KENYATTA UNIVERSITY

EXAMINATION FOR THE DEGREE OF BACHELOR OF ARTS, EDUCATION AND SCIENCE

SMA 301: REAL ANALYSIS II

2ND SEMESTER 2021/2022 TIME: 2 HOURS

INSTRUCTIONS: Answer Question ONE and any other TWO.

QUESTION ONE (30 MARKS)

(a) Prove that if (X,d) is a metric space then

$|d(x,z) - d(y,z)| \leq d(x,y) \forall\, x,y,z \in X$ (3 marks)

(b) Let (X,d) be a metric space.

Define the following terminologies:

(i) An open sphere centered at $x_0 \in X$ with radius r.

(ii) A neighbourhood of a point $p \in X$

(iii) An open subset O of X.

(iv) A limit point of a subset A of X.

(7 marks)

(c) Let (X,d) be a metric space with the usual metric.

(i) Define a closed set A in X. (2 marks)

(ii) Show by use of an example that a set which fails to be closed need not be open.

(3 marks)

(d) Let (X,d) be a metric space.

(i) Prove that every convergent sequence is a cauchy sequence. (3 marks)

(ii) Let Q be the set of rational numbers in which the metric d is defined by

$d(x,y) = |x-y| \forall x,y \in Q$. Then (Q,d) is a metric space. Show by use of a counter-example that the converse of 1d) (i) above need not be true. (3 marks)

(e) Consider the interval I = [a,b]

(i) Define a partition of the interval I = [a,b] (2 marks)

(ii) Let a function f be defined on as interval I = [a, b] and P be a partition of [a, b] when is a function f said to be of bounded variation. (2 marks)

(iii) Show clearly whether or not the function given by

$$f(x) = \begin{cases} x\sin\frac{\pi}{x}, o < x \leq 1 \\ o, x = 0 \end{cases}$$ is of bounded variation. (5 marks)

QUESTION TWO (20 MARKS)

(a) Let ($\mathbb{R}$, d) be a metric space. Show that on the real line with the usual metric, the singleton set {x} is NOT open in $(\mathbb{R}, d)$. (2 marks)

(b) (i) Let (X,d) be a metric space. Give the definition of a complete metric space. (2 marks)

(ii) Let X be the set of all continuous real valued functions defined on [-1,1] and let the metric d be defined by

$$= d(f,g) = \int_{-1}^{1} |f(x) - g(x)| dx, \forall f, g \in X$$

Show that (X,d) is NOT a complete metric space. (6 marks)

(c) Let (X,d_x) and (Y,d_y) be metric spaces and $f: X \to Y$ be a function.

(i) Define pointwise continuity in terms of open spheres i.e. when if f said to be continuous at $x_o \in X$. (2 marks)

(ii) Prove that f is continuous on X if f f^{1} (o) is open in X for each set O open in Y. (8 marks)

QUESTION THREE (20 MARKS)

(a) Let (X,d) be a metric space.

Let $\{A_\alpha\}$ be a family of subsets in X

(i) Define an open cover of a set A in X. (2 marks)

(ii) Define a finite subcover of a set A in X. (2 marks)

(iii) When is a subset A of a metric space (X,d) said to be compact? (2 marks)

(iv) When is a metric space (X,d) said to be compact? (2 marks)

(b) Prove that every closed subset of a compact metric space is compact. (4 marks)

(c) Let f be a function of bounded variation on [a, b]. Prove that f is bounded.

(3 marks)

(d) Let f be a continuous function on [a,b] and f' exist on [a,b]. Let f' be bounded

(i.e. $|f'(x)| \leq M, M > 0, m \in \mathbb{R}) \forall x \in [a, b]$

Prove that f is of bounded variation. (5 marks)

QUESTION FOUR (20 MARKS)

(a) Prove that the function $f(x) = x^3$ is Riemann integrable on [0, 1] and show that

$$\int_0^1 x^3 dx = \frac{1}{4}$$ (9 marks)

(b) Evaluate the following Riemann-Stieltjes integrals

(i) $\int_0^2 x^2 dx^2$ (2 marks)

(ii) $\int_0^2 [x] dx^2$ where [x] denotes the greatest integer less than x. (3 marks)

(c) Let f: [0,1]→ ℝ be defined by f(x) = x and take α (x) = x. Given the evenly spaced partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$, show that $L(p, f, \alpha) = \frac{1}{2} - \frac{1}{2n}$ (6 marks)

QUESTION FIVE

(a) Find the radius of convergence of the following series and state for what values of x the series converges.

(i) $\sum_{n-1}^{\infty} \frac{x^{2n}}{(-3)^n}$ (5 marks)

(ii) $x + \frac{x^{2n}}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ (5 marks)

(iii) $1 + x + 2!x^2 + 3!x^3 + 4!x^4 + \ldots = \sum_{n=o}^{\infty} n!x^n$ (5 marks)

(b) (i) State without proof the Welerstrass's M-test of uniform convergence of series.

(2 marks)

(ii) Test for uniform convergence the series

$\frac{2s}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^5}{1+x^6} + \cdots, \frac{-1}{2} \leq x \leq \frac{1}{2}$ (3 marks)