0.1 normal form

Equation

$$U + \mathcal{K} * U = \mathcal{N}(U;\mu), \tag{0.1}$$

Introduce H^ℓ_Γ in the notation section, this is the subsace of H^ℓ which respect the symmetry of Γ: $u \in H^\ell_\Gamma$ if $u(\gamma x) = u(x)$ for all $\gamma \in \Gamma$.

Set $\mathcal{T} := I_k + \mathcal{K}*$ so $\widehat{\mathcal{T}}(\xi) = I_k + \widehat{\mathcal{K}}(\xi)$,

First, let T_0 be the coordinate transformation which normalizes the second moment matrix S_{ij}, so that in the coordinate $y = T_0^{-1}x$, the kernel $\tilde{\mathcal{K}}(y) = |\det T_0|\mathcal{K}(T_0 y)$ satisfies

$$\int x_i x_j \langle \mathcal{E}_1^*, \tilde{\mathcal{K}}(x)\mathcal{E}\rangle dx = 2\delta_{ij}$$

note in these coordinates $U(x)$ has been transfromed into $U(T_0 y) := \tilde{U}(y)$, the linear part is $\tilde{U} + \tilde{\mathcal{K}} * \tilde{U}$, keep in mind $\widehat{\tilde{\mathcal{K}}}(\xi) = \widehat{\mathcal{K}}((T_0^{-1})^T\xi)$, so $I_k + \int \tilde{\mathcal{K}}$ is the same as $I_k + \int \mathcal{K}$, in particular we still choose $\mathcal{E}_1, \mathcal{E}_1^*$ to span the corresponding kernels.

note also that the first moment of $\tilde{\mathcal{K}}$ vanishes, $\int y\tilde{\mathcal{K}}(y) = |\det T_0| T_0^{-1} \int x\mathcal{K}(x)dx = 0$ if $x = T_0 y$.

Hence for the rest of this outline we work with the coordinates $y = T_0 x$ and the kernel $\tilde{\mathcal{K}}$, In which case we transform the operator $\widehat{\mathcal{T}}$ to a "normal form" via the following lemma.

(I drop tildes in order to write Fourier transform as hat..., so now $\widehat{\mathcal{T}}(\xi) = I_k + \widehat{\tilde{K}}(\xi)$, also identify U with $\tilde{U} = U \circ T_0$)

Lemma 0.1. *There exist invertible $k \times k$ matrices P, Q, and a multiplier operator L whose symbol $\widehat{L}(\xi) \in L^\infty(\mathbb{R}^n, \mathbb{R}^{k\times k})$ such that*

$$\widehat{L}(\xi)P[I_k + \widehat{\mathcal{K}}(\xi)]Q = \mathrm{diag}\{\frac{|\xi|^2}{1+|\xi|^2}, I_{n-1}\}.$$

Proof. we divide our construction in 2 steps.

Step 1. When $\xi = 0$, the rank of $\widehat{\mathcal{T}}(0) = I_k + \widehat{\mathcal{K}}(0)$ is equal to $k-1$ since it has a one dimensional kernel spanned by $\mathcal{E}_0$, it is standard that there exist invertible matrices P and Q such that

$$P\widehat{\mathcal{T}}(0)Q = \mathrm{diag}\{0, I_{k-1}\}.$$

By the assumption on finite second moment, $\widehat{\mathcal{K}}(\xi)_{i,j}$ are $\mathscr{C}^2$ functions. Taylor expand near $\xi = 0$, we find

$$P\widehat{\mathcal{T}}(\xi)Q = \begin{pmatrix} A(\xi) & B(\xi) \\ C(\xi) & D(\xi) \end{pmatrix},$$

where $A(\xi) = |\xi|^2 + o(|\xi|^2)$ exactly due to the normalization condition $\int x_i x_j \langle \mathcal{E}_1^*, \tilde{\mathcal{K}}(x)\mathcal{E}\rangle dx = 2\delta_{ij}$. $B(\xi), C(\xi)$ are $k-1$ sized row, column vectors respectively, both satisfies $B(\xi), C(\xi) = \mathcal{O}(|\xi|^2)$, and $D(\xi) = I_{k-1} + \mathcal{O}(|\xi|^2)$.

Step 2. Set $H(\xi) = \operatorname{diag}\{\frac{1+|\xi|^2}{|\xi|^2}, I_{k-1}\}$ for $\xi \neq 0$. We find

$$
\begin{aligned}
P\widehat{\mathcal{T}}(\xi)QH(\xi) &= \left(\begin{array}{c|c} |\xi|^2 + o(|\xi|^2) & \cdots \mathcal{O}(|\xi|^2) \cdots \\ \hline \vdots & \\ \mathcal{O}(|\xi|^2) & I_{k-1} + \mathcal{O}(|\xi|^2) \\ \vdots & \end{array}\right) \left(\begin{array}{c|c} \frac{1+|\xi|^2}{|\xi|^2} & 0 \cdots 0 \\ \hline 0 & \\ \vdots & I_{k-1} \\ 0 & \end{array}\right) \\
&= \left(\begin{array}{c|c} 1 + \frac{o(|\xi|^2)}{|\xi|^2} + |\xi|^2 + o(|\xi|^2) & \cdots \mathcal{O}(|\xi|^2) \cdots \\ \hline \vdots & \\ \mathcal{O}(1) & I_{n-1} + \mathcal{O}(|\xi|^2) \\ \vdots & \end{array}\right).
\end{aligned}
$$

Therefore, for $|\xi| \neq 0$ small, $P\widehat{\mathcal{T}}(\xi)QH(\xi)$ is invertible with uniform bounds on the inverse.

We then define $\widehat{L}(\xi)$ to be the inverse of $P\widehat{\mathcal{T}}(\xi)QH(\xi)$. Then $\widehat{L}(\xi) \in L^\infty$. Despite $\widehat{L}$ does not necessarily have a limit at 0, it is still uniformly invertible on $\mathbb{R}^n \backslash \{0\}$, exactly by the invertiblity for the nonzero wavenumbers and the convergence $\widehat{\mathcal{T}}(\xi) \to I_k$ as $|\xi| \to \infty$.

Hence it follows

$$
L(\xi)P\widehat{\mathcal{T}}(\xi)Q = H(\xi)^{-1}[P\widehat{\mathcal{T}}(\xi)Q]^{-1}P\widehat{\mathcal{T}}(\xi)Q = H(\xi)^{-1} = \operatorname{diag}\{\frac{|\xi|^2}{1+|\xi|^2}, I_{k-1}\},
$$

which is as stated in the lemma. ∎

Since $\widehat{L} \in L^\infty$, we know the multiplier L maps H^ℓ in H^ℓ and is bounded. Denote by M the multiplier operator with symbol $|\xi|^2/(1+|\xi|^2) := m(\xi)$, set $V(y) = Q^{-1}U(y)$, with standard coordinates $V(y) = (v_c(y), v_h(y))^T$. Then after precondition (0.1) with LP we obtain an equivalent equation

$$
\operatorname{diag}(M, I_{n-1})V(y) = LP\mathcal{N}(QV(y);\mu). \tag{0.2}
$$

0.2 Change of coordinates and rescaling, Lyapunov on hyperbolic part

Transformed equations. Write $v_h = (v_2, \ldots, v_k)^T$ in the standard coordinate in $\mathbb{R}^{k-1}$. Set $\mathcal{H}(V;\mu) := P\mathcal{N}(QV;\mu)$. Then with respect to the standard basis in $\mathbb{R}^k$, we denote by $\mathcal{H}_c(V;\mu)$ the first component of the nonlinearity $\mathcal{H}$ and $\mathcal{H}_h(V;\mu)$ the remaining $k-1$ components. Then, also with respect to the usual basis, we write the multiplier operator L and its symbol in matrix form

$$
L = \begin{pmatrix} L_{cc} & L_{ch} \\ L_{hc} & L_{hh} \end{pmatrix}, \quad \widehat{L}(\xi) = \begin{pmatrix} \widehat{L}_{cc}(\xi) & \widehat{L}_{ch}(\xi) \\ \widehat{L}_{hc}(\xi) & \widehat{L}_{hh}(\xi) \end{pmatrix}.
$$

Here terms with subscript cc denote a scalar, terms with subscript ch a $(k-1)$ dimensional row vector, terms with subscript hc a $(k-1)$ dimensional column vector, terms with subscript hh a $(k-1)\times(k-1)$ matrix.

In this notation, system (0.1) becomes

$$
Mv_c + L_{cc}\mathcal{H}_c(v_c, v_h;\mu) + L_{ch}\mathcal{H}_h(v_c, v_h;\mu) = 0, \tag{0.3}
$$
$$
v_h + L_{hc}\mathcal{H}_c(v_c, v_h;\mu) + L_{hh}\mathcal{H}_h(v_c, v_h;\mu) = 0. \tag{0.4}
$$

By part (i) of Hypothesis (TC), we may use Taylor's theorem to write $\mathcal{H}_j$ with $j = c, h$ as

$$\begin{aligned}\mathcal{H}_j(v_c, v_h; \mu) &= \Big(a^j_{101}\mu v_c + a^j_{011}\mu v_h + a^j_{110} v_c v_h + a^j_{200} v_c^2 + a^j_{020}[v_h, v_h]\Big) + \mathcal{R}_j(v_c, v_h; \mu)\\ &:= \mathcal{B}_j(v_c, v_h; \mu) + \mathcal{R}_j(v_c, v_h; \mu),\end{aligned}$$

where for the multiindex $\omega = (l, m, n)$ with $|\omega| = 2$, we denoted $a^j_\omega = \dfrac{1}{\omega!} D^\omega \mathcal{H}_j(0, 0; 0)$, and the remainder R_j satisfies the pointwise estimate

$$|R_j(v_c, v_h; \mu)| = |R_j(V; \mu)| = \mathcal{O}(\mu^2|V| + \mu|V|^2 + |V|^3) \tag{0.5}$$

for $(V; \mu)$ bounded.

We are in particular interested in the terms μv_c and v_c^2. In equation (0.3), the term μv_c is preconditioned by $L_{cc}a^c_{101} + L_{ch}a^h_{101}$, and the coefficient of v_c^2 is preconditioned by $L_{cc}a^c_{200} + L_{ch}a^h_{200}$. Using Hypothesis (TC), we claim that

$$\alpha = a^c_{101} = \widehat{L}_{cc}(0)a^c_{101} + \widehat{L}_{ch}(0)a^h_{101}, \quad \text{and} \quad \beta = a^c_{200} = \widehat{L}_{cc}(0)a^c_{200} + \widehat{L}_{ch}(0)a^h_{200}.$$

Indeed, to verify the first assertion, use the definition of L in lemma 0.1, we find $\widehat{L}_{cc}(0) = 1$ and $\widehat{L}_{ch}(0) = (0, \ldots, 0)$, thus verifying the second equality $a^c_{101} = \widehat{L}_{cc}(0)a^c_{101} + \widehat{L}_{ch}a^h_{101}$. To verify the first equality, let e_1 denote the standard coordinate vector $(1, 0, \ldots, 0)^T \in \mathbb{R}^k$, then the derivative $a^c_{101} = \dfrac{\partial^2}{\partial\mu\partial v_c}\mathcal{H}_c(0, 0; 0)$ is given by

$$\langle D_{\mu,V}\mathcal{H}(0; 0)e_1, e_1\rangle = \langle D_{\mu,U} P\mathcal{N}(0; 0)Qe_1, e_1\rangle = \langle PD_{\mu,U}\mathcal{N}(0; 0)\mathcal{E}_1, e_1\rangle = \langle D_{\mu,U}\mathcal{N}(0; 0)\mathcal{E}_1, \mathcal{E}_1^*\rangle = \alpha,$$

which verifies the first equality $\alpha = a^c_{101}$. The computations for β is similar.

In the next paragraph, we shall make a series of rescalings to simplify the equation.

Rescaling. Recall we have assumed $\alpha\mu > 0$, set now $\tilde{\mu} = \alpha\mu$ and write $\sqrt{\tilde{\mu}} = \varepsilon$. We then rescale the functions v_c, v_h to $\tilde{v}_c, \tilde{v}_h$ through

$$v_c(\cdot) = \frac{-1}{\beta}\varepsilon^2\tilde{v}_c(\varepsilon\cdot), \quad v_h(\cdot) = \varepsilon^2\tilde{v}_h(\varepsilon\cdot).$$

We substitute these variables in equation (0.3) and (0.4), divide the first equation by $(-1/\beta)\varepsilon^4$, the second by ε^2, and then obtain

$$\varepsilon^{-2}M^\varepsilon\tilde{v}_c + \sum_{j=c,h} L^\varepsilon_{cj}[\tilde{\mathcal{B}}_j(\tilde{v}_c, \tilde{v}_h) + \varepsilon^{-4}\tilde{\mathcal{R}}_j(\tilde{v}_c, \tilde{v}_h; \varepsilon)], \tag{0.6}$$

$$\tilde{v}_h + \sum_{j=c,h} L^\varepsilon_{hj}[\varepsilon^2\tilde{\mathcal{B}}_j(\tilde{v}_c, \tilde{v}_h) + \varepsilon^{-2}\tilde{\mathcal{R}}_j(\tilde{v}_c, \tilde{v}_h; \varepsilon)] = 0. \tag{0.7}$$

Note that here equation (0.6) and (0.7) holds pointwise in $z = \sqrt{\tilde{\mu}}y$. Since y is arbitrary, they hold for all $z \in \mathbb{R}^n$. We rather view them as functional equation in $\tilde{v}_c(\cdot)$ and $\tilde{v}_h(\cdot)$.

The rescaled linear operators M^ε and L^ε_j for $j = cc, ch, hc, hh$ have symbols $m(\varepsilon s)$, $\widehat{L}_j(\varepsilon s)$ respectively.

The rescaled nonlinear terms $\tilde{\mathcal{B}}_j, \tilde{\mathcal{R}}_j$ for $j = c, h$ are defined through

$$\tilde{\mathcal{B}}_j(u,v) = \frac{a^j_{101}}{\alpha}u + \frac{a^j_{011}}{\alpha}v + a^j_{110}uv + \frac{a^j_{200}}{-\beta}u^2 + a^j_{020}(-\beta)v^2,$$
$$\tilde{R}_j(u,v;\varepsilon) = \mathcal{R}_j\left(\frac{\varepsilon^2 u}{-\beta}, \varepsilon^2 v; \frac{\varepsilon^2}{\alpha}\right).$$

In particular, the coefficient of the term $\tilde{v}_c$ now equals $a^c_{101}/\alpha = 1$, and the coefficient of $\tilde{v}_c^2$ now equals $a^h_{200}/(-\beta) = \beta/(-\beta) = -1$ by the computation we have done in the previous step. As a consequence, we have

$$\tilde{B}_c(\tilde{v}_c, \tilde{v}_h) = \tilde{v}_c - \tilde{v}_c^2 + \mathcal{O}(|\tilde{v}_h| + |\tilde{v}_h|^2 + \tilde{v}_c|\tilde{v}_h|).$$

From the multiplication algebra property and Sobolev embedding of $H^\ell(\mathbb{R}^n, \mathbb{R}^k)$ with $\ell > n/2$, for any $u \in H^\ell(\mathbb{R}^n), v \in H^\ell(\mathbb{R}^n, \mathbb{R}^{k-1})$, and $j = c, h$ we have the following estimates

$$\|\tilde{\mathcal{B}}_j(u,v)\|_{H^\ell} \leq C\left(\|u\|_{H^\ell} + \|v\|_{H^\ell} + \|u\|_{H^\ell}\|v\|_{H^\ell} + \|u\|^2_{H^\ell} + \|v\|^2_{H^\ell}\right), \tag{0.8}$$

with some constant C. For the remainder terms $\tilde{\mathcal{R}}_j$, we have

$$\|\tilde{\mathcal{R}}_j(u,v;\varepsilon)\|_{H^\ell} = \mathcal{O}(\varepsilon^6) \tag{0.9}$$

as $\varepsilon \to 0$ from (0.5).

For the rescaled linear operator L^ε, using the definition of L and the Fourier transform characterization of H^ℓ, we find the following estimates hold

$$\|(L^\varepsilon_{cc} - 1)u\|_{H^\ell} \to 0, \quad \|L^\varepsilon_{ch}v\|_{H^\ell} \to 0, \quad \|(L^\varepsilon_{hh} - I_{k-1})w\|_{H^\ell} \to 0, \quad \text{and } \|L^\varepsilon_{hc}v\|_{H^\ell} \leq C. \tag{0.10}$$

for $u \in H^\ell(\mathbb{R}^n), v \in H^\ell(\mathbb{R}^n; \mathbb{R}^{k-1}), w \in H^\ell(\mathbb{R}^n; \mathbb{R}^k)$ and C is some constant independent of ε.

Now it remains to understand the behavior of the term $\varepsilon^{-2}M^\varepsilon v_c$ as $\varepsilon \to 0$. But before that, we first reduce (0.6) and (0.7) to a scalar equation using a fixed point argument in the next subsection. The main result will be proved shortly after.

To further ease notations, we drop the tildes, and still use $v_j, \mathcal{B}_j, \mathcal{R}_j$ $(j = c, h)$ for the same variables after the rescaling.

0.3 Lyapunov-Schmidt reduction and proof of the main result

We first solve (0.7) to obtain v_h as a function of v_c by a fixed point argument. We then substitute this function back into equation (0.6) to obtain a scalar equation for v_c and ε, which will be solved again using a fixed point argument.

We write the left hand side of (0.7) as $\mathcal{G}(v_h; v_c, \varepsilon)$ with $\mathcal{G}$ defined so that

$$\mathcal{G}(v; u, \varepsilon) = v + \sum_{j=c,h} L^\varepsilon_{hj}\left(\varepsilon^2\mathcal{B}_j(u,v;\varepsilon) + \varepsilon^{-2}\mathcal{R}_j(u,v;\varepsilon)\right),$$

using estimates (0.8) and (0.9), we have $\mathcal{G} : H^\ell(\mathbb{R}^n, \mathbb{R}^{k-1}) \times H^\ell(\mathbb{R}^n) \to H^\ell(\mathbb{R}^n, \mathbb{R}^{k-1})$ for each $\varepsilon > 0$. Note that we are treating v_c as an additional (Banach space-valued) parameter. The following lemma accomplishes what we were planning to do.

Lemma 0.2. *Fix $r>0$ not necessarily small, let B_r denote the ball centered at 0 with radius r in $H^\ell(\mathbb{R}^n)$, there then exist $\varepsilon_0>0$ sufficiently small and a map $\psi(u,\varepsilon): B_r\times(0,\varepsilon_0)\to H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$ such that $v=\psi(u,\varepsilon)$ solves $\mathcal{G}(v;u,\varepsilon)=0$. Moreover, the map $u\mapsto\psi(u,\varepsilon)$ is smooth for $u\in B_r$, and we have*

$$\|\psi(u,\varepsilon)\|_{H^\ell}=\mathcal{O}(\varepsilon^2),\quad \|D_u\psi(u,\varepsilon)\|_{H^\ell\to H^\ell}=\mathcal{O}(\varepsilon^2),$$

as $\varepsilon\to 0$, uniformly for $u\in B_r$ where $D_u\psi(u,\varepsilon)$ denotes the Frechet derivative of ψ with respect to u at the point (u,ε).

Proof. We will solve $\mathcal{G}(v;u,\varepsilon)=0$ using a Newton iteration scheme. For $u\in B_r$ and ε_0 small, we claim the following properties hold for $\mathcal{G}$:

(i) $\|\mathcal{G}(0;u,\varepsilon)\|_{H^\ell}=\mathcal{O}(\varepsilon^2)$, uniformly in $u\in B_r$ and $\varepsilon<\varepsilon_0$.

(ii) $\mathcal{G}$ is smooth in v, and $D_v\mathcal{G}(0;u,\varepsilon): H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})\to H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$ is bounded invertible with uniform bounds on the inverse for $|\varepsilon|<\varepsilon_0$ and $u\in B_r$.

For (i), since L^ε is uniformly bounded in ε, there exist a constant C such that $\|L^\varepsilon_{hc}\|_{H^\ell\to H^\ell}+\|L^\varepsilon_{hh}\|_{H^\ell\to H^\ell}\le C$, we then have

$$\begin{aligned}\|\mathcal{G}(0,u;\varepsilon)\|_{H^\ell}\le{}&\varepsilon^2 C(\|\mathcal{B}_c(u,0;\varepsilon)\|_{H^\ell}+\|\mathcal{B}_h(u,0;\varepsilon)\|_{H^\ell})\\&+\varepsilon^{-4}C(\|\mathcal{R}_c(u,0;\varepsilon)\|_{H^\ell}+\|\mathcal{R}_h(u,0;\varepsilon)\|_{H^\ell}).\end{aligned}$$

Using estimates (0.8) and (0.9), we have $\|\mathcal{G}(0;u,\varepsilon)\|_{H^\ell}\le C(r)\varepsilon^2$ uniformly in $u\in B_r$ and ε small.

For (ii), we conclude the smoothness of $\mathcal{G}$ in v using [Smoothness of sup operator] and the fact that L^ε_j are bounded linear operators. We compute the Frechet derivative of $\mathcal{G}$ to obtain

$$D_v\mathcal{G}(v;u,\varepsilon)w=w+\sum_{j=c,h}L^\varepsilon_{hj}(\varepsilon^2 D_v\mathcal{B}_j(u,v;\varepsilon)+\varepsilon^{-2}D_v\mathcal{R}_j(u,v;\varepsilon))w$$

for $w\in H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$. Using estimate (0.9), we see that $D_v\mathcal{G}(0;u,\varepsilon)$ is an $\mathcal{O}(\varepsilon^2)$ perturbation of the identity as an operator on $H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$ uniformly for $u\in B_r$. Thus, if ε_0 is small enough, then for all ε with $|\varepsilon|<\varepsilon_0$, we have that $D_v\mathcal{G}(0;u,\varepsilon)$ is bounded invertible with uniform bounds in ε.

After establishing (i) and (ii), fix $\delta>0$ and $u\in B_r$. Let N_δ denote the closed ball of radius δ around 0 in $H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$, we introduce a map $\mathcal{S}(\cdot;u,\varepsilon): H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})\to H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$ as follows

$$\mathcal{S}(v;u,\varepsilon)=v-D_v\mathcal{G}(0;u,\varepsilon)^{-1}[\mathcal{G}(v;u,\varepsilon)].$$

We then find

$$\|\mathcal{S}(0;u,\varepsilon)\|_{H^\ell}\le\|D_v\mathcal{G}(0;u,\varepsilon)^{-1}\|_{H^\ell\to H^\ell}\|\mathcal{G}(0;u,\varepsilon)\|_{H^\ell}=\mathcal{O}(\varepsilon^2).$$

Also, $D_v\mathcal{S}(0;u,\varepsilon)=0$ by definition, and $\mathcal{S}$ is smooth in v by (ii). Therefore, if δ is small and $v\in N_\delta$, it then follows that $\|D_vS(v;u,\varepsilon)\|_{H^\ell\to H^\ell}\le C\delta$ for some constant C independent of δ.

Then we start our iteration with $v_0=0$, $v_{n+1}=\mathcal{S}(v_n;u,\varepsilon)$, $n\ge 0$. Suppose by induction $v_k\in N_\delta$ for $1\le k\le n$, then

$$\|v_{n+1}-v_n\|_{H^\ell}\le C\delta\|v_n-v_{n-1}\|_{H^\ell},$$

by the mean value theorem. Therefore

$$\|v_{n+1}\|_{H^\ell} \le \frac{C}{1-C\delta}\|v_1 - v_0\|_{H^\ell} = \frac{C}{1-C\delta}\|\mathcal{S}(0;u,\varepsilon)\|_{H^\ell}.$$

This implies that for ε small and $u \in B_r$, we have $v_{n+1} \in N_\delta$, and that $\mathcal{S}$ is a contraction for δ sufficiently small. Then, as in the proof of Banach's fixed point theorem, we conclude that $v_n \to v = \psi(u,\varepsilon)$ as $n \to \infty$ and v is a fixed point of $\mathcal{S}$. Note that we automatically get $\|\psi(u,\varepsilon)\|_{H^\ell} = \mathcal{O}(\varepsilon^2)$ uniformly in $u \in B_r$.

To show the smooth dependence of $\psi(u,\varepsilon)$, we note that $\mathcal{G}(v;u,\varepsilon)$ is also smooth in u by [smoothness of superposition operators...]. By choosing ε small, the contraction constant for $\mathcal{S}$ can be chosen uniformly in $u \in B_r$. Hence by adopting the proof of the uniform contraction principle (see [Chicone ODE], Theorem 1.244), we conclude that ψ depends smoothly on u as well.

Finally, to get the estimate $\|D_u\psi\|_{H^\ell \to H^\ell} = \mathcal{O}(\varepsilon^2)$, we differentiate the equation $0 = \mathcal{G}(\psi(u,\varepsilon);u,\varepsilon)$ in u for $u \in B_r$ to see $D_u\psi$ satisfies the equation

$$D_v\mathcal{G}(\psi(u,\varepsilon);u,\varepsilon)D_u\psi(u,\varepsilon) + D_u\mathcal{G}(\psi(u,\varepsilon);u,\varepsilon) = 0.$$

Now, $D_u\mathcal{G}(v;u,\varepsilon)$ is of the form

$$D_u\mathcal{G}(v;u,\varepsilon)w = \sum_{j=c,h} L^\varepsilon_{hj}(\varepsilon^2 D_u\mathcal{B}_j(u,v;\varepsilon) + \varepsilon^{-2}D_u\mathcal{R}_j(u,v;\varepsilon))w,$$

hence, for $u \in B_r$ and $v = \psi(u,\varepsilon) \in N_\delta$, we have $\|D_u\mathcal{G}(v;u,\varepsilon)\|_{H^\ell \to H^\ell} = \mathcal{O}(\varepsilon^2)$ again by estimate (0.8) and (0.9).

On the other hand, $D_v\mathcal{G}(v;u,\varepsilon)$ is uniformly invertible in ε for $v = \psi(u,\varepsilon) \in N_\delta$ and $u \in B_r$ as calculated previously. Therefore we can write $D_u\psi(u,\varepsilon) = -[D_v\mathcal{G}]^{-1}D_u\mathcal{G}$ and conclude that

$$\|D_u\psi(u,\varepsilon)\|_{H^\ell \to H^\ell} \le C(r,\delta)\varepsilon^2.$$

This finishes the proof. ∎

Remark. Because of the dependence of the convolution operator $L^\varepsilon_{hc}, L^\varepsilon_{hh}$ on ε is not smooth at $\varepsilon = 0$, we cannot use the usual implicit function theorem directly to solve the equation $\mathcal{G}(v;u,\varepsilon) = 0$. We follow the Newton iteration scheme as in [Faye Scheel, advance paper] to circumvent this problem.

Using lemma 0.2, we substitute $v_h = \psi(v_c,\varepsilon)$ into equation (0.6). We obtain the following scalar equation

0.4 Precondition with $\mathcal{M}^\varepsilon$ and solve the reduced scalar equation

We get the scalar equation in $\tilde{v}_c$ after rescaling,

$$0 = \varepsilon^{-2}M^\varepsilon\tilde{v}_c + \sum_{j=c,h} L^\varepsilon_{cj}\left[B_j(\tilde{v}_c,\psi(\tilde{v}_c,\varepsilon)) + \varepsilon^{-4}\mathcal{R}_j(\tilde{v}_c,\psi(\tilde{v}_c,\varepsilon);\varepsilon)\right]. \tag{0.11}$$

It is now crucial to understand the behavior of the operator M^ε as $\varepsilon \to 0$. Recall that by definition

$$\widehat{M^\varepsilon v}(\xi) = m(\varepsilon\xi)\widehat{v}(\xi) = \frac{|\varepsilon\xi|^2}{1+|\varepsilon\xi|^2}\widehat{v}(\xi),$$

for any $v \in H^\ell(\mathbb{R}^n, \mathbb{R}^k)$. We then define a new operator $\mathcal{M}^\varepsilon$ through

$$\widehat{\mathcal{M}^\varepsilon v}(\xi) = \frac{m(\varepsilon\xi)}{|\varepsilon\xi|^2}\widehat{v}(\xi) = \frac{1}{1+|\varepsilon\xi|^2}\widehat{v}(\xi).$$

Since $1/(1+|\varepsilon\xi|^2)$ is a bounded function on $\mathbb{R}^n$, $\mathcal{M}^\varepsilon$ maps $H^\ell(\mathbb{R}^n, \mathbb{R}^k)$ into itself.

For $v \in H^\ell(\mathbb{R}^n, \mathbb{R}^k)$, $(\mathcal{M}^\varepsilon)^{-1}$ is defined through

$$\widehat{(\mathcal{M}^\varepsilon)^{-1}} v(\xi) = \frac{|\varepsilon\xi|^2}{m(\varepsilon\xi)}\widehat{v}(\xi) = (1+|\varepsilon\xi|^2)\widehat{v}(\xi),$$

moreover we have:

$$\begin{aligned}
\|((\mathcal{M}^\varepsilon)^{-1} - 1)v\|_{H^{\ell-2}} &= \left\| \left(1+|\varepsilon\xi|^2 - 1\right)\widehat{v}(\xi)(1+|\xi|^2)^{\frac{\ell-2}{2}} \right\|_{L^2} \\
&\leq \sup_{\ell} \left| \frac{|\varepsilon\xi|^2}{1+|\xi|^2} \right| \|\widehat{v}(\xi)(1+\xi^2)^{\frac{\ell}{2}}\|_{L^2} \\
&\leq \varepsilon^2 \|v\|_{H^\ell}.
\end{aligned}$$

Therefore, considered as an operator from $H^\ell(\mathbb{R}^n, \mathbb{R}^k)$ to $H^{\ell-2}(\mathbb{R}^n, \mathbb{R}^k)$, $(\mathcal{M}^\varepsilon)^{-1}$ is well-defined, and $\|(\mathcal{M}^\varepsilon)^{-1}v - v\|_{H^{\ell-2}} \to 0$ as $\varepsilon \to 0$ for $v \in H^\ell(\mathbb{R}^n, \mathbb{R}^k)$. This simple observation is central to identifying the leading-order terms and we state it as a lemma.

Lemma 0.3. *The multiplier operator $(\mathcal{M}^\varepsilon)^{-1}$ with symbol $\dfrac{|\varepsilon\xi|^2}{m(\varepsilon\ell)} = 1+|\varepsilon\xi|^2$ is well defined, maps from $H^\ell(\mathbb{R}^n, \mathbb{R}^k)$ into $H^{\ell-2}(\mathbb{R}^n, \mathbb{R}^k)$, and satisfies the estimate*

$$\|(\mathcal{M}^\varepsilon)^{-1} - I\|_{H^\ell \to H^{\ell-2}} = \mathcal{O}(\varepsilon^2).$$

Of course, $(M^\varepsilon)^{-1} - I$ is just a rescaled version for the usual Laplacian. In particular, it respects the symmetry of Γ, that is $(\mathcal{M}^\varepsilon)^{-1}$ takes H^ℓ_Γ into $H^{\ell-2}_\Gamma$.

Let now v_* be the unique positive ground sate solution of the equation $\Delta v - v + v^2 = 0$, we have

Proposition 0.4. *Assume $d > 0, n < 6$ and $\ell > n/2$. If $\varepsilon_1 > 0$ is sufficiently small, then for $0 < \varepsilon < \varepsilon_1$, there exist a family of solutions to (0.11) of the form $v_c(\cdot;\varepsilon) = v_*(\cdot) + w(\cdot;\varepsilon)$. Here $w = w(\cdot,\varepsilon) \in H^\ell_\Gamma(\mathbb{R}^n, \mathbb{R}^k)$ is a family of correctors parametrised by ε such that $\|w(\cdot,\varepsilon)\|_{H^\ell} \to 0$ as $\varepsilon \to 0$.*

Proof. We substitute the ansatz $v_c = v_* + w$ into (0.11), where v_* is as stated in the lemma and $w \in H^\ell_\Gamma$. We will determine an equation for w and ε and show that it can be solved using Newton iteration scheme near $(w, \varepsilon) = (0, 0)$. First, for the term $\varepsilon^{-2}M^\varepsilon v_c$ with $v_c \in H^\ell$, we apply Fourier transform to obtain

$$\varepsilon^{-2} m(\varepsilon\xi)\widehat{v}_c(\xi) = -\frac{m(\varepsilon\xi)}{|\varepsilon\xi|^2}(-|\xi|^2)\widehat{v}_c(\xi) = -\widehat{\mathcal{M}^\varepsilon \Delta v_c},$$

and equation (0.11) becomes

$$0 = -\mathcal{M}^\varepsilon \Delta v_c + \left(L^\varepsilon_{cc} + L^\varepsilon_{ch} a^h_{101}\right) v_c + \left(L^\varepsilon_{cc} a^c_{110} + L^\varepsilon_{ch} a^h_{110}\right) v_c^2 + \mathcal{R}(v_c, \psi; \varepsilon),$$

where $\mathcal{R}(v_c,\psi;\varepsilon)$ contains all the terms of order ε^2 and higher,

$$\mathcal{R}(v_c,\psi;\varepsilon) = \sum_{j=c,h} L^\varepsilon_{cj}\left[\frac{a^j_{011}}{\alpha}\psi + a^j_{110}v_c\psi + a^j_{020}(-\beta)[\psi,\psi] + \varepsilon^{-4}\mathcal{R}_j(v_c,\psi;\varepsilon)\right].$$

Indeed, for w with $v_*+w\in B_r$, we claim that $\mathcal{R}$ satisfies the estimate $\|\mathcal{R}\|_{H^\ell} = \mathcal{O}(\varepsilon^2)$. To see this, we first apply Lemma 0.2 with $r = 2\|v_*\|_{H^\ell}$ to obtain $\psi = \psi(v_*+w,\varepsilon)$ which satisfies $\|\psi(v_*+w,\varepsilon)\|_{H^\ell} = \mathcal{O}(\varepsilon^2)$.

The linear operators $L^\varepsilon_{cc}, L^\varepsilon_{ch}$ are uniformly bounded in ε, so that we have

$$\left\|\sum_{j=c,h} L^\varepsilon_{cj}\left(\frac{a^j_{011}}{\alpha}\psi + a^j_{110}v_c\psi + a^j_{020}(-\beta)[\psi,\psi]\right)\right\|_{H^\ell} \le K(\|\psi\|_{H^\ell} + \|\psi\|^2_{H^\ell}) = \mathcal{O}(\varepsilon^2),$$

for some constant K from Lemma 0.2.

On the other hand, the remainders $\mathcal{R}_c$ and $\mathcal{R}_h$ satisfy $\|\mathcal{R}_c\|_{H^\ell} = \mathcal{O}(\varepsilon^6), \|\mathcal{R}_h\|_{H^\ell} = \mathcal{O}(\varepsilon^6)$ uniformly for v_* and w such that $v_*+w\in B_r$ as $\varepsilon\to 0$ by estimates (0.8) and (0.9). Therefore we conclude that $\|\mathcal{R}(v_c,\psi;\varepsilon)\|_{H^\ell} = \mathcal{O}(\varepsilon^2)$ for $v_c = v_*+w\in B_r$.

Next, add the equation $d\Delta v_* - v_* + v_*^2 = 0$ to the right hand side of (0.11) and precondition with the operator $(\mathcal{M}^\varepsilon)^{-1}$. Set $\alpha^\varepsilon = L^\varepsilon_{cc} + \frac{a^h_{101}}{\alpha}L^\varepsilon_{ch}, \beta^\varepsilon = -L^\varepsilon_{cc} + \frac{a^h_{200}}{-\beta}L^\varepsilon_{ch}$ and we find

$$\begin{aligned} 0 &= (\mathcal{M}^\varepsilon)^{-1}\left[(d-\mathcal{M}^\varepsilon)\Delta v_* - \mathcal{M}^\varepsilon\Delta w + \alpha^\varepsilon(v_*+w) - v_* + \beta^\varepsilon(v_*+w)^2 + v_*^2 + \mathcal{R}\right] \\ &= [(\mathcal{M}^\varepsilon)^{-1} - d^{-1}]\mathcal{M}^\varepsilon d\Delta v_* + (\mathcal{M}^\varepsilon)^{-1}\left[(\alpha^\varepsilon - 1)v_* + (\beta^\varepsilon+1)v_*^2 + \mathcal{R}\right] + \\ &\qquad -\Delta w + (\mathcal{M}^\varepsilon)^{-1}\left[\alpha^\varepsilon w + \beta^\varepsilon(2v_*w + w^2)\right], \\ &\qquad := F_1(w;\varepsilon) + F_2(w;\varepsilon) := F(w;\varepsilon). \end{aligned} \tag{0.12}$$

By Lemma 0.3, we have that F maps $H^\ell_\Gamma(\mathbb{R}^n)$ to $H^{\ell-2}_\Gamma(\mathbb{R}^n)$. Our goal is to set up a Newton iteration scheme to solve $F(w,\varepsilon)=0$ for w in terms of ε as a fixed point problem.

Following the strategy of Lemma 0.2, we shall show

(i) $\|F(0,\varepsilon)\|_{H^{\ell-2}} \to 0$ as $\varepsilon\to 0$.

(ii) $F(w,\varepsilon)$ is continuously differentiable in w and $D_wF(0,\varepsilon): H^\ell_\Gamma(\mathbb{R}^n)\to H^{\ell-2}_\Gamma(\mathbb{R}^n)$ is uniformly invertible in ε.

For (i), we note that

$$F(0,\varepsilon) = F_2(0;\varepsilon) = [(\mathcal{M}^\varepsilon)^{-1} - d^{-1}]\mathcal{M}^\varepsilon d\Delta v_* + (\mathcal{M}^\varepsilon)^{-1}[(\alpha^\varepsilon-1)v_* + (\beta^\varepsilon+1)v_*^2 + \mathcal{R}(v_*,\psi;\varepsilon)].$$

By [reference on ground state], $\Delta v_*\in H^\ell(\mathbb{R}^n)$ for all ℓ, since $\mathcal{M}^\varepsilon$ take $H^\ell(\mathbb{R}^n)$ into itself and is uniformly bounded in ε, we conclude from Lemma 0.3 that $\|[(\mathcal{M}^\varepsilon)^{-1} - d^{-1}]\mathcal{M}^\varepsilon d\Delta v_*\|_{H^\ell}\to 0$ as $\varepsilon\to 0$.

Moreover, by (0.10), it holds that $\|\alpha^\varepsilon v - v\|_{H^\ell}\to 0$ and $\|\beta^\varepsilon v + v\|_{H^\ell}\to 0$ as $\varepsilon\to 0$ for any $v\in H^\ell(\mathbb{R}^n)$, and the remainder $\mathcal{R}(v_*,\psi;\varepsilon)$ satisfies $\|\mathcal{R}\|_{H^\ell} = \mathcal{O}(\varepsilon^2)$ as proved earlier. Hence, we conclude that

$$\|F(0;\varepsilon)\|_{H^{\ell-2}} = \|F_2(0;\varepsilon)\|_{H^{\ell-2}} \to 0$$

as $\varepsilon \to 0$, which proves (i).

For (ii), we first verify that F is continuously differentiable in w from $H^\ell(\mathbb{R}^n)$ to $H^{\ell-2}(\mathbb{R}^n)$. Indeed, take $h, w_0 \in H^\ell(\mathbb{R}^n)$ with w_0 fixed. We observe that $D_wF(w_0;\varepsilon)h : H^\ell(\mathbb{R}^n) \to H^{\ell-2}(\mathbb{R}^n)$ is given by

$$D_wF(w_0;\varepsilon)h = -\Delta h + (\mathcal{M}^\varepsilon)^{-1}\left[(a^\varepsilon h) + 2v_*\beta^\varepsilon h + 2w_0 h) + D_w\mathcal{R}h\right].$$

The smooth dependence on w_0 comes from Proposition [smoothness of superposition operator].

Now, at $w_0 = 0$, we see that, $D_wF(0;\varepsilon)h \to -\Delta h + h - 2v_*h = \mathcal{L}h$ in $H^{\ell-2}(\mathbb{R}^n)$ as $\varepsilon \to 0$ for $h \in H^\ell$ because $\|D_w\mathcal{R}h\|_{H^\ell} = \mathcal{O}(\varepsilon^2)$ as remarked earlier. By [the nondegenracy in H^ℓ_Γ lemma], the operator $\mathcal{L} : H^\ell_\Gamma(\mathbb{R}^n) \to H^{\ell-2}_\Gamma(\mathbb{R}^n)$ is bounded invertible. We notice that $D_wF(0;\varepsilon)$ respects the symmetry and is a small perturbation of $\mathcal{L}$, therefore invertible with uniform bounds on the inverse for ε small enough. This shows (ii).

We now set up the Newton iteration scheme, define $\tilde{\mathcal{S}}$ through

$$\tilde{\mathcal{S}}(w;\varepsilon) = w - D_wF(0;\varepsilon)^{-1}[F(w;\varepsilon)].$$

Note that $\tilde{\mathcal{S}}$ respects the symmetry as well: $\tilde{\mathcal{S}} : H^\ell_\Gamma(\mathbb{R}^n) \to H^{\ell-2}_\Gamma(\mathbb{R}^n)$. Therefore we can proceed as in Lemma 0.2 to obtain $w = w(\varepsilon)$ which solves $F(w(\varepsilon);\varepsilon) = 0$ for ε small enough and satisfies $\|w(\varepsilon)\|_{H^\ell} \to 0$ as $\varepsilon \to 0$. ∎

Finally, we prove Theorem (main result).

Proof of Theorem. We now write the tildes for the rescaled variables. From proposition 0.4, we know that (0.11) has a solution of the form $\tilde{v}_c(\cdot) = v_*(\cdot) + w(\cdot;\varepsilon)$. Together with $\tilde{v}_h = \psi(\tilde{v}_c, \varepsilon)$, reverting the rescaling, we obtain $v_c(\cdot) = -\frac{\alpha}{\beta}\mu\tilde{v}_c(\sqrt{\alpha\mu}\cdot)$ and $v_h(\cdot) = \alpha\mu\tilde{v}_h(\sqrt{\alpha\mu}\cdot)$ as solutions to (0.3) and (0.4).

Now, recall that $V = (v_c, v_h)^T$ and the original variable U are related by $U = QV$ where Q is defined in the proof of Lemma 0.1. We conclude that $U(\cdot) = v_c(\cdot)\mathcal{E}_0 + v_\perp(\cdot)$, where $v_\perp$ takes values in the complement of $\mathcal{E}_0$. The behavior of $v_c, v_\perp$ as $\mu \to 0$ is a direct consequence of Lemma 0.2 and Proposition 0.4. Lastly, we restore to the original variable $x = T_0 y$, thus getting the desired form of the solution.

[1] ∎

References

[1] Peter W. Bates, Paul C. Fife, Xiaofeng Ren, and Xuefeng Wang. Traveling waves in a convolution model for phase transitions. *Archive for Rational Mechanics and Analysis*, 138(2):105–136, 1997.