computations

authors

2017

Let $b(\ell)$ be a bounded continuous function on $\mathbb{R}$, consider the multiplier operator M_b defined through

$$\widehat{M_b u}(\ell) = b(\ell)\widehat{u}(\ell),$$

for $u \in L^2$.

Let b^ε denote the function $b(\varepsilon\ell)$. Take for example $b(\ell) = \frac{\ell^2}{\ell^2+1}$.

Given $\delta > 0$, we have

$$\begin{aligned}\|M_{b^\varepsilon}u - M_{b(0)}u\|_{L^2} = \|(b(\varepsilon\ell) - b(0))\widehat{u}(\ell)\|_{L^2} &= \int |b(\varepsilon\ell) - b(0)|^2 |\widehat{u}(\ell)|^2 d\ell \\ &= \int_{|\ell|\leq R} (\cdots) + \int_{|\ell|>R} (\cdots) := I + II\end{aligned}$$

where R is chosen so that $\int_{|\ell|>R} |\widehat{u}|^2 < \delta$.

Now we can take ε small so that $|b(\varepsilon\ell) - b(0)| < \delta$ for all $|\ell| \leq R$ as b is continuous at 0. As a result, we have $I \leq \delta^2 \int |\widehat{u}|^2$ and $II \leq 2\max_{\ell\in\mathbb{R}} b(\ell)\delta$. Therefore we conclude that $M_{b^\varepsilon}u \to b(0)u$ in L^2.

But $\sup_{\ell\in\mathbb{R}} |b(\varepsilon\ell) - b(0)|$ does not converge to 0.