Euler's Formula

(1)

$$i = \sqrt{-1}$$

$$(sin\,x)^{(4n)} = sin\,x \qquad sin\,0 = 0 \qquad i^{4n} = 1$$

$$(sin\,x)^{(4n+1)} = cos\,x \qquad cos\,0 = 1 \qquad i^{4n+1} = i$$

$$(sin\,x)^{(4n+2)} = -\,sin\,x \qquad -\,sin\,0 = 0 \qquad i^{4n+2} = -1$$

$$(sin\,x)^{(4n+3)} = -\,cos\,x \qquad -\,cos\,0 = -1 \qquad i^{4n+3} = -i$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

(2)

$$sin\,x = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!} - \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!}$$

$$cos\,x = \frac{d}{dx} sin\,x = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} - \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!}$$

(3)

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} + i \sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!} - \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!} - i \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!}$$

$$= \left(\sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} - \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n+2)!} \right) + i \left(\sum_{n=0}^{\infty} \frac{x^{4n+1}}{(4n+1)!} - \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)!} \right)$$

$$= cos\,x + i\,sin\,x$$

$$\therefore e^{ix} = cos\,x + i\,sin\,x$$