Bifurcation of coherent structures in nonlocally coupled system

Arnd Scheel & Tianyu Tao
University of Minnesota

Coherent structures

1. spikes, fronts, wave trains... (Maybe some pictures?)

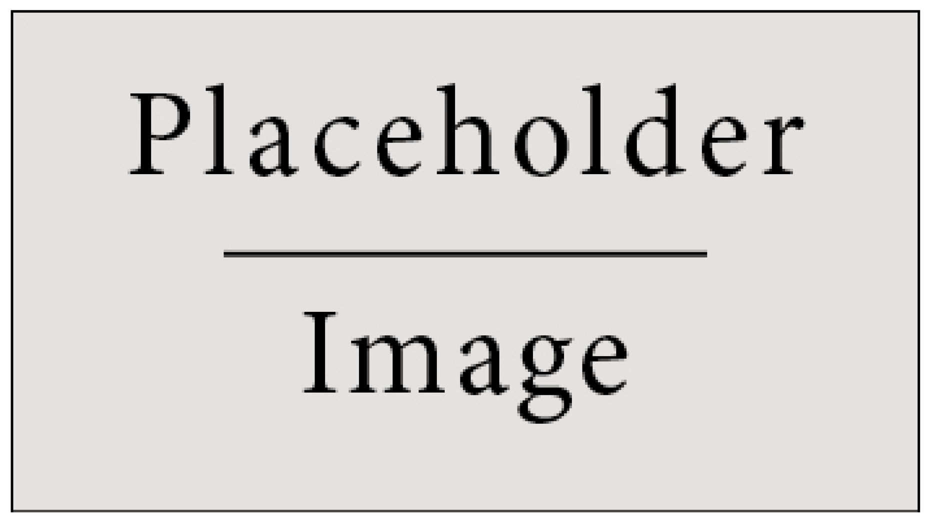

Figure 1: Figure caption

Reaction-Diffusion Models

Bifurcation to coherent structures in RD system

$$U_t = D\Delta_x U + F(U;\mu).$$

Assume

- $U = U(x) \in \mathbb{R}^k, x \in \mathbb{R}^n$;
- $\mu \in \mathbb{R}^p$ parameter, nonlinearity N satisfies $N(0;\mu) = 0$.

Radially symmetric patterns are studied in [1] using spatial dynamic techniques.

1. rewrite the stationary equation as a nonautonomous system of ODE;
2. carefully construct a center manifold;
3. study reduced dynamics on the center manifold, get complete classification of small bounded solutions.

Nonlocal Model Equations

Nonlocal diffusion are ubiquitous in modeling of natural phenomenas [2]

1. neural field models: $u_t = K * S(u)$,
2. water wave equations: $u_t = (Mu - u^2)_x$.

Can we find similar patterns in these models? will they have different properties due to nonlocality?

A Bifurcation Problem

We study the following system of equations for $U(x) \in \mathbb{R}^k$ with $x \in \mathbb{R}^n$

$$\boldsymbol{U + K * U - N(U;\mu) = 0}. \qquad (1)$$

Effective linear diffusive coupling (L):

- K is a matrix of convolution kernel with finite second moments $K(x), |x|^2 K(x) \in L^1(\mathbb{R}^n)$
- K is symmetric, $K(\gamma x) = K(x)$ for all $\gamma \in \Gamma \subset O(n)$. The fixed point set of Γ is $\{0\}$ only.
- the Fourier determinant $D(\xi) = \det(I + K(\xi))$ has $D(0) = 0, D'(0) = 0, D''(0) \neq 0$.

Let e span the kernel of $I + K(0)$, choose e^* span the cokernel.

A Bifurcation Problem-Continued

Transcritical bifurcation in kinetics (TC):

- We assume a transcritical bifurcation scenario:

$$N(0;\mu) = 0,$$
$$\langle e^*, D_{u\mu} N(0;0)e \rangle \neq 0,$$
$$\langle e^*, D_{uu} N(0;0)[e,e] \rangle \neq 0$$

- N is smooth, so the superposition operator $U(\cdot) \mapsto N(U(\cdot))$ is smooth.

We then find a pseudo-differential operator L and invertible matrix P, Q so that

$$LP(I + K*)Q = \mathrm{diag}(M, I_{k-1}),$$

with $M = (1-\Delta)^{-1}\Delta$ and system decouples into two equations for a scalar function v_c and a $\mathbb{R}^{k-1}$ valued function v_h.

Main Result

Fix $n < 6$ and $\ell > n/2$. Assume Hypotheses (L) and (TC), a solution of the form

$$\boldsymbol{U(x;\mu) = -\beta^{-1}\alpha\mu[v_*(\sqrt{\alpha\mu}x) + w(x;\mu)]e + v_\perp(x;\mu)}$$

where v_* is the unique positive ground state of $\Delta v - v + v^2 = 0$, $w \in H^\ell(\mathbb{R}^n)$ is a corrector which converges to 0 as $\mu \to 0$, lastly, $v_\perp$ satisfy $\langle e, v_\perp \rangle = 0$ and $\|v_\perp\| = O(\mu^2)$.

Sketch of Proof

- Work with function space

$$H^\ell_\Gamma = \{u \in H^\ell, u(\gamma x) = u(x), \gamma \in \Gamma\};$$

- Set $\varepsilon = \sqrt{\alpha\mu}$, then **rescale** by $v_c = -\beta^{-1}\varepsilon^2 \tilde{v}_c(\varepsilon x)$, $v_h = \varepsilon^2 \tilde{v}_h(\varepsilon x)$, reduce equation

$$\varepsilon^{-2} m^\varepsilon \Delta \tilde{v}_c = \tilde{v}_c - \tilde{v}_c^2 + O(\varepsilon^2) \qquad (2)$$
$$\tilde{v}_h = O(\varepsilon^2) \qquad (3)$$

with m^ε the rescaled pseudo-differential operator, with symbol $(1 + \varepsilon^2|\xi|^2)^{-1}$;

- solve (3) to express $\tilde{v}_h$ in terms of $\tilde{v}_c$ by a fixed point argument. Get a reduced scalar bifurcation equation;
- substitute the ansatz $\tilde{v}_c = v_* + w$, get an equation in w.

Sketch of Proof-Continued

- Observe $(m^\varepsilon)^{-1} : H^\ell \to H^{\ell-2}$ is well-defined, with $\|(m^\varepsilon)^{-1} - 1\| = O(\varepsilon^2)$ as $\varepsilon \to 0$;
- **precondition** equation $\tilde{v}_c$ by the operator $(m^\varepsilon)^{-1}$, simplify, get

$$0 = \Delta w - (w - 2v_* w - w^2)$$
$$- ((m^\varepsilon)^{-1} - 1)(w - 2v_* w - w^2 + \Delta v_*) + O(\varepsilon^2) \qquad (4)$$

denote the right hand of (4) by $F(w;\varepsilon)$, note $F(w;\varepsilon) \to 0$ in L^2 as $(w;\varepsilon) \to (0;0)$, also, $D_w F$ is continuous near $(0,0)$, with

$$D_w F(0;0) = \Delta - 1 + 2v_*,$$

which is nondegenerate and invertible from H^ℓ_Γ to $H^{\ell-2}_\Gamma$ ([3]), these facts allow the set up of an Newton iteration scheme to continue $F(w;\varepsilon) = 0$ from H^ℓ_Γ to $H^{\ell-2}_\Gamma$ near $(0;0)$.

Properties of the Spike

1. The spikes constructed here are not necessarily exponentially localized, it depends on the localization of the convolution kernel K, for K algebraically localized we get algebraically localized spikes due to the corrector w.
2. Typical examples of Γ can be all of $O(n)$ or subgroups generated by reflection across a hyperplane invariant under the action of Γ. Generalize the studies on radial symmetry in the local case.

Further Directions

1. Stability: probably unstable, notice no Evans function techniques available due to nonlocality. It is possible to use the information on the spectrum of the ground state and a perturbative argument.
2. Transition to large μ?: For μ large, in some examples it can be shown many discontinuous solutions exist by implicit function theorem. How do things look in between large μ and small μ?

References

[1] Arnd Scheel.
Radially symmetric patterns of reaction-diffusion systems.
Mem. Amer. Math. Soc., 165(786):viii+86, 2003.

[2] Paul C. Bressloff.
Spatiotemporal dynamics of continuum neural fields.
J. Phys. A, 45(3):033001, 109, 2012.

[3] Juncheng Wei and Matthias Winter.
Mathematical aspects of pattern formation in biological systems, volume 189 of *Applied Mathematical Sciences*.
Springer, London, 2014.