(1)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} x^{2n+1}$$

$$erf\,x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} x^{2n+1}$$

(2)

$$\left(\int_{-\infty}^{\infty} e^{-\frac{1}{a}x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{1}{a}x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{1}{a}y^2}\,dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{a}x^2} e^{-\frac{1}{a}y^2}\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{a}(x^2+y^2)}\,dy\,dx = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{a}r^2} r\,dr\,d\theta$$

$$-\frac{1}{a}r^2 = t \qquad -\frac{2}{a}r\,dr = dt \qquad r\,dr = -\frac{a}{2}dt$$

$$= \int_0^{2\pi}\int_0^{-\infty} e^t\left(-\frac{a}{2}\right)dt\,d\theta = -\frac{a}{2}\int_0^{2\pi}\int_0^{-\infty} e^t\,dt\,d\theta = -\frac{a}{2}\int_0^{2\pi} d\theta \int_0^{-\infty} e^t\,dt$$

$$= -\frac{a}{2}\cdot 2\pi \cdot [e^t]_0^{-\infty} = -a\pi\cdot(0-1) = a\pi$$

$$\therefore \int_{-\infty}^{\infty} e^{-\frac{1}{a}x^2}\,dx = \sqrt{a\pi}$$

(3)

$$e^{-\frac{1}{a}x^2} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{a}x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,a^n} x^{2n}$$

$$\int_0^x e^{-\frac{1}{a}t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,a^n(2n+1)} x^{2n+1}$$

(4)

$$\frac{1}{\sqrt{a\pi}}\int_0^x e^{-\frac{1}{a}t^2}\,dt = \frac{1}{\sqrt{a}\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,a^n(2n+1)} x^{2n+1} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,a^{n+\frac{1}{2}}(2n+1)} x^{2n+1}$$

$$= \frac{1}{2}\frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\sqrt{a}^{2n+1}(2n+1)} x^{2n+1} = \frac{1}{2}\frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} \left(\frac{x}{\sqrt{a}}\right)^{2n+1}$$

$$\therefore \frac{1}{\sqrt{a\pi}}\int_0^x e^{-\frac{1}{a}t^2}\,dt = \frac{1}{2} erf\,\frac{x}{\sqrt{a}}$$