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MODULATIONAL STABILITY OF GROUND STATES OF NONLINEAR SCHRÖDINGER EQUATIONS*

MICHAEL I. WEINSTEIN†

Abstract. The modulational stability of ground state solitary wave solutions of nonlinear Schrödinger equations relative to perturbations in the equation and initial data is studied. In the "subcritical case" ground states are shown by variational methods to be stable modulo time-dependent adjustments (modulations) of free parameters. These parameters satisfy the *modulation equations*, a coupled system of nonlinear ODE's governing the amplitude, phase, position and speed of the dominant solitary wave part of the solution.

1. Introduction. The initial-value problem (IVP) for the nonlinear Schrödinger equation (NLS)

$$2i\phi_t+\Delta\phi+|\phi|^{2\sigma}\phi=0, \qquad 0<\sigma<\frac{2}{N-2}, \tag{1.1}$$
[1]

$$\phi(x,0)=\phi_0(x), \qquad x\in\mathbb{R}^N \tag{1.2}$$

arises in the mathematical description of a diverse set of physical phenomena. Some of these are

(a) the propagation of a narrow electromagnetic beam through a medium with an index of refraction dependent on the field intensity [1], [7], [18], [27],

(b) electromagnetic (Langmuir) waves in a plasma [31], [32], and

(c) the motion of a vortex filament for the Euler equations of fluid mechanics [15].

NLS has nonlinear bound states which are "localized" finite energy solutions. These are believed to describe wave phenomena that are observed in the above physical contexts. Such solutions of (1.1) can be found in the form

$$\psi^0(x,t)=u(x)e^{it/2}. \tag{1.3}$$
[2]

Substitution of (1.3) into (1.1) implies

$$\Delta u-u+|u|^{2\sigma}u=0, \qquad 0<\sigma<\frac{2}{N-2}. \tag{1.4}$$

The existence of infinitely many H^1 solutions of (1.4) follows from work of Strauss [24] (see also [4]). Among them is a real, positive, and radial solution which we call the *ground state* and denote by $R(x)$. To describe a physical phenomenon, a nonlinear bound state should be stable. In this paper we study the stability of the ground state relative to small perturbations in the equation and initial data.

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† Department of Mathematics, Stanford University, Stanford, California 94305. Present address: Department of Mathematics, Princeton University, Princeton, New Jersey 08544. This research was supported in part by the National Science Foundation, the Office of Naval Research, the Air Force Office of Scientific Research and the Army Research Office.

[1] For $N=1$ or 2 we allow $0<\sigma<\infty$.

[2] If we seek solutions of the form $\psi^0=ue^{iEt/2}$, E real, then the rescaling $W(x)=E^{1/2\sigma}u(Ex)$ leads to (1.4).

We now discuss the sense in which we expect the ground state to be stable and then state our results in this direction. Let a, η_0, $\xi=(\xi_1,\cdots,\xi_N)$ and $\theta_0=(\theta_{0,1},\cdots,\theta_{0,N})$ be real constants. By the scaling properties of (1.1) or by direct verification it can be seen that the functions

$$\psi(x,t;a,\theta_0,\xi,\eta_0)=a^{-1/\sigma}R\left(a^{-1}[\theta-\theta_0]\right)\exp\left[i\left(\xi\cdot(\theta-\theta_0)+(\eta-\eta_0)\right)\right] \tag{1.5}$$

form a $(2N+2)$-parameter family of solutions of (1.1) for $0<\sigma<2/(N-2)$ provided the following relations hold:

$$\text{(1.6)}\qquad \begin{aligned} &\text{(a)}\quad \frac{\partial\theta_i}{\partial t}=-\xi_i, &&\text{(c)}\quad \frac{\partial n}{\partial t}=\frac{a^{-2}+\xi\cdot\xi}{2},\\ &\text{(b)}\quad \frac{\partial\theta_i}{\partial x_j}=\delta_{ij}, &&\text{(d)}\quad \frac{\partial\eta}{\partial x_j}=0,\qquad 1\leqq i,j\leqq N. \end{aligned}$$

In the "critical" case $\sigma=2/N$, there is at least one more scale invariance of (1.1). In particular, for $b\in\mathbb{R}$ the functions [20]

$$\begin{aligned} \psi(x,t;a,b,\theta_0,\xi,\eta_0) &= (a+bt)^{-N/2}R\left((a+bt)^{-1}[\theta-\theta_0]\right)\\ &\quad\cdot\exp\left[i\left(\xi\cdot(\theta-\theta_0)+\frac{b}{2}(a+bt)^{-1}|\theta-\theta_0|^2+\eta-\eta_0\right)\right] \end{aligned} \tag{1.7}$$

form a $(2N+3)$-parameter family of solutions of (1.1) provided (1.6a, b, d) are satisfied together with the following extension of (1.6c):

$$\frac{\partial\eta}{\partial t}=\frac{(a+bt)^{-2}+\xi\cdot\xi}{2}. \tag{1.6c$'$}$$

Note that (1.7) reduces to (1.5) when $b=0$. We will refer to the functions (1.5) when $\sigma\neq 2/N$ and (1.7) when $\sigma=2/N$ as the *ground state traveling wave family* or simply *ground state family* of NLS.

To study the stability of the ground state family we consider the perturbed IVP

$$2i\phi_t^\varepsilon+\Delta\phi^\varepsilon+|\phi^\varepsilon|^{2\sigma}\phi^\varepsilon=\varepsilon F\left(|\phi^\varepsilon|\right)\phi^\varepsilon,\qquad \phi^\varepsilon(x,0)=R(x)+\varepsilon S(x). \tag{1.8}$$

In general, the solution ϕ^ε of (1.8) will not evolve in the simple form

$$\phi^\varepsilon(x,t)=\left[R(x)+\varepsilon w_1+\varepsilon^2 w_2+\cdots\right]e^{it/2} \tag{1.9}$$

with $\varepsilon w_1+\varepsilon^2 w_2+\cdots$ genuinely small for large times (say of order $1/\varepsilon$). The possibility of the *linearized perturbation*, εw_1, becoming nonnegligible for large time can be displayed as follows. Differentiation with respect to the free parameters of ψ in (1.5) or (1.7) generates solutions of the linearized (about the ground state) equation with polynomial growth in time. Since in general εw_1 will contain these solutions of the linearized problem or "secular modes", εw_1 will not remain small for large times.

When $\sigma<2/N$ there are $2N+2$ secular modes associated with the $2N+2$ parameter family of solutions (1.5). When $\sigma=2/N$, 2 more secular modes arise giving $2N+4$. One of these new modes is associated with the new parameter b in (1.7). The other, however, has not been associated with a classical symmetry of equation (1.1). This is further discussed in [20].

In §2 we show that these secular modes are the only source of linear instability of the ground state. In particular, we show that by constraining the evolution of the linearized perturbation w_1 to the space H^1 with the secular modes removed, w_1 is controllable in H^1 (Theorem 2.12). We call this space M. All theorems of §2 are stated and proofs carried out in an arbitrary spatial dimension N, although the technical point about $N(L_+)$ in part b of Proposition 2.8 has been completely proved, only in dimension $N=1$ for all σ and in dimension $N=3$ for $0<\sigma\leqq 1$ (see appendix A). Since the stability analysis of the ground state concerns the case $\sigma \leqq 2/N$, our results are completely rigorous in dimensions $N=1$ and 3, and are lacking only in the above mentioned technical point in other dimensions.

A natural remedy to the growth of w_1 in (1.9) is the use of the ground state *family* with slowly varying parameters: $a(\varepsilon t)$, $\xi(\varepsilon t)$ etc. as the leading order Ansatz. The idea is to choose the slow functions a, ξ etc. to constrain the evolution of w_1 to M, thereby ensuring that $\varepsilon w_1(t)$ is genuinely small for times of order $1/\varepsilon$. More precisely, in §3 we prove our main result which we state now as (later as Theorem 1′)

THEOREM 1. *Let $\sigma<2/N$. Expand the solution $\phi^\varepsilon(x,t)$ of* (1.8) *as*

$$\phi^\varepsilon(x,t)=\left(\lambda^{1/\sigma}R(\lambda(\theta-\theta_0))+\varepsilon w_1+\varepsilon^2 w_2+\cdots\right)e^{i[\xi\cdot(\theta-\theta_0)+\eta-\eta_0]} \tag{1.10}$$

with

$$w_1=w_1\left(\lambda(\theta-\theta_0),\ \int_0^t \lambda^2\,ds\right).$$

For a class of perturbations F, if the $2N+2$ parameters λ, ξ, θ_0 and η_0 evolve as functions of $T\equiv\varepsilon t$, according to the coupled system of $2N+2$ ordinary differential equations (3.5) (*the modulation equations*), *then*

(i) *$w_1\in M$ for $t>0$, and*

(ii) *for any $T_0>0$*

$$\sup_{0\leqq t\leqq T_0/\varepsilon} \|\varepsilon w_1(t)\|_{H^1}=\alpha(\varepsilon), \tag{1.11}$$

where $\alpha(\varepsilon)\downarrow 0$ as $\varepsilon\to 0$.

The modulation equations have been derived previously for various one-dimensional formal perturbation theories [16], [17], [19], [22]. An aim of this paper is to present some justification for these perturbation techniques. Results on the nonlinear stability of ground states were obtained by Cazenave [5] for a logarithmic NLS and by Cazenave and Lions [6] for (1.1) with $\sigma<2/N$. In these works, perturbations of the initial data alone are considered. They prove H^1 stability of the ground state modulo "adjustments in the free phase and centering parameters." These adjustments are incorporated in the norm and are not explicitly constructed. (This idea has been used for other equations as well [2], [3], [10].) Thus, their work shows that initial data near a solitary wave evolves nearly as a solitary wave with unspecified, possibly different position and phase for times $t>0$. The modulation theory presented and analyzed here provides an approximate and constructive answer to the questions (a) *where the solitary wave is located* and (b) *what its phase is* for $t>0$ (see Example 3.1). When small perturbations of the equation (e.g. dissipation) occur, the phase and centering parameters are, in general, not sufficient to describe the dominant part of the solution for $t>0$. Modulation theory gives the additional explicit information on the *amplitude* and *speed* of the solitary wave needed to track the solution closely (see Example 3.2).

In the mathematical theory of NLS, the case $\sigma=2/N$ has been understood to play a distinguished role. It is also of physical interest since the case $\sigma=1$, $N=2$ arises in modelling self-focusing optical beams [18]. The cases $\sigma<2/N$, $\sigma=2/N$ and $\sigma>2/N$ are called, respectively, *subcritical*, *critical* and *supercritical* cases. A consequence of work of Ginibre and Velo [13] is that if $\sigma<2/N$, (1.1)–(1.2) has global solutions in $C([0,\infty);H^1(\mathbb{R}^N))$ for all $\phi_0\in H^1(\mathbb{R}^N)$. In the case $\sigma\geqq 2/N$ Glassey [14] has displayed a class of initial data for which the solution of the IVP "blows up" in finite time in H^1, i.e. there is a finite time T, such that $\int|\nabla_x\phi(x,t)|^2\,dx\to\infty$ as $t\to T$ (see also [28]).

Numerical observations in certain critical cases (Zakharov–Synakh [33] for $(\sigma,N)=(1,2)$ and later Sulem–Sulem–Patera [26] for $(\sigma,N)=(1,2)$ and $(2,1)$) indicate that when $\sigma=2/N$, the ground state plays an important role in the structure of solutions developing singularities. This phenomenon is discussed in detail in the articles [20], [26].

Related analytical results were obtained in [29] where it was proved using a particular variational characterization of the ground state (exploited further in §2) that when $\sigma=2/N$, a sharp sufficient condition for global existence in (1.1)–(1.2) is

$$\int|\phi_0(x)|^2\,dx<\int R^2(x)\,dx. \tag{1.12}$$

The linearized stability results of §2 for the case $\sigma=2/N$ also give information on the special role of the ground state. In this case the techniques of [6] do not apply.

Notation. All integrals are understood to be taken over $\mathbb{R}^N$, the N-dimensional Euclidean space, unless otherwise indicated.

1) $W=(u,v)^t=\binom{u}{v}\in\mathbb{R}^2$,

2) $\|f\|_p=\left(\int|f(x)|^p\,dx\right)^{1/p}$, $\|W\|_p^p=\|u\|_p^p+\|v\|_p^p$,

$L^p(\mathbb{R}^N)=L^p=\{f\colon \|f\|_p<\infty\}$,

3) $\|f\|_{H^s}^2=\int|\hat{f}(\xi)|^2(1+|\xi|^2)^s\,d\xi$,

$H^s(\mathbb{R}^N)=H^s=\{f\colon \|f\|_{H^s}<\infty\}$,

4) $N(A)=$ null space of an operator A,

5) for functions $f,g\in\mathbb{R}^N$, $(f,g)=\sum_{i=1}^{N}\int f_ig_i\,dx$.

2. The linearized NLS operator. We first consider the stability of the gound state $\psi^0(x,t)=R(x)e^{it/2}$ of (1.1) by seeking a solution to the perturbed equation (1.8) of the form (1.9). Small perturbations of initial data may be incorporated in the equation through a redefinition of ϕ^ε (see Example 3.1, (3.7)). Substitution of (1.9) into (1.8) and linearization yields the following IVP for the linearized perturbation w:

$$2iw_t+\Delta w-w+(\sigma+1)R^{2\sigma}w+\sigma R^{2\sigma}\bar{w}=F(R)R,\qquad w(x,0)=0. \tag{2.1}$$

We will also study the homogeneous IVP, (2.1) with $F\equiv 0$ and $w(x,0)=w_0(x)$, which we denote by (2.1H). IVP (2.1H) has a useful conserved quantity given in the following:

THEOREM 2.1 *Let $w(x,t)$ be an H^1-solution of* (2.1H). *Then*

$$\int\left(|\nabla w|^2+|w|^2-(\sigma+1)R^{2\sigma}|w|^2-\frac{\sigma}{2}R^{2\sigma}(w^2+\bar{w}^2)\right)dx \tag{2.2}$$

is independent of time.

Proof. We multiply (2.1) by $\bar{w}_t$, take the real part and then integrate by parts. This gives (d/dt) (2.2) $=0$. □

It is expedient to work with the real and imaginary parts of w. We set $w \equiv u + iv$ and make the following definitions:

$$(2.3)\quad \text{(a)}\quad L_+ \equiv -\Delta + 1 - (2\sigma+1)R^{2\sigma}, \qquad \text{(c)}\quad L = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix},$$

$$\text{(b)}\quad L_- = -\Delta + 1 - R^{2\sigma}, \qquad \text{(d)}\quad W = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Now (2.1) can be written as the real system:

$$(2.4)\qquad 2W_t = LW + G, \qquad W|_{t=0} = 0,$$

where G is a 2-vector with components $\operatorname{im} F\cdot R$ and $-\operatorname{re} F\cdot R$. We denote the homogeneous IVP ($G\equiv 0$ and $W|_{t=0} = W_0$) by (2.4H). Let Ω_t denote the propagator or solution operator for (2.4H). Thus $W(t) \equiv \Omega_t W_0$. Theorem 2.1 can now be expressed as:

COROLLARY 2.2. *Let $W \in H^1 \times H^1$ be a solution of* (2.4H). *Then, for $t > 0$,*

$$(2.5)\qquad Q(W) \equiv Q(u,v) \equiv (L_+u,u) + (L_-v,v) = Q(u_0,v_0).$$

We would like to use $Q^{1/2}$ as a norm, measuring the size of the perturbation W. However, Q is not positive definite on $H^1 \times H^1$. We now introduce a subspace on which we will see that $Q^{1/2}$ is equivalent to the $H^1 \times H^1$ norm.

DEFINITION 2.3. For $\sigma \leqq 2/N$ we set

$$(2.6)\qquad M \equiv H^1 \times H^1 \cap [N_g(L^*)]^{\perp},$$

where $B^{\perp} = \{a = (a_1, a_2) \mid \int a_1 b_1 + a_2 b_2\, dx = 0 \text{ for all } b = (b_1, b_2) \in B\}$ and $N_g(A) =$ generalized null space of $A = \bigcup_{j=1}^{\infty} N(A^j)$.

In appendix B we derive and display explicitly the elements of $N_g(L)$ and $N_g(L^*)$ (Theorems B.2–3). This explicit information and Definition 2.3 imply the following orthogonality relations which we require in the coming analysis:

PROPOSITION 2.4. *Let $(f,g)^t \in M$. Then* (i) *For $\sigma < 2/N$, the following $2N+2$ orthogonality relations hold*:

$$(2.7)\quad \text{(a)}\quad (f,R) = 0, \qquad \text{(c)}\quad (g, R_{x_j}) = 0,$$

$$\text{(b)}\quad (f, x_j R) = 0 \qquad \text{(d)}\quad \left(g, \frac{1}{\sigma}R + x\cdot\nabla R\right) = 0, \qquad 1 \leqq j \leqq N.$$

(ii) *For $\sigma = 2/N$, we have the $2N+4$ orthogonality conditions*: (2.7) *and the two new relations*

$$(2.8)\quad \text{(a)}\quad \left(f, |x|^2 R\right) = 0,$$

$$\text{(b)}\quad (g,\rho) = 0, \quad \textit{where } L_+\rho = -|x|^2 R.$$

The following result shows that under suitable restrictions the linearized energy controls a classical norm.

THEOREM 2.5. *Let $\sigma \leqq 2/N$ and $(f,g)^t \in M$. There exist positive constants K and K', independent of f and g, such that*

$$(2.9)\qquad K\left(\|f\|_{H^1}^2 + \|g\|_{H^1}^2\right) \leqq Q(f,g) \leqq K'\left(\|f\|_{H^1}^2 + \|g\|_{H^1}^2\right).$$

Thus for $\sigma \leqq 2/N$, M is a closed linear subspace on which the functional $Q^{1/2}$ defines a norm equivalent to the $H^1 \times H^1$ norm.

The upper bound in (2.9) is simple and holds for any f and g in H^1. The difficult part is the lower estimate. We base the proof on several propositions. The first is a particular characterization of the ground state R, introduced in [29, Thm. B]. For $u \in H^1$ we define the functional

$$J^{\sigma,N}(u) = \frac{\|\nabla u\|_2^{\sigma N} \|u\|_2^{2+\sigma(2-N)}}{\|u\|_{2\sigma+2}^{2\sigma+2}}, \qquad 0 < \sigma < \frac{2}{N-2}. \tag{2.10}$$

PROPOSITION 2.6. *For* $0 < \sigma < 2/(N-2)$

$$\alpha \equiv \inf_{u \in H^1(\mathbb{R}^N)} J^{\sigma,N}(u) \tag{2.11}$$

is attained at a function R with the following properties:

(1) $R > 0$ *and* $R = R(|x|)$,

(2) $R \in H^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$,

(3) R *is a solution of* (1.4).

In [29] the functional $J^{\sigma,N}$ was minimized to obtain the optimal constant of a classical interpolation estimate of Nirenberg [23] and Gagliardo [11], [12]. There, the optimal constant, which is expressible in terms of α, is used in an a priori estimate derived from the conserved quantities of (1.1) to obtain the condition (1.12) for global existence when $\sigma = 2/N$.

We now use $J^{\sigma,N}$ to prove the following result which is at the heart of Theorem 2.5:

PROPOSITION 2.7. *Let* $\sigma \leqq 2/N$. *Then*

$$\inf_{(f,R)=0} (L_+ f, f) = 0. \tag{2.12}$$

Proof I.[3] First, note that $L_+ \nabla R = 0$ and $(\nabla R, R) = 0$. Therefore the infimum in (2.12) is nonpositive. Since $J^{\sigma,N}$ attains its minimum at R,

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} J^{\sigma,N}(R + \varepsilon\eta) \geqq 0 \tag{2.13}$$

for all $\eta \in C_0^\infty(\mathbb{R}^N)$. Now (2.13) can be written as

$$(T\eta, \eta) \geqq 0, \quad \text{where} \tag{2.14}$$

$$\begin{aligned}(2+\sigma(2-N))^{-1} Tz \equiv L_+ z + (2-N) a_{\sigma N}(R,z) R \\ - b_{\sigma N}(R,z)\Delta R + (\sigma N - 2) c_{\sigma N}(\Delta R, z)\Delta R.\end{aligned} \tag{2.15}$$

Here, $a_{\sigma N}$ $b_{\sigma N}$ and $c_{\sigma N}$ are constants, dependent on σ and N, that are positive for $0 < \sigma < 2/(N-2)$. Thus, (2.14) and (2.15) imply

$$(L_+ f, f) \geqq (2 - \sigma N) c_{\sigma N} (\Delta R, f)^2 \tag{2.16}$$

for any f with $(f, R) = 0$. Since the right-hand side of (2.16) is nonnegative for $\sigma \leqq 2/N$ the result follows. □

[3] A second, more general, proof is given in Appendix E (see also [30]).

In what follows we will need the next result, part (b) of which has been proved completely only in dimension $N=1$ for all $\sigma>0$ and in dimension $N=3$ for $0<\sigma\leqq 1$ (see appendix A for the proofs). This suffices for our modulational stability analysis of the ground state for $\sigma\leqq 2/N$ to be complete in dimensions $N=1$ and 3.

PROPOSITION 2.8.

(a) *L_- is a nonnegative self-adjoint operator in L^2 with null space $N(L_-)=\operatorname{span}\{R\}$.*

(b) *L_+ is a self-adjoint operator in L^2 with null space $N(L_+)=\operatorname{span}\{R_{x_i}; 1\leqq i\leqq N\}$.*

The next two results show that by imposing the additional constraints defining M, $Q^{1/2}$ controls the H^1 norm.

PROPOSITION 2.9. *Let $\sigma\leqq 2/N$. There exists a positive constant $C_{\sigma N}^{+}$ such that for any f satisfying orthogonality conditions* (2.7a, b) (*and* (2.8a) *when $\sigma=2/N$*)

$$(L_+f,f)\geqq C_{\sigma N}^{+}(f,f). \tag{2.17}$$

PROPOSITION 2.10. *Let $\sigma\leqq 2/N$. There exists a positive constant $C_{\sigma N}^{-}$ such that for any g satisfying orthogonality relations* (2.7c, d) (*and* (2.8b) *when $\sigma=2/N$*)

$$(L_-g,g)\geqq C_{\sigma N}^{-}(g,g). \tag{2.18}$$

Proof of Proposition 2.9. We first consider the case $\sigma<2/N$. Let $\tau\equiv\inf_{\|f\|_2=1}(L_+f,f)$, where f is constrained by (2.7a, b). We will prove $\tau>0$ by showing that the assumption $\tau=0$ leads to a contradiction. This will suffice by Proposition 2.7. We first show that $\tau=0$ implies the minimum is attained in the admissible class. We then can consider an associated Lagrange multiplier problem to deduce $\tau>0$.

Let $\{f_\nu\}$ be a minimizing sequence i.e. $\|f_\nu\|_2=1$, $(L_+f_\nu,f_\nu)\downarrow 0$ and f_ν satisfies (2.7a, b). Then for any $\eta>0$ we can choose f_ν so that

$$0<\int(\nabla f_\nu)^2\,dx+\int f_\nu^2\,dx\leqq(2\sigma+1)\int R^{2\sigma}f_\nu^2\,dx+\eta. \tag{2.19}$$

Since $\|f_\nu\|_2$ is finite, (2.19) implies $\|f_\nu\|_{H^1}$ are uniformly bounded. Thus a subsequence f_ν exists that converges weakly to some H^1 function f_*. By weak convergence f_* satisfies (2.7a, b). We also have $\int R^{2\sigma}f_\nu^2\,dx\to\int R^{2\sigma}f_*^2\,dx$ by Hölder's inequality, interpolation, and the uniform decay of R. Thus $f_*\not\equiv 0$, by (2.19) since η is arbitrary.

We now show that the minimum is attained at f_* and $\|f_*\|_2=1$. By Fatou's lemma $\|f_*\|_2\leqq 1$. Suppose $\|f_*\|_2<1$. Then, define $g_*=f_*/\|f_*\|_2$ which is admissible. Let $\zeta\in L^2$, $\|\zeta\|_2=1$. By weak convergence of f_ν to f_*, $(\zeta,\nabla f_*)=\liminf_{\nu\to\infty}(\zeta,\nabla f_\nu)\leqq\liminf_{\nu\to\infty}\|\nabla f_\nu\|_2$. Maximizing over all such ζ, we obtain

$$\|\nabla f_*\|_2\leqq\liminf_{\nu\to\infty}\|\nabla f_\nu\|_2.$$

Since $(R^{2\sigma}f_\nu,f_\nu)\to(R^{2\sigma}f_*,f_*)$, we have

$$(L_+f_*,f_*)\leqq\liminf_{\nu\to\infty}(L_+f_\nu,f_\nu)=0.$$

Hence, $(L_+g_*,g_*)\leqq 0$. By Proposition 2.7, $(L_+g_*,g_*)=0$. Thus we can take $\|f_*\|_2=1$ and the minimum to be attained there.

Since the minimum is attained at an admissible function $f_* \not\equiv 0$, there exists $(f_*, \lambda, \beta, \gamma)$ among the critical points of the Lagrange multiplier problem

$$(2.20)\qquad \begin{aligned} &\text{(a)}\quad (L_+ - \lambda)f = \beta R + \gamma\cdot xR, \qquad \beta\in\mathbb{R},\quad \gamma\in\mathbb{R}^N,\\ &\text{(b)}\quad \|f\|_2 = 1,\\ &\text{(c)}\quad f \text{ satisfies (2.7a, b)}.\end{aligned}$$

By (2.20) $\lambda = (L_+f, f)$, so $\lambda=\tau=0$ is a critical value. Therefore, we need to conclude that

$$(2.21)\qquad L_+f_* = \beta R + \gamma\cdot xR$$

has no nontrivial solutions (f_*, β, γ) satisfying the side constraints. Taking the inner product of (2.21) with ∇R, integrating by parts, and using that $R_{x_i}\in N(L_+)$, we find $\gamma\equiv 0$. Therefore $f_* = -1/2\beta((1/\sigma)R + x\cdot\nabla R) + \theta\cdot\nabla R$, $\theta\in\mathbb{R}^N$. Here we use that $L_+((1/\sigma)R + x\cdot\nabla R) = -2R$ (by inspection) and part (b) of Proposition 2.8. Now $\theta = 0$ by (2.7b). Also, since $(f_*, R) = -1/2\beta(1/\sigma - N/2)\,\|R\|_2^2$, (2.7a) is violated when $\sigma < 2/N$. Thus $f_*\equiv 0$, a contradiction. We now conclude that $\tau>0$. This settles the case $\sigma<2/N$.

In the case $\sigma = 2/N$, we show that $\tau' \equiv \inf_{\|f_2\|=1}(L_+f, f)$ where f is constrained by (2.7a, b) and (2.8a) is strictly positive. The proof of part (i) adapts and we conclude that if $\tau'=0$, then the minimum is attained at an admissible function f_*. We are thus led to the Lagrange multiplier problem

$$(2.22)\qquad \begin{aligned} &\text{(a)}\quad L_+f_* = \beta R + \gamma\cdot xR + \delta|x|^2R,\\ &\text{(b)}\quad \|f_*\|_2 = 1,\\ &\text{(c)}\quad f_* \text{ satisfies (2.17) and (2.19)}.\end{aligned}$$

We now argue as before: (2.22a) implies that $f_* = -1/2\beta((N/2)R + x\cdot\nabla R) - \delta\rho + \theta\cdot\nabla R$ since $L_+((N/2)R + x\cdot\nabla R) = -2R$, $L_+\nabla R = 0$ and $L_+\rho = -|x|^2R$ (see (B.15)). Since $((N/2)R + x\cdot\nabla R, |x|^2R) = -\int|x|^2R^2\,dx \neq 0$, $(\rho, R)\neq 0$ (see (B.16)), and $(\nabla R, xR)\neq 0$, the constraints (2.7a, b) and (2.8a) imply $f_*\equiv 0$. Hence $\tau'>0$. This completes the proof of Proposition 2.9. □

Proof of Proposition 2.10. We first consider the case $\sigma<2/N$. Let $\mu = \inf_{\|g\|_2} = 1(L_-g, g)$, where g is constrained by (2.7a, b). By a proof similar to that in part (i) of Proposition 2.9, if $\mu = 0$, then the minimum is attained at an admissible function $g_*\not\equiv 0$. Since L_- is nonnegative (Proposition 2.8a), $g_* = R/\|R\|_2$. But g_* does not satisfy (2.7d) since $(R, (1/\sigma)R + x\cdot\nabla R) = (1/\sigma - N/2)\|R\|_2^2$, which does not vanish for $\sigma<2/N$. Thus $g_*\equiv 0$, a contradiction. We conclude $\mu>0$ in the case $\sigma<2/N$.

For $\sigma = 2/N$, let $\mu' = \inf_{\|g\|_2=1}(L_-g, g)$, where g is constrained by (2.7c, d) and (2.8b). Although $g_* = R/\|R\|_2$ now satisfies (2.7d), by (B.17) it violates (2.8b) since $(R, \rho) = -\int|x|^2R^2\,dx\neq 0$. Hence, $\mu'>0$. □

Proof of Proposition 2.5. For $\sigma\leqq 2/N$, $(f, g)^t\in M$ implies by Proposition 2.4 and Propositions 2.9–2.10

$$Q(f, g)\equiv (L_+f, f) + (L_-g, g) \geqq C^+(f, f) + C^-(g, g) \geqq K_1\left(\|f\|_2^2 + \|g\|_2^2\right).$$

The lower estimate of (2.9) now follows easily. □

That the space M is "natural" for the linearized evolution (2.4H) is clear from

PROPOSITION 2.11. *Let $\sigma\leqq 2/N$. Then, Ω_t maps M into itself.*

The proof of this proposition is a simple but lengthy computation which we give in appendix C. The idea is to compute the evolution of the $2N+2$ modes defining M, for $\sigma<2/N$ ($2N+4$ modes, for $\sigma=2/N$). These satisfy a simple linear system of ODE's in time ((C.3) or (C.7)). Thus if we assume that W_0 has a vanishing component in M, then so will $W(t)$.

We conclude this section with the following result on the "H^1 control" of Ω_t on M. It is the main analytical tool of §3.

THEOREM 2.12. *Let $\sigma\leqq 2/N$, and consider* (2.4) *with $W_0\in M$ and $G\in M$. Then, $W(t)\in M$ for $t>0$ and*

$$K\|W(t)\|_{H^1}^2\leqq Q(W(t))\leqq K'\|W(t)\|_{H^1}^2. \tag{2.23}$$

In addition under the above hypothesis, for (2.4H) *we have*

$$K\|W(t)\|_{H^1}^2\leqq Q(W_0)\leqq K'\|W_0\|_{H^1}^2. \tag{2.24}$$

Proof. Since Ω_t: $M\to M$ (Proposition 2.11) estimate (2.9) holds with $(f,g)^t=W(t)$ for all t (Theorem 2.5). This is precisely (2.23). For (2.4H) $Q(W(t))$ is conserved (Corollary 2.2). Thus (2.24) follows. □

3. Modulational stability. In this section we will prove Theorem 1. Our first aim is to derive the modulation equations referred to in the statement of the theorem. Let $\tau\equiv\int_0^t\lambda^2(\varepsilon s)\,ds$, $\Theta\equiv\lambda(\theta-\theta_0)$. We now let the $2N+2$ parameters vary slowly, i.e. $a^{-1}\equiv\lambda=\lambda(T)$, $\xi_i=\xi_i(T)$, $\theta_{0,i}=\theta_{0,i}(T)$, and $\eta_0=\eta_0(T)$, $1\leqq i\leqq N$ where $T\equiv\varepsilon t$. We will determine the slow time evolution of the functions λ,ξ, θ_0, and η_0 that will ensure (1.11). Substitution of (1.10) into (1.8), use of (1.6), and the balance of terms of order ε yields the IVP

$$W_\tau=LW+G,\quad W_0=0,\quad W=(u,v)^t. \tag{3.1}$$

We display the source term, $G=G(\varepsilon t)=(f^\varepsilon(\varepsilon t),g^\varepsilon(\varepsilon t))^t$ explicitly:

$$\begin{aligned}&\text{(a)}\quad f^\varepsilon=A^\varepsilon+\operatorname{im}F(R)R,\\ &\text{(b)}\quad g^\varepsilon=B^\varepsilon-\operatorname{re}F(R)R;\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\text{(a)}\quad A^\varepsilon=-2\lambda^{1/\sigma-3}\dot\lambda\left(\frac{1}{\sigma}R+\Theta\cdot\nabla R\right)+2\lambda^{1/\sigma-1}\dot\theta_0\cdot\nabla R,\\ &\text{(b)}\quad B^\varepsilon=-2\lambda^{1/\sigma-3}\dot\xi\cdot\Theta R+2(\xi\cdot\dot\theta_0+\dot\eta_0)\lambda^{1/\sigma-2}R.\end{aligned} \tag{3.3}$$

Our aim is to control the large τ (or t) behavior of W in a suitable norm. In §2 the natural norm with which to study (3.1) was found to be the "linearized energy" Q (see 2.5). By Theorem 2.12, if $G^\varepsilon\in M$ for $\tau>0$, then

$$K\|W(\tau)\|_{H^1}^2\leqq Q(W(\tau)). \tag{3.4}$$

By Proposition 2.4, $G^\varepsilon\in M$ for $\tau>0$ implies that f^ε and g^ε satisfy the $2N+2$-orthogonality conditions (2.7). These relations are called the *modulation equations*. After some simplification we find that the conditions (2.7) reduce to the following system of

$2N+2$ ordinary differential equations in time for the $2N+2$ parameters λ, ξ, θ_0, and η_0:

$$\text{(3.5)}\qquad \begin{aligned} &\text{(a)}\quad 2\left(\frac{1}{\sigma}-\frac{N}{2}\right)\|R\|_2^2\dot{\lambda}=\lambda^{-1/\sigma+3}(f^\varepsilon, R),\\ &\text{(b)}\quad \|R\|_2^2\dot{\theta}_{0,j}=\lambda^{-1/\sigma+1}(f^\varepsilon, \Theta_j R), \qquad 1\leqq j\leqq N,\\ &\text{(c)}\quad \|R\|_2^2\dot{\xi}_j=\lambda^{-1/\sigma+3}(g^\varepsilon, R_{\Theta_j}), \qquad 1\leqq j \leqq N,\\ &\text{(d)}\quad 2\left(\frac{1}{\sigma}-\frac{N}{2}\right)\|R\|_2^2(\xi\cdot\dot{\theta}_0+\dot{\eta}_0)=\lambda^{-1/\sigma+2}\left(g^\varepsilon, \frac{1}{\sigma}R+\Theta\cdot\nabla R\right).\end{aligned}$$

Note that in the critical case $\sigma=2/N$, (3.5) becomes singular. This is a manifestation of the need to incorporate the two additional generalized eigenmodes (recall there are $2N+4$ when $\sigma=2/N$) with which one can constrain the evolution to M. Since we have not found a $(2N+4)$-parameter family of solutions (we only have the $(2N+3)$-parameter family (1.7)) we have not carried this out for $\sigma=2/N$.

We now restate Theorem 1 more precisely and give the proof.

THEOREM 1′. *Consider the* IVP (1.8), *where* ϕ^ε *the solution is expanded as in* (1.10). *Suppose the* $2N+2$ *"slow" functions* $\lambda(T)$, $\xi(T)$, $\theta_0(T)$, *and* $\eta_0(T)$ *solve* (3.5) *and are such that* $|f^\varepsilon(\varepsilon t)|$ *and* $|g^\varepsilon(\varepsilon t)|$ *of* (3.2) *are uniformly bounded* (*independently of* ε) *on any t-interval* $0\leqq t\leqq T_0/\varepsilon$. *Then*, (1.11) *holds*.

Proof of Theorem 1′. By (3.1), we have

$$W^\varepsilon(\tau)=\Omega_\tau\int_0^\tau \Omega_{-s}G(\varepsilon s)\,ds,$$

where $\Omega_\tau=\exp[\frac{1}{2}L\tau]$ is a unitary group acting in the space M with norm $Q^{1/2}$ (Theorem 2.5). The modulation equations (3.5) imply $G(\varepsilon\tau)\in M$ for all τ corresponding to the t-interval $[0, T_0/\varepsilon]$. Thus, by Theorem 2.12, $W^\varepsilon(\tau)\in M$ and

$$K\|\varepsilon W^\varepsilon(\tau)\|_{H^1}^2\leqq Q\left[\varepsilon\int_0^\tau \Omega_{-s}G(\varepsilon s)\,ds\right]. \tag{3.6}$$

We now want to study the behavior of the right-hand side of (3.6) for fixed time intervals of order $1/\varepsilon$ as $\varepsilon\to 0$. Heuristically, since G is "almost constant" the mean ergodic theorem [9] should imply that $\varepsilon\int_0^\tau \Omega_{-s}G(\varepsilon s)\,ds$ tends to the projection of G onto the null space of L in the space M. But the null space of L in M is empty since the only possible members, $(0,R)^t$ and $(0,\nabla R)^t$, do not lie in M. Thus we should have $Q(\varepsilon\int_0^\tau \Omega_{-s}G(\varepsilon s)ds)\downarrow 0$ for $0\leqq\tau\leqq 0(1/\varepsilon)$ as $\varepsilon\to 0$. This argument is made rigorous by invoking Lemma D.1. The assertion (1.11) now follows from this and (3.6). □

We now study (1.8) for two specific perturbations. We apply Theorem 1′, the hypotheses of which are easily verified.

Example 3.1. Initial data near the ground state.

Consider, for $0<\sigma<2/N$, (1.1) with

$$\phi^\varepsilon(x,0)=\psi(x,0;\lambda^{in},0,x_0,\eta_0^{in})+\varepsilon S(x), \qquad S\in H^1.$$

Expand $\phi^\varepsilon(x,t)$ as in (1.10) with w_1, the linearized perturbation, defined by

$$w_1=(u+\operatorname{re} S)+i(v+\operatorname{im} S). \tag{3.7}$$

Then, $W=(u,v)^t$ satisfies (3.1) with

$$(3.8)\quad \begin{aligned} &(\mathrm{a})\quad f^\varepsilon=A^\varepsilon+L_-\operatorname{im}S,\\ &(\mathrm{b})\quad g^\varepsilon=B^\varepsilon-L_+\operatorname{re}S \end{aligned}$$

and A^ε, B^ε defined in (3.3). The modulation equations (3.5) reduce to

$$(3.9)\quad \begin{aligned} &(\mathrm{a})\quad \dot\lambda=0\Rightarrow\lambda(\varepsilon t)\equiv\lambda^{in},\\ &(\mathrm{b})\quad \dot\xi_i=0\Rightarrow\xi_i\equiv 0; \end{aligned}$$

$$(3.10)\quad (\mathrm{a})\quad \|R\|_2^2\dot\theta_{0,i}=-2\int \operatorname{im}S(x)R_{\Theta_i}\big(\lambda^{in}(x-\theta_0)\big)\,dx,\qquad 1\leqq i\leqq N,$$

$$(\mathrm{b})\quad (1/\sigma-N/2)\|R\|_2^2\dot\eta_0=\int \operatorname{re}S(x)R\big(\lambda^{in}(x-\theta_0)\big)\,dx.$$

Therefore, to leading order, perturbations of ground state initial values induce modulations of the parameters θ_0 and η_0 alone. This is consistent with the nonlinear stability results in [6].

Example 3.2. Simple dissipation.
Consider the IVP (1.8) with

$$(3.11)\quad \begin{aligned} &(\mathrm{a})\quad F^\varepsilon(z)=-i\kappa,\ \kappa>0\\ &(\mathrm{b})\quad \phi^\varepsilon(x,0)=\psi\big(x,0;\lambda^{in},0,x_0,\eta_0^{in}\big). \end{aligned}$$

This has been considered in one spatial dimension in [17], [19]. Following the procedure outlined earlier, we obtained the following modulation equations:

$$(3.12)\quad \begin{array}{lll} \dot\lambda=-\kappa\dfrac{\sigma}{2-\sigma N}\lambda, & \lambda(0)=\lambda^{in}, & \\ \dot\theta_{0,i}=0, & \theta_{0,i}(0)=x_{0,i}, & 1\leqq i\leqq N,\\ \dot\xi_i=0, & \xi_i(0)=0, & 1\leqq i\leqq N,\\ \dot\eta_0=-\xi\cdot\dot\theta_0, & \eta_0(0)=\eta_0^{in}. & \end{array}$$

These equations are easily solved:

$$(3.13)\quad \begin{array}{ll} \lambda(\varepsilon t)=\lambda^{in}\exp\left[-\kappa\dfrac{\sigma}{2-\sigma N}\varepsilon t\right], & \xi_i(\varepsilon t)\equiv 0,\\ \theta_{0,i}(\varepsilon t)\equiv x_{0,i}, & \eta_0(\varepsilon t)\equiv\eta_0^{in}. \end{array}$$

The Ansatz (1.10) and (3.13) imply that the effect of small simple dissipation is a slow exponential decrease in the amplitude of the ground state profile, accompanied by a simultaneous broadening of the profile due to the changing spatial scale, $\lambda(\varepsilon t)(x-x_0)$.

Appendix A. Partial proof of Proposition 2.8.

Proof of Proposition 2.8a. Since $L_-R=0$ and $R\in L^2$, we have $0\in\sigma(L_-)$. Since $R>0$, it is the ground state. Thus L_- is nonnegative. Finally, $N(L_-)=\operatorname{span}\{R\}$ since the ground state is nondegenerate.

Proof of Proposition 2.8b *for* $N=1$. Since $R''-R+R^{2\sigma+1}=0$ $\big(R(x)=(\sigma+1)^{1/2\sigma}\operatorname{sech}^{1/\sigma}(\sigma x)>0\big)$, $R'\in N(L_+)$. Define $w=R'/R''(0)$. Then $w(0)=0$ and $w'(0)=1$. Let v be the solution of $L_+v=0$ with $v(0)=1$ and $v'(0)=0$. We then have the

Wronskian relation $w(x)v'(x)-w'(x)v(x)=-1$. Therefore, $R'(x)v'(x)-R''(x)v(x)=-R''(0)=\sigma(\sigma+1)^{1/2\sigma}$. We then have that $(v/R')'=\sigma(\sigma+1)^{1/2\sigma}/(R')^2$. To prove that $v\notin N(L_+)$ and therefore that $N(L_+)=\operatorname{span}\{R'\}$ it will suffice to show that $|v|\to\infty$ as $x\to\infty$. For $\varepsilon>0$ and $x>0$,

$$v(x)=\frac{v(\varepsilon)}{R'(\varepsilon)}R'(x)+\int_\varepsilon^x \frac{ds}{[R'(s)]^2}\sigma(\sigma+1)^{1/2\sigma}R'(x).$$

Since $R'(x)=O(e^{-x})$ as $x\to\infty$, i.e., $|R'(x)|\leqq Ce^{-x}$, and $R'<0$ for $x>0$,

$$v(x)\leqq v(\varepsilon)\frac{R'(x)}{R'(\varepsilon)}+\int_\varepsilon^x \frac{ds}{C^2e^{-2s}}R'(x)\to-\infty \quad \text{as } x\to\infty.$$

Proof of Proposition 2.8b *for* $N=3$, $0<\sigma\leqq1$ *and partial proof in all other* ($N>1$) *cases*. Since the equation $\Delta R-R+R^{2\sigma+1}=0$ is translation invariant in space, $\nabla R=R'(|x|)x/|x|\in N(L^+)$. Now L_+ is an operator with a potential dependent only on $r=|x|$. Hence an L^2 solution of $L_+v=0$ can be expanded in a series with functions of the form $f(r)Y(\theta)$, where $f\in L^2(0,\infty;r^{N-1}dr)$ and $Y\in L^2(S^{N-1})$. These functions must satisfy

$$A_kf\equiv\left(-\frac{d^2}{dr^2}-\frac{N-1}{r}\frac{d}{dr}+1-(2\sigma+1)R^{2\sigma}+\frac{\lambda_k}{r^2}\right)f=0, \tag{A.1}$$

$$-\Delta_{S^{N-1}}Y=\lambda_kY, \tag{A.2}$$

where $\lambda_k=k(N-2+k)$.

The null functions ∇R correspond to $k=1$. Therefore to prove the theorem it will suffice to show that there is no solution of (A.1) satisfying the correct boundary conditions at $r=0$ and $r=\infty$ for $k\in\{0,2,3,4,\cdots\}$.

We handle the case $k\geqq2$ as follows. The function R' is an eigenfunction of A_1 corresponding to the eigenvalue zero. Since R' has no interior zeros it is the ground state of A_1. Hence A_1 is a nonnegative operator. Setting $k\equiv1+\delta$, $\delta>0$, we find $A_k=A_1+\delta(N+\delta)r^{-2}$ which is a positive operator. Hence $f\in L^2$ and $A_kf=0$ for $k\geqq2$ implies $f\equiv0$.

The case $k=0$ has not yet been completely resolved. Coffman, in proving the uniqueness of the positive decaying solution of $R''+(2/r)R'-R+R^3=0$ ($\sigma=1,N=3$), first shows that $|(\partial u/\partial r)(r,\alpha_0)|\to\infty$ as $r\to\infty$, where $u(r,\alpha)$ is the solution of the IVP

$$u''+\frac{2}{r}u'-u+u^3=0, \qquad u(0,\alpha)=\alpha$$

and α_0 is the initial value generating the ground state $u(r,\alpha_0)=R(r)$ [8, Lemma 4.2]. This lemma holds as well, with minor changes in the proof, for the range $0<\sigma\leqq1$, $N=3$.[4] This eliminates the $k=0$ mode in the case $0<\sigma\leqq1$, which includes the subcritical and critical cases in dimension three. (More general results on the uniqueness of the ground state of (1.4) were obtained by McLeod and Serrin [21].) For general σ and N, since the problem is reduced to consideration of the ODE (A.1) with $\lambda_k=0$, we can say that $\dim N(L_+)$ is either N or $N+1$.

[4] C. V. Coffman, personal communication.

Appendix B. Generalized null spaces. In this appendix we display explicitly the generalized null spaces of L and L^*, denoted by $N_g(L)$ and $N_g(L^*)$. The results of this section depend on Proposition 2.8b which has been completely proved in dimension one for all σ, and in dimension three for $0<\sigma\leqq 1$.

We begin with the following useful observation which follows by direct verification, using (1.4).

PROPOSITION B.1.

$$\text{(B.1)}\quad \begin{aligned} &\text{(a)}\quad L_-R=0, &&\text{(c)}\quad L_+\nabla R=0,\\ &\text{(b)}\quad L_-xR=-2\nabla R, &&\text{(d)}\quad L_+\left(\frac{1}{\sigma}R+x\cdot\nabla R\right)=-2R.\end{aligned}$$

We are interested in the null spaces of L, L^2, $L^3,\cdots$. For example, the equation $LW=0$, where $W=(u,v)^t$ reduces to $L_-v=0$ and $L_+u=0$. By Proposition 2.9, $(0,R)^t$ and $(\nabla R,0)^t$ span $N(L)$. The equation $L^2W=0$ implies $L_-L_+u=0$ and $L_+L_-v=0$. By Proposition B.1 we find that $N(L^2)-N(L)$ is spanned by $((1/\sigma)R+x\cdot\nabla R,0)^t$ and $(0,xR)^t$.

Continuing, we consider $L^3W=0$ which implies

$$\text{(B.2)}\quad \begin{aligned}&\text{(a)}\quad L_-L_+L_-v=0,\\ &\text{(b)}\quad L_+L_-L_+u=0.\end{aligned}$$

We seek to construct functions in $N(L^3)-N(L^2)$. Equation (B.2a) implies $L_+L_-v=cR$, c constant. By Proposition B.1,

$$\text{(B.3)}\qquad L_-v=-\frac{c}{2}\left(\frac{1}{\sigma}R+x\cdot\nabla R\right).$$

Now (B.3) will have an H^1 solution if the following solvability condition holds:

$$\text{(B.4)}\qquad 0=\left(R,\frac{1}{\sigma}R+x\cdot\nabla R\right)=\left(\frac{1}{\sigma}-\frac{N}{2}\right)\|R\|_2^2.$$

Thus for $\sigma\neq 2/N$, (B.3) has no solutions giving rise to an element of $N(L^3)-N(L^2)$.

Consider now (B.2b). This implies

$$\text{(B.5)}\qquad L_-L_+u=c\cdot\nabla R,\qquad c\in\mathbb{R}^N.$$

By (B.1b)

$$\text{(B.6)}\qquad L_+u=-\frac{c}{2}\cdot xR.$$

Since $(\nabla R,xR)\neq 0$, (B.6) has no solution generating an element of $N(L^3)-N(L^2)$.

It follows that for $\sigma\neq 2/N$, $N(L^3)=N(L^2)$. Similarly, for $\sigma\neq 2/N$ $N(L^k)=N(L^2)$, $k\geqq 3$. The same procedure can be followed to deduce $N_g(L^*)$. We summarize these observations.

THEOREM B.2. *Let $\sigma \neq 2/N$. $N_g(L)=\cup_{j=1}^2 N(L^j)$ and $N_g(L^*)=\cup_{j=1}^2[N(L^*)^j]$ are spanned by the following two $2N+2$-dimensional biorthogonal sets*:

(B.7)
$$\text{(a)}\quad a_1=\alpha_1^{-1}=(0,-R)^t,$$
$$\text{(b)}\quad a_{2,j}=\alpha_2^{-1}(-R_{x_j},0)^t, \qquad 1\leqq j\leqq N,$$
$$\text{(c)}\quad a_{3,j}=\alpha_2^{-1}(0,x_jR)^t, \qquad 1\leqq j\leqq N,$$
$$\text{(d)}\quad a_4=\alpha_1^{-1}\left(\frac{1}{\sigma}R+x\cdot\nabla R,0\right)^t, \quad \textit{where}$$
$$\alpha_1=\left(\frac{N}{2}-\frac{1}{\sigma}\right)\|R\|_2^2, \qquad \alpha_2=\frac{1}{2}\|R\|_2^2.$$

(B.8)
$$\text{(a)}\quad b_1=\left(0,\frac{1}{\sigma}R+x\cdot\nabla R\right)^t,$$
$$\text{(b)}\quad b_{2,j}=(x_jR,0)^t, \qquad 1\leqq j\leqq N,$$
$$\text{(c)}\quad b_{3,j}=(0,-R_{x_j})^t, \qquad 1\leqq j\leqq N,$$
$$\text{(d)}\quad b_4=(-R,0)^t,$$

where

(B.9)
$$(a_i,b_k)=\delta_{ik} \quad \textit{and} \quad (a_{i,m},b_{k,l})=\delta_{im}\delta_{mk}\delta_{kl}.$$

In the critical case, $\sigma=2/N$, (B.4) implies that (B.3) has a solution generating an element of $N(L^3)-N(L^2)$. In fact (B.3) can be solved explicitly since

(B.10)
$$L_-|x|^2R=-4\left(\frac{N}{2}R+x\cdot\nabla R\right).$$

Consider now the equation $L^4W=0$, when $\sigma=2/N$. This implies

(B.11)
$$\text{(a)}\quad L_-L_+L_-L_+u=0,$$
$$\text{(b)}\quad L_+L_-L_+L_-v=0.$$

We seek a solution generating an element of $N(L^4)-N(L^3)$. (B.11a) and (B.1a) imply

(B.12)
$$L_+L_-L_+u=cR, \qquad c \text{ constant}.$$

(B.1d) implies

(B.13)
$$L_-L_+u=-\frac{c}{2}\left(\frac{N}{2}R+x\cdot\nabla R\right).$$

(B.10) implies

(B.14)
$$L_+u=\frac{c}{8}|x|^2R.$$

(B.14) has a solution since the inhomogeneous term $|x|^2R$ satisfies the solvability condition $(|x|^2R,\nabla R)=0$. We define a particular solution of (B.12) $u=-(c/8)\rho$, where ρ is the unique radial and even solution of

(B.15)
$$L_+\rho=-|x|^2R.$$

Thus $(\rho,0)^t\in N(L^4)-N(L^3)$.

It is easily seen that (B.11b) has no solution giving rise to an element of $N(L^4)-N(L^3)$. Similarly it can be checked that $N(L^k)=N(L^4)$ $k\geqq 5$, when $\sigma=2/N$. $N_g(L^*)$ is similarly deducible.

We summarize the structure of the generalized null spaces when $\sigma=2/N$ in

THEOREM B.3. *Let* $\sigma=2/N$. $N_g(L)=\bigcup_{j=1}^4 N(L^j)$ *and* $N_g(L^*)=\bigcup_{j=1}^4 N[(L^*)^j]$ *are spanned by the following* $2N+4$*-dimensional biorthogonal sets*:

$$\text{(B.16)}\quad \begin{aligned}
&\text{(a)}\quad n_1=\beta_1^{-1}(0,-R)^t,\\
&\text{(b)}\quad n_{2,j}=\beta_2^{-1}(-R_{x_j},0)^t,\qquad 1\leqq j\leqq N,\\
&\text{(c)}\quad n_{3,j}=\beta_2^{-1}(0,x_jR)^t,\qquad 1\leqq j\leqq N,\\
&\text{(d)}\quad n_4=\beta_1^{-1}\left(\frac{N}{2}R+x\cdot\nabla R,0\right)^t,\\
&\text{(e)}\quad n_5=\beta_1^{-1}\left(0,|x|^2R\right)^t+\gamma_1(0,R)^t,\\
&\text{(f)}\quad n_6=\beta_1^{-1}(\rho,0)^t,\quad \textit{where}
\end{aligned}$$

$$\text{(B.17)}\quad \begin{aligned}
&\beta_1=-(R,\rho)=\frac{1}{2}\left(|x|^2R,R\right),\qquad \beta_2=\frac{1}{2}\|R\|_2^2,\\
&\gamma_1=\alpha_1^{-2}\left(|x|^2R,\rho\right),\qquad \gamma_2=(2\beta_1)^{-1}\left(|x|^2R,\rho\right).
\end{aligned}$$

$$\text{(B.18)}\quad \begin{aligned}
&\text{(a)}\quad m_1=(0,\rho)^t,\\
&\text{(b)}\quad m_{2,j}=(x_jR,0)^t,\qquad 1\leqq j\leqq N,\\
&\text{(c)}\quad m_{3,j}=\left(0,-R_{x_j}\right)^t,\qquad 1\leqq j\leqq N,\\
&\text{(d)}\quad m_4=\left(-\frac{1}{2}|x|^2R,0\right)^t+\gamma_2(-R,0)^t,\\
&\text{(e)}\quad m_5=\left(0,\frac{N}{2}R+x\cdot\nabla R\right)^t,\\
&\text{(f)}\quad m_6=(-R,0)^t.
\end{aligned}$$

Appendix C. The secular evolution. We set $S\equiv N_g(L)$ for $\sigma\leqq 2/N$. Recalling Definition 2.3 of M, we have by the biorthonormality of $N_g(L^*)$ and $N_g(L)$ the following:

PROPOSITION C.1. *For* $\sigma\leqq 2/N$, $H^1\times H^1\cong M\oplus S$.

Proposition 2.11 stated that M is mapped to itself by Ω_t. The following result describes the evolution in the complementing space S.

THEOREM C.2. *Consider* (2.4) *with* $W_0\equiv(u_0,v_0)^t\in S$ *and* $G\equiv(f,g)^t\in S$. *Then*, $W(t)=(u(t),v(t))^t\in S$ *and has the following form.*

(a) *For* $\sigma<2/N$

$$W(t)=\sum_{j=1}^4\mu_j(t)\cdot a_j,\quad \textit{where} \tag{C.1}$$

$$\mu_j(t)=\left(b_j,W(t)\right),\qquad 1\leqq j\leqq 4. \tag{C.2}$$

The functions μ_j satisfy the system of ODE's

(C.3)
$$\begin{aligned}
&\text{(a)}\quad 2\dot{\mu}_1(t) = -2\mu_4(t) + c_1, \\
&\text{(b)}\quad 2\dot{\mu}_{2,k}(t) = 2\mu_{3,k}(t) + c_{2,k}, \qquad 1 \leqq k \leqq N, \\
&\text{(c)}\quad 2\dot{\mu}_{3,k}(t) = c_{3,k}, \qquad 1 \leqq k \leqq N, \\
&\text{(d)}\quad 2\dot{\mu}_4(t) = c_4, \qquad \textit{where}
\end{aligned}$$

(C.4)
$$c_j = (b_j, G).$$

(b) *For* $\sigma = 2/N$

(C.5)
$$W(t) = \sum_{j=1}^{6} \nu_j(t) n_j, \quad \textit{where}$$

(C.6)
$$\nu_j(t) = (m_j, W(t)), \qquad 1 \leqq j \leqq 6.$$

The functions $\nu_j(t)$ satisfy the system of ODE's

(C.7)
$$\begin{aligned}
&\text{(a)}\quad 2\dot{\nu}_1(t) = -2\nu_4(t) + \beta_1\gamma_1\nu_6(t) + d_1, \\
&\text{(b)}\quad 2\dot{\nu}_2(t) = 2\nu_3(t) + d_2, \\
&\text{(c)}\quad 2\dot{\nu}_3(t) = d_3, \\
&\text{(d)}\quad 2\dot{\nu}_4(t) = -4\nu_5(t) + d_4, \\
&\text{(e)}\quad 2\dot{\nu}_5(t) = \nu_6(t) + d_5, \\
&\text{(f)}\quad 2\dot{\nu}_6(t) = d_6, \quad \textit{where}
\end{aligned}$$

(C.8)
$$d_j = (m_j, G).$$

Proof. We substitute the expansion (C.1) ((C.5) when $\sigma = 2/N$) into (2.4) and equate coefficients of like modes. This yields system (C.3) ((C.7) when $\sigma = 2/N$). □

Appendix D. A mean ergodic theorem for slowly varying functions. To prove Theorem 1′ we use the following lemma:

LEMMA D.1. *Let L be a skew-adjoint operator on a separable Hilbert space H, and* $\exp(Ls)$ *be a corresponding unitary group of transformations. Let $J^\varepsilon = J(\varepsilon s)$ be a continuous H-valued function with H-norm bounded, independently of ε, for $0 \leqq s \leqq T/\varepsilon$, where $T > 0$ is fixed. Furthermore, assume*

$$J(\varepsilon s) \in N^{\perp}(L) \quad \textit{for } 0 \leqq s \leqq T/\varepsilon, \quad \varepsilon > 0.$$

Then,

(D.1)
$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{T} \left\| \int_0^{T/\varepsilon} e^{Ls} J(\varepsilon s)\, ds \right\|_H = 0.$$

We prove (D.1) in the form

(D.2)
$$\lim_{\varepsilon \to 0} \frac{1}{T} \left\| \int_0^{T} e^{Ls/\varepsilon} f(s)\, ds \right\|_H = 0$$

if $f(s) \in N^{\perp}(L)$ for $0 \leqq s \leqq T$.

Let $\{\phi_j\}$ be a countable orthonormal basis of H. Let $\delta>0$ be arbitrary. Choose $n=n(\delta,T)$ so that

$$\sup_{0\leqq t\leqq T}\|f(t)-f^n(t)\|_H<\delta/2, \tag{D.3}$$

where $f^n(t)=\sum_{j=1}^n c_j(t)\phi_j$ and $c_j(t)=(\phi_j,f(t))$. For every j, $1\leqq j\leqq n$, we select $k_j(t)$, a piecewise constant approximation to $c_j(t)$ such that

$$\sup_{0\leqq t\leqq T}|c_j(t)-k_j(t)|<\frac{\delta}{2n}. \tag{D.4}$$

We then have

$$\begin{aligned}\int_0^T e^{Ls/\varepsilon}f(s)\,ds&=\int_0^T e^{Ls/\varepsilon}[f(s)-f^n(s)]\,ds\\&\quad+\int_0^T e^{Ls/\varepsilon}\sum_{j=1}^n\left[c_j(s)-k_j(s)\right]\phi_j ds\\&\quad+\int_0^T e^{Ls/\varepsilon}\sum_{j=1}^n k_j(s)\phi_j ds.\end{aligned} \tag{D.5}$$

Estimating in H,

$$\begin{aligned}\left\|\int_0^T e^{Ls/\varepsilon}f(s)\,ds\right\|_H&\leqq\frac{T\delta}{2}+Tn\frac{\delta}{2n}+\left\|\int_0^T e^{Ls/\varepsilon}\sum_{j=1}^n k_j(s)\phi_j ds\right\|_H\\&=T\delta+\left\|\int_0^T e^{Ls/\varepsilon}\sum_{j=1}^n\sum_{m=1}^{p_j}\chi_{j,m}k_{j,m}\phi_j ds\right\|_H\\&\leqq T\delta+\sum_{j=1}^n\sum_{m=1}^{p_j}\left\|\int_0^T e^{Ls/\varepsilon}\chi_{j,m}k_{j,m}\phi_j ds\right\|_H.\end{aligned} \tag{D.6}$$

Here, $k_j(t)=\sum_{m=1}^{p_j}\chi_{j,m}k_{j,m}$ is a piecewise constant approximation of $c_j(t)$ on $[0,T]$, where $\{\chi_{j,m}\}$ are characteristic functions of subintervals $\{I_{j,m}\}$ that can be chosen to be uniformly distributed in $[0,T]$. We note that the double-sum is finite and independent of ε. Since $f(s)\varepsilon N^{\perp}(L)$ for $0\leqq s\leqq T$, $\phi_j\varepsilon N^{\perp}(L)$. By the mean ergodic theorem [9], each term in this double-sum tends to zero as $\varepsilon\to 0$. Thus,

$$\frac{1}{T}\left\|\int_0^T e^{Ls/\varepsilon}f(s)\,ds\right\|_H\to\delta\quad\text{as }\varepsilon\to 0. \tag{D.7}$$

Since δ was arbitrary, the proof is complete.

Appendix E. A second proof of Proposition 2.7. This proof is based on the following general lemma.

LEMMA E.1. *Let A be a self-adjoint having exactly one negative eigenvalue, λ_0 with corresponding ground state eigenfunction $f_0\geqq 0$. Define*

$$-\infty<\alpha\equiv\min_f(Af,f),\quad\textit{where } \|f\|_2=1 \textit{ and } (f,R)=0. \tag{E.1}$$

We assume $(R,f_0)\neq 0$ *and* $R\in N^{\perp}(A)$. *Then* $\alpha\geqq 0$ *if*

$$(A^{-1}R,R)\leqq 0. \tag{E.2}$$

Proof of Lemma E.1. If α is attained for the function f_*, then by the theory of Lagrange multipliers there is a triple (f_*,λ_*,β_*) satisfying

$$Af_*=\lambda_*f_*+\beta_*R,\quad \|f_*\|_2=1,\quad (f_*,R)=0. \tag{E.3}$$

Taking the inner product of (E.3) with f_* we get

$$\lambda_*=(Af_*,f_*). \tag{E.4}$$

Therefore, to prove that $\alpha\geqq 0$ it suffices to preclude $\lambda_*\leqq 0$. First, if $\lambda_*=\lambda_0$, then taking the inner product of (E.3) with f_0 we conclude that either $\beta_*=0$ or $(R,f_0)=0$. Neither is possible since $(R,f_0)\neq 0$. If $\lambda\in(\lambda_0,0]$, we get from (E.3)

$$f_*=\beta_*(A-\lambda_*)^{-1}R. \tag{E.5}$$

Now λ_* is a critical point if

$$g(\lambda_*)=0, \tag{E.6}$$

where

$$g(\lambda)=\big((A-\lambda)^{-1}R,R\big). \tag{E.7}$$

Note that

$$g'(\lambda)=\big\|(A-\lambda)^{-1}R\big\|_2^2 \tag{E.8}$$

since A is self-adjoint. Therefore, g is increasing on $(\lambda_0,0]$. Moreover,

$$g(0)=(A^{-1}R,R). \tag{E.9}$$

It follows that if (E.2) holds, then $g(\lambda_*)\neq 0$ in $(\lambda_0,0)$. This proves the lemma.

Proof II *of Proposition* 2.7. We identify A with L_+. That L_+ has exactly one negative eigenvalue can be seen as follows (see also [30]):

Case (i) $N=1$: $R'\in N(L_+)$ has exactly one node at $x=0$ implying, by ODE oscillation theorems, that 0 is the second eigenvalue.

Case (ii) $N\geqq 2$, $\sigma<2/(N-2)$: Since the coefficient of the second term in (2.15) is nonpositive, we have that

$$\tilde{T}=L_++\big[(\sigma N-2)c_{\sigma N}(\Delta R,\cdot)-b_{\sigma N}(R,\cdot)\big]\Delta R \tag{E.10}$$

is a nonnegative operator. Therefore, L_+ is a rank one perturbation of a nonnegative operator. By the min-max principle, L_+ has at most one negative eigenvalue. Since $\nabla R\in N(L_+)$ is not positive it is not the ground state, implying that L_+ has exactly one negative eigenvalue.

As was remarked at the beginning of Proof I, we have that the infimum in (2.12) is nonpositive. By techniques used to prove Proposition 2.9, the infimum in (2.12) is actually a minimum. Therefore, L_+ and R satisfy the hypotheses of Lemma E.1 and it suffices to calculate $(L_+^{-1}R,R)$. By (B.1),

$$\big(L_+^{-1}R,R\big)=\left(\frac{N}{2}-\frac{1}{\sigma}\right)\|R\|_2^2. \tag{E.11}$$

This is nonpositive if $\sigma \leqq 2/N$, Proposition 2.7 now follows from Lemma E.1 □

We note that in the supercritical case $\sigma > 2/N$, the infimum in (2.12) will be negative since $(L_+^{-1}R, R) > 0$. This suggests modulational instability of the ground state for $\sigma > 2/N$.

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Note added in proof. The author has proved, using the results presented here on the linearized NLS operator, nonlinear Lyapunov stability of ground states relative to small perturbations in initial data: *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, to appear.

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