KENYATTA UNIVERSITY

EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE (STATISTICS AND PROGRAMMING, ACTUARIAL SCIENCE, MATHEMATICS AND COMPUTER SCIENCE, GENERAL), BACHELOR OF EDUCATION, BACHELOR OF ARTS (GENERAL) AND BACHELOR OF ECONOMICS AND STATISTICS

SST 205: PROBABILITY AND STATISTICS II

2ND SEMESTER 2020/2021 TIME: 2HOURS

INSTRUCTIONS

Answer Question **ONE** and any other **TWO** questions

QUESTION ONE [30 MARKS]

a) State two conditions for the function $f(x, y)$ to be a joint probability distribution of the continuous random variables X and Y. [2 Marks]

b) Given that $f(x,y) = \begin{cases} k(2x+4y), & x = 1,2\,; y = 1,2,3 \\ 0 & otherwise \end{cases}$

is a joint probability distribution of the discrete random variables X and Y. Find the value of the constant k, hence determine $P(Y \leq 2)$. [4 Marks]

c) The joint probability density function of continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{3}{10}(x^2+y^2), & 0 \leq x \leq 2\,; 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

Obtain the joint distribution function (c.d.f), $F(x, y)$, of X and Y. Hence or otherwise calculate $P\left(\frac{1}{2} \leq X \leq 1,\ \frac{1}{4} \leq Y \leq \frac{1}{2}\right)$. [6 Marks]

d) If X and Y are random variables, show that

$$Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2ab\, Cov(X, Y),$$

where a and b are constants. [4 Marks]

e) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size $n \geq 2$ from a distribution that is normally distributed with mean μ and variance σ^2 $i.e$ $Y \sim N(\mu, \sigma^2)$. Show that $\bar{Y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$. [3 Marks]

f) The joint probability distribution of two discrete random variables X and Y is given in the table below:

Y \ X	1	2	3
1	0.2	0	0.1
2	0	0.1	0.3
3	0.1	0	0.2

i) Find the marginal probability distribution functions of X and Y [2 Marks]

ii) Are X and Y independent? [2 Marks]

iii) Determine the conditional probability distribution function of X given Y=1. [4 Marks]

g) If X, X_2, X_3 are independent random variables which are normally with means 0 and variances 1 $i.e$ $X_i \sim N(0,1), i = 1,2,3$. State the distributions of the following random variables:

i) $U = X_1 + X_2 - X_3$ [1 Mark]

ii) $V = X_1^2 + X_2^2$ [1 Mark]

iii) $W = \frac{(X_1^2+X_2^2)/2}{X_3^2}$ [1 Mark]

QUESTION TWO [20 MARKS]

The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} k(x+y), & 0 < x+y < 2, \quad 0 < x < 2, 0 < y < 2 \\ 0 & otherwise \end{cases}$$

a) Find the value of the constant k. [4 Marks]

b) Find the marginal densities of X and Y. [6 Marks]

c) Are X and Y independent? [2 Marks]

d) Obtain the conditional mean of Y given X=x. [4 Marks]

e) Determine the conditional variance of the conditional variance of Y given X=x. [4 Marks]

QUESTION THREE [20 MARKS]

a) Let the independent random variables X_1 and X_2 have the same probability distribution

$$f(x) = \begin{cases} \frac{1}{3} & , \quad x = 1,2,3 \\ 0 & \quad otherwise \end{cases}$$

Determine the moment generating function of a new random variable $V = X_1 - X_2$, hence obtain the mean and variance of V. [10 Marks]

b) If Y_1 and Y_2 are independent continuous random variables and each has probability density of the form

$$f(y) = \begin{cases} \theta e^{-\theta y} & , \quad y > 0 \\ 0 & \quad otherwise \end{cases}$$

Find the joint probability density function of the new random $V = \frac{Y_1}{Y_2}$ and $U = Y_1 + Y_2$. Hence find the marginal densities of U and V. [10 Marks]

QUESTION FOUR [20 MARKS]

a) Suppose that X and Y have the bivariate normal density with $\mu_x = 3, \mu_y = 2, \sigma_x^2 = 4, \sigma_y^2 = 2, Cov(X,Y) = 1$. Determine

   i) the conditional distribution of Y given X=x, [3 Marks]

   ii) the regression of Y on X, [2 Marks]

   iii) the conditional variance of Y given X=x [2 Marks]

b) Given that a random variable X is normally distributed with mean 0 and variance 1 $i.e\ X \sim N(0,1)$, and U has a Chi-squared distribution with n degrees of freedom $i.e\ U \sim \chi^2_{(n)}$. If X and U are stochastically independent, find the mean and variance of a new random variable

$$T = \frac{X}{\sqrt{U/n}}$$

[13 Marks]

QUESTION FIVE [20 MARKS]

a) Let the discrete random variables X and Y have a joint probability distribution given by

$$f(x,y) = \begin{cases} \frac{1}{9} & , \ x = 1,2,3\ , \ y = 1,2,3 \\ 0 & otherwise \end{cases}$$

Find the joint moment generating function of X and Y. Hence or otherwise obtain the correlation coefficient of random variables X and Y. [12 Marks]

b) Let Y be a continuous random variable having a Chi-squared distribution with n degrees of freedom $i.e$

$$g(y) = \begin{cases} \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} y^{n/2 - 1} e^{-y/2}, & y > 0 \\ 0 & otherwise \end{cases}$$

Determine the moment generating function, hence or otherwise find the mean and variance of Y. [8 Marks]