Derivative of Exponential Functions

(1)

$$\frac{d}{dx}a^x = \lim_{m\to 0}\frac{a^{x+m}-a^x}{m} = \lim_{m\to 0}\frac{a^x(a^m-1)}{m} = a^x\cdot\lim_{m\to 0}\frac{a^m-1}{m}$$

$$t = a^m - 1 \qquad a^m = 1+t \qquad m = log_a(1+t)$$

$$m \to 0 \qquad a^m \to 1 \qquad t = a^m - 1 \to 0$$

$$= a^x\cdot\lim_{t\to 0}\frac{t}{log_a(1+t)} = a^x\cdot\lim_{t\to 0}\frac{1}{\frac{1}{t}log_a(1+t)} = a^x\cdot\lim_{t\to 0}\frac{1}{log_a(1+t)^{\frac{1}{t}}}$$

$$= a^x\cdot\frac{1}{log_a e} = a^x\cdot log_e a = a^x \ln a$$

(2)

$$\frac{d}{dx}e^x = e^x \ln e = e^x\, log_e e = e^x$$

$$\frac{d^n}{dx^n}e^x = e^x$$

(3)

$$e^x = \sum_{n=0}^{\infty}\frac{[e^x]_{x=0}}{n!}x^n = \sum_{n=0}^{\infty}\frac{e^0}{n!}x^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$e = e^1 = \sum_{n=0}^{\infty}\frac{1^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}$$