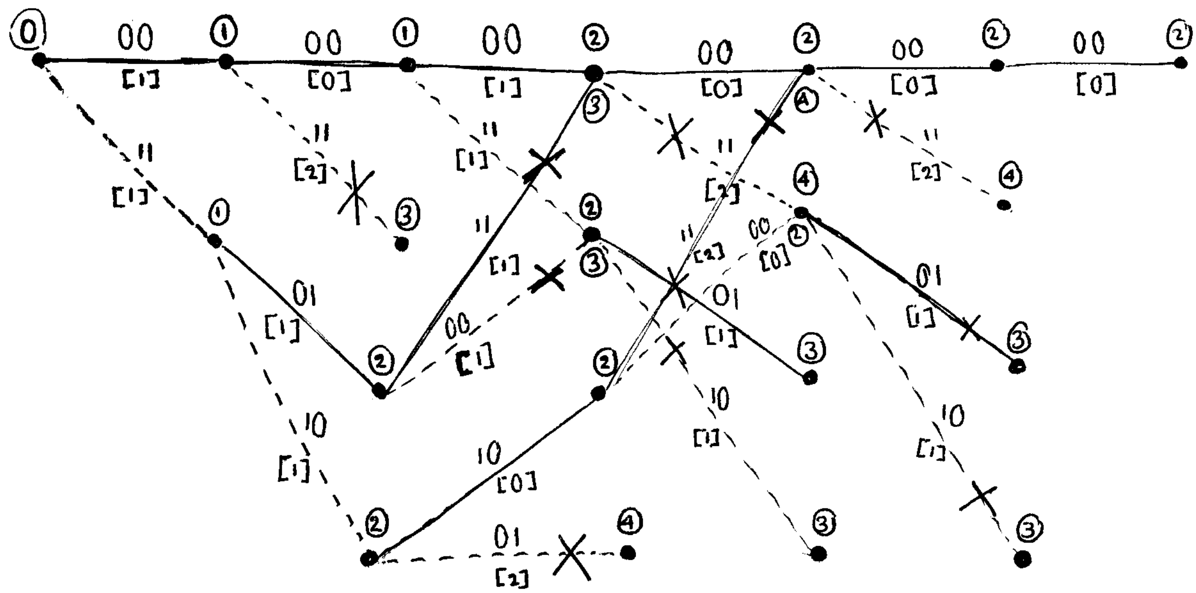


The trellis diagram for MSK is as follows:



Since $d_{min} = 5$ and the number of errors in the received sequence is 2, it should be possible to decode the correct sequence. This is readily demonstrated by applying the Viterbi algorithm.

Problem 10.25



Received bits 10 00 10 00 00 00

Notations

(n) path metric

[n] branch metric

—— message bit 0

- - - - message bit 1

—×— deleted path

From the above figure we see that the decoded sequence is 00 00 00 00 0000..., thereby correcting for the two errors in the received sequence.

Problem 10.26

(a) Coding gain for binary symmetric channel is

$$G_a = 10 \log_2 \left(\frac{10 \text{ x } 1/2}{2}\right)$$
$$= 10 \log_{10} 2.5$$
$$= 4 \text{ dB}$$

(b) Coding gain for additive white Gaussian noise channel is

$$G_a = 10 \log_{10} \left(10 \text{ x } \frac{1}{2}\right)$$
$$= 10 \log_{10} 5$$
$$= 7 \text{ dB}$$

Problem 10.27

The trellis of Fig. P10.27 corresponds to binary data transmitted through a dispersive channel, viewed as a finite-state (i.e., two-state) machine. There are two states representing the two possible values of the previous channel bit. Each possible path through the trellis diagram of Fig. P10.27 corresponds to a particular data sequence transmitted through the channel.

To proceed with the application of the Viterbi algorithm to the problem at hand, we first note that there are two paths of length 1 through the trellis; their squared Euclidean distances are as follows:

$$d_{1,1}^2 = (1.0 - 1.1)^2 = 0.01$$

$$d_{1,2}^2 = (1.0 - (-.9))^2 = 3.61$$

Each of these two paths is extended in two ways to form four paths of length 2; their squared Euclidean distances from the received sequence are as follows:

(a)

$$d_{2,1}^2 = 0.01 + (0.0 - 1.1)^2 = 1.22$$

$$d_{2,2}^2 = 3.61 + (0.0 - 0.9)^2 = 4.42$$

(b)

$$d_{2,3}^2 = 0.01 + (0.0 - (-0.9))^2 = 0.82$$

$$d_{2,4}^2 = 3.61 + (0.0 - (-1.1))^2 = 4.82$$

Of these four possible paths, the first and third ones (i.e., those corresponding to squared Euclidean distances $d^2_{2,1}$ and $d^2_{2,3}$) are selected as the "survivors", which are found to be in agreement. Accordingly, a decision is made that the demodulated symbol $a_0=1$.

Next, each of the two surviving paths of length 2 is extended in two ways to form four new paths of length 3. The squared Euclidean distances of these four paths from the received sequence are as follows:

(a)

$$d_{3,1}^2 = 1.22 + (0.2 - 1.1)^2 = 2.03$$

$$d_{3,2}^2 = 0.82 + (0.2 - 0.9)^2 = 1.31$$

(b)

$$d_{3,3}^2 = 1.22 + (0.2 - (-0.9))^2 = 2.43$$

$$d_{3,4}^2 = 0.82 + (0.2 - (-1.1))^2 = 2.51$$

This time, the second and third paths (i.e., those corresponding to the squared Euclidean distances $d^2_{3,2}$ and $d^2_{3,3}$) are selected as the "survivors". However, no decision can be made on the demodulated symbol a_1 as the two paths do not agree.

To proceed further, the two surviving paths are extended to form two paths of length 4. The squared Euclidean distances of these surviving paths are as follows:

(a)

$$d^2_{4,1} = 1.31 + (-1.1 - 1.1)^2 = 6.15$$

$$d^2_{4,2} = 2.43 + (-1.1 - 0.9)^2 = 6.43$$

(b)

$$d^2_{4,3} = 1.31 + (-1.1 - (-0.9))^2 = 1.35$$

$$d^2_{4,4} = 2.43 + (-1.1 - (-1.1))^2 = 2.43$$

The first and third paths are therefore selected as the "survivors", which are now found to agree in their first three branches. Accordingly, it is decided that the demodulated symbols are $a_0 = +1$, $a_1 = -1$, and $a_2 = +1$. It is of interest to note that although we could not form a decision on a_1 after the third iteration of the Viterbi algorithm, we are able to do so after the fourth iteration.

Figure 1 shows, for the problem at hand, how the trellis diagram is pruned as the application of the Viterbi algorithm progresses through the trellis of Fig. P11.5

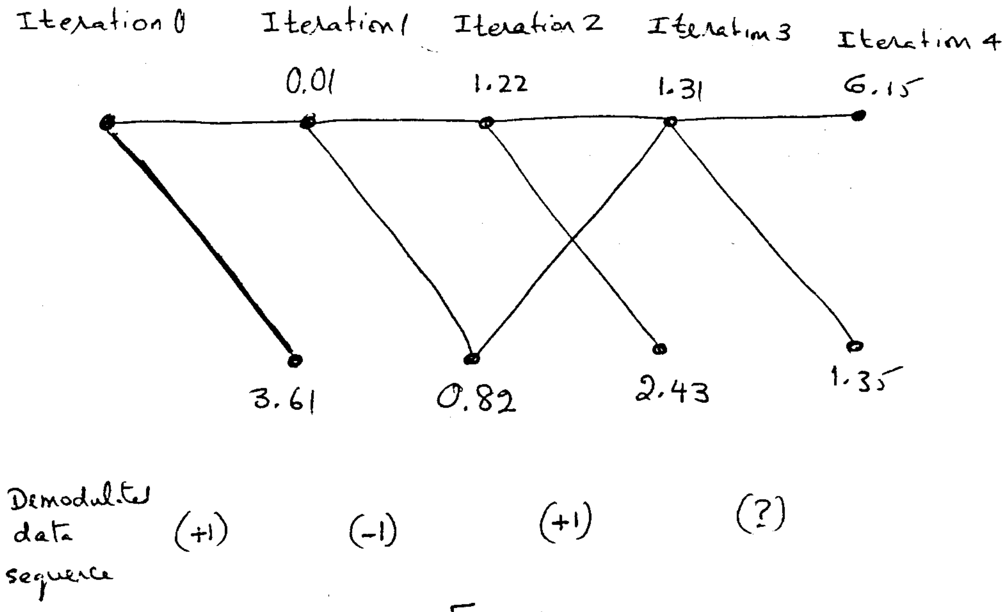
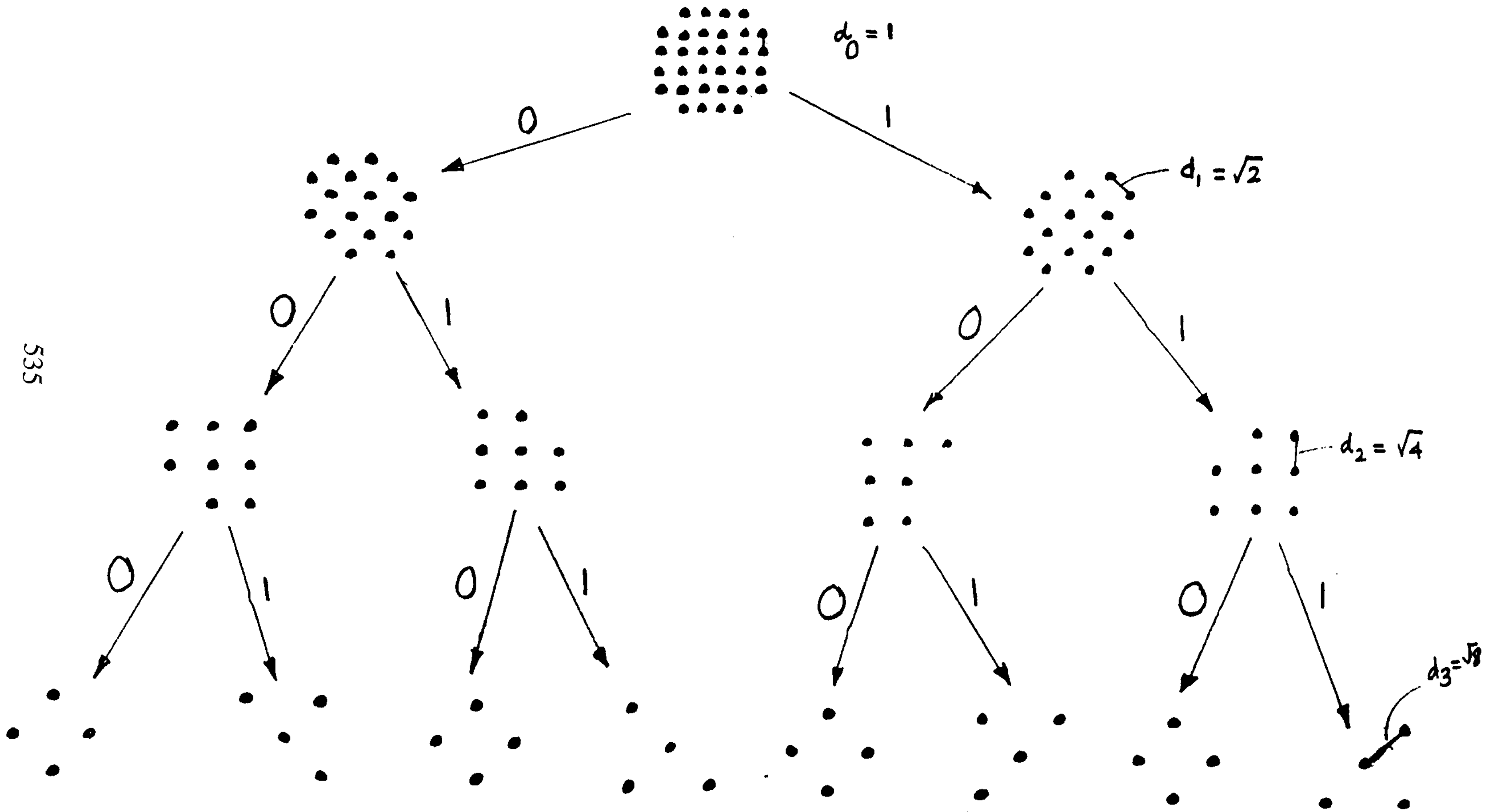


Fig. 1

(This problem is taken from R.E. Blahut, "Digital Transmission of Information", Addison-Wesley, 1990, pp. 144-149.The interested reader may consult this book for a more detailed treatment of the subject.)


$d_0 = 1$
$d_1 = \sqrt{2}$
$d_2 = \sqrt{4}$
$d_3 = \sqrt{8}$
0
1

Problem 10.29

(a) Without coding, the required E_b/N_0 is 12.5 dB. Given a coding gain of 5.1 dB, the required E_b/N_0 is reduced to

$$\left(\frac{E_b}{N_0}\right)_{\text{req}} = 12.5 - 5.1$$

$$= 7.4 \text{ dB}$$

For the downlink, the equation for C/N_0 is

$$\left(\frac{C}{N_0}\right)_{\text{downlink}} = \text{EIRP} + \frac{G_r}{T} - L_{\text{free-space}} + k$$

(b) By definition, the formula for receive antenna gain is

$$G_r = \frac{4\pi A_r}{\lambda^2}$$

where A_r is the receive antenna aperture and λ is the wavelength. Let $(A_r)_{\text{coding}}$ denote the receive antenna aperture that results from the use of coding. Hence

$$10\log_{10}\left(\frac{4\pi A_r}{\lambda^2}\right) - 10\log_{10}\left(\frac{4\pi (A_r)_{\text{coding}}}{\lambda^2}\right) = 5.1 \text{ dB}$$

or, equivalently,

$$10\log_{10}\left(\frac{A_r}{(A_r)_{\text{coding}}}\right) = 5.1 \text{ dB}$$

Hence,

$$\frac{A_r}{(A_r)_{\text{coding}}} = \text{antilog } 0.51 = 3.24$$

The antenna aperture is therefore reduced by a factor of 3.24 through the use of coding. Expressing this result in terms of the antenna dish diameter, d,we may write

$$\frac{\pi d^2/4}{\pi (d_{\text{coding}})^2/4} = \left(\frac{d}{d_{\text{coding}}}\right)^2 = 3.24$$

which yields

$$\frac{\text{Diameter of antenna without coding}}{\text{Diameter of antenna with coding}} = \frac{d}{d_{\text{coding}}} = \sqrt{3.24} = 1.8$$

That is, the antenna diameter is reduced by a factor of 1.8 through the use of coding.

Problem 10.30

Nonlinearity of the encoder in Fig. P10.30 is determined by adding (modulo-2) in a bit-by-bit manner a pair of sets of values of the five input bits $\{I_{1,n}, I_{2,n-1}, I_{1,n-2}, I_{2,n}, I_{2,n-1}\}$ and the associated pair of sets of values of the three output bits $Y_{0,n}$, $Y_{1,n}$ and $Y_{2,n}$. If the result of adding these two sets of values of input bits, when it is treated as a new set of values of output-bits, does not always give a set of values of input bits identical to the result of adding the two sets of values of the aforementioned output bits, then the convolutional encoder is said to be nonlinear, For example, consider two sets of values for the sequence $\{I_{1,n}, I_{1,n-1}, I_{1,n-2}, I_{2,n}, I_{2,n-1}\}$, that are given by $\{0,0,1,1,1\}$ and $\{0,1,0,0,0\}$. The associated sets of values of the three output bits $Y_{0,n}$, $Y_{1,n}$, $Y_{2,n}$, are $\{0,1,1\}$ and $\{1,0,0\}$, respectively. If the 5-bit sets are passed through the Exclusive OR (i.e., mod-2 adder) bit-by-bit, the result is $\{0,1,1,1,1\}$. If the resulting set $\{0,1,1,1,1\}$ is input into the encoder, then the associated output bits are $\{1,1,0\}$. However, when the sets of output bits $\{0,1,1\}$ and $\{1,0,0\}$ are passed through the Exclusive OR, bit-by-bit, the result is $\{1,1,1\}$. Since the two results $\{1,1,0\}$ and $\{1,1,1\}$ are different, it follows that the convolutional encoder of Fig. P10.30 is nonlinear.

Problem 10.31

Let the code rate of turbo code be R. We can write

$$\left(\frac{1}{R} - 1\right) = \left(\frac{1}{r_c^{(1)}} - 1\right) + \left(\frac{1}{r_c^{(2)}} - 1\right)$$

$$\frac{1}{R} = \left(\frac{1}{r_c^{(1)}}\right) + \left(\frac{1}{r_c^{(2)}} - 1\right)$$

$$= \left(\frac{q_1}{p} + \frac{q_2}{p} - 1\right)$$

$$= \frac{q_1 + q_2 - p}{p}$$

Hence

$$R = p/(q_1 + q_2 - p)$$

Problem 10.32

Figure 1 is a reproduction of the 8-state RSC encoder of Figure 10.26 used as encoder 1 and encoder 2 in the turbo encoder of Fig. 10.25 of the textbook. For an input sequence consisting of symbol 1 followed by an infinite number of symbols 0, the outputs of the RSC encoders will contain an infinite number of ones as shown in Table 1.

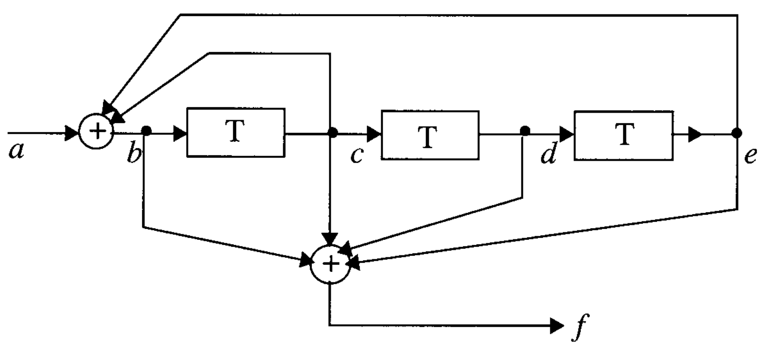


Fig. 1

$b = a \oplus c \oplus e$

$f = b \oplus c \oplus d \oplus e$

Initial conditions: $c = d = e = 0$ {empty}

(Input)	Intermediate inputs				(output)
a	b	c	d	e	f
1	1	0	0	0	1
0	1	1	0	0	0
0	1	1	1	0	1
0	0	1	1	1	1
0	1	0	1	1	1
0	0	1	0	1	0
0	0	0	1	0	1
0	1	0	0	1	0
0	1	1	0	0	0

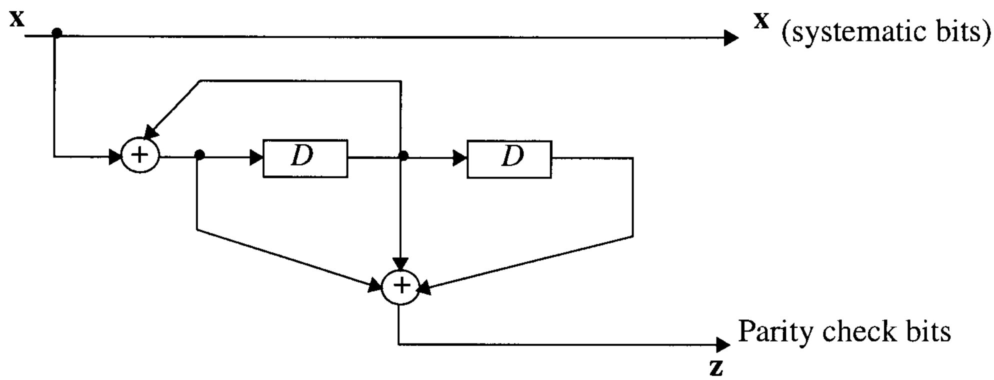
The output is 1011101001110100111...

Therefore, an all zero sequence with a single bit error (1) will cause an infinite number of channel errors.

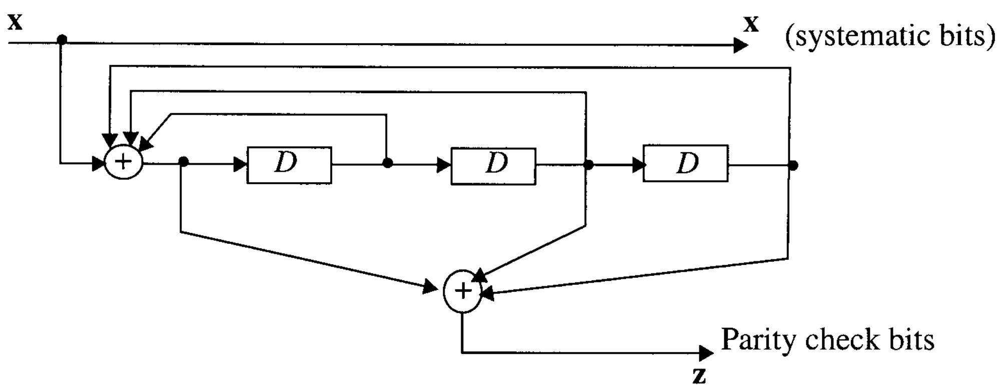
[Note: The all zero input sequence produces an all zero output sequence.]

Problem 10.33

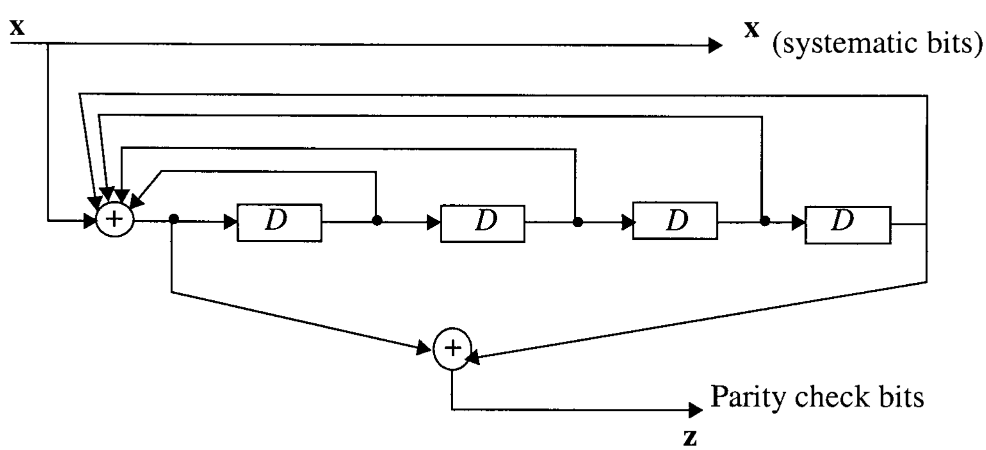
(a) 4-state encoder



8-state encoder



16-state encoder



(b) 4-state encoder

$$\mathbf{g}(D) = \left[1, \frac{1+D+D^2}{1+D^2}\right]$$

By definition, we have

$$\left(\frac{B(D)}{M(D)}\right) = \frac{1+D+D^2}{1+D^2}$$

where $B(D)$ denotes the transform of the parity sequence $\{b_i\}$ and $M(D)$ denotes the transform of the message sequence $\{m_i\}$. Hence,

$$(1+D^2)B(D) = (1+D+D^2)M(D)$$

The parity-check equation is given by

$$(m_i + m_{i-1} + m_{i-2}) + (b_i + b_{i-2}) = 0$$

where the addition is modulo-2.

Similarly for the 8-state encoder, we find that the parity-check equation is

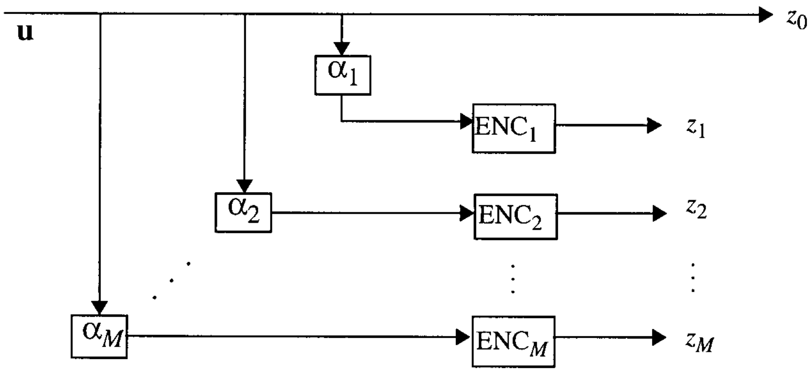
$$m_i + m_{i-2} + m_{i-3} + b_i + b_{i-1} + b_{i-2} + b_{i-3} = 0$$

For the 16-state encoder, the parity-check equation is

$$m_i + m_{i-4} + b_i + b_{i-1} + b_{i-2} + b_{i-3} + b_{i-4} = 0$$

Problem 10.34

(a) Encoder



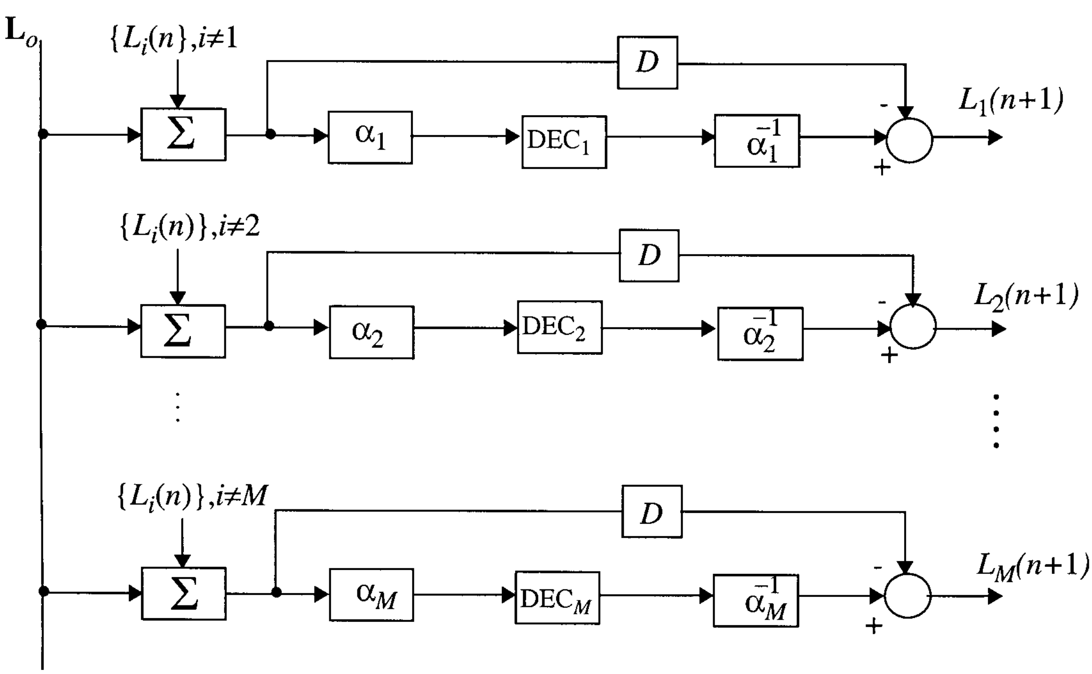
$\alpha_1, \alpha_2, \cdots, \alpha_M$ are M interleavers

$\mathrm{ENC}_1, \mathrm{ENC}_2, \cdots, \mathrm{ENC}_M$ are M recursive systematic convolutional (RSC) encoders

z_o is the message sequence

$z_1, z_2, \cdots, z_M$ are the resulting M parity sequences

(b) Decoder



$\alpha_1^{-1}, \alpha_2^{-1}, \cdots, \alpha_m^{-1}$ are de-interleavers.

The generalized encoder and decoder presented here are described in Valenti (1998); see the Bibliography.

Problem 10.35

The decoding scheme used for turbo codes relies on the assumption that the bit probabilities remain independent from one iteration to the next. To maintain as much independence as possible from one iteration to the next, only extrinsic information is fed from one stage to the next, since the input and the output of the same stage will be highly correlated. However, this correlation decreases as $|t_1 - t_2|$ increases, where t_1, t_2 are any two time instants. The interleaving is utilized to spread correlation information outside of the memory of subsequent decoder stages.

Problem 10.36

The basic idea behind the turbo principle is to use soft information from one stage as input to the next stage in an iterative fashion. For a joint demodulator/decoder, this could be arranged as shown in Fig. 1.

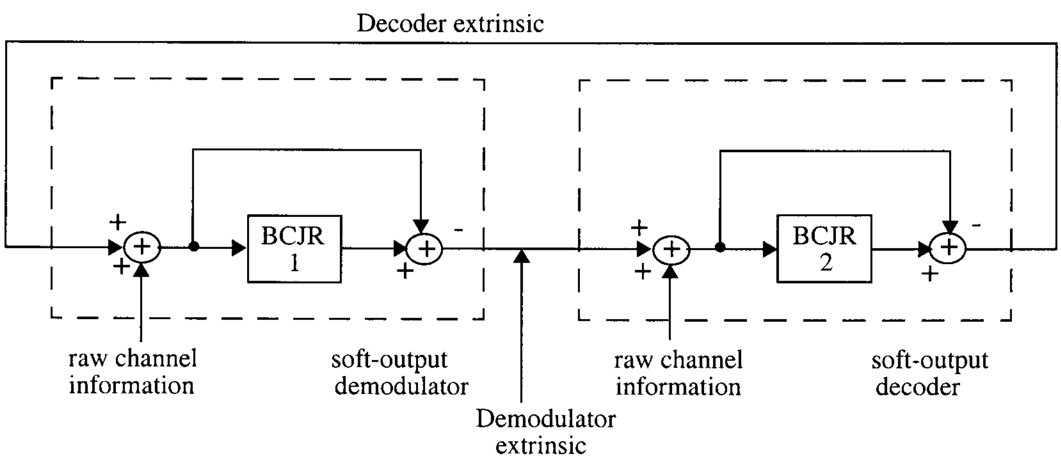


Figure 1

In this figure, BCJR 1 is a MAP decoder corresponding to the Markov model of the modulator and channel; and BCJR 2 is a MAP decoder corresponding to the Markov model of the forward error correction code. The raw channel information is fed into the soft demodulator on the first iteration; this is combined with the extrinsic information from the previous decoding stage on subsequent iterations. The extrinsic information from the soft-output demodulation stage plus the raw channel information is the input to the decoding stage. Feeding back the extrinsic information from the latter stage closes the loop. At any stage the output from the decoder can be used to estimate the data. (Figure 1 shows a symmetric implementation. Other arrangements are possible.)

Problem 10.37

Matlab codes

```
% Probelm 10.37 , CS: Haykin
% Turbo coding
%M. Sellathurai

clear all

% Block size
block_size = 400; % 200 and 400

% Convolutional code polynomial
code_polynomial = [ 1 1 1; 1 0 1 ];
[n,K]=size(code_polynomial);
m=K-1;

% Code rate for  punctured code
code_rate = 1/2;

% Number of iterations
no_of_iterations = 5;

% Number of blocks in error for termination
block_error_limit = 15;

% signal-to-noise-ratio in db
 SNRdb = [1];
 snr = 10^(SNRdb/10);

%  channel reliability value and variance of AWGN channel
 channel_reliability_value = 4*snr*code_rate;
  noise_var = 1/(2*code_rate*snr);

%initializing the error counters
block_number = 0;
block_errors(1,1:no_of_iterations) = zeros(1, no_of_iterations);
bit_errors(1,1:no_of_iterations) = zeros(1, no_of_iterations);
total_errors=0;

while block_errors(1,  no_of_iterations)< block_error_limit
   block_number=block_number+1;

% Transmitter end
% generating random data
```

```
    Data = round(rand(1, block_size-m));
% random scrambler
   [dummy, Alpha] = sort(rand(1,block_size));
% turbo-encorder output
     turbo_encoded = turbo_encorder( Data, code_polynomial, Alpha) ;

% Receiver end
% AWGN+turbo-encorder out put
     received_signal = turbo_encoded+sqrt(noise_var)*randn(1,(block_size)*2);
% demultiplexing the signals
    demul_output = demultiplexer(received_signal, Alpha );
%scaled received signal
    Datar=  demul_output *channel_reliability_value/2;

% Turbo decoder
extrinsic = zeros(1, block_size);
apriori = zeros(1, block_size);

for  iteration = 1: no_of_iterations

% First decoder
    apriori(Alpha) = extrinsic;
    LLR = BCJL1(Datar(1,:), code_polynomial, apriori);
    extrinsic = LLR - 2*Datar(1,1:2:2*(block_size)) - apriori;

% Second decoder
   apriori = extrinsic(Alpha);
   LLR = BCJL2(Datar(2,:), code_polynomial, apriori);
   extrinsic = LLR - 2*Datar(2,1:2:2*(block_size)) - apriori;

% Hard decision of information bits
  Datahat(Alpha) = (sign(LLR)+1)/2;

% Number of bit errors
         bit_errors(iteration) = length(find(Datahat(1:block_size-m)~=Data));

% Number of block errors
         if bit_errors(iteration )>0
            block_errors(iteration) = block_errors(iteration) +1;
         end
end

%Total bit errors
total_errors=total_errors+ bit_errors;

% bit error rate
```

```
          if block_errors(no_of_iterations)==block_error_limit
        BER(1:no_of_iterations)= total_errors(1:no_of_iterations)/...
        block_number/(block_size-m);
          end
end
```

```
function output = turbo_encorder( Data, code_g, Alpha)
% Turbo code encorder
% Used in Problem 10.36, CS: Haykin
%M. Sellathurai

[n,K] = size(code_g);
m = K - 1;
block_s = length(Data);

state = zeros(m,1);
y=zeros(3,block_s+m);

% encorder 1
for i = 1: block_s+m
   if  i <= block_s
      d_k = Data(1,i);
   elseif  i >  block_s
      d_k = rem( code_g(1,2:K)*state, 2 );
   end
a_k = rem( code_g(1,:)*[d_k ;state], 2 );
v_k = code_g(2,1)*a_k;

   for j = 2:K
      v_k = xor(v_k, code_g(2,j)*state(j-1));
   end;
state = [a_k;state(1:m-1)];
y(1,i)=d_k;
y(2,i)=v_k;
end

%encorder 2
% interleaving the data
for i = 1: block_s+m
   ytilde(1,i) = y(1,Alpha(i));
end

state = zeros(m,1);
% encorder 2

for i = 1: block_s+m
  d_k = ytilde(1,i);
a_k = rem( code_g(1,:)*[d_k ;state], 2 );
v_k = code_g(2,1)*a_k;
 for j = 2:K
      v_k = xor(v_k, code_g(2,j)*state(j-1));
   end;
```

```
state = [a_k; state(1:m-1)];
y(3,i)=v_k;
end
% inserting odd and even parities
 for i=1: block_s+m
       output(1,n*i-1) = 2*y(1,i)-1;
       if rem(i,2)
          output(1,n*i) = 2*y(2,i)-1;
       else
          output(1,n*i) = 2*y(3,i)-1;
       end
end
```

```
function [nxt_o, nxt_s, lst_o, lst_s] = cnc_trellis(code_g);
%used in Problem10.36.
% code trellis for  RSC;
% Mathini Sellathurai



% code properties
[n,K] = size(code_g);
m = K - 1;
no_of_states = 2^m;

for s=1: no_of_states
dec_cnt_s=s-1; i=1;

% decimal to binary state
while  dec_cnt_s >=0 & i<=m
       bin_cnt_s(i) = rem( dec_cnt_s,2) ;
      dec_cnt_s  = (dec_cnt_s-  bin_cnt_s(i))/2;
        i=i+1;
end
bin_cnt_s=bin_cnt_s(m:-1:1);

% next state when input is 0
   d_k = 0;
   a_k = rem( code_g(1,:)*[0 bin_cnt_s ]', 2 );
   v_k = code_g(2,1)*a_k;
   for j = 1:K-1
      v_k = xor(v_k, code_g(2,j+1)*bin_cnt_s(j));
   end;
nstate0 = [a_k  bin_cnt_s(1:m-1)];
y_0 = [0 v_k];


% next state when input is 1
   d_k = 1;
   a_k = rem( code_g(1,:)*[1 bin_cnt_s]', 2 );
   v_k = code_g(2,1)*a_k;
   for j = 1:K-1
      v_k = xor(v_k, code_g(2,j+1)*bin_cnt_s(j));
   end;
nstate1 = [a_k bin_cnt_s(1:m-1)];
y_1=[1 v_k];
% next output when input 0 1
nxt_o(s,:) = [y_0 y_1];
```

```
% binary to decimal state
d=2.^(m-1:-1:0);
dstate0=nstate0*d'+1; dstate1=nstate1*d'+1;
% next state when input 0 1
nxt_s(s,:) = [ dstate0  dstate1 ];

% finding the possible previous state frm the trellis
lst_s(nxt_s(s, 1), 1)=s;
lst_s(nxt_s(s, 2), 2)=s;
lst_o(nxt_s(s, 1), 1:4) =  nxt_o(s, 1:4) ;
lst_o(nxt_s(s, 2), 1:4) =  nxt_o(s, 1:4) ;

end
```

```
function output = demultiplex(Data, Alpha);
% demultiplexing the received signal
% used in problem 10.36, CS: Haykin
% Mathini Sellathurai

block_s = fix(length(Data)/2);
output=zeros(2,block_s);

for i = 1: block_s
       Dataf(i) = Data(2*i-1);
         if rem(i,2)>0
          output(1,2*i) = Data(2*i);
       else
          output(2,2*i) = Data(2*i);
       end
end

for i = 1:block_s
   output(1,2*i-1) = Dataf(i);
   output(2,2*i-1) = Dataf(Alpha(i));
end
```

```
function L = BCJL1(Datar, code_g ,apriori)
% log-BCJL (LOG-MAP algorithm) for decoder 1
% Used in Problem 10.36, CS: Haykin

% states, memory, constraint length and block size
block_s = fix(length(Datar)/2);
[n,K] = size(code_g);
m = K - 1;
no_of_states = 2^m;
infty = 1e10;
zero=1e-300;


% forward recursion
alpha(1,1) = 0;
alpha(1,2:no_of_states) = -infty*ones(1,no_of_states-1);

% code-trellis
[nxt_o, nxt_s, lst_o, lst_s] = cnc_trellis(code_g);
nxt_o = 2*nxt_o-1;
lst_o = 2*lst_o-1;

for i = 1:block_s
    for cnt_s = 1:no_of_states
      branch = -infty*ones(1,no_of_states);
      branch(lst_s(cnt_s,1)) = -Datar(2*i-1)+Datar(2*i)*...
      lst_o(cnt_s,2)-log(1+exp(apriori(i)));
      branch(lst_s(cnt_s,2)) = Datar(2*i-1)+Datar(2*i)*...
      lst_o(cnt_s,4)+apriori(i)-log(1+exp(apriori(i)));
    if(sum(exp(branch+alpha(i,:)))>zero)
        alpha(i+1,cnt_s) = log( sum( exp( branch+alpha(i,:))));
     else
    alpha(i+1,cnt_s) =-1*infty;
    end
 end
    alpha_max(i+1) = max(alpha(i+1,:));
    alpha(i+1,:) = alpha(i+1,:) - alpha_max(i+1);
end

% backward recursion
beta(block_s,1)=0;
beta(block_s,2:no_of_states) = -infty*ones(1,no_of_states-1);


for i = block_s-1:-1:1
  for cnt_s = 1:no_of_states
```

```
      branch = -infty*ones(1,no_of_states);
      branch(nxt_s(cnt_s,1)) = -Datar(2*i+1)+Datar(2*i+2)*...
      nxt_o(cnt_s,2)-log(1+exp(apriori(i+1)));
      branch(nxt_s(cnt_s,2)) = Datar(2*i+1)+Datar(2*i+2)*...
      nxt_o(cnt_s,4)+apriori(i+1)-log(1+exp(apriori(i+1)));
      if(sum(exp(branch+beta(i+1,:)))<zero)
         beta(i,cnt_s)=-infty;
      else
         beta(i,cnt_s) = log(sum(exp(branch+beta(i+1,:))));
     end
   end
beta(i,:) = beta(i,:) - alpha_max(i+1);
end

for k = 1:block_s
   for cnt_s = 1:no_of_states
      branch0 = -Datar(2*k-1)+Datar(2*k)*lst_o(cnt_s,2)-log(1+exp(apriori(k)));
      branch1 = Datar(2*k-1)+Datar(2*k)*lst_o(cnt_s,4)+apriori(k)-log(1+exp(apriori(k)));
      den(cnt_s) = exp( alpha(k,lst_s(cnt_s,1))+branch0+  beta(k,cnt_s));
      num(cnt_s) = exp( alpha(k,lst_s(cnt_s,2))+branch1+ beta(k,cnt_s));
   end
L(k) = log(sum(num)) - log(sum(den));
end
```

```
function L = BCJL1(Datar, code_g ,apriori)
% log-BCJL (LOG-MAP algorithm) for decoder 1
% Used in Problem 10.36, CS: Haykin

% states, memory, constraint length and block size
block_s = fix(length(Datar)/2);
[n,K] = size(code_g);
m = K - 1;
no_of_states = 2^m;
infty = 1e10;
zero=1e-300;


% forward recursion
alpha(1,1) = 0;
alpha(1,2:no_of_states) = -infty*ones(1,no_of_states-1);

% code-trellis
[nxt_o, nxt_s, lst_o, lst_s] = cnc_trellis(code_g);
nxt_o = 2*nxt_o-1;
lst_o = 2*lst_o-1;

for i = 1:block_s
    for cnt_s = 1:no_of_states
      branch = -infty*ones(1,no_of_states);
      branch(lst_s(cnt_s,1)) = -Datar(2*i-1)+Datar(2*i)*...
      lst_o(cnt_s,2)-log(1+exp(apriori(i)));
      branch(lst_s(cnt_s,2)) = Datar(2*i-1)+Datar(2*i)*...
      lst_o(cnt_s,4)+apriori(i)-log(1+exp(apriori(i)));
    if(sum(exp(branch+alpha(i,:)))>zero)
         alpha(i+1,cnt_s) = log( sum( exp( branch+alpha(i,:))));
     else
    alpha(i+1,cnt_s) =-1*infty;
    end
 end
    alpha_max(i+1) = max(alpha(i+1,:));
    alpha(i+1,:) = alpha(i+1,:) - alpha_max(i+1);
end

% backward recursion
beta(block_s,1)=0;
beta(block_s,2:no_of_states) = -infty*ones(1,no_of_states-1);


for i = block_s-1:-1:1
  for cnt_s = 1:no_of_states
```

```
      branch = -infty*ones(1,no_of_states);
      branch(nxt_s(cnt_s,1)) = -Datar(2*i+1)+Datar(2*i+2)*...
      nxt_o(cnt_s,2)-log(1+exp(apriori(i+1)));
      branch(nxt_s(cnt_s,2)) = Datar(2*i+1)+Datar(2*i+2)*...
      nxt_o(cnt_s,4)+apriori(i+1)-log(1+exp(apriori(i+1)));
      if(sum(exp(branch+beta(i+1,:)))<zero)
         beta(i,cnt_s)=-infty;
      else
         beta(i,cnt_s) = log(sum(exp(branch+beta(i+1,:))));
     end
  end
beta(i,:) = beta(i,:) - alpha_max(i+1);
end

for k = 1:block_s
  for cnt_s = 1:no_of_states
     branch0 = -Datar(2*k-1)+Datar(2*k)*lst_o(cnt_s,2)-log(1+exp(apriori(k)));
     branch1 = Datar(2*k-1)+Datar(2*k)*lst_o(cnt_s,4)+apriori(k)-log(1+exp(apriori(k)));
     den(cnt_s) = exp( alpha(k,lst_s(cnt_s,1))+branch0+  beta(k,cnt_s));
     num(cnt_s) = exp( alpha(k,lst_s(cnt_s,2))+branch1+ beta(k,cnt_s));
  end
L(k) = log(sum(num)) - log(sum(den));
end
```

Answer to Problem 10.37

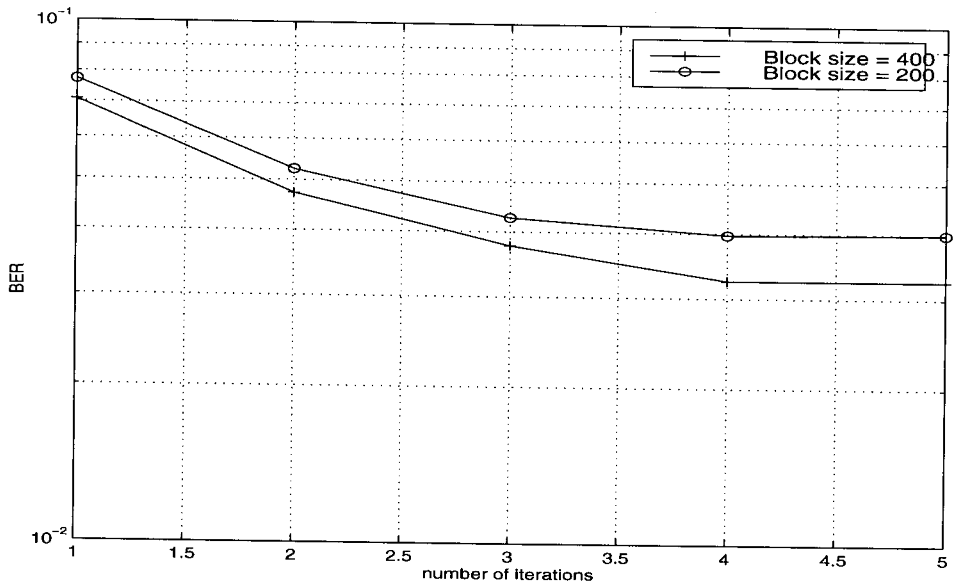


Figure 1: bit error rate Vs. the number of iterations for Block sizes: 200, and 400