SMA 201

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ADVANCED CALCULUS

CONTENTS

CHAPTER 1
Rolle's Theorem

1.1 Introduction

You have studied, in Differential Calculus, the definition of the continuity and differentiability of a function at a point x = c and in an interval [a, b]. The graph of a function, which is continuous in a closed interval, $a \le x \le b$ consists of an unbroken curve over this interval. The tangent drawn to a curve y = f(x) at a point P on the curve has gradient $\frac{dy}{dx}$ or $f'(x)$. Thus $f'(c)$ represents the gradient to the curve at x = c.

Mathematicians like, Rolle, Lagrange, Taylor, Maclaurin and others observed some important properties of continuous and differentiable functions f (x) in an interval. $a \le x \le b$. They have established those properties in the form of Theorems or important results. We are going to study these theorems in the following chapters. Among those theorems, Rolle's Theorem is the most fundamental in mathematical analysis. Michael Rolle is a French mathematician who had published this theorem in 1691.

1.2 Objectives of this Chapter.

By the end of this chapter you should be able to:

i. state and prove Rolle's Theorem
ii. apply Rolle's Theorem in any given differentiable function

1.3 Rolle's Theorem.

If a function f(x) is;

i. continuous in a closed interval [$a \le x \le b$]
ii. differentiable in open interval [$a < x < b$]
iii. and if f(a) = f(b) then there exists at least one point c in the open interval $a < x < b$ such that $f'(c) = 0$.

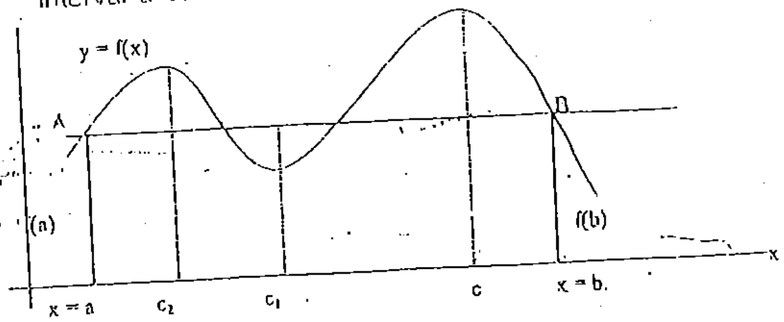

Figure 1

Proof
By virtue of the continuity of the function f (x) in [a, b] the graph is an unbroken curve

Case 1
As a particular case, the curve may be a straight line, y = f (a) or y = f (b) = constant. Then f (a) is identically equal to a constant for all x in [a, b] and hence $f'(x) = 0$ at all points in [a, b] and any point in [a, b] may be called as c and the theorem is true in this case.

Case 2
If $f(x) \neq$ constant everywhere between a and b, then either the curve is above the line AB in some place or below the line AB in some other place or both above and below the line AB as in the figure. Since the function is continuous it is bounded and it will have a maximum value M and a minimum value m or both. That is the function f(x) has an extreme value at a point c where f(c) is maximum or minimum at a point c and c is neither a nor b.

Therefore c is in the open interval $[a < c < b]$ and the derivative of $f'(c)$ must be zero at the point c or $f'(c)$ for some c in $a < c < b$ where f (x) is a maximum or a minimum and $f'(x) = 0$.

Note: There may be more than one place between a and b where the derivative is zero. In the figure 1, for example, $\frac{dy}{dx}$ is zero at c_1 and c_2 also and $f'(c_1) = 0 = f'(c_2)$. Hence we say that there is at least one point c in $a < x < b$

The following examples illustrate the truth of Rolle's theorem.

Example 1
Consider the function f (x) = $x^2 - 7x + 10$ or f (x) = (x – 2) (x – 5) in the interval
[2 ≤ x ≤ 5].

We will see that there is a point c in the interval [2 < c ≤ 5] such that f^1 (c) = 0.
Now, (i) f (x) is continuous in the closed interval [2 ≤ x ≤ 5], since it is a polynomial.

Proof

By virtue of the continuity of the function f (x) in [a, b] the graph is an unbroken curve

Case 1

As a particular case, the curve may be a straight line, y = f (a) or y = f (b) = constant. Then f (a) is identically equal to a constant for all x in [a, b] and hence $f'(x) = 0$ at all points in [a, b] and any point in [a, b] may be called as c and the theorem is true in this case.

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If $f(x) \neq$ constant everywhere between a and b, then either the curve is above the line AB in some place or below the line AB in some other place or both above and below the line AB as in the figure. Since the function is continuous it is bounded and it will have a maximum value M and a minimum value m or both. That is the function f(x) has an extreme value at a point c where f(c) is maximum or minimum at a point c and c is neither a nor b.

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Rolles Theorem

→ f(x) is Continuous on a closed interval [a, b]

→ f(x) is differentiable on open interval (a, b)

→ The value of this function at a point c should be equal to value of b

* Continuity - A function is said to be continuous at x=0 if

$\lim_{x \to a} f(x) = f(a)$

(ii) $f(x)$ is differentiable in the open interval $[2 < x < 5]$, since any polynomial is continuous and differentiable in the interval from $-\infty$ to ∞.

(iii) $f(2) = f(5)$ since $f(2) = 0 = f(5)$.

Thus $f(x) = x^2 - 7x + 10$ satisfies all the three conditions of Rolle's Theorem, Hence there should be a point c in $[2 \le x < 5]$ such that $f^1(c) = 0$

Now $f(x) = x^2 - 7x + 10$

$f^1(x) = 2x - 7$

$f^1(x) = 0$ gives $x = 3.5$

Hence $c = 3.5$ in $[2 < x < 5]$

Example 2

Verify Rolle's Theorem, forth function $f(x) = (x - 3)(x - 8)$ in the interval $[3 \le x \le 8]$

Solution

$f(x)$ is continuous in $[3 \le x \le 8]$ since it is a polynomial.
$f(x)$ is differentiable in $[3 < x < 8]$ since it is a polynomial.
$f(3) = f(8)$ since each equals to zero.

We must prove that there exists a point c in $[3 < x < 8]$ such that $f^1(c) = 0$.
Let us find the value c.
$f(x) = (x - 3)(x - 8) = x^2 - 11x + 24$
$f^1(x) = 2x - 11$
$f^1(x) = 0$ gives $x = 5.5$
hence c is the point ($c = 5.5$) in the interval $[3 < x < 8]$

Example 3

Verify Rolle's Theorem for the function

$f(x) = \dfrac{\sin x}{e^x}$ in the interval $[0, \pi]$

Solution

Since both sin x and e^x are continuous and differentiable functions, $\frac{\sin x}{e^x}$ is also continuous and differentiable in the interval $[0, \pi]$.

$$f(o) = \frac{\sin 0}{e^0} = \frac{0}{1} = 0$$

$$f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$$

Hence f (0) = f (π) since each equals to zero. Rolle's theorem is applicable and we can find in $[0 < x < \pi]$ such that $f^1(c) = 0$.

$f(x) = \sin x\, e^{-x}$
$f^1(x) = -\sin x\, e^{-x} + e^{-x} \cos x$.
$f^1(x) = 0$ gives, $e^{-x} (\cos x - \sin x) = 0$

Therefore $\cos x - \sin x = 0$ since $e^{-x} \neq 0$.

$\cos x = \sin x$ gives $x = \frac{\pi}{4}$ and other values.

There is at least one value of $c = \frac{\pi}{4}$ such that $f^1 = (\frac{\pi}{4}) = 0$ and $\frac{\pi}{4}$ lies in the open interval $[0 < = \frac{\pi}{4} < \pi]$

Hence Rolle's Theorem is verified.

Example 4

Verify Rolle's theorem for the function, $f(x) = (x-2)^6 (x-5)^{11}$ in the interval $[2 < x < 5]$

Solution

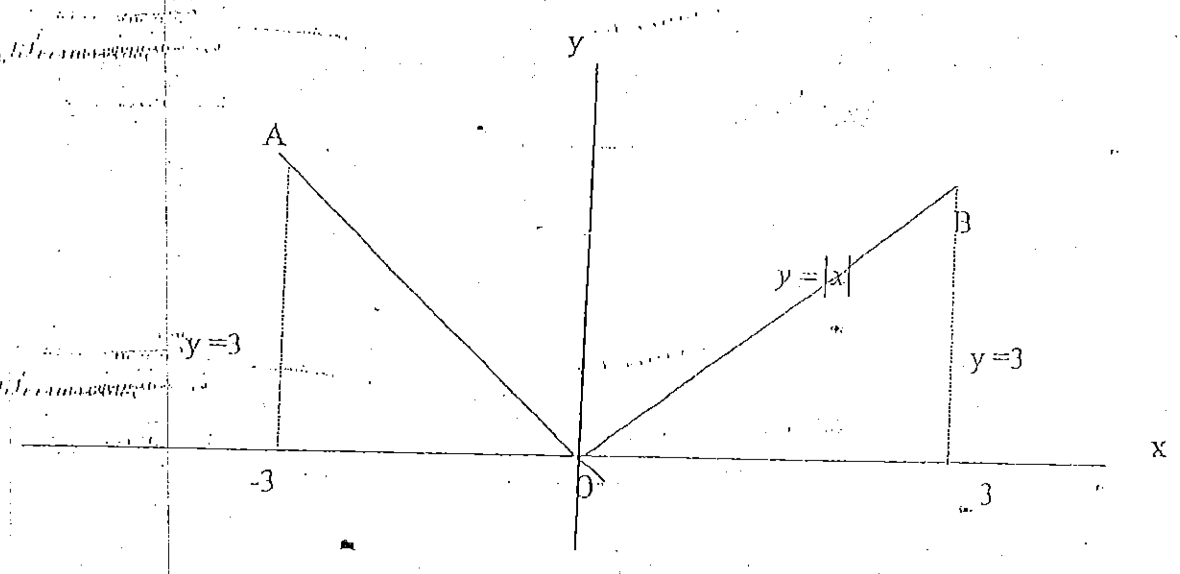
Since f (x) is a polynomial, it is continuous $[2 \leq x \leq 5]$ and differentiable in the interval $[2 < x < 5]$

Also f (2) = f (5) since each equals to zero.

$f(x) = (x-2)^6 (x-5)^{11}$

$$=(x-2)^5 (x-5)^{10} [11 (x-2) + 6 (x-5)]$$
$$= (x-2)^5 (x-5)^{10} (11x - 22 + 6x - 30)$$
$$= (x-2)^5 (x-5)^{10} (17x - 52)$$

$f^1 (x) = 0$ gives $x = 2$, $x = 5$, $x = \frac{52}{17} = 3\frac{1}{17}$ here the point c of Rolle's theorem is $c = 3\frac{1}{17}$ in the open interval $[2 < x < 5]$. Hence there is at least one point $c = 3\frac{1}{17}$ in the open interval such that $f^1 (c) = 0$



Example 5

In the figure above, the graph represents the curve $y=|x|$.

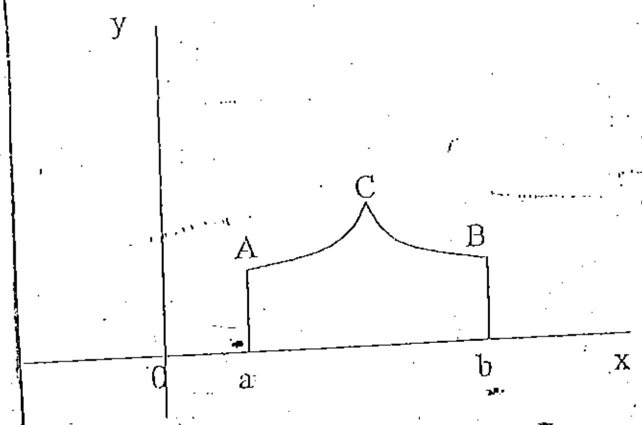
a. Is the function $y=|x|$ continuous? Give reasons.
b. What is the value of f (x) at x = -3 and at x = 3?
c. Does the function satisfy all the three conditions of Rolle's theorem?
d. Does there exist a point c in the open interval $-3 < x < 3$ such that $f'(c) = 0$.

Solut

a. it is continuous since the graph is unbroken
b. the value of f (x) at x = -3 and 3 or equal to 3
c. no since it is not diferentiable at the point o ($\frac{dy}{dx}$ does not exist)
d. No

Exam

In tire aA = bB



Check whether the function satisfies all the three conditions of Rolle's theorem.
Does there exist a point c in $a < x < b$ such that $f'(c) = 0$.

Solutio

a. f continuous in the closed interval [a, b] since the curve is uen but f (x) is not differentiable at the point C, There is no ta line at the point C.
b. Nce $f'(x)$ does not exist at C.

Summa

1. In hapter we have looked at Rolle's Theorem which is funatal in Differential Calculus.

2. We can apply Rolle's Theorem for a function f (x) if and only if f (x) satisfies the following 3 conditions

i). f (x) should be continuous in a closed interval $[a \le x \le b]$

ii). f (x) is differentiable in the open interval $[a < x < b]$

iii). f (a) = f (b)

3. If all the three conditions are satisfied we can be assured that there is at least one point c in $[a < x < l]$ such that $f^1(c) = 0$.

4. Geometrically the curve y = f (x) has at least one point where the curve has a maximum or minimum value or both, when the 3 conditions are satisfied.

Exercise 1

1. Verify Rolle's theorem for the function, $f(x) = x^2 - 11x + 24$ in the interval $[3 \le x \le 8]$

2. Verify Rolle's theorem for the function, $f(x) = (x + 2)^2 (x - 7)$ in the interval $[-2 \le x \le 7]$

Verify Rolle's theorem for the following functions:

3. $f(x) = x^2 - 3x - 40$ in the interval $[-5 < x < 8]$
4. $f(x) = (x - 1)^5 (x - 5)^6$ in the interval $[1 < x < 5]$
5. $f(x) = (x - a)^m (x - b)^n$ in the interval $[a \le x \le b]$

6. Verify Rolle's theorem for the function, $f(x) = e^x (\sin x - \cos x)$ in the interval

$$\left[\frac{\pi}{4} \le x \le \frac{5\pi}{4}\right]$$

7. Verify Rolle's theorem for the function $f(x) = \ln\left(\frac{(x^2+12)}{7x}\right)$ in the interval $[3 \le x \le 4]$

8. If $f(x) = \ln\frac{x^2+ab}{(a+b)x}$ show that f (x) satisfies Role's Theorem. Find the point c. in the interval $[a \le x \le b]$

9. If $f(x)=\frac{2\sin x}{e^x}$ verify Role's Theorem in the interval $(0, \pi)$.

10. If $f(x)=(x+2)^5(x-4)^{15}$ in the interval (-2, 4) find the value of c in Role's Theorem.

Further Reading

1. Advanced Calculus
   By Watson Fulks
   University of Colorado
   John Wiley and Sons
   New York Tolonto Singapore.

2. Mathematical Methods for Science Students
   By G. Stephenson
   Addison Wesley Longman Limited
   Edinburg Gate, Harlow, London.

CHAPTER 2
Mean Value Theorems

2.1 Introduction

In the previous chapter we have seen Rolle's Theorem and its applications. In this chapter we are going to see the extension of Rolle's Theorem, namely the **Mean Value Theorems**. In Rolle's theorem, one of the three conditions for f (x) is that f (a) = f (b). Suppose f (x) does not satisfy this condition or f (a) ≠ f (b), what will happen to f^1 (c)? The mean value theorem answers this question. There are different forms of Mean Value Theorem, namely Lagrange's, Cauchy's and Taylor's mean value Theorems. First let us see Lagrange's mean value Theorem.

2.2 Objectives of the chapter

By the end of this chapter we will learn

i). Lagrange's Mean Value Theorem and its Geometrical meaning.

ii). Cauchy's Mean Value Theorem.

iii). Other forms of Mean value Theorem.

iv). Taylor's extended Mean Vale Theorem

2.3 Lagrange's Mean Value Theorem

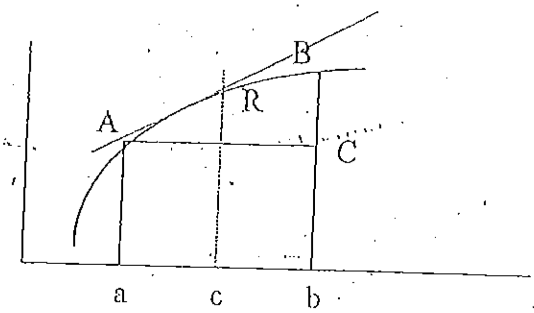
If a function f (x) is

i) continuous in a close interval [a ≤ x ≤ b], and

ii) differentiable in the open interval [a < x < b]

then these exists at least one point c in the open interval [a < x < b] such that

$f^1(c) = \dfrac{f(b) - f(a)}{b - a}$. Can you deduce Rolle's theorem from the above result if f (a) = f (b) is a further condition?



Proof

We shall prove this theorem using Rolle's Theorem

$\dfrac{f(b)-f(a)}{b-a}$ is a constant

Let us call it as R.

Then $\dfrac{f(b)-f(a)}{b-a} = R$ .... (1)

or f (b) – f (a) = (b – a)R

or f (b) – f (a) – (b – a)R = 0 .... (2)

Consider a function ϕ (x) where ϕ (x) = f (b) – f (x) – (b – x) R .... (3)

Now (i) ϕ (x) is continuous in [a ≤ x ≤ b] since f (x) is continuous in [a ≤ x ≤ b]

(ii) ϕ (x) is differentiable in [a < x < b] since f (x) is differentiable.

(iii) ϕ (a) = f (b) – f (a) – (b – a) R = 0 from the choice of R in (1) or (2)

ϕ (b) = f (b) – f (b) – (b – b) = 0 from he choice of ϕ (x) in (3)

hence

ϕ (a) = ϕ (b). since both are equal to zero.

Now ϕ (x) satisfies all the three conditions of Rolle's Theorem. Then there exists a point c in [a < x < b] such that ϕ^1 (c) = 0.

Now ϕ (x) = f (b) – f (x) – (b – x) R from (3).

Then ϕ^1 (x) = - f^1 (x) + R

ϕ^1 (c) = - f^1 (c) + R = 0 gives,

f^1 (c) = R

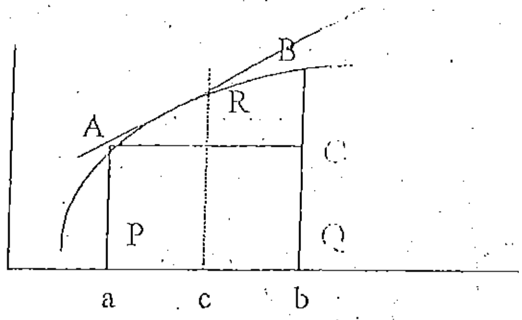
or f^1 (c) = $\dfrac{f(b)-f(a)}{b-a}$ from (1)

Thus Lanrange's Mean Value Theorem is proved. It is generally expressed as,

f (b) – f (a) = (b – a) f^1 (c)

or f (b) = f (a) + (b – a) f^1 (c).

2.4 Geometrical meaning of Lagrange's Mean Value Theorem



Let P and Q be two points on the x axis such that their x coordinates are a and b respectively. If f (x) is continuous in $[a \le x \le b]$ the curve y = f (x) is continuous. Also

$f^1(x)$ exists in $[a < x < b]$.

Let PA = f (a), QB = f (b)

Then BC = f (b) – f (a) and CA = b – a.

What is the meaning of $\dfrac{f(b)-f(a)}{b-a}$?

$$\frac{f(b)-f(a)}{b-a} = \frac{QB-PA}{PQ} = \frac{BC}{AC} = \tan\theta$$

= gradient of the chord AB.

Let c be a point in $[a < x < b]$ and R be the point on the curve so that CR = f (c)

Draw a tangent line at the point R on the curve y = f (x). The gradient of the tangent line

at R $= \dfrac{dy}{dx} = f^1(x) = f^1(c)$.

Lagrange's Mean Value Theorem states that the gradient of the tangent line at R, $f^1(c)$ is equal to the gradient of the chord AB. In other words the theorem assures that there exists a point R on the curve y = f (x) the tangent at which is **parallel** to the chord AB.

2.5 Cauchy's Mean Value Theorem

You all might have known the famous French Mathematician, Cauchy, 'the intellectual giant' of his time. He is the Father of the valuable branch of mathematics, namely Complex Analysis. We shall see how Cauchy's Mean Value Theorem for two functions
f (x) and g (x).

If f (x) and g (x) are
continuous in the closed interval [$a \le x \le b$] and differentiable in the open

interval
[$a < x < b$] then there exists at least one point c in the open interval [$a <$
~~x~~ $< b$] such that
$[f(b) - f(a)]\, g^1(c) = [g(b) - g(a)]\, f^1(c)$

or $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f^1(c)}{g^1(c)}$ if g (b) – g (a) ≠ 0 and $f^1(x)$ and $g^1(x)$ are not simultaneously zero in [$a < x < b$].

Proof

Cauchy's Mean Value Theorem is proved using the fundamental Rolle's Theorem. The proof is simple if the correct function $\phi(x)$ is selected appropriately.

Let $\phi(x) = [f(b) - f(a)]\, g(x) - [g(b) - g(a)]\, f(x)$.

Now (i) $\phi(x)$ is continuous in [$a \le x \le b$]

(ii) $\phi(x)$ is differentiable in [$a < x < b$]

(iii) $\phi(b) = \phi(a)$, (You can verify easily).

Hence Rolle's Theorem is applicable for $\phi(x)$ and there exists a point c in [$a < x < b$] such that $\phi^1(c) = 0$.

Now; $\phi^1(x) = [f(b) - f(a)]\, g^1(x) - [g(b) - g(a)]\, f^1(x)$

$\phi^1(c) = [f(b) - f(a)]\, g^1(c) - [g(b) - g(a)]\, f^1(c) = 0$

Hence [f (b) – f (a)] g^1 (c) = [g (b) – g (a)] f^1 (c). Divide both sides by [g (b) – g (a)] g^1 (c)

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f^1(c)}{g^1(c)}$$ if g (b) – g (a) ≠ 0 and f^1 (x) and g^1 (x) are not zero simultaneously in [a < x < b]

2.6 Derivation of Lagrange's Theorem from Cauchy's Mean Value Theorem

Lagrange's mean value Theorem is only a particular case of Cauchy's Mean Value Theorem.

Proof:

Cauchy's Mean Value Theorem states that if f (x) and g (x) are continuous in [a ≤ x ≤ b] and differentiable in [a < x < b] then there exists a point c in [a < x < b] such that

[f (b) – f (a)]g^1 (c) = [g (b) – g (a)] f^1 (c)

Now take g (x) = x and f (x) = any differentiable function in any interval [a < x < b], g^1 (x) = 1.

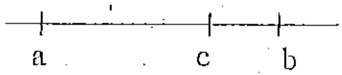
If f (x) is continuous a ≤ x ≤ b and differentiable in [a < x < b], then by Cauchy's theorem there exists a point c such that [f (b) – f (a)] g^1 (c) = [g (b) – g (a)] f^1 (c). Since g (x) = x,

g^1 (x) = 1, g^1 (c) = 1, g^1 (b) – g (a) = (b – a) we have f (b) – f (a) = (b – a) . f^1 (c). This is Lagrange's Theorem.

2.7 Other forms of Mean Value Theorems

Different forms of mean value theorems appear depending on the consideration of the interval [a ≤ x ≤ b].

Suppose we take the range of the interval



$(b - a)$ as h. When $b - a = h$, b is replaced by $a + h$, c is strictly in the open interval so that $c = a + \theta h$ where $0 < \theta < 1$.

According to this notations, Lagrange's Mean Value Theorem, $f(b) - f(a) = (b - a) f^1(c)$ becomes $f(a + h) = f(a) + h f^1(a + \theta h)$.

2.8 Taylor's extended mean value Theorem

If $f(x)$ and its first $(n - 1)$ derivatives $f^1(x), f^{11}(x), \ldots\ldots f^{n-1}(x)$ are continuous in the closed interval $[a \le x \le b]$ and $f^n(x)$ exists in $[a < x < b]$ then there exists a point c in

$[a < x < b]$ such that $f(b) = f(a) + (b - a) f^1(a) + \dfrac{(b-a)^2}{2!} f^{11}(a)$

$$+ \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c) \qquad (1)$$

This is one form of Taylor's Mean Value Theorems.

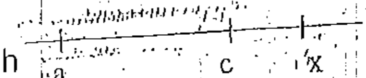
Other forms of Taylor's Mean Value Theorem

i). If we put $b - a = h$ in (1) then $b = a + h$ then $c = a + \theta h$ where $0 < \theta < 1$.

Then $f(a + h) = f(a) + h f^1(a) + \dfrac{h^2}{2!} f^{11}(a) + \ldots\ldots\ldots$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h)$$

ii). If we put $b = x$, in (1) and $c = a + \theta h$



$$f(x) = f(a) + (x-a) f^1(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots$$

$$+ \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n[a + \theta(x-a)]$$

where $0 < \theta < 1$

Example 1

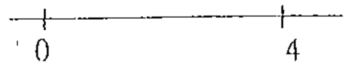
If $f(x) = x^3 + 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$ and the interval is $[0 \le x \le 4]$ find c of the Lagrange mean value Theorem.

Solution

$F(x) = x^3 - 6x^2 + 11x - 6$

Now, $f(x)$ is continuous in $[0 \le x \le 4]$ and differentiable in $[0 < x < 4]$ since it is a polynomial. Hence there exists a point c in $[o < x < 4]$ such that $f(b) - f(a) = (b - a) f^1(c)$ (Lagrange's Mean Value theorem)

Put $b = 4$, $a = 0$



$F(4) - f(0) = (4 - 0) f^1(c)$

$(6) - (-6) = 4 f^1(c)$.

$4 f^1(c) = 12, \quad f^1(c) = 3$

Now $f^1(x) = 3x^2 - 12x + 11$

$f^1(c) = 3c^2 - 12c + 11$

$3c^2 - 12c + 11 = 3$

$3c^2 - 12c + 8 = 0$

solving $c = \dfrac{6 + 2\sqrt{3}}{3}$ or $\dfrac{6 - 2\sqrt{3}}{3}$

$c = 3.155$ or $c = 0.845$.

hence there are two values of c in the open interval $[0 < x < 4]$

Example 2

Verify Lagrange's Mean Value Theorem for the function $f(x) = e^x$ in $[0 \le x \le 1]$

Solution

Lagrange's Theorem states that:

$f(b) - f(a) = (b - a) f^1(c)$

where $[0 < c < 1]$

$f(1) - f(0) = (1 - 0) f^1(c)$.

$e^1 - e^0 = f^1(c)$

$e - 1 = f^1(c)$

Now $f^1(x) = e^x$ and $f^1(c) = e^c$.

$$e^c = (e - 1)$$

Tataing logarithm

$c = \ln(e - 1)$ we can see that $\ln(e - 1)$ lies in $[0 < x < 1]$.

Example 3

Verify Lagrange's Mean Value Theorem for the function $f(x) = \sqrt{x^2 - 4}$ for the closed interval $[2 \le x \le 3]$.

Solution

$f(x) = \sqrt{x^2 - 4}$ is continuous in $[2 \le x \le 3]$ and differential in $[2 < x < 3]$

Then, $f(b) - f(a) = (b - a) f^1(c)$

$f(3) - f(2) = (3 - 2) f(c)$

$\sqrt{5} - \sqrt{0} = f^1(c)$..................(1)

$f(x) = \sqrt{x^2 - 4} \Rightarrow f^1(x) = \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}}(2x)$

$$f^1(x) = \frac{x}{\sqrt{x^2-4}}$$

$$f^1(c) = \frac{c}{\sqrt{c^2-4}} \quad \text{.......................................(2)}$$

Substitute (2) in (1),

$$\frac{c}{\sqrt{c^2-4}} = \sqrt{5}$$

$$\frac{c^2}{c^2-4} = 5$$

$4c^2 - 20 = 0$ $\qquad c = +\sqrt{5}, -\sqrt{5}$

$c = \sqrt{5}$ is applicable in $[2 < c < 3]$

Exercise 2

1. If $f(x) = x^3 - 6x^2 + 9x + 3$, verify Lagrange's Mean Value Theorem for f (x) in
   $[0 \le x \le 4]$ {Hint: f (x) satisfies the conditions for the theorem. a = 0, b = 4 and $\frac{f(b)-f(a)}{b-a} = \frac{7-3}{4-0} = f^1(c)$
   $f^1(c) = 3c^2 - 12c + 9 = 1$ gives c = 3.155 or c = 0.845. There exists two values of c in $[0 \le x \le 4]$}

2. Verify Lagrange's Mean Value Theorem for f (x) = ln x in the interval $[1 \le x \le e]$.

3. Show that there exists at least one value of c for $f(x) = e^{2x}$ in the interval $[o < x < 1]$ such that $\frac{f(1)-f(0)}{1-0} = f^1(c)$.

4. Find c so that $f^1(c) = \dfrac{f(b) - f(a)}{b - a}$ for the function f (x) = x^2 – 3x – 1 when the interval is $a = \dfrac{-11}{7}$ and $b = \dfrac{13}{7}$.

5. Verify the mean value theorem for the function f (x) = x^2 in any interval [a ≤ x ≤ b]

6. Find c of the Lagrange's Theorem for the function f (x) = $\sqrt{x^2 - 4}$ for the interval [2 ≤ x ≤ 3]. Hence verify the theorem.

Summary

In this chapter we have seen the following:

1. Lagrange's Mean Value Theorem: If f (x) is continuous in [a ≤ x ≤ b], and differentiable in [a < x < b] then there exists as least one point c in the open interval such that
f (b) = f (a) + (b – a) f^1 (c).

2. The geometrical meaning of the theorem is: There exists a point R on the curve
y = f (x), the tangent at which is parallel to the chord AB

3. Cauchy's Mean Value Theorem: If f (x) and g (x) are continuous in [a ≤ x ≤ b] and differentiable in [a < x < b] then there exists at least one point in the open interval such that [f (b) – f (a)] g^1 (c) = [g (b) – g (a)] f^1 (c).

4. Lagrange's Mean Value Theorem is deduced from Cauchy's Mean Value Theorem.

5. By taking the interval [a ≤ x ≤ b] in the form of [a ≤ x ≤ a + h] a different form of Lagrange's Theorem is obtained. In this notation the final result in the theorem becomes, f (a + h) = f (a) + h f^1 (a + θ h) where 0 < θ < 1.

6. The Mean Value Theorem is extended by Taylor for the interval [a ≤ x ≤ a + h] if f (x) and its first (n – 1) derivatives are continuous in [a ≤ x ≤ b] and differentiable in

[a < x < b] then there exists a point c in [a < x < b] such that

$$f(a+h) = f(a) + h f^1(a) + \frac{h^2}{2!} f^{11}(a) + \ldots\ldots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h) \text{ where } 0 < \theta < 1.$$

Further Reading

1. Advanced Calculus
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2. Mathematical Methods for Science Students
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CHAPTER 3

Taylors and Maclaurin's Series

3.1 Introduction

In Chapter two, we have seen Taylor's extended mean value theorem.

If f (x) and its first (n - 1) derivatives are continuous in [a ≤ x ≤ a + h] and f^n (x) is differentiable in [a < x < a + h] then there exists a number θ, [0 < θ < 1] such that,

$$f(a+h) = f(a) + h f^1(a) + \frac{h^2}{2!} f^{11}(a) + \dots\dots\dots + \frac{h^{n-1}}{(n-1)} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h) \qquad (3.1)$$

In the following section we are going to obtain Taylor's Series from this Taylor's extended mean value Theorem.

3.2 Objectives of the Chapter:

In this Chapter we shall learn Taylor's and Maclaurine's power series for a given f (x). We shall also learn Cauchy's higher mean value theorem.

3.3 Taylor's Series

In Taylor's extended mean value Theorem, we have considered the interval consisting of a and a + h. Suppose we consider the interval as a and x. In other words we take

a + h = x so that h = x − a.

Substitute a + h = x and h = x – a in Taylor's extended mean value Theorem (3.1). We get f (x) = f (a) + (x – a) f^{1} (a) + $\frac{(x-a)^2}{2!} f^{11}(a) + \ldots\ldots\ldots\ldots$

$$+ \frac{(x-a)^{n-1}}{(n-1)} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n[a+\theta(x-a)]$$

Let S_n (x) = f (a) + (x – a) f^1 (a) + ... $\frac{(x-a)^{n-1}}{(n-1)} f^{n-1}(a)$

and R_n (x) = $\frac{(x-a)^n}{n!} f^n[a+\theta(x-a)]$

Then f (x) = S_n (x) + R_n (x)

f (x) - S_n (x) = R_n (x)

$$\lim_{n\to\infty}|f(x)-S_n(x)| = \lim_{n\to\infty} R_n(x)$$

If

$\lim_{n\to\infty} R_n(x) \to 0$, then f (x) – S_n (x) = 0

F (x) = S_n x

Hence

$$f(x) = f(a) + \frac{(x-a)}{1!} f^1(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \frac{(x-a)^3}{3!} f^{111}(a) + \ldots\ldots\ldots\ldots$$

This infinite series is called Taylors series of the function f (x)

Note: The term R_n (x) = $\frac{(x-a)}{n!} f^n[a+\theta(x-a)]$ is called the remainder according to Taylor. The infinite series is convergent and valid only if $R_n \to 0$ when $n \to \infty$.

3.4 Maclaurin's Series

To expand a function f (x) into an infinite series, Maclaurin's series is more useful than Taylor's series. Consider Taylor's Extended mean Value Theorem in the interval a and a + h.

$$f(a+h) = f(a) + hf^{1}(a) + \frac{h^2}{2!}f^{11}(a) + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h) \ldots\ldots (1)$$

$$0 < \theta < 1.$$

Maclaurin took the interval as 0 and x. In other words he put a = 0, a + h = x so that h = x; Substituting these values in the extended mean value Theorem (1).

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots\ldots\ldots\ldots\ldots +$$

$$\frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n[\theta x]$$

If the last term $R_n(x) = \frac{x^n}{n!}f^n(\theta x) \to 0$

When $n \to \infty$, we get Maclaurin series.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots \text{to } \infty$$

This infinite series is called Maclaurin's series of f (x).

3.5 Cauchy's Higher Mean Value Theorem

The French Mathematician Cauchy, "the intellectual Giant", has developed the higher mean value theorem in a different form. The difference is only in the last term in Taylor's theorem. It is given below:

If a function f (x) is such that its (n – 1)th derivative f^{n-1} is

(i) continuous in a closed interval [a ≤ x ≤ a + h]
(ii) differentiable in the open interval [a < x < a + h] then there exists a number θ in [0 < θ < 1] such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots\ldots\ldots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)$$

$$0 < h < 1.$$

This theorem can also be proved in a similar way as in Taylor's Mean Value Theorem.

Select a constant R and a function ϕ (x) such that,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots\ldots\ldots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + hR$$

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x)$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + (a+h-x)A.$$

Then use Rolle's theorem. This proof is to be completed by you as an exercise. It is not difficult!

The last term in Cauchy's Higher Mean Value Theorem is

$$\frac{h^{n}}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)$$

This term is called Cauchy's Form of Remainder. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots\ldots\ldots +$$

$$\frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h)$$

Taking the interval as a and x or putting a + h = x and h = x – a, we get

f (x) = f (a) + (x – a) f^{1} (a) + $\frac{(x-a)^2}{2!} f''(a) + \ldots\ldots\ldots$

$$+\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(x-a)^{n}}{(n-1)!}(1-\theta)^{n-1}f^{n}[a+\theta(x-a)]$$

If $R_n \to 0$ when $n \to \infty$ we get Cauchy's series namely

$$f(x) = f(a) + (x-a)f^{1}(a) + \frac{(x-a)^2}{2!}f^{11}(a) + \ldots\ldots\ldots\ldots \text{to } \infty$$

3.6 Expansion of functions

Example 1

Expand e^x as an infinite series.

Solution

Let $f(x) = e^x \ldots\ldots\ldots\ldots f(0) = 1$

$f'(x) = e^x \ldots\ldots\ldots\ldots f'(0) = 1$

$f''(x) = e^x \ldots\ldots\ldots\ldots f''(0) = 1$

$f'''(x) = e^x \ldots\ldots\ldots\ldots f'''(0) = 1$

In general $f^n(x) = e^x$

$$R_n(x) = \frac{x^n}{n!}f^n(\theta x) = \frac{x^n}{n!}e^{\theta x}$$

$$\lim_{n\to\infty} R_n(x) = e^{\theta x}\lim_{n\to\infty}\frac{x^n}{n!}$$

$$= e^{\theta x}(0) \quad \text{since } \lim_{n\to\infty}\frac{x^n}{n!} = 0$$

Hence $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots to\ldots\infty$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots to\ldots\infty$$

Example 2

Expand sin x as an infinite series.

Solution

Let $f(x) = \sin x$ .... $f(0) = 0$

$f'(x) = \cos x$ .... $f'(0) = 1$

$f''(x) = -\sin x$ .... $f''(0) = 0$

$f'''(x) = -\cos x$ .... $f'''(0) = -1$

In general $f^n(x) = f^n \sin(x + \frac{n\pi}{2})$

$$R_n(x) = \frac{x^n}{n!} f^n(\theta x)$$

$$R_n(x) = \frac{x^n}{n!} \sin(\theta x + \frac{n\pi}{2})$$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{x^n}{n!} \sin(\theta x + \frac{n\pi}{2})$$

$$= 0, \text{since } \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Hence Maclaurin's series is valid.

Therefore $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \text{..........} to \ \infty$

$$\sin x = 0 + x(1) + 0 - \frac{x^3}{3!} + \text{..........} to \ldots \infty$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \text{..........} to \ldots \infty$$

Example 3

Expand ln (1 + x) as an infinite series.

Solution

Let $f(x) = \ln(1+x)$ ……………… $f(0) = 0$

$f'(x) = \dfrac{1}{1+x}$ ………………… $f'(0) = 1$

$f''(x) = \dfrac{-1}{(1+x)^2}$ ………………… $f''(0) = -1$

$f'''(x) = \dfrac{1.2}{(1+x)^3}$ ……………… $f'''(0) = 1x2 = 2$

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \ldots f^n(\theta x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

$$R_n(x) = \frac{x^n}{n!}\frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$= 0, \text{since} \lim_{n\to\infty}\left(\frac{x}{1+\theta x}\right)^n = 0$$

$$\lim_{n\to\infty}\frac{n-1}{n} = \lim_{n\to\infty} 1 - \frac{1}{n} = 0$$

Example 4

Expand $e^x \sin x$ using Maclaurin's series up to x^3 term. (Assume the possibility of expansion).

Let $f(x) = e^x \sin x$ ……………………………………… $f(0) = 0$

$f'(x) = e^x \cos x + \sin x\, e^x$ ………………………… $f'(0) = 1$

$f''(x) = e^x(-\sin x) + \cos x\, e^x + \sin x\, e^x + e^x \cos x$ $f''(0) = 2$

$= 2e^x \cos x$ $f''(0) = 2$

$f'''(x) = 2e^x(-\sin x) + \cos x\,(2e^x)$ ………………………

$f''''(x) = 2e^x(-\cos x) + 2e^x(-\sin x) + \cos x\,(2e^x) + 2e^x(-\sin x)$ $f'''(0) = 0$

$= 0$

By Maclaurin's series,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!} + f'''(0) + \frac{x^4}{4!}f''''(0) + \ldots\ldots\ldots\ldots$$

$$= \frac{x}{1!} + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \ldots\ldots\ldots$$

Exercise 3

Using Maclaurin's series expand the following:

1). $f(x) = \tan x$

2). $f(x) = \cos x$

3). $f(x) = \ln(1 + e^x)$

4). $f(x) = e^x \cos x$

5). $f(x) = e^{\sin x}$

6). $f(x) = x \cos x$

7). $f(x) = \tan\left(x + \frac{\pi}{4}\right)$.

8). $f(x) = \ln(1 + \sin x)$.

9). Prove that $x \cos x = 1 - x + \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} \ldots\ldots$.

10). Show that $\cos^2 x = 1 - x^2 - \frac{x^4}{3} - \frac{2x^6}{45} + \ldots\ldots$.

Summary

In this chapter we have seen the following

1. Taylor's series,

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots\ldots\ldots\ldots$$

provided that f (x) is continuous in any interval a and x and $f^n(x)$ is differentiable and $\lim_{n\to\infty} \frac{(x-a)^n}{n!} f^n[a + \theta(x-a)$ tends to zero when $n \to \infty$.

2. Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots to \infty$$

provided that f (x) and its (n-1) derivatives are continuous, $f^n(x)$ is differentiable and $\lim_{n \to \infty} \frac{x^n}{n!} f^n(\theta x) \to 0$ when $n \to \infty$.

3. Cauchy's form of Remainder. When

$$f(x) = f(a) + \frac{(x-a)}{1!} f^1(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \ldots$$

$$+ \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{(n-1)!} (1-\theta)^{n-1} f^n[a + \theta(x-a)]$$

The last term is called Cauchy's form of Remainder R_n.

If $R_n \to 0$ when $n \to \infty$ Cauchy's series is obtained.

Further Reading

1. Advanced Calculus
   By Watson Fulks
   University of Colorado
   John Wiley and Sons
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2. Mathematical Methods for Science Students
   By G. Stephenson
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   Edinburg Gate, Harlow, London.

CHAPTER 4

Indeterminate Forms and L'Hospital Rule

4.1 Introduction

What is the colour of a chameleon? We cannot answer this question correctly. Its colour may be green, yellow, blue or white. If I ask you the colour of a chameleon that is sitting inside the green leaves, you can answer that it is green.

What is the value of $\frac{0}{0}$ or $\frac{\infty}{\infty}$?

You are wrong if you say that the value is one. Just like the colour of a chameleon is indeterminate, the value of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ is also indeterminate. The value can be anything from $-\infty$ to $+\infty$. In this Chapter we are going to learn the values of certain indeterminate quantities such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0 which are all indeterminate.

4.2 Objectives of the Chapter:

By the end of this Chapter you should be able to:

evaluate the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

4.3 Limits of functions

You have already studied the limits of functions in calculus I. Let us revise the limits of functions.

i). $\lim_{x \to 2} \frac{x^2+5x+10}{x+10} = \frac{4+10+10}{12} = 2$

Here we just substitute x = 2, though it is not given that x = 2. It is given that x is nearly two. x may be 2.0001 or x may be 1.999 and so on.

ii). Consider

$$\lim_{x\to 3}\frac{x^2-5x+6}{x^2-4x+3}.$$

$x \to 3$

If we substitute x = 3, we get the limit = $\frac{0}{0}$. Hence we cannot find the value by actual substitution.

$$\lim_{x\to 3}\frac{x^2-5x+6}{x^2-4x+3}=\lim_{x\to 3}\frac{(x-3)(x-2)}{(x-3)(x-1)}$$

$$=\lim_{x\to 3}\frac{x-2}{x-1}$$

$$=\frac{1}{2} \text{ by putting } x = 3.$$

According to the problem, x is not exactly 3 and so x – 3 is not zero. Hence we can cancel the factor (x – 3).

iii). Find the $\lim_{x\to 4}\frac{x^3-7x^2+14x-8}{x^2-2x+8}$.

$$\lim_{x\to 4}\frac{x^3-7x^2+14x-8}{x^2-2x+8}=\lim_{x\to 4}\frac{(x-4)(x-1)(x-2)}{(x-4)(x+2)}$$

$$=\lim_{x\to 4}\frac{(x-1)(x-2)}{x+2}$$

$$=\frac{(4-1)(4-2)}{(4+2)}$$

$$=\frac{6}{6}$$

$$=1$$

By actual substitution we find that the limit becomes $\frac{0}{0}$. As we have seen earlier, $\frac{0}{0}$ may have any value from $-\infty$ to ∞ including unity.

For large functions factorization may be difficult or the function may not be factorisable. How will you find the limit in such cases? We are going to find easier ways in this Chapter. In fact we are going to use L' Hospital rule to find limits of the forms, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

In the following section we will establish L' Hospital Rule for the indeterminate form: $\frac{0}{0}$.

4.4 Theorem: L' Hospital Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f^1(x)}{g^1(x)},$$

if f (a) = 0, g (a) = 0

Proof

Let f (a) = 0 and g (a) = 0
Consider the interval [a, x] for Cauchy's Mean Value Theorem.

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f^1(c)}{g^1(c)}$$

If f (a) and g (a) = 0,

$$\frac{f(x)}{g(x)}=\frac{f^1(c)}{g^1(c)}$$

where a < c < x

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f^1(c)}{g^1(c)}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^1(x)}{g^1(x)}$$

since when $x \to a$, the interval $\to 0$ and $c \to a$ or $c \to x$

4.5 Generalisation of L' Hospital Rule for $\frac{0}{0}$ form.

Using the extension of Cauchy's Higher Mean Value Theorem L' Hospital Rule can be extended:

i). If f (a) = 0, g (a) = 0 then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^1(x)}{g^1(x)}$$

ii). If f^1 (a) = 0, g^1 (a) = 0

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{11}(x)}{g^{11}(x)}$$

iii). If f^{11} (a) = g^{11} (a) = 0 then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{111}(x)}{g^{111}(x)}$$ and so on.

Example 1 .

Find $\lim_{x \to 4} \frac{x^3 - 7x^2 + 14x - 8}{x^2 - 2x - 8}$

Solution

If we substitute x = 4 we get limit as $\frac{0}{0}$

By L' Hospital Rule,

$$\lim_{x\to 4}\frac{x^3-7x^2+14x-8}{x^2-2x-8}=\lim_{x\to 4}\frac{3x^2-14x+14}{2x-2}$$

When we put x = 4 now, $=\dfrac{48-56+14}{6}$

$= 1$

Example 2

Find $\displaystyle\lim_{x\to 2}\frac{x^3+6x^2-36x+40}{x^3-12x+16}$

Solution

If we put x = 2, we get the indeterminate form $\frac{0}{0}$ using L' Hospital Rule,

Find $\displaystyle\lim_{x\to 2}\frac{x^3+6x^2-36x+40}{x^3-12x+16}=\lim_{x\to 2}\frac{3x^2+12x-36}{3x^2-12}$

If we put x = 2, we again get the indeterminable form $\frac{0}{0}$.

Therefore $\displaystyle\lim_{x\to 2}\frac{x^3+6x^2-36x+40}{x^3-12x+16}=\lim_{x\to 2}\frac{6x+12}{6x}$

$=\dfrac{24}{12}$

$= 2.$

You can try to do the same problem using factorization.

Example 3

Find $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Solution

If we substitute x = 0, we get $\frac{\sin 0}{0} = \frac{0}{0}$

Using L' Hospital Rule,

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1$$

Example 4

Determine the $\lim_{x \to 1} \frac{1 + \ln x - x}{1 - 2x + x^2}$

When x = 1 the function is of the form $\frac{0}{0}$

$$\lim_{x \to 1} \frac{1 + \ln x - x}{1 - 2x + x^2} = \lim_{x \to 1} \frac{\frac{1}{x} - 1}{-2 + 2x} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{-\frac{1}{x^2}}{2}$$

$$= \frac{-1}{2}$$

4.6 L' Hospital Rule for the indeterminate form $\frac{\infty}{\infty}$

If $\lim_{x \to a} f(a)$ be ∞ and $\lim_{x \to a} g(a)$ be ∞ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} = \frac{0}{0}$ form

Thus the indeterminant form $\frac{\infty}{\infty}$ can be converted to the form $\frac{0}{0}$ and L' Hospital Rule may be applied. Sometimes we can use $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^1(x)}{g^1(x)}$ and then apply.

Example 5

Evaluate $\lim_{x \to 0} \frac{\ln x^r}{\cos ecx}$

Solution

In 0 cosec 0 become both ∞

Therefore $\lim_{x \to 0} \frac{\ln x}{\cos ecx} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\cos ecx \cot x}$

$$= \lim_{x \to 0} \frac{1}{-x.\frac{1}{\sin x}.\frac{\cos x}{\sin x}}$$

$$= \lim_{x \to 0} \frac{-\sin^2 x}{x \cos x} \text{ (is of the form } \frac{0}{0}\text{)}.$$

$$= \lim_{x \to 0} \frac{-2 \sin x \cos x}{(\cos x - x \sin x)}$$

$$= \lim_{x \to 0} \frac{0}{1} = 0$$

Example 6

Determine $\lim_{x \to a} \frac{\ln(x-a)}{\ln(e^x - e^a)}$

Solution

This function is of the form $\frac{\infty}{\infty}$.

If we use $\lim\limits_{x\to a}\frac{\ln(x-a)}{\ln(e^x-e^a)} = \lim\limits_{x\to a}\frac{1/x-a}{e^x/(e^x-e^a)}$

$$= \lim_{x\to a}\frac{e^x-e^a}{e^x(x-a)} \ldots\left(=\frac{0}{0}\right)$$

$$= \lim_{x\to a}\frac{e^x}{e^x(1)+(x-a)e^x}$$

$$= \frac{e^a}{e^a}$$

$$= 1$$

4.7 L' Hospital Rule for the indeterminate form 0 . ∞

If $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=\infty$

then $\lim\limits_{x\to a} f(x)\times xg(x)$ takes the form $0 \times \infty$

We shall write $\overset{\lim}{x\to a} f(x)\times g(x) = \overset{\lim}{x\to a}\frac{f(x)}{1/g(x)} = \overset{\lim}{x\to a}\frac{g(x)}{1/f(x)}$

So the new forms are of the type $\frac{0}{0}$ and $\frac{\infty}{\infty}$ respectively and the limit can therefore be obtained as in 4.5 and 4.6.

Example 7

Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\sec 3x}{\sec 5x}$

Solution

$$\lim_{x\to\frac{\pi}{2}}\frac{\sec 3x}{\sec 5x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos 5x}{\cos 3x}\left(=\frac{0}{0}\,form\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{-5\sin 5x}{-3\sin 3x}$$

$$=\frac{-5(1)}{-3(-1)}$$

$$=-\frac{5}{3}$$

4.8 L' Hospital Rule for the indeterminate form $\infty-\infty$

If $\lim_{x\to a}f(x)=\infty$ and $\lim_{x\to a}g(x)=\infty$

$\lim_{x\to a}[f(x)-g(x)]$ can be transformed into a fraction which assumes either

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

Using 4.5 and 4.6 the limit is obtained.

Example 8

Evaluate $\lim\left(\frac{1}{x}-\cot x\right)$

$x\to 0$

Solution

The limit is of the form $\infty-\infty$

$$\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right) = \lim_{x\to 0}\left[\frac{1}{x}-\frac{\cos x}{\sin x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin x - x\cos x}{x\sin x}\right]\ldots\left(\frac{0}{0}\,form\right)$$

$$= \lim_{x\to 0}\left[\frac{\cos x-\cos x+x\sin x}{x\cos+\sin x}\right]$$ using L'Hospital's Rule, we have

$$= \lim_{x\to 0}\left[\frac{x\sin x}{x\cos x+\sin x}\right]\ldots\left(\frac{0}{0}\,form\right)$$

$$= \lim\left[\frac{x\cos x+\sin x}{\cos x - x\sin x+\cos x}\right]$$
$$x\to 0$$

$$= \frac{0}{2}$$

$$=0$$

4.9 L' Hospital Rule for indeterminate forms $0^0, 1^\infty, \infty^0, 0^\infty$

In the above forms 0 and ∞ occur in the index.

We assume the limit as A and take logarithm on both sides. *l*n A will take the indeterminate form 0 x ∞. We can use 4.7 to evaluate the limit.

Example 9

Evaluate $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$

Solution

This is of the form 1^∞ in which ∞ comes in index

Let A = $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$

ln A = $\lim_{x\to\frac{\pi}{4}} \tan 2x.\ln(\tan x)$ $(\infty\times 0\,form)$

$$= \lim_{x\to\frac{\Pi}{4}} \frac{\ln(\tan x)}{\cot 2x} \ldots \left(\frac{0}{0}\,form\right)$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{\sec^2 x/\tan x}{-2\operatorname{cosec}^2 2x}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{-\sin^2 2x}{2\tan x\cos^2 x}$$

$$= \frac{-(1)}{2(1)(\frac{1}{\sqrt{2}})^2}$$

$= -1$

Thus ln A = -1

Therefore A = $e^{-1} = \frac{1}{e}$

Therefore = $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x} = \frac{1}{e}$

Exercise 4

Determine the limits of the following:

1. $\lim_{x\to 2}\frac{x^2-2x-15}{x^2+5x+4}\ \frac{0}{0}$).

2. $\lim_{x\to 2}\frac{x^2-10x+16}{x^2-5x+16}$

3. $\lim_{x\to 1}\frac{1+\ln x-x}{1-2x+x^2}$

4. $\lim_{x\to 2}\frac{x^3+4x^2-28x+32}{x^3-x^2-8x+12}$

5. $\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}$

6. $\lim_{x\to 0}\frac{\sin x}{x}$

7. $\lim_{x\to 0}\frac{\sin 8x}{x}$

8. $\lim_{x\to\infty}\frac{12x+3}{2x-5}$

==[Hint: $\frac{12x+3}{2x-5}=\frac{12+\frac{3}{x}}{2-\frac{5}{x}}=\frac{12}{2}$

f $x\to\infty$

9. $\lim_{x\to\infty}\frac{6x^2+3x+1}{2x^2-4x+8}$

10. $\lim_{x\to 0}\frac{\ln x}{\cot x}$

11. $\lim_{x\to 0}\frac{\cot 2x}{\cot x}$

12. $\lim_{x\to\infty}\frac{[illegible]}{x^3}$

13. $\lim \frac{1-x}{\cot \pi x}$

14. $\lim_{x\to\pi}\sec 3x\cos 5x$

15. $\lim_{x\to 0}\tan x.\ln x$ [write: $\frac{\ln x}{\cot x}=\frac{\infty}{\infty}$ form. Apply L' Hospital Rule]

16. $\lim\left(\frac{1}{x}-\cot x\right)$ [Hint: write $x\to 0$
$\frac{\sin x-x\cos x}{x\sin x}=\frac{0}{0}$ form].

17. $\lim(\sec x-\tan x)$
$x\to\frac{\Pi}{2}$

18. $\lim\left[\frac{1}{x}-\frac{\ln(1+x)}{x^2}\right]$
$x\to 0$

19. $\lim\left[x\tan x-\frac{\Pi}{2}\sec x\right]$
$x\to\frac{\Pi}{2}$

20. $\lim x^x$
$x\to 0$

21. $\lim_{x \to \infty} x^{\sqrt{x}}$

22. $\lim_{x \to 0} (\cos x)^{\frac{1}{x}}$

23. $\lim_{x \to 0} (\cos x)^{\sin x}$

24. $\lim_{x \to 0} (\cos x)^{\operatorname{cosec}^2 x}$

25. $\lim_{x \to 0} \left(\frac{x}{e^x - 1}\right)^{\frac{1}{x}}$

26. $\lim_{x \to 1} (2 - x)^{\tan\frac{\Pi x}{2}}$

27. $\lim_{x \to \frac{\Pi}{2}} (\sin x)^{\sec^2 x}$

28. $\lim_{x \to 0} (\sin x)^{\tan x}$

29. $\lim_{x \to 0} (\tan x)^{\cos x}$

30. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$

Summary

In this Chapter we have learnt the following:

1. Indeterminate forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

   To evaluate $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms we use conveniently L' Hospital Rule:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^1(x)}{g^1(x)} \text{ if } \frac{f(x)}{g(x)} = \frac{0}{0} or \frac{\infty}{\infty},$$

$$= \lim_{x \to a} \frac{f^{11}(x)}{g^{11}(x)} \text{ if } \frac{f^1(x)}{g^1(x)} = \frac{0}{0} or \frac{\infty}{\infty},$$

$$= \lim_{x \to a} \frac{f^{111}(x)}{g^{111}(x)} \text{ if } \frac{f^{11}(x)}{g^{11}(x)} = \frac{0}{0} or \frac{\infty}{\infty},$$

....

2. To evaluate $\lim_{x \to a}$ f (x). g (x) if this is of the form $0 \times \infty$,

   we write $\lim_{x \to a}$ f (x). g (x) = $\lim_{x \to a} \frac{f(x)}{1/g(x)}$ so that it will be

   of the form: $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

3. To evaluate $\lim_{x \to a}$ [f (x) – g (x)] if this is of the form $\infty - \infty$,

   we write $\lim_{x \to a}$ [f (x) – g (x)] = $\lim_{x \to a} \frac{\phi(x)}{\psi(x)}$ such that $\lim_{x \to a} \frac{\phi(x)}{\psi(x)}$

   will be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then use L' Hospital Rule.

4. When $\lim_{x \to a}[f(x)]^{g(x)}$ takes any of the form $0^0 or 1^\infty or \infty^0 or 0^0$

We assume $\lim_{x \to a}[f(x)]^{g(x)} = k$ so that $\ln k = \lim_{x \to a}[g(x)]\ln(x)$. Right hand side may take any of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\infty - \infty$ from which we can use any of the previous methods namely L' Hospital Rule or method stated in (4.5) or (4.6).

Further Reading

1. Advanced Calculus
By Watson Fulks
University of Colorado
John Wiley and Sons
New York Toronto Singapore.

2. Mathematical Methods for Science Students
By G. Stephenson
Addison Wesley Longman Limited
Edinburg Gate, Harlow, London.

CHAPTER 5

Functions of Many Variables and Partial Differentiation

5.1 Introduction

We have considered till now only functions of one independent variable x and a dependent variable y.
For example consider.

$$y = 2x^2 + 5x + 6$$

Here x is an independent variable and it can take any value. The value of y depends on the value of x:

If $x = 2$, $y = 2(4) + 5(2) + 6 = 24$

If $x = 5$, $y = 2(25) + 5(5) + 6 = 81$

In this Chapter, we extend the ideas to functions of two or more variables and a single dependent variable.

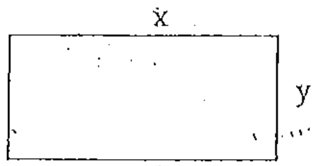
5.2 Objectives of the Chapter:

By the end of this Chapter you should be able to

i). Define functions of many variables.
ii). Define partial derivatives and higher order partial derivatives.
iii). Define homogeneous functions.
iv). Know Euler's theorem on homogeneous functions.

5.3 Functions of many variables

Consider the area of a rectangle.



If the length of a rectangle is x and its breadth is y, then x and y are independent variables that can take any values. The area of a rectangle say z depends on the two variables x and y.

$\therefore$ Therefore $z = f(x,y) = xy$.

When x = 8 cm and y = 5 cm, the area z = 8(5) = 40 cm^2

5.4 Functions of three or more variables

Consider $f(x,y,z) = 2x^2 + 3xy + y^2 + z^2$. This is an example of a function of three variables. For example, the volume of a cuboid of length x, breadth y and height z is given by

$V = f(x,y,z) = xyz.$

Here x,y,z are three independent variables and V is the dependent variable. This idea of three variables can be extended to many variables.

For example if $x_1, x_2, x_3 \ldots x_n$ are n variables then

$u = f(x_1, x_2, \ldots x_n)$ is a function of n variables. It is customary to write this as

$u = u(x_1, x_2, \ldots x_n)$

5.5 Partial differentiation.

Let u = f (x, y) be a function of two variables. The derivative of u with respect to x, when x varies and y remains constant is called the partial differentiation of u with respect to x and is denoted by the symbol, $\frac{\partial u}{\partial x}$.

We may write this symbolically,

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

Similarly when x remains constant and y varies, the partial derivative (or differentiation) of u with respect to y is,

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y,) - f(x,y)}{\Delta y}$$

For example.

If $u = 2x^2 + 3y^2$ find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}2x^2 + \frac{\partial}{\partial x}3y^2$$
$$= 2(2x) + 0 \text{ since, in the term}$$
$$= 4x$$

Here, in $\frac{\partial}{\partial x}3y^2$, there is no x in $3y^2$ and y is treated just as a constant with respect to x.

$$\text{Similarly, } \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2x^2 + 3y^2)$$
$$= \frac{\partial}{\partial y}(2x^2) + 3\frac{\partial}{\partial y}y^2$$
$$= 0 + 3(2y)$$
$$= 6y$$

x is treated as a constant in $\frac{\partial}{\partial y}(2x^2)$.

Example 1.

If $u = 3x^2y + 8xy^3 + 10x^2y^2$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x^2y + 8xy^3 + 10x^2y^2)$$
$$= 3y\frac{\partial x^2}{\partial x} + 8y^3\frac{\partial x}{\partial x} + 10y^2\frac{\partial x^2}{\partial x}$$
$$= 3y(2x) + 8y^3(1) + 10y^2(2x)$$
$$= 6xy + 8y^3 + 20xy^2.$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}(3x^2y + 8xy^3 + 10x^2y^2)$$

$$=3x^2\frac{\partial}{\partial y}y+8x\frac{\partial}{\partial y}y^3+10x^2\frac{\partial}{\partial y}y^2$$

$$=3x^2+8x(3y^2)+10x^2(2y)$$

$$=3x^2+24xy^2+20x^2y$$

Example 2

If u(x,y,z) = $x^3+2y^3+5z^3+8xyz$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$

Solution.

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(x^3+2y^3+5z^3+8xyz)$$

$$=3x^2+8yz$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}(x^3+2y^3+5z^3+8xyz)$$

$$=6y^2+8xz$$

$$\frac{\partial u}{\partial z}=\frac{\partial}{\partial z}(x^3+2y^3+5z^3+8xyz)$$

$$=15z^2+8xy$$

Example 3

If f (x,y) = 2x sin y + 3y^{2}cos x, find $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}[2x\sin y+3y^2\cos x]$$

$$=\sin y\frac{\partial}{\partial x}2x+3y^2\frac{\partial}{\partial x}\cos x$$

$$=2\sin y-3y^2\sin x$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}[2x\sin y+3y^2\cos x]$$

$$=2x\frac{\partial}{\partial y}\sin y+\cos x\frac{\partial}{\partial x}3y^2$$

$$=2x\cos y+6y\cos x$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=2\sin y-3y^2\sin x+2x\cos y+6y\cos x$$

5.6 Higher order partial derivatives

If u = u (x, y), $\frac{\partial u}{\partial x}$ is called first partial derivative of u with respect to x and $\frac{\partial u}{\partial y}$ is called the first partial derivative of u with respect to y.

Suppose $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are differentiable, we can obtain the second partial derivatives.

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \text{ or } \frac{\partial^2 u}{\partial x^2}$$

$\frac{\partial^2 u}{\partial x^2}$ is called the second partial derivative of u with respect to x.

Similarly

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right]=\frac{\partial^2 u}{\partial y^2}$$

is called the second partial derivative of u with respect to y. In the same way, – — — –

$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]$ is written as $\frac{\partial^2 u}{\partial x \partial y}$

$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]$ is written as $\frac{\partial^2 u}{\partial y \partial x}$

generally, if u is a continuous function, then $\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$. This means that the order of differentiation is not important if the function is continuous.

Example 4

If $u=(x^2+y^2+z^2)^2$ find $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$

Solution

let $u=(x^2+y^2+z^2)^2$

Apply function of function rule,

$$\frac{\partial u}{\partial x}=2(x^2+y^2+z^2)\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$=4x(x^2+y^2+z^2)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=4(x^2+y^2+z^2)(1)+8x^2$$

$$\frac{\partial^2 u}{\partial x^2}=4(x^2+y^2+z^2)+8x^2$$

since the function is symmetric,

$$\frac{\partial^2 u}{\partial y^2} = 4(x^2 + y^2 + z^2) + 8y^2$$

$$\frac{\partial^2 u}{\partial z^2} = 4(x^2 + y^2 + z^2) + 8z^2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 20(x^2 + y^2 + z^2)$$

5.7 Homogeneous functions of many variables

i. A function f(x, y) is said to be homogeneous of degree n if $F(k\,x, k\,y) = k^n f(x, y)$.

Thus when we replace x with k x and y with k y, we will get the same function f(x, y) multiplied by k^n, if f(x,y) is a homogeneous function.

ii. Similarly a function f(x,y,z) is said to be homogeneous of degree n if
$f(kx, ky, kz) = k^n f(x,y,z)$

iii. In general a function $f(x_1, x_2, x_3 \ldots x_n)$ is said to be homogeneous of degree n if $f(kx_1, kx_2, kx_3 \ldots kx_n) = k^n f(x_1, x_2, x_3 \ldots x_n)$

Example 5

Show that $2x^3 + 3x^2y - 2xy^2$ is a homogeneous function of degree 3.

Solution

$$f(x,y) = 2x^3 + 3x^2 - 2xy^2$$

$$f(kx,ky) = 2k^3x^3 + 3k^2x^2ky - 2kxk^2y^2$$

$$= 2k^3x^3 + 3k^3x^2y - 2k^3xy^2$$

$=k^3(2x^3 + 3x^2y - 2xy^2)$

$=k^3\, f(x,y)$

Hence $2x^3 + 3x^2y - 2xy^2$ is a homogeneous function of x and y of degree 3.

Example 6

Show that $2x^3 + 3y^2z - 5y^3 + 9xyz + z^3$ is a homogeneous function. State its degree.

Solution

Let $f(x,y,z) = 2x^3 + 3y^2z - 5y^3 + 9xyz + z^3$.

Put $x \to kx$, $y \to ky$, $z \to kz$

$$f(kx,ky,kz) = 2(kx)^3 + 3(ky)^2kz - 5(ky)^3 + 9kx.kykz + (kz)^3$$

$$= 2k^3x^3 + 3k^3y^2z - 5k^3y^3 + 9k^3xyz + k^3z^3$$

$$= k^3\left[2x^3 + 3y^2z - 5y^3 + 9xyz + z^3\right]$$

$$= k^3 f(x,y,z)$$

Hence the given function is homogeneous of degree 3.

Example 7

Show that $\dfrac{\sqrt{x}+\sqrt{y}}{x+y}$ is a homogeneous function of degree - ½

Solution.

Let $f(x,y) = \dfrac{\sqrt{x}+\sqrt{y}}{x+y}$

Replacing x by kx, y by ky

$$F(kx, ky) = \frac{\sqrt{kx}+\sqrt{ky}}{kx+ky}$$

$$= \frac{\sqrt{k}\left(\sqrt{x}+\sqrt{y}\right)}{k(x+y)}$$

$$= k^{-\frac{1}{2}}\,\frac{\left(\sqrt{x}+\sqrt{y}\right)}{k(x+y)}$$

$$= k^{-\frac{1}{2}} f(x,y)$$

Hence by definition f(x,y) is homogeneous of degree - ½

Example 8

Show that $\frac{x^2+y^2}{x+y}$ is a homogeneous function of degree 1

Solution

Consider $f(x,y) = \frac{x^2+y^2}{x+y}$

$$f(kx,ky) = \frac{k^2x^2+k^2y^2}{kx+ky}$$

$$= \frac{k^2\left(x^2+y^2\right)}{k(x+y)}$$

$$= k^1\left(\frac{x^2+y^2}{x+y}\right)$$

$$= k^1 f(x,y)$$

Hence $\frac{x^2+y^2}{x+y}$ is a homogeneous function of degree 1

5.8 Euler's theorem on homogeneous functions

Euler is a famous French Mathematician who has contributed a lot to mathematics. He is the one who introduced the notation $e^{i\theta}$ for $\cos\theta + i\sin\theta$ in complex analysis. Euler has discovered a famous theorem in partial differentiation, which we shall see now.

If f(x,y) is a homogeneous differentiable function of degree n then,

i. $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

ii. $$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

Example 9

Verify Euler's Theorem on the function $f(x,y) = 2x^3 + 3x^2y - 4y^3$

Solution

$$f = f(x,y) = 2x^3 + 3x^2y - 4y^3$$

$$\frac{\partial f}{\partial x} = 6x^2 + 6xy; \quad \frac{\partial^2 f}{\partial x^2} = 12x + 6y$$

$$\frac{\partial f}{\partial y} = 3x^2 - 12y^2, \quad \frac{\partial^2 f}{\partial y^2} = -24y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6x$$

it is verified that f is a homogeneous function

Solution

Consider f(x,y) = $\dfrac{x^3+y^3}{x+y}$

$$f(kx,ky) = \frac{kx^3+ky^3}{kx+ky}$$

$$= \frac{k^3}{k}\left[\frac{x^3+y^3}{x+y}\right]$$

Hence $\dfrac{x^3+y^3}{x+y}$ is a homogeneous function of degree n = 2 and so also

$\ln\dfrac{x^3+y^3}{x+y}$

By Euler's Theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = 2\frac{x^3+y^3}{x+y}$$

Example 11

a. Show that f(x,y) = $\dfrac{\sqrt{x}+\sqrt{y}}{y+x}$ is homogeneous of degree -½

b. Find $x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}$

c. Determine $x^2\dfrac{\partial^2 f}{\partial x^2}+2xy\dfrac{\partial^2 f}{\partial x\partial y}+y^2\dfrac{\partial^2 f}{\partial y^2}$

Solution.

a. f(x,y) = $\dfrac{\sqrt{x}+\sqrt{y}}{x+y}$ is homogeneous of degree -½ (see example 7)

b. By Euler's Theorem on homogeneous functions, Now

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=x(6x^2+6xy)+y(3x^2-12y^3)$$

$$=6x^3+6x^2y+3x^2y-12y^3$$

$$=6x^3+9x^2y-12y^3$$

$$=3(2x^3+3x^2y-4y^3)$$

$$=3f$$

$$=nf$$

according to Euler's Theorem.

ii. $$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}$$

$$=x^2[12x+6y]+2xy[6x]+y^2[-24y]$$

$$=12x^3+6x^2y+12x^2y-24y^3$$

$$=12x^3+18x^2y-24y^3$$

$$=6(2x^3+3x^2y-4y^3)$$

$$=3x2\ f(x,y)$$ since n=3, n(n-1) = 6

Hence Euler's Theorem on homogeneous function is verified.

Example 10

a. Show that f(x,y) = $\frac{x^3+y^3}{x+y}$ is homogeneous of degree 2.

b. Without differentiation find,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = nf$$

$$= -½\,\frac{\sqrt{x}+\sqrt{y}}{x+y}$$

c. $$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

$$=\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{\sqrt{x}+\sqrt{y}}{x+y}$$

$$=\frac{3}{4}\,\frac{\sqrt{x}+\sqrt{y}}{x+y}$$

Example 12

If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u$.

Solution

Let us avoid the inverse function

$$u = \tan^{-1}\frac{x^3+y^3}{x-y} \Rightarrow \frac{x^3+y^3}{x-y} = \tan u$$

$$f = \tan u = \frac{x^3+y^3}{x-y}$$

since $\frac{x^3+y^3}{x-y}$ is a homogeneous function of order 2,

Hence $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = n.f$

$$x\frac{\partial \tan u}{\partial u} + y\frac{\partial \tan u}{\partial u} = 2.\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + \sec^2 u\frac{\partial u}{\partial x} = 2\tan u$$

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\sec^2 u = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u}$$

$$= 2\frac{\sin u}{\cos u} \times \cos^2 u$$

$$= 2\sin u\cos u$$

$$= \sin 2u$$

Exercise 5

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in the following cases:

   a. $u = x^2 + y^2$

   b. $u = 2x^3 + 3x^2 y + 5y^3$

   c. $u = e^{2x}\sin 3y$

   d. $u = e^{x-y}$

2. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ in the following cases:

   a. $f(x, y) = 5x^4 + 3y^4 - 2x^2y^3$

   b. $f(x, y) = e^{x+y}$

3. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ where $u = u(x, y) = \ln(y \sin x + x \sin y)$.

4. If $f(x, y, z) = 2x^3 - 5x^2y + 3xy^2 + y^3 + z^3$

a. Show that $f(x, y, z)$ is homogenous of degree 3.

b. Using Euler's theorem or otherwise find

i). $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z}$

ii). $x^2\dfrac{\partial^2 f}{\partial x^2} + y^2\dfrac{\partial^2 f}{\partial y^2} + z^2\dfrac{\partial^2 f}{\partial z^2} + 2xy\dfrac{\partial^2 f}{\partial x\partial y} + 2yz\dfrac{\partial^2 f}{\partial y\partial z} + \dfrac{2zx\partial^2 f}{\partial x\partial z}$

5. If $u = \sin^{-1}\dfrac{x^2 + y^2}{x + y}$

show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

[Hint: $\sin u = \dfrac{x^2 + y^2}{x + y}$. Let $f = \sin u = \dfrac{x^2 + y^2}{x + y}$ be homogeneous of order 1. Hence $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 1f$

$$x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = \sin u,$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos\frac{\partial u}{\partial y} = \sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}$$]

6. If $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$, show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u[1 - 4\sin^2 u].$$

7. If $u = \ln(x^2 + y^2 + z^2)$ show that

$$x\frac{\partial^2 u}{\partial y\partial z} = y\frac{\partial^2 u}{\partial z\partial x} = z\frac{\partial^2 u}{\partial x\partial y}$$

8. If u = $\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

9. If z = $\tan^{-1}\frac{y}{x}$, show that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$

10. If u = $\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$ verify Euler's Theorem.

Summary

In this Chapter we have learnt the following:

1. $u = f(x)$

   $u = f(x, y)$

   $u = f(x, y, z)$

   $u = f(x_1, x_2, x_3 \ldots\ldots\ldots x_n$ are examples of functions of one, two, three and many variables.

2. The definition of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, are seen.

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y)}{\Delta y}$$

3. Higher derivatives were explained

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\partial^2 u}{\partial x}\right]$$

4. If $f(x, y, z)$ is any function and if $f(kx, ky, kz) = k^n f(x, y, z)$ then $f(x, y, z)$ is called a homogeneous function of degree n.

5. If $f(x,y)$ is a homogeneous differentiable function of degree n then,

   i. $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

   ii. $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} = n(n-1)f$

Further Reading

1. Advanced Calculus
   By Watson Fulks
   University of Colorado
   John Wiley and Sons
   New York Tolonto Singapore.

2. Mathematical Methods for Science Students
   By G. Stephenson
   Addison Wesley Longman Limited
   Edinburg Gate, Harlow, London.

CHAPTER 6
Maxima and Minima of Functions of Many Variables

6.1 Introduction

You have studied in the previous units the maximum and the minimum values of a function of a single independent variable x. If y = f(x) is a function of x, you know that $f'(x) = 0$ gives the values of x at which there may be maximum or minimum values for the function f (x).

In this chapter, we shall study the maximum or the minimum values of a function of many variables such as x, y, z and so on or the variables may be $x_1, x_2, x_3 \ldots\ldots x_n$.

If u = f(x, y, z), we wish to find the values of x, y, z for which u will be local maximum or a minimum.

Maximum or minimum of a function of many variables occur in many fields in our every day life. For example a company manufacturing a particular product (a car, a television, a radio) has different categories of cost such as the raw material cost (x), the labour cost (y) and the overhead cost (z). The profit for the company depends on the selling price of the product and the total cost which consist of many variables x, y, z and so on. Every company expects that the profit should be maximum and the total cost should be minimum. Hence there is a need for finding the maximum and the minimum for functions of many variables.

6.2 Objectives of the Chapter

By the end of this Chapter, you should be able to ...

- find the maximum or the minimum values of a function of many variables.
- find the maximum or minimum values of a function when there are additional constraints using **Lagrange's Multipliers.**

6.3 Conditions for Maximum or Minimum values of the function f(x, y).

Let $u = f(x, y)$ be a function of two independent variables x and y. First, we state the condition required for the existence of maximum or minimum values.

The following two conditions are necessary and sufficient for f(x, y) to attain an extreme value (maximum or minimum value).

i). $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$, **at a point x = a, y = b.**

ii) $\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ **at the same point**

6.4 To distinguish between maximum or minimum values.

a) If $\frac{\partial^2 f}{\partial x^2}$ = **negative** (-ve) at x =a, y = b; **f(x, y)** attains a **maximum** value at x = a, y = b.

b) If $\frac{\partial^2 f}{\partial x^2}$ = **positive** (+ve) at x = a, y = b, **f(x, y)** attains a **minimum** value at x = a, y = b.

You must be careful in the following cases.

Case 1

If $\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) < \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ there will be no maximum or minimum value for **f(x, y)** even though $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

Case 2

If $\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

Then nothing can be said about the maxima or minima. It requires further investigation.

Example 1

Find the maximum or the minimum values of the function $2(x^2 - y^2) - x^4 + y^4$

Solution

Let ; $f = 2(x^2 - y^2) - x^4 + y^4$

$f = 2x^2 - 2y^2 - x^4 + y^4$

Now

$$\frac{\partial f}{\partial x} = 4x - 4x^3$$

$$\frac{\partial f}{\partial y} = -4y + 4y^3$$

$$\frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = -4 + 12y^2$$

Two conditions are necessary for the existence of maxima or minima.

Condition 1

$\partial f/\partial x = 0$

$4x - 4x^3 = 0$
$4x(1- x^2) = 0$
$4x (1 + x) (1 - x) = 0$
$= 0$

$x = 0, x = 1, x = -1$
1.

$\partial f/\partial y = 0$

$-4y + 4y^3 = 0$
$4y(y2 - 1) = 0$
$4y(y + 1) (y - 1)$

$y = 0, y = 1, y = -1$

Here x takes 3 values and y can take 3 values.
Hence there are 3 x 3 pairs of values to be examined for maximum or minimum values. They are;
(x, y) = (0, 0), (0, 1), (0, -1), (1, 0), (1, 1), (1, -1), (-1, 0), (-1, 1), (-1, -1)

Condition 2

$(\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2) > (\partial^2 f/\partial x\partial y)^2$

(2)

$(4 - 12x^2)(-4 + 12y^2) > (0)^2$

$16(1 - 3x^2)(3y^2 - 1) > 0$ ....

(3)

At x = 0, y = 0, (3) becomes –16 > 0 is not satisfied
x = 0, y =1, (3) becomes 32 > 0 is satisfied
x = 0, y = -1, (3) becomes 32 > 0 is satisfied
x = 1, y = 0, (3) becomes 32 > 0 is satisfied
x = 1, y = 1, (3) becomes – 64 > 0 is not satisfied
x = 1, y = -1, (3) becomes – 64 > 0 is not satisfied

$x = -1,\ y = 0$, (3) becomes $32 > 0$ is satisfied

$x = -1,\ y = 1$, (3) becomes $-64 > 0$ is not satisfied

$x = -1,\ y = -1$, (3) becomes $-64 > 0$ is not satisfied

The function attains maximum or minimum only at;

(0, 1), (0, -1), (1, 0), (-1, 0).

Now, let us check for maximum or minimum points

$$\partial^2 f/\partial x^2 = 4 - 12x^2$$

If $x = 0,\ y = 1$, $\partial^2 f/\partial x^2 = 4$ which is positive

Then (0, -1) is a minimum point for the function

If $x = 0,\ y = -1$, $\partial^2 f/\partial x^2 = 4$ which is positive

Then (0, 1) is also a minimum point for the function

If $x = 1,\ y = 0$, $\partial^2 f/\partial x^2 = -8$ which is negative

Then (1, 0) is a maximum point for the function

If $x = -1,\ y = 0$, $\partial^2 f/\partial x^2 = -8$ which is negative

Then (-1, 0) is a maximum point for the function

Conclusion:

For the function;

$$f(x, y) = 2x^2 - 2y^2 - x^4 + y^4,$$

(0, 1), (0, -1) are the minimum points and the minimum values of the function at these points are -1 and -1 respectively.

(1, 0), (-1, 0) are the maximum points and the maximum values of the function at these points are 1 and 1 respectively.

Example 2

Find the maximum or minimum values of
$xy(6 - x - y)$

Solution

Let $f = xy(6 - x - y)$.

$f = 6xy - x^2y - xy^2$

$\partial f/\partial x = 6y - 2xy - y^2 = y(6 - 2x - y)$

$\partial f/\partial y = 6x - x^2 - 2xy = x(6 - 2y - x)$

$\partial^2 f/\partial x^2 = -2y$

$\partial^2 f/\partial y^2 = -2x$

$\partial^2 f/\partial x\partial y = 6 - 2x - 2y = 2(3 - x - y)$

Two conditions are necessary for the existence of the maxima or minima.

Condition 1

$\partial f/\partial x = 0$ $\quad\quad$ $\partial f/\partial y = 0$

$y(6 - 2x - y) = 0$ $\quad\quad$ $x(6 - 2y - x)$

$y = 0$ or $6 - 2x - y = 0$ $\quad\quad$ $x = 0$, or 6

$- 2y - x = 0$

We must solve the simultaneous equations,

$6 - 2x - y = 0$ and $6 - 2y - x = 0$

On solving $2x + y = 6$ and $x + 2y = 6$, $x = 2$, $y = 2$.

$x = 0, x = 2; y = 0, y = 2$

Hence there are 2x2 points. They are

$(x, y) = (0, 0), (0, 2), (2, 0)$ and $(2, 2)$

Condition 2

$(\partial^2 f/\partial x^2)\ (\partial^2 f/\partial y^2) > (\partial^2 f/\partial x\partial y)^2$

(2)

$(-2y)\ (-2x) > 2^2\ (3 - x - y)^2$

$4xy > 4(3 - x - y)^2$

(3)

At the point (0, 0) equation (3) becomes $0 > 4$, not satisfied

At the point (0, 2) equation (3) becomes $0 > 4$, not satisfied

At the point (2, 0) equation (3) becomes $0 > 4$, not satisfied

At the point (2, 2) equation (3) becomes $16 > 4$, satisfied.

Hence, there is only one point (2, 2), which can be either a maximum or a minimum.

$\partial^2 f/\partial x^2 = -2y$

at the point (2, 2), $\partial^2 f/\partial x^2 = -4$, which is negative.
Then (2, 2) is a maximum point.

Conclusion

The function $xy(6 - x - y)$ has only one maximum point at (2, 2) and there is no minimum point for the function.

The maximum value of the function at (2, 2) is $(2)\ (2)\ (6 - 2 - 2) = 8$

6.5 Maxima and Minima subject to constraints

Sometimes we have to determine the maxima or minima values of a function $f(x, y)$ subject to certain constraints. Such problems are solved using Lagrange multipliers. The method is explained in the following:

Lagrange multipliers

Suppose $f(x, y)$ is to be examined for stationary points subject to the constraints,

$g(x, y) = 0$

For $f(x, y)$ to be stationary, $\partial f = 0$

$df = \partial f/\partial x\,(dx) + \partial f/\partial y\,(dy) = 0$

(2)

$dg = \partial g/\partial x\,(dx) + \partial g/\partial y\,(dy) = 0$, since $g(x, y) = 0$

(3)

Multiply equation 3 by Lagrange multiplier (λ),

$\lambda dg = \lambda \partial g/\partial x\,(dx) + \lambda \partial g/\partial y\,(dy) = 0$,

(4)

Adding (2) and (4),

$d(f + \lambda g) = (\partial f/\partial x + \lambda \partial g/\partial x)\,dx + (\partial f/\partial y + \lambda \partial g/\partial y)\,dy = 0$

We choose λ such that

$\partial f/\partial x + \lambda \partial g/\partial x = 0$

(5)

then automatically,

$\partial f/\partial y + \lambda \partial g/\partial y = 0$

(6)

The three unknown values x, y and λ are obtained from the three equations
(1), (5) and (6).

This method of obtaining the maximum or the minimum values with constrains is called **Lagrange's method of undetermined multipliers.**

Example 3

Find the stationary points of $f(x, y) = x^2 + y^2$ subject to the constraint
$3x + 2y = 6.$

Solution

Let $f = x^2 + y^2$

$g = 3x + 2y - 6 = 0.$

$df = \partial f/\partial x\,(dx) + \partial f/\partial y\,(dy) = 2xdx + 2ydy = 0$

$dg = \partial g/\partial x\,(dx) + \partial g/\partial y\,(dy) = 3\partial x + 2dy = 0$

$df + \lambda dg = (2x + 3\lambda)dx + (2y + 2\lambda)dy = 0$

Solving the three equations,

$2x + 3\lambda = 0$ (1)

$2y + 2\lambda = 0$ (2)

$3x + 2y - 6 = 0$ (3)

for the three unknowns x, y, λ, we get

from (1) $x = -3/2\lambda$

from (2) $y = -\lambda$

Substitute these values of x and y in equation (3)

$3(-3/2\lambda) + 2(-\lambda) - 6 = 0$

$\lambda = -12/13$

Hence,

$x = -3/2\lambda = 18/13$

$y = -\lambda = 12/13$

therefore, the extreme point is (18/13, 12/13).

$\partial^2 f/\partial x^2 = 2$, which is positive.

Hence, the point is a **minimum.**

$f = x^2 + y^2$

$\partial f/\partial x = 2x$, $\partial f/\partial y = 2y$, $\partial^2 f/\partial x^2 = 2$, $\partial^2 f/\partial y^2 = 2$, $\partial^2 f/\partial x\partial y = 0$

$(\partial^2 f/\partial x^2)$ $(\partial^2 f/\partial y^2)$ should be greater than $(\partial^2 f/\partial x\partial y)^2$

$2 \times 2 > 0$, hence satisfied.

Therefore the point (18/13, 12/13) could be either maximum or minimum.

Now $\partial^2 f/\partial x^2 = 2$ which is positive.

Hence, (18/13, 12/13) is a minimum point.

The minimum value of the function $f(x, y) = x^2 + y^2$ subject to the condition

$3x + 2y - 6 = 0$ is $(18/13)^2 + (12/13)^2 = 36/13$.

Example 4

Find the minimum value of $x^2 + y^2 + z^2$ when $x + 2y + 4z = 21$ is the constraint.

Solution

Let $f = x^2 + y^2 + z^2$

$g = x + 2y + 4z - 21 = 0$

$\partial f = \partial f/\partial x\ (dx) + \partial f/\partial y\ (dy) + \partial f/\partial z\ (dz) = 2xdx + 2ydy + 2zdz = 0$

$\partial g = \partial g/\partial x\ (dx) + \partial g/\partial y\ (dy) + \partial g/\partial z\ (dz) = \partial x + 2dy + 4dz = 0$

$\partial f + \lambda\partial g = (2x + \lambda)dx + (2y + 2\lambda)dy + (2z + 4\lambda)dz = 0$

We must solve for x, y, z and λ from the equations:

$2x + \lambda = 0$

(1)

$2y + 2\lambda = 0$

(2)

$2z + 4\lambda = 0$

(3)

$x + 2y + 4z - 21 = 0$

(4)

From (1), (2) and (3),

$x = -\lambda/2$

$y = -\lambda$

$z = -2\lambda$

Substitute these values of λ in equation (4)

$-\lambda/2 - 2\lambda - 8\lambda - 21 = 0$

$\lambda = -2$

Hence,

$x = 1$

$y = 2$

$z = 4$

therefore, the extreme point may be (1, 2, 4).

if,

$(\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2) > (\partial^2 f/\partial x \partial y)^2$

$\partial^2 f/\partial x^2 = 2$, $\partial^2 f/\partial y^2 = 2$, $\partial^2 f/\partial x \partial y = 0$

$2 \times 2 > 0$, the condition is satisfied.

Now, $\partial^2 f/\partial x^2 = 2$, which is positive.

Hence (1, 2, 4) is a **minimum** point.

The minimum value of $f = x^2 + y^2 + z^2$ when $x = 1$, $y = 2$, $z = 4$ is 21 units.

Exercise 6

1 (a) Let $f = f(x, y)$ be a function of two independent variables. State two conditions required for the function f to have extreme values.

b) Explain how you will distinguish between the maximum and minimum point of the function f(x, y).

Examine the following for maxima and minima values

2. $xy(12 - x - y)$

3. $24xy - x^2y - xy^2$

4. $4x^2 + 6xy + 9y^2 - 8x - 24y + 4$

5. $y^2 + 2yx^2 + 4x - 3$

6. $x^3 + y^3 - 9xy$

7. Find the stationary points of $f(x, y) = 2(x^3 + y^3) - 3(x^2 + y^2) + 1$, examining
whether they are maxima or minima

8. Find the maximum value of $x^2y^2z^2$ subject to the restriction $x^2 + y^2 + z^2 = 25$

9. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $x + 2y + 4z = 42$.

10. Find the minimum value of $x^2 + y^2 + z^2$ subject to the constraint $x - 2y + 5z = 60$

11. Find the minimum sum of the three positive numbers x, y and z whose product is 216.

12. The temperature T at any point (x, y, z) in space is $T = 3xyz^2$. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = 4$

13. Find the minimum value of $x^2 + y^2 + z^2$ when $xy + yz + zx = 75$

14. If $x^2 + y^2 + z^2 = 25$, show that the maximum value of $yz + zx + xy$ is 25 and the minimum value is -12.5.

Summary

In this chapter, we have learnt the following:

Conditions for maximum and minimum values of a function:

If f(x, y) is a function, it must satisfy two conditions to have a maximum or minimum point.

(i) $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ **at a point x = a, y = b.**

(ii) $\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ **at the same point**

To distinguish between maximum and minimum values of the function:

a) If $\frac{\partial^2 f}{\partial x^2}$ = negative (-ve) at x =a, y = b; f(x, y) attains a maximum value at x = a, y = b.

b) If $\frac{\partial^2 f}{\partial x^2}$ = positive (+ve) at x =a, y = b; f(x, y) attains a minimum value at x = a, y = b.

Lagrange multipliers

If f(x, y) is to be examined subject to the constraint

g (x, y) = 0 (1)

we use

$$df = \partial f/\partial x\,(dx) + \partial f/\partial y\,(dy) = 0$$

$$dg = \partial g/\partial x\,(dx) + \partial g/\partial y\,(dy) = 0$$

Hence

$$d(f + \lambda g) = (\partial f/\partial x + \lambda \partial g/\partial x)\,dx + (\partial f/\partial y + \lambda \partial g/\partial y)\,dy = 0$$

Using

$$\partial f/\partial x + \lambda \partial g/\partial x = 0 \quad (2)$$

$$\partial f/\partial y + \lambda \partial g/\partial y = 0 \quad (3)$$

we solve the three unknown values x, y and λ from the three equations
(1), (2) and (3). (x, y) will be either a maximum or a minimum value.

Further Reading

1. Advanced Calculus
   By Watson Fulks
   University of Colorado
   John Wiley and Sons
   New York Tolonto Singapore.

2. Mathematical Methods for Science Students
   By G. Stephenson
   Addison Wesley Longman Limited
   Edinburg Gate, Harlow, London.

CHAPTER 7

Multiple Integrals

7.1 Introduction

In the previous courses in Calculus you have studied integrals of real-valued functions of one variable x. we now study integrals of real- valued functions of many variables. In this Chapter we study mainly the double and triple integrals which can be extended to any number of variables $x_1, x_2, x_3, \ldots x_n$

7.2 Objectives of the Chapter

By the end of this Chapter you should be able to;

(i) define a double and triple integral.

(ii) evaluate double and triple integrals.

(iii) apply the double and triple integrals for finding the areas and volumes of bounded regions.

7.3 Double integrals

Suppose $f(x,y)$ is continuous and single valued function of x and y inside and on the boundary **C** of a rectangular region R of the xy- plane. Let R be bounded by the lines x =a, x =b, y =c y =d; a,b,c,d being constants (see fig. 7.1)

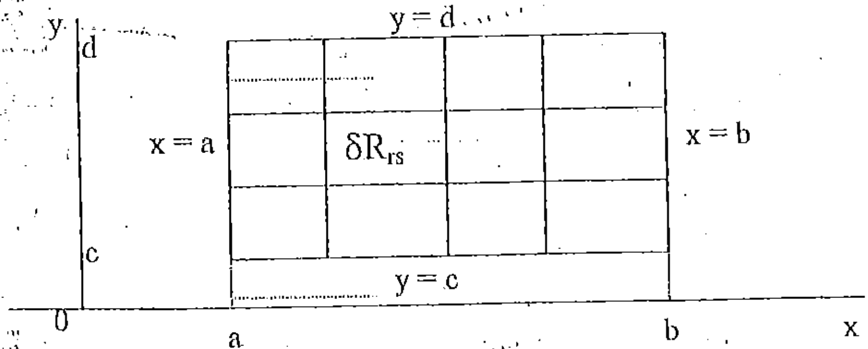


Fig. 7.1

If the interval [a,b] is divided into m parts such that,

$a(=x_0) < x_1 < x_2 \ldots x_{m-1} < b(=x_m)$

and the interval [c,d] is divided into n parts such that,

$c(=y_0) < y_1 < y_2 \ldots < y_{n-1} < d(=y_n)$

The region **R** is divided into $m\,n$ rectangular elements of area

δR_{rs}

$(r = 1,2 \ldots m,\ s = 1,2 \ldots n)$

The area $\delta R_{rs} = (x_r - x_{r-1})(y_s - y_{s-1}) = \delta x_r\, \delta y_s$

We now choose an arbitrary point with coordinates (x,y) lying in the **rs** th rectangle such that,

$$x_{r-1} \le x \le x_r$$

$$y_{s-1} \le y \le y_s$$

Now $f(x,y)\delta R_{rs}$ is very small since δR_{rs} is small.

Consider the double sum

$$\sum_{r=1}^{m}\sum_{s=1}^{n} f(x,y)\,\delta R_{rs}$$

If the limit of this sum as m and n both tend to infinity, is independent of the way of subdividing [a,b] and [c,d] and also the way of choosing the point (x,y) inside δR_{rs} where $\delta R_{rs} \to 0$ then

$$I = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{r=1}^{m}\sum_{s=1}^{n} f(x,y)\,\delta R_{rs}$$

is called the double integral of the function f over the region R and is written as

$$I = \int_R f(x,y)\,dR_{rs} = \iint_R f(x,y)\,dx\,dy$$

7.4 Triple integrals

In 7.3, the double integral over a bounded region R in two dimension is defined. We can extend the same idea to a bounded region R in three dimension. Suppose f(x,y,z) is a continuous and single valued function of x, y and z bounded inside and on the boundary S of a region R in three dimension.

Let R be bounded by the planes

$x = a_1,\ x = a_2,\ y = b_1,\ y = b_2,\ z = c_1,\ z = c_2$

$a_1, a_2, b_1, b_2, c_1, c_2$ being constants.

If the interval $[a_1, a_2]$, $[b_1, b_2]$ $[c_1, c_2]$ are divided into m, n and q parts respectively such that

$a_1 (= x_0) < x_1 < x_2 \ldots x_{m-1} < x_m (= a_2)$

$b_1 (= y_0) < y_1 < y_2 \ldots y_{n-1} < y_n (= b_2)$

$c_1 (= z_0) < z_1 < z_2 \ldots z_{q-1} < z_q (= c_2)$

The region R is divided into mnq volume elements of volume δR_{rst}

$(r = 1, 2, 3, \ldots, m,\ s = 1, 2, 3, \ldots n,\ t = 1, 2, 3, \ldots q)$

As in double integral,

$f(x,y,z)\, \delta R_{rst}$ is very small.

Consider

$$\sum_{r=1}^{m} \sum_{s=1}^{n} \sum_{t=1}^{q} f(x,y,z)\delta R_{rst}$$

If the limit of the sum as m, n, q tend to infinity, is independent of the way of subdividing $[a_1, a_2]$, $[b_1, b_2]$ $[c_1, c_2]$ and also the way of choosing the point inside δR_{rst} then

$$I = \lim_{\substack{m\to\infty \\ n\to\infty \\ q\to\infty}} \sum_{r=1}^{m}\sum_{s=1}^{n}\sum_{t}^{q} f(x, y, z)\delta R_{rst}$$

is called the triple integral of the function f(xyz) over the region R and is written as

$$I = \int_R f(x, y, z)dR = \iiint_R f(x, y, z)dxdydz$$

where $[a_1 < x < a_2]$, $[b_1 < y < b_2]$ $[c_1 < z < c_2]$

Worked examples

Example 1

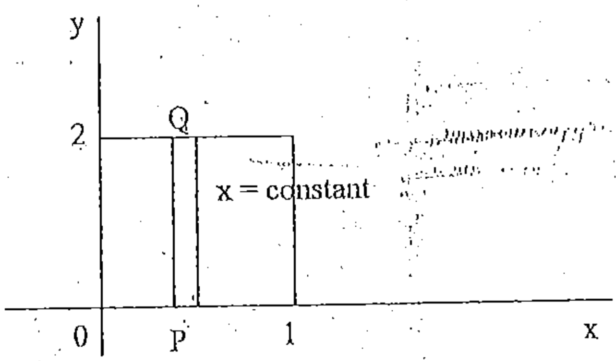
Evaluate the double integral

$$\iint_R (6x^2 + 3y^2 + 2)dxdy$$

where R is the region bounded by the lines

$0 \le x \le 1$, $0 \le y \le 2$

Solution



First we integrate the function with respect to y along the strip PQ treating x as constant and next we integrate with respect to x so that PQ moves from left to right to cover the whole of the region R.

Hence

$$I = \int_{x=0}^{1} dx \int_{y=0}^{2} (6x^2 + 3y^2 + 2)dy$$

$$= \int_{x=0}^{1} dx \left[6x^2 y + \frac{3y^3}{3} + 2y \right]_{y=0}^{2}$$

$$= \int_{x=0}^{1} dx \left[12x^2 + 8 + 4 \right]$$

$$= \int_{x=0}^{1} (12x^2 + 12)dx$$

$$= \left[\frac{12x^3}{3} + 12x \right]_{x=0}^{1}$$

$$= (4 + 12) - (0)$$

$$= 16$$

Example 2

Evaluate the double integral

$$\iint_R \frac{x}{y} dxdy$$

where R is the rectangle $|x| \leq 1,\ 1 \leq y \leq 2$

Solution

First we integrate the function with respect to y (we can take in any order if the boundaries are constants. If the

boundaries are not constants we cannot take in any order)
First we can integrate (say) with respect to y treating x as constant. Then

$$I = \int_{x=-1}^{1} dx \int_{y=1}^{2} \frac{x}{y} dy$$

$$= \int_{x=-1}^{1} dx [x \ln y]_{y=1}^{2}$$

$$= \int_{x=-1}^{1} dx (x\ln 2 - x\ln 1)$$

$$= \int_{x=-1}^{1} x\ln 2 dx$$

$$= \ln 2 \int_{x=-1}^{1} x dx$$

$$= \ln 2 \left[\frac{x^2}{2}\right]_{x=-1}^{1}$$

$$= \ln 2 \left[\frac{1}{2} - \frac{1}{2}\right]$$

$$= 0$$

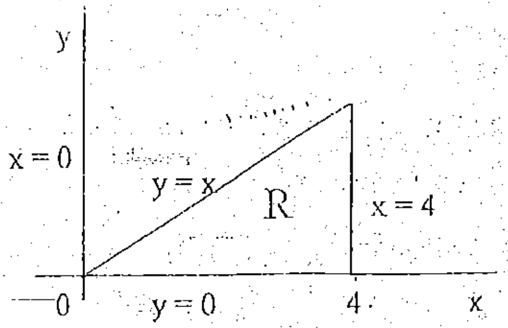
7.5 Double and triple integrals when the boundaries are variables

When the boundaries of the double or triple integrals are not constants, the order of integration is very important. If the boundaries of y contains variables x we must first integrate the function with respect to y and then only with respect to x.

Example 3

Consider

$$\int_{x=0}^{4}\int_{y=0}^{x}(x^2+y^2)dxdy$$



Solution

Here the boundary of y contains the variable x. hence we must first integrate with respect to y treating x as constant and then we must integrate with respect to x whose boundaries are constants.

The region R consists of the boundaries

x = 0 to x = 4

y = 0 to y = x

Then

$$I = \int_{x=0}^{4} dx \int_{y=0}^{x}(x^2+y^2)dy$$

$$= \int_{0}^{4} dx\left[x^2 y + \frac{y^3}{3}\right]_0^x$$

$$= \int_{x=0}^{4}\left[(x^3 + \frac{x^3}{3})\right]dx$$

$$= \int_{x=0}^{4} \left(\frac{4x^3}{3}\right)dx$$

$$= \left[\frac{x^4}{3}\right]_0^4$$

$$= \frac{256}{3}$$

Example 4

Consider

$$I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy^2\,dxdy$$

a) sketch the region R

b) state the order of integration, giving reason.

c) Evaluate I

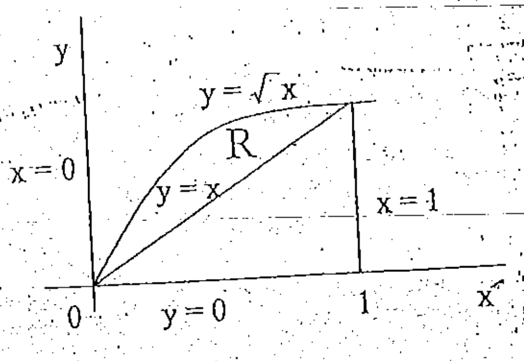
Solution

a) the lower limit (boundary) for y is the straight line $y = x$.

the upper boundary for y is $y = \sqrt{x}$ or $y^2 = x$

the upper and lower limits for x are $x = 0$ and $x = 1$.

Hence the region R is shown in figure below:



b) we must integrate with respect to y first since the boundaries of y contain the variable x.

c) $I = \int_{x=0}^{1} dx \int_{y=x}^{\sqrt{x}} xy^2\, dy$

$$= \int_{x=0}^{1} dx \left[\frac{xy^3}{3}\right]_{y=x}^{y=\sqrt{x}}$$

$$= \int_{x=0}^{1} dx \left[\frac{x.x^{\frac{3}{2}}}{3} - \frac{x.x^3}{3}\right]$$

$$= \left[\frac{2x^{\frac{7}{2}}}{7\times 3} - \frac{x^5}{15}\right]_0^1$$

$$= \frac{2}{21} - \frac{1}{15}$$

$$= \frac{1}{35}$$

7.6 Application of multiple integrals

Double integrals are very useful in evaluating areas and volumes.

For example $\iint_R dxdy$ represents the area of R since dxdy is the area of the infinitesimal rectangular element in R.

Similarly if f (x, y) is a continuous and single valued function of x and y **the volume under the surface z = f (x, y)** in a three dimensional Cartesian Coordinate system and standing vertically above a region R in xy – plane is equals to

$$V = \iint_R z\,dxdy = \iint_R f(x, y)\,dxdy$$

(Volume above the xy – plane being counted as positive and those below negative).

7.7 Transformation to polar coordinates in double integral.

Generally it is convenient, sometimes, to transform the Cartesian Coordinates (x, y) to Polar coordinates (r cos θ , r sin θ). In this case the elementary area dxdy becomes r dθ dr. Hence $\iint_{Rxy} f(x,y)\,dxdy = \iint_{Rr\theta} f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr$

Example 6

Evaluate $I = \iint_R \left(1-\sqrt{x^2+y^2}\right)dxdy$ where R is the region bounded by the circle $x^2 + y^2 = 1$

Solution

Using the formula

$$\iint_R f(x,y)\,dxdy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr$$

$$\iint\left(1-\sqrt{x^2+y^2}\right)dxdy = \iint_R (1-r)\,r\,d\theta\,dr$$

$$= \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{1} (r-r^2)\,dr$$

$$= \int_0^{2\pi} d\theta\left(\frac{1}{6}\right)$$

$$= \frac{\pi}{3}$$

Exercise 7

1. Let $I = \int_{x=0}^{5}\int_{y=0}^{5-x} dxdy$

   a) Sketch the region of integration R.

   b) State the order of integration

$\iint\limits_R f(x,y)dxdy = \int\limits_{x=a}^{b}\int\limits_{y=c}^{d} f(xy)dxdy$ when R is the region bounded by $a \le x \le b$, $c \le y \le d$

2. The definition of a triple integral, $\iiint\limits_R f(x,y,z)dxdydz = \int\limits_{x=a}^{a_2}\int\limits_{y=b_1}^{b_2}\int\limits_{z=c_1}^{c_2} f(x,y,z)dxdydz$ where $R:\{a_1 \le x \le a_2, b_1 \le y \le b_2, c_1 \le z \le c_2\}$
3. When the boundaries are not constants we must first integrate with respect to one letter whose boundary contains a variable.
4. Application of double integral to find areas of regions was seen $A = \iint\limits_R dxdy$
5. Application of triple integrals to evaluate the volume under the surface z = f (x,y) was explained. $V = \iint\limits_R zdxdy = \iint\limits_R f(x,y)dxdy$ when z = f (x, y) and R are known.
6. Sometimes it is easier to evaluate double integrals using polar coordinates. We use the formula, $\iint\limits_R f(x,y)dxdy = \iint\limits_R f(r\cos\theta, r\sin\theta)rd\theta dr$ for the conversion of Cartesian Coordinates to polar coordinates.

Further Reading

1. Advanced Calculus
   By Watson Fulks.
   University of Colorado
   John Wiley and Sons
   New York Tolonto Singapore.

2. Mathematical Methods for Science Students
   By G. Stephenson
   Addison Wesley Longman Limited
   Edinburg Gate, Harlow, London.

c) Evaluate I

2. Consider $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy^2 dy$

a) Sketch the region of integration R.
b) Which integration will you do first? Why?
c) Evaluate I

3. Evaluate $I = \int_{\theta=\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{2\cos\theta} r^2 \cos\theta drd\theta$

4. Let $I = \int_{x=0}^{4} \int_{y=0}^{x} (x^2 + y^2)dxdy$

a) Sketch the region R of the Integration.
b) Evaluate I

5. Evaluate $I = \iint_R xdxdy$ where R is the region bounded by the curve $r = 4(1+\cos\theta)$

6. Show that $\iint_R (2x^2 + y)dxdy = \frac{1}{6}$ when R is the region bounded by the line y = x and the curve $y = x^2$

7. Evaluate $\iint_R xy\sqrt{x^2 + y^2}\,dxdy$ where $R:\{0 \le x \le 1; 0 \le y \le 2\}$

8. Evaluate $\iint_R (x+y)e^{x+y}dxdy$ $R:\{0 \le x \le 1; 2 \le y \le 4\}$

9. Evaluate $\iiint_R \cos(x+2y+3z)dxdydz$ where $R:\left\{0 \le x \le \pi, 0 \le y \le \frac{\pi}{2}, 0 \le z \le \frac{\pi}{6}\right\}$

Summary

In this chapter we have seen the following:

1. The definition of a double integral,

CHAPTER 8

Stoke's Theorem

8.1 Introduction

Stoke's theorem is used to convert a surface integral to a line integral. In other words a double integral is transformed into a line integral and vice versa.

The theorem deals with an open surface whose boundary is a simple closed curve C. For example a hemispherical shell (a coconut – shell) has a surface with a simple closed curve as the boundary. A simple closed curve C has the property that any straight line parallel to any one of the coordinate axes will cut the curve in utmost two points. The curve does not intersect itself.

S is an open two sided surface bounded by a closed non-intersecting curve. Stoke's theorem is applicable only for such a surfaces bounded by a simple closed curve C.

8.2 The objectives of the Chapter:

By the end of this chapter you should be able to: -

i). give the meaning of a simple closed curve C

ii). state and prove Stoke's theorem.

iii). apply Stoke's theorem to find the area and volumes of closed figures.

8.3 Review of some important Results in vector Analysis.

We now give a review of some important results that we need in discussing Stoke's theorem.

i). **Representation of a vector r**

In three dimension a vector **r** is represented as **r** =x i + y j + z k ii). A vector function F is represented as

$\overline{F}$ (x,y,z) = F_1 (x, y, z) i + F_2 (x, y, z)j + F_3 (x, y, z)k or simply $\overline{F} = F_1 i + F_2 j + F_3 k$.

iii). Dot product of two vectors (or scalar product of two vector,

If $\overline{A} = a_1 i + a_2 j + a_3 k$ and $\overline{B} = b_1 i + b_2 j + b_3 k$

$$\overline{A}.\overline{B} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_2$$

iv). The length of a vector or modulus of a vector

If $\overline{A} = a_1 i + a_2 j + a_3 k$ then the length of the vector (or modulus of the vector) $\overline{A}$ is given by $\left|\overline{A}\right| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

v). Cross product or vector product of two vectors

If $\overline{A} = a_1 i + a_2 j + a_3 k$ and $\overline{B} = b_1 i + b_2 j + b_3 k$

$\overline{A} \times \overline{B}$ is represented as the determinant

$$\overline{A} \times \overline{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i\,(a_2 b_3 - a_3 b_2) - j\,(a_1 b_3 - a_3 b_1) + k\,(a_1 b_2 - a_2 b_1)$$

vi). The notation ∇ and $\nabla \times \overline{F}$ (curl $\overline{F}$)

We denote $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ as ∇

If $\overline{F} = F_1 i + F_2 j + F_3 k$

$$\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

or curl $\overline{F}$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}$$

$$= i\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - j\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + k\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

Example 1

If $\overline{F} = xz\,i - xy^2\,j + yz^2\,k$ and $r = x\,i + y\,j + z\,k$

Find (i) $\overline{F}\,.\,\mathbf{r}$ (ii) $\overline{F} \times \mathbf{r}$ (iii) $\nabla \times \overline{F}$ and (iv) $\overline{F}\,.\,dr$

Solution

i). $\overline{F}\,.\,\mathbf{r} = \begin{pmatrix} xz \\ -xy^2 \\ yz^2 \end{pmatrix} . \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2z - xy^3 + yz^3$ (this is a scalar).

ii). $\overline{F} \times \mathbf{r} = \begin{vmatrix} i & j & k \\ xz & -xy^2 & yz^2 \\ x & y & z \end{vmatrix}$

$$= i\begin{vmatrix} -xy^2 & yz^2 \\ y & z \end{vmatrix} - j\begin{vmatrix} xz & yz^2 \\ x & z \end{vmatrix} + k\begin{vmatrix} xz & -xy^2 \\ x & y \end{vmatrix}$$

$= i\,(-xy^2z - y^2z^2) - j\,(xz^2 - xyz^2) + k\,(xyz - x^2y)$

(this is a Vector)

iii). $\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -xy^2 & yz^2 \end{vmatrix}$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xy^2 & yz^2 \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xz & yz^2 \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz & xy^2 \end{vmatrix}$$

$=$

$$i\left[\frac{\partial}{\partial y}yz^2 + \frac{\partial}{\partial z}xy^2\right] - j\left[\frac{\partial}{\partial x}yz^2 - \frac{\partial}{\partial z}xz\right] + k\left[\frac{\partial}{\partial x}xy^2 + \frac{\partial}{\partial y}xz\right]$$

$= i\,[z^2 + 0] - j\,[0 - x] - k[y^2 + 0]$

$= i\, z^2 + j\, x - k\, y^2$

iv). $\overline{F} = x\,z\,i - x\,y^2\,j + yz^2\,k$

$\mathbf{r} = x\,i + y\,j + z\,k$

$d\mathbf{r} = d\,(x\,i + y\,j + z\,k)$

$= i\,d\,x + j\,d\,y + k\,d\,z$

$\overline{F} \,.\, d\mathbf{r} = \begin{pmatrix} xz \\ -xy^2 \\ yz^2 \end{pmatrix} . \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$ writing in column vector form.

$= x\,z\,d\,x - x\,y^2\,d\,y + yz^2\,d\,z$ (dot product is a scalar)

8.4 Statement of Stokes Theorem in Vector form

Let S be an open and simply connected surface whose boundary is a simple closed curve C.

If a Vector function,

$\overline{F} = i\ F_1\ (x, y, z) + j\ F_2\ (x, y, z) + k\ F_3\ (x, y, z)$ is a continuous and has continuous partial derivatives in S and on C, then

$$\int_c \overline{F}.d\mathrm{r} = \iint_S \left(\nabla \times \overline{F}\right).n\ dS.$$

8.5 Stokes theorem in Cartesian form:

Using the definition of $F.dr$ and $(\nabla x F).n\ d\ S$ Stokes theorem in Cartesian form, becomes:

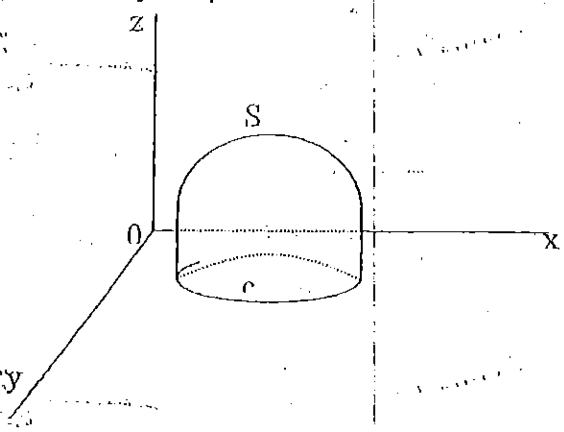
$$\int_c \left(F_1dx + F_2dy + F_3dz\right)$$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right]ds$$

where the unit normal to S is $\overline{n} = i\cos\alpha + j\cos\beta + k\cos r$

8.6 Proof of Stoke's theorem

The projection of the surface S on coordinate planes are regions bounded by simple closed curves.



Let the surface S have the equation (in three dimension), $z = f(x, y)$

Consider $\iint_C (\nabla x F_1 i).\bar{n}\, ds$

$$\nabla \times (F_1 i) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix}$$

$$= \frac{\partial F_1}{\partial z} j + \frac{\partial F_1}{\partial y} k$$

$$= [\nabla \times (F_1 i)].\ ndS = \left[\frac{\partial}{\partial z} n.j - \frac{\partial F_1}{\partial y} n.k\right] dS$$

(1)

The position vector to any point of S is

$$\mathbf{r} = xi + yj + zk = xi + yj + f(xy)k$$

$$\frac{\partial \mathbf{r}}{\partial y} = j + \frac{\partial f}{\partial y} k$$

But $\frac{\partial \mathbf{r}}{\partial y}$ is a vector tangent to S and so perpendicular to n.

Hence

$$n.\frac{\partial \mathbf{r}}{\partial y} = n.j + \frac{\partial z}{\partial y} n.k = 0$$

$$n.j = \frac{-\partial z}{\partial y} n.k$$

(2)

Substitute (2) in (1) to obtain,

$$\nabla \times (F_1 i).ndS = \left[-\frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} n.k - \frac{\partial F_1}{\partial y} n.k\right] dS$$

$$= \left[-\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial y} \right] n.kdS$$

$$=$$

$$-\frac{\partial \mathbf{F}}{\partial y} n.kdS = \frac{\partial F}{\partial y} n.kdxdy$$

[since on S, F_1 (x, y, z) = F(x, y)]

$$= -\frac{\partial \mathbf{F}}{\partial y} n.kdxdy$$

Then $\iint_S \nabla \times (F_1 i).ndS = \iint_R -\frac{\partial F}{\partial y} dxdy$ where R is the projection of S on the x, y – plane

$$= \int_F \mathbf{F}dx$$

$$= \int_C F_1 dx \qquad (3)$$

Similarly, by projection on y z and z x planes

$$\iint_S [\nabla \times F_2 j] . nds = \int_C F_2 dy \qquad (4)$$

$$\iint_S [\nabla \times F_3 k] . \text{nds} = \int_C F_3 dz \qquad (5)$$

Adding (3), (4) and (5)

$$\iint_S (\nabla \times \mathbf{F}).nds = \int_C \mathbf{F}.dr.$$

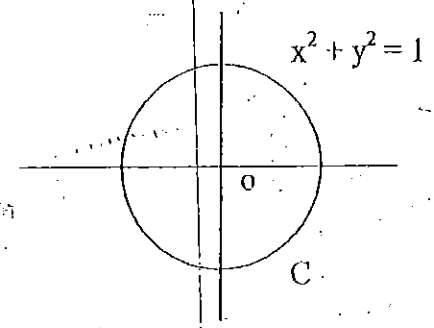
Hence the result.

Example 2

Verify Stoke's Theorem for the vector function
F = y i + z j + x k, for the surface S where S is the upper half surface of the sphere
$x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution

The boundary C of S is a circle, $x^2 + y^2 = 1$ in the x y plane.



Suppose x = cos t, y = sin t, z = 0

Where $0 \leq t \leq 2\pi$ are parametric equation of C.

Now, $\int_C \mathbf{F}.dr = \int_C (yi + zj + xk).(dxi + dyj + dzk)$

$= \int_C (ydx + zdy + kdz)$

$= \int_C ydx$, since on C, z = 0 = d z

$= \int_0^{2\pi} \sin t . \frac{dx}{dt} . dt$

$$= \int_0^{2\pi} -\sin^2 t dt$$

$$= -\frac{1}{2}\int_0^{2\pi} (1-\cos 2t).dt$$

$$= -\frac{1}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{2\pi}$$

$= -\pi$

Let us now find

$\int (\nabla \times \mathbf{F}).\overline{n}\, ds$

$$\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -i - j - k$$

$$\iint (\nabla \times \mathbf{F}).\overline{n}\, ds = \iint_{S_1} (\nabla \times F).k ds$$

$$= \iint_{S_1} (-i - j - k).k ds$$ ($\overline{n}$ =k for x y plane)

$$= \iint_{S_1} (-1) ds$$

$$= -\iint_{S_1} ds$$

$= S_1$ where S_1 is the area of the circle of radius 1 unit

$= -\pi \times 1^2$

$= -\pi$

Hence we find that

$$\int_C \mathbf{F}.d\mathbf{r} = \iint (\nabla \times F).n\, ds = -\pi$$

Hence Stoke's Theorem is verified.

Example 3

Verify Stoke's Theorem for the vector function

$F = (2x - y)\, i - y z^2 j - y z\, k$ when S is the upper half surface of the sphere

$x^2 + y^2 + z^2 = 9$ and C is its boundary.

Solution

The boundary C of S is a circle $x^2 + y^2 = 9$ in the x y – plane. It is a circle with center origin and radius 3 units. The parametric equations of C is $x = 3 \cos t$, $y = 3 \sin t$, $z = o$,

$0 \le t \le 2\pi$.

We must verify that

$$\int_C \mathbf{F}.d\mathbf{r} = \iint (\nabla \times F).n\, ds$$

$$\int_C \mathbf{F}.d\mathbf{r} = \int_C \left[(2x - y)i - yz^2 j - y^2 zk \right].(dxi + dyj + dzk)$$

$$= \int_C \left[(2x - y)\,dx - yz^2\,dy - y^2 z\,dz \right]$$

$$= \int_C (2x - y)\,dx$$ since z = 0 on the x y plane.

$$= \int_C [2(3\cos t) - 3\sin t]\frac{dx}{dt}\,dt$$

$$= \int_C (6\cos t - 3\sin t)(-3\sin t)\,dt$$

$$= \int_C (-18\cos t\sin t + 9\sin^2 t)\,dt$$

$$= \int_C \left[-9\sin 2t + 9\left(\frac{1-\cos 2t}{2}\right) dt. \right]$$

$$= -9\int_{t=0}^{2\pi} \sin 2t\,dt + 9\int \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) dt.$$

$$= -9\left[-\frac{\cos t}{2}\right]_0^{2\pi} + 9\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= 9\pi$$

Right side of (1) is

$$\iint_S (\nabla \times \mathbf{F}).\overline{n}\,ds$$

$$= \iint_{S_1} (\nabla \times \mathbf{F}).k\,ds$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = k$$

$\iint_{S_1} (\nabla \times \mathbf{F}).n\,ds = \iint_{S_1} k.k\,ds = \iint_{S_1} dS$ = S = area of the circle of radius 3 units

$= 9\pi$

Hence $\int_C \mathbf{F}.d\mathbf{r} = \iint (\nabla \times \mathbf{F}).\overline{n}\,ds = 9\pi$

8.7 Evaluation of Line Integrals

Some times the evaluation of line integrals over a curve C may be difficult. Such cases we can use stoke's Theorem

$$\int_C \mathbf{F}.d\mathbf{r} = \iint (\nabla \times \mathbf{F})\overline{n}\,ds$$

to evaluate $\int_C \mathbf{F}.d\mathbf{r}$.

The following example illustrates this principle.

Example 4

Evaluate $\int_C (xydx + xy^2dy)$ when C is the square on the x y – plane with vertices (1, 0), (-1, 0), (0, 1), (0, -1).

Solution

Since the surface integral over the area is easier we shall find

$$\iint_S (\nabla \times \mathbf{F}).\overline{n}\,\overline{ds}$$

Here F = x y i + x y^2 j and n = k for the plane

$$\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = (y^2 - x)k$$

$$\nabla \times \mathbf{F.n} = (y^2 - x)k.k = y^2 - x$$

$$\iint (y^2 - x)ds = \int_{y=-1}^{1} dy \int_{x=-1}^{1} (y^2 - x)dx$$

$$= \int_{y=-1}^{1} dy \left[y^2 x - \frac{x^2}{2} \right]_{-1}^{1}$$

$$= \int_{y=-1}^{1} 2y^2 dy$$

$$= \left[2\frac{y^3}{3} \right]_{y=-1}^{1}$$

$$= \frac{4}{3}$$

Hence $= \int_C (xydx + xy^2dy) = \iint_S (\nabla \times \mathbf{F}).\mathbf{n}ds = \frac{4}{3}$

8.8 Evaluation of surface integrals.

Sometimes the evaluation of surface integrals may be difficult. In such case we can use Stoke's Theorem to evaluate the surface integral. The following example illustrates this principle.

Example 5

Evaluate $= \iint_S (\nabla \times \mathbf{F}).\mathbf{n}ds$ where F = y i + (x – 2 x z) j – x y k and S is the surface of the sphere $x^2 + y^2 + z^2 = 25$ above the x y plane.

Solution

The boundary C of the hemisphere $x^2 + y^2 + z^2 = 25$ is the circle $x^2 + y^2 = 25$ on x y – plane. We shall use the corresponding line integral in this case where

F = y i + (x – 2x z)j – x y k

$$\int \mathbf{F}.d\mathbf{r} = \int_C [yi + (x - 2xz)j - xyk].\ [idx + jdy + kdz]$$

since r = x i + y j + z k

$$= \int_C (ydx + xdy) \text{ since } z = 0 \text{ on } x\,y - \text{plane}$$

$$= \int [5\sin t d(5\cos t) + 5\cos t d(\sin t)]$$

$$= \int_0^{2\pi} [5 \sin t(-5 \sin t) + 5 \cos t(5 \cos t)]\, dt$$

$$= 25 \int_0^{2\pi} (\cos^2 t - \sin^2 t) dt$$

$$= 25 \int_0^{2\pi} \cos\ 2t dt$$

$$= 25 \left[\frac{\sin 2t}{2} \right]_0^{2\pi}$$

$= 0.$

Hence $\iint_S (\nabla \times \mathbf{F}).\mathbf{n} ds = \oint_C \mathbf{F}.d\mathbf{r} = 0$

Example 6

Evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}).\, n\, ds$ by transforming into a line integral, S being that part of the surface of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$ and $F = y\, i + z\, j + x\, k$.

Solution

The projection of the surface on $z = 0$ is the circle, $x^2 + y^2 = 1$ putting $z = 0$

$$\iint (\nabla \times \mathbf{F}).\mathbf{n}\, ds = \int_C \mathbf{F}.d\mathbf{r}$$

$$= \int_C (yi + zi + xk).(idx + jdy + kdz)$$

since $r = x\, i + y\, j + zk$

$$= \int_C (ydx + zdy + kdz).$$

$= \int_C y\,dx$ since z = 0 on the x y plane

$= \int_0^{2\pi} \sin t(-\sin t)\,dt$ when x = cos t,

y = sin t for the circle $x^2 + y^2 = 1$

$= \int_0^{2\pi} \sin^2 t\,dt$

$= -\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) dt$

$= -\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{2\pi}$

$= -\pi$

Exercise 8

1. State Stoke's Theorem in vector form and in Cartesian form
2. Verify Stokes Theorem for the function $\overline{F}$ = y i + z j + x k if the surface S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 36$ and C is its boundary.
3. Verify Stoke's Theorem for $\overline{F}$ = (2x – y) i – y z^2 j – y^2 z k where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 16$ and C is its boundary which is the circle $x^2 + y^2 = 16$.
4. If $\overline{F}$ = (x^3 j – y^3 i) and S is the circular disc C $x^2 + y^2 \le 1$, z = 0, verify Stoke's Theorem.

5. Evaluate by Stoke's Theorem

$$\int_C (e^x dx + 2y dy - dz)$$

6. Evaluate by Stoke's Theorem

$$\int_C (\sin z dx - \cos x dy + \sin dz)$$

where C is the rectangle $0 \le x \le \pi$, $0 \le x \le 1$, $z = 3$

7. Evaluate the surface integral;

$\iint_S curl \ \mathbf{F} . n \ ds$ by transforming into a line integral, S being the surface $z = 1 - x^2 - y^2$ for which $z \ge 0$ and $\overline{F} = y\,i + z\,j + x\,k$.

8. Verify Stoke's Theorem for the function $\overline{F} = x^3 i - x y j$, integrated along the rectangle, on the plane z = 0 and sides are along the lines, x = 0, y = 0, x = 4, y = 3.

9. Evaluate $\iint_S (\nabla \times F).nds$ where $\overline{F} = (y - z + 2)i + (y z + 4) j - x z k$ and S is the surface of the cube, x = y = z = 0 x= y = z= 2 above the x y plane. [Hint : the curve C bounding the surface S is the square on the x y plane, given by x = 0, x = 2, y = 0, y = 2].

10. Verify Stoke's Theorem for the function $\overline{F} = x^2 i + x y j$ taken round the square x = 0, y = 0, x = a, y = b on the plane z = 0

Summary

In this chapter we have learnt the following :

1. A simple closed curve C has the property that any one of the coordinate axes cuts the curve in at most two points.

2. Stoke's theorem in vector form:
   Let S be an open and simply connected surface whose boundary is a simple closed curve C. If a vector function $\overline{F} = F_1(x, y, z)i + F_2(x, y, z)j + F_3(x, y, z)k$ is continuous and has partial derivatives in S and on C, then

$$\int_C \mathbf{F}.d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}).\mathbf{n}\,ds$$

3. Application of Stokes Theorem to line integrals and surface integrals

Further Reading

1). Advanced Calculus
By Watson Fulks
University of Colorado
John Wiley and Sons
New York, Toronto. Singapore.

2). Mathematical Methods for Science Students
By G. Stephenson
Addison Wesley Longman Limited
Edinburgh Gate. Harlow London.

CHAPTER 9
Green's Theorem

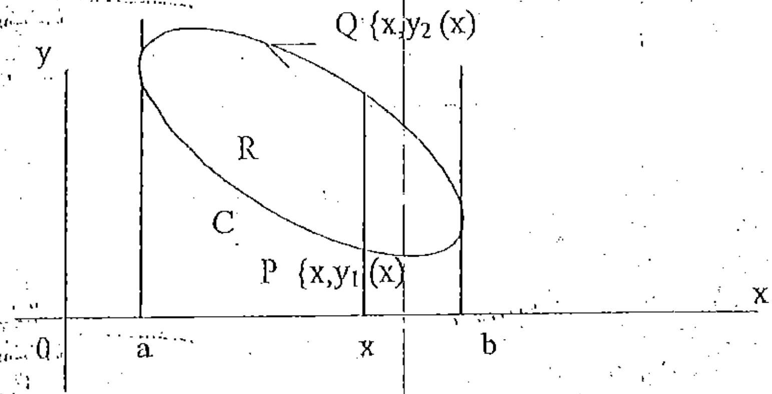
9.1 Introduction

In the last chapter we have learnt the evaluation of double and triple integrals over an area and volume respectively. Suppose a region R on x y- plane is bounded by a simple closed curve C. Green's theorem gives a relationship between the double integral over the region R and the line integral around the closed curve C.

9.2 Objectives of the chapter

In this lesson we shall study the statement and proof of Green's Theorem and its application to find areas bounded by simple closed curves.

9.3 Green's Theorem



Let P = P(x,y) and Q = Q (x,y) be two functions such that

P,Q, $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$, are finite and continuous inside and on the boundary C of some region R of the xy – plane then,

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_c (Pdx + Qdy),$$

Proof

$$\iint_R \frac{\partial P}{\partial y} dxdy = \int_{x=a}^{b} dx \int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial P}{\partial y} dy$$

$$= \int_{x=a}^{x=b} [P]_{y=y_1(x)}^{y=y_2(x)} dx$$

$$= \int_{x=a}^{x=b} [P\{x, y_2(x)\} - P\{x, y_1(x)\}] dx$$

$$= - \int_{x=b}^{a} P\{x, y_2(x)\} dx - \int_{x=a}^{b} P\{x, y_1(x)\} dx$$

$$= - \left[\int_a^b P\{x, y_1(x)\} dx + \int_b^a P\{x, y_2(x)\} dx \right]$$

$$= - \int_C P(x, y) dx$$

Similarly

$$\iint_R \frac{\partial Q(x, y)}{\partial x} dxdy = - \int Q(x, y) dy$$

hence, $\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dxdy = - \int_C (Pdx + Qdy)$

or $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_C (Pdx + Qdy)$

where the time integral is in the anticlockwise (positive) direction.

Example 1

Using Green's Theorem, evaluate

$$\int_C [y(2xy-1)dx + x(2xy+1)dy]$$

where c is the circle $x^2 + y^2 = 9$

1.1. Solution

By Green's Theorem,

$$\int_C \{Pdx + Qdy\} = \iint_R \left\{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\} dxdy$$

Here $P = 2xy^2 - y$, $Q = 2x^2y + x$

$$\frac{\partial P}{\partial y} = 4xy - 1, \quad \frac{\partial Q}{\partial x} = 4xy + 1$$

$$\int_c \{y(2xy-1)dx + x(2xy+1)dy\}$$

$$= \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy$$

$$= \iint [(4xy+1) - (4xy-1)] dxdy$$

$$= \iint 2dxdy$$

$$= 2\iint dxdy$$

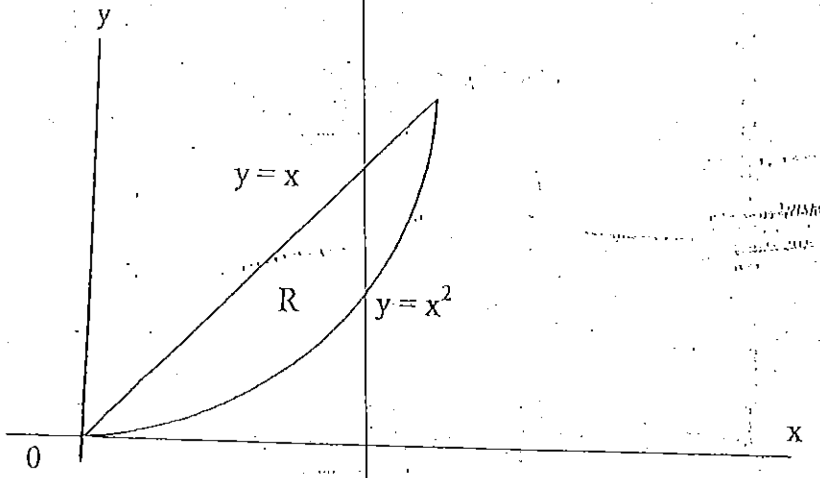
$= 2$ [area of the region R which is the area of the circle $x^2 + y^2 = 9$]

$= 2(\pi r^2)$ where r = 3

$= 18\pi$

1.2. *Example 2*

Verify Green's Theorem in the plane for the functions P (x, y) = xy + y^2 and Q (x, y) = x^2



1.3. *Solution*

We should verify Green's Theorem:

$$\int_C (Pdx + Qdy) = \iint_R \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dxdy$$

The line integral consists of the curve y = x^2 and the line y = x as in the figure.

Along the curve y = x^2, dy = 2xdx.

$$\int_{y=x^2} (Pdx + Qdy) = \int \left[(xy + y^2)dx + x^2 dy \right]$$

$$= \int_0^1 \left[\left\{ x \cdot x^2 + (x^2)^2 \right\} dx + x^2(2x)dx \right]$$

$$= \int_0^1 \left[(x^3 + x^4)dx + 2x^3 dx \right]$$

$$= \int_0^1 (3x^3 + x^4)dx$$

$$= \iint (2x - x - 2y)\,dx\,dy$$

$$= \iint (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} dx \int_{y=x^2}^{x} (x - 2y)\,dy$$

$$= \int_{x=0}^{1} dx \left[xy - y^2 \right]_{x^2}^{x}$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1$$

$$= -\frac{1}{20}$$

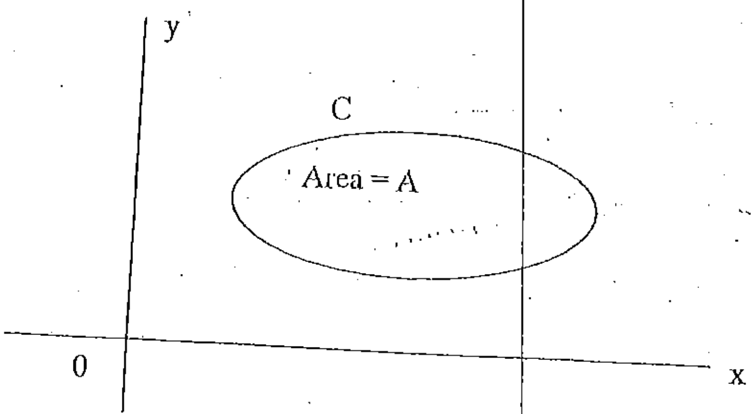
Thus we proved that

$$\int_c (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = -\frac{1}{20}$$

9.4 Expression for area bounded by a simple closed curve C.

Using Green's Theorem in plane we can show that the area of a region on xy – plane bounded by a closed curve C is,

$$A = \frac{1}{2} \int_c (x\,dy - y\,dx)$$



$$=\left[\frac{3x^4}{4}+\frac{x^5}{5}\right]_0^1$$

$$=\frac{19}{20}$$

Along y = x, dy = dx.

$$\int_{y=x}(Pdx+Qdy)=\int\left[(xy+y^2)dx+x^2dy\right]$$

$$=\int_1^0\left[(x^2+x^2)dx+x^2dx\right]$$

$$=\int_1^0 3x^2dx$$

$$=\left[x^3\right]_1^0$$

$$=-1$$

Hence $\int_c(Pdx+Qdy)=\frac{19}{20}+(-1)=\frac{-1}{20}$

Now $Q = x^2$ and $P = xy + y^2$

$\frac{\partial Q}{\partial x}=2x$ and $\frac{\partial P}{\partial y}=x+2y$

Then $\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

Proof

By Green's Theorem,

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

where P and Q are any functions of x and y provided that P, Q, $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$, are finite and continuous inside and on the curve C.

Consider the functions, P = -y and Q = x which are finite and continuous.

Therefore $\int_C (-ydx + xdy) = \iint_R \left[\frac{\partial(x)}{dx} - \frac{\partial(-y)}{\partial y} \right] dxdy$

$$= \iint_R 2dxdy$$

$$= 2\iint_R dxdy$$

$= 2$ [area of the bounded Region R]

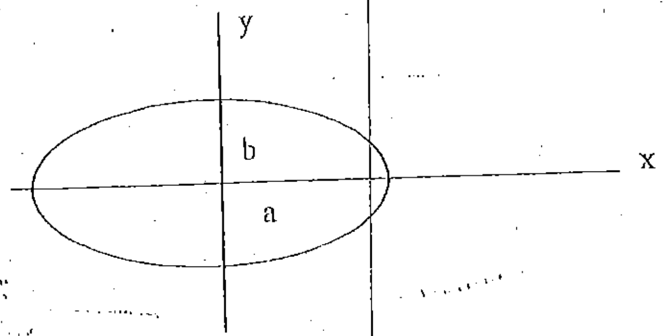
Hence $\int_c (xdy - ydx) = 2A$

$$A = \frac{1}{2}\int_c (xdy - ydx)$$

Example 3

a) What are the parametric equations of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with semi axes a and b.

b) Prove that the area of an ellipse whose semi major and minor axes are a and b is πab.



Solution

a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the Cartesian equation of an ellipse.

$x = a\cos\theta$, $y = b\sin\theta$ are the parametric equations of the ellipse whose semi-major and semi minor axes are a and b respectively.

b) Area of the closed region $= \frac{1}{2}\int_c (xdy - ydx)$

$$= \frac{1}{2}\int_0^{2\pi}\left[a\cos\theta d(b\sin\theta) - b\sin\theta d(a\cos\theta)\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[ab\cos^2\theta + ab\sin^2\theta\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta+\sin^2\theta)d\theta$$

$$= \frac{1}{2}ab\left[\theta\right]_0^{2\pi}$$

$$= \pi ab$$

Example 4

Evaluate by green's theorem

$$\int_c\left[(\cos x\sin y - xy)dx + (\sin x\cos y)dy\right]$$

where c is the circle $x^2 + y^2 = 16$

Solution

Since area integral over the circle is easier that line integral we use green's theorem to evaluate this problem.

Let P cos x sin y – xy, Q = sin x cos y

$\frac{\partial P}{\partial y} = \cos x\cos y - x$, $\frac{\partial Q}{\partial x} = \cos x\cos y$

$$\int_c\left[(\cos x\sin y - xy)dx + \sin x\cos y dy\right]$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$$= \iint_R (\cos x \cos y - \cos x \cos y + x) dxdy$$

$$= \iint_R x dxdy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{4} (r\cos\theta)(rd\theta dr)$$

since area $dxdy = rd\theta dr$ in polar coordinates.

$$= \int_0^{2\pi} \cos\theta d\theta \left[\frac{r^3}{3} \right]_0^4$$

$$= \frac{64}{3} \int_0^{2\pi} \cos\theta d\theta$$

$$= \frac{64}{3} [\sin\theta]_0^{2\pi}$$

$$= \frac{64}{3}(0)$$

$$= 0$$

9.5 Green's theorem in the plane in vector form

Prove that Green's theorem in vector form is $\int_c \overline{F}.d\overline{r} = \iint_R (\nabla \times \overline{F}).kdR$

Proof

Green's theorem in Cartesian form is,

$$\int_c (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

..... (1)

where R is a closed region on xy – plane bounded by the curve C.

Let

$\overline{F} = Pi + Qj$

$\overline{r} = xi + yj$

$\overline{F}.d\overline{r} = (Pi + Qj).(idx + jdy)$

$\quad = \text{Pdx + Qdy}$

$$\int_c (Pdx + Qdy) = \int_c \overline{F}.d\overline{r}$$

(2)

$$\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & O \end{vmatrix}$$

$$= -\frac{\partial Q}{\partial z} i + \frac{\partial P}{\partial z} j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

$$\nabla \times \overline{F}.k = \left[-\frac{\partial Q}{\partial z} i + \frac{\partial P}{\partial z} j + \left(\frac{\partial Q}{\partial x} - \frac{\partial m}{\partial y}\right) k\right].k$$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Hence $= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R \left(\nabla \times \overline{F}\right).kdR$

(3)

From (1) and (2) and (3)

$$\int_c \overline{F}.d\overline{r} = \iint_R (\nabla \times \overline{F}).kdr$$

Example 5

If $\overline{F} = (x^2 - y^2)i + xyj$ and r = xi + yj find the value of $\int \overline{F}.d\overline{r}$ around the rectangle boundary x =0, x = a, y = 0, y = b

$\oint_c [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where C is the boundary of the region defined by $y = \sqrt{x}$, $y = x^2$

3. Apply Green's theorem on the plane to evaluate $\oint_c [(y - \sin x)dx + \cos x dy]$ where C is the triangle enclosed by the lines y = 0, $x = \pi$, $x = \frac{\pi y}{2}$

4. Apply Green's theorem on the plane to evaluate $\oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy]$ where C is the boundary enclosed by the x axis and the semi circle $y^2 = 1 - x^2$

5. Verify Green's theorem on the plane for $\oint_c [(x^2 - xy^3)dx + (y^2 + 2xy)dy]$ where C is the square with vertices (0,0), (2,0), (2,2), (0,2).

6. Evaluate $\oint_c [(3x^2 - 2y)dx - (x + 3\cos y)dy]$ around the parallelogram having vertices at (0,0), (2,0), (3,1) and (1,1).

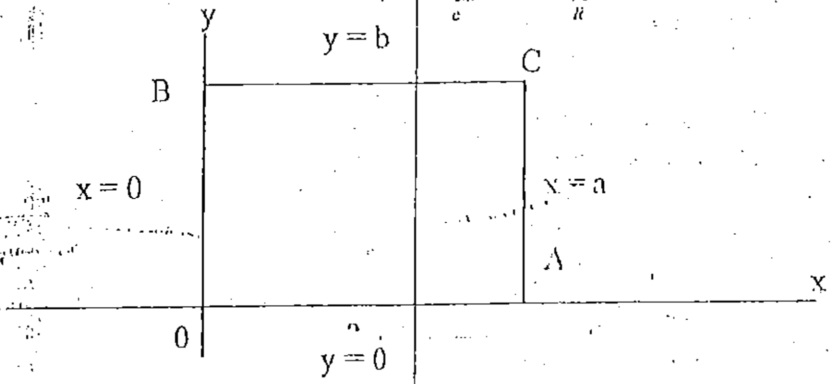
7. Verify Green's theorem in the plane $\oint_c [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where C is the boundary of the region defined by x = 0, y = 0, x + y = 1

8. Find the area bounded by one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ a > 0 and x axis.

9. Verify Green's theorem in the plane for $\oint_c [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where C is the square with vertices (0,0), (4,0), (4,4) and (0,4).

Solution

By Green's theorem in vector form, $\int_c \overline{F}.d\overline{r} = \iint_R (\nabla\times\overline{F}).k\,dr$



$$\nabla\times\overline{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$= 4\,ky$

$$(\nabla\times\overline{F}).k = (4ky).k = 4y$$

$$\int \overline{F}.dr = \iint_R 4\,ydxdy$$

$$= 4\int dx\int ydy$$

$$= 4\int_{x=0}^{a} dx\left[\frac{y^2}{2}\right]_0^b$$

$$= 4\frac{b^2}{2}[x]_0^a$$

$$= 2ab^2$$

Exercise 9

1. Verify Green's theorem on the plane for $\oint_c \left[(2xy-x^2)dx+(x^2+y^2)dy\right]$ where c is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$.
2. Verify Green's theorem on the plane for

Summary

In this lesson we have seen the following:

1. Green's theorem in the plane in Cartesian form:

$$\int_c (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$ if P,Q, $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$, are finite and continuous inside and on the boundary C of some region R of the xy – plane.

2. The Area A of a region on xy – plane bounded by a closed curve C is $A = \frac{1}{2}\int_c (xdy - ydx)$

3. Green's theorem in the plane in vector form is $\int_c \overline{F}.d\overline{r} = \iint_R (\nabla \times \overline{F}).kdr$ if $\overline{F}$ is continuous function inside and on the boundary C of the region R of the xy – plane.

Further Reading.

1. Advanced Calculus
   By Watson Fulks
   University of Colorado
   John Wiley and Sons
   New York. Toronto. Singapore

2. Mathematical Methods for Science Students
   By G. Stephenson
   Addison Wesley Longman Limited
   Edinburgh Gate, Harlow, London.

Chapter 10
Fourier Series for period 2L

10.1 Introduction

In Chapter three we have learnt how certain functions may be represented as power series by means of the Taylor's and Maclaurin's series. As you will see, a trigonometric series has some advantages, over a power series for the representation of functions. For example, the condition of infinite differentiability that is the case of power series is not needed. Moreover, it is possible to differentiate or integrate term wise without having satisfied the uniform convergence criteria, that form the usual requirements for such operation in the case of power series. The only requirement for a function f(x) to expand as a Fourier series is that it should be a periodic function. First we shall see the definition of a periodic function.

10.2 Objectives of the Chapter

By the end of this chapter, you should be able to,

i) Define Fourier series for the interval (-L, L) so that period is 2L
ii) Find the Fourier series for a given function
iii) Define odd and even functions and know some properties of definite integrals of odd and even functions.

10.3 Periodic Functions.

If a function f(x) satisfies the functional equation.

$f(x+P)=f(x)$

then it is called a periodic function with period P.

For example, sin x and cos x are periodic functions with period 2π,

Tan x is a periodic function with period π since,

$$Sin(x+2\pi = \sin x$$

$$\cos(x=2\pi)=\cos x$$

$$\tan(x+\pi)=\tan x$$

10.4. Graphs of Periodic Functions

In the figure the graph represents a periodic function with period 4units

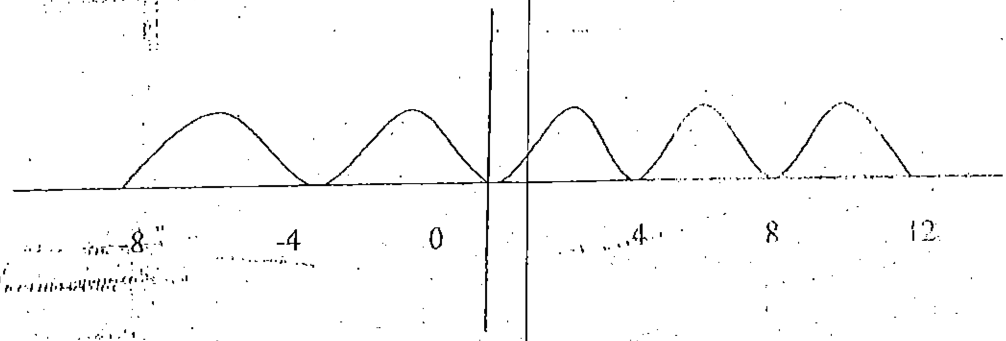


Figure 10.1

You will observe that the shape of the graph is repealed after an interval of 4 units

10.5 Definition of Fourier Series for Period L

Consider a function f(x) defined in the interval (-L , L) and determined outside this interval by
f(x+2L) = F(x)
The Fourier series corresponding to f (x) is defined as

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right) \quad (1)$$

where the Fourier constants a_n and 6_n are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi}{L} \, dx. \quad n = 0, 1, 2 \ldots\ldots\ldots \quad (2)$$

$$b_n = \frac{1}{L} \int_{-l}^{l} f=(x) \sin\frac{n\pi x}{L} \, dx. \quad n = 0, 1, 2 \ldots\ldots\ldots \quad (3)$$

If f(x) has the period 2 L, the coefficient a_n and b_n can be determined from

$$a_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \cos\frac{n\pi x}{L} \, dx \quad (4)$$

$$b_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \sin \frac{n\pi x}{L} dx \qquad (5)$$

where c is any real number
In the special case c = -L
(4) and (5) become the same as (2) and (3)

From $a_n = \frac{1}{L} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{L} dx$

We have $a_0 = \frac{1}{L} \int_{-l}^{l} f(x)\, dx$, (Putting n = 0

Hence $\frac{a^0}{2} = \frac{1}{2L} \int_{-l}^{l} f(x)\, dx.$

10. 6 Fourier series for period 2π

Let the period 2L =2π or L = π
In this case

$$f(x) = \frac{a^0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx$$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n x\, dx$

and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n x\, dx.$

$$\frac{a^0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx.$$

10.7 Convergence of Fourier series.

We expand a function in an infinite Trigonometric series. This series may converge to f (x) or converge to another function Φ (x) or diverge. Even if a point x is a discontinuity for the function f(x), the function f(x) could be convergent. Dirichlet had found that the function should satisfy **three** conditions which are sufficient but not necessary. If the following three conditions are satisfied the convergence is guaranteed. Even though these three conditions are not satisfied the series may be convergent.

10.8 Dirichlet's condition for convergence of a Fourier series.

(i) If f(x) is defined and single valued except at a **finite** number of points in the interval (-L, L)].

(ii) f (x) is periodic with period 2L.

(iii) f (x) and f^1 (x) are piecewise continuous in (-L, L) then the Fourier series converges to f(x).
if x is a point of continuity.
If x is a point of discontinuity.

F (x) converges to $\frac{f(x+\varepsilon)+f(x-\varepsilon)}{2}$

where ε is a positive quantity tending to zero.

10.9 Definition of odd and even function.

Knowledge of odd and even functions will be very helpful in expanding a function in a Fourier series.

(i) A function f(x) is called odd if f(-x) = -f (x).
For example consider the function

$F(x) = x^3 + \sin x$
$F(-x) = (-x)^3 + \sin(-x)$
$= -x^3 - \sin x$
$= -(x^3 + \sin x)$
$= -f(x)$

Hence $f(x) = x^3 + \sin x$ is an odd function.

(ii) A function f (x) is called even if f(-x) = f(x)
For example consider the function

$$f(x) = x^2 + \cos x + 8$$
$$f(-x) = (-x^2) + \cos(-x) + 8$$
$$= x^2 + \cos x + 8$$
$$= f(x)$$

Since $f(-x) = f(x)$ the function $f(x) = x^2 + \cos x + 8$ is an example of an even function.

Note: A function may be neither odd nor even. It a mixed function.
For example

$F(x) = x^2 + x^3 + \sin x$ is neither odd nor even. Check yourself by considering $f(-x)$.

10.10 Properties of odd and even functions

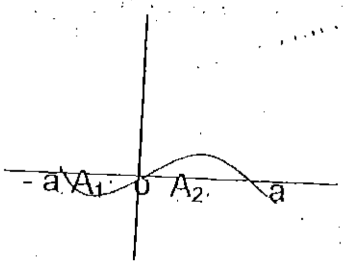
(i) If $f(x)$ is an odd function then

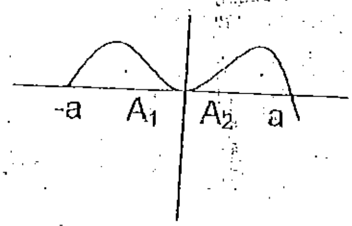
$$\int_{-a}^{a} f(x)\,dx = 0$$

(ii) If $f(x)$ is an even function then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Consider the graph of an odd function and an even function given below in the interval $(-a, a)$.





y= f(x), an odd function

y = f(x) an even function

$\int_{-a}^{a} f(x)dx$ represents the area under a curve

a) if $f(x)$ is an odd function

$\int_{-a}^{a}$ f(x) dx = $-A_1 + A_2 = 0$ since A_1 and A_2 are equal in magnitude but opposite in sign.

b) if f(x) is an even function.

$\int_{-a}^{a}$ f(x) dx = $A_1 + A_2 = 2A_2$ since $A_1 = A_2$ in magnitude and sign.

Thus $\int_{-a}^{a}$ f(x) dx = 0 if f(x) is odd

$$= 2 \int_{0}^{a} f(x)\, dx \text{ if } f(x) \text{ is even.}$$

10.11 New Theorem for integration by parts:

You all know that the formula for integration by parts namely,

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \quad (1)$$

When there is a repeated integration two times, three times.....and n times , I am giving a new Theorem which will be very useful and simple.

New Theorem for extension of integration by parts.

$$\int u \frac{dv}{dx} dx = u\,v - u^{1}v_1 + u^{11}v_2 - u^{111}v_3 + \ldots\ldots\ldots\ldots\ldots$$

where u^{1}, u^{11}, u^{111} .... are successive derivates of u and v_1 , v_2, v_3..... are successive integration of v we can prove this theorem easily.

Proof of the extension theorem

You have a formula

$$\int u \frac{dv}{dx} dx = u\,v - \int v \frac{du}{dx} dx$$

$$= uv - \int \frac{du}{dx} v\, dx$$

$$= uv - \left[\frac{du}{dx}v_1 - \int v_1 \frac{d^2u}{dx^2}dx\right]$$

$$= uv - u^1v_1 + \int u^{11}v_1\,dx$$

$$= uv - u^1v_1 + \left[u^{11}v_2 - \int u^{111}v_2\,dx\right]$$

$$= uv - u^1v^1 + u^{11}v_2 - \int u^{111}v_2\,dx$$

Thus we can extend this procedure until one of the functions a or v will be terminate. Then

$$\int \frac{dv}{dx}dx = u\,v - u^1v_1 + u^{11}v_2 - u^{111}v_3 + \ldots\ldots\ldots\ldots\ldots\ldots$$

The following examples will illustrate the application of the new Theorem.

Example

Find $\int x^3 \cos x\,dx$.

Solution

Let $I = \int x^3 \cos x\,dx = \int u\frac{dv}{dx}dx$

Here $u = x^3$ $\quad v = \sin x,\ \left(\int \cos x dx = \sin x\right)$

$u^1 = 3x^2$ $\quad v_1 = -\cos x$

$u^{11} = 6x$ $\quad v_2 = -\sin x$

$u^{111} = 6$ $\quad v_3 = \cos x$

$u^{1111} = 0$ $\quad v_4 = \ldots\ldots\ldots$

using the new theorem,

$$\int u\frac{dv}{dx}dx = uv - u^1v_1 + u^{11}v_2 - u^{111}v_3 + u^{1111}v_4 + \ldots\ldots\ldots\ldots$$

$$\int x^3\cos x\,dx = (x^3\sin x) - (-3x^2\cos x) + (-6x\sin x) - (6\cos x)$$

$$= x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x$$

Example

Find $\int x^4 e^{2x}\,dx$

Solution

By New Theorem,

$$\int u\frac{dv}{dx^n}dx = uv - u^1v_1 + u^{11}v_2 - u^{111}v_3 + u^{1111}v_4 \ldots\ldots\ldots\ldots$$

Here $u = x^4$ $\quad v = \frac{e^{2x}}{2}$ $\quad \int e^{2x} = \frac{e^{2x}}{2}$

$u^1 = 4x^3 \qquad v_1 = \frac{e^{2x}}{4}$

$u^{11} = 12x^2 \qquad v_2 = \frac{e^{2x}}{8}$

$u^{111} = 24x \qquad v_3 = \frac{e^{2x}}{16}$

$u^{1111} = 24 \qquad v_4 = \frac{e^{2x}}{32}$

$u^v = 0 \qquad v^5 = \ldots\ldots\ldots$

Hence we have

$$\int x^4e^{2x}dx = \left(\frac{x^4e^{2x}}{2}\right) - \left(x^3e^{2x}\right) + \left(\frac{3}{2}x^2e^{2x}\right) - \left(\frac{3}{2}xe^{2x}\right) + \left(\frac{3}{2}e^{2x}\right)$$

$$= e^{2x}\left(\frac{x^4}{2} - x^3 + \frac{3}{2}x^2 - \frac{3}{2}x + \frac{3}{4}\right)$$

This new Theorem given by me will be very useful, in the chapter, on Fourier series.

Example 3

(a) Expand $f(x) = x^2, -\pi\angle x\angle\pi$ in a Fourier series.

(b) using your Result

prove that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$

Solution

By definition of Fourier series in $[-L, L]$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + bn \sin\frac{n\pi x}{L} \right) \quad (1)$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, n = 0,1,2........$ (2)

And $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, n = 0,1,2.....$ (3)

In this problem $f(x) = x^2$ and $L = \pi$

Then $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx$ (4)

We use extension formula for integration by parts for $u = x^2$ and $\frac{dv}{dx} = \cos nx$

or $v = \frac{\sin nx}{n}$, we have

$u = x^2 \qquad v = \frac{\sin nx}{n}$

$u^1 = 2x \qquad v_1 = \frac{-\cos nx}{n^2}$

$u^{11} = 2 \qquad v_2 = \frac{-\sin nx}{n^3}$

$u^{111} = 0 \qquad v_3 = ...$

Then $\int_{-\pi}^{\pi} x^2 \cos nx dx$

$= 2\int_0^{\pi} x^2 \cos ndx$ since x^2 cosnx is an even function

$$= 2\left[\frac{x^2 \sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= 2\left[\left(0 + \frac{2\pi}{n^2}con\pi - 0\right) - (0 + 0 - 0)\right]$$

$$= \frac{4\pi}{n2}\cos n\pi \quad (5)$$

Substituting (5) in (4) we have

$$a_n = \frac{1}{\pi}\left(\frac{4\pi}{n^2}\cos n\pi\right)$$

$$= \frac{4}{n^2}\cos n\pi \text{ if } n \neq 0 \qquad (6)$$

$$\text{If } n = 0,\ a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}$$

$$\text{or } \frac{a^o}{2} = \frac{2\pi^2}{6},$$

$$\text{now } b_n = \frac{i}{\pi}\int_{-\pi}^{\pi} x^2 \sin\frac{n\pi x}{\pi}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$$

$= \frac{1}{\pi}(o) = o$ since x^2 sin nx is an odd function

(7)

Substituting (6) and (7) in (1) we have

$$\int(x) = x^2 = \frac{2\pi^2}{6} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi\cos nx$$

or

$$x^2 = \frac{2\pi^2}{6} + 4\left(\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\ldots\right)$$

(8)

6) substituting x= 0 in (8) to get

$$0 = \frac{2\pi^2}{6} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots\ldots\ldots\ldots\right)$$

then $4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots\right) = \frac{2\pi^2}{6}$

or $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots = \frac{\pi^2}{12}$

Example 4

a) Expand $f(x)=x$, for the interval $-\pi\angle x\angle\pi$ in a Fourier series

b) Show that for $-\pi\angle x\angle\pi$,

$$x^2=\frac{\pi^2}{3}-4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\text{..........}of\int(x)\right)$$

using $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\text{............}=\frac{\pi^2}{12}$

Solution

By the formula for Fourier series for 2L period or interval [-L, L] we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+bn\sin\frac{n\pi x}{L}\right)$$

where $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}dx, n=0,1,2\text{.................}$ (2)

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}, n=0,1,2\text{..........} \quad (3)$$

in this problem $f(x)=x$ and $[-L,L]=[-\pi,\pi]$

then $$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\frac{\cos n\pi x}{\pi}dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos nx dx$$

since (x cosn x) is an odd function that is

if f(x) = x cos nx

f(-x) = - xcos nx -F(x)

Now $$a_o=\frac{1}{\pi}\int_{-\pi}^{\pi}xdx=\left[\frac{x^2}{2\pi}\right]_{-\pi}^{\pi}=0$$

$$bn=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin\frac{n\pi x}{\pi}dx$$

$\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx dx(4)$ since $f(x)=x\sin nx$ is the function as $f(-x)=f(x)$

$$\frac{2}{\pi}\int_0^\pi x\sin nx\,dx \qquad (4)$$

since f(x)=xsin nx is an even function

let $u = x$ and $\frac{dv}{dx} = \sin nx$

then u =x $\qquad v = -\frac{\cos nx}{n}$

$u^1 = 1 \qquad v1 = -\frac{\sin nx}{n^2}$

$u^{11} = 0 \qquad v_2 = \ldots\ldots\ldots\ldots$

$$\int u\frac{dv}{dx}dx = uv - u^1v_1 + u^{11}v_2 \ldots\ldots\ldots$$

$$\int_0^\pi x\sin nx\,dx = \left[-x\frac{-\cos nx}{n} + \frac{\sin nx}{n^2} + 0\right]_0^\pi$$

$$= \left[\frac{-\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi - 0\right]$$

$$= \frac{-\pi}{n}\cos\pi \qquad (5)$$

substituting (5) in (4) we have

$$bn = \frac{-2}{\pi}\left[\frac{\pi}{n}\cos n\pi\right] = \frac{-2}{n}\cos n\pi \qquad (6)$$

substituting the values of a_n and bn in (1) we have

$$f(x) = \sum_{n=1}^{\infty}\frac{-2}{n}(\cos n\pi)\sin\frac{n\pi x}{\pi}$$

$$x = -2\sum_{n=1}^{\infty}\frac{-1^n\sin nx}{n}$$

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\ldots\ldots\right) \quad (7)$$

Let $x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\ldots\ldots\ldots\right)$

Since f(x) =x is convergent by Dirichlet's conditions we can intergrate term by term

Integrating both sides of (1) we have

$$\int x dx = 2\int\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\ldots\ldots\right) 8$$

then

$$\frac{x^2}{2} + A = 2\left(\frac{-\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2}\ldots\ldots\ldots\right)$$

or

$$x^2 + 2A = 4\left(\frac{-\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2}\ldots\ldots\ldots\right)$$

put x= 0, 2A =

$$-4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right.$$

$$2A = -4\left(\frac{\pi^2}{1^2}\right)$$ using

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{12} \qquad (10)$$

$$2A = \frac{\pi^2}{3}$$

substituting (10) in (9) we have

$$x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\ldots\right)$$

Exercise 10.

1. Find the Fourier series for the function
$f(x) = \begin{cases} 0 & -\pi\angle x\angle 10 \\ 2 & 0\angle x\angle \pi \end{cases}$ period 2 π

2. $f(x) = \begin{cases} -x & -4\le x\le 0 \\ x & 0\le x\le 4 \end{cases}$ period 8
find the Fourier series for f(x)

3. $f(x) = \begin{cases} 0 & 3\angle x\angle 0 \\ 2x & 0\angle x\angle 3 \end{cases}$ period 6
find the Fourier series for f(x)

4.a) show that for $-\pi \angle x \angle \pi$

$$x\cos x = -\frac{1}{2}\sin x + 2\left(\frac{2}{1.3}\sin 2x - \frac{3}{2.4}\sin 3x + \frac{4}{3.5}\sin 4x........\right)$$

b) use (a) to show that for $-\pi \le x \le \pi$

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} + ...\right)$$

5. Obtain the Fourier series for the function $f(x) = \begin{cases} 0 & -\pi \angle x \angle 0 \\ \sin x & 0 \le x \le \pi \end{cases}$

6. Obtain the Fourier series for x in $-\pi \angle x \angle \pi$

7. Find the trigonometric Fourier series of the function f defined by f(x) =|x| $-\pi \le x \le \pi$ on the interval $\pi \le x \le \pi$

8. find the geometric Fourier series of the function f defined by

f(x) =x $\quad -4 \le x \le 4$

9. Find the Trigonometric Fourier series of the function f defined by

$f(x) = \begin{cases} 0 & -\pi \le x \angle 0 \\ x2 & 0 \le x \le \pi \end{cases}$

on the interval $-\pi \le x \le \pi$

10. Find the Trigonometric Fourier series of f given by

$f(x) = |x|$ in the interval $-6 \le x \le 6$

Exercise 10 Fourier series for a period 2L

1. $f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$

2. $2 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(1-\cos n\pi}{n2}\cos\frac{n\pi x}{4}$

3. $\frac{3}{2}+\sum_{n=1}^{\infty}\left[\frac{6(\cos n\pi-1)}{n^2\pi^2}\cos\frac{n\pi x}{3}-\frac{6\cos n\pi}{n\pi}\sin\frac{n\pi x}{3}+\ldots\ldots\right]$

4. $\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{11}\sum_{1}^{\infty}\frac{\cos 2nx}{4n^2-1}$

5. $x=\sum_{n=1}^{\infty}\frac{2}{n}\left(-1^{n+1}\right)\sin nx$

6. $|x|=\frac{\pi}{2}-\frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$

7. $x=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{4}; -4\le x\le 4$

8. $\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx+\frac{1}{\pi}\left[\left(\frac{2}{n^3}-\frac{\pi^2}{n}\right)(-1)^n-\frac{2}{n^3}\right]\sin nx;$

9. $3-\frac{24}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n-1)^2}\frac{\cos(2n-1)}{6}\pi x$

Further Reading

1. Advanced Calculus
   By Watson Fulks
   University of Colorodo
   John Wiley and sons
   New York. Toronto. Singapore.

2. Mathematical Methods for Science students
   By G. Stephenson
   Addison Wesley
   Longman Gate, Harrow, London.

CHAPTER 11
Half Range Fourier Series

11.1 Introduction

Fourier series has been defined in 10.4 for the interval [-L, L] or for the period 2 L. Outside this interval f (x + 2 L) = f (x). Consider half of this interval namely (0,L). We can find a Fourier series in this interval [0 L] either as a series of sine only or a series of cosine only. Such series are called Half – Range Fourier series.

11.2 Objectives of the chapter.

By the end of this chapter you will be able to determine f (x) as a

(i) cosine series only in the interval (0, L) or

(ii) sine series only in the interval (0, L)

11.3 Half – Range Fourier Cosine Series

Suppose f (x) is defined in the interval [0, L] and we are required to expand f (x) in a Fourier cosine series in this interval [0, L]. Then we use the formula,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$ and $b_n = 0$, n = 0, 1, 2 ....

There is no b_n in this case.

11.4 Half – Range Fourier sine series

Suppose f (x) is defined in the interval [0, L] and we are required to expand f (x) in a Fourier sine series in this interval [0, L]. Then we use the formula $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}$$

and $a_n = 0$ for n = 1, 1, 2.....

Note that the limits for the integration for a_n or b_n is from 0 to L for Half – Range Fourier cosine and sine series whereas in Fourier series of period 2 L we use the limit from – L to L.

Example 1

Expand f (x) = x, 0< x < 2 in a half – range sine series.

Solution

By formula for sine series in 0 < x < L

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad (1)$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}$

In this problem f (x) = x which is defined in [0, 2] and L = 2

Hence $b_n = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}dx$

Using integration by parts we have

$$b_n = \left[\frac{-2}{n\pi}\cos\frac{n\pi x}{2} - \frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{-4}{n\pi}\cos n\pi$$

Using (2) in (1) we have

$$f(x) = x = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi \sin\frac{n\pi x}{2}$$

$$x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2}.....\right)$$

Example 2

Expand f (x) = x, 0 < x < 2 in half – range cosine series.

Solution

By formula for cosine series in the interval 0 < x < L

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad (1)$$

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$ (2)

In this problem f (x) = x which is defined in [0, 2] and L = 2.

Hence $f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$

and $a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$ (3)

Using integration by parts, for

u = x and $\frac{dv}{dx} = \cos\left(\frac{n\pi x}{2}\right)$ or $v = \frac{2}{n\pi}\sin\frac{n\pi x}{2}$

we have

$u \qquad v = \frac{2}{n\pi}\sin\frac{n\pi x}{2}$

$u' = 1 \quad v_1 = \frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}$

$u'' = 0 \quad v_2 = \ldots\ldots\ldots\ldots$

By extension Theorem for integration by parts, we have,

$\int u\frac{dv}{dx}dx = uv - u'v_1 + u''v_2 - \ldots\ldots\ldots$ (See Sengo's presentation of the theorem)

$$\int x\cos\frac{n\pi x}{2}\,dx = \frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2} \qquad (4)$$

Using (4) in (3) we have

$$a_n = \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_0^2$$

$$= \left(\frac{4}{n\pi}\sin n\pi + \frac{4}{n^2\pi^2}\cos n\pi\right) - \left(0 + \frac{4}{n^2\pi^2}\right)$$

$$= \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

if n = 0, $a_0 = \frac{2}{2}\int_0^2 x dx = \left[\frac{x^2}{2}\right]_0^2 = 2$

Then $x = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$

or $x = 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\ldots\right)$

Example 3

Expand f (x) = sin x in a Fourier cosine series in $0 < x < \pi$

Solution

By formula for half – Range Fourier cosine series for f (x), in [0 L], we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$

Here f (x) = sin x and L = π

Then $a_n = \frac{2}{\pi}\int_0^{\pi}\sin x\cos\frac{n\pi x}{\pi}dx$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}2\sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(x+nx) + \sin(x-nx)]dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x + \sin(1-n)x]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{(n+1)} - \frac{-\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x-1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1+\cos n\pi}{n+1} - \frac{1+\cos n\pi}{n-1}\right]$$

$$= \frac{-2(1+\cos n\pi)}{\pi(n^2-1)} \text{ if } n \neq 1$$

When n = 1, $a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\right]$$

$$= 0$$

Hence $f(x) = \sin x = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1+\cos \pi x)}{n^2-1}\cos nx$

or $\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \dots\right)$

Example 4

Expand $f(x) = \begin{cases} x & 0 < x < 4 \\ 8-x & 4 < x < 8 \end{cases}$ in a series of sines.

Solution

By formula for Half – Range sine series in [0, L] for f (x) we have

$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

where $b_n = \frac{2}{L}\int_0^{L} f(x)\sin\frac{n\pi x}{L}$ (1)

Here f (x) = x in 0 < x < L and f (x) = 8 – x in 4 < x < 8 and the interval is [0, 8]
or L = 8.

Then $b_n = \frac{2}{8}\int_0^8 f(x)\sin\frac{n\pi x}{8}\,dx$

$$= \frac{1}{4}\int_0^4 f(x)\sin\frac{n\pi x}{8}\,dx + \frac{1}{4}\int_4^8 f(x)\sin\frac{n\pi x}{8}\,dx$$

$$= \frac{1}{4}\int_0^4 x\sin\frac{n\pi x}{8}\,dx + \frac{1}{4}\int_4^8 (8-x)\sin\frac{n\pi x}{8}\,dx \qquad (2)$$

Using integration by parts,

$$\frac{1}{4}\int_0^4 x\sin\frac{n\pi x}{8}\,dx$$

$$= \left[\frac{-8}{n\pi}x\cos\frac{n\pi x}{8} + \sin\frac{n\pi x}{8}\frac{8^2}{n^2\pi^2}\right]_0^4$$

$$= \frac{-32}{n\pi}\cos\frac{n\pi}{2} + \frac{8^2}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad (3)$$

and $\frac{1}{4}\int_4^8 (8-x)\sin\frac{n\pi x}{8}$

$$= \frac{1}{4}\left[-(8-x)\frac{8}{n\pi}\cos\frac{n\pi x}{8} - \frac{8^2}{n^2\pi^2}\sin\frac{n\pi x}{8}\right]_4^8 \text{ using Integration by parts}$$

$$= \frac{1}{4}\left[\frac{-32}{n\pi}\cos\frac{n\pi}{2} + \frac{8^2}{n^2\pi^2}\sin\frac{n\pi}{2} + 4\cos\frac{n\pi}{2}\left(\frac{8}{n\pi}\right)\right] + \frac{8^2}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad (4)$$

Substituting (3) and (4) in (2) we have

$$b_n = \frac{1}{4}\left[\frac{-32}{n\pi}\cos\frac{n\pi}{2} + \frac{8^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{32}{n\pi}\cos\frac{n\pi}{2} + \frac{8^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{4}\left[\frac{128}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{32}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Then $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{8}$

or $f(x)=\frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{8}$

Exercise 11

1. a). Expand f (x) = cos x, $0 < x < \pi$ in a Fourier sine series.

   b). How will you define f (x) at x = 0 and x = π

2. a). Prove that for $0 \le x \le =\pi$

$$x(\pi - x)=\frac{\pi^2}{6}-\left(\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 6x}{3^2}+\ldots\right)$$

   b). Hence show that $\sum_{1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

3. a). Prove that for $0 \le x \le =\pi$

$$x(\pi - x)=\frac{8}{\pi}\left(\frac{\sin x}{1^3}+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+\ldots\right)$$

   b). By differentiating the above series

   prove that for $0 \le x < =\pi$

$$x=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right)$$

4. Expand f (x) = x, $0 < x < 2\pi$ as a Fourier sine series.

5. Expand $f(x)=\begin{cases} x & 0 < x < \pi \\ 2\pi - x & \pi < x < 2\pi \end{cases}$ as a Fourier cosine series.

6. If $f(x)=\begin{cases} x & 0 < x < \pi \\ x-2\pi & \pi < x < 2\pi \end{cases}$ as a Fourier sine series.

Summary

You have learnt the following from this chapter

Let f (x) be defined in the Interval [0, L]

(i) If you need Half – Range Fourier cosine series in [0, L] for f (x)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$

(ii) If you need Half range Fourier sine series in [0, L] for f (x)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

Further Reading

3. Advanced Calculus
   By Watson Fulks
   University of Colorodo
   John Wiley and sons
   New York. Toronto. Singapore.

4. Mathematical Methods for Science students
   By G. Stephenson
   Addison Wesley
   Longman Gate. Harrow, London.

5. Fourier Series
   Schaum's Outline Series
   Mc. Graw Hill Company
   New York. Sydney.