8.04 Problem Set 1 Solutions

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June 2014

(15 points) Radiative collapse of a classical atom

Suppose the world was actually governed by classical mechanics. In such a classical universe, we might try to build a Hydrogen atom by placing an electron in a circular orbit around a proton. However, we know from 8.03 that a non-relativistic, accelerating electric charge radiates energy at a rate given by the Larmor formula,

$$\frac{dE}{dt} = -\frac{2}{3}\frac{q^2a^2}{c^3}$$

(in cgs units) where q is the electric charge and a is the magnitude of the acceleration. So the classical atom has a stability problem. How big is this effect?

(a) Show that the energy lost per revolution is small compared to the electron's kinetic energy. Hence, it is an excellent approximation to regard the orbit as circular at any instant, even though the electron eventually spirals into the proton.

(b) Using the typical size of an atom (1Å) and a nucleus (1 fm), calculate how long it would take for the electron to spiral into the proton.

(c) Compare the velocity of the electron (assuming an orbital radius of 0.5 Å) to the speed of light ? will relativistic corrections materially alter your conclusions?

(d) As the electron approaches the proton, what happens to its energy? Is there a minimum value of the energy the electron can have?

(a) The coulomb force in cgs units is given by $F_c = \frac{Zq^2}{r^2}$ so from $F = ma$ we get that for coulomb force the acceleration is given by

$$a = \frac{q^2}{mr^2}$$

from centripetal acceleration $F = \frac{mv^2}{r}$ and using the value for the coulomb force above we get

$$F = \frac{mv^2}{r} = \frac{q^2}{r^2} \rightarrow E_k = \frac{1}{2}mv^2 = \frac{1}{2}\frac{q^2}{r}$$

from the problem statement above the Larmor formula provides a value for the energy loss with time:

$$\frac{dE}{dt} = -\frac{2}{3}\frac{q^2a^2}{c^3}$$

from the equation derived for the acceleration above this becomes

$$\frac{dE}{dt} = -\frac{2}{3}\frac{q^2a^2}{c^3} = -\frac{2}{3}\frac{q^2}{c^3}(\frac{q^2}{mr^2})^2$$

to get the amount of energy lost in one orbit we need the period which for a circular orbit is given by

$$T = -\frac{2r\pi}{v} = \frac{2r\pi}{\sqrt{(\frac{q^2}{mr})}} = 2\pi\sqrt{(\frac{mr^3}{q^2})}$$

from this $E_{orbit} = \int_o^T \frac{dE}{dt}dt = \frac{dE}{dt}T$ assuming energy is approximatly constant for a single orbit

$$\frac{dE}{dt}T = (-\frac{2}{3}\frac{q^2}{c^3}(\frac{q^2}{mr^2})^2)(2\pi\sqrt{(\frac{mr^3}{q^2})}) = \frac{16\sqrt{(2)}\pi}{3c^3m^{3/2}}(\frac{q^2}{2r})^{5/2}$$

Thus the ratio of the orbital and the kinetic energies is given by

$$\frac{E_{orbit}}{E_k} = \frac{\frac{r\pi}{3c^3m^{3/2}}(\frac{q^2}{2r})^{5/2}}{\frac{1}{2}\frac{q^2}{r}} = \frac{16\sqrt(2)\pi}{3}(\frac{\frac{q^2}{2r}}{mc^2})^{3/2}$$

given that the energy for the ground state of hydrogen is $\frac{e^2}{2r} = 13.6ev$, and the rest energy is $mc^2 = .511Mev$, we can compute a value for the ratio:

$$\frac{E_{orbit}}{E_k} = \frac{16\sqrt(2)\pi}{3}(\frac{13.6}{.511*10^6})^{3/2} = 3.253*10^{-6}$$

(b) From above we can show that the total energy of an electron in orbit is

$$E = E_k - \frac{q^2}{r} = -\frac{q^2}{2r}$$

differentiating

$$\frac{dE}{dt} = \frac{q^2}{2r^2}\frac{dr}{dt} = -\frac{2}{3}\frac{e^2}{c^3}(\frac{e^2}{mr^2})^2$$

simplifying we get

$$\frac{dr}{dt} = -\frac{4q^2}{3m^2r^2c^3} \to -\frac{3m^2c^3}{4q^4}r^2dr\ t = \int_0^t dt = -\frac{m^2c^3}{4q^4}\int_{r_i}^{r_f} 3r^2dr = \frac{m^2c^3}{4q^4}(r_i^3 - r_f^3)$$

r_f approximatly zero compared to the initial radius so drop this term from the above equation

$$t = \frac{m^2c^3}{4q^4}r_i^e = 1.05*10^{-10}seconds$$

(c) From above we can see that the velocity of the electron is given by

$$v = \sqrt(\frac{q^2}{mr}) = \sqrt(\frac{(1.602*10^{-19})^2}{9.11*10^{-19}(.5*10^{-10})}) = 23.7m/s$$

(d) as the radius approaches r_f is about 10^{-15} the energy lost per orbit gets very large

Not sure whats going to happen here.

(25 points) Dimensional Analysis: Two Kinds of Quantum Gravity

(a) Gravitational bound states Consider a particle sitting on a table which is kept from floating away only by the force of gravity. This system is characterized by just three physical parameters, the mass of the particle, m, the acceleration of gravity on Earth, g = 9.8 $\frac{m}{s^2}$, and Planck's constant, $\hbar = \frac{1}{2\pi}h$. The energy given by $E = 12mv^2 + mgx$.

i. Using only dimensional analysis, find the product of powers of m, g, $\hbar$ which give a characteristic energy, E. (i.e., write $E = m^\alpha g^\beta \hbar^\gamma$ and solve for α, β, γ) Can you find such a characteristic energy without using the Planck constant?

ii. Repeat to find characteristic length, time, and speeds (l, t, v) for this system

iii. Classically, putting the system in its lowest energy configuration ($E = 0$) would require the particle to sit perfectly still ($v = 0$) precisely on the surface ($x = 0$). Use the uncertainty relation, $\delta x \delta p \geq \frac{\hbar}{2}$, to argue (briefly!) that the particle cannot have $E = 0$ while respecting the uncertainty principle.
ASIDE: Quantum mechanically, then, there must be some minimum energy this system can have which cannot be predicted classically! For a particle on a table, this may not seem so important - but for Hydrogen, which you've just shown to be classically unstable, this is absolutely key. We will soon learn how to calculate the minimum ("ground state") energy of such systems.

iv. Use your dimensional analysis results to give a simple estimate for the ground state energy of this system. How does your estimate behave as $h \to 0$? Does this make sense? Explain why or why not.

v. Evaluate E, l, t and v numerically for a neutron ($m_N = 1.7 * 10^{27} kg$). How high above the surface will the particle typically be found?

(b) The Planck Scale The scale at which gravity (characterized by the Newton constant, G_N), quantum mechanics ($\hbar$), and relativity (c) are all important is called the Planck scale.

i. Using dimensional analysis, find the combination of powers of $G_N, \hbar$ and c which make a length - we call this the Planck length, L_P .

ii. Evaluate L_P numerically, and compare to a typical scale for nuclear or particle physics, namely $1F = 10^{-15}$ m.

iii. Repeat to find the Planck mass, M_P , evaluate it numerically, and compare to the mass of a typical nuclear constituent (like the proton mass). Do we need to understand Quantum Gravity to study nuclear physics?

(a)

i. $E \approx m^{\alpha} g^{\beta} \hbar^{\gamma}$
$[kg]^1[m]^2[s]^{-2} = [kg]^{\alpha}[\frac{m}{s^2}]^{\beta}[\frac{kgm^2}{s}]^{\gamma}$
so set up equations of exponents
$kg \to 1 = 1\alpha + 0\alpha + 1\alpha$
$m \to 2 = 0\beta + 1\beta + 2\beta$
$sec \to 1 = 1\gamma + 0\gamma + 1\gamma$

solve simultaneous equations and this gives $\alpha = \frac{1}{3}, \beta = \frac{2}{3}, \gamma = \frac{2}{3}$

ii. $l \approx m^{\alpha} g^{\beta} \hbar^{\gamma}$ $[m] = [kg]^{\alpha}[\frac{m}{s^2}]^{\beta}[\frac{kgm^2}{s}]^{\gamma}$
so set up equations of exponents
$kg \to 0 = 1\alpha + 0\alpha + 1\alpha$
$m \to 1 = 0\beta + 1\beta + 2\beta$
$sec \to 0 = 1\gamma + 0\gamma + 1\gamma$

solve simultaneous equations and this gives $\alpha = \frac{2}{3}, \beta = -\frac{1}{3}, \gamma = \frac{2}{3}$
repeat for t $\to \alpha = -\frac{1}{3}, \beta = -\frac{2}{3}, \gamma = \frac{1}{3}$
repeat for v $\to \alpha = -\frac{1}{3}, \beta = \frac{1}{3}, \gamma = \frac{1}{3}$

iii. If E= 0 requires V=0 and x=0, then $\delta x \delta p = 0$ which violates the uncertainty principle (note p = mv)

iv. from above using parameters for mass of electron and g

$$E \approx (mg^2\hbar^2)^{1/3} = (9.11 * 10^{-31} * (9.8)^2 * (1.054 * 10^{-34})^2)^{1/3} = 9.9 * 10^{-33} J = 6.18 * 10^{-14} eV$$

note that the ground state energy for hydrogen is 13.6eV

as $\hbar \to 0$ then $E \to 0$ as well. As stated above E can never be exactly zero by the uncertainty principle so no this does not hold up.

v. Substituting into the approximations above gives

$E = 1 * 10^{-12}$ eV
$l = 10 * 10^{-6}$ m
$t = 1 * 10^{-3}$ seconds
$v = 0.01 \frac{m}{s}$

(b)

i. using same process as above $L_P = \sqrt{(\frac{\hbar G}{c^3})}$

ii. substituting in values to the last equations gives $L_P = 1.616 * 10^{-35}$ m

iii. using same process as above $M_P = \sqrt{(\frac{\hbar c}{G})} = 2.18 * 10^{-8}$ kg

length is much less than typical nuclear sizes and nuclear masses are much less than M_P so no this is not a factor in nuclear physics.
(anyone have any better ideas??)

(20 points) de Broglie Relations and the Scale of Quantum Effects

(a) Light Waves as Particles The Photoelectric effect suggests that light of frequency ν can be regarded as consisting of photons of energy $E = h\nu$, where $h = 6.63 \cdot 10^{-27}$ erg*s.

i. Visible light has a wavelength in the range of 400-700 nm. What are the energy and frequency of a photon of visible light?

ii. The microwave in my kitchen operates at roughly 2.5 GHz at a max power of $7.5 \cdot 10^9 \frac{erg}{s}$. How many photons per second can it emit? What about a low-power laser ($10^4 \frac{erg}{s}$ at 633 nm), or a cell phone ($4 \cdot 10^6 \frac{erg}{s}$ at 850 MHz)?

iii. How many such microwave photons does it take to warm a 200ml glass of water by $10C$? (The heat capacity of water is roughly $4.18 \cdot 10^7 \frac{erg}{gK}$)

iv. At a given power of an electromagnetic wave, do you expect a classical wave description to work better for radio frequencies, or for X-rays?

(b) Matter Particles as Waves If a wavelength can be associated with every moving particle, then why are we not forcibly made aware of this property in our everyday experience? In answering, calculate the de Broglie wavelength $\lambda = hp$ of each of the following particles

$p = mv$ and $\lambda = \frac{h}{p} = \frac{h}{mv}$

i. an automobile of mass 2 metric tons (2000 kg) traveling at a speed of 50 mph (22 ms)

ii. a marble of mass 10 g moving with a speed of $10\frac{cm}{s}$

iii. a smoke particle of diameter 10^{-5} cm and a density of, say, ($.2\ \frac{g}{cm^3}$) being jostled about by air molecules at room temperature (T=300K) (assume that the particle has the same translational kinetic energy as the thermal average of the air molecules, $KE = \frac{3}{2}k_BT$, with $k_B = 1.38 \cdot 10^{-16}\frac{erg}{k}$)

iv. An ^{87}Rb atom that has been laser cooled to a temperature of $T = 100\mu K$. Again, assume $KE = \frac{3}{2}k_BT$

(a)

i. $\nu = \frac{c}{\lambda} \rightarrow E = h\nu = \frac{h \cdot c}{\lambda}$
for $\lambda = 400$ nm then
$\nu = \frac{3 \cdot 10^8}{400 \cdot 10^{-9}} = 750$ THz and
$E = \frac{(6.63 \cdot 10^{-27}) * (3 \cdot 10^8)}{400 \cdot 10^{-9}} \approx = 5 \cdot 10^{-12}$ ergs

for $\lambda = 700$ nm then
$\nu = \frac{3 \cdot 10^8}{700 \cdot 10^{-9}} = 430$ THz and
$E = \frac{(6.63 \cdot 10^{-27}) * (3 \cdot 10^8)}{700 \cdot 10^{-9}} \approx = 2.9 \cdot 10^{-12}$ ergs

ii. $E = (6.626 \cdot 10^{-34}) \cdot (2.5 \cdot 10^9) = 1.66 \cdot 10^{-24}$ Joules/photon
$N = \frac{750}{1.66 \cdot 10^{-24}} = 4.5 \cdot 10^{26}$ photons/sec

$\lambda = 644$ nm then $E = \frac{h \cdot c}{\lambda} = \frac{6.626 \cdot 10^{-34} \cdot 2.988 \cdot 10^8}{633 \cdot 10^{-9}} = 2.96 \cdot 10^{-19}$ J. and
$N_{laser} = \frac{1 \cdot 10^-3}{2.96 \cdot 10^{-19}} = 3.38 \cdot 10^{15}$ photons/s.

Cell phone $\nu = 850 \cdot 10^6$ Hz
$E = (6.626 \cdot 10^{-34}) \cdot (850 \cdot 10^6) = 5.63 \cdot 10^{-25}$ Joules/photon
$N = \frac{0.4}{5.63 \cdot 10^{-25}} = 7.10 \cdot 10^{23}$ photons/s

iii. 1 liter water weights 1kg so or in this problem 0.2kg
$E = Cm\Delta T = 4.18 \cdot 10^3 * 0.2 * 10 = 8.36 \cdot 10^3$ Joules
$N = \frac{8.36 \cdot 10^3}{1.62 \cdot 10^{-24}} = 5.16 \cdot 10^{27}$ photons.

iv. (My answer) Classical Theory works best for radio frequencies as the number of photons is larger for radio frequencies by about 10^{11} with large numbers of photons classical statistics describes behavior very well with low numbers quantum effects become larger.

(MIT OCW Solutions answer) We expect a classical wave description to work better for radiofrequencies. The classical electromagnetic description of photons works fine when a number of photons is large. However, this description breaks down when we try to describe a single photon. At a given power of an electromagnetic wave, a number of photons for radiofrequencies in the detection window is much larger than that for X-rays. Therefore, the wave description of photons is adequate for radiofrequencies but is not adequate for X-rays. We can estimate a ratio of a number of photons for radiofrequencies and for X-rays at a given power (i.e. 1 W). Energy of radiofrequencies is about 10^{-7} eV and that of X-rays is about 10^4 eV.

$R = \frac{N_{radio}}{N_{x-ray}} = \frac{10^4}{10^{-7}} = 10^{11}$.

Therefore, at a given power, for every X-ray photon, there are about 10^{11} radiofrequency photons. Assume that a relaxation time of a photon detector is about 1 ps (10^{-12} s). Our detector can detect a single photon if it arrives at the detector at a rate of 1 photon per 1

ps. This time scale determines our detection-window time scale. Therefore, the power of the He-Ne laser in which we expect quantum effect to become important is:
$P = \frac{E_{photon}}{1ps} = \frac{2.96 \cdot 10^{-19}}{10^{-12}} = 2.96 \cdot 10^{-7}$ W.

(b)

i. $\lambda = \frac{6.626 \cdot 10^{-34}}{2 \cdot 10^3 \cdot 22} = 1.5 \cdot 10^{-38}$

ii. $\lambda = \frac{6.626 \cdot 10^{-34}}{0.01 \cdot .1} = 6.6 \cdot 10^{-31}$

iii. Volume of the particle is $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(1 \cdot 10^{-7})^3 = 4.2 \cdot 10^{-21} m^3$
the mass is $m = V \cdot \rho = (4.2 \cdot 10^{-21} m^3) \cdot (200 \frac{kg}{m^3}) = 8.4 \cdot 10^{-19}$
given that $E_{KE} = \frac{3}{2} k_B T \rightarrow E_{KE} = \frac{3}{2} \cdot (1.38 \cdot 10^{-23})(300) = 6.24 \cdot 10^{-21}$ so $\frac{1}{2} m v^2 = 6.24 \cdot 10^{-21} \rightarrow v = \sqrt{(\frac{2 \cdot (6.24 \cdot 10^{-21})}{8.4 \cdot 10^{-19}})} = .122 \frac{m}{s}$
from this $p = (8.4 \cdot 10^{-19}) \cdot (.122) = 1.02 \cdot 10^{-19}$ and then $\lambda = \frac{h}{p} = \frac{6.602 \cdot 10^{-34}}{1.02 \cdot 10^{-19}} = 5.9 \cdot 10^{-15}$ m

iv. $m = 1.44 \cdot 10^{-25}$ kg
$E_{KE} = \frac{3}{2} k_B T \rightarrow E_{KE} = \frac{3}{2} \cdot (1.38 \cdot 10^{-23})(100 \cdot 10^{-6}) = 2.07 \cdot 10^{-27}$ so $\frac{1}{2} m v^2 = 2.07 \cdot 10^{-27} \rightarrow$
$v = \sqrt{(\frac{2(2.07 \cdot 10^{-27})}{1.44 \cdot 10^{-25}}} = .17$ m
from this $p = (1.44 \cdot 10^{-25}) \cdot (.17) = 2.44 \cdot 10^{-26}$ and then
$\lambda = \frac{h}{p} = \frac{6.602 \cdot 10^{-34}}{2.44 \cdot 10^{-26}} = 2.7 \cdot 10 - 8$

(15 points) Double-slit interference of electrons

(a) Electrons of momentum p fall normally on a pair of slits separated by a distance d. What is the distance, w, between adjacent maxima of the interference fringe pattern formed on a screen a distance D beyond the slits? note: You may assume that the width of the slits is much less than the electron de Broglie wavelength.

(b) In an experiment performed by Jonsson in 1961(!!!), electrons were accelerated through a 50kV potential towards two slits separated by a distance $d = 2 \cdot 10^{-4}$ cm, then detected on a screen D = 35 cm beyond the slits. Calculate the electron's de Broglie wavelength, λ, and the fringe spacing, w.

(c) What values would d, D, and w take if Jonsson?s apparatus were simply scaled up for use with visible light rather than electrons?

(a) $\Delta x = x_2 - x_1 = m\lambda$ when $m = 0, 1, 2, \ldots$
$\theta = sin(\frac{m\lambda}{d}) = \frac{m\lambda}{d}$
$H = Lsin(\theta) \approx L\theta = \frac{m\lambda L}{d}$ $w = H_{m+1} - H_m = (m+1)\frac{\lambda L}{d} - m\frac{\lambda L}{d} = \frac{\lambda L}{d}$
for an electron $\lambda = \frac{h}{p}$ so $w = \frac{hL}{pd}$

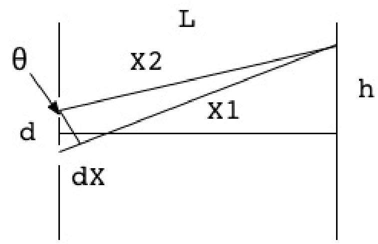

Figure 1: Double Slit Geometry

(b) $v = \sqrt{(\frac{2e\Delta V}{m})} = \sqrt{(\frac{2(1.6 \cdot 10^{-19}(50)}{9.11 \cdot 10^{-31}})} = 4.19 \cdot 10^6$

de Broglie Wavelength $\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.63 \cdot 10^{-34}}{(9.11 \cdot 10^{-31})(4.19 \cdot 10^6} = 1.74 \cdot 10^{-10}$

$w = \frac{hL}{pd} = \frac{(6.63 \cdot 10^{-34})0.35}{(9.11 \cdot 10^{-31})(4.19 \cdot 10^6)(2 \cdot 10^{-7})} = 3.04 \cdot 10^{-4}$

(c) Not sure whats being asked for here. there are three unknowns with only one equation

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