Need a new formulation

$$\log p(x)_{\forall i} \approx \mathbf{E}_{\mathbf{z}\leftarrow q(z|x)}\left[\log p(x^{(i)})\right]$$ Maximize!

$$= \mathbf{E}_q\left[\log \frac{p(x^{(i)}\,|\,z)p(z)}{p(z\,|\,x^{(i)})}\frac{q(z\,|\,x^{(i)})}{q(z\,|\,x^{(i)})}\right]$$

Variational + multiply by one

$p(z\,|\,x^{(i)})$ this is still a problem

$$= \mathbf{E}_q\left[\log p(x^{(i)}\,|\,z)\right] + \mathbf{E}_q\left[\log \frac{p(z)}{q(z\,|\,x^{(i)})}\right] + \mathbf{E}_q\left[\log \frac{q(z\,|\,x^{(i)})}{p(z\,|\,x^{(i)})}\right]$$

$$= \mathbf{E}_q\left[\log p(x^{(i)}\,|\,z)\right] - \mathbf{E}_q\left[\log \frac{q(z\,|\,x^{(i)})}{p(z)}\right] + \mathbf{E}_q\left[\log \frac{q(z\,|\,x^{(i)})}{p(z\,|\,x^{(i)})}\right]$$

$$= \mathbf{E}_q\left[\log p(x^{(i)}\,|\,z)\right] - D_{KL}\left[q(z\,|\,x^{(i)})\|p(z)\right] + \cancel{D_{KL}\left[q(z\,|\,x^{(i)})\|p(z\,|\,x^{(i)})\right]}$$

always non-negative

$$\log p(x)_{\forall i} \geq \mathbf{E}_q\left[\log p(x^{(i)}\,|\,z)\right] - D_{KL}\left[q(z\,|\,x^{(i)})\|p(z)\right]$$ Will Maximize Lower Bound

Can we motivate this in a different way?

The Loss Function

Maximize through
Error of Reconstruction
Same as minimizing cross entropy

want $p(z)$ to be $\mathcal{N}(\mu = 0, \Sigma = I)$
because it makes nice latent space

$$q(z \mid x^{(i)}) \rightarrow (\mu_{z|x}, \Sigma_{z|x}) \qquad p(z) \rightarrow \mathcal{N}(0,1)$$

$$D_{KL}\left((\mu, \Sigma) \| \mathcal{N}(0,1)\right) = \frac{1}{2}\left(\mathrm{tr}(\Sigma) + \mu \cdot \mu^T - \underbrace{k}_{|z|} - \log\left(\det(\Sigma)\right)\right)$$

Can get this by manipulating the KL for normal distribution

Determinant of diagonal matrix is simple. Motivates diagonal covariance…

$$= \frac{1}{2}\left(\sum_k \Sigma_{k,k} + \sum_k \mu_k^2 - \sum_k 1 - \log\left(\prod_k \Sigma_{k,k}\right)\right)$$

$$= \frac{1}{2}\left(\sum_k \Sigma_{k,k} + \sum_k \mu_k^2 - \sum_k 1 - \sum_k \log \Sigma_{k,k}\right)$$

$$\geq \mathbf{E}_{q(z|x^{(i)})}\left[\log p(x^{(i)} \mid z)\right] - D_{KL}\left[q(z \mid x^{(i)}) \| p(z)\right]$$

$$= \frac{1}{2}\sum_k \left(\Sigma_{k,k} + \mu_k^2 - 1 - \log \Sigma_{k,k}\right)$$

The Covariance Output

$$\geq \mathbf{E}_{q(z|x^{(i)})}\left[\log p(x^{(i)}\,|\,z)\right] - D_{KL}\left[q(z\,|\,x^{(i)})\|p(z)\right]$$

Maximize through
Error of Reconstruction
Same as minimizing cross entropy

want $p(z)$ to be $\mathcal{N}(\mu = 0, \Sigma = I)$
because it makes nice latent space

$q(z\,|\,x^{(i)}) \rightarrow (\mu_{z|x}, \Sigma_{z|x}) \quad p(z) \rightarrow \mathcal{N}(0,1)$

$$= \frac{1}{2}\sum_k \left(\Sigma_{k,k} + \mu_k^2 - 1 - \log \Sigma_{k,k}\right)$$

raw covariance is not numerically stable because of underflow

$$\log \Sigma_{k,k} = \widehat{\Sigma_{k,k}}$$

predicted by
$q(z\,|\,x^{(i)})$

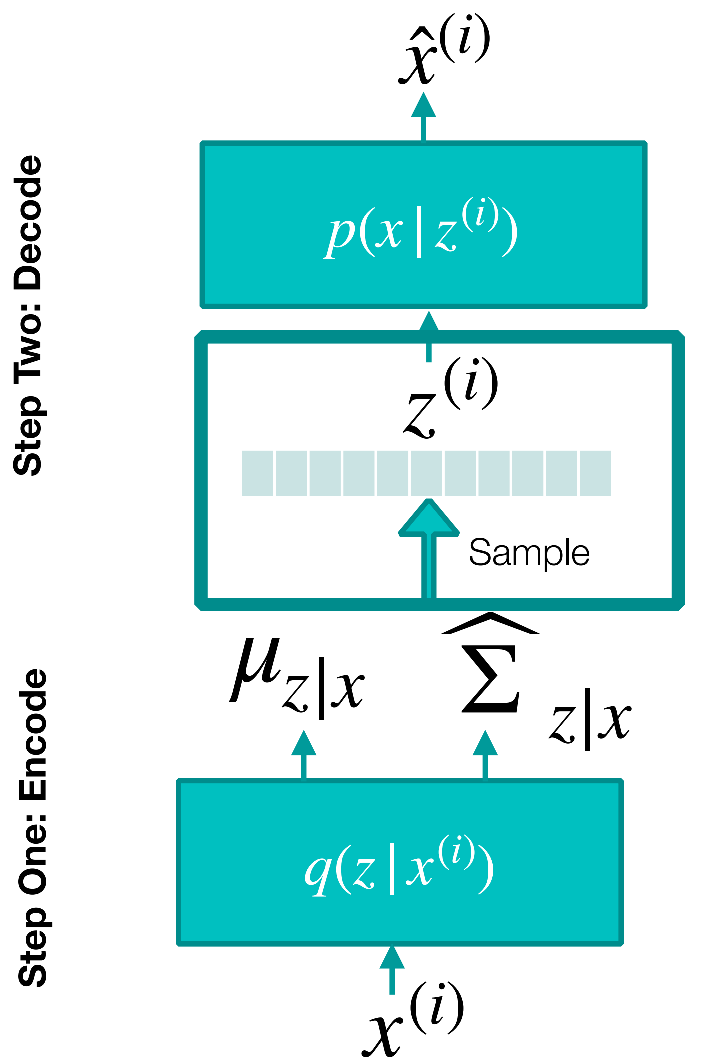
$$= \frac{1}{2}\sum_k \left(\exp\left(\widehat{\Sigma_{k,k}}\right) + \mu_k^2 - 1 - \widehat{\Sigma_{k,k}}\right)$$

so we will have the neural network output log variance

Also, remember we assume **diagonal covariance**, so z's are not correlated
This means covariance is only a vector of variances (the diagonal of Σ)

Back Propagating

$$\geq \mathbf{E}_{q(z|x^{(i)})}\left[\log p(x^{(i)}\,|\,z)\right] - D_{KL}\left[q(z\,|\,x^{(i)})\|p(z)\right]$$



This is partially differentiable by chain rule…

$$\mathcal{N}(\mu_{z|x}, \exp(\widehat{\Sigma_{z|x}})) = z$$

$$= \mu(x^{(i)}) + \exp(\widehat{\Sigma(x^{(i)})}) \cdot \mathcal{N}(0,1)$$

To update q,
we need to back propagate
through sampling layer. How?

The Loss Function Implementation

```
# Encode the input into a mean and variance parameter
z_mean, z_log_variance = encoder(input_img)
```

$\mu(x^{(i)})$ $\widehat{\Sigma(x^{(i)})}$ $q(z \mid x^{(i)})$

```
# Draw a latent point using a small random epsilon
z = z_mean + exp(z_log_variance) * epsilon
```

$$z = \mu(x^{(i)}) + \exp(\widehat{\Sigma(x^{(i)})}) \cdot \mathcal{N}(0,1)$$

```
# Then decode z back to an image
reconstructed_img = decoder(z)
```

$\hat{x}^{(i)} = p(x^{(i)} \mid z)$

```
# Instantiate a model
model = Model(input_img, reconstructed_img)
```

```
def vae_loss(self, x, z_decoded):
    x = K.flatten(x)
    z_decoded = K.flatten(z_decoded)
    xent_loss = keras.metrics.binary_crossentropy(x, z_decoded)
    kl_loss = -5e-4 * K.mean(
        1 + z_log_var - K.square(z_mean) - K.exp(z_log_var), axis=-1)
    return K.mean(xent_loss + kl_loss)
```

$$-\mathbf{E}_{q(z|x^{(i)})}\left[\log p(x^{(i)} \mid z)\right]$$

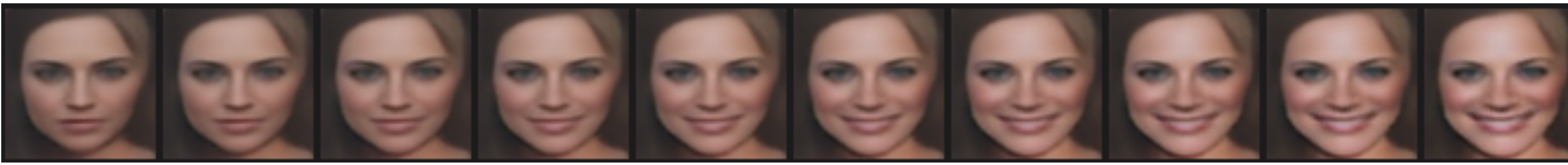
$$-\lambda \sum_k 1 + \widehat{\Sigma(x^{(i)})} - \mu(x^{(i)})^2 - \exp(\widehat{\Sigma(x^{(i)})})$$

Note:
Flipped from maximization to minimization
and added lambda for tradeoff in reconstruction, normal latent space

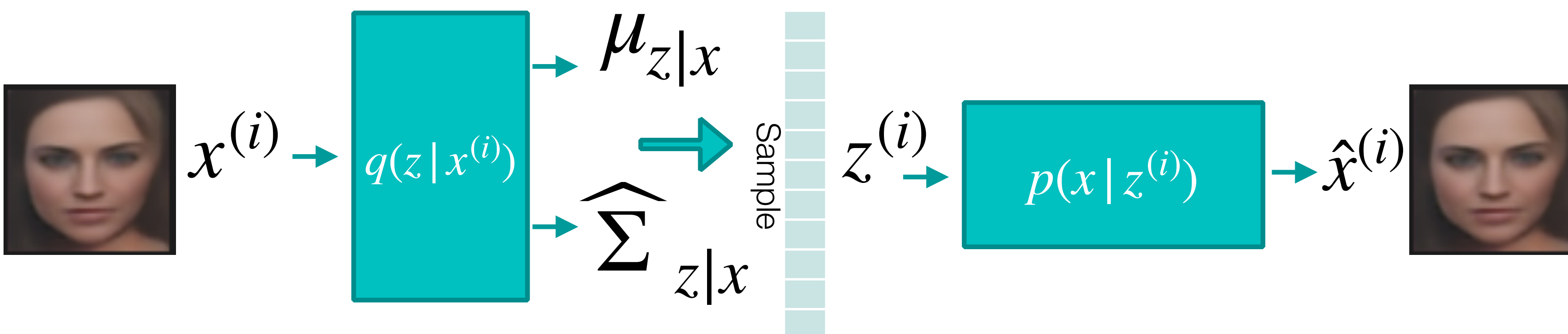
$$= -\mathbf{E}_{q(z|x^{(i)})}\left[\log p(x^{(i)} \mid z)\right] - \lambda \sum_k 1 + \widehat{\Sigma(x^{(i)})} - \mu(x^{(i)})^2 - \exp(\widehat{\Sigma(x^{(i)})})$$

Now that its trained, so what?

Encoding faces, then adjust the "z" that relates to smiling.

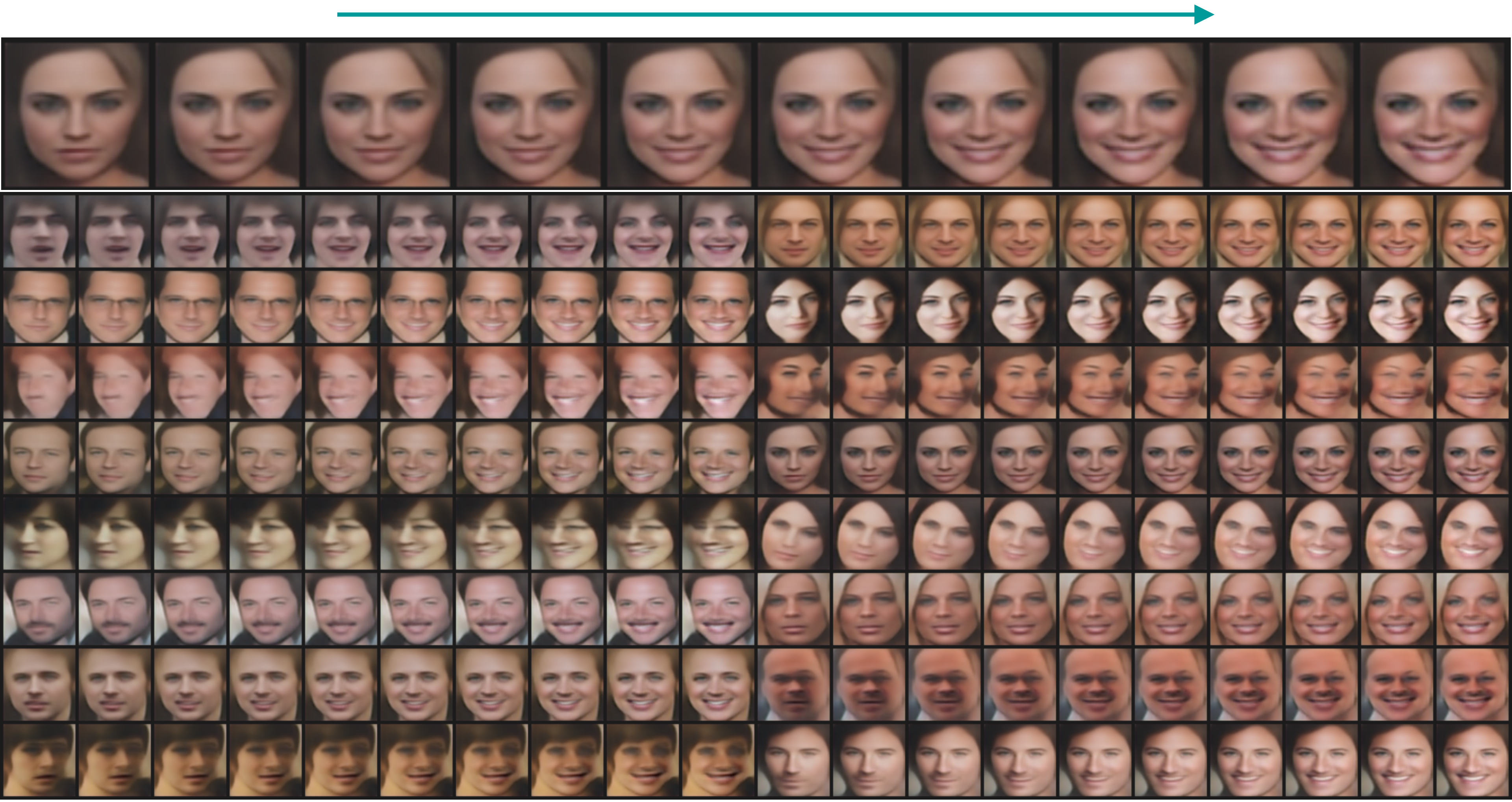


Investigate what happens by moving around each z_i



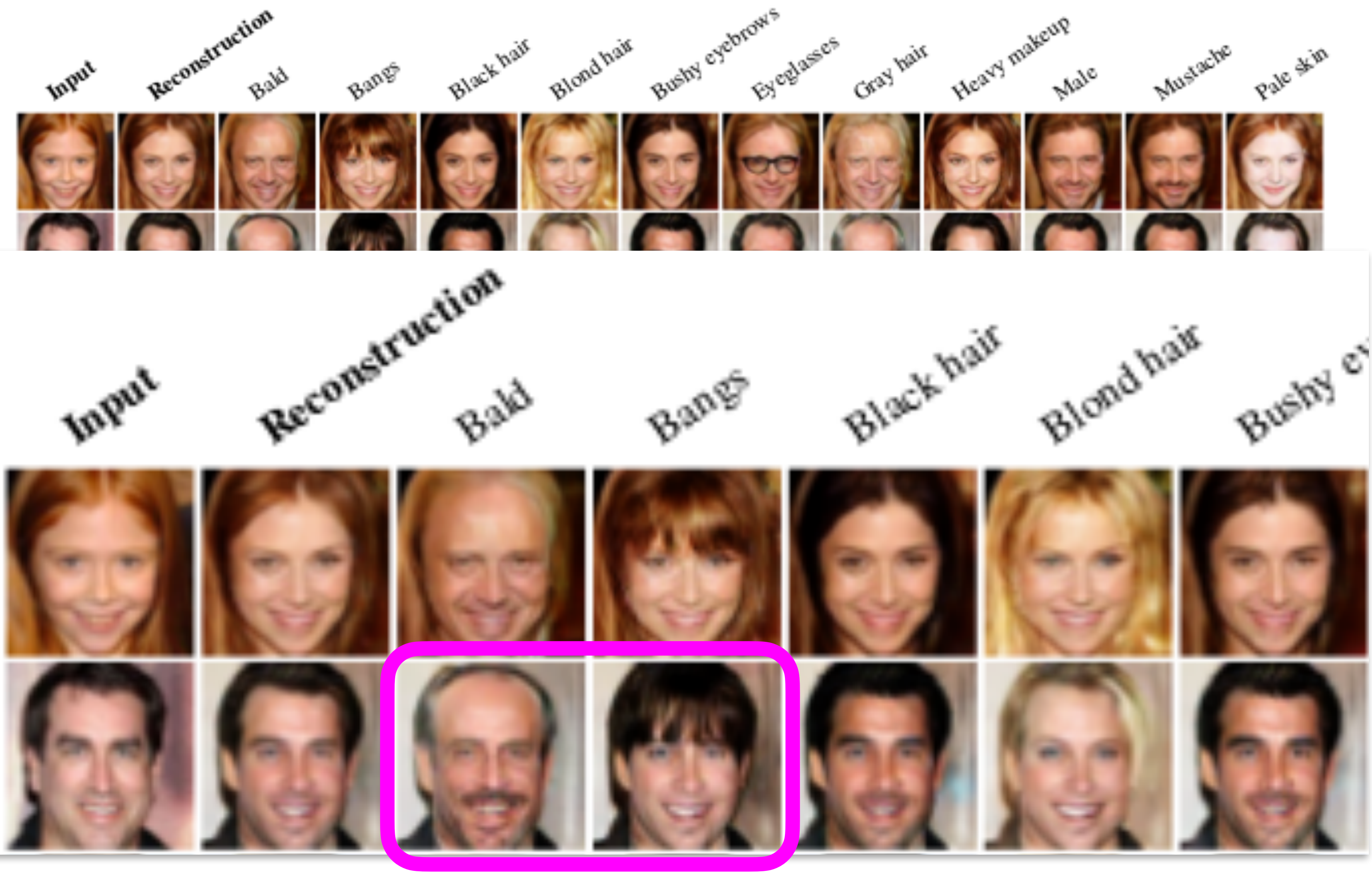
VAE Examples

Encoding faces, then adjust the “z” that relates to smiling.



VAE Examples

Different, automatically found z, latent variables



VAEs in Keras

Sampling from variational auto encoder using MNIST

Demo by Francois Chollet

In Master Repo: `07a VAEs in Keras.ipynb`

Follow Along: https://github.com/fchollet/deep-learning-with-python-notebooks/blob/master/8.4-generating-images-with-vaes.ipynb

Lecture Notes for

Neural Networks and Machine Learning

Generative Networks

Next Time:
General GANs

Reading: Chollet CH8