Bifurcation of coherent structures in nonlocally coupled system

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Abstract

Motivated by models for neural fields, we study the existence of pulses bifurcating from a spatially homogeneous state in nonlocally coupled systems of equations. More specifically, we look at equations of the form $AU + K * U = N(U;\mu)$, where N encodes nonlinear terms, A is an invertible matrix, and K an even matrix convolution kernel. Assuming the presence of neutral modes, that is, solutions of the form $u \sim \exp(i\ell x)$ to the linear part, we show under appropriate assumptions on the nonlinearity and the unfolding in μ that pulses bifurcate. Such an anlysis is carried out using center manifold reduction, when coupling is local, say, $K = \delta''$. Here, we rely on functional analytic methods using predictors from formal expansions and correctors obtained after preconditioning the nonlinear system.

1 Main Result

Our interest in this note concerning the equation

$$AU + K * U = N(U;\mu) \tag{1.1}$$

where $U = U(x)$ is a vector valued ($\mathbb{R}^m$) function defined on $\mathbb{R}$, $\mu > 0$ is a real parameter, A is a m by m matrix and $K = K(x)$ is a matrix of convolution kernels, we assume the following on A and K:

Hypothesis (On linear operator $A + K*$)

(i). We require A is invertible, the entries of K are even, $K(x) = K(-x)$, and belongs to $L^1(\mathbb{R})$, and is exponentially localized, i.e. there is $\tau > 0$ such that $\int_{\mathbb{R}} e^{\tau|x|}K(x)dx$ is finite.

(ii). if $\hat{K}(\ell)$ denotes the Fourier transform of the convolution kernel $K(x)$, which is a matrix depending on ℓ, we assume that $\det(A + \widehat{K}(\ell)) = D\ell^2 + O(\ell^4)$ with $D \neq 0$ as $\ell \to 0$, more over we require for all $\ell \neq 0$, $A + \hat{K}(\ell)$ is invertible.

From (ii), we know $A + \widehat{K}(\ell)$ is invertible for all ℓ close to 0, while at $\ell = 0$, as a consequence of the determinant assumption, the kernel of $A + \widehat{K}(0)$ is spanned by a vector unique up to scalar multiplication, we denote $e_0 \in \mathbb{R}^m$ to be this kernel so that $\langle e_0, e_0\rangle = 1$.

Hypothesis (On nonlinearity N)

We assume $N(u;\mu):\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^m$ is a smooth nonlinearity, with $N(0;\mu)=0$ for all μ, and if e_0^* denotes the dual for e_0, we require: $\langle e_0^*, D_{u\mu}N(0;0)e_0\rangle\neq 0$, $\langle e_0^*, D_{uu}N(0;0)[e_0,e_0]\rangle\neq 0$.

We can prove the following lemma which "diagonalize" the operator $A+K*$ in the following sense:

Lemma 1.1. *There exists m by m invertible matrix M_1, M_2 so that*

$$M_1(A+\widehat{K}(\ell))M_2=\begin{pmatrix} A_{00}+\widehat{K}_{00}(\ell) & A_{0h}+\widehat{K}_{0h}(\ell)\\ A_{h0}+\widehat{K}_{h0}(\ell) & A_{hh}+\widehat{K}_{hh}(\ell)\end{pmatrix}$$

where $A_{00}+\widehat{K}_{00}(\ell)=d\ell^2+O(\ell^4)$ ($d\neq 0$) is scalar valued, $A_{0h}+\widehat{K}_{0h}(\ell)=O(\ell^2)$ and $A_{h0}+\widehat{K}_{h0}(\ell)=O(\ell^2)$ are $1\times(m-1)$ and $(m-1)\times 1$ matrix while $A_{hh}+\widehat{K}_{hh}(\ell)$ is an invertible $m-1$ by $m-1$ matrix for all ℓ with uniform bounds on the inverse in ℓ.

Proof. When $\ell=0$, there exist M_2 so that

$$[A+\widehat{K}(0)]M_2=\left(\begin{array}{c|c} 0 & *\cdots* \\ \hline 0 & \\ \vdots & H \\ 0 & \end{array}\right)$$

where $*$ denotes generic numbers, and H is an invertible $m-1$ by $m-1$ matrix, now apply elementary matrix successively from the left reduce the first row to $(1,0\cdots 0)$, let M_1 denote the product of all such elementary matrix.

For ℓ close to 0, we expand $\widehat{K}(\ell)$ around 0, the first component must be equal to $d\ell^2$ for some $d\neq 0$ due to the determinant requirement, the other entries are of order ℓ^2 as the entries of K are assumed to be even.

Hence

$$\begin{pmatrix} A_{00} & A_{0h}\\ A_{h0} & A_{hh}\end{pmatrix}=M_1AM_2 \text{ and } \begin{pmatrix} \widehat{K}_{00}(\ell) & \widehat{K}_{0h}(\ell)\\ \widehat{K}_{h0}(\ell) & \widehat{K}_{hh}(\ell)\end{pmatrix}=M_1\widehat{K}(\ell)M_2$$

are the desired matrix, we remark that the Taylor expansion of the Fourier coefficients says that we have: $\int_{\mathbb{R}}K_{00}(x)+A_{00}=0$, $\int_{\mathbb{R}}K_{0h}+A_{0h}=(0,\cdots,0)$, $\int_{\mathbb{R}}K_{h0}+A_{h0}=(0,\cdots,0)^T$, and $\int_{\mathbb{R}}x^2K_{00}(x)=2d\neq 0$, moreover, by the integralbility assumption on K, we see the $M_1\widehat{K}(\ell)M_2\to 0$ as $\ell\to\infty$.

□

With lemma above we introduce a new variable $V(x)=M_2^{-1}U(x)$, we may write $V(x)=(u_0(x),u_h(x))^T$ where $u_0(x)$ is scalar-valued and $u_h(x)$ is such that $\hat{u}_h(\ell)\perp e_0$ for all ℓ.

Put $\tilde{N}(V;\mu)=M_1N(M_2V;\mu)$, in the new variable V, and write $\tilde{N}_0=\langle\tilde{N},e_0\rangle$ with $\tilde{N}_h$ similarly defined, equation (1.1) becomes

$$(A_{00}+K_{00}*)u_0+(A_{0h}+K_{0h}*)u_h+\tilde{N}_0(u_0,u_h;\mu)=0 \tag{1.2}$$

$$(A_{h0}+K_{h0}*)u_0+(A_{hh}+K_{hh}*)u_h+\tilde{N}_h(u_0,u_h;\mu)=0 \tag{1.3}$$

Denote $L_j = A_j + K_j*$ ($j = 00, 0h, h0, hh$ respectively) for brevity, then write out the nonlinearity explicitly, we have

$$0 = L_{00}u_0 + L_{0h}u_h + a_{101}\mu u_0 + a_{200}u_0^2 + a_{011}\mu u_h + a_{020}u_h^2 + a_{110}u_0u_h + R_1(u_0, u_h; \mu) \tag{1.4}$$
$$0 = L_{hh}u_h + L_{h0}u_0 + b_{101}\mu u_0 + b_{011}\mu u_h + b_{200}u_0^2 + b_{110}u_0u_h + b_{020}u_h^2 + R_2(u_0, u_h; \mu) \tag{1.5}$$

where the remainder R_1, R_2 are of order $O(u_0^2u_h, u_0u_h^2, u_0^3, u_h^3, \mu u_0^2, \mu u_h^2, \mu u_0u_h, \mu^2u_0, \mu^2u_h)$ as $(u_0, u_h) \to 0$.

We change variable from u_h to u_h^1 where u_h^1 is defined to be satisfy the relation:

$$u_h = -L_{hh}^{-1}L_{h0}u_0 + u_h^1 := \phi(u_0, u_h^1),$$

the existence of L_{hh}^{-1} follows from Lemma 1.1, so in the variable u_h^1, (1.5) is

$$0 = L_{hh}u_h^1 + b_{101}\mu u_0 + b_{011}\mu\phi + b_{200}u_0^2 + b_{110}u_0\phi + b_{020}\phi^2 + R_2(u_0, \phi; \mu)$$

We then rescale the variables according to $u_0(x) = \mu\tilde{u}_0(\sqrt{\mu}x)$ and $u_h^1(x) = \mu\tilde{u}_h^1(\sqrt{\mu}x)$, and put $\varepsilon = \sqrt{\mu}$, $L_j^\varepsilon = A + \varepsilon^{-1}K_j(\varepsilon^{-1}\cdot)*$ for the rescaled linear operator, note that u_h has been rescaled to

$$\varepsilon^2[-(L_{hh}^\varepsilon)^{-1}L_{h0}^\varepsilon\tilde{u}_0 + \tilde{u}_h^1] := \varepsilon^2\phi^\varepsilon(\tilde{u}_0, \tilde{u}_h^1),$$

to ease notations, we still use u_0, u_h^1 for the same variables after the rescaling, and we abbreviate u_h by ϕ^ε whenever it is convenient to do so.

We then get two rescaled equations, after dividing both sides of the equation by $\mu = \varepsilon^2$, we have the equation for u_0:

$$\begin{aligned} 0 &= L_{00}^\varepsilon u_0 + L_{0h}^\varepsilon[\phi^\varepsilon] + \varepsilon^2(a_{101}u_0 + a_{200}u_0^2 + a_{011}[\phi^\varepsilon] + a_{020}[\phi^\varepsilon]^2 + a_{110}u_0[\phi^\varepsilon]) + \varepsilon^4R_1(u_0, u_h^1; \varepsilon) \\ &= L^\varepsilon u_0 + L_{0h}^\varepsilon u_h^1 + \varepsilon^2B_0(u_0, u_h^1; \varepsilon) + \varepsilon^4R_1(u_0, u_h^1; \varepsilon) \end{aligned} \tag{1.6}$$

where we abbreviated $L = L_{00} - L_{0h}L_{hh}^{-1}L_{h0}$ so that $L^\varepsilon = L_{00}^\varepsilon - L_{0h}^\varepsilon(L_{hh}^\varepsilon)^{-1}L_{h0}^\varepsilon$, and the term $B_0(u_0, u_h^1; \varepsilon)$ is equal to $a_{101}u_0 + a_{200}u_0^2 + a_{011}[\phi^\varepsilon] + a_{020}[\phi^\varepsilon]^2 + a_{110}u_0[\phi^\varepsilon]$.

and the equation for u_h^1:

$$0 = L_{hh}^\varepsilon u_h^1 + \varepsilon^2B_h(u_0, u_h^1; \varepsilon) + \varepsilon^4R_2(u_0, u_h^1; \varepsilon), \tag{1.7}$$

where $B_h(u_0, u_h^1; \varepsilon) = b_{101}u_0 + b_{200}u_0^2 + b_{011}[\phi^\varepsilon] + b_{020}[\phi^\varepsilon]^2 + b_{110}u_0[\phi^\varepsilon]$.

Now our plan is to first solve (1.7) to get u_h^1 as a function of u_0, then plug in this function back to equation (1.6), and solve the resulting scalar equation in u_0 to get our main result.

To do so, write the right hand of (1.7) as $L_{hh}^\varepsilon G(u_h^1; u_0, \varepsilon)$, so that the function $G = G(v; u, \varepsilon)$ is given by $G(v; u, \varepsilon) = v + (L_{hh}^\varepsilon)^{-1}\varepsilon^2(B_h(u, v; \varepsilon) + \varepsilon^2R_2(u, v; \varepsilon))$, we want to solve $G = 0$ for u_h^1, viewing u_0 and ε as parameters, the following lemma gives the precise properties we want:

Lemma 1.2. *Fix k and $r > 0$, let $B_r \subset H^k(\mathbb{R})$ denote the closed ball of radius r in $H^k(\mathbb{R})$, there is $\varepsilon_0 > 0$ sufficiently small so that if $|\varepsilon| < \varepsilon_0$ and for any function u_0 in B_r, there exists a map $\psi(u, \varepsilon) : B_r \times (-\varepsilon_0, \varepsilon_0) \to [B_r]^{m-1}$ such that $u_h^1 = \psi(u_0, \varepsilon)$ solves $G(\psi(u_0, \varepsilon); u_0, \varepsilon) = 0$, moreover, we have $\|\psi(u_0, \varepsilon)\|_{[H^k]^{m-1}} = O(\varepsilon^2)$ as $\varepsilon \to 0$, and ψ is smoothly dependent on the parameter u, if $D_u\psi : H^k \to H^k$ denotes the Frechet derivative of ψ with respect to u, then we also get $\|D_u\psi\| = O(\varepsilon^2)$ as $\varepsilon \to 0$.*

Proof. We use a Newton iteration scheme: for $u_0 \in B_r$ and ε_0 small to be chosen, We show the following properties holds for G:

- $\|G(0; u_0, \varepsilon)\|_{[H^k(\mathbb{R})]^{m-1}} = O(\varepsilon^2)$
- G is smooth in v, and $D_v G(0; u_0, \varepsilon) : [H^k(\mathbb{R})]^{m-1} \to [H^k(\mathbb{R})]^{m-1}$ is invertible with uniform bounds for the inverse for $|\varepsilon| < \varepsilon_0$ and $u_0 \in B_r$.

To see the first bullet point, simply notice

$$G(0; u_0; \varepsilon) = \varepsilon^2 (L^\varepsilon_{hh})^{-1} \left(B_h(u_0, 0; \varepsilon) + \varepsilon^2 R_2(u_0, 0; \varepsilon) \right),$$

use $H^k(\mathbb{R})$ is an algebra for $k \geq 2$ and the fact that $(L^\varepsilon_{hh})^{-1}, L^\varepsilon_{h0}$ are uniformly bounded in ε, as

$$B_h(u_0, 0; \varepsilon) = b_{101} u_0 + b_{200} u_0^2 + b_{011} \phi^\varepsilon(u_0; 0) + b_{020} [\phi^\varepsilon(u_0; 0)]^2 + b_{110} u_0 [\phi^\varepsilon(u_0; 0)]$$

where now $\phi^\varepsilon(u_0; 0) = -(L^\varepsilon_{hh})^{-1} L^\varepsilon_{h0} u_0$, we have therefore

$$\|B_h(u_0, 0; \varepsilon)\|_{[H^k]^{N-1}} \leq C \|u_0\|^2_{[H^k]^{N-1}} \leq C r^2,$$

for some constant C independent of ε. We get a similar bound for the remainder term R_2, hence $\|G(0; u_0, \varepsilon)\|_{[H^k]^{N-1}}$ is $O(\varepsilon^2)$ as claimed.

For the second bullet point, it is clear that G is smooth in v, we compute the Frechet derivative of G with respect to v, we get $D_v G(0; u_0, \varepsilon)$ is of the form

$$I + \varepsilon^2 (L^\varepsilon_{hh})^{-1} \left(b_{011} + b_{020} [2 L^\varepsilon_{hh} L^\varepsilon_{h0} u_0] + b_{110} u_0 + \varepsilon^2 D_v R_2(0; u_0, \varepsilon) \right)$$

again use $(L^\varepsilon_{hh})^{-1}, L^\varepsilon_{h0}$ are uniformly bounded in ε and $\|u_0\|_{H^k} \leq r$, we conclude that $D_v G(0; u_0, \varepsilon)$ is a $O(\varepsilon^2)$ pertrubation of the identity as an operator from $[H^k(\mathbb{R})]^{m-1}$ to itself, thus for ε small enough, we have $D_v G(0; u_0, \varepsilon)$ is uniformly invertible in ε.

After establishing these two points, fix $\delta > 0$, let X be the ball of raidus δ around 0 in $[H^k(\mathbb{R})]^{m-1}$, we introduce a map $S(\cdot; u_0, \varepsilon) : X \to X$ as follows:

$$S(v; u_0, \varepsilon) = v - D_v G(0; u_0, \varepsilon)^{-1} [G(v; u_0, \varepsilon)]$$

then, we see

$$\|S(0; u_0, \varepsilon)\|_{[H^k]^{m-1}} \leq \|D_v G(v; u_0, \varepsilon)^{-1}\| \|G(v; u_0, \varepsilon)\|_{[H^k]^{m-1}} = O(\varepsilon^2).$$

Also, $D_v S(0; u_0, \varepsilon) = 0$ by definition, by continuity, if δ is small, then for $\|v\| \in X$ we have $\|D_v S\| \leq C\delta$ for some constant C.

Then we start our iteration, with $v_0 = 0$, $v_{n+1} = S(v_n; u_0, \varepsilon)$, $n \geq 0$. Suppose by induction $v_k \in X$ for $1 \leq k \leq n$, then

$$\|v_{n+1} - v_n\| \leq C\delta \|v_n - v_{n-1}\|$$

by the mean value theorem, so

$$\|v_{n+1}\| \leq \frac{C}{1 - C\delta} \|v_1 - v_0\| = \frac{C}{1 - C\delta} \|S(0; u_0, \varepsilon)\|$$

so for ε small and $u_0 \in B_r$, we get $v_{n+1} \in X$, and we that S is a contraction for δ sufficiently small, apply Banach fixed point theorem, we get $v = \psi(u_0, \varepsilon)$ as a fixed point of S, so that $\psi(u_0, \varepsilon) = S(\psi(u_0, \varepsilon); u_0, \varepsilon)$, and consequently $G(\psi(u_0, \varepsilon); u_0, \varepsilon) = 0$, note we get the estimate $\|\psi\|_{[H^k]^{m-1}} = O(\varepsilon^2)$ from the iteration.

The smooth dependence of $\psi(u_0, \varepsilon)$ on u_0 is a consequence of the uniform contraction theorem: note $G(v; u, \varepsilon)$, and therefore the map S is smooth in u by assumptions on the nonlinearity, by choosing ε small, the contraction constant for S can be chosen uniformly in $u \in B_r$, hence we get ψ depends smoothly on u_0 as well.

To get the estimate on the derivative $\|D_u\psi\|_{H^k \to H^k}$, we differentiate the equation $0 = G(\psi(u_0, \varepsilon); u_0, \varepsilon)$ in u_0, we see $D_u\psi = -[D_vG]^{-1}D_uG$, but

$$D_uG(v; u_0, \varepsilon) = O(\varepsilon^2)$$

for $u_0 \in B_r$, as can be computed directly, and D_vG is uniformly invertible for ε small, hence $\|D_u\psi\|$ is $O(\varepsilon^2)$ as claimed. $\square$

Use this lemma, we plug in $u_h^1 = \psi(u_0, \varepsilon)$ into equation (1.6), after dividing both sides by ε^2, we arrived at the following scalar equation in u_0:

$$0 = \varepsilon^{-2} \left(L_\varepsilon u_0 + L_{0h,\varepsilon}\psi(u_0, \varepsilon)\right) + B_0(u_0, \psi(u_0, \varepsilon); \varepsilon) + \varepsilon^2 \tilde{R}_1(u_0, \psi(u_0, \varepsilon); \varepsilon) \tag{1.8}$$

Therefore, we need to study the behaviors of the operators L_j^ε as $\varepsilon \to 0$, let M^ε, denote the operator whose Fourier symbol is $\widehat{L}^\varepsilon(\ell)/(\varepsilon\ell)^2$, the next lemma summarizes the properties we need,

Lemma 1.3. *Fix k, with M^ε defined above, and L_j^ε for $j = h0, 0h$, we have the following:*

(i) $M^\varepsilon : H^k \to H^{k-2}$ *is bounded with* $\|M^\varepsilon\|_{H^k \to H^{k-2}}$ *independent in* ε*;*

(ii) $M^\varepsilon : H^{k-2} \to H^k$ *is invertible, with* $(M^\varepsilon)^{-1} : H^k \to H^{k-2}$ *satisfies the estimate*

$$\|(M^\varepsilon)^{-1} - d^{-1}\|_{H^k \to H^{k-2}} = O(\varepsilon^2)$$

as $\varepsilon \to 0$*, we denote the operator* $(M^\varepsilon)^{-1} - d^{-1}$ *by* $\mathcal{J}^\varepsilon$*;*

(iii) $L_j^\varepsilon : H^k \to H^k$ *(*$j = h0, 0h$*) is bounded with* $\|L_j^\varepsilon\|_{H^k \to H^k}$ *independent in* ε*; and* $L_j^\varepsilon : H^k \to H^{k-2}$ *is bounded with* $\|L_j^\varepsilon\|_{H^k \to H^{k-2}} = O(\varepsilon^2)$ *as* $\varepsilon \to 0$*.*

Proof. The symbol for L_ε equals $\widehat{L}(\varepsilon\ell) = \widehat{L}_{00}(\varepsilon\ell) - \widehat{L}_{0h}(\varepsilon\ell)(\widehat{L}_{hh}(\varepsilon\ell))^{-1}\widehat{L}_{h0}(\varepsilon\ell) = d(\varepsilon\ell)^2 + O((\varepsilon\ell)^4)$ by lemma 1.1 on the expansions about the symbols of L_{00}, L_{0h}, L_{h0} and L_{hh}. Hence $m_1(\varepsilon\ell) := \widehat{L}(\varepsilon\ell)/(\varepsilon\ell)^2 = d + O((\varepsilon\ell)^2)$ is the symbol of M_ε. We then define $m_2(\varepsilon\ell) := (\widehat{L}(\varepsilon\ell) - d(\varepsilon\ell)^2)/(\varepsilon\ell)^4$, again by lemma 1.1 we know m_1, m_2 are analytic, in particular, $m_1(0) = d \neq 0$.

Then $\|M^\varepsilon\|_{H^2 \to L^2} = \sup_\ell |m_1(\varepsilon\ell)|$, which is independent of ε by change of variable $\varepsilon\ell = \eta$, this is (i).

(ii), this is the key computation, we have:

$$
\begin{aligned}
\|\mathcal{J}^\varepsilon\|_{H^k\to H^{k-2}} = \sup_\ell \left|\frac{1}{1+\ell^2}\widehat{\mathcal{J}_\varepsilon}(\ell)\right| &= \sup_\ell \left|\left(\frac{d-M_\varepsilon(\ell)}{dM_\varepsilon(\ell)}\right)\frac{1}{1+\ell^2}\right| \\
&= \sup_\ell \left|\frac{d^{-1}(\varepsilon\ell)^2 m_2(\varepsilon\ell)}{m_1(\varepsilon\ell)}\frac{1}{1+\ell^2}\right| \\
&= \sup_\eta \left|\varepsilon^2\frac{d^{-1}\eta^2}{\eta^2+\varepsilon^2}\left(\eta^{-2}+d\widehat{L}(\eta)^{-1}\right)\right|
\end{aligned}
$$

where in the last step we changed variable by $\eta = \varepsilon\ell$, we shall carefully study the quantity

$$\frac{m_2(\eta)}{m_1(\eta)} = \frac{\widehat{L}(\eta)-d\eta^2}{\eta^2\widehat{L}(\eta)} = \eta^{-2} - d\widehat{L}(\eta)^{-1}.$$

First, notice that $m_1(0)\neq 0$, hence at $\eta = 0$ this is a finite number.

Further, if at some finite η, $m_1(\eta) = 0$, this can only happen if $\widehat{L}(\eta) = 0$ for some $0 < |\eta| < \infty$, this is impossible, as a 0 of $\widehat{L}(\eta)$ corresponds to a η for which the matrix $\begin{pmatrix}\widehat{L}_{00}(\eta) & \widehat{L}_{0h}(\eta)\\ \widehat{L}_{h0}(\eta) & \widehat{L}_{hh}(\eta)\end{pmatrix}$ is singular, because of the formula

$$\det\begin{pmatrix}\widehat{L}_{00}(\eta) & \widehat{L}_{0h}(\eta)\\ \widehat{L}_{h0}(\eta) & \widehat{L}_{hh}(\eta)\end{pmatrix} = \det(\widehat{L}_{hh}(\eta))\det(\widehat{L}_{00}(\eta) - \widehat{L}_{0h}(\eta)L_{hh}(\eta)^{-1}\widehat{L}_{h0}(\eta)) = \det(\widehat{L}_{hh}(\eta))\widehat{L}(\eta)$$

holds, as $\widehat{L}_{hh}(\eta)$ is invertible for all η.

But our assumption says the point $\eta = 0$ is the only value this can happen, hence $\widehat{L}(\eta)\neq 0$ for all finite η.

Finally, the matrix $\begin{pmatrix}\widehat{L}_{00}(\eta) & \widehat{L}_{0h}(\eta)\\ \widehat{L}_{h0}(\eta) & \widehat{L}_{hh}(\eta)\end{pmatrix}$ converges to M_1AM_2 as $\eta\to\infty$, which invertible by assumption and lemma 1.1, so again by the determinant formula we conclude $\widehat{L}(\eta)\neq 0$ at $\eta = \infty$.

Therefore, the quantity $\eta^{-2} - d\widehat{L}(\eta)^{-1}$ is bounded on $\eta\in\mathbb{R}$, so

$$\sup_\eta \left|\varepsilon^2\frac{d^{-1}\eta^2}{\eta^2+\varepsilon^2}\left(\eta^{-2}+d\widehat{L}(\eta)^{-1}\right)\right| \leq C\varepsilon^2$$

for some constant C, this shows $\|\mathcal{J}^\varepsilon\|_{H^2\to L^2}$ is of order ε^2.

(iii). The first claim is clear, since $\sup_\ell |L_j^\varepsilon(\ell)| = \sup_\ell |L_j(\varepsilon\ell)|$, the conclusion follows as in (i);

The second claim follows from the straightforward computation:

$$\|L_j^\varepsilon\|_{H^2\to L^2} \leq \sup\left\|\frac{\widehat{L}_j(\varepsilon\ell)}{1+\ell^2}\right\| = \sup_\eta\left\|\frac{\varepsilon^2}{\varepsilon^2+\eta^2}\widehat{L}_j(\eta)\right\| \leq \varepsilon^2\sup_\eta\left\|\frac{\widehat{L}_j(\eta)}{\eta^2}\right\|$$

and we know $\widehat{L}_j(\eta) = O(\eta^2)$ as $\eta\to 0$ and $\widehat{L}_j(\eta)\to A_j$ as $\eta\to\infty$, so the supremum is finite and $\|L_j^\varepsilon\|_{H^k\to H^{k-2}}$ is of order ε^2. □

We are ready to state and prove our main results:

Theorem 1.4. *Fix k so that $k-2 \geq 0$, then there exist ε_0 small so that for $0<|\varepsilon|<\varepsilon_0$, a solution of the form $u_0 = u_*(\cdot) + w$ to (1.8) exists, here $w = w(\varepsilon) \in H^k_{even}(\mathbb{R})$ is an ε-dependent perturbation term such that $\|w(\varepsilon)\|_{H^2} = O(\varepsilon^2)$ as $\varepsilon \to 0$, and u_* is the unique solution to*

$$du_*^{''} - u_* + u_*^2 = 0$$

which decays at both ends of the real line: $u_*(\pm\infty) = \lim_{x\to\pm\infty} u_*(x) = 0$.

Proof. We assume $a_{101} = -1, a_{200} = 1$ after possibly another rescaling Let $u_0 = u_* + w$ where u_* is the unique bounded solution of the equation $du'' - u + u^2 = 0$ satisfies $u_*(\pm\infty) = 0$. We determine the equation satisfied by w:

Substitute $u_0 = u_* + w$, subtract the equation $0 = du_*^{''} - u_* + u_*^2$ from equation (1.8), we have

$$0 = (M^\varepsilon - d)v_*^{''} + M^\varepsilon w^{''} - w + 2u_*w + w^2 + \mathcal{R}$$

where $\mathcal{R}$ contains all the "ε^2 term"

$$\begin{aligned}\mathcal{R} = \mathcal{R}(u_0, \psi(u_0,\varepsilon);\varepsilon) &= \varepsilon^{-2} L^\varepsilon_{0h}\psi(u_0,\varepsilon) + a_{011}[-(L^\varepsilon_{hh})^{-1}L^\varepsilon_{h0}u_0 + \psi(u_0,\varepsilon)] \\ &+ a_{020}[-(L^\varepsilon_{hh})^{-1}L^\varepsilon_{h0}u_0 + \psi(u_0,\varepsilon)]^2 + a_{110}u_0[-(L^\varepsilon_{hh})^{-1}L^\varepsilon_{h0}u_0 + \psi(u_0,\varepsilon)] \\ &+ \varepsilon^2 R_1(u_0, \psi(u_0,\varepsilon);\varepsilon)\end{aligned}$$

Now precondition the operator $(M^\varepsilon)^{-1}$ to both sides of this equation:

$$0 = (1 - d(M^\varepsilon)^{-1})u_*^{''} + w^{''} - (M^\varepsilon)^{-1}(w - 2u_*w - w^2) + (M^\varepsilon)^{-1}\mathcal{R},$$

We write the term $-(M^\varepsilon)^{-1}(w - 2u_*w - w^2)$ as $(-d^{-1} + d^{-1} - (M^\varepsilon)^{-1})(w - 2u_*w - w^2)$, and the equation becomes

$$0 = w'' - d^{-1}(w - 2u_*w - w^2) + (d^{-1} - (M^\varepsilon)^{-1})(w - 2u_*w - w^2 + du_*'') + (M^\varepsilon)^{-1}\mathcal{R} \tag{1.9}$$

We denote the right hand side of (1.9) as $F(w,\varepsilon)$, and the plan is to set up an Newton iteration scheme to solve the equation $F(w,\varepsilon) = 0$ for w in terms of ε as a fixed point point problem.

As in lemma 1.2, we plan to do the following:

- $\|F(0,\varepsilon)\|_{H^{k-2}} \to 0$ as $\varepsilon \to 0$.
- $D_wF(0,\varepsilon) : H^k_{even} \to H^{k-2}_{even}$ is invertible with uniform bounds in ε on the inverse.

(i) To show the continuity condition $\|F(0,\varepsilon)\|_{H^{k-2}} \to 0$ as $\varepsilon \to 0$ holds, we plug in $w = 0$, and get:

$$F(0,\varepsilon) = (d^{-1} - (M^\varepsilon)^{-1})du_*^{''} + (M^\varepsilon)^{-1}\mathcal{R}$$

since $\|d^{-1} - (M^\varepsilon)^{-1}\|_{H^k \to H^{k-2}} = \|\mathcal{J}^\varepsilon\|_{H^2\to L^2} = O(\varepsilon^2)$, we focus on $\mathcal{R}$,

First, the remainder $\varepsilon^2\|(M^\varepsilon)^{-1}R_1\|_{H^{k-2}}$ is of order ε^2, just because $(M^\varepsilon)^{-1}$ is uniformly bounded from $H^k \to H^{k-2}$.

To estimate $\varepsilon^{-2}(M^{\varepsilon})^{-1}L^{\varepsilon}_{0h}\psi(u_*+w,\varepsilon)$, we write it as

$$\varepsilon^{-2}\mathcal{J}^{\varepsilon}L^{\varepsilon}_{0h}\psi(u_*+w,\varepsilon)+\varepsilon^{-2}d^{-1}L^{\varepsilon}_{0h}\psi(u_*+w,\varepsilon)$$

for the first summand, we have

$$\|\varepsilon^{-2}\mathcal{J}^{\varepsilon}L^{\varepsilon}_{0h}\psi(u_*+w,\varepsilon)\|_{H^{k-2}}\leq\|\mathcal{J}^{\varepsilon}\|_{H^k\to H^{k-2}}\|\|L^{\varepsilon}_{0h}\|_{H^k\to H^k}\|\varepsilon^{-2}\psi\|_{H^k}=O(\varepsilon^2)$$

since $\|\psi\|_{H^k}=O(\varepsilon^2)$ by lemma (1.2);

while for the second summand, use lemma (1.3) (iii), we have

$$\|d^{-1}\varepsilon^{-2}L^{\varepsilon}_{0h}\psi\|_{H^{k-2}}\leq d^{-1}\|L^{\varepsilon}_{0h}\|_{H^k\to H^{k-2}}\|\varepsilon^{-2}\psi\|_{H^k}=O(\varepsilon^2)$$

the other terms are dealt similarly, note when estimating the H^{k-2} norm of the nonlinear term $\|-(L^{\varepsilon}_{hh})^{-1}L^{\varepsilon}_{h0}(u_*+w)+\psi(u_*+w,\varepsilon)]^2\|_{H^{k-2}}$, we use the fact that H^k embedds into L^{∞}, if we denote the term $-(L^{\varepsilon}_{hh})^{-1}L^{\varepsilon}_{h0}(u_*+w)+\psi(u_*+w,\varepsilon)$ by A, then

$$\begin{aligned}\|(M^{\varepsilon})^{-1}[A]^2\|_{H^{k-2}}&\leq\|(M^{\varepsilon})^{-1}-d^{-1}\|_{H^k\to H^{k-2}}\|A^2\|_{H^k}+d^{-1}\|A^2\|_{H^{k-2}}\\&\leq\|\mathcal{J}^{\varepsilon}\|_{H^k\to H^{k-2}}\|A\|^2_{H^k}+d^{-1}\|A\|_{\infty}\|A\|_{H^{k-2}}\end{aligned}$$

and $\|A\|_{H^{k-2}}$ is $O(\varepsilon^2)$ because since $\|\psi\|_{H^{k-2}}$ is $O(\varepsilon^2)$ and

$$\|-(L^{\varepsilon}_{hh})^{-1}L^{\varepsilon}_{h0}(u_*+w)\|_{H^{k-2}}\leq\|-(L^{\varepsilon}_{hh})^{-1}\|_{H^{k-2}\to H^{k-2}}\|L^{\varepsilon}_{h0}\|_{H^k\to H^{k-2}}\|u_*+w\|_{H^{k-2}}.$$

Therefore we see $\|[-(L^{\varepsilon}_{hh})^{-1}L^{\varepsilon}_{h0}(u_*+w)+\psi(u_*+w,\varepsilon)]^2\|_{H^{k-2}}$ is of $O(\varepsilon^2)$ as well, similarly we have the same estimate for the term $u_0[-(L^{\varepsilon}_{hh})^{-1}L^{\varepsilon}_{h0}(u_*+w)+\psi(u_*+w,\varepsilon)]$.

(ii) We check F is continuously differentiable in w, first note the term

$$(M^{\varepsilon})^{-1}\mathcal{R}=(M^{\varepsilon})^{-1}\mathcal{R}(u_*,\psi(u_*+w,\varepsilon);\varepsilon)$$

is smooth in w simply because R_1 is smooth in all its variables, and $\psi(u_0,\varepsilon)$ is smooth in u_0 by lemma 1.2, and L^{ε}_j are just linear operators.

Then we only need to find the derivative of the term

$$\tilde{F}(w,\varepsilon)=w''-d^{-1}(w-2u_*w-w^2)+\mathcal{J}^{\varepsilon}(w-2u_*w-w^2+du_*'')$$

we compute the Frechet derivative, for $h\in H^k$, we find

$$D_w\tilde{F}(w,\varepsilon)h:=h''-d^{-1}(h-2(u_*+w)h)-2\mathcal{J}^{\varepsilon}(u_*+w)h$$

we have:

$$\|\tilde{F}(w+h,\varepsilon)-\tilde{F}(w,\varepsilon)-D_w\tilde{F}(w,\varepsilon)h\|=O(\|h\|^2)$$

So $D_wF(w,\varepsilon):H^k\to H^{k-2}$ is given by

$$D_wF(w,\varepsilon)h=h''-d^{-1}(h-2(u_*+w)h)-2\mathcal{J}^{\varepsilon}(u_*+w)h+(M^{\varepsilon})^{-1}D_w\mathcal{R}$$

The continuity of $D_wF(w,\varepsilon)$ in w follows from the following: Since $\mathcal{R}$ is smooth in w, we only need to check $\tilde{F}$ is continuously differentiable in w, for $h \in H^k$ with $\|h\|_{H^k} = 1$, we have:

$$\begin{aligned}\|D_w\tilde{F}(w_1,\varepsilon)h - D_w\tilde{F}(w_2,\varepsilon)h\|_{H^{k-2}} &= \|2d^{-1}h(w_1-w_2) - \mathcal{J}^\varepsilon 2h(w_1-w_2)\|_{H^{k-2}} \\ &\le \|2d^{-1}h(w_1-w_2)\|_{H^{k-2}} + \|\mathcal{J}^\varepsilon\|_{H^k\to H^{k-2}}\|2h(w_1-w_2)\|_{H^k} \\ &\le (d^{-1} + \|\mathcal{J}_\varepsilon\|_{H^k\to H^{k-2}})\|2h(w_1-w_2)\|_{H^k} \\ &\le C\|h\|_{H^k}\|w_1-w_2\|_{H^k} = C\|w_1-w_2\|_{H^k}\end{aligned}$$

where again we used the fact that H^k is an algebra. We therefore conclude that $D_wF(w,\varepsilon)$ is continuous in w.

Finally we claim that the remainder term $\mathcal{R}$ satisfies

$$\|(M^\varepsilon)^{-1}D_w\mathcal{R}(u_*+w,\psi(u_*+w,\varepsilon);\varepsilon)\|_{H^k\to H^{k-2}} = O(\varepsilon^2)$$

as $\varepsilon \to 0$.

To see this, recall from lemma 1.2 we have that

$$\|D_u\psi(u_*+w,\varepsilon)\|_{H^k\to H^k}$$

is $O(\varepsilon^2)$ as well, hence, for the first term in $(M^\varepsilon)^{-1}D_w\mathcal{R}$, which is $\varepsilon^{-2}(M^\varepsilon)^{-1}L^\varepsilon_{0h}D_u\psi(u_*+w,\varepsilon)$ we have

$$\begin{aligned}&\|(M^\varepsilon)^{-1}L^\varepsilon_{0h}\varepsilon^{-2}D_u\psi(u_*+w,\varepsilon)\|_{H^k\to H^{k-2}} \\ &\le \|\mathcal{J}^\varepsilon\|_{H^k\to H^{k-2}}\|L^\varepsilon_{0h}\|_{H^k\to H^k}\|\varepsilon^{-2}D_u\psi\|_{H^k\to H^k} + d^{-1}\|L^\varepsilon_{0h}\|_{H^k\to H^{k-2}}\|\varepsilon^{-2}D_u\psi\|_{H^k\to H^k} \\ &= O(\varepsilon^2)\end{aligned}$$

this is essentially the same estimate we did to show $\|\mathcal{R}\|_{H^k} = O(\varepsilon^2)$, the other terms in $\mathcal{R}$ is dealt similarly.

(iii) Now, by above estimate, we find D_wF is continuous at $(w,\varepsilon) = (0,0)$, therefore we have $D_wF(0,0)h = h'' - d^{-1}(h - 2u_*h) := Lh$, we show this is an invertible operator from $H^k_{even} \to H^{k-2}_{even}$. This will show that $D_wF(0,\varepsilon)$ is invertible with inverse bounded uniformly in ε as $\varepsilon \to 0$.

To show the invertiblity I show first that the kernel of L consists of bounded solutions (since $H^k(\mathbb{R})$ functions are bounded continuous by embedding) for the differential equation $dh'' - h + 2u_*h = 0$.

This equation is satisfied by the function $v_*'(x)$, and we claim that it is the unique element which spans the kernel of L, to see this, we rewrite the equation as a nonautonoumous linear first order system

$$\dot{Y}(x) = A(x)Y(x), \text{ wtih } Y(x) = \begin{pmatrix} h(x) \\ h'(x)\end{pmatrix} \text{ and } A(x) = \begin{pmatrix} 0 & 1 \\ d^{-1}(1-2u_*(x)) & 0\end{pmatrix}$$

Now, as $A(x)$ converges to the hyperbolic matrix $A_\infty = \begin{pmatrix} 0 & 1 \\ d^{-1} & 0\end{pmatrix}$ as $x \to \pm\infty$, by the robustness of exponential dichotomy, we see $A(x)$ possess an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}-$, let E^s_+ denote the image of the stable projection of the exponential dichotomy on $\mathbb{R}^+$ and E^u_- denote the unstable projection of the exponential dichotomy on $\mathbb{R}^-$. The kernel of L is isomorphic to $E^s_+ \cap E^u_-$, but E^u_- and E^s_+ are one-dimensional since the stable and unstable eigenspace of A_∞ are both 1, hence the kernel is at most one-dimensional, since u_*' already lies in the kernel, we see the dimension of the kernel is exactly one.

We show that L is Fredholm with index zero from H^k_{even} to H^{k-2}_{even}, by writing it as the compact perturbation of an invertible operator: write

$$dLh = (dh'' - h) + (2u_*)h := L_1 h + L_2 h,$$

We need to show $L_2 : h \mapsto (2u_*)h$ is compact: write $L_2 h$ as the limit of $2\chi_{(-L,L)}(x)u_*(x)h(x) := K_L h$ as $L \to \infty$, and show that K_L is compact. Note first $H^k(\mathbb{R})$ functions continuously differentiable by Sobolev embedding, so if h_i is a bounded sequence in H^k, they are in particular a bounded sequence in C^1, then we need to show $K_L h_i$ possess a convergent subsequence, but h_i has an convergence subsequence on $[-L, L]$ since it is a compact interval where h_i has continuous hence bounded derivative, so by Arezela-Ascoli, we see K_L is compact from H^k to H^k, finally $K_L \to L_2$ in operator norm since

$$\sup_{x\in\mathbb{R}} |2u_*(x)\chi_{(-L,L)}(x) - 2u_*(x)| \to 0$$

as $L \to \infty$ because $u_*(x) \to 0$ as $x \to \pm\infty$. Hence L_2 is the limit of the sequence of compact operators K_L in the operator norm, we conclude that L_2 is compact.

Then we need to show the second order differential operator $L_1 h = dh'' - h$ is invertible as an operator from the space $H^k_{even} \to H^{k-2}_{even}$, simply solve this equation in the Fourier side:

$$-(1 + d\ell^2)\hat{h}(\ell) = \hat{f}(\ell)$$

we get $h = \mathcal{F}^{-1}\{-(1 + d\ell^2)^{-1}\hat{f}\}$, this is clearly bounded from H^{k-2}_{even} to H^k_{even} and is an inverse for L_2.

With these facts of L_1 and L_2, we conclude that L is Fredholm with zero index from H^k_{even} to H^{k-2}_{even}, since the kernel consists precisely of the odd function u_*', it is invertible from H^k_{even} to H^{k-2}_{even}.

Finally we claim $D_w F(0, \varepsilon)$ is continuous at $\varepsilon = 0$, indeed, a simple calculation shows $D_w F(0, \varepsilon) = L + \mathcal{J}^\varepsilon u_* h + (M^\varepsilon)^{-1} D_w \mathcal{R}$, which is an ε^2 perturbation of L, so for ε small, we conclude that $D_w F(0; \varepsilon)^{-1}$ exists and is uniformly bounded by some constant independent of ε.

(iv) Here we set up our Newton iteration scheme again, this is analogous to lemma 1.2, we introduce a map $S(\cdot; \varepsilon) : H^k \to H^k$ as

$$S(w; \varepsilon) = w - D_w F(0; \varepsilon)^{-1}[F(w; \varepsilon)]$$

based on previous calculation, we have

$$\|S(0; \varepsilon)\|_{H^k} \leq \|D_w F(0; \varepsilon)^{-1}\|_{H^{k-2}_{even} \to H^k_{even}} \|F(0; \varepsilon)\|_{H^{k-2}_{even}} = O(\varepsilon^2)$$

as $\varepsilon \to 0$.

Also, S is continuously differentiable in w, and $D_w S(0; \varepsilon) = 0$ by definition of S, hence, there exists $\tilde{\delta}$ and a constant $\tilde{C}$ such that if $\|w\|_{H^k} \leq \tilde{\delta}$, then $\|D_w S(w; \varepsilon)\| \leq \tilde{C}\tilde{\delta}$, simply by continuity of $D_w S$ in w.

Then, we set up the iteration:

$$w_{n+1} = S(w_n; \varepsilon)$$

exactly as in lemma 1.2, again for ε small S maps $B_{\tilde{\delta}} \subset H^k(\mathbb{R})$ into itself and is a contraction, and we find $w = w(\varepsilon) = \lim_{n\to\infty} w_n$ and $w(\varepsilon) = S(w(\varepsilon); \varepsilon)$, so $F(w(\varepsilon); \varepsilon) = 0$ and $\|w(\varepsilon)\| = O(\varepsilon^2)$, which is the desired properties.

$\square$