Properties of ∑

Symbol ∑ is used to express series of sequences.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Instead of writing a long statement with tons of '+', ∑ simplifies the expression. Because ∑ literally means to sum up all the terms of the sequence it has some important properties as follows:

$$\sum_{k=1}^{n} (a_k \pm b_k) = (a_1 \pm b_1) + (a_2 \pm b_2) + \cdots + (a_n \pm b_n)$$

$$= (a_1 + a_2 + \cdots + a_n) \pm (b_1 + b_2 + \cdots + b_n) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k$$

Any constants inside ∑ which are not a sigma variable can move in and out of ∑ expression freely. Also addition and subtraction of ∑ of the same range can be merged or split as needed. This is called linearity.

$$\sum_{k=1}^{n} 1 = 1 + 1 + \cdots + 1 = n$$

Expression above means to add all terms which are 1 regardless of k values. Based on the definition of ∑, the series has n terms from k=1 to k=n. Therefore it equals to the number of terms in the series.

Using these properties we can derive some formulae related to positive integers.

$$\sum_{k=1}^{n}(k+1)^2 = \sum_{k=1}^{n}k^2 + 2\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$$

$$\sum_{k=1}^{n}(k+1)^2 - \sum_{k=1}^{n}k^2 = 2\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$$

$$\sum_{k=2}^{n+1}k^2 - \sum_{k=1}^{n}k^2 = 2\sum_{k=1}^{n}k + n$$

$$(n+1)^2 - 1^2 = n^2 + 2n + 1 - 1 = n^2 + 2n = 2\sum_{k=1}^{n}k + n$$

$$n^2 + 2n - n = n^2 + n = n(n+1) = 2\sum_{k=1}^{n}k$$

$$\therefore \sum_{k=1}^{n}k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n}(k+1)^3 = \sum_{k=1}^{n}k^3 + 3\sum_{k=1}^{n}k^2 + 3\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$$

$$\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n}k^3 = 3\sum_{k=1}^{n}k^2 + \frac{3n(n+1)}{2} + n$$

$$\sum_{k=2}^{n+1}k^3 - \sum_{k=1}^{n}k^3 = 3\sum_{k=1}^{n}k^2 + \frac{3n^2+3n}{2} + n$$

$$(n+1)^3 - 1^3 = n^3 + 3n^2 + 3n + 1 - 1 = n^3 + 3n^2 + 3n = 3\sum_{k=1}^{n}k^2 + \frac{3n^2+5n}{2}$$

$$n^3 + 3n^2 + 3n - \frac{3n^2+5n}{2} = \frac{2n^3+6n^2+6n-3n^2-5n}{2} = 3\sum_{k=1}^{n}k^2$$

$$\frac{2n^3+3n^2+n}{2} = \frac{n(2n^2+3n+1)}{2} = \frac{n(n+1)(2n+1)}{2} = 3\sum_{k=1}^{n}k^2$$

$$\therefore \sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$$