Multi-D case

discussion

2017

Outlines for multi-Dimensions

We consider

$$u + \mathcal{K} * u = \mathcal{N}(u; \mu) \tag{0.1}$$

with $u = u(x) : \mathbb{R}^n \to \mathbb{R}^p$, $n < 6$, need $(\frac{n+2}{n-2} > 2)$.

(i) Linear assumptions on matrix convolution operator $\mathcal{K}$ requires it to be **radially symmetric**, similar to 1-d case, $W^{2,1}$ with finite 4th moment. By symmetry and smoothness, the Fourier transform $\widehat{u}(\xi)$ is a radially symmetric function of class $\mathscr{C}^4(\mathbb{R}^n; \mathbb{R}^p)$. Write $|\xi| = s$ for the radial variable, then $\widehat{\mathcal{K}}(\xi) = k(s)$ with $k(s) = k(0) + (k''(0)/2)s^2 + o(s^2)$ as $s \to 0$. Similarly define $\mathcal{D}(s) = \det(I_p + k(s))$, require $\mathcal{D}(0) = 0, \mathcal{D}''(0) \neq 0$. We then get $\mathcal{E}_0, \mathcal{E}_0^*$ as before. Set $d = \langle \frac{1}{2}k''(0)\mathcal{E}_0, \mathcal{E}_0^* \rangle$.

(ii) Nonlinear assumptions is unaffected by considering $x \in \mathbb{R}^n$. Except that I now need to work with $H^\ell(\mathbb{R}^n; \mathbb{R}^p)$ with $\ell > n/2$ in order to take advantage of the Banach algebra property and embedding results. I think I should add a short proof (maybe in the appendix) of the fact that the superposition operator $\tilde{\mathcal{N}}$ defined by $\tilde{\mathcal{N}}(u)(\cdot) = \mathcal{N}(u(\cdot); \mu)$ takes H^ℓ into itself and is as smooth as $\mathcal{N}$ is, provided $\mathcal{N}(0; \mu) = 0$. In particular, α, β can be computed using the same formula.

(iii) We then construct $S(s), P, Q$ that brings $I_n + k(s)$ into the diagonal form $\operatorname{diag}\left(\dfrac{ds^2}{1+s^2}, I_{p-1}\right)$ exactly as before. Define new variable v by $Qv = u$, write $v = (v_c, v_h)$ in the standard coordinates. Use the same scaling $v(\cdot) \to \mu v(\sqrt{\mu}\cdot)$, write $\varepsilon = \sqrt{\mu}$. We need to solve the two equation

$$\varepsilon^{-2} M^\varepsilon v_c \ = (L\mathcal{N})_c, \tag{0.2}$$

$$v_h \ = (L\mathcal{N})_h \tag{0.3}$$

with $\widehat{L} = S(s)$ and M^ε has symbol $m(\varepsilon s) = d(\varepsilon s)^2/(1 + (\varepsilon s)^2)$.

(iv) We may solve $v_h = \psi(v_c, \varepsilon)$ with $\|\psi\|_{H^2}, \|D_u\psi\|$ of order ε^2. The proof is unchanged. We get the scalar equation

$$\varepsilon^{-2} M^\varepsilon v_c = (L\mathcal{N}(v_c, \psi(v_c, \varepsilon); \varepsilon)_c \tag{0.4}$$

Set $\mathcal{M}^\varepsilon$ so that $\widehat{\mathcal{M}^\varepsilon v} = \frac{d}{1+(\varepsilon s)^2}\widehat{v}(\xi) = \frac{d}{1+(\varepsilon|\xi|)^2}\widehat{v}(\xi)$. Fix $\ell > n/2$, then

$$\|(\mathcal{M}^\varepsilon)^{-1}v - d^{-1}v\|_{H^\ell} \le \varepsilon^2 \|v\|_{H^{\ell+2}}$$

we then substitute the ansatz $v_c = v_* + w$ where now $v_* = v_*(x)$ is the unique "ground state" solution to the equation $\Delta v = v - v^2$. The existence, uniqueness and nondegenracy of ground state to equation $\Delta v = v - v^q$ need q lies in the range $1 < q < \frac{n+2}{n-2}$, so we require $n < 6$. These results say also that v_* is radial, decays to 0 as $|x| \to \infty$ exponentially fast and smooth. After precondition with $(\mathcal{M}^\varepsilon)^{-1}$, we have

$$0 = -\Delta w + d^{-1}[\alpha^\varepsilon w + \beta^\varepsilon(2v_* w + w^2)] + \mathcal{R}(w, \varepsilon) := F(w, \varepsilon)$$

with $\|\mathcal{R}\|_{H^{\ell+2}} \to 0$. We need to show $F(w, \varepsilon) : H^{\ell+2} \to H^\ell$ satisfy

- $F(0, 0) = 0$ (continuity at $\varepsilon = 0$);
- $D_w F(0, 0) = -\Delta + d^{-1}[1 - 2v_*]$ is Fredholm with index 0 from $H^{\ell+2} \to H^\ell$, and its kernel is spanned by the n partial derivatives of v_*, this comes from the nondegenracy of v_*.

If we then denote X, Y to be the subspace of $H^{\ell+2}, H^\ell$ that is orthogonal to $\partial_{x_i} v_*$, $i = 1, \ldots, n$. We apply implicit function theorem/Newton iteration on X, Y to solve for $F(w(\varepsilon), \varepsilon) = 0$ for ε small.