SMA 203: LINEAR ALGEBRA II

Course Outline

Linear Transformations. Linear mappings and their matrices with respect to an arbitrary basis, change of basis, conjugation of matrices. Theory of determinants, characteristic polynomials, eigen values, eigen vectors and diagonalization. Minimal polynomial and cayley-Hamilton theorem.

References:

1. Linear algebra by Hoffman and Kunze.
2. Shaum's series of Theory and Problems in Linear Algebra.

Linear Transformations

Def: Let V and W be vector spaces over $\Re$ (or a field F). A linear transformation/mapping from V into W is a function $T : V \to W$ such that

1) $T(x+y) = T(x) + T(y)\ \forall x, y \in V$
2) $T(\alpha x) = \alpha T(x) \forall\ x \in V, \alpha \in \Re$

Domain of T , $dom\,T = \{x \in V : Tx \quad is\ defined\}$

Co domain of T or Range of T or image of T , $imT = \{Tx : x \in V \quad or\ x \in domT\}$.

A linear transformation preserves the vector space operation of addition and multiplication.

Examples:

1. Show that the zero function $T : V \to W, Tx = 0 \forall x \in V$ is linear.

Soln:

$T(x+y) = T(x) + T(y) = 0$, since $T(x) = 0$ and $T(y) = 0$
$T(\alpha x) = \alpha Tx = \alpha(0) = 0$

2. Show that the identity function $I : V \to V$, $Ix = x$ is a linear transformation.

Soln:

$I(x_1) = x_1 \quad I(x_2) = x_2$

$I(x_1 + x_2) = x_1 + x_2 = I(x_1) + I(x_2)$

$I(\alpha x) = \alpha x = \alpha(Ix)$

3. Show that $\forall \alpha \in \Re, T : V \to V, Tx = \alpha x$ is a linear transformation

Soln:

$T(x+y) = \alpha(x+y) = \alpha x + \alpha y = Tx + Ty$

$T(\beta x) = \alpha(\beta x) = \beta(\alpha x) = \beta Tx$

4. Determine whether or not $T_1(x, y, z) = (2x + y, xy, x - z)$ is a linear transformation.

Soln:

$T_1((x_1,y_1,z_1)+(x_2,y_2,z_2))$
$=T_1(x_1+x_2\ ,y_1+y_2,\ z_1+z_2)$
$=(2(x_1+x_2)+(y_1+y_2),\ (x_1+x_2)(y_1+y_2),\ (x_1+x_2)-(z_1+z_2))$...(1)

On the other hand,

$T_1(x_1,y_1,z_1)+T_2(x_2,y_2,z_2)$
$=(2x_1+y_1,\ x_1y_1, x_1-z_1)+(2x_2+y_2,\ x_2y_2,\ x_2-z_2)$
$=(2(x_1+x_2)+(y_1+y_2),\ x_1y_1+x_2y_2,\ (x_1+x_2)-(z_1+z_2))$... (2)

Comparing equations (1) and (2), we see that

$(x_1+x_2)(y_1+y_2)\neq x_1y_1+x_2y_2$

$\Rightarrow T_1$ is not linear.

5. Determine whether or not $T_2(x,y,z)=\left(2x,3y,\dfrac{x}{z}\right)$

Soln:

$$T_2(x_1+x_2,\ y_1+y_2,\ z_1+z_2)=\left(2x_1+2x_2,\ 3y_1+3y_2,\ \frac{x_1+x_2}{z_1+z_2}\right)\text{...(1)}$$

$T_2(x_1,y_1,z_1)+T_2(x_2,y_2,z_2)$

$$=\left(2x_1,3y_1,\frac{x_1}{z_1}\right)+\left(2x_2,3y_2,\frac{x_2}{z_2}\right)$$

$$=\left(2x_1+2x_2,3y_1+3y_2,\ \frac{x_1}{z_1}+\frac{x_2}{z_2}\right)\text{...} \qquad (2)$$

$$\frac{x_1+x_2}{z_1+z_2}\neq\frac{x_1}{z_1}+\frac{x_2}{z_2}$$

$\Rightarrow T_2$ is not linear.

6. Determine whether or not $T(x,y,z)=(3,2x,4y-z)$ is a linear transformation.

Soln:

$T_3(\alpha x,\alpha y,\alpha z)=(3,2\alpha x,4\alpha y-\alpha z)$
$\neq\alpha(3,\ 2x,\ 4y-z)=\alpha T_3(x,y,z)$

$\Rightarrow T_3$ is not linear.

Exercise 1

1.Let A be an $m\times n$ matrix. Show that the map $T:\Re^n\to R^m$ defined by $T_A=Ax$ is linear.

2.Show that :

$(a)T(x,y)=x^2+y^2$ is not linear
$(b)T(x,y)=7x-5y$ is linear
$(c)T(x,y)=2x^2-4y$ is not linear
$(d)T(x,y)=(e^x,e^y)$ is not linear

3.Determine which of the following mappings are linear

(a) $T(x,y)=(x-2y,\ 2x+5y)$

b) $T(x,y)=(\alpha x+\beta y,\ \gamma x+\delta y), \alpha,\beta,\gamma,\delta$ fixed scalars

c) $T(x,y)=7e^{x}e^{y}$

d) $T:\Re^2\to\Re^3$ defined by

$Tx=(e_1\cdot x,\ e_2\cdot x,\ 0\cdot x), e_1,e_2\in\Re^2$

Theorem:

(a) If $T:V\to W$ is a linear transformation, then

(i) $T0=0$ (ii) $T(-x)=-Tx$

(b) $T:V\to W$ is linear iff $T(\alpha x+\beta y)=\alpha Tx+\beta Ty \quad \forall x,y\in V \alpha,\beta\in\Re$

Proof:

(a) (i) $\forall x\in V\ 0x=0$, since T is linear

$$T(0)=T(0x)=0Tx=0$$

$(ii)\ T(-x)=T(-1x)=-1Tx=-Tx$

(b) $\Rightarrow$ Suppose T is linear

$$T(\alpha x+\beta y)=T(\alpha x)+T(\beta y)$$
$$=\alpha Tx+\beta Ty$$

$\Leftarrow$ Suppose $T(\alpha x+\beta y)=\alpha Tx+\beta Ty$

Take $\alpha=\beta=1$; $\quad T(x+y)=Tx+Ty$

Take $\beta=0$;

$$T(\alpha x)=T(\alpha x+0y)=\alpha Tx+0Ty=\alpha Tx$$

Thus T is linear.

Using an induction process we can show that if $T:V\to W$ is linear then

$T(\alpha_1V_1+\alpha_2V_2+\ldots+\alpha_nV_n)=\alpha_1TV_1+\alpha_2TV_2+\ldots+\alpha_nTV_n$

Remark: To show that $T:V\to W$ is linear, it is enough to show that $T(\alpha x+\beta y)=\alpha Tx+\beta Ty$

Example: Show that $T:\Re^2\to\Re^2; T(x,y)=(x-2y,2x+3y)$ is linear.

Solution:

Let $U=(x_1,y_1)$ and $V=(x_2,y_2)$

$$\begin{aligned}T(\alpha U+\beta V)&=T\{(\alpha x_1,\alpha y_1)+(\beta x_2,\beta y_2)\}\\
&=T(\alpha x_1+\beta x_2,\alpha y_1+\beta y_2)\\
&=(\alpha x_1+\beta x_2-2\alpha y_1-2\beta y_2,2\alpha x_1+2\beta x_2+3\alpha y_1+3\beta y_2)\\
&=(\alpha x_1-2\alpha y_1+\beta x_2-2\beta y_2,2\alpha x_1+3\alpha y_1+2\beta x_2+3\beta y_2)\\
&=(\alpha x_1-2\alpha y_1,2\alpha x_1+3\alpha y_1)+(\beta x_2-2\beta y_2,2\beta x_2+3\beta y_2)\\
&=\alpha(x_1-2y_1,2x_1+3y_1)+\beta(x_2-2y_2,2x_2+3y_2)\\
&=\alpha T(U)+\beta T(V)\end{aligned}$$

Theorem: If $T:V\to W$ is a linear transformation

a)If U is a subspace of V, $T(U)$ is a subspace of W.

b)If U is a subspace of W, $T^{-1}(U)$ is a subspace of V.

Proof:

a) Let $Tx, Ty \in T(U), \alpha \in \Re$ i.e. $x, y \in U$

$$Tx + Ty = T(x+y) \in T(U)$$
$$\alpha Tx = T(\alpha x) \in T(U) \text{ since } \alpha x \in U$$
,

$$\Rightarrow T(U) \text{ is a subspace of W}$$

(b) $T^{-1}(U) = \{x \in V : Tx \in U\}$

Let $x, y \in U, \alpha \in \Re$

$T^{-1}x, T^{-1}y \in T^{-1}(U)$; $T^{-1}x + T^{-1}y = T^{-1}(x+y) \in T^{-1}(U)$

Note:

$x + y \in U$ and

$$T(T^{-1}x + T^{-1}y) = T(T^{-1}x) + T(T^{-1}y)$$
$$= x + y \in U$$
$$\Rightarrow T^{-1}x + T^{-1}y \in T^{-1}(U)$$

$$\alpha x \in U, T(\alpha T^{-1}x) = \alpha T(T^{-1}x) = \alpha x \in U$$
$$= \alpha T^{-1}x \in T^{-1}(U)$$

Rank &Nullity ; Kernel &Image of a Linear Transformation

Def: A linear transformation $T : V \to W$ is said to be <u>surjective/onto</u> if $T(V) = W$ i.e. $W = imT$ or Range $T = W$.

T is said to be <u>injective/one-to-one</u> if any two different vectors in V have different images in W i.e. $T(x) = Ty$ iff $x=y$.

T is said to be <u>bijective/bijection</u> if T is **one-one** and **onto**.

Examples:

1. Show that $T : \Re^2 \to \Re^2$, defined by $Tx = T(x_1, x_2) = (x_1, -x_2)$ is one-to-one and onto.

Solution:

$T(x_1, -x_2) = (x_1, x_2) \Rightarrow$ and therefore T is onto.

$$T(x) = T(x_1, -x_2) = (x_1, x_2)$$
$$T(y) = (y_1, -y_2) = (y_1, y_2)$$

$T(x) = T(y) \Rightarrow (x_1, -x_2) = (y_1, -y_2) \Rightarrow x_1 = x_2,\ y_1 = y_2$

$(x_1, -x_2) = (y_1, -y_2) \Rightarrow x_1 = y_1, x_2 = y_2$, hence one-one.

2. Show that $T : \Re^2 \to \Re$, defined by $T(x, y) = x^2 + y^2$, is not one-one.

Solution:

$$T(x, y) = x^2 + y^2$$
$$T(x, -y) = x^2 + y^2$$
$$T(-x, y) = x^2 + y^2$$
$$T(-x, -y) = x^2 + y^2$$
$$\therefore T(x, y) = T(x, -y) = T(-x, y) = (-x, -y) = T(y, x)$$

$\Rightarrow T$ is not one-one

T is not onto; since

$x^2+y^2\geq 0$
$-5\neq T(x,y)=x^2+y^2$

3.Show that $T:\Re^2\to\Re^2$, defined by $T(x,y)=(x-2y,2x+3y)$
is one-one and onto

Solution:

Suppose T(x) = T(y), where $x=(x_1,x_2)$, $y=(y_1,y_2)$

$(x_1-2x_2,2x_1+3x_2)=(y_1-2y_2,2y_1+3y_2)$

then $x_1=y_1$ & $x_2=y_2$

Therefore T is one-one.

To check whether it is onto,

Simply solve for x and y in the following:

$x_1-2x_2=x$, $\Rightarrow x_1=\frac{1}{7}(3x+2y)$

$2x_1+3x_2=y\Rightarrow$ $x_2=\frac{1}{7}(-2x+y)$ $\forall(x,y)\in\Re^2$

i.e. $(x,y)=T\left(\frac{1}{7}(3x+2y),\frac{1}{7}(-2x+y)\right)$

4. Show that $T:P^n\to P^{n-1}$ defined by $T(p(x))=\frac{d}{dx}(p(x))$ is not one-one but it is onto.

Solution:

Is not one-one since

$\frac{d}{dx}a=\frac{d}{dx}b=0\Rightarrow$ but $a\neq b$

It is onto since $\frac{d}{dx}\left[\int_a^x p(x)\right]=p(x)\Rightarrow$

5.Let $e_1\in\Re^2$, $e_1=(1,0)$ and let $T:\Re^2\to R^2$ be defined by
$Tx=(x\cdot e_1)e_1$. Show that T is not one-one but it is onto.

Solution:

$T(2,1)=(2,1).(1,0)(1,0)=(2,0)$
$T(2,5)=(2,5)\cdot(1,0)(1,0)=(2,0)$
$(2,1)\neq(2,5)$

General $T(a,b)=T(a,c)$, T is not onto.

$\operatorname{Im}T=\{(a,0):a\in\Re\}\neq\Re^2$

Definition: Let $F:V\to U$ be a linear mapping. The image of F ,written $\operatorname{Im}F$,is the set of image points in U .

$\operatorname{Im}F=\{u\in U:F(v)=u \quad \textit{for some } v\in V$

The **Kernel of** F ,written $KerF$,is the set of elements of V which map into $0\in U$
i.e. $KerF=\{v\in V;F(v)=0\}$.

Theorem:

Let $T:V\to W$ be a linear transformation. Then
1] $\operatorname{Im}T$ is a subspace of W

2] $KerT$ is a subspace of V

Proof:

1] Already done in previous theorem

2]Let $x, y \in KerT \quad \alpha \in \Re$

$T(x+y) = Tx + Ty = 0 + 0 = 0$

$\Rightarrow x + y \in KerT$

$T(\alpha x) = \alpha Tx = \alpha 0 = 0$

$\Rightarrow \alpha x \in KerT$ and so $KerT$ is a subspace of V.

Theorem: Let V be of finite dimension and let $F : V \to U$ be a linear mapping. Then $\dim V = \dim(KerF) + \dim(imF)$

i.e. the sum of the dimensions of the image and Kernel of a linear mapping is equal to the dimension of its domain.

Remark: Let $F : V \to U$ be a linear mapping. The **rank** of F is defined to be the **dimension of its image** and the **nullity** of F is defined to be the **dimension of its Kernel.**

i.e. $rank(F) = \dim(\operatorname{Im} F)$ *and* $nullity(F) = \dim(KerF)$

The preceding theorem thus yields the following formula for F when V has finite dimension:

$rank(F) + nullity(F) = \dim(V)$

Recall that the rank of a matrix A was originally defined to be the dimension of its column space and of its row space. Observe that if we now view A as a linear mapping, then both definitions correspond since the image of A is precisely its column space.

Corollary:

Let $T : V \to W$ be linear with $\dim V = \dim W = n$ .The following statements are equivalent.

1] T is a bijection

2] Nullity $T = 0$

3] Rank $T = n$

Remark: A linear transformation $T : V \to W$ is injective (one-one) iff nullity $T = 0$,and is surjective (onto) iff $rankT = \dim W$

Examples:

1. Let $T : \Re^4 \to \Re^3$ be a linear transformation linear defined by

$Te_1 = (1,3,2),\ Te_2 = (1,0,0),\ Te_3 = (2,3,2),\ Te_4 = (4,3,2)$ Where e_1, e_2, e_3, e_4 are the standard unit vectors in $\Re^4$

a) Determine $rankT$ and a basis of $\operatorname{Im} T$

b) Find nullity T.

Solution:

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 0 & 0 \\ 2 & 3 & 2 \\ 4 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 2 \\ 2 & 3 & 2 \\ 4 & 3 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dim \operatorname{Im} T = rankT = 2$$

Basis= $\{(1,0,0),(1,3,2)\}$

or $\{(1,0,0),(2,3,2)\}$

or $\{(1,0,0,),(4,3,2)\}$

(or any other two vectors)

(b) $\dim \Re^4 = rankT + nullityT;\ 4 = 2 + nullityT \Rightarrow nullityT = 2$

2. Let $T:\Re^2 \to \Re^2$ be defined by $T(x,y)=(x+y,x+y)$ Find the rank, nullity, Kernel and basis of Ker T and Im T.

Solution:

$\operatorname{Im} T = \{(x+y,x+y) \mid x,y \in \Re\}$
$= \{(a,a): a \in \Re\}$
$\{a(1,1): a \in \Re\}$

$rankT = 1 \quad basis = \{(1,1)\}$
$nullityT = \dim \Re^2 - rankT = 2-1 = 1$
$KerT = \{(x+y,x+y) = (0,0)\}$
$= \{(x,y) \mid x = -y\}$
$= \{(x,-x) \mid x \in \Re\}$
$= \{x(1,-1) \mid x \in \Re\}$
$nullityT = 1 \quad , basis = \{(1,-1)\}$

3. $T:\Re^3 \to \Re^3$ defined by $T(x_1,x_2,x_3) = (x_3, x_1+x_2, x_1+x_2+x_3)$. Find the rank, nullity, Kernel and basis of KerT and Im T.

Solution:

$T(x_1,x_2,x_3) = (x_3, x_1+x_2, x_1+x_2+x_3)$

$KerT = \{x \in \Re^3 \mid Tx = 0\}$
$= \{(x_1,x_2,x_3) \mid (x_3, x_1+x_2, (x_1+x_2+x_3) = 0)\}$
$= \{(x_1,x_2,0) \mid x_1+x_2 = 0\}$
$= \{(x_1,-x_1,0) \mid x_1 \in \Re\}$
$= \{x(1,-1,0) \mid x \in \Re\}$

$nullityT = 1, basis\{(1,=1,0)\}$

$rankT = \dim \Re^3 - nullityT = 3-1 = 2$

$T(1,0,0) = (0.1.1)$

$T(0,1,0) = (0,1,1)$

$T(0,0,1) = (1,0,1)$

$Basis\ for\ \operatorname{Im} T = \{(1,0,1),(0,1,1)\}$

Recall

Linear mappings

Let V and U be vector spaces over the same field K.

A mapping $T:V \to U$ is called a linear mapping (or a linear transformation) if:

1. For any $k,w \in V, T(v+w) = T(v) + T(w)$
2. For any scalar $k \in K$ and $v \in V, T(kv) = kTv$

Examples:

1. Let $T:R^2 \to \Re^2$ be defined by $T(x,y) = (x+y,x)$.Show that T is a linear transformation.

Soln:

Condition 1:

Let $(x_1,y_1),(x_2,y_2) \in \Re^2$ and scalar $k \in \Re$,

$$T[(x_1, y_1),+(x_2, y_2)] = T(x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2 + y_1 + y_2, x_1 + x_2)$$
$$T(x_1, y_1) + T(x_2, y_2) = (x_1 + y_1, x_1) + (x_2 + y_2, x_2)$$
$$= (x_1 + x_2 + y_1 + y_2, x_1 + x_2)$$
$$\therefore T[(x_1, y_1) + (x_2, y_2)] = T(x_1, y_1) + T(x_2, y_2)$$

Condition 2:

$$T[\lambda(x_1, y_i)] = T(\lambda x_1, \lambda y_1) = (\lambda x_1 + \lambda y_1, \lambda x_1)$$
$$= \lambda(x_1 + y_1, x_1)$$
$$= \lambda T(x_1, y_1)$$

Hence T is linear.

2. Let $T : \Re^2 \to \Re$ be defined by $T(x, y) = x + y + 1$.Show that T is not linear.

Soln:

Condition 1:

Now Let u=$(1,0)$, v=$(0,1) \in \Re^2$

$T(u + v)$

$= T[(1,0) + (0,1)] = T(1,1) = 1 + 1 + 1 = 3$

But *T(u)* + *T(v)* =

$$T(1,0) + T(0,1) = (1 + 0 + 1) + (0 + 1 + 1) = 4$$
$$\therefore T[(1,0) + (0,1)] \neq T(1,0) + T(0,1)$$

Hence T is not linear.

3. Let T be the linear mapping $T : \Re^2 \to \Re^2$ defined by $T(3,1) = (2,-4)$ and $T(1,1) = (0,2)$.Find $T(x, y)$. In particular find $T(7,4)$.

Soln:

First write (x, y) as a linear combination of (3,1) and (1,1)

$$(x, y) = \lambda_1(3,1) + \lambda_2(1,1) \Rightarrow x = 3\lambda_1 + \lambda_2 \cdots (i)$$
$$y = \lambda_1 + \lambda_2 \cdots (ii)$$

Solving for λ_1 and λ_2 in terms of x and y, we get

$$\lambda_1 = \frac{x}{2} - \frac{y}{2} \text{ and } \lambda_2 = \frac{3}{2}y - \frac{x}{2}$$
$$\therefore (x, y) = \left(\frac{x}{2} - \frac{y}{2}\right)(3,1) + \left(\frac{3}{2}y - \frac{x}{2}\right)(1,1)$$

Now,

$$T(x,y)=T\left[\left(\frac{x}{2}-\frac{y}{2}\right)(3,1)\right]+\left[\left(\frac{3}{2}y-\frac{x}{2}\right)(1,1)\right]$$
$$=\left(\frac{x}{2}-\frac{y}{2}\right)T(3,1)+\left(\frac{3}{2}y-\frac{x}{2}\right)T(1,1)$$
$$=\left(\frac{x}{2}-\frac{y}{2}\right)(2,-4)+\left(\frac{3}{2}y-\frac{x}{2}\right)(0,2)$$
$$=(x-y,\ -2x+2y)+(0,\ 3y-x)$$
$$=(x-y,\ -3x+5y)$$

$$\therefore T(7,4)=(3,-1)$$

Exercise:

1. Show that the mapping defined by $T(x,y,z)=2x+y-4z$ where $T:\Re^2\to\Re$ is a linear map.
2. Show that the mapping $T:\Re^2\to\Re^3$ defined by $T(x,y)=(x+1,2y,x+y)$ is not linear.
3. Let T be the linear mapping $T:\Re^2\to\Re$ defined by $T(1,1)=3$ and $T(0,1)=2$ .Find $T(x,y)$.
4. Let $T:\Re^4\to\Re^3$ be defined by $T(1,1,1,1)=(1,0,-1),T(0,1,1,1)=(2,1,1),T(0,0,1,1)=(1,-2,3)$ and $T(0,0,0,1)=(-2,3,2)$. Find $T(x,y,z,w)$ and hence determine $T(2,3,4,5)$.

Matrices and linear transformations

Suppose $\{e_1,e_2,\ldots e_n\}$ is a basis for a vector space V over a field K and for $v\in V$ .Suppose that $v=a_1e_1+a_2e_2+\cdots+a_ne_n$ .Then the coordinate vector of v relative to the basis $\{e_i\}$ is the column vector

$$[v]_e=\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$$

Example: Let $v=(4,3)\in\Re^2$ and $\{f_1=(1,3),\ f_2=(2,5)\}$ be a basis of $\Re^2$. Find the coordinate vector of v relative to the basis $\{f_i\}$.

Solution:

$(4,3)=\lambda_1(1,3)+\lambda_2(2,5)$ so

$$\lambda_1+2\lambda_2=4\ldots(i)$$
$$3\lambda_1+5\lambda_2=3\ldots(ii)$$

$$\Rightarrow 3\lambda_1+6\lambda_2=12$$
$$3\lambda_1+5\lambda_2=3$$
$$\lambda_2=9$$

From (i) $\lambda_1=4-18=-14$

So $(4,3)=-14f_1+9f_2$, and so $[v]_e=\begin{pmatrix}-14\\9\end{pmatrix}$

Matrix representation of linear operators

Let T be a linear mapping from a vector space V to itself $(T:V\to V)$ .Suppose $\{e_1,e_2\ldots,e_n\}$ is a basis for V .Now,

$T(e_1),T(e_2),\ldots,T(e_n)$ are vectors in V and each is a linear combination of $\{e_i\}$:

$T(e_1) = a_{11}e_1 + a_{12}e_2 + \ldots + a_{1n}e_n$
$T(e_2) = a_{21}e_1 + a_{22}e_2 + \ldots + a_{2n}e_n$
$\ldots$
$T(e_n) = a_{n1}e_1 + a_{n2}e_2 + \ldots + a_{nn}e_n$

Definition: The transpose of the above matrix of coefficients, denoted by $[T]_e$ is called the matrix representation of T relative to the basis of $\{e_1, e_2, \ldots e_n\}$ or simply the matrix of T in the basis $\{e_i\}$.

Examples

1. Find the matrix representation of $T : \Re^2 \to \Re^2$ defined by $T(x, y) = (2y,\ 2x - y)$ relative to the standard basis $e_1 = (1,0)$, $e_2 = (0,1)$ of $\Re^2$.

Soln:

If $(a,b) \in \Re^2$; then $(a,b) = ae_1 + be_2$

$T(e_1) = T(1,0) = (0,2) = 0e_1 + 2e_z$

$T(e_2) = T(0,1) = (2,-1) = 2e_1 - e_2 \qquad \therefore [T]_e = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}$

2. Let $T : \Re^2 \to \Re^2$ be a linear mapping defined by $T(x, y) = (4x - 2y,\ 2x + y)$ .Compute the matrix of T in the basis $\{e_1 = (1,1),\ e_2 = (-1,0)\}$.

Soln:

Let $(a,b) \in \Re$, we write (a,b) as a linear combination of e_1 and e_2,

$$(a,b) = \lambda_1(1,1) + \lambda_2(-1,0)$$

$$\Rightarrow \begin{matrix} \lambda_1 - \lambda_2 = a \\ \lambda_1 \quad = b \end{matrix} \quad \Rightarrow \begin{matrix} \lambda_1 = b \\ \lambda_2 = b - a \end{matrix}$$

So $(a,b) = b(1,1) + (b-a)(-1,0)$.

Now,

$$T(1,1) = (2,3) = 3(1,1) + 1(-1,0) \qquad = 3e_1 + 1e_2$$
$$T(-1,0) = (-4,-2) = -2(1,1) + 2(-1,0) = -2e_1 + 2e_2 \qquad ;\text{So } [T]_e = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

3. Let V be the vector space of polynomials in t over $\Re$ of degree ≤ 3 and let $D : V \to V$ be the differential linear mapping defined by $D(f(t)) = \frac{d}{dt}(f(t))$ .Compute the matrix of D with respect to the basis $\{1, t, t^2, t^3\}$.

Solution:

$$D(1) = 0 = 0.1 + o.t + o.t^2 + o.t^3$$
$$D(t) = 1 = 1.1 + 0.t + o.t^2 + o.t^3$$
$$D(t^2) = 2t = 0.1 + 2.t + 0.t^2 + 0.t^3$$
$$D(t^3) = 3t^2 = 0.1 + 0.t + 3.t^2 + o.t^3$$

$$\text{So } [D] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise:

1. Find the matrix representation of each of the following linear mappings $T : \Re^3 \to \Re^3$ relative to the standard basis $e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)$

(i) $T(x, y, z) = (2x - 3y + 4z, 5x - y + 2z, 4x + 2y)$

(ii) $T(x,y,z)=(2y+z,\ \ x-4y,3x)$

2. Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and T be a linear mapping from $\Re^3$ to $\Re^3$ defined by $T(v)=Av$ where v is written as a column vector.

Find the matrix of T in each of the following basis

(i) $e_1=(1,0), e_2=(0,1)$

(ii) $e_1=(1,3), e_2=(2,5)$

3.Each of the sets (i) $\{1,t,e^t,te^t\}$ and (ii) $\{e^{3t},te^{3t},t^2e^{3t}\}$ is a basis of a vector space V of functions $f:\Re\to\Re$ .Let D be the differential mapping defined by $D(f(t))=\dfrac{d}{dt}(f(t))$ .Find the matrix of D in each of the given basis.

Ans:

(i) $[D]=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}$ (ii) $[D]=\begin{bmatrix}3 & 0 & 0\\ 1 & 3 & 0\\ 0 & 2 & 3\end{bmatrix}$

Theorem: Let $\{e_1,e_2,\ldots,e_n\}$ be a basis of a vector space V and T be any linear mapping from V to itself. Then for any vector $V\in v,$ $[T]_e[v]_e=[T(v)]_e$

That is, if we multiply the coordinate vector of V by the matrix representation of T , then we obtain the coordinate vector of $T(v)$.

Example: Let T be the linear mapping $T:\Re^3\to\Re^3$ defined by $T(x,y,z)=(2y+z,x-4y,3x)$.

(i)Find the matrix of T in the basis $\{f_1=(1,1,1), f_2=(1,1,0), f_3=(1,0,0)\}$

(ii)Verify that $[T]_f[v]_f=[T(v)]_f$ for any vector $v\in\Re^3$

Solution:

(i)Let ***v***= (a,b,c) be any vector in $\Re^3$ .Write (a,b,c) as a linear combination of $\{f_i\}$.

$(a,b,c)=\lambda_1(1,1,1)+\lambda_2(1,1,0)+\lambda_3(1,0,0)$. So,

$$\left.\begin{aligned}\lambda_1+\lambda_2+\lambda_3&=a\\ \lambda_1+\lambda_2\quad&=b\\ \lambda_1\quad&=c\end{aligned}\right\}\Rightarrow\lambda_1=c,\ \lambda_2=b-c,\ \lambda_3=a-b$$

And so $(a,b,c)=cf_1+(b-c)f_2+(a-b)f_3$. Since

$T(x,y,z)=(2y+z,x-4y,3x)$

$T(1,1,1)=(3,-3,3)=3f_1-6f_2+6f_3$

$T(1,1,0)=(2,-3,3)=3f_1-6f_2+5f_3$

$T(1,0,0)=(0,1,3)=3f_1-2f_2-1f_3$

and $[T]_f=\begin{bmatrix}3 & 3 & 3\\ -6 & -6 & -2\\ 6 & 5 & -1\end{bmatrix}$

(ii)Suppose $v=(a,b,c)\in\Re^3$,then

$$v=(a,b,c)=cf_1+(b-c)f_2+(a-b)f_3 \quad \text{And so } [v]_f=\begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$$

Also,

$$T(v)=T(a,b,c)=(2b+c,\ a-4b,\ 3a)=3af_1+(-2a-4b)f_2+(-a+6b+c)f_3$$

And so

$$[T(v)]_f=\begin{bmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{bmatrix}$$

Thus

$$[T]_f[v]_f=\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}\begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}=\begin{bmatrix} 3c+3b-3c+3a-3b \\ -6c-6b+6c-2a+2b \\ 6c+5b-5c-a+b \end{bmatrix}=\begin{bmatrix} 3a \\ -2a-4b \\ -a+c+6b \end{bmatrix}=[T(v)]_f$$

Exercise:

Let T be the linear mapping $T:\Re^2\to\Re^2$ defined by $T(x,y)=(5x+y,\ 3x-2y)$. Find the matrix of T with respect to the basis $\{f_1=(1,2), f_2=(2,3)\}$. Verify that $[T]_f[v]_f=[T(v)]_f$ for any $v\in\Re^2$

Ans: $[V]_f=\begin{bmatrix} -3a+2b \\ 2a-b \end{bmatrix}$ $\quad [T]_f=\begin{bmatrix} -23 & -39 \\ 15 & 26 \end{bmatrix}$; $\ [T(v)]_f=\begin{bmatrix} -a-7b \\ 7a+4b \end{bmatrix}$

Change of basis

Definition: Let $\{e_1,e\ ,\ldots,e_n\}$ be a basis of V and let $\{f_1,f\ ,\ldots,f_n\}$ be another basis. Suppose

$$f_1=a_{11}e_1+a_{12}e_2+\cdots+a_{1n}e_n$$
$$f_2=a_{21}e_1+a_{22}e_2+\ldots+a_{2n}e_n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_n=a_{n1}e_1+a_{n2}e_2+\ldots+a_{nn}e_n$$

The transpose P of the above matrix of coefficients is termed the transition matrix from the 'old' basis $\{e_i\}$ to the new basis $\{f_i\}$:

$$P=\begin{bmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{bmatrix}$$

Since the vectors $\{f_1,f\ ,\ldots,f_n\}$ are linearly independent, the matrix P is invertible. In fact the inverse P^{-1} is the transition matrix from the basis $\{f_i\}$ back to the basis $\{e_i\}$.

Examples:

1. Consider the following basis of $\Re^2$

$e_1=\{(1,0), e_2=(0,1)\}$ and $f_1=\{(1,1),\ f_2=(-1,0)\}$.

Then
$$f_1=(1,1)\ =\ 1(1,0)+1(0,1)=\ e_1+e_2$$
$$f_2=(-1,0)=-1(1,0)+0(0,1)=-e_1+0e_2$$

Hence the transition matrix P from $\{e_i\}$ to $\{f_i\}$ is $P=\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

Recall: If $(a,b)\in\Re^2, (a,b)=bf_1+(b-a)f_2$ so

$$e_1=(1,0)=0(1,1)-1(-1,0)=0f_1-f_2$$
$$e_2=(0,1)=1(1,1)+1(-1,0)=f_1+f_2$$

Hence the transition matrix Q from the basis $\{f_i\}$ back to $\{e_i\}$ is $Q=\begin{bmatrix}0&1\\-1&1\end{bmatrix}$

Observe that Q and P are inverses:

$$PQ=\begin{bmatrix}1&-1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\-1&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$$

2. Consider the following basis of $\Re^3$

$\{e_1=(1,0,0), e_2=(0,1,0), e_3=(0,0,1)\}$
$f_1=\{(1,1,1), f_2=(1,1,0), f_3=(1,0,0)\}$

(i)For $v=(a,b,c)\in\Re^3$, find $[v]_e$ and $[v]_f$

(ii)Find the transition matrix P from $\{e_i\}$ to $\{f_i\}$ and Q from $\{f_i\}$ to $\{e_i\}$ .Verify that $Q=P^{-1}$

(iii)Show that $[T]_f=P^{-1}[T]_e P$ where T is defined by $T(x,y,z)=(2y+z, x-4y, 3x)$

Solution:

(i) $v=(a,b,c)=a(1,0,0)+b(0,1,0)+c(0,1,0)=ae_1+be_2+ce_3 \quad \therefore [v]_e=\begin{bmatrix}a\\b\\c\end{bmatrix}$

Also

$$v=(a,b,c)=\lambda_1(1,1,1)+\lambda_2(1,1,0)+\lambda_3(1,0,0)$$

$$\begin{aligned}&\lambda_1+\lambda_2+\lambda_3=a && \lambda_1=c\\ \Rightarrow\ &\lambda_1+\lambda_2 \qquad\quad =b\}\Rightarrow && \lambda_2=b-c\\ &\lambda_1 \qquad\qquad\quad\ =c && \lambda_1=a-b\end{aligned}$$

So $v=cf_1+(b-c)f_2+(a-b)f_3 \quad \therefore [v]_f=\begin{bmatrix}c\\b-c\\a-b\end{bmatrix}$

(ii) $\begin{aligned}f_1&=(1,1,1)=1e_1+1e_2+1e_3\\ f_2&=(1,10)=e_1+e_2+oe_3\\ f_3&=(1,0,0)=e_1+oe_2+oe_3\end{aligned}$ And $P=\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}$

$\begin{aligned}e_1&=(1,0,0)=0f_1+0f_2+1f_3\\ e_2&=(0,10)=0f_1+1f_2-1f_3\\ e_3&=(0,0,1)=1f_1-1f_2+0f_3\end{aligned}$ $\quad Q=\begin{bmatrix}0&0&1\\0&1&-1\\1&-1&0\end{bmatrix}$

$PQ=\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&-1\\1&-1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. So $Q=P^{-1}$

(iii)

$T(x,y,z)=(2y+z,x-4y,3x)$
$T(e_1)=T(1,0,0)=(0,1,3) \quad =0e_1+e_2+3e_3$
$T(e_2)=T(0,1,0)=(2,-4,0) \quad =2e_1-4e_2+0e_3$
$T(e_3)=T(0,0,1)=(1,0,0) \quad =e_1+oe_2+oe_3$

$$\therefore [T]_e=\begin{bmatrix}0 & 2 & 1\\ 1 & -4 & 0\\ 3 & 0 & 0\end{bmatrix}$$

$T(f_1)=T(1,1,1)=(3,-3,3)=3f_1-6f_2+6f_3$
$T(f_2)=T(1,10)=(2,-3,3)=3f_1-6f_2+5f_3$
$T(f_3)=T(1,0,0)=(0,1,3)= \quad 3f_1-2f_2-f_3$

$$\therefore [T]_f=\begin{bmatrix}3 & 3 & 3\\ -6 & -6 & -2\\ 6 & 5 & -1\end{bmatrix}$$

Now,

$$P^{-1}[T]_e P=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & -1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}0 & 2 & 1\\ 1 & -4 & 0\\ 3 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$$
$$=\begin{bmatrix}3 & 0 & 0\\ -2 & -4 & 0\\ -1 & 6 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}=\begin{bmatrix}3 & 3 & 3\\ -6 & -6 & -2\\ 6 & 5 & -1\end{bmatrix}=[T]_f$$

Theorem: Let P be the transition matrix from a basis $\{e_i\}$ to a basis $\{f_i\}$ in a vector space V. Then for any vector $v\in V, P[v]_f=[v]_e$ .Hence $[v]_f=P^{-1}[v]_e$

Thus the effect of P is to transform the coordinate vector in the new basis $\{f_i\}$ back to the coordinate vector in the old basis $\{e_i\}$

Example: Let $v=(a,b)\in\Re^2$ and consider the following basis of $\Re^2$ $\{e_1=(1,0),e_2=(0,1)\}$ and $\{f_1=(1,1), f_2=(-1,0)\}$

$$(a,b)=b(1,1)+(b-a)(-1,0)$$
$$=bf_1+(b-a)f_2$$

So $[v]_f=\begin{bmatrix}b\\ b-a\end{bmatrix}$. Also $(a,b)=a(1,0)+b(0,1)=ae_1+be_2$. So $[v]_e=\begin{bmatrix}a\\ b\end{bmatrix}$

From the previous example,

$$P=\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix} \text{and } Q=P^{-1}=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}$$

$$P[v]_f = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ b-a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = [v]_e$$
$$P^{-1}[v]_e = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ b-a \end{bmatrix} = [v]_f$$

Theorem: Let P be the transition matrix from a basis $\{e_i\}$ to a basis $\{f_i\}$ in a vector space V .Then for any linear operator T on V , $[T]_f = P^{-1}[T]_e P$

Example: Let T be the linear mapping $T:\Re^2 \to \Re^2$ defined by $T(x,y) = (4x-2y, 2x+y)$, $\{e_1 = (1,0), e_2 = (0,1)\}$ and $\{f_1 = (1,1), f_2 = (-1,0)\}$ be two basis of $\Re^2$ .Verify that $[T]_f = P^{-1}[T]_e P$

Solution:
Recall: $(a,b) = ae_1 + be_2$ and $(a,b) = bf_1 + (b-a)f_2$
So
$T(e_1) = T(1,0) = (4,2) = 4e_1 + 2e_2$
$T(e_2) = T(0,1) = (-2,1) = -2e_1 + e_2 \qquad \therefore [T]_e = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$
Also
$T(f_1) = T(1,1) = \qquad (2,3) = 3f_1 + f_2$
$T(f_2) = T(-1,0) = (-4,-2) = -2f_1 + 2f_2 \qquad$ So $[T]_f = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$
From the previous example, $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$
Now,
$$P^{-1}[T]_e P = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} = [T]_f$$

Exercise:
Consider the following basis of $\Re^3$
$\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$
$\{f_1 = (1,-1,0), f_2 = (1,0,-1), f_3 = (0,0,1)\}$
(i)For v= $(a,b,c) \in \Re^3$,find $[v]_e$ and $[v]_f$
(ii)Find the transition matrix P from $\{e_i\}$ to $\{f_i\}$ and Q from $\{f_i\}$ to $\{e_i\}$.
(iii)Verify that $Q = P^{-1}$
(iv)Show that $P^{-1}[v]_e = [v]_f$,for any vector $v \in \Re^3$
(v)Show that the matrix representation $[T]_f = P^{-1}[T]_e P$ where T is defined by $T(x,y,z) = (-2x+y+z, x-2y+z, x+y-2z)$

So far we have been dealing with linear mappings from one vector space to the same vector space i.e $T:V \to V$ .These linear mappings are normally called **linear operators**.
Next we consider the general case of linear mappings from one vector space into another.
Let V and U be vector spaces over the same field K and $\dim V = m$ and $\dim U = n$ .Further more let $\{e_1, e_2, \ldots, e_m\}$ be a basis of V and $\{f_1, f_2, \ldots, f_n\}$ be a basis of U. Suppose $T:V \to U$ is a linear mapping. Then

$T(e_1), T(e_2), \ldots, T(e_m)$ are vectors in U and so each of them can be written as a linear combination of $\{f_1, f_2, \ldots, f_n\}$
.

$$T(e_1) = a_{11} f_1 + a_{12} f_2 + \cdots + a_{1n} f_n$$
$$T(e_2) = a_{21} f_1 + a_{22} f_2 + \ldots + a_{2n} f_n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$T(e_m) = a_{m1} f_1 + a_{m2} f_2 + \ldots + a_{mn} f_n$$

The transpose of the above matrix of coefficients is denoted by $[T]_e^f$ and is called the matrix representation of T relative to the basis $\{e_1, e_2, \ldots, e_n\}$ and the basis $\{f_1, f_2, \ldots, f_n\}$

$$\text{i.e } [T]_e^f = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{m1} \\ a_{12} & a_{22} & \ldots & a_{m2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n} & a_{2n} & \ldots & a_{mn} \end{bmatrix}$$

Theorem: For any vector $\boldsymbol{v} \in V$ **,** $[T]_e^f [v]_e = [T(v)]_f$

That is, if we multiply the coordinate vector of v in the basis $\{e_i\}$ by $[T]_e^f$ we obtain the coordinate vector of $T(v)$ with respect to the basis $\{f_i\}$

Examples:

1. Let $T : \Re^3 \to \Re^2$ be the linear mapping defined by $T(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$

(i)Find the matrix representation of T relative to the following basis of $\Re^3$ and $\Re^2$:

$\{f_1 = (1,1,1), f_2(1,1,0), f_3(1,0,0)\}$ and $\{g_1 = (1,3), g_2 = (2,5)\}$

(ii)Verify that for any vector $\mathbf{v} \in \Re^3$

$$[T]_f^g [v]_f = [T(v)]_g$$

Solution:

Let $(a,b) \in \Re^2$,then

$$(a,b) = \lambda_1 (1,3) + \lambda_2 (2,5)$$

$$\left.\begin{aligned} \lambda_1 + 2\lambda_2 &= a \\ 3\lambda_1 + 5\lambda_2 &= b \end{aligned}\right\} \Rightarrow \quad \begin{aligned} 3\lambda_1 + 6\lambda_2 &= 3a \\ 3\lambda_1 + 5\lambda_2 &= b \\ \hline \lambda_2 &= 3a - b \\ \lambda_1 &= 2b - 5a \end{aligned}$$

So $(a,b) = (2b - 5a) g_1 + (3a - b) g_2$

Since $T(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$,

$$\begin{aligned} T(f_1) &= T(1,1,1) = (1,-4) = -7g_1 + 4g_2 \\ T(f_2) &= T(1,1,0) = (5,-4) = -33g_1 + 19g_2 \\ T(f_3) &= T(1,0,0) = (3,1) = -13g_1 + 8g_2 \end{aligned} \qquad \text{And} \qquad [T]_f^g = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

(ii)Let $v = (a,b,c)$,then $(a,b,c) = cf_1 + (b-c) f_2 + (a-b) f_3$ So $[v]_f = \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$

$T(v) = T(a,b,c) = (3a+2b-4c, a-5b+3c)$
$= \left[2(a-5b+3c) - 5(3a+2b-4c)\right] g_1 + \left[3(3a+2b-4c) - (a-5b+3c)\right] g_2$
$= (-13a-20b+26c) g_1 + (8a+11b-15c) g_2$

So $[T(v)]_g = \begin{bmatrix} -13a-20b+26c \\ 8a+11b-15c \end{bmatrix}$

Now

$$[T]_f^g [v]_f = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix} \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix} = \begin{bmatrix} -7c-33b+33c-13a+13b \\ 4c+19b-19c+8a-8b \end{bmatrix} = \begin{bmatrix} -13a-20b+26c \\ 8a+11b-15c \end{bmatrix} = [T(v)]_g$$

2.Let $T:\Re^2 \to \Re^2$ be defined by $T(x,y) = (2x-3y, x+4y)$.Find the matrix of T in the basis $\{e_1 = (1,0), e_2 = (0,1)\}$ and $\{f_1 = (1,3), f_2 = (2,5)\}$ of $\Re^2$ respectively.

Solution:

If $(a,b) \in \Re^2, (a,b) = (2b-5a) f_1 + (3a-b) f_2$

Since $T(x,y) = (2x-3y, x+4y)$,

$T(e_1) = T(1,0) = (2,1) = -8f_1 + 5f_2$
$T(e_2) = T(0,1) = (-3,4) = 23f_1 - 13f_2$

So $[T]_e^f = \begin{bmatrix} -8 & 23 \\ 5 & -13 \end{bmatrix}$

3.Let P_n denote the vector space of all polynomials in x of degree $\le n$ over $\Re$.

Let $T: P_2 \to P_3$ be the linear mapping defined by $T(p(x)) = xp(x-3)$

that is, $T\left(c_0 + c_1 x + c_2 x^2\right) = x\left(c_0 + c_1(x-3) + c_2(x-3)^2\right)$

Find the matrix representation $[T]_B^{B'}$ where $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$

Solution:

$T(1) = x = 0.1 + 1.x + 0.x^2 + 0.x^3$
$T(x) = x(x-3) = x^2 - 3x = 0.1 - 3.x + 1.x^2 + 0.x^3$
$T(x^2) = x(x-3)^2 = x(x^2-6x+9) = x^3 - 6x^2 + 9x$
$= 0.1 + 9.x - 6x^2 + 1.x^3$

So $[T]_B^{B'} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$

4. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 3 \end{bmatrix}$ be the matrix of $T: P_2 \to P_2$ with respect to the basis $B = \{v_1, v_2, v_3\}$ where

$v_1 = 3x + 3x^2, v_2 = -1 + 3x + 2x^2, v_3 = 3 + 7x - 2x^2$

(a)Find $(i)\left[T(v_1)\right]_B, (ii)\left[T(v_2)\right]_B, (iii)\left[T(v_3)\right]_B$

(b)Find $(i) T(v_1), (ii) T(v_2)$ and $(iii) T(v_3)$

Solution:

(a) $(i)\left[T(v_1)\right]_B = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$, $(ii)\left[T(v_2)\right]_B = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $(iii)\left[T(v_3)\right]_B = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$

(b)

$(i) T(v_1) = 1v_1 + 2v_2 + 6v_3$
$= 1(3x + 3x^2) + 2(-1 + 3x + 2x^2) + 6(3 + 7x - 2x^2)$
$= 16 + 51x - 5x^2$

$(ii) T(v_2) = 3v_1 + 0v_2 - 2v_3$
$= 3(3x + 3x^2) + 0(-1 + 3x + 2x^2) - 2(3 + 7x - 2x^2)$
$= -6 - 5x + 13x^2$

$(iii) T(v_3) = -1v_1 + 5v_2 + 4v_3$
$= -1(3x + 3x^2) + 5(-1 + 3x + 2x^2) + 4(3 + 7x - 2x^2)$
$= 7 + 40x - x^2$

5.Let $A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}$.Recall that A determines a linear mapping $F : \Re^3 \to \Re^2$ defined by $F(v) = Av$ where v is written as a column vector.

(i)Show that the matrix representation of F relative to the standard basis of $\Re^3$ and $\Re^2$ is the matrix A itself: $[F] = A$

(ii)Find the matrix representation of F relative to the following basis of $\Re^3$ and $\Re^2$
$\{f_1 = (1,1,1), f_2 = (1,1,0), f_3 = (1,0,0)\}, \{g_1 = (1,3), g_2 = (2,5)\}$

Recall that A determines a linear mapping $F : \Re^3 \to \Re^2$ defined by $F(v) = Av$, v column.

Solution

(i)

$$F(1,0,0) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2e_1 + 1e_2$$

$$F(0,1,0) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} = 5e_1 - 4e_2$$

$$F(0,0,1) = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{pmatrix} -3 \\ 7 \end{pmatrix} = -3e_1 + 7e_2$$

So $[F] = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} = A$

(ii)Recall:If $(a,b) \in \Re^2, (a,b) = (2b - 5a)g_1 + (3a - b)g_2$

$$F(f_1)=\begin{bmatrix}2 & 5 & -3\\ 1 & -4 & 7\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{pmatrix}4\\4\end{pmatrix}=-12g_1+8g_1$$

$$F(f_2)=\begin{bmatrix}2 & 5 & -3\\ 1 & -4 & 7\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{pmatrix}7\\-3\end{pmatrix}=-41g_1-24g_2 \qquad \text{So } [F]_f^g=\begin{bmatrix}-12 & -41 & -8\\ 8 & 24 & 5\end{bmatrix}$$

$$F(f_3)=\begin{bmatrix}2 & 5 & -3\\ 1 & -4 & 7\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{pmatrix}2\\1\end{pmatrix}=-3g_1+5g_2$$

Exercise

1. Find the matrix representation of each of the following linear mappings relative to the basis of $\Re^n$

(i) $T:\Re^2\to\Re^3$ defined by $T(x,y)=(3x-y,2x+4y,5x-6y)$

(ii) $T:\Re^4\to\Re^2$ defined by $T(x,y,s,t)=(3x-4y+2s-5t,5x+7y-s-2t)$

2. Let $T:P_2\to P_2$ be the linear mapping defined by $T(P(x))=P(2x+1)$ that is $T(c_0+c_1x+c_2x^2)=c_0+c_1(2x+1)+c_2(2x+1)^2$

(i)Find $[T]_B$ with respect to the basis $B=\{1,x,x^2\}$

(ii)Compute $T(2-3x+4x^2)$

3. Let $T:\Re^3\to\Re^2$ be the linear mapping which is represented by the matrix $\begin{bmatrix}-7 & -33 & -13\\ 4 & 19 & 8\end{bmatrix}$

With respect to the following basis of

$\Re^3: e_1=(1,1,1), e_2=(1,1,0), e_3=(1,0,0)$

$\Re^2: g_1=(1,3), g_2=(2.5)$

Find

(i) $[T(e_1)]_g, [T(e_2)]_g, [T(e_3)]_g$

(ii) $T(e_1), T(e_2), T(e_3)$

4.Let $T:\Re^4\to\Re^3$ be a linear mapping such that

$T(1,0,0,0)=(2,3,6), T(0,1,0,0)=(1,2,0), T(0,0,1,0)=(-1,2,-3), T(0,0,0,1)=(0,2,-1)$

Find the matrix representation of T relative to the basis

$\Re^4: (1,1,0,0),(0,1,1,0),(0,0,1,1),(1,0,0,1)$

$\Re^3: (1,0,0),(0,1,0),(0,0,1)$

Similarity matrices/Conjugation of matrices

Suppose A and B are square matrices for which there exists an invertible matrix P such that $B=P^{-1}AP$ .Then B is said to be similar to A .Note that if B is similar to A thus A is similar to B .

Theorem: Similarity of matrices is an equivalence relation

Proof:

(i)The identity I is invertible and $I=I^{-1}$.Since $A=I^{-1}AI$, A is similar to A *(reflexibility)*

(ii)Suppose A is similar to B .Then there exists an invertible matrix P such that $A=P^{-1}BP$. Hence $B=(P^{-1})^{-1}AP^{-1}$ and P^{-1} is invertible. Thus B is similar to A . *(symmetric)*

(iii) Suppose A is similar to B and B is similar to C. Then there exists invertible matrices P and Q such that $A = P^{-1}BP$ and $B = Q^{-1}CQ$. Hence $A = P^{-1}BP = P^{-1}(Q^{-1}CQ)P = (QP)^{-1}C(QP)$ and QP is invertible. Thus A is similar to C. *(transitivity).*

Theorem: If A and B are similar, then they have the same determinant.

Proof:

Since A and B are similar, there exists an invertible matrix P such that $B = P^{-1}AP$

So

$$\begin{aligned}
\det B &= \det\left(P^{-1}AP\right) \\
&= \det\left(P^{-1}\right)\det A \det P \\
&= \det\left(P^{-1}\right)\det P \det A \\
&= \frac{1}{\det P}\det P \det A \\
&= \det A
\end{aligned}$$

So A and B have the same determinant.

Determinant of a Linear Mapping

The determinant of a linear mapping T, $\det T$, is the determinant of any matrix representation of T.

Example: Let $T : \Re^2 \to \Re^2$ be defined by $T(x, y) = (x + y, -2x + 4y)$. Find the determinant of T.

Solution:

We can choose any basis B of $\Re^2$ and compute the determinant of $[T]_B$

If we choose $B = \{e_1 = (1,0), e_2 = (0,1)\}$

$T(1,0) = (1,-2) = 1e_1 - 2e$

$T(0,1) = (1,4) = 1e_1 + 4e_2$

So $[T]_B = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ And $\det T = 1 \times 4 - (-2) \times 1 = 6$

Had we choosen $B' = \{(1,1), (1,2)\}$. Let $(a,b) \in \Re^2$, then $(a,b) = \lambda_1(1,1) + \lambda_2(1,2)$

$$\begin{aligned}
\lambda_1 + \lambda_2 &= a \\
\lambda_1 + 2\lambda_2 &= b \\
\hline
-\lambda_2 &= a - b \quad \Rightarrow \lambda_2 = b - a, \quad \lambda_1 = 2a - b
\end{aligned}$$

So $(a,b) = (2a - b)(1,1) + (b - a)(1,2)$

$T(1,1) = (2,2) = 2(1,2) + 0(1,2)$

$T(1,2) = (3,6) = 0(1,1) + 3(1,2)$

So $[T]_{B'} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ And $\det T = (2 \times 3) - (0 \times 0) = 6$

Definition: A linear mapping T is said to be **diagonalizable** if for some basis $\{e_1, e_2, \ldots, e_n\}$ it can be represented as a diagonal matrix.

Similarly, a square matrix A is said to be diagonalizable if it is similar to a diagonal matrix i.e. if there exists a matrix P such that $D = P^{-1}AP$ or $A = P^{-1}DP$.

Theorem: Let A be a matrix representation of a linear mapping T. Then T is diagonalizable if and only if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Example: Let $T(x, y) = (x + 4y, -2x + 3y)$ and let $\{e_1 = (1,0), e_2 = (0,1)\}$ and $\{f_1 = (1,1), f_2 = (2,-1)\}$ be two basis of $\Re^2$.

(a) Find the transition matrix P from $\{e_i\}$ to $\{f_i\}$ and the transition matrix $Q = P^{-1}$ from $\{f_i\}$ to $\{e_i\}$, (b)Show that $P^{-1}[T]_e P=[T]_f$ is a diagonal matrix of T i.e T is diagonalizable.

Solution

(a)

$$\begin{aligned} f_1 &= (1,1)=1(1,0)+1(0,1)=1e_1+1e_2 \\ f_2 &= (2,-1)=2(1,0)-1(0,1)=2e_1-1e_2 \end{aligned} \qquad \text{So } P=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}$$

Next, let $(a,b)\in\Re^2$ and let $(a,b)=\lambda_1(1,1)+\lambda_2(2,-1)$

$$\text{So } \begin{aligned} \lambda_1+2\lambda_2&=a\\ \lambda_1-\lambda_2&=b \end{aligned}$$

$$\overline{\lambda_2=a-b};\quad \lambda_2=\frac{a-b}{3};\quad \lambda_1=\frac{a+2b}{3}\;\therefore(a,b)=\left(\frac{a+2b}{3}\right)f_1+\left(\frac{a-b}{3}\right)f_2$$

$$\begin{aligned} e_1&=(1,0)=\frac{1}{3}f_1+\frac{1}{3}f_2\\ e_2&=(0,1)=\frac{2}{3}f_1-\frac{1}{3}f_2 \end{aligned} \qquad \text{So } Q=P^{-1}=\begin{bmatrix}\frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{bmatrix}$$

(b)Since $T(x,y)=(x+4y,2x+3y)$

$$\begin{aligned} T(e_1)&=T(1,0)=(1,2)=e_1+2e_2\\ T(e_2)&=T(0,1)=(4,3)=4e_1+3e_2 \end{aligned} \qquad \text{So } [T]_e=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$$

Also

$$\begin{aligned} T(f_1)&=T(1,1)=(5,5)=5f_1+0f_2\\ T(f_2)&=T(2,-1)=(-2,1)=0f_1-f_2 \end{aligned} \qquad \therefore [T]_f=\begin{bmatrix}5 & 0\\ 0 & -1\end{bmatrix}\text{, a diagonal matrix.}$$

Now,

$$P^{-1}[T]_e P=\begin{bmatrix}\frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{5}{3} & \frac{10}{3}\\ -\frac{1}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}=\begin{bmatrix}5 & 0\\ 0 & -1\end{bmatrix}=[T]_f$$

Hence T is diagonalizable.

Example: Let T be the vector space of 2×2 matrices over $\Re$ and let $M=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$.Let T be a linear mapping on$V$ defined by $T(A)=M(A)$.Find the matrix of T with respect to the standard basis of V.

Hint: Standard basis of V is $\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\}$

Solution:

$$T(e_1)=T\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\3&0\end{pmatrix}$$
$$=1e_1+0e_2+3e_3+0e_4$$
$$T(e_2)=T\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&3\end{pmatrix}$$
$$=0e_1+1e_2+0e_3+3e_4$$
$$T(e_3)=T\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}2&0\\4&0\end{pmatrix}$$
$$=2e_1+0e_2+4e_3+0e_4$$
$$T(e_4)=T\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&2\\0&4\end{pmatrix}$$
$$=0e_1+2e_2+0e_3+4e_4$$
$$\therefore T=\begin{bmatrix}1&0&2&0\\0&1&0&2\\3&0&4&0\\0&3&0&4\end{bmatrix}$$

Theory of Determinants

Recall:

(a)Determinant of 2×2 matrices

If $A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$,Then $|A|=a_{11}a_{22}-a_{12}a_{21}$

(b)Determinant of 3×3 matrices

If $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$, then $|A|=a_{11}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}$

Properties of determinants

Let A be a square matrix.

1. If A has a row or column of zeros only then $\det A=0$.
2. If A has two identical rows/columns then $\det A=0$ .If one row/column is a scalar multiple of another, $\det A=0$.
3. If a multiple of a row of A is added to another row to produce matrix B , then $\det B=\det A$.
4. If two rows of A are interchanged to produce matrix B , then $\det B=-\det A$.
5. If one row of A is multiplied by scalar k to produce matrix B ,then $\det B=k\det A$.
6. $\det A=\det A^T$.
7. If B is another square matrix of the same order as A , then $\det(AB)=\det A\cdot\det B$.
8. If A is a diagonal or a triangular matrix, then $\det A$ is equal to the product of the entries in the main diagonal.

Determinant of matrices of order 4×4 and above

There are two main techniques of computing the determinant of matrices of order 4×4 and above.

(a)Cofactor expansion-define minors and cofactors

Recall:(Laplace expansion theorem)

The determinant of the matrix $A=[a_{ij}]$ is equal to the sum of the products obtained by multiplying elements of any row(or column) of A by their respective cofactors.

$$|A|=a_{i1}A_{i1}+a_{i2}A_{i2}+\ldots+a_{in}A_{in}=\sum_{j=1}^{n}a_{ij}A_{ij}$$

and

$|A| = a_{ij}A_{ij} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}$

Note: The trick is to expand along a row(or column)that has a lot of zeros.

Examples

1. Compute $\det A$ where

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & 3 \end{bmatrix}$$

Solution:

Expand along row1 to get,

$$|A| = (-1)^2(4)\begin{vmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & 3 \end{vmatrix} = 4\begin{vmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ 8 & 4 & 3 \end{vmatrix}$$

Next, expand along row1 to get

$$|A| = 4(-1)(-1)^2\begin{vmatrix} 3 & 0 \\ 4 & 3 \end{vmatrix} = -4\begin{vmatrix} 3 & 0 \\ 4 & 3 \end{vmatrix} = (-4)(9-0) = -36$$

Note: A is a lower triangular matrix, therefore $|A| = 4 \times (-1) \times 3 \times 3 = -36$

2. Compute $\det A$,where

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Solution:

Expand along column 1 to get,

$$|A| = 3(-1)^2\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix}$$

Next expand the 4×4 sub matrix along column1

$$|A| = 3 \cdot 2(-1)^2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 6\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 6\{1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}\} = 6\{-2-0+0\} = -12$$

3.Compute the determinant of

$$A = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$$

Solution:

Expand along column 2 to have,

$$|A| = 3(-1)^5 \begin{vmatrix} 4 & -7 & 3 & -5 \\ 0 & 2 & 0 & 0 \\ 5 & 5 & 2 & -3 \\ 0 & 9 & -1 & 2 \end{vmatrix} = -3 \begin{vmatrix} 4 & -7 & 3 & -5 \\ 0 & 2 & 0 & 0 \\ 5 & 5 & 2 & -3 \\ 0 & 9 & -1 & 2 \end{vmatrix}$$

Next expand along row 2,

$$|A| = (-3)(2)(-1)^4 \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} = -6 \begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} = -6\{4\begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 5 & -3 \\ 0 & 2 \end{vmatrix} - 5\begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix}\}$$

$$= -6\{4(1) - 3(10) - 5(-5)\} = -6\{4 - 30 + 25\} = -6(-1) = 6$$

(b)Gauss elimination method

In this technique, multiples of rows and columns are added to other rows and columns, creating zeros which simplifies subsequent calculations. As we saw earlier, this operation does not change the value of the determinant. The strategy is to reduce the matrix to echelon form and use the fact that determinant of a triangular matrix is the product of diagonal entries.

Examples:

1. Compute $\det A$,where

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

Solution: We first factor 2 from the top row,

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

Next we perform elementary operations,

$$2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 + 3R_1 \\ R_4 - R_1 \end{matrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} R_3 + 4R_2$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix} R_4 - \frac{1}{2}R_3 = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \times 1 \times 3 \times -6 \times 1 = 36$$

At times we can combine cofactor expansion and Gauss-Jordan Elimination

2. Compute the determinant of

$$A=\begin{bmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{bmatrix}$$

Solution: Make use of 1 in the (second row, third column) to create zeros below and above it.

$$|A|=\begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}\begin{matrix} R_1-2R_2 \\ \\ R_3+3R_2 \\ R_4+R_2 \end{matrix}=\begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix}$$

Next expand along column 3 to get,

$$|A|=(1)(-1)^5\begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}=-\begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}=-1\{1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}+2\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}+5\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\}=-1\{+(1)+2(-7)+5(-5)\}=\{1-14-25\}=-1(1$$

2. Compute $\det A$ where,

$$A=\begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$$

Solution: Make use of 2 in column 1 to create a zero below it.

$$|A|=\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix}\underset{=}{R_4+R_2}\;=\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & -1 \end{vmatrix}$$

Next expand along column 1 to obtain,

$$|A|=2(-1)^3\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}=-2\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}=-2\{1\begin{vmatrix} 6 & 2 \\ -3 & 1 \end{vmatrix}-2\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix}-1\begin{vmatrix} 3 & 6 \\ 0 & -3 \end{vmatrix}\}=-2\{12-2(3)-1(-9)\}=-2\{12-6+9\}=-2$$

Exercise

1. Compute the determinant of each of the following matrices.

$$(a)\begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}(b)\begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}(c)\begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{bmatrix}(d)\begin{bmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{bmatrix}(e)\begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{bmatrix}(f)\begin{bmatrix} 2 & 5 \\ 4 & 7 \\ 6 & -2 \\ -6 & 7 \end{bmatrix}$$

Eigen values and eigen vectors

Polynomials of matrices and linear operators

Consider a polynomial $f(t)$ over a field K, $f(t)=a_nt^n+a_{n-1}t^{n-1}+a_{n-2}t^{n-2}+\ldots+a_1t+a_0$.If A is a square matrix over K,we define $f(A)=a_nA^n+a_{n-1}A^{n-1}+a_{n-2}A^{n-2}+\ldots+a_1A+a_0I$,where I is the identity matrix of the same order as A.

In particular, we say that A is a zero or root of the polynomial $f(t)$ if $f(A)=0$ (zero matrix of the same order as A).

Examples: Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $f(t)=2t^2-3t+7$ and $g(t)=t^2-5t-2$ .Find $f(A)$ and $g(A)$.

Solution:

$$P(A)=2A^2-3A+7I$$
$$=2\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}^2-3\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$=2\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}-3\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}14 & 20\\ 30 & 44\end{bmatrix}-\begin{bmatrix}3 & 6\\ 9 & 12\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$
$$=\begin{bmatrix}18 & 14\\ 21 & 39\end{bmatrix}$$

$$g(A)=A^2-5A-2I$$
$$=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}^2-5\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}-\begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$
$$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

So A is a root or zero of $g(t)$.

Exercise:

1. Find $f(A)$where $A=\begin{bmatrix}1 & -2\\ 4 & 5\end{bmatrix}$and $f(t)=t^2-3t+7$.
2. Show that $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$ is a zero of $f(t)=t^2-4t+5$

Suppose $T:V\to V$ is a linear operator on a vector space V over a field K .If $f(t)=a_nt^n+a_{n-1}t^{n-1}+a_{n-2}t^{n-2}+\ldots+a_1t+a_0$,we define $f(T)$in the same way as we did for matrices.

That is $f(T)=a_nT^n+a_{n-1}T^{n-1}+a_{n-2}T^{n-2}+\ldots+a_1T+a_0I$,where I is the identity operator.

We also say that T is the zero or root of polynomial $f(t)$ if $f(T)=O$ (the zero operator on V).

Eigen values and eigen vectors

Let $T:V\to V$ be a linear operator on a vector space V over a field K .A scalar $\lambda\in K$ is an eigen value of T if there exists a non-zero vector $\mathbf{v}\in V$ for which $T(\mathbf{v})=\lambda\mathbf{v}$.

Every vector satisfying this relation is then called an eigen vector of T belonging to the eigen value λ.
The set of all such vectors is a subspace of V called the eigen space of λ.
The terms characteristic value and characteristic vector (or proper value and proper vector) are also frequently used instead of eigen value and eigen vector.

Examples:

1.Let $I:V\to V$ be the identity operator on V .Then for any vector $\mathbf{v}\in V,\quad I(\mathbf{v})=1\cdot\mathbf{v}$,so 1 is an eigen value of I & any vector in V is an eigen vector belonging to the eigen value 1.

2.Let D be the differential operator in the vector space V of differential functions i.e $D:V\to V$ .We have $D\left(e^{5t}\right)=5\cdot e^{5t}$ .Hence 5 is an eigen value of D with eigen vector e^{5t}.

If A is an $n\times n$ matrix over a field K ,then $\lambda\in K$ is an eigen value of A if for some non-zero vector(column) $\mathbf{v}\in K^n$ then $A\mathbf{v}=\lambda\mathbf{v}$.

In this case v is an eigen vector of A belonging to the eigen value λ.

Example: Find the eigen values and eigen vectors associated with the matrix $\begin{bmatrix}1 & 2\\ 3 & 2\end{bmatrix}$

Solution: We are seeking for a scalar λ and a non-zero vector $\mathbf{v}=\begin{pmatrix}x\\ y\end{pmatrix}$ such that $A\mathbf{v}=\lambda\mathbf{v}$

$$\begin{bmatrix}1 & 2\\ 3 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\lambda\begin{bmatrix}x\\ y\end{bmatrix}$$

i.e
$$\begin{matrix}x+2y=\lambda x\\ 3x+2y=\lambda y\end{matrix}\Rightarrow\begin{matrix}\lambda x-(x+2y)=0\\ \lambda y-(3x+2y)=0\end{matrix}$$

so $\left.\begin{matrix}(\lambda-1)x-2y=0\\ -3x+(\lambda-2)y=0\end{matrix}\right\}$ (eqn 1) Homogenous system of linear equations

A homogenous system has a non-zero solution if and only if the determinant of the matrix of coefficients is zero:
Now ,

$$\begin{vmatrix}\lambda-1 & -2\\ -3 & \lambda-2\end{vmatrix}=0;\quad (\lambda-1)(\lambda-2)-6=0$$
$$\lambda^2-3\lambda-4=0;\ \lambda^2-4\lambda+\lambda-4=0$$
$$\lambda(\lambda-4)+1(\lambda-4)=0$$
$$(\lambda+1)(\lambda-4)=0\Rightarrow\lambda=-1\ or\ \lambda=4$$

For $\lambda=4$,setting $\lambda=4$ in eqn (1), we get the eigen vectors belonging to the eigen value $\lambda=4$,i.e.

$$\begin{matrix}3x-2y=0\\ -3x+2y=0\end{matrix}\Rightarrow 3x-2y=0\ or\ y=\frac{3}{2}x$$

Thus the dimension of the eigen space of $\lambda=4$ is 1 and a basis for this eigen space is{(2,3)}

So $\mathbf{v}=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix}$ and any other vector belonging to the eigen space of $\lambda=4$ is a multiple of $\mathbf{v}=\begin{pmatrix}2\\ 3\end{pmatrix}$.

For $\lambda=-1$,setting $\lambda=-1$ in (eqn1) we get,

$$\begin{matrix}-2x-2y=0\\ -3x-3y=0\end{matrix}\to -2x-2y=0\to x+y=0$$

Thus the dimensions of the eigen space of $\lambda=-1$ is 1 and a basis for this eigen space is {(1,-1)}

Thus $\mathbf{v}=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}$ is the eigen vector of $\lambda=-1$ and any multiple of $\mathbf{v}=\begin{bmatrix}1\\ -1\end{bmatrix}$

Theorem: Let $T:V\to V$ be a linear operator on V. Then $\lambda\in K$ is an eigen value if T if and only if the mapping $\lambda I-T$ is singular(not invertible)i.e $Ker(\lambda I-T)\neq\{0\}$,so there exists a vector $v\neq 0$ such that $(\lambda I-T)\mathbf{v}=\mathbf{0}$. The set of all eigen vectors of T belonging to λ (eigen space of λ)is then the $Ker(\lambda I-T)$.

Proof: λ is an eigen value of T if and only if there exists a non-zero vector $v\in V$ such that $T(\mathbf{v})=\lambda\mathbf{v}$.

So $(\lambda I)\mathbf{v}-T(\mathbf{v})=\mathbf{0}$ and $(\lambda I-T)\mathbf{v}=\mathbf{0}\Rightarrow\mathbf{v}\neq\mathbf{0}\in Ker(\lambda I-T)$. Thus $\lambda I-T$ is singular.

Example: Find all eigen values and a basis for each eigen space of $T:\Re^3\to\Re^3$ defined by $T(x,y,z)=(2x+y,y-z,2y+4z)$

Solution:

First find the matrix representation of T ,say relative to the standard basis of $\Re^3$

$$T(e_1)=T(1,0,0)=(2,0,0)=2e_1+oe_2+oe_3$$
$$T(e_2)=T(0,1,0)=(1,1,2)=e_1+e_2+2e_3$$
$$T(e_3)=T(0,0,1)=(0,-1,4)=oe_1-e_2+4e_3$$

So $[T]_e=A=\begin{bmatrix}2&1&0\\0&1&-1\\0&2&4\end{bmatrix}$

By the theorem above, λ is an eigen value of A if $|\lambda I-A|=0$. So

$$|\lambda I-A|=\left|\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}-\begin{bmatrix}2&1&0\\0&1&-1\\0&2&4\end{bmatrix}\right|=0\Rightarrow\begin{vmatrix}\lambda-2&-1&0\\0&\lambda-1&1\\0&-2&\lambda-4\end{vmatrix}=0$$

$$(\lambda-2)\begin{vmatrix}\lambda-1&1\\-2&\lambda-4\end{vmatrix}=0\Rightarrow(\lambda-2)\left[(\lambda-1)(\lambda-4)+2\right]=0$$
$$(\lambda-2)(\lambda^2-5\lambda+6)=0$$
$$(\lambda-2)\left[(\lambda-2)(\lambda-3)\right]=0$$
$$(\lambda-2)^2(\lambda-3)=0\Rightarrow\lambda=2\ or\ \lambda=3$$

Therefore the eigen values are $\lambda=2,2$ and $\lambda=3$. To obtain the eigen vectors belonging to $\lambda=2$, solve $(\lambda I-A)\mathbf{v}=0,\lambda=2$:

$$\begin{bmatrix}0&-1&0\\0&1&1\\0&-2&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

So $\left.\begin{matrix}-y=0\\y+z=0\\-2y-2z=0\end{matrix}\right\}\Rightarrow y+z=0$; Thus $\begin{matrix}y=0\\y+z=0\end{matrix}$

This system has one independent solution, say $(1,0,0)$, so the dimension of the eigen space of $\lambda=2$ is one and the basis is $\{(1,0,0)\}$

Eigen space of $\lambda=3$

$$(\lambda I-A)\mathbf{v}=\begin{bmatrix}1&-1&0\\0&2&1\\0&-1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\mathbf{0}\Rightarrow\begin{matrix}x-y=0\\2y+z=0\end{matrix}\Rightarrow\begin{matrix}x-y=0\\2y+z=0\end{matrix}$$

The number of independent solution in the above system is 1,so the dimension of this eigen space is 1 and its basis is $\{(1,1,-2)\}$

2. Let $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$.Find all the eigen values of A and a basis for each eigen space.

Solution:

$|\lambda I - A| = 0$ if λ is an eigen value of A

So

$$\begin{vmatrix}\lambda-1 & -4\\ -2 & \lambda-3\end{vmatrix} = 0 \Rightarrow (\lambda-1)(\lambda-3)-8=0 \Rightarrow \lambda^2-4\lambda-5=0$$

$$(\lambda+1)(\lambda-5)=0 \Rightarrow \lambda=5 \text{ or } \lambda=-1$$

i.e. eigen values of A are $\lambda=5$ and $\lambda=-1$

Eigen space of $\lambda=5$,

$$(\lambda I - A)\mathbf{v} = \begin{bmatrix}4 & -4\\ -2 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{matrix}4x-4y=0\\ -2y+2y=0\end{matrix} \rightarrow -x+y=0$$

This eigen space has dimension 1 and its basis is $\{(1,1)\}$

Eigen space of $\lambda=-1$,

$$\begin{bmatrix}-2 & -4\\ -2 & -4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \qquad \text{So} \quad \left.\begin{matrix}-2x-4y=0\\ -2x-4y=0\end{matrix}\right\} \Rightarrow x+2y=0$$

The system has 1 independent solution; hence the dimension of this eigen space is 1.A basis of this eigen space is $\{(2,-1)\}$

Exercise

1.Find all the eigen values and a basis of each eigen space of the operator $T:\Re^3 \rightarrow \Re^3$ defined by $T(x,y,z)=(x+y+z,2y+z,2y+3z)$

2.For each matrix,find all the eigen values and a basis of each eigen space.

$$i)\ \begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix} \quad ii)\ \begin{bmatrix}2 & -1\\ 1 & 4\end{bmatrix} \quad iii)\ \begin{bmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{bmatrix} \quad iv)\ \begin{bmatrix}-3 & 1 & -1\\ -7 & 5 & -1\\ -6 & 6 & -2\end{bmatrix}$$

Theorem: Non-zero eigen vectors of a matrix belonging to distinct eigen values are linearly independent.

Diagonalization of matrices

Let $T:V \rightarrow V$ be a linear operator on a vector space V with a finite dimension n .Then T can be represented by a diagonal matrix.

$$\begin{bmatrix}\lambda_1 & 0 & 0 & \dots & 0\\ 0 & \lambda_2 & 0 & \dots & 0\\ 0 & 0 & \lambda_3 & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & \lambda_n\end{bmatrix}$$

if and only if there exists a basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of V for which

$$\begin{aligned}T(\mathbf{v}_1) &= \lambda_1\mathbf{v}_1\\ T(\mathbf{v}_2) &= \quad \lambda_2\mathbf{v}_2\\ &\dots\dots\dots\dots\\ T(\mathbf{v}_n) &= \dots \quad \lambda_n\mathbf{v}_n\end{aligned}$$

That is the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ are eigen vectors of T belonging to the eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$.

Theorem: A linear operator $T:V \to V$ can be represented by a diagonal matrix D if and only if V has a basis of eigen vectors of T. In this case the diagonal elements of D are the corresponding eigen values.

In the above theorem, if we let P denote the matrix whose columns are the n linearly independent eigen vectors of A, then $D = P^{-1}AP$

If a linear operator or a matrix can be represented by a diagonal matrix we say that it is diagonalizable and if $P^{-1}AP = D$ is a diagonal matrix, then we say that P diagonalizes A.

Example: Diagonalize

$$a) A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad b) A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

Solution:

$|\lambda I - A| = 0$, if λ is an eigen value of A

$$\begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) - 6 = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

Eigen values of A are $\lambda = 4$ and $\lambda = -1$

Eigen space of $\lambda = 4$, $\begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \begin{matrix} 3x - 2y = 0 \\ -3 + 2y = 0 \end{matrix} \} \Rightarrow 3x - 2y = 0$$

A basis of the eigen space of $\lambda = 4$ is $\{(2,3)\}$

Eigen space of $\lambda = -1$, $\begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \begin{matrix} -2x - 2y = 0 \\ -3x - 3y = 0 \end{matrix} \} \Rightarrow -2x - 2y = 0 \Rightarrow x + y = 0$$

A basis for eigen space of $\lambda = -1$ is $\{(1,-1)\}$

Now, $\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{-2}{5} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{4}{5} \\ \frac{-3}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}, \text{a diagonal matrix.}$$

2. For each matrix find all the eigen values and a basis of each eigen space.

$$a) A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \quad b) B = \begin{bmatrix} -3 & 1 & -1 \\ 7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$$

Which matrix can be diagonalized and why?

Solution:

(a) $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-1 & 3 & -3 \\ -3 & \lambda+5 & -3 \\ -6 & 6 & \lambda-4 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)\begin{vmatrix} \lambda+5 & -3 \\ 6 & \lambda-4 \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ -6 & \lambda-4 \end{vmatrix} - 3\begin{vmatrix} -3 & \lambda+5 \\ -6 & 6 \end{vmatrix} = 0$$

$$(\lambda-1)[(\lambda+5)(\lambda-4)+18]-3[-3(\lambda-4)-18]-3[-18+6(\lambda+5)]=0$$

$$(\lambda-1)(\lambda^2+\lambda-2)-3(-3\lambda-6)-3(6\lambda+12)=0$$

$$\lambda^3+\lambda^2-2\lambda-\lambda^2-\lambda+2+9\lambda+18-18\lambda-36=0$$

$$\lambda^3-12\lambda-16=0$$

Using factor theorem:

$\lambda=-2$ is a zero of the above equation, hence $\lambda+2$ is a factor of $\lambda^3-12\lambda-16$

So

$$\lambda^3-12\lambda-16=0 \Leftrightarrow (\lambda+2)(\lambda^2-2\lambda-8)=0$$

$$(\lambda+2)(\lambda+2)(\lambda-4)=0 \Rightarrow (\lambda+2)^2(\lambda-4)=0$$

Eigen values of A are $\lambda=-2$ and $\lambda=4$

Eigen space of $\lambda=-2$,

$$\begin{bmatrix} 3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Leftrightarrow \left.\begin{matrix} -3x+3y-3z=0 \\ -3x+3y-3z=0 \\ -6x+6y-6z \end{matrix}\right\} \rightarrow -3x+3y-3z=0 \rightarrow x-y+z=0$$

There are two independent solutions.

Hence the dimension of this eigen space is 2, its basis is $\{(1,1,0),(0,1,1)\}\ or(1,0,-1)$

Eigen space of $\lambda=4$,

$$\begin{bmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Leftrightarrow \left.\begin{matrix} 3x+3y-3z=0 \\ -3x+9y-3z=0 \\ -6x+6y=0 \end{matrix}\right\} \rightarrow \begin{matrix} x+y-z=0 \\ -x+3y-z=0 \\ -x+y=0 \end{matrix} \rightarrow \begin{matrix} x+y-z=0 \\ 2x-2y=0 \\ -x+y=0 \end{matrix} \rightarrow \begin{matrix} x+y-z=0 \\ 2x-2y=0 \end{matrix} \rightarrow \begin{matrix} x+y-z=0 \\ x-y=0 \end{matrix}$$

A basis for the eigen space of $\lambda=4$ is $\{(1,1,2)\}$

Since $\{(1,1,2),(1,1,0),(0,1,1)\}$ span $\mathfrak{R}^3$ then A is diagonalizable.

$$P=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad and \quad P^{-1}AP=\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(ii) $B=\begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

$|\lambda I - B| = 0$

$$\begin{vmatrix} \lambda+3 & -1 & 1 \\ 7 & \lambda-5 & 1 \\ 6 & -6 & \lambda+2 \end{vmatrix} = 0$$

$$\lambda+3\begin{vmatrix} \lambda-5 & 1 \\ -6 & \lambda+2 \end{vmatrix} + +1\begin{vmatrix} 7 & 1 \\ 6 & \lambda+2 \end{vmatrix} + 1\begin{vmatrix} 7 & \lambda-5 \\ 6 & -6 \end{vmatrix} = 0$$

$$(\lambda+3)\left[(\lambda-5)(\lambda+2)+6\right]+1\left[7(\lambda+2)-6\right]+1\left(-42-6(\lambda-5)\right)=0$$

$$(\lambda+3)\left(\lambda^2-3\lambda-4\right)+(7\lambda+8)+1(-12-6\lambda)=0$$

$$(\lambda+3)\left(\lambda^2-3\lambda-4\right)+7\lambda+8-12-6\lambda=0$$

$$\lambda^3-3\lambda^2-4\lambda+3\lambda^2-9\lambda-12+7\lambda+8-12-6\lambda=0$$
$$\lambda^3-12\lambda-16=0 \Rightarrow (\lambda+2)^2(\lambda-4)=0$$

Eigen values are $\lambda=-2$ and $\lambda=4$

Eigen space of $\lambda=-2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Leftrightarrow \begin{matrix} x-y+z=0 \\ 7x-7y+z=0 \\ 6x-6y=0 \end{matrix} \to \begin{matrix} x-y+z=0 \\ 7x-7y+z=0 \\ x-y=0 \end{matrix} \Rightarrow \begin{matrix} x-y+z=0 \\ -6x+6y=0 \\ x-y=0 \end{matrix} \to \begin{matrix} x-y+z=0 \\ -6x+6y=0 \end{matrix} \to \begin{matrix} x-y+z=0 \\ -x+y=0 \end{matrix}$$

There is 1 independent solution. A basis for the eigen space of $\lambda=-2$ is $\{(1,1,0)\}$

Eigen space of $\lambda=4$,

$$\begin{bmatrix} 7 & -1 & 1 \\ 7 & -1 & 1 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Leftrightarrow \begin{matrix} 7x-y+z=0 \\ 7x-y+z=0 \\ 6x-6y+6z=0 \end{matrix} \to \begin{matrix} 7x-y+z=0 \\ x-y+z=0 \end{matrix} \to \begin{matrix} 7x-y+z=0 \\ 6x=0 \end{matrix} \Rightarrow x=0$$

There is 1 independent solution.

A basis for the eigen space of $\lambda=4$ is $\{(0,1,1)\}$

Now, $\{(1,1,0),(0,1,1)\}$ are not enough to span $\Re^3$ and so B is not diagonalizable.

Exercise: Diagonalize

1. $A=\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

Note: If one is interested only in determining whether a matrix is diagonalizable and is not concerned with actually finding a diagonalizing matrix P then it suffices to find the dimensions of the eigen spaces.

Let $f:\Re^3 \to \Re^2$ be a linear mapping with $A=\begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}$ as the matrix of f with respect to the standard basis of $\Re^3$ and $\Re^2$. Find the matrix of f with respect to the basis $B=\{(1,1,1),(1,1,0),(1,0,0)\}$ of $\Re^3$ and the basis $B^{'}=\{(1,3),(2,5)\}$ of $\Re^2$.

Solution:

Since A has been obtained using standard basis of $\Re^3$ and $\Re^2$, the images of f and the images of A are exactly the same.

Now,

$$(xy,z)=x(1,0,0)+y(0,1,0)+z(0,0,1)$$

$$\begin{aligned}\therefore\ f(x,y,z) &= xf(1,0,0)+yf(0,1,0)+zf(0,0,1)\\ &= x(2,1)+y(5,-4)+z(-3,7)\\ &= (2x+5y-3z,x-4y+7z)\end{aligned}$$

Let $(a,b)\in\Re^2$, then

$$(a,b)=\lambda_1(1,3)+\lambda_2(2,5)$$

$$\begin{matrix} a=\lambda_1+2\lambda_2 \\ b=3\lambda_1+5\lambda_2 \end{matrix} \Rightarrow \begin{matrix} 3a=3\lambda_1+6\lambda_2 \\ \underline{b=3\lambda_1+5\lambda_2} \\ 3a-b=\lambda_2 \end{matrix}$$

$$\lambda_1=a-6a+2b$$

$$=2b-5a$$

$$\therefore (a,b)=(2b-5a)(1,3)+(3a-b)(2,5)$$

$$f(1,0,0)=(4,4)\ \ =-12(1,3)+8(2,5)$$

$$f(1,1,0)\ =(7,-3)=-41(1,3)+24(2,5)$$

$$\therefore [f]=\begin{pmatrix} -12 & -41 & -8 \\ 8 & 24 & 5 \end{pmatrix}$$

4.Let $V=P_2[x]$ be the vector space of all polynomials of degree less than or equal to 2. Define a linear operator $T:V\to V$ by $T(p(x))=p(2x+3)$. Find the matrix of T relative to the basis $\{1+x+x^2,x+x^2,x^2\}$.

Solution:

$$T(1+x+x^2)=1+(2x+3)+(2x+3)^2=1+2x+3+4x^2+12x+9=13+14x+4x^2=\lambda_1(1+x+x^2)+\lambda_2(x+x^2)+\lambda_2(x^3)$$

$$\begin{matrix} \lambda_1=13 \\ \Rightarrow\ \ \lambda_1+\lambda_2=14\ \ \} \\ \lambda_1+\lambda_2+\lambda_3=4 \end{matrix} \qquad \Rightarrow \begin{matrix} \lambda_1=13 \\ \lambda_2=1 \\ \lambda_3=-10 \end{matrix} \qquad \therefore T(e_1)=13e_1+1e_2-10e_3$$

$$T(x+x^2)=(2x+3)+(2x+3)^2=2x+3+4x^2+12x+9=12+14x+4x^2$$

$$\lambda_1+(\lambda_1+\lambda_2)x+(\lambda_1+\lambda_2+\lambda_3x)=12+14x+4x^2$$

$$\lambda_1=12,\lambda_2=2,\lambda_3=-10; \qquad T(e_2)=12e_1+2e_2-10e_3$$

$$T(x^2)=(2x+3)^2=4x^2+12x+9; \quad \lambda_1=9;\lambda_1+\lambda_2=12\Rightarrow\lambda_2=3;\lambda_3=2\Rightarrow T(e_3)=9e_1+3e_2+2e_3$$

$$\therefore T=\begin{bmatrix} 13 & 12 & 9 \\ 1 & 2 & 3 \\ -10 & -10 & 2 \end{bmatrix}$$

Determinants

Example: Compute $\det A$, where $A=\begin{pmatrix}1 & -4 & 2\\ -2 & 8 & -9\\ -1 & 7 & 0\end{pmatrix}$

Solution:

1.The strategy is to reduce A to echelon form then use the fact that the determinant of a triangular matrix is the product of the diagonal entries.

$$|A|=\begin{vmatrix}1 & -4 & 2\\ -2 & 8 & -9\\ -1 & 7 & 0\end{vmatrix}\begin{matrix}\\ R_2+2R_1\\ R_3+R_1\end{matrix}=\begin{vmatrix}1 & -4 & 2\\ 0 & 0 & -5\\ 0 & 3 & 2\end{vmatrix}R_2\Leftrightarrow R_3=-\begin{vmatrix}1 & -4 & 2\\ 0 & 3 & 2\\ 0 & 0 & -5\end{vmatrix}=(-1)(1)(3)(-5)=15$$

2.Evaluate the determinant of

$$A=\begin{pmatrix}3 & -2 & -5 & 4\\ -5 & 2 & 8 & -5\\ -2 & 4 & 7 & -3\\ 2 & -3 & -5 & 8\end{pmatrix}$$

Solution:

First reduce A to a matrix which has 1 as an entry, such as adding twice the first row to the second row.Use thus 1 to create zeros below and above it.

$$|A|=\begin{vmatrix}3 & -2 & -5 & 4\\ -5 & 2 & 8 & -5\\ -2 & 4 & 7 & -3\\ 2 & -3 & -5 & 8\end{vmatrix}R_2+2R_1=\begin{vmatrix}3 & -2 & -5 & 4\\ 1 & -2 & -2 & 3\\ -2 & 4 & 7 & -3\\ 2 & -3 & -5 & 8\end{vmatrix}\begin{matrix}R_1-3R_2\\ \\ R_3+2R_2\\ R_4-2R_2\end{matrix}=\begin{vmatrix}0 & 4 & 1 & -5\\ 1 & -2 & -2 & 3\\ 0 & 0 & 3 & 3\\ 0 & 1 & -1 & 2\end{vmatrix}$$

Expand along column 1 to get

$$|A|=(1)(-1)^3\begin{vmatrix}4 & 1 & -5\\ 0 & 3 & 3\\ 1 & -1 & 2\end{vmatrix}=-1\begin{vmatrix}4 & 1 & -5\\ 0 & 3 & 3\\ 1 & -1 & 2\end{vmatrix}-1\{4\begin{vmatrix}3 & 3\\ -1 & 2\end{vmatrix}-1\begin{vmatrix}0 & 3\\ 1 & 2\end{vmatrix}-5\begin{vmatrix}0 & 3\\ 1 & -1\end{vmatrix}\}=-1\{4(9)-1(-3)-5(-3)\}=-1\{36+3+15\}$$

Block matrices

Using a system of horizontal and vertical lines we can partition a matrix A into smaller matrices called blocks(or cells).The matrix A is then called a block matrix,clearly a given matrix may be divided into blocks in different ways.

Example: $$A=\left(\begin{array}{ccccc}1 & -2 & 0 & 1 & 3\\ 2 & 3 & 5 & 7 & -2\\ \cline{4-5} 3 & 1 & 4 & 5 & 9\end{array}\right)=\left(\begin{array}{c|cccc}1 & -2 & 0 & 1 & 3\\ \cline{4-5} 2 & 3 & 5 & 7 & -2\\ \hline 3 & 1 & 4 & 5 & 9\end{array}\right)=\left(\begin{array}{c|cccc}1 & -2 & 0 & 1 & 3\\ \cline{1-3} 2 & 3 & 5 & 7 & -2\\ \cline{1-3} 3 & 1 & 4 & 5 & 9\end{array}\right)$$

The convenience of the partition into blocks is that the result of operations on block matrices can be obtained by carrying out the computations with blocks, just as if they were actual elements of the matrix .

Determinant of block matrices

Theorem: Consider the block matrix $M=\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}$ where A and C are square matrices.Then $|M|=|A||C|$.

More generally,if M is a triangular block matrix with square matrices $A_1,\ldots,A_m$ on the diagonal.

$$M=\begin{pmatrix} A_1 & B & \dots & C \\ 0 & A_2 & \dots & D \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_m \end{pmatrix} \qquad |M|=|A_1||A_2|\dots|A_m|$$

Example: Find the determinant of the matrices

$$A=\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix} \quad ,B=\begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

Solution:

a)Since A is a triangular block matrix

$$|A|=\begin{vmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{vmatrix} \text{(triangular block matrix)}$$

$$=\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix}$$
$$=(2\times 2)-(0\times 1)\bullet(1\times 4)-(-2\times 1)$$
$$=(4-0)(4+2)=4\bullet 6=24$$

(b)

$$|B|=\begin{vmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} \begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix} |7| = (4-0)\times(20-6)\times 7 = 4\times 14\times 7 = 392$$

Exercise

Find the determinant of the following matrices.

(i) $A=\begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$ (ii) $B=\begin{pmatrix} 2 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (iii) $C=\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Theorem: Non-zero eigen vectors of a matrix belonging to distinct eigen values are linearly independent.

Proof:

Let $v_1, v_2, \ldots, v_n$ be non-zero eigen vectors of an operator $T: V \to V$ belonging to distinct eigen values.

$\lambda_1, \lambda_2, \ldots, \lambda_n$ .We need to show that if $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0 \Rightarrow a_1 = a_2 = \ldots = a_n = 0$

We proof by induction.

If $n = 1$,then v_1 is linearly independent since $v_1 \neq 0$.

Assume $n > 1$.Suppose $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$ (…eqn 1) where a_i are scalars.

Applying T to the above relation,we obtain by linearlity $a_1T(v_1) + a_2T(v_2) + \ldots + a_nT(v_n) = T0 = 0$

But by hypothesis $T(v_i) = \lambda_i v_i$;hence $a_1\lambda_1v_1 + a_2\lambda_2v_2 + \ldots + a_n\lambda_nv_n = 0$ …(eqn 2)

On the other hand multiplying (eqn 1) by λ_n we have

$a_1\lambda_nv_1 + a_2\lambda_nv_2 + \ldots + a_n\lambda_nv_n = 0$ (…eqn 3)

Subtracting 3 from 2 gives,

$a_1(\lambda_1 - \lambda_n)v_1 + a_2(\lambda_2 - \lambda_n) + \ldots + a_{n-1}(\lambda_{n-1} - \lambda_n)v_{n-1}$

By induction, each of the above coefficients is D.Since λ_i are distinct $\lambda_i - \lambda_n \neq 0$ for $i \neq n$ .Hence $a_1 = \ldots = a_{n-1} = 0$ .Substituting into (eqn1),we get $a_nv_n = 0$ and hence $a_n = 0$ .Thus the v_i are linearly independent.

Example: In the previous example,

$A_{11} = -18 \quad A_{12} = 2 \quad A_{13} = 4$

$A_{21} = -11 \quad A_{22} = 14 \quad A_{23} = 5$

$A_{31} = -10 \quad A_{32} = -4 \quad A_{33} = -8$

Row 1: $|A| = -46 = 2(-18) + 3(2) + (-4)(4)$

Row 3 : $|A| = 1(-10) + (-1)(-4) + 5(-8) = -46$

Column 1: $|A| = 2(-18) + 0(-11) + 1(-10) = -46$

Column 2 : $|A| = 3(2) + (-4)(14) + (-1)(-4) = -46$

Exercise

Suppose $\{e_1, e_2\}$ is a basis of V and $T: V \to V$ is a linear operator for which $T(e_1) = 3e_1 - 2e_2$ and $T(e_2) = e_1 + 4e_2$ .Suppose $\{f_1, f_2\}$ is a basis of V for which $f_1 = e_1 + e_2$ and $f_2 = 2e_1 + 3e_2$ .Find the matrix of T in the basis $\{f_1, f_2\}$.

Ans $\begin{pmatrix} 8 & 11 \\ -2 & -2 \end{pmatrix}$

Trace: The trace of a square matrix $A = (a_{ij})$,written $tr(A)$,is defined to be the sum of its diagonal elements.

$tr(A) = a_{11} + a_{22} + \ldots + a_{nn}$

Theorem: Similar matrices have the same trace.

Thus the trace of a linear operator T is the trace of any one of its matrix representations $tr(T) = tr([T]_e)$

Example: Let V be the space of 2×2 matrices over $\Re$ and let $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.Let T be the linear operator on V defined by $T(A) = MA$ .Find the trace of T .

Solution:

We must find a matrix representation of T .Choose the usual of V .

$\{E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$

Then

$$T(E_1) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = 1 \cdot E_2 + 0 \cdot E_2 + 3 \cdot E_3 + 0.E_1$$

$$T(E_2) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = 0 \cdot E_1 + 1 \cdot E_2 + 0 \cdot E_3 + 3 \cdot E_4$$

$$T(E_3) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = 2 \cdot E_1 + 0 \cdot E_2 + 4 \cdot E_3 + 0 \cdot E_4$$

$$T(E_4) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = 0 \cdot E_1 + 2 \cdot E_2 + 0 \cdot E_3 + 4 \cdot E_4$$

Hence $[T]_E = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}$ and $tr[T] = 1 + 1 + 4 + 4 = 10$

Characteristic Polynomial and Cayley-Hamilon Theorem.

Let A be a square matrix,the determinant $D(x) = \det(xI - A) = |xI - A|$ is called the characteristic polynomial of A .We also call $|xI - AI = 0|$ the characteristic equation of A.

Example: Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 2 & -1 \\ 4 & 0 & -2 \end{bmatrix}$$

Solution:

$$|xI - A| = \begin{vmatrix} x-1 & -3 & 0 \\ 2 & x-2 & 1 \\ -4 & 0 & x+2 \end{vmatrix}$$

$$= (x-1)\begin{vmatrix} x-2 & 1 \\ 0 & x+2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ -4 & x+2 \end{vmatrix}$$

$$= (x-1)(x-2)(x+2) + 3[2(x+2)+4]$$

$$= (x-1)(x^2-4) + 6x + 24$$

$$= x^3 - 4x - x^2 + 4 + 6x + 24$$

$$= x^3 - x^2 + 2x + 28$$

which is the characteristic polynomial of A.

Cayley-Hamilton Theorem: Every matrix is a root(or zero)of its characteristic polynomial.

Example: Verify the Cayley-Hamilton theorem given that

(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

Solution:

$$D(x)=|xI-A|=\begin{vmatrix} x-1 & -2 \\ -3 & x-2 \end{vmatrix}=(x-1)(x-2)-6=x^2-3x-4$$

Thus $f(x)=x^2-3x-4$ is the characteristic polynomial of A .Now,

$$f(A)=A^2-3A+4I$$
$$=\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}^2-3\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}-\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$
$$=\begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix}-\begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix}-\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus A is a zero of its characteristic polynomial.

Theorem: Similar matrices have the same characteristic polynomial.

Proof:

Let A and B be similar matrices.Then there exists an invertible matrix P such that $B=P^{-1}AP$

Now,

$$\begin{aligned} |xI-B| &= |xI-P^{-1}AP| \\ &= |P^{-1}(xI-A)P|, \quad (I=P^{-1}P) \\ &= |P^{-1}||xI-A||P| \\ &= |P^{-1}||P||xI-A| \\ &= \frac{1}{|P|}|P||xI-A| \\ &= |xI-A| \end{aligned}$$

Theorem:Let A be a square matrix over a field K .A scalar $\lambda \in K$ is an eigen value of A if and only if λ is a root of the characteristic polynomial of A .

Theorem: Suppose $M=\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$ where A_1 and A_2 are square matrices. Then the characteristic polynomial of M is the product of the characteristic polynomials of A_1 and A_2 .

Example: Find the characteristic polynomial of

$$A=\left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{array}\right)$$

Solution:

$$D(x)=|xI-A|=\begin{vmatrix} x-2 & -1 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-1 & -1 \\ 0 & 0 & 2 & x-4 \end{vmatrix}=\begin{vmatrix} x-2 & -1 \\ 0 & x-2 \end{vmatrix}\begin{vmatrix} x-1 & -1 \\ 2 & x-4 \end{vmatrix}$$
$$=(x-2)^2\left[(x-1)(x-4)+2\right]=(x-2)^2(x^2-5x+6)=(x-2)^3(x-3)$$

The eigen values of A are $\lambda=2,2,2$ and $\lambda=3$ .We can extend the above theorem to triangular block matrices:

If $M=\begin{pmatrix} A_1 & B & \dots & C \\ 0 & A_2 & \dots & D \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_n \end{pmatrix}$

Where A_i are square matrices, then the characteristic polynomial of M is the product of the characteristic polynomial of the A_i .

Example: Find the characteristic polynomial of

$$A=\left(\begin{array}{cc:cc:c} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 7 \end{array}\right)$$

Solution:

$$|xI-A|=\begin{vmatrix} x-2 & -5 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 & 0 \\ 0 & 0 & x-4 & -2 & 0 \\ 0 & 0 & -3 & x-5 & 0 \\ 0 & 0 & 0 & 0 & x-7 \end{vmatrix}$$
$$=\begin{vmatrix} x-2 & -5 \\ 0 & x-2 \end{vmatrix}\begin{vmatrix} x-4 & -2 \\ -3 & x-5 \end{vmatrix}(x-7)$$
$$=(x-2)^2\left[(x-4)(x-5)-6\right](x-7)$$
$$=(x-2)^2(x^2-9x+14)(x-7)$$
$$=(x-2)^2(x-2)(x-7)(x-7)$$
$$=(x-2)^3(x-7)^2$$

Exercise

Find the characteristic polynomial of each of the following matrices.

(a) $A=\begin{pmatrix} 3 & -5 & 5 \\ 5 & -7 & 5 \\ 5 & -5 & 3 \end{pmatrix}$ (b) $B=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $C=\begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$

Minimal polynomial

Definition: Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_2x^2+a_1x+a_0$ be a polynomial of degree n .Then a_n is called the leading coefficient and and $f(x)$is monic if $a_n=1$

Example: $x^3+x^2+1, x^2+1, x^4+2x^3+3x+1$ are all monic polynomials.

Definition: Let A be a square matrix. Then minimal polynomial of A is a monic polynomial $m(x)$ of lowest degree such that $m(A)=0$.

Theorem: The minimal polynomial of a matrix divides the characteristic polynomial.
Theorem: The minimal polynomial and the characteristic polynomial have the same irreducible factors(factors that cannot be factorized further)

e.g. if $\begin{array}{l}\Delta(x)=(x-1)^2(x-2)(x-3)\\ M(x)=(x-1)(x-2)(x+3)\\ M(x)=(x-1)^2(x-2)(x+3)\end{array}$

Examples

1.Find the characteristic polynomial, minimal polynomial and eigen values of the matrix.

$$A=\begin{pmatrix}3 & -5 & 5\\ 5 & -7 & 5\\ 5 & -5 & 4\end{pmatrix}$$

Solution:

$$|xI-A|=\begin{vmatrix}x-3 & 5 & -5\\ -5 & x+7 & -5\\ -5 & 5 & x-3\end{vmatrix}$$
$$=(x-3)\begin{vmatrix}x+7 & -5\\ 5 & x-3\end{vmatrix}-5\begin{vmatrix}-5 & -5\\ -5 & x-3\end{vmatrix}-5\begin{vmatrix}-5 & x+7\\ -5 & 5\end{vmatrix}$$
$$=(x-3)\left[(x+7)(x-3)+25\right]-5(-5(x-3)-25)-5(-25+5x+35)$$
$$=(x-3)(x^2+4x+4)-5(-5x-10)-5(10+5x)$$
$$=(x-3)(x^2+4x+4)+25x+50-50-25x$$
$$=(x-3)(x+2)^2$$

$\therefore$ the characteristic polynomial of A is $\Delta(x)=(x-3)(x+2)^2 \Rightarrow$ the eigen values of A are $\lambda=3$ and $\lambda=-2,-2$.
The possibilities of minimal polynomial are

$M_1(x)=(x-3)(x+2)$ or $M_2(x)=(x-3)(x+2)^2$

$$M_1(A)=(A-3I)(A+2I)=\begin{pmatrix}0 & -5 & 5\\ 5 & -10 & 5\\ 5 & -5 & 0\end{pmatrix}\begin{pmatrix}5 & -5 & 5\\ 5 & -5 & 5\\ 5 & -5 & 5\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=0$$

$M_2(A)=0$ by Cayley-Hamilton Theorem
$\Rightarrow M_1(x)=(x-3)(x+2)$ is the minimal polynomial.
2. Find the characteristic polynomial, minimal polynomial and eigen values of

$$A=\left(\begin{array}{cc|cc}2 & 1 & 0 & 0\\ 0 & 2 & 0 & 0\\ \hline 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 5\end{array}\right)$$

Solution:

$$|xI - A| = \begin{vmatrix} x-2 & -1 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-2 & 0 \\ 0 & 0 & 0 & x-5 \end{vmatrix} = \begin{vmatrix} x-2 & -1 \\ 0 & x-2 \end{vmatrix} \begin{vmatrix} x-2 & 0 \\ 0 & x-5 \end{vmatrix} = (x-2)^2(x-2)(x-5) = (x-2)^3(x-5)$$

Therefore the characteristic polynomial of A is $f(x) = (x-2)^3(x-5)$

The eigen values of A are $\lambda = 2,2,2$ and $\lambda = 5$,possibilities of minimal polynomials are:

$M_1(x) = (x-2)(x-5); M_2(x) = (x-2)^2(x-5); M_3(x) = (x-2)^3(x-5)$

$$M_1(A) = (A-2I)(A-3I)$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

Hence $M_1(x)$ is not the minimal polynomial A

$$M_2(A) = (A-2I)^2(A-3I)$$
$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}^2 \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$$

$M_3(A) = 0$ by Cayley-Hamilton Theorem. Hence $M_2(x) = (x-2)^2(x-3)$ is the minimal polynomial of A .

Exercise

1. For each matrix, find the polynomial having the matrix as a root.

$$i) A = \begin{pmatrix} 2 & 5 \\ 1 & -3 \end{pmatrix} \quad ii) B = \begin{pmatrix} 2 & -3 \\ 7 & -4 \end{pmatrix} \quad iii) C = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{pmatrix}$$

2. Find the characteristic polynomial, minimal polynomial and eigen values of each matrix.

(i) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}$ (iii) $C = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$

Minimal polynomial theorem

Theorem: The minimal polynomial $m(x)$ of a matrix A divides every polynomial which has A as a zero.In particular , $m(x)$ divides the characteristic polynomial of A .

Proof:

Suppose $f(x)$ is a polynomial for which $f(A) = 0$ .By division algorithm there exists polynomial $q(x)$ and $r(x)$ for which $f(x) = m(x)q(x) + r(x)$ and $r(x) = 0$ or $\deg r(x) < \deg m(x)$ .Substituting $x = A$ in this equation and using the fact that $f(A)=0$ and $m(A)=0$,we obtain $r(A)=0$ .If $r(x)\neq 0$,then $r(x)$ is a polynomial degree less than $m(x)$ which has A as a zero; this contradicts the definition of the minimal polynomial. Thus $r(x)=0$ and so $f(x)=m(x)q(x)$ i.e $m(x)$ divides $f(x)$.

Theorem: Let $m(x)$ be the minimal polynomial of an *n*-square matrix A.Then the characteristic polynomial of A divides $m(x)^n$

Theorem: The characteristic polynomial $\Delta(x)$ and the minimal polynomial $m(x)$ of a matrix A have the same irreducible factors.

Proof:

Suppose $f(x)$is an irreducible polynomial .If $f(x)$divides $m(x)$then since $m(x)$ divides $\Delta(x)$, $f(x)$divides $\Delta(x)$.On the other hand ,if $f(x)$divides $\Delta(x)$, then by the preceding theorem, $f(x)$ divides $(m(x))^n$.But $f(x)$is irreducible. Hence $f(x)$ also divides $m(x)$.Thus $m(x)$and $\Delta(x)$,have the same irreducible factors.