Uniqueness and Nonuniqueness for Positive Radial Solutions of $\Delta u + f(u, r) = 0$

WEI-MING NI
University of Minnesota

AND

ROGER D. NUSSBAUM
Rutgers University and Courant Institute

1. Introduction

In this paper we continue a study, which was begun in [17], of the question of uniqueness of positive solutions of the nonlinear Dirichlet problem

$$\begin{aligned} \Delta u + f(u, |x|) &= 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{1.1}$$

In equation (1.1), Ω denotes a ball or an annulus in $\mathbb{R}^n$ and $f \geqq 0$ is, roughly speaking, superlinear in u and satisfies $f(0, |x|) = 0$. Problem (1.1) arises in various circumstances, for instance, in the "fast diffusion" phenomenon in plasma physics (see [2], or see Remark 4 in Section 4 of [17] for a brief discussion). The uniqueness question for (1.1) is answered affirmatively in [10] in the case when $f(u, |x|) = u^p$, $p > 1$, and Ω is a ball. There the proof consists of two steps: first, prove that all positive solutions of (1.1) have to be radially symmetric; then, show that the positive radial solution is unique. It turns out that the first step is quite general, namely, it holds for any Lipschitz continuous $f(u)$. The second step is, however, very special—it makes use of both the hypothesis that $f(u) = u^p$ is homogeneous of degree $p > 1$ and the assumption that Ω is a ball in order to apply a scaling argument. Thus we shall continue the study initiated in [17] of the uniqueness of *positive*, *radial* solutions of (1.1) when f may not be homogeneous and Ω may be a ball or an annulus. (We should mention that, on an annulus, there are non-radially symmetric positive solutions of (1.1) even when $f(u) = u^p$; see [3] for $f(u) = u^p$ and [7], [22] and [8] for other nonlinearities).

In this paper we shall extend almost all of the results in [17], answer many of the problems left open in [17] and, perhaps more surprisingly, obtain some nonuniqueness results for "nice" nonlinearities f. We shall now describe some of our main results. For the sake of simplicity, we shall limit ourselves here to the case $f(u, |x|) \equiv f(u)$ and shall defer the more general case to the main body

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of the paper. Thus for the remainder of this section we shall be describing results for

$$\begin{aligned} &\Delta u+f(u)=0 \quad \text{in} \quad \Omega, \\ &u|_{\partial\Omega}=0, \end{aligned} \tag{1.1$'$}$$

THEOREM 1.2. *Let u be a positive radial solution of*

$$\begin{aligned} &\Delta u+u^p=0 \quad \text{in} \quad \Omega, \qquad p>1, \\ &u|_{\partial\Omega}=0, \end{aligned} \tag{1.3}$$

where Ω is an annulus in $\mathbb{R}^n$, $n\geqq 2$. Then u is unique in the class of all positive radial functions.

This is a special case of Theorem 3.1 in Section 3 below. Working with more general nonlinearities, we also extend Theorems 1.4 and 1.6 in [17] as follows.

THEOREM 1.4. *Let Ω be an annulus in $\mathbb{R}^n$, $n\geqq 3$, and u be a positive radial solution of (1.1)$'$. Then u is unique in the class of all positive radial functions provided f satisfies*

$$\frac{n}{n-2}f(t)\geqq tf'(t)>f(t)>0 \tag{1.5}$$

in $t>0$.

THEOREM 1.6. *Let Ω be a ball in $\mathbb{R}^n$, $n\geqq 3$, and u be a positive solution of (1.1)$'$. Then u is radially symmetric and is unique in the class of all positive functions provided f satisfies (1.5) in $t>0$.*

The case $n=2$ is somewhat special, but for $n=2$ we also obtain analogues to Theorems 1.4 and 1.6 which we shall present in Section 2 below.

For the case of an annulus, we can actually do better than Theorem 1.4. If $\Omega=\{x\in\mathbb{R}^n \mid 0<a<|x|<b\}$, we have

THEOREM 1.7. *If b/a is small (say, $b/a\leqq(n-1)^{1/(n-2)}$ for $n\geqq 3$, and $b/a\leqq e$ for $n=2$), then (1.1)$'$ can have at most one positive radial solution provided $tf'(t)>f(t)>0$ in $t>0$.*

Note that there is no growth condition imposed on f in Theorem 1.7 except the requirement that f be superlinear. In general, there is a "balance" between the ratio b/a and the growth of f, as the following result shows.

THEOREM 1.8. *Assume that $n\geqq 3$. Then equation (1.1)$'$ has at most one positive radial solution if, for $t>0$, we have*

$$\left(\frac{n}{n-2}+\frac{2a^{n-2}}{b^{n-2}-a^{n-2}}\right)f(t)\geqq tf'(t)>f(t)>0. \tag{1.9}$$

In view of Theorem 1.2, one might suspect that condition (1.9) in Theorem 1.8 is superfluous. It is therefore somewhat surprising that this is *not* the case.

THEOREM 1.10. *Assume that* $n \geqq 3$. *For each pair* (p, q) *with* $1 < p < (n+2)/(n-2) < q < \infty$, *there exists an* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0)$, $(1.1)'$ *has at least three positive radial solutions for* $f(t) = t^p + \varepsilon t^q$ *and* $\Omega = \{x \in \mathbb{R}^n \mid 0 < a < |x| < 1\}$, *for all a sufficiently small.*

In case $n = 2$, we have obtained similar nonuniqueness results for both of the cases when Ω is a ball and when Ω is an annulus (see Section 4). It is perhaps worthwhile to make a few comments about how Theorem 1.10 is proved. It is well known that $(1.1)'$ has no positive solution if $f(u) = u^p$, $p \geqq (n+2)/(n-2)$, and Ω is a ball in $\mathbb{R}^n$, $n \geqq 3$. However, it is an interesting and curious fact proved by P. R. Rabinowitz [21] that $(1.1)'$ *does have a positive solution in a ball if* $f(t) = t^p + \varepsilon t^q$, *where* ε, p, q *are given in Theorem* 1.10. It is this fact which plays an essential role in our proof of Theorem 1.10. In this way, the Sobolev cut-off exponent $(n+2)/(n-2)$, or the "Pohozaev phenomenon" enters the uniqueness question also! Curiously, we shall see that if we replace the outer radius of Ω in Theorem 1.10 by b and let b tend to ∞, *we obtain uniqueness again* (for each fixed $a > 0$)!

As for the proofs of Theorems 1.4, 1.6, 1.7 and 1.8, we essentially follow the approach used in [17], i.e., some comparison identities first used by Kolodner [14], and later by Coffman [5], [6].

The proof of Theorem 1.2 is special, we transform it into an autonomous equation, then apply Sturm's comparison and separation theorems. This line of approach has also been used before in the study of the Emden–Fowler equation.

A common preliminary step of all our proofs will be to bring the problem (1.1) (or $(1.1)'$) into more desirable form by making the following kind of change of variables:

$$\theta(r)w(s) = u(r) \quad \text{and} \quad s = \rho(r). \tag{1.11}$$

Here $r = |x|$, and θ and ρ are sufficiently smooth and must be properly chosen under different circumstances.

In Section 2, we prove Theorems 1.4, 1.6, 1.7 and 1.8, the corresponding results for $n = 2$ and their extensions. In Section 3, we prove Theorem 1.2 and its generalizations. In Section 4, we give the proofs of Theorem 1.10 and various related results described above. Some concluding remarks are included in Section 5. And, in the appendix, we prove a version of an oscillation-type result generalizing Theorem 2.5 in [17] for the case $1 < p < (n+2)/(n-2)$.

2. The Basic Uniqueness Theorems

We first present an extension of Theorem 3.8 in [17]; this generalization will be the fundamental tool in our proofs in this section. In certain singular situations (see subsections 2.3 and 2.4), Theorem 2.4 below will not be directly applicable; however, the ideas of the proof of Theorem 2.4 will still apply.

Consider the initial value problem

$$
\begin{aligned}
& w''+g(w,s)=0 \quad \text{for} \quad s\geqq \alpha, \\
& w(\alpha)=0, \qquad w'(\alpha)=d>0,
\end{aligned}
\tag{2.1}
$$

where $\alpha \in \mathbb{R}$ and $'$ denotes differentiation with respect to s. If we assume that $g:\mathbb{R}\times[\alpha,\infty)\to\mathbb{R}$ is C^1, then for each $d>0$ there is a unique solution $w(s,d)$ of (2.1) which is defined on a maximal s-interval $J_d \subset [\alpha,\infty)$. The function $w(s,d)$ has the property that $w(s,d)$ is a C^1 function of d and $\varphi \equiv \partial w/\partial d$ satisfies

$$
\begin{aligned}
& \varphi''+g_w(w,s)\varphi=0 \quad \text{for} \quad s\geqq \alpha, \\
& \varphi(\alpha)=0, \qquad \varphi'(\alpha)=1.
\end{aligned}
\tag{2.2}
$$

If we assume that $g(0,s)\equiv 0$ for $s\geqq\alpha$, then the existence and uniqueness theorem for initial value problems for ordinary differential equations implies that if $w(s,d)=0$ for some $s>\alpha$, then $w'(s)\neq 0$. For each $d>0$, define $z(d)$ to be the first $s>\alpha$ such that $w(s,d)=0$, if such an s exists. Thus the domain D of $d\to z(s)$ is the set of d for which such an s exists. The fact that $w'(z(d),d)<0$ if $z(d)$ is defined and the fact that w is C^3 in s and C^1 in d imply that D is open and $d\to z(d)$ is continuous on D. Furthermore, if $d\notin D$, $w(s,d)>0$ for all $s>\alpha$, and if $d_n\in D$ and $d_n\to d$ as $n\to\infty$, then $z(d_n)\to+\infty$. The fact that $d\to z(d)$ may not be defined for all d requires caution in some of our proofs.

THEOREM 2.4. *Let $\alpha\in\mathbb{R}$ and suppose that $g:\mathbb{R}\times[\alpha,\infty)\to\mathbb{R}$ is a C^1 map such that $g(0,s)\equiv 0$ for $s\geqq\alpha$. Suppose that if w is any solution of the boundary value problem*

$$
\begin{aligned}
& w''(s)+g(w,s)=0, \\
& w(\alpha)=w(\beta)=0, \qquad \alpha<\beta,
\end{aligned}
\tag{2.3}
$$

such that $w(s)>0$ for $\alpha<s<\beta$, then $\max_{\alpha\leqq s\leqq\beta} w(s)\leqq M$ (where we allow $M=\infty$). Finally assume that g satisfies the following conditions:

$$wg_w>g \quad \text{for} \quad 0<w\leqq M \quad \text{and} \quad \alpha\leqq s\leqq\beta, \tag{2.5}$$

$$g_s\equiv\frac{\partial g}{\partial s}\leqq 0 \quad \text{for} \quad 0<w\leqq M \quad \text{and} \quad \alpha\leqq s\leqq\beta, \tag{2.6}$$

$$(s-\alpha)g_s+2g\geqq 0 \quad \text{for} \quad 0<w\leqq M \quad \text{and} \quad \alpha\leqq s\leqq\beta. \tag{2.7}$$

Then equation (2.3) *has at most one positive solution.*

Proof: The proof is similar to that of Theorem 3.8 in [17].

If $z(d)$ is as defined before, we know that $w'(z(d),d)<0$, so the implicit function theorem implies that $z'(d)$ is defined and continuous. Suppose we can prove that whenever $z(d)=\beta$ for some $d>0$, then $z'(d)<0$. We claim then that (2.3) has at most one positive solution. If not, there are positive reals $d_1<d_2$ such that $z(d_1)=z(d_2)=\beta$. Suppose first that $z(d)$ is defined for all $d \geqq d_1$, and let d_3 be the first value of $d>d_1$ such that $z(d)=\beta$; d_3 exists because of our assumption that $z(d_2)=\beta$. But then we have a contradiction: by our construction we have $z'(d_3)\geqq 0$, while we assume that $z'(d_3)<0$. Thus there must exist a value $d_4>d_1$ such that $w(s, d_4)>0$ for all $s>\alpha$; in fact, given the existence of d_4, we can define d_4 by

$$d_4=\sup\{\delta>d_1:[d_1, \delta]\subset D=\text{the domain of } z(d)\}<\infty.$$

With this choice of d_4 one still has $d_4 \notin D$, but also

$$\lim_{d\to d_4^-} z(d)=\infty.$$

Since we know that $z(d)<\beta$ for d near d_1 and $d>d_1$, there must exist a value of d such that $d_1<d<d_4$ and $z(d)=\beta$; and if we define d_5 to be the smallest such value of d, we again obtain a contradiction as above.

Thus it suffices to prove that $z'(d_0)<0$ if $z(d_0)=\beta$. Differentiating the relationship $w(z(d), d)\equiv 0$ with respect to d, we find that, if $z(d_0)=\beta$,

$$z'(d_0)=-\varphi(\beta, d_0)/w'(\beta, d_0);$$

thus it suffices to prove $\varphi(\beta, d_0)<0$.

For convenience assume $d=d_0$, where $z(d_0)=\beta$; we shall write $\varphi(s)$ and $w(s)$ instead of $\varphi(s, d)$ and $w(s, d)$. Notice that (2.6) and (2.7) imply that $g(w, s)\geqq 0$ for $0\leqq w\leqq M$ and $\alpha\leqq s\leqq\beta$ and that (2.5) then implies that $g(w, s)>0$ for $0<w\leqq M$ and $\alpha\leqq s\leqq\beta$. We shall use this below. By (2.5) and the Sturm comparison theorem, φ must vanish in (α, β). Let ζ_0 be the first zero of φ in (α, β). We claim that $\varphi(s)<0$ for $\zeta_0<s\leqq\beta$.

Define $x(s)=(s-\alpha)w'(s, d_0)$ and $y(s)=w'(s, d_0)$. It is easy to verify that the Wronskians of y and φ and of x and φ satisfy the following identities (which are similar to identities used in [14], [5] and [6]):

$$\frac{d}{ds}(y\varphi'-y'\varphi)=g_s\varphi, \tag{2.8}$$

$$\frac{d}{ds}(x'\varphi-\varphi'x)=-\varphi[(s-\alpha)g_s+2g]. \tag{2.9}$$

Since $g>0$ for $w>0$ and $\alpha\leqq s\leqq\beta$, w is concave in (α, β). Therefore there exists $\gamma\in(\alpha, \beta)$ such that $w'(s)>0$ for $\alpha\leqq s<\gamma$ and $w'(s)<0$ for $\gamma<s\leqq\beta$. Integrating (2.9) from α to ζ_0 we obtain

$$-(\zeta_0-\alpha)w'(\zeta_0)\varphi'(\zeta_0)\leqq 0.$$

The uniqueness theorem for initial value problems implies that $\varphi'(\zeta_0)<0$, so we conclude that

$$w'(\zeta_0)\leqq 0$$

and

$$\zeta_0 \geqq \gamma.$$

Next suppose that φ vanishes at $\zeta_1 \in (\zeta_0, \beta]$ with $\varphi<0$ in (ζ_0, ζ_1). Then, integrating (2.8) from ζ_0 to ζ_1, we obtain

$$w'(\zeta_1)\varphi'(\zeta_1)\geqq w'(\zeta_0)\varphi'(\zeta_0)\geqq 0.$$

Because $\zeta_1>\zeta_0\geqq\gamma$, we have $w'(\zeta_1)<0$ and, of course, $\varphi'(\zeta_1)>0$; thus the previous inequality gives a contradiction. Therefore we have $\varphi<0$ in $(\zeta_0, \beta]$, and the theorem is proved.

We divide the rest of this section into four subsections, subsection 2.1–2.4. In subsection 2.1, we prove Theorems 1.4, 1.7 and 1.8 in the case Ω is an annulus in $\mathbb{R}^n$, $n\geqq 3$. In subsection 2.2, we establish corresponding results in the case Ω is an annulus in $\mathbb{R}^2$. In subsection 2.3, we prove Theorem 1.6 and in subsection 2.4, we establish the corresponding result in the case Ω is a ball in $\mathbb{R}^2$. In all cases we shall transform our differential equation to one of the form (2.1) and then either use Theorem 2.4 or, if the transformed equation is singular at $s=\alpha$, as in subsections 2.3 and 2.4, use the ideas of the proof of Theorem 2.4. In this section we shall take the constant M in Theorem 2.4 to be $M=\infty$, but we shall need the more general case for an application in Section 4.

2.1. Observe that Theorem 1.4 is a special case of Theorem 1.8. We shall present a more general form of Theorem 1.8 by allowing $r(=|x|)$ dependence of f, i.e., $f=f(u, r)$. Using the idea introduced in (1.11), we set

$$u(r)=\theta(r)w(s),$$

and

$$s=\rho(r),$$

where $\theta(r)=r^{2-n}$, $\rho(r)=r^{n-2}$. Then the equation in (1.1),

$$u_{rr}+\frac{n-1}{r}u_r+f(u, r)=0,$$

is transformed into the following:

$$w_{ss}+\frac{1}{(n-2)^2 s^{1-2/(n-2)}}f\left(\frac{w}{s}, s^{1/(n-2)}\right)=0, \tag{2.10}$$

and the boundaries $r=a$ and $r=b$ are transformed into $\alpha=a^{n-2}$ and $\beta=b^{n-2}$, respectively. Suppose w is a solution of (2.10) with initial values $w(\alpha)=0$ and

$w_s(\alpha)=d>0$, and let $z=z(d)$ be the first zero of w in (α,∞) (if it exists). We examine what conditions on f correspond to (2.5), (2.6) and (2.7).

In the present setting we have

$$g(w,s)=\frac{1}{(n-2)^2}s^{-1+2/(n-2)}f\left(\frac{w}{s},s^{1/(n-2)}\right). \tag{2.11}$$

Theorem 2.4 shows that uniqueness of positive solutions of (2.3) will follow if $g(w,s)$ satisfies (2.5)-(2.7) for $w>0$ and $\alpha\leqq s\leqq\beta$, where $\alpha=a^{n-2}$ and $\beta=b^{n-2}$ and $\Omega=\{x\,|\,a<|x|<b\}$. Straightforward computation shows that (2.5) holds if and only if

$$f(u,r)-uf_u(u,r)<0 \quad \text{for} \quad u>0 \quad \text{and} \quad a\leqq r\leqq b. \tag{2.12}$$

Inequality (2.6) holds if and only if

$$f(u,r)-uf(u,r)-2\left(\frac{n-3}{n-2}\right)f(u,r)+\left(\frac{1}{n-2}\right)rf_r(u,r)\leqq 0 \tag{2.13}$$

for $u>0$ and $a\leqq r\leqq b$. Inequality (2.7) holds if and only if

$$\left[\frac{n}{n-2}+\frac{2a^{n-2}}{r^{n-2}-a^{n-2}}\right]f(u,r)-uf_u(u,r)+\left(\frac{1}{n-2}\right)rf_r(u,r)\geqq 0 \tag{2.14}$$

for $u>0$ and $a\leqq r\leqq b$, and (2.7) will certainly be true for g if we strengthen (2.14) to

$$\left[\frac{n}{n-2}+\frac{2a^{n-2}}{b^{n-2}-a^{n-2}}\right]f(u,r)-uf_u(u,r)+\left(\frac{1}{n-2}\right)rf_r(u,r)\geqq 0 \tag{2.14$'$}$$

for $u>0$ and $a\leqq r\leqq b$. (Note that (2.12), (2.13) and (2.14)$'$ imply that $f(u,r)>0$ for $u>0$ and $a\leqq r\leqq b$, so (2.14)$'$ is a strengthening of (2.14).)

We thus have obtained the following generalization of Theorem 1.8:

THEOREM 2.15. *Assume that $n\geqq 3$ and that $\Omega=\{x\in\mathbb{R}^n: 0<a<|x|<b\}$. If $f:[0,\infty)\times[a,b]\to[0,\infty)$ is a C^1 function which satisfies* (2.12), (2.13), *and* (2.14)$'$, *problem* (1.1) *has at most one positive, radial solution.*

We now prove Theorem 1.7 in the case $n\geqq 3$. Again we shall allow f's which may have an r dependence. Consider (1.1), in which $\Omega=\{x\in\mathbb{R}^n: 0<a<|x|<b\}$ and introduce the change of variables

$$U(s)=u(r), \qquad s=1/r^{n-2}, \tag{2.16}$$

(notice that this change of variables reverses orientation). Problem (1.1) now takes the form

$$\begin{gathered} U_{ss}+\frac{1}{(n-2)^2}s^{-2-2/(n-2)}f(U,s^{-1/(n-2)})=0,\\ U(\alpha)=U(\beta)=0, \end{gathered} \tag{2.17}$$

where $\alpha = 1/b^{n-2}$ and $\beta = 1/a^{n-2}$. As in the proof of Theorem 2.15, we apply Theorem 2.4 to conclude that (2.17) has a unique positive solution if the following conditions hold for $u > 0$, $\alpha \leqq s \leqq \beta$ (or, $a \leqq r \leqq b$):

$$uf_u(u, r) > f(u, r), \tag{2.18}$$

$$2(n-1)f(u, r) + rf_r(u, r) \geqq 0, \tag{2.19}$$

$$2((n-1)\alpha - s)f(u, r) \geqq (s-\alpha)rf_r(u, r). \tag{2.20}$$

Conditions (2.18)-(2.20) imply that the corresponding function $g(w, s)$ satisfies (2.5)-(2.7); hence $g(w, s) > 0$ for $w > 0$ and $f(u, r) > 0$ for $u > 0$.

The above calculations yield

THEOREM 2.21. *Assume that* $n \geqq 3$, $\Omega = \{x \in \mathbb{R}^n \mid a < |x| < b\}$ and $f: [0, \infty) \times [a, b] \to [0, \infty)$ *is a* C^1 *map. Problem* (1.1) *has at most one positive, radial solution in* Ω *if* f *satisfies* (2.18), (2.19) *and* (2.20)′ *for* $u > 0$ *and* $a \leqq r \leqq b$, *where* (2.20)′ *is the condition*

$$2\left(\frac{n-1}{b^{n-2}} - \frac{1}{a^{n-2}}\right)f(u, r) \geqq \left(\frac{1}{r^{n-2}} - \frac{1}{b^{n-2}}\right)rf_r(u, r). \tag{2.20$'$}$$

Theorem 1.7 follows immediately from the above theorem if $f_r \equiv 0$.

Remark 2.22. In conditions (1.5), (1.9), (2.5), (2.12) and (2.18), the strict inequalities may be relaxed somewhat. We shall not pursue this further here, however (see, for example, Remark 3.14 in [17]).

2.2. We first take up the proof of Theorem 1.7 in the case $n = 2$. As before, we make a transformation of the form given by (1.11). Set

$$U(s) = u(r) \quad \text{and} \quad s = -\log r \qquad \text{(i.e., } r = e^{-s}\text{)}.$$

Then it is straightforward to verify that (1.1) with $\Omega = \{x \in \mathbb{R}^2 \mid 0 < a < |x| < b\}$ is transformed into

$$\begin{aligned} &U_{ss} + e^{-2s}f(U, e^{-s}) = 0, \\ &U(\alpha) = U(\beta) = 0, \end{aligned} \tag{2.23}$$

where $\alpha = -\log b$ and $\beta = -\log a$. Now, applying Theorem 2.4 to (2.23) in the same way as we did in subsection 2.1, we have

THEOREM 2.24. *Problem* (1.1) *has at most one positive radial solution in* $\Omega = \{x \in \mathbb{R}^2 \mid 0 < a < |x| < b\}$ *if the following conditions hold for* $u > 0$, $a \leqq r \leqq b$:

$$uf_u(u, r) > f(u, r), \tag{2.24}$$

$$2f(u, r) + rf_r(u, r) \geqq 0, \tag{2.25}$$

$$(2.26)\qquad 2f(u,r)\left[1-\log\frac{b}{a}\right]\geqq \log\frac{b}{r}\cdot rf_r(u,r).$$

In case $f_r \equiv 0$, we have $f>0$ in $u>0$ and (2.26) is satisfied if $b/a \leqq e$. We leave the detailed computations to the reader.

We now take up the question of extending Theorems 1.4 and 1.8 to the $\mathbb{R}^2$ case. Unfortunately, this extension is not straightforward, and the results we obtain are not as elegant as one might hope. The condition we obtain replacing (1.5) or (1.9) is rather peculiar and we have no explanation for it. Again, we begin with the change of variables (1.11). Define

$$u(r)=r^{-\delta/2}w(s),$$

where

$$s=r^{\delta},$$

and $\delta>0$ is to be chosen later. Equation (1.11) is transformed to

$$(2.27)\qquad w_{ss}+\frac{1}{s^2}w+\frac{1}{\delta^2}s^{(4+\delta)/2\delta-2}f\left(\frac{w}{s^{1/2}}, s^{1/\delta}\right)=0,$$

with boundary conditions $w(\alpha)=w(\beta)=0$ where $\alpha=a^{\delta}$, $\beta=b^{\delta}$. We wish to apply Theorem 2.4 to equation (2.27). Some tedious calculations show that (2.5) is equivalent to

$$(2.28)\qquad f(u,r)-uf_u(u,r)<0 \quad \text{for} \quad u>0 \quad \text{and} \quad a\leqq r\leqq b.$$

Inequality (2.6) holds if we have, for $u>0$ and $a\leqq r\leqq b$,

$$(2.29)\qquad \frac{4-3\delta}{\delta}f(u,r)-uf_u(u,r)+\frac{2}{\delta}rf_r(u,r)\geqq 0.$$

And (2.7) is implied by

$$(2.30)\qquad \left[\frac{4+\delta}{\delta}+\frac{4a^{\delta}}{r^{\delta}-a^{\delta}}\right]f(u,r)-uf_u(u,r)+\frac{2}{\delta}rf_r(u,r)\geqq 0,$$

for $u>0$ and $a\leqq r\leqq b$. We may replace (2.30) by

$$(2.30)'\qquad \left[\frac{4+\delta}{\delta}+\frac{4a^{\delta}}{b^{\delta}-a^{\delta}}\right]f(u,r)-uf_u(u,r)+\frac{2}{\delta}rf_r(u,r)\geqq 0.$$

Notice that (2.29) and (2.30)′ imply that $f\geqq 0$ for $u>0$ and $a\leqq r\leqq b$, and (2.28) then implies that the inequality is strict. Thus (2.30)′ is a strengthening of (2.30). Arguing as in the proof of Theorem 2.15, we obtain.

THEOREM 2.31. *Assume that* $f:[0,\infty)\times[a,b]\to[0,\infty)$ *is* C^1. *Problem* (1.1) *has at most one positive radial solution in* $\Omega=\{x\in\mathbb{R}^2 \mid 0<a<|x|<b\}$ *if conditions* (2.28), (2.29) *and* (2.30)′ *are satisfied in* $u>0$, $a\leqq r\leqq b$, *for some* $\delta>0$.

To make this result a little more transparent, we look at two special cases.

COROLLARY 2.32. *Problem* (1.1)′ *has at most one positive radial solution in* $\Omega=\{x\in\mathbb{R}^2 \mid 0<a<|x|<b\}$ *if there exists* $\delta>0$ *such that the following condition holds:*

$$\left(\frac{4+\delta}{\delta}+\frac{4a^\delta}{b^\delta-a^\delta}\right)f(u)\geqq uf_u(u)>\max\left(1,\frac{4-3\delta}{\delta}\right)f(u)>0 \tag{2.33}$$

for $u>0$ *and* $a<r<b$.

As another example, we set $f(u,r)=r^l(u^p+u^q)$, where $l\in\mathbb{R}$ and $q\geqq p>1$ are two real numbers. Then we have the uniqueness of the positive radial solutions of (1.1) in $\Omega=\{x \mid 0<a<|x|<b\}$ if, for some $\delta>0$,

$$\frac{4+\delta+2l}{\delta}+\frac{4a^\delta}{b^\delta-a^\delta}\geqq q\geqq p\geqq\frac{4-3\delta+2l}{\delta}.$$

In particular, if for some $\delta>0$

$$\frac{4+\delta+2l}{\delta}\geqq q\geqq p\geqq\frac{4-3\delta+2l}{\delta}$$

and $q\geqq p>1$, then (1.1) has at most one positive radial soluton for every annulus in $\mathbb{R}^2$. It is easy to see that there exists $\delta>0$ such that the preceding inequality is satisfied if and only if $l>-2$ and $q\leqq p+4$.

2.3. The purpose of this subsection is to prove an extension of Theorem 1.6. However, we begin with some general remarks about radial solutions of (1.1). Consider the equation

$$\begin{aligned}&(r^{n-1}u_r(r))_r+r^{n-1}f(u(r),r)=0, && r>0,\\ &u(0)=d>0, \qquad u_r(0)=0.\end{aligned} \tag{2.34}$$

PROPOSITION 2.35. *Assume that* $n\geqq 1$ *and that* $f:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ *is* C^1. *For each* $d>0$ *there exists a unique* C^2 *solution* $u(r)=u(r,d)$ *of* (2.34). *The function* $u(r)$ *is defined on a maximal interval* $[0,r_d)$ *and* $\mathcal{O}=\{(r,d): d>0,\ 0\leqq r<r_d\}$ *is open in* $[0,\infty)\times(0,\infty)$. *The function* u *is* C^2 *in the* r *variable and* C^1 *in the* d *variable.*

Proof: Of course "uniqueness" of the solution means that if u_1 and u_2 are solutions of (2.34) for the same d and u_i is defined on $[0,\rho_i)$, then u_1 and u_2 agree on their common domain. Given this, one easily obtains a solution defined on a maximal interval.

Suppose we can prove that given $d_0>0$ there exist positive numbers ε and ε_1 such that for $|d-d_0|<\varepsilon_1$ there exists a unique solution $u(r,d)$ of (2.34) defined for $0\leqq r\leqq\varepsilon$ and that $u(r,d)$ is C^2 in r and C^1 in d. The rest of the proposition will then follow from standard forms of the existence and uniqueness theorem for the initial value problem for ordinary differential equations.

Thus select $d_0 > 0$ and for $\varepsilon > 0$ (ε will be specified later) define a Banach space X by

$$X = \{(u, v) \mid u, v : [0, \varepsilon] \to \mathbb{R} \text{ are continuous and } v(0) = 0\}.$$

The norm is given by

$$\|(u, v)\| = \max\left(\max_{0 \leqq t \leqq \varepsilon} |u(t)|, \max_{0 \leqq t \leqq \varepsilon} |v(t)|\right).$$

If $x = (u, v) \in X$ and $d > 0$, define $\Phi(u, v, d) \in X$ by

$$\Phi(u, v, d)(r) = \left(d + \int_0^r v(\rho)\, d\rho, \left(\frac{1}{r^{n-1}}\right) \int_0^r \rho^{n-1} f(u(\rho), \rho)\, d\rho \right).$$

We want a fixed point of $(u, v) \to \Phi(u, v, d)$. Select $B > 0$ and $\varepsilon_0 > 0$ and set

$$M = \sup\{|f(u, \rho)|, |f_u(u, \rho)| : |u - d_0| \leqq 2B, 0 \leqq \rho \leqq \varepsilon_0\}.$$

Define a closed, convex bounded set $C_d \subseteq X$ by

$$C_d = \{(u, v) \in X \mid \max_{0 \leqq t \leqq \varepsilon} |u(t) - d| \leqq B, \max_{0 \leqq t \leqq \varepsilon} |v(t)| \leqq B\}.$$

If $\varepsilon \leqq \varepsilon_0$, $|d - d_0| \leqq B$ and $(u, v) \in C_d$, then we find

$$\|\Phi(u, v, d) - (d, 0)\| \leqq \max\left(\max_{0 \leqq r \leqq \varepsilon} \int_0^r |v(\rho)|\, d\rho, \max_{0 \leqq r \leqq \varepsilon} \frac{1}{r^{n-1}} \int_0^r \rho^{n-1} |f(u(\rho), \rho)|\, d\rho \right).$$

The right-hand side of the previous inequality is dominated by

$$\max(B\varepsilon, M\varepsilon/n);$$

thus if $\Phi_d(u, v) \equiv \Phi(u, v, d)$, $\Phi_d(C_d) \subset C_d$ for $\varepsilon > 0$ and small and $|d - d_0| \leqq B$.

Next notice that if $x_j = (u_j, v_j) \in C_d$ for $j = 1, 2$ and $\|x_1 - x_2\| = \delta$, then

$$\|\Phi_d(x_1) - \Phi_d(x_2)\| \leqq \max_{0 \leqq r \leqq \varepsilon} \int_0^r |v_1(\rho) - v_2(\rho)|\, d\rho, \tag{2.36}$$

or

$$\|\Phi_d(x_1) - \Phi_d(x_2)\| \leqq \max_{0 \leqq r \leqq \varepsilon} \frac{1}{r^{n-1}} \int_0^r \rho^{n-1} |f(u_1(\rho), \rho) - f(u_2(\rho), \rho)|\, d\rho. \tag{2.37}$$

The right-hand side of (2.36) is dominated by $\delta\varepsilon$, and the right-hand of (2.37) is dominated by $M\delta\varepsilon/n$. It follows that, for $\varepsilon > 0$ small enough and $|d - d_0| \leqq B$, Φ_d is a Lipschitz map of C_d into itself with Lipschitz constant less than one. Thus the contraction mapping principle implies Φ_d has a unique fixed (u, v) in C_d.

It follows immediately that if $\Phi_d(u, v) = (u, v)$, then u is C^1 on $[0, \varepsilon]$ and $r^{n-1}u_r(r)$ is C^1 on $[0, \varepsilon]$ and u satisfies (2.34) on $[0, \varepsilon]$. With a little more care one can prove (we leave this to the reader) that u is C^2 on $[0, \varepsilon]$. The uniqueness of a solution u of (2.34) on some small interval $[0, \varepsilon]$ follows from the contraction mapping principle applied to Φ_d (note that if $v(t) \equiv u_r(t)$, then (u, v) is a fixed point of Φ_d and (u, v) lies in C_d for ε small).

It remains to prove that u is C^1 in d. Define $\Psi(u, v, d)$ by

$$\Psi(u, v, d) = (u, v) - \Phi(u, v, d).$$

One can show that Ψ is continuously Frechet differentiable on $X \times (0, \infty)$. The Frechet derivative with respect to the x variable at a point $x = (u, v)$ is given by the linear map $(h_1, h_2) \to (g_1, g_2)$, where

$$g_1(t) \equiv h_1(t) - \int_0^t h_2(\rho)\, d\rho,$$

$$g_2(t) \equiv h_2(t) - \frac{1}{t^{n-1}} \int_0^t \rho^{n-1} f_u(u(\rho), \rho) h_1(\rho)\, d\rho.$$

One can easily show that (for fixed u) this linear map is one-to-one and onto for ε small. The implicit function theorem for Banach spaces now implies that if $\Psi(u_0, v_0, d_0) = 0$, then for $|d - d_0|$ small enough there is $(u_d, v_d) \in X$ with $(u_{d_0}, v_{d_0}) = (u_0, v_0)$, $d \to (u_d, v_d)$ a C^1 map and $\Psi(u_d, v_d, d) = 0$. One easily derives from this that $u(r, d) = u_d(r)$ and that $u(r, d)$ is C^1 in d, which proves Proposition 2.35.

Now consider the boundary value problem

$$\begin{aligned} &(r^{n-1}u_r(r))_r + r^{n-1}f(u(r), r) = 0, \\ &u_r(0) = 0, \qquad u(R) = 0. \end{aligned} \tag{2.38}$$

Let $u(r, d)$ denote the solution of (2.34), let $\theta(r, d) = \partial u/\partial d$, and let $z(d)$ denote the first positive r value such that $u(r, d) = 0$ (if $z(d)$ exists). To prove uniqueness of positive solutions of (2.38) it suffices, as before, to prove $z'(d) < 0$ whenever $0 < z(d) \leqq R$. Again, to prove $z'(d) < 0$ it suffices to prove that $\theta(z(d), d) < 0$.

As before we make a change of variables:

$$u(r, d) \equiv \frac{w(s, d)}{s} \quad \text{and} \quad s = r^{n-2}, \tag{2.39}$$

and we define $\varphi(s, d) = \partial w/\partial d$. A calculation gives

$$\begin{aligned} &w_{ss} + g(w, s) = 0 \quad \text{for} \quad 0 < s, \\ &w(0) = 0, \qquad w_s(0) = d, \end{aligned} \tag{2.40}$$

where $g(w, s)$ is singular at $s=0$ and is given by

$$g(w, s) \equiv \frac{1}{(n-2)^2 s^{(n-4)/(n-2)}} f\left(\frac{w}{s}, s^{1/(n-2)}\right). \tag{2.41}$$

Differentiating φ with respect to s shows that φ satisfies

$$\begin{gathered} \varphi_{ss} + g_w(w, s)\varphi = 0 \quad \text{for} \quad 0 < s, \\ \varphi(0) = 0, \qquad \varphi_s(0) = 1. \end{gathered} \tag{2.42}$$

If $c = z(d)^{n-2}$ it suffices to prove that $\varphi(c) < 0$. The proof is like that of Theorem 2.4, so we shall be sketchy. Inequalities (2.5)–(2.7) will be satisfied for our function g if f is C^1 and the following conditions are satisfied for $u > 0$ and $0 < r \leqq R$:

$$f(u, r) - uf_u(u, r) < 0, \tag{2.43}$$

$$-\frac{n-4}{n-2} f(u, r) - uf_u(u, r) + \frac{1}{n-2} rf_r(u, r) \leqq 0, \tag{2.44}$$

$$\frac{n}{n-2} f(u, r) - uf_u(u, r) + \frac{1}{n-2} rf_r(u, r) \geqq 0. \tag{2.45}$$

(Notice that when $f_r \equiv 0$, these conditions reduce to (1.5).) Of course w and φ are C^3 on $(0, c]$ in s, but the equation

$$sw'(s) = \frac{1}{n-2} su_r(r)r + w(s)$$

shows that w is C^1 on $[0, c]$ and also that $x(s, d) = sw'(s)$ is C^1 on $[0, c]$.

The function φ has a first zero $\zeta_0 \in (0, c)$. The Sturm comparison does not apply (g is singular), but if φ were positive on $(0, c)$ one would obtain

$$\int_\varepsilon^c \frac{d}{ds}(\varphi w' - w\varphi')\, ds = \int_\varepsilon^c \varphi[wg_w - g]\, ds. \tag{2.46}$$

Taking lim inf of both sides as $\varepsilon \to 0^+$, the right-hand side would be positive and the left-hand side nonpositive, a contradiction.

The function $x(s) = x(s, d)$ satisfies (2.9) for $0 < s \leqq c$. If one integrates (2.9) from ε to ζ_0 and then lets ε approach 0 (using that $x'(s)$ is continuous at $s = 0$), one obtains as in Theorem 2.4 that $\zeta_0 \geqq \gamma$, where w achieves its maximum on $[0, c]$ at γ.

Finally, the same argument as in Theorem 2.4 shows that $\varphi(s, d)$ is negative on $(\zeta_0, c]$.

Thus we have proved the following theorem:

THEOREM 2.47. *Assume that* $f:[0, \infty) \times [0, R] \to [0, \infty)$ *is* C^1 *and that* f *satisfies inequalities* (2.43), (2.44) *and* (2.45). *Then for* $n > 2$ *equation* (2.38) *has at* *most one C^2 solution which is positive on $[0, R)$, and for $n \geq 3$ and $\Omega = \{x \in \mathbb{R}^n \mid |x| < R\}$ problem (1.1) has at most one positive, radial solution.*

2.4. We consider the result corresponding to Theorem 2.47 in the case that Ω is a ball in $\mathbb{R}^2$. The discussion follows that in subsection 2.3, but there are some additional complications.

Let $u(r, d)$ be the solution of (2.34) for $n = 2$ and define $\theta(r, d) = \partial u/\partial d$. To prove uniqueness of positive solutions of (2.38), it suffices to prove that if $z(d)$ is defined as in subsection 2.3, thên $z'(d) < 0$.

Following subsection 2.2 we assume that $f(u, r)$ is C^1 and that there exists $\delta > 0$ such that, for all $u > 0$ and $0 < r \leqq R$,

$$f(u, r) - uf_u(u, r) < 0, \tag{2.48}$$

$$\frac{4-3\delta}{\delta} f(u, r) - uf_u(u, r) + \frac{2}{\delta} rf_r(u, r) \leqq 0, \tag{2.49}$$

$$\frac{4+\delta}{\delta} f(u, r) - uf_u(u, r) + \frac{2}{\delta} rf_r(u, r) \geqq 0. \tag{2.50}$$

As usual, (2.48)-(2.50) imply that $f(u, r) > 0$ for $u > 0$. If $f_r \equiv 0$, (2.48)-(2.50) will hold if, for all $u > 0$, $0 < r \leqq R$,

$$0 < f(u, r) < uf_u(u, r), \tag{2.51}$$

and if there exists $A > -3$ such that

$$Af(u, r) \leqq uf_u(u, r) \leqq (A+4)f(u, r). \tag{2.52}$$

If δ is as in (2.48)-(2.50), make the change of variables

$$u(r, d) = \frac{w(s, d)}{\sqrt{s}}, \qquad s = r^\delta, \tag{2.53}$$

so that w satisfies

$$w_{ss} + g(w, s) = 0, \qquad s > 0, \tag{2.54}$$

where $g(w, s)$ is given by

$$g(w, s) = \frac{1}{s^2} w + \frac{1}{\delta^2} s^{(4+\delta)/2\delta - 2} f\left(\frac{w}{\sqrt{s}}, s^{1/\delta}\right).$$

As in subsection 2.2 one can verify that if f satisfies (2.48)-(2.50), then g satisfies (2.5)-(2.7) for all $w > 0$ and for $0 < s \leqq R^\delta$. As before, one has that $\varphi = \partial w/\partial d$ satisfies

$$\varphi_{ss} + g_w(w, s)\varphi = 0, \qquad s > 0. \tag{2.55}$$

If $c \equiv (z(d))^{\delta}$, we must show that $\varphi(c) < 0$. The problem is that while φ and w are C^3 on $(0, c]$ and continuous on $[0, c)$ with $w(0, d) = 0$ and $\varphi(0, d) = 1$, their derivatives with respect to s are not continuous at $s = 0$. In fact, if $s = r^{\delta}$ we have (suppressing d in our notation)

$$
\begin{aligned}
w(s, d) &\equiv w(s) = r^{\delta/2}u(r, d) \equiv r^{\delta/2}u(r), \\
\varphi(s, d) &\equiv \varphi(s) = r^{\delta/2}\theta(r, d) \equiv r^{\delta/2}\theta(r),
\end{aligned}
\tag{2.56}
$$

and

$$
\begin{aligned}
\frac{dw}{ds} &= \frac{1}{2} r^{-\delta/2}u(r) + \frac{1}{\delta} r^{-\delta/2+1}u_r(r), \\
\frac{d\varphi}{ds} &= \frac{1}{2} r^{-\delta/2}\theta(r) + \frac{1}{\delta} r^{-\delta/2+1}\theta_r(r).
\end{aligned}
\tag{2.57}
$$

However, a calculation gives

$$
\varphi(s)w'(s) - w(s)\varphi'(s) = \frac{1}{\delta} r[\theta(r)u_r(r) - \theta_r(r)u(r)],
\tag{2.58}
$$

so that, even though $\varphi'(s)$ and $w'(s)$ are not continuous at $s = 0$, we still have

$$
\lim_{s \to 0^+} \varphi(s)w'(s) - w(s)\varphi'(s) = 0.
\tag{2.59}
$$

With the aid of (2.59) we can use the same argument as in the proof of Theorem 2.47 to conclude that φ has a zero ζ_0 with $0 < \zeta_0 < c$.

Since g satisfies (2.5)-(2.7), $g(w, s) > 0$ for $w > 0$, and there exists a unique point $\gamma \in (0, c)$ such that $w'(\gamma) = 0$. As in Theorem 2.4 we claim $\zeta_0 \geqq \gamma$. To see this define $x(s) = sw'(s)$. A calculation gives

$$
\begin{aligned}
x'(s)\varphi(s) - \varphi'(s)x(s) = {} & \frac{1}{\delta^2}(\tfrac{1}{2}\delta + 1) r u_r(r)\theta(r) + \frac{1}{\delta^2} r^2 u_{rr}(r)\theta(r) \\
& - \frac{1}{2\delta} r\theta_r(r)u(r) - \frac{1}{\delta^2} r^2 u_r(r)\theta_r(r),
\end{aligned}
\tag{2.60}
$$

so that

$$
\lim_{s \to 0^+} x'(s)\varphi(s) - \varphi'(s)x(s) = 0.
\tag{2.61}
$$

Using equation (2.9) we see that

$$
\int_{\varepsilon}^{\zeta_0} \frac{d}{ds}(x'(s)\varphi(s) - \varphi'(s)x(s))\, ds = -\int_{\varepsilon}^{\zeta_0} \varphi(s)[sg_s(w, s) + 2g(w, s)]\, ds.
\tag{2.62}
$$

Taking the lim inf as ε approaches 0^+ of both sides of (2.62) we find that

$$
-\zeta_0 w'(\zeta_0)\varphi'(\zeta_0) \leqq 0.
$$

Just as in Theorem 2.4 we conclude that $w'(\zeta_0) \leqq 0$ and $\zeta_0 \geqq \gamma$.

The proof that $\varphi(s)<0$ for $\zeta_0<s\leqq c$ is the same as in Theorem 2.4, because (2.54) is nonsingular away from $s=0$. Thus we have proved the following theorem:

THEOREM 2.63. *Let $\Omega=\{x\in\mathbb{R}^2:|x|<R\}$ and let $f:[0,\infty)\times[0,R]\to[0,\infty)$ be a C^1 map which satisfies* (2.48), (2.49) *and* (2.50) *for some $\delta>0$. Then equation* (1.1) *has at most one positive, radial solution.*

The following is a typical corollary of Theorem 2.63.

COROLLARY 2.64. *Assume that*

$$f(u)=\sum_{j=1}^{k} a_j u^{p_j},$$

where $1<p_1<p_2<\cdots<p_k\leqq p_1+4$ and $a_j>0$ for $1\leqq j\leqq k$. Then if Ω is a ball in $\mathbb{R}^2$, equation (1.1) *has exactly one positive, radial solution (existence is well known).*

It seems likely that the condition $p_k\leqq p_1+4$ in Corollary 2.64 is unnecessary.

3. A Special Case: $f(u,|x|)=|x|^l u^p$, $l\in\mathbb{R}$, $p>1$

In this section, we shall prove the following extension of Theorem 1.2.

THEOREM 3.1. *For each integer $k>0$, there exists at most one radial solution of the problem*

$$\begin{aligned}&\Delta u+|x|^l|u|^{p-1}u=0,\\ &u|_{\partial\Omega}=0,\end{aligned}\tag{3.2}$$

where $l\in\mathbb{R}$, $p>1$, $\Omega=\{x\in\mathbb{R}^n\,|\,0<a<|x|<b\}$, $n\geqq 2$, $u_r(a)>0$ and u has exactly $k-1$ zeros in (a,b).

Proof: Applying (1.11) with $\theta(r)=r^{\varepsilon}$, $s=\rho(r)=\log r$ to the initial value problem

$$\begin{aligned}&u_{rr}+\frac{n-1}{r}u_r+r^l|u|^{p-1}u=0,\\ &u(a)=0,\qquad u_r(a)>0,\end{aligned}\tag{3.2$'$}$$

we get

$$\begin{aligned}&w_{ss}+(n+2\varepsilon-2)w_s+\varepsilon(n+\varepsilon-2)w+r^{2-\varepsilon+\varepsilon p+l}|w|^{p-1}w=0,\\ &w(\alpha)=0,\qquad w_s(\alpha)=d>0,\end{aligned}$$

where $\alpha = \log a$. Choosing $\varepsilon \doteq -(l+2)/(p-1)$, the previous equation becomes

$$(3.3)\qquad \begin{gathered} w_{ss} + (n+2\varepsilon-2)w_s + \varepsilon(n+\varepsilon-2)w + |w|^{p-1}w = 0, \\ w(\alpha) = 0, \qquad w_s(\alpha) = d > 0. \end{gathered}$$

Set $\varphi \equiv \partial w/\partial d$; then φ satisfies

$$(3.4)\qquad \begin{gathered} \varphi_{ss} + (n+2\varepsilon-2)\varphi_s + \varepsilon(n+\varepsilon-2)\varphi + p|w|^{p-1}\varphi = 0, \\ \varphi(\alpha) = 0, \qquad \varphi_s(\alpha) = 1, \end{gathered}$$

and, if we set $v = w_s$, v satisfies

$$(3.5)\qquad \begin{gathered} v_{ss} + (n+2\varepsilon-2)v_s + \varepsilon(n+\varepsilon-2)v + p|w|^{p-1}v = 0, \\ v(\alpha) = d > 0, \qquad v_s(\alpha) = -(n+2\varepsilon-2)d. \end{gathered}$$

Note that φ and v satisfy the same differential equation, but the initial conditions in (3.4) and (3.5) insure that φ and v are linearly independent. Thus the Sturm separation theorem applies, and we conclude that the zeros of v and φ are interlacing. Next, we claim that v vanishes only once between two consecutive zeros of w. One first observes that if $v(\sigma) = w_s(\sigma) = 0$, then $w_{ss}(\sigma) \neq 0$. For if $w_{ss}(\sigma) = 0$, then the constant function $\tilde{w}(s) \equiv w(\sigma)$ satisfies the differential equation in (3.3) and $\tilde{w}'(\sigma) = w'(\sigma) = 0$ and $\tilde{w}(\sigma) = w(\sigma)$. Thus the uniqueness theorem for the initial value problem for ordinary differential equations implies that $w(s)$ is a constant, which contradicts the assumption that $w'(\alpha) = d > 0$. It follows that whenever $w_s(\sigma) = 0$ one has $w_{ss}(\sigma) \neq 0$. From this one sees that every critical point in $(0, \infty)$ of a solution w of (3.3) is isolated and is a local maximum or local minimum; furthermore, the critical points are alternately local maxima and local minima, beginning with a local maximum. Now suppose that v vanishes more than once between two consecutive zeros of w; by the above requirements w has a local maximum and a local minimum on the interval between its zeros. For definiteness assume that w is positive on the interval between its zeros and let ξ be the first point in the interval where w achieves a local minimum. Select points $\eta < \xi < \zeta$ such that $w(s) > w(\xi)$ for s in $(\eta, \xi) \cup (\xi, \zeta)$ and $w(\eta) = w(\xi) = w(\zeta)$. Observe that we must then have $w_s(\eta) > 0$ and $w_s(\zeta) \leqq 0$. Multiplying the equation in (3.3) by w_s we get

$$(3.6)\qquad \tfrac{1}{2}(w_s^2)_s + (n+2\varepsilon-2)w_s^2 + \varepsilon(n+\varepsilon-2)\tfrac{1}{2}(w^2)_s + \frac{1}{p+1}(|w|^{p+1})_s = 0.$$

Integrating (3.6) from η to ξ gives

$$-\tfrac{1}{2}w_s^2(\eta) + \int_\eta^\xi (n+2\varepsilon-2)w_s^2 = 0,$$

so we must have $(n+2\varepsilon-2)>0$. Integrating (3.6) from ξ to ζ, we obtain

$$\tfrac{1}{2}w_s^2(\zeta)+\int_\xi^\zeta (n+2\varepsilon-2)w_s^2=0,$$

which leads to a contradiction since $(n+2\varepsilon-2)>0$ and $w>w(\xi)$ in (ξ, ζ). This proves our assertion that $v=w_s$ vanishes exactly once between two consecutive zeros of w.

Let $z_k(d)$ be the k-th zero of w in (α, ∞), $y_k(d)$ be the k-th zero of φ in (α, ∞) and $x_k(d)$ be the k-th zero of v in (α, ∞); we have

$$x_1(d)<z_1(d)<x_2(d)<z_2(d)<\cdots<x_k(d)<z_k(d),$$

from the above discussion. By the interlacing property of the zeros of v and φ, we have $x_1(d)<y_1(d)$. By the Sturm comparison theorem, $y_1(d)<z_1(d)$. Therefore, we see that $x_1(d)<y_1(d)<z_1(d)$. Repeating this argument, we obtain

$$x_1(d)<y_1(d)<z_1(d)<x_2(d)<\cdots<z_{k-1}(d)<x_k(d)<y_k(d)<z_k(d).$$

Therefore, we have $(-1)^k\varphi(z_k)>0$. Differentiating $w(z_k(d), d)\equiv 0$ with respect to d, we find that

$$w_s(z_k(d), d)\frac{\partial z_k}{\partial d}+\varphi(z_k(d))\equiv 0.$$

Since $(-1)^k w_s(z_k(d), d)>0$, we conclude that $\partial z_k/\partial d<0$ for every k, i.e., $z_k(d)$ is strictly monotonically decreasing in $d>0$ (if $z_k(d)$ exists), and this fact proves our theorem.

We should remark that if, in case $n>2$, we set

$$\theta(r)=r^{2-n}, \qquad s=\rho(r)=r^{n-2},$$

then (3.2)′ becomes

$$\begin{aligned} &w_{ss}+\frac{1}{(n-2)^2}s^{(l+4-n)/(n-2)-p}|w|^{p-1}w=0, \\ &w(\alpha)=0, \qquad w_s(\alpha)=d>0, \end{aligned} \tag{3.7}$$

where $\alpha=a^{n-2}$. Denote by $z_k(d)$ again the k-th zero of w in (3.7); the conclusion that $\partial z_k/\partial d<0$ follows directly from a result of Coffman in case $p\geqq 3$. (Theorem 2.1 in [5].) Note that the proof of Theorem 3.1 above slightly improves Coffman's theorem to the case $p>1$, namely, we have the following.

THEOREM 3.8. *The two-point boundary value problem*

$$\begin{aligned} &y_{rr}+r^\nu|y|^{2n}y=0, \\ &y(a)=y(b)=0, \qquad 0<a<b<\infty, \end{aligned} \tag{3.8}$$

possesses, for each integer $k>0$, a unique solution with $k-1$ zeros on (a, b) with $y'(a)>0$ if $\nu\in\mathbb{R}$, $n>0$.

Proof: Set $y=x^m w$, where $m=-(\nu+2)/2n$ and $s=\log r$; then the equation in (3.8) becomes

$$w_{ss}+(2m-1)w_s+m(m-1)w+|w|^{2n}w=0.$$

We then proceed just as we did in the proof of Theorem 3.1 above.

4. Nonuniqueness and Related Results

In this section, we prove Theorem 1.10 and those results discussed in Section 1 following the statement of Theorem 1.10. As before, we first take up the case when Ω is an annulus in $\mathbb{R}^n$, $n\geqq 3$.

4.1. Let u be a positive, radial solution of

$$\begin{aligned} &\Delta u+f(u)=0 \quad \text{in} \quad \Omega, \\ &u|_{\partial\Omega}=0, \end{aligned} \tag{4.1}$$

where $\Omega=\{x\in\mathbb{R}^n \,|\, 0<a<|x|<b\}$, $n\geqq 3$. Applying (1.11) to (4.1) with

$$w(s)=u(r), \qquad s=\frac{1}{r^{n-2}},$$

where $r=|x|$, we obtain

$$\begin{aligned} &w_{ss}+\frac{1}{(n-2)^2}\frac{f(w)}{s^{2+2/(n-2)}}=0, \\ &w(\alpha)=0=w(\beta), \end{aligned} \tag{4.2}$$

where $\alpha=1/b^{n-2}$, $\beta=1/a^{n-2}$ (note that the transform $s=1/r^{n-2}$ reverses orientation). To analyze (4.2) it seems best to consider the corresponding initial value problem instead:

$$\begin{aligned} &w_{ss}+\frac{1}{(n-2)^2}\frac{f(w)}{s^{2+2/(n-2)}}=0, \\ &w(\alpha)=0, \qquad w_s(\alpha)=\xi>0, \end{aligned} \tag{4.3}$$

(this corresponds for u in (4.1) to "shooting backwards" from the outer boundary of Ω) and study the first zero $\beta=\beta(\xi)$ of the solution w.

We shall analyze a more general problem than (4.3), namely

$$\begin{aligned} &w_{ss}+h(w,s)=0, \\ &w(\alpha)=0, \qquad w_s(\alpha)=\xi>0, \end{aligned} \tag{4.4}$$

and denote the solution by $w(s;\xi)$. Set

$$A=\{\xi>0 \,|\, \text{there exists } \beta=\beta(\xi)>\alpha \text{ such that } w(\beta;\xi)=0 \text{ and } w>0 \text{ in } (\alpha,\beta)\}.$$

Then it is easy to see that if $h>0$ for $w>0$, $s \geqq \alpha$, we have

$$A=\{\xi>0 \mid \text{there exists } \gamma=\gamma(\xi)>\alpha \text{ such that } w_s(\gamma,\xi)=0 \\ \text{and } w_s>0 \text{ in } (\alpha,\gamma)\}. \tag{4.5}$$

We shall study the set A under the following hypotheses on h:

(h_1) $h(w,s)>0$ for $w>0$, $s \geqq \alpha$, and h is C^1;

(h_2) there exists a $\tau>0$ such that $h(w,s)/w^{1+\tau}$ is increasing in $w>0$ for every fixed s;

(h_3)

$$\limsup_{x\to\infty}\left(x\int_x^\infty \frac{1}{\sqrt{t}}h(c\sqrt{t},t)\,dt\right)=\infty \quad \text{for all} \quad c>0$$

and

$$\int_x^\infty h(1,s)\,ds<\infty.$$

LEMMA 4.6. *A is an open set.*

LEMMA 4.7. *Let (η,ζ) be a connected component of A and $\xi\in(\eta,\zeta)$. Then $\gamma(\xi)$, and thus $\beta(\xi)$, tends to ∞ as $\xi\to\eta$ (or, as $\xi\to\zeta$) provided $\eta\in(0,\infty)$ (or $\zeta\in(0,\infty)$, respectively).*

Both lemmas follow from the continuous dependence of solutions on initial values.

LEMMA 4.8. *A is unbounded if* (h_1) *and* (h_2) *hold.*

Proof: By a result of Nehari (see [16] p. 113, Theorem IV) we simply set

$$F(w^2,s)=\frac{h(w,s)}{w}$$

and observe that F satisfies the hypothesis in [16]; we conclude that A is nonempty. Moreover, Nehari's theorem implies that, for each $\gamma>\alpha$, the boundary value problem

$$w_{ss}+h(w,s)=0, \\ w(\alpha)=w_s(\gamma)=0, \tag{4.9}$$

has a solution $w>0$ in $(\alpha,\gamma]$. Let w be such a solution and denote $\xi=w_s(\alpha)$.

Claim: $\xi\to\infty$ as $\gamma\to\alpha$. For we have

$$\xi=w_s(\alpha)=-\int_\alpha^\gamma w_{ss}(s)\,ds=\int_\alpha^\gamma h(w(s),s)\,ds \\ \leqq \int_\alpha^\gamma h(\xi(s-\alpha),s)\,ds,$$

where we have used the facts that $h>0$, $h(w,s)$ is increasing in w, and $w(s)\leqq \xi(s-\alpha)$ (because w is concave). Suppose there is a sequence $\gamma_n\to\alpha$ for which a corresponding ξ_n remains bounded, say, by M. Then there exists a constant M_1 (independent of n) such that

$$\xi_n(\gamma_n-\alpha)\leqq M_1.$$

Because h is continuously differentiable and $h(0,s)=0$, there exists a constant C (independent of n) such that

$$h(w,s)\leqq Cw$$

for $0\leqq w\leqq M_1$ and $\alpha\leqq s\leqq\alpha+1$. It follows that for n large enough we have

$$\begin{aligned}\xi_n &\leqq \int_\alpha^{\gamma_n} h(\xi_n(\gamma_n-\alpha),s)\,ds\\ &\leqq \int_\alpha^{\gamma_n} C\xi_n(\gamma_n-\alpha)\,ds\leqq C(\gamma_n-\alpha)^2\xi_n.\end{aligned}\tag{4.10}$$

From (4.10) we conclude that for all n

$$1\leqq C(\gamma_n-\alpha)^2,$$

a contradiction.

REMARK 4.11. Actually, we can prove a much more general result than Lemma 4.8. Weaken (h_2) as follows:

$(h_2)'$ There exists a constant $\sigma>1$ such that

$$\lim_{w\to+\infty}\frac{h(w,s)}{w^\sigma}=+\infty$$

and the convergence is uniform for s in any compact interval in $[\alpha,\infty)$. Also one has

$$\lim_{w\to0^+}\frac{h(w,s)}{w}=0$$

and the convergence is uniform for s in any compact interval in $[\alpha,\infty)$.

We can prove the following generalization of Lemma 4.8.

LEMMA 4.8′. *Assume that h satisfies* (h_1) *and* $(h_2)'$. *Then A contains an unbounded interval* (ζ,∞).

Lemma 4.8 will be adequate for our purposes, and for reasons of length we shall not prove Lemma 4.8′ here.

LEMMA 4.12. $0\in\bar{A}$ *if* (h_1), (h_2) *and* (h_3) *hold.*

Proof: Define F as in the proof of Lemma 4.8; by the result of Nehari quoted there, we conclude that (4.9) has a positive solution (for each $\gamma > \alpha$). Note that (h_3) is equivalent to

$$x\int_x^\infty \frac{1}{\sqrt{t}} h(c\sqrt{t}, t)\, dt = x\int_x^\infty \frac{1}{\sqrt{t}} c\sqrt{t} F(c^2 t, t)\, dt$$
$$= x\int_x^\infty cF(c^2 t, t)\, dt = \infty$$

for all $c > 0$. Thus, by Theorem VIII, p. 118 in [16] (in fact, the necessary condition there), we have

$$\lambda(\alpha, \gamma) \to 0 \quad \text{as} \quad \gamma \to \infty,$$

where

$$\lambda(\alpha, \gamma) = \min_{y \in E} J(y) \tag{4.13}$$

and

$$J(y) = \int_\alpha^\gamma [y_s^2 - G(y^2, s)]\, ds,$$

$$G(z, s) = \int_0^z F(t, s)\, dt,$$

$$E = \left\{ y(\alpha) = 0, y \not\equiv 0 \text{ is continuous and piecewise } C^1, \text{ and } \int_\alpha^\gamma y_s^2 = \int_\alpha^\gamma y^2 F(y^2, s)\, ds \right\}.$$

From (h_2) we obtain

$$J(y) \geqq \frac{\tau}{2+\tau} \int_\alpha^\gamma y_s^2,$$

(see equation (22) on p. 111 of [16] and note that our τ corresponds to Nehari's 2ε). We conclude that

$$\int_\alpha^\gamma (w')^2\, ds \to 0 \tag{4.14}$$

and

$$\int_\alpha^\gamma w^2 F(w^2, s)\, ds = \int_\alpha^\gamma wh(w, s)\, ds \to 0 \tag{4.15}$$

as $\gamma \to \infty$, where $w \equiv w_\gamma$ is a minimizer in E of J (in (4.13)). The existence of w_γ is insured by a result of Nehari (Theorem IV, p. 113 in [16]); Nehari also shows that w_γ can be taken to satisfy (4.9).

Next, we show that $w_\gamma(x)\to 0$ as $\gamma\to\infty$ (for each fixed $x>\alpha$). To see this observe that for x fixed and $\gamma>x$

$$w_\gamma(x)=\int_\alpha^x w'_\gamma\,ds \leqq (x-\alpha)^{1/2}\left(\int_\alpha^\gamma (w'_\gamma)^2\,ds\right)^{1/2}\to 0 \tag{4.16}$$

as $\gamma\to\infty$. For convenience, we extend $w_\gamma\equiv 0$ outside $[\alpha,\gamma]$. Then

$$\begin{aligned}\xi_\gamma=w'_\gamma(\alpha)&=\int_\alpha^\gamma(-w''_\gamma)=\int_\alpha^\infty h(w_\gamma(s),s)\,ds\\ &=\left(\int_{T_\gamma}+\int_{S_\gamma}\right)(h(w_\gamma(s),s))\,ds,\end{aligned}$$

where $S_\gamma=\{s\geqq\alpha\,|\,w_\gamma(s)\geqq 1\}$, $T_\gamma=\{s\geqq\alpha\,|\,w_\gamma(s)<1\}$. Moreover,

$$\int_{S_\gamma} h(w_\gamma(s),s)\,ds\leqq\int_{S_\gamma} w_\gamma h(w_\gamma(s),s)\,ds\to 0$$

as $\gamma\to\infty$, by (4.15). For the other term,

$$\int_{T_\gamma} h(w_\gamma(s),s)\,ds=\int_\alpha^\infty h(w_\gamma(s),s)\chi_{T_\gamma}(s)\,ds\leqq\int_\alpha^\infty h(1,s)\,ds<\infty,$$

where χ_{T_γ} is the characteristic function of T_γ. Since $h(w_\gamma(s),s)\chi_{T_\gamma}(s)\leqq h(w_\gamma(s),s)\to 0$ pointwise (by (4.16)), Lebesgue dominated convergence theorem implies that

$$\int_{T_\gamma} h(w_\gamma(s),s)\,ds\to 0$$

as $\gamma\to\infty$. Thus, we have

$$\xi_\gamma=w'_\gamma(\alpha)\to 0 \quad\text{as}\quad \gamma\to\infty. \tag{4.17}$$

Since $\xi_\gamma\in A$, for all $\gamma>\alpha$, $0\in\bar{A}$.

We are now ready for the

Proof of Theorem 1.10: It is easy to check that if $f(w)=w^p+\varepsilon w^q$, where $\varepsilon>0$, $1<p<(n+2)/(n-2)<q<\infty$, then

$$h(w,s)\equiv\frac{f(w)}{(n-2)^2 s^{2+2/(n-2)}}$$

satisfies (h_1), (h_2) and ($\hat{h}_3$). Thus, the set A is unbounded and its closure contains 0. If we can show that $A\neq(0,\infty)$, then A is disconnected and Lemma 4.7 implies that for β sufficiently large, there exist two solutions of (4.4) with $\beta=\beta(\xi_1)=\beta(\xi_2)$ and $\xi_1\neq\xi_2\in A$. Thus, (4.2) has at least two positive solutions if β is sufficiently large. The existence of the third solution of (4.2) (again, for β large) follows from the proof of (4.17) in the proof of Lemma 4.12. Therefore, to complete the proof of Theorem 1.10, we only have to show that $A \neq (0, \infty)$ for this choice of f. This is essentially due to P. R. Rabinowitz [21]. Here we use *a priori* bounds and the fixed point index as in [9] in order to obtain a generalization of Rabinowitz's theorem.

PROPOSITION 4.18. *Let Ω be a bounded, smooth domain in $\mathbb{R}^n$, $n \geqq 2$, and $g: \mathbb{R}^+ \times \bar{\Omega} \to \mathbb{R}^+$ a locally Lipschitzian map. Consider the elliptic boundary value problem*

$$\Delta u + u^p + \varepsilon g(x, u) = 0, \qquad u|_{\partial\Omega} = 0. \tag{4.19}$$

If $1 < p < (n+2)/(n-2)$, there exists $\varepsilon_0 > 0$ such that, for $0 \leqq \varepsilon < \varepsilon_0$, (4.19) has a solution $u = u_\varepsilon$ which is positive on Ω.

Proof: For fixed δ, $0 < \delta < 1$, $-\Delta$ defines a one-to-one, bounded linear map from $\{u \in C^{2,\delta}(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ onto $C^\delta(\bar{\Omega})$; let $B_0: C^\delta(\bar{\Omega}) \to C^{2,\delta}(\bar{\Omega})$ denote the inverse of this map. It is known that B_0 has an extension to a continuous linear map $B: C(\bar{\Omega}) \to C^1(\bar{\Omega})$. Let K denote the cone of non-negative functions in $C(\bar{\Omega})$. We replace the problem of finding a positive solution u of (4.19) by the problem of finding a fixed point $u_\varepsilon \in K$, $u_\varepsilon \neq 0$, of

$$u = B(u^p + \varepsilon g(x, u)) \equiv \Phi_\varepsilon(u). \tag{4.20}$$

It is not hard to see that any such fixed point is actually in $C^{2,\delta}(\bar{\Omega})$ and (by the maximum principle) strictly positive on Ω. It is proved in [9] that there exist positive constants $m < M$ such that any nonzero solution $u \in K$ of

$$\Phi_0(u) = u$$

satisfies

$$m < \|u\| < M.$$

Furthermore it is proved in [9] that if $W = \{u \in K : m < \|u\| < M\}$, then $i_K(\Phi_0, W)$, the fixed point index of Φ_0 on W, satisfies

$$i_K(\Phi_0, W) = -1.$$

By the compactness of Φ_0, there exists $c > 0$ such that

$$\|\Phi_0(u) - u\| > c$$

whenever $u \in K$ and $\|u\| = M$ or $\|u\| = m$. If d is defined by

$$d = \max\{g(x, w) : x \in \bar{\Omega}, 0 \leqq w \leqq M\}$$

and $\|B\|$ denotes the norm of B considered as a map of $C(\bar{\Omega})$ into itself, then, for any $u \in \bar{W}$,

$$\|\varepsilon B(g(x, u(x)))\| \leqq \varepsilon \|B\| d.$$

It follows that if $\varepsilon_0 = c/\|B\|d$, then $\|\Phi_\varepsilon(u) - u\| > 0$ whenever $0 \leqq \varepsilon \leqq \varepsilon_0$, $u \in K$ and $\|u\| = M$ or $\|u\| = m$. The homotopy property for the fixed point index implies that

$$i_K(\Phi_\varepsilon, W) = -1$$

for $0 \leqq \varepsilon \leqq \varepsilon_0$, so Φ_ε has a fixed point in W for $0 \leqq \varepsilon \leqq \varepsilon_0$.

In case Ω is a ball with radius b and $g(x, u) = u^q$, the solution in Proposition 4.18 must be radial (by results in [10]). Then for $s = 1/r^{n-2}$, $w(s) = u(r)$ is a solution of (4.3) such that $w(s) \uparrow u(0) > 0$ as $s \to \infty$. This proves that $A \neq (0, \infty)$ and is disconnected.

Remark 4.21. Proposition 4.18 remains true if u^p is replaced by a function $f(u)$ for which *a priori* bounded theorems were obtained in [9], e.g., $f(u) = \sum_{j=1}^{k} a_j u^{p_j}$ with $a_j > 0$ and $1 < p_j < (n+2)/(n-2)$.

Actually, we only need Proposition 4.18 when Ω is a ball and when $g(x, u) = g(|x|, u)$. In this case it is not necessary to use [9] or [10]; simply use exactly the same proof as in Proposition 4.18, but replace K by K_r, the cone of non-negative, radial functions in $C(\bar{\Omega})$. This cone is mapped into itself by Φ_ε, and it is proved in [20] that there exist positive constants m and M such that every positive, radial solution of $\Phi_0(u) = u$ satisfies $m < \|u\| < M$ and $i_{K_r}(\Phi_0, W_r) = -1$, where $W_r = \{u \in K_r : m < \|u\| < M\}$. This proof relies on more elementary results than those in [9] or [10].

4.2. Now we take up the case of $\mathbb{R}^2$. Consider the following problem:

$$\Delta u + f(u) = 0, \quad u > 0 \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = 0, \tag{4.22}$$

where Ω is a ball or an annulus in $\mathbb{R}^2$. For conciseness, we shall assume $\Omega = \{x \in \mathbb{R}^2 \,|\, |x| < b\}$ or $\Omega = \{x \in \mathbb{R}^2 \,|\, 0 < a < |x| < b\}$ in this subsection, i.e., we always assume that the radius of the ball or the outer radius of the annulus under consideration are of the same length. As the reader will easily see, there is no loss of generality in doing so. The main results of this subsection are contained in the following theorem:

THEOREM 4.23. *With Ω given above, there exist an f and a b which have the following properties:*

(i) $f(0) = f'(0) = 0$ *and* $f \geqq 0$ *in* $\mathbb{R}^+$;
(ii) f *is smooth and convex*;
(iii) (4.22) *possesses at least two positive solutions if Ω is a ball of radius b*;
(iv) (4.22) *possesses at least three positive radial solutions if Ω is an annulus with sufficiently small inner radius $a > 0$ and outer radius b.*

To prove this, we again apply (1.11) with $w(s) = u(r)$, $s = -\log r$ and consider the corresponding initial value problem

$$(4.24) \qquad \begin{gathered} w_{ss} + e^{-2s} f(w) = 0, \\ w(\alpha) = 0, \qquad w_s(\alpha) = \xi > 0, \end{gathered}$$

where $\alpha = -\log b$. (Note that this change of variables again reverses the order of r and converts the original problem (4.22) into an exterior problem.) Similarly, set

$$A = \{\xi > 0 \mid \text{there exists } \gamma = \gamma(\xi) \text{ such that } w_s(\gamma(\xi), \xi) = 0 \text{ and } w_s > 0 \text{ in } (\alpha, \gamma)\}.$$

Suppose f satisfies

(f_1)
$$\sup_{t>0} \frac{f(t)\log f(t)}{tf'(t)} < \infty;$$

(f_2) there exists $\tau > 0$ such that $f(t)/t^{1+\tau}$ is increasing in $t > 0$;

(f_3)
$$\limsup_{x\to\infty} x \int_x^\infty e^{-t} t^{-1/2} f(ct^{1/2}) = \infty \quad \text{for all} \quad c > 0.$$

Then, we conclude from Hempel's work [11] that

(I) A is open and contains an unbounded component,

(II) there exists a sequence $\xi_n > 0$ such that $\gamma(\xi_n) \to \infty$ as $\xi_n \to 0$,

(III) if (4.24) has a solution $w(s, \eta) \uparrow \infty$ as $s \to \infty$ (with initial speed η), then (4.24) must have at least two solutions which tend to some finite limits at ∞.

(We should remark that Hempel's results also depend on the previously mentioned work of Nehari.)

From Section 6 of [11], one sees easily that there exists a function f satisfying (f_1), (f_2), (f_3) and properties (i) and (ii) (in Theorem 4.23). For such an f, equation (4.24) possesses a solution which goes to ∞ at $s = \infty$ for some initial point α (determined implicitly in such a construction for f). Then $b = e^{-\alpha}$ and from (III) above, we have two solutions of (4.24) increasing to some finite limits as $s \to \infty$. These solutions are transformed back (by $u(r) = w(s)$, $r = e^{-s}$) into two positive solutions of (4.22) with $\Omega = \{x \in \mathbb{R}^2 \mid |x| < b\}$. This settles part (iii) of Theorem 4.23. For part (iv) of Theorem 4.23, one uses part (iii) and (II) above to argue just as we did in subsection 4.1 (in the proof of Theorem 1.10); one concludes for small $a > 0$ (for which $\beta = -\log a$ is large) that there are at least three positive radial solutions of (4.22). One last remark: such an f constructed in Section 6 of [11] actually grows like $\exp\{\exp u\}$ at $u = \infty$; this of course lies outside the range of uniqueness for $\mathbb{R}^2$ (cf. Section 2).

4.3. In this subsection we shall show that if $f(z)=Az^p+Bz^q$, where $1<p<n/(n-2)$, $1<q$, and A and B are positive, then for fixed a and for b large enough equation (4.1) has a unique, positive, radial solution. (The existence of such a solution is well known and easy.) Thus we are studying (for $n\geqq 3$)

$$(4.25)\qquad \begin{aligned} &(r^{n-1}u_r(r))_r+r^{n-1}f(u(r))=0, \qquad a\leqq r\leqq b,\\ &u(a)=u(b)=0, \end{aligned}$$

where $0<a<b$ and where $f(z)$ is given by

$$(4.26)\qquad f(z)=Az^p+Bz^q.$$

We shall always assume that $A>0$, $B>0$ and $1<p$ and $1<q$, but for the moment we make no other assumptions.

Define constants α, β, γ, c and d by

$$(4.27)\qquad \begin{aligned} &\alpha=a^{n-2}, \quad \beta=b^{n-2}, \quad \gamma=\frac{\beta}{\alpha}, \quad c=\alpha\left(\frac{A}{B}\right)^{1/(q-p)},\\ &d=\left(\frac{1}{n-2}\right)^2 a^2A\left(\frac{A}{B}\right)^{(p-1)/(q-p)}. \end{aligned}$$

If we make the change of variables

$$w(s)=\frac{1}{c}r^{n-2}u(r), \qquad s=\left(\frac{r}{a}\right)^{n-2},$$

equation (4.25) is transformed to

$$(4.28)\qquad \begin{aligned} &w_{ss}+s^{2/(n-2)-1}h\left(\frac{w}{s}\right)=0,\\ &w(1)=w(\gamma)=0, \end{aligned}$$

where

$$h(z)=d(z^p+z^q).$$

The key to establishing the previously mentioned uniqueness turns out to be obtaining fairly explicit *a priori* bounds for any positive solution of (4.28) and proving that, if $1<p<n/(n-2)$, these bounds approach zero as $\gamma\to\infty$.

PROPOSITION 4.29. *Assume that w is a positive solution of* (4.28) *and that $M=\max_{1\leqq s\leqq\gamma}w(s)$. Define $\varepsilon=(\gamma-1)^{-1}$.* Then M satisfies

$$(4.30)\qquad \begin{aligned} 2\pi M\varepsilon \geqq\; & dM^p\varepsilon^{p-2/(n-2)}\left[c_1\left(\frac{1}{1+4\varepsilon}\right)^{p+1}+c_2\left(\frac{1}{1+\varepsilon}\right)^{p+1-2/(n-2)}\right]\\ &+dM^q\varepsilon^{q-2/(n-2)}\left[c_1\left(\frac{1}{1+4\varepsilon}\right)^{q+1}+c_3\left(\frac{1}{1+\varepsilon}\right)^{q+1-2/(n-2)}\right], \end{aligned}$$

where

$$
(4.31) \qquad
\begin{aligned}
c_1 &= \int_{1/4}^{1/2} t^{2/(n-2)-1} \sin(\pi t)\, dt, \\
c_2 &= \int_0^{1/2} t^p \sin(\pi t)\, dt, \\
c_3 &= \int_0^{1/2} t^q \sin(\pi t)\, dt.
\end{aligned}
$$

Proof: Define $v(s) = \sin(\pi(s-1)/(\gamma-1))$ and note that v is positive for $1 < s < \gamma$ and satisfies (for $\lambda = (\pi/(\gamma-1))^2$)

$$
\begin{aligned}
v''(s) + \lambda v(s) &= 0, \\
v(1) = v(\gamma) &= 0.
\end{aligned}
$$

Multiplying (4.28) by v and integrating by parts one obtains

$$
(4.32) \qquad \int_1^\gamma s^{2/(n-2)-1} h\left(\frac{w}{s}\right) v\, ds = \lambda \int_1^\gamma wv\, ds.
$$

Because M is the maximum of w one immediately has

$$
(4.33) \qquad \lambda \int_1^\gamma wv\, ds \leqq \lambda M \int_1^\gamma v\, ds = 2M\pi\varepsilon.
$$

Relation (4.32) and (4.33) yield

$$
(4.34) \qquad
\begin{aligned}
2M\pi\varepsilon &\geqq d\left(\int_1^\gamma s^{2/(n-2)-1-p} w^p v\, ds\right) + d\left(\int_1^\gamma s^{2/(n-2)-1-q} w^q v\, ds\right) \\
&\equiv dI_1 + dI_2.
\end{aligned}
$$

We need lower bounds for the integrals I_1 and I_2 in (4.34). The function w is concave down on $[1, \gamma]$; if w achieves its maximum on the interval at τ, concavity implies that w lies above $w_1(s)$, where $w_1(s)$ has a graph consisting of the line segment from the point $(1, 0)$ to (τ, M) and a line segment from the point (τ, M) to $(\gamma, 0)$. One easily sees that $w_1(s) \geqq w_2(s)$, where

$$
(4.35) \qquad w_2(s) = \begin{cases} M\left(\dfrac{s-1}{\gamma-1}\right) & \text{for } \quad 1 \leqq s \leqq \dfrac{\gamma+1}{2}, \\[2ex] M\left(\dfrac{\gamma-s}{\gamma-1}\right) & \text{for } \quad \dfrac{\gamma+1}{2} \leqq s \leqq \gamma. \end{cases}
$$

If we use the fact that $w(s) \geqq w_2(s)$, we find that

$$I_1 = \int_1^\gamma s^{2/(n-2)-1-p} w^p v\, ds \geqq M^p \left[\int_1^{(\gamma+1)/2} s^{2/(n-2)-1-p} \left(\frac{s-1}{\gamma-1} \right)^p v\, ds \right] + M^p \left[\int_{(\gamma+1)/2}^{\gamma} s^{2/(n-2)-1-p} \left(\frac{\gamma-s}{\gamma-1} \right)^p v\, ds \right].$$

Changing variables we obtain

$$\int_1^{(\gamma+1)/2} s^{2/(n-2)-1-p} \left(\frac{s-1}{\gamma-1} \right)^p v\, ds = \varepsilon^{-1} \int_0^{1/2} \left(\frac{\varepsilon}{\varepsilon+t} \right)^{p+1-2/(n-2)} t^p \sin(\pi t)\, dt \equiv J_1(\varepsilon, p).$$

To estimate $J_1(\varepsilon, p)$ observe that

$$\begin{aligned} J_1(\varepsilon, p) &\geqq \varepsilon^{p-2/(n-2)} \int_{1/4}^{1/2} \left(\frac{t}{\varepsilon+t} \right)^{p+1} (\varepsilon+t)^{2/(n-2)} \frac{\sin(\pi t)}{t}\, dt \\ &\geqq \varepsilon^{p-2/(n-2)} \left(\frac{1}{1+4\varepsilon} \right)^{p+1} \int_{1/4}^{1/2} t^{2/(n-2)-1} \sin(\pi t)\, dt. \end{aligned} \tag{4.36}$$

By another change of variables we find that

$$\begin{aligned} \int_{(\gamma+1)/2}^{\gamma} s^{2/(n-2)-1-p} \left(\frac{\gamma-s}{\gamma-1} \right)^p v\, ds &= \varepsilon^{-1} \int_0^{1/2} \left(\frac{\varepsilon}{1+\varepsilon-t} \right)^{p+1-2/(n-2)} t^p \sin(\pi t)\, dt \\ &\geqq \varepsilon^{p-2/(n-2)} \left(\frac{1}{1+\varepsilon} \right)^{p+1-2/(n-2)} \int_0^{1/2} t^p \sin(\pi t)\, dt. \end{aligned} \tag{4.37}$$

Inequalities (4.36) and (4.37) imply that

$$dI_1 \geqq dM^p \varepsilon^{p-2/(n-2)} \left[c_1 \left(\frac{1}{1+4\varepsilon} \right)^{p+1} + c_2 \left(\frac{1}{1+\varepsilon} \right)^{p+1-2/(n-2)} \right]. \tag{4.38}$$

The same argument gives an analogous estimate for dI_2, yielding (4.30).

We are now in a position to prove our theorem.

THEOREM 4.39. *Assume that* $f(z)$ *is given by* (4.26), *where* $1 < p < n/(n-2)$, $1 < q$ *and* $n \geqq 3$. *If* $a > 0$ *is fixed, equation* (4.25) *has exactly one positive solution for all* b *sufficiently large.*

Proof: The existence of a positive solution is well known. To prove uniqueness it suffices to prove that (4.28) has at most one positive solution for γ sufficiently large, where γ and d in (4.28) are as in (4.27). By using (4.30) and dropping the term corresponding to q we find

$$2\pi \varepsilon^{n/(n-2)-p} \geqq dM^{p-1} \left[c_1 \left(\frac{1}{1+4\varepsilon} \right)^{p+1} + c_2 \left(\frac{1}{1+\varepsilon} \right)^{p+1-2/(n-2)} \right], \tag{4.40}$$

where $\varepsilon=(\gamma-1)^{-1}$. Since we assume that $1<p<n/(n-2)$, it follows that if we define M_γ by $M_\gamma=\sup\{w(s)|1\leqq s\leqq\gamma$ and w is a positive solution of (4.28)$\}$, then $\lim_{\gamma\to\infty}M_\gamma=0$ and M_γ can be estimated by (4.40).

Now set $g(w,s)=s^{2/(n-2)-1}h(w/s)$ for $s\geqq 1$, $w>0$. To complete the proof we need only observe that (2.5), (2.6) and (2.7) will be satisfied for $0<w\leqq M_\gamma$ and $1\leqq s\leqq\gamma$ because $1<p<n/(n-2)$ and because $M_\gamma\to 0$ as $\gamma\to\infty$; thus Theorem 2.4 implies uniqueness.

Remark 4.41. If $p=n/(n-2)$, it follows from (4.40) that M_γ is bounded as $\gamma\to\infty$. Depending on the original constants a, b, A, B and q, Theorem 2.4 may still apply in this case and give uniqueness.

5. Concluding Remarks

1. Consider problem (1.1)$'$ and assume that $uf'(u)>f(u)$ for $u>0$ and that $f(u)$ grows like u^p for large u but is not equal to u^p. Theorem 1.4 says that if $p\leqq n/(n-2)$, then one has uniqueness of positive radial solutions in any annulus, while Theorem 1.10 shows that for $p>(n+2)/(n-2)$ uniqueness is, in general, lost. If $n/(n-2)<p<(n+2)/(n-2)$, we do not have a complete answer, although Theorem 1.8 provides partial results for the case of an annulus. For the case of a ball we know even less. Theorem 1.6 gives uniqueness in the range $1\leqq p\leqq n/(n-2)$, and we suspect nonuniqueness if $p>(n+2)/(n-2)$ (although we have no proof). There have been some numerical studies (see Remark 2.3 in [3]) which suggest nonuniqueness for $f(u)=u^5+\mu u^q$, $1<q<3$, $\mu>\mu_0$ (μ_0 some positive number) and Ω a ball in $\mathbb{R}^3$.

We shall now briefly describe a result which implies, in particular, that for $f(t)=\lambda t+\sum_{i=1}^k A_i t^{p_i}$, $1<p_i<p_2<\cdots<p_k<(n+3)/(n-1)$, $\lambda>0$, $A_i>0$, and $\Omega=\{x\in\mathbb{R}^n|0<a<|x|<b\}$, problem (1.1)$'$ has at most one positive, radial solution provided a is sufficiently large. This result suggests that there is room for improvement in our previous theorems. The proof is similar to arguments elsewhere in this paper, so we shall be sketchy.

We begin with a variant of Theorem 2.4.

THEOREM 5.1. *Consider the following boundary value problem:*

$$(5.2)\qquad \begin{gathered} w''+Bw'+Cw+e^{(2-\gamma)s}f(e^{\gamma s}w)=0, \qquad \alpha\leqq s\leqq\beta, \\ w(\alpha)=w(\beta)=0. \end{gathered}$$

Assume that $B<0$, $C\leqq 0$ and $\gamma\leqq 0$. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a C^1 function such that, for all $u>0$,

$$(5.3)\qquad 0<f(u)<f'(u)u,$$

$$(5.4) \qquad (2-\gamma)f(u)+\gamma f'(u)u>0,$$

and

$$(5.5) \qquad -2BC\,e^{-2\alpha}u-2Bf(u)-[1-e^{B(\beta-\alpha)}][(2-\gamma)f(u)+\gamma f'(u)u]\geqq 0.$$

Then problem (5.2) *has at most one positive solution.*

Proof: Consider the initial value problem

$$(5.6) \qquad \begin{gathered} w''+Bw'+Cw+e^{(2-\gamma)s}f(e^{\gamma s}w)=0, \\ w(\alpha)=0, \qquad w'(\alpha)=d>0. \end{gathered}$$

If $w(s)=w(s,d)$ is the solution of (5.6), $\varphi(s)=\varphi(s,d)=\partial w/\partial d$ and $z(d)$ is the first zero of $w(s)$ (if it exists), it suffices (as in Theorem 2.4) to prove that $\varphi(\beta)<0$ whenever $z(d)=\beta$.

Thus suppose w satisfies (5.2), $w'(\alpha)=d$ and w is positive on (α,β). We first claim that there is a number τ, $\alpha<\tau<\beta$, such that $w'(s)>0$ for $\alpha\leqq s<\tau$ and $w'(s)<0$ for $\tau<s\leqq\beta$. This follows by essentially the same argument used to prove Lemma 6.14 in the appendix; we leave it, therefore, to the reader.

An application of the Sturm comparison theorem implies that φ has at least one zero on (α,β). If σ_1 denotes the first zero of φ on (α,β) and $y(s)\equiv w'(s)$ and $\xi(s)\equiv e^{\gamma s}w(s)$, a calculation gives

$$(5.7) \qquad \frac{d}{ds}(\varphi\,e^{Bs}y'-y\,e^{Bs}\varphi')=-e^{(2-\gamma)s}((2-\gamma)f(\xi)+\gamma f'(\xi)\xi)\varphi\,e^{Bs}.$$

If one assumes that $\sigma_1\leqq\tau$, one obtains a contradiction by integrating (5.7) from α to σ_1. Thus we have $\sigma_1>\tau$.

Suppose the second zero, σ_2, of φ satisfies $\sigma_2\leqq\beta$. Define $\zeta(s)$ by

$$\zeta(s)=(e^{Bs}-e^{B\beta})w'(s).$$

An unpleasant calculation gives

$$(5.8) \qquad \begin{aligned} &\int_{\sigma_1}^{\sigma_2}\frac{d}{ds}(\varphi\,e^{Bs}\zeta'-\zeta\,e^{Bs}\varphi') \\ &\quad =\int_{\sigma_1}^{\sigma_2}\varphi\,e^{2Bs}\,e^{(2-\gamma)s}\{-2BC\xi\,e^{-2s}-2Bf(\xi)-[1-e^{B(\beta-s)}] \\ &\qquad\qquad \times[(2-\gamma)f(\xi)+\gamma f'(\xi)\xi]\}\,ds. \end{aligned}$$

The left-hand side of (5.8) is positive, but by using (5.5) one sees that the right-hand side is negative, a contradiction. It follows that $\varphi(\beta)<0$, and the theorem is proved.

As an immediate application of Theorem 5.1 we have

COROLLARY 5.9. *Suppose that* $f:\mathbb{R}\to\mathbb{R}$ *is given by*

$$f(v)=\lambda v+\sum_{j=1}^{k} A_j v^{p_j},$$

where $\lambda>0$, $A_j>0$, $1<p_1<p_2<\cdots<p_k$ *and* $n/(n-2)<p_k<(n+2)/(n-2)$ *for an integer* $n>2$. *Consider the boundary value problem*

$$(5.10)\qquad \begin{aligned}&(r^{n-1}u_r(r))_r+r^{n-1}f(u(r))=0,\\ &u(a)=u(b)=0, \qquad 0<a<b,\end{aligned}$$

and define constants γ, B *and* C *by*

$$(5.11)\qquad \gamma=-\frac{2}{p_k-1}<0, \qquad B=n+2\gamma-2 \quad \text{and} \quad C=\gamma(n+\gamma-2).$$

Then, if

$$(5.12)\qquad \left[|B|-1+\left(\frac{a}{b}\right)^{|B|}\right]\lambda+|B|C\left(\frac{1}{a^2}\right)\geqq 0,$$

the boundary value problem (5.10) *has at most one solution.*

Proof: The transformation $r^\gamma w(s)=u(r)$, $s=\log r$ takes (5.10) to (5.2) with $\alpha=\log a$, $\beta=\log b$, and γ, B and C as in (5.11). One verifies that $\gamma<0$, $B<0$ and $C<0$, and that (5.3) and (5.4) hold without the assumption (5.12). A little thought shows that (5.12) implies (5.5).

If one wants a uniqueness statement in Corollary 5.9 for every $b>a$, then one must have

$$(5.13)\qquad |B|>1$$

in (5.12), and (5.13) implies that

$$(5.14)\qquad p_k<\frac{n+3}{n-1}.$$

Furthermore, (5.12) is true for every $b>a$ (assuming (5.13)) if

$$(5.15)\qquad \lambda a^2\geqq-\frac{|B|C}{|B|-1}.$$

Thus we obtain

COROLLARY 5.16. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be as in Corollary* 5.9 *and assume that* (5.14) *and* (5.15) *are satisfied. Then for every* $b>a$ *equation* (5.10) *has at most one positive solution.*

We should make one remark about Corollary 5.16. If W is a bounded, open subset of $\mathbb{R}^n$ with smooth boundary, the eigenvalue problem

$$
\begin{aligned}
&\Delta v+\lambda v=0 \quad \text{on} \quad W, \\
&v=0 \quad \text{on} \quad \partial W,
\end{aligned} \tag{5.17}
$$

has a positive eigenvalue $\lambda_1=\lambda_1(W)$ with corresponding positive eigenvector v_1. It is well known and trivial (multiply (5.18) below by v_1 and integrate by parts) that the semilinear problem

$$
\begin{aligned}
&\Delta u+f(u)=0 \quad \text{on} \quad W, \\
&u=0 \quad \text{on} \quad \partial W,
\end{aligned} \tag{5.18}
$$

has no positive solution if $f(v)>\lambda_1 v$ for all $v>0$, where $\lambda_1=\lambda_1(W)$. If we take $W=\Omega_{a,b}=\{x\in\mathbb{R}^n \mid 0<a<|x|<b\}$, the above remark shows that Corollary 5.16 is trivial (there are no positive solutions) if $\lambda \geqq \lambda_1(\Omega_{a,b})$. However, it is well known that, for fixed a, $\lambda_1(\Omega_{a,b})$ approaches zero as b approaches ∞, and in fact (under the hypotheses of Corollary 5.16) there exists a unique positive solution of (5.10) whenever $\lambda_1(\Omega_{a,b})>\lambda$.

2. We wish to describe an interesting phenomenon concerning uniqueness and nonuniqueness of positive radial solutions on ring-shaped domains. Let $\Omega_{a,b}=\{x\in\mathbb{R}^n|0<a<|x|<b\}$, $n\geqq 3$, and $f(u)=u^p+\varepsilon u^q$, $1<p<n/(n-2)$, $(n+2)/(n-2)<p<\infty$. Theorem 1.10 says there is an $\varepsilon>0$ such that $(1.1)'$ has at least three positive radial solutions in $\Omega_{a,1}$ for all sufficiently small positive a. Now, fix such an a. Then Theorem 1.8 implies that if $b>a$ is sufficiently close to a, $(1.1)'$ has at most one positive radial solution in $\Omega_{a,b}$. Therefore, we see that if we fix $a>0$ and $\varepsilon>0$ as above and let $b>a$ vary, then for b close to a problem $(1.1)'$ has exactly one positive radial solution, for $b=1$ problem $(1.1)'$ has at least three positive radial solutions, and for b very large problem $(1.1)'$ again has (according to Theorem 4.39) exactly one positive radial solution. Our results also show that the uniqueness question is rather subtle and does not depend only on the ratio of the outer and inner radii, as Theorem 1.8 might lead one to suspect.

3. We ought to say a few words about the existence of positive (radial) solutions for (1.1). An existence theorem for the case that $f(u,r)$ grows strictly less rapidly than u^p, $p=(n+2)/(n-2)$, at ∞ and Ω is a ball can be found in Section 4 of [20]. More general existence theorems for positive solutions on general bounded domains can be found, for example, in [1] and [9]. The existence problem for the annulus is easy (it is equivalent to a nonsingular ordinary differential equation); a discussion is given in [13], where no restriction (other than polynomial growth) is imposed on the growth of f at ∞. If Ω is a ball, we may still have existence even if $f(u,r)$ grows faster than u^p, $p=(n+2)/(n-2)$.

For instance, if $f(u, r) \sim r^l u^p$, $l > 0$, and $1 < p < (n+2+2l)/(n-2)$, we still have existence (see [18]).

4. Problem (1.1) arises in various branches of applied mathematics, for example the question of fast diffusion in some plasmas (see [2]). If $f(u) = u^p$, $p > 1$, (1.1)$'$ has arisen in the study of stellar structures in astrophysics and is known as the Lane–Emden equation (see [4]). More recently, Henon [12] has proposed another model in which $f(u, r) = r^l u^p$ for various ranges of l and p; the corresponding equation is known as a generalized Lane–Emden equation. So-called E-solutions (solutions in balls), F-solutions (solutions in annuli) and M-solutions (singular solutions in balls) have been studied in [4] for the case $f(u) = u^p$. M-solutions for more general nonlinearities f, and some applications to parabolic equations have been studied in [19].

5. We do not investigate the uniqueness question for superlinear elliptic equations on $\mathbb{R}^n$. We refer the reader to [6] and [15], where the equation

$$\Delta u - u + u^p = 0$$

is studied for some p and n.

Appendix

Oscillation Theorems for Radial Solutions of Equation (1.1)$'$

We shall say that a function $u(r)$ oscillates if, given any number r_1, there exists a number $r_2 > r_1$ such that $u(r_2) = 0$. We are interested in determining conditions on a function $f(u)$ such that every solution of

$$(6.1)\qquad \begin{aligned} &(r^{n-1}u_r(r))_r + r^{n-1}f(u(r)) = 0, && r \geqq a,\\ &u(a) = 0, \qquad u_r(a) = d > 0, && a > 0, \end{aligned}$$

oscillates and also every C^2 solution of

$$(6.2)\qquad \begin{aligned} &(r^{n-1}u_r(r))_r + r^{n-1}f(u(r)) = 0, && r > 0,\\ &u(0) = d > 0, \qquad u_r(0) = 0, && \end{aligned}$$

oscillates. In terms of our previous notation, such theorems will imply that the map $d \to z(d) =$ the first zero of a solution $u(r, d)$ of (6.1) or (6.2) is defined for all $d > 0$. If we can find conditions on $f(u)$, for $u \geqq 0$, which insure that every solution of (6.1) or (6.2) has a first zero, and if we then assume that $g(u) \equiv -f(-u)$ satisfies the same conditions for $u \geqq 0$, then every solution of (6.1) or (6.2) will oscillate. Thus we shall concentrate on finding conditions on f which insure that every solution of (6.1) or (6.2) has a first zero $z(d)$. Our main theorem gives such a result and generalizes, except in the case $f(u) = |u|^p \operatorname{sgn} u$ and $p = (n+2)/(n-2)$, Theorem 2.5 of [17].

THEOREM 6.3. *Assume that* $f:\mathbb{R}\to\mathbb{R}$ *is* C^1, *that* $f(v)>0$ *for all* $v>0$, *that* $f'(v)>0$ *for* $0<v<v_0$, $v_0>0$, *and that*

$$\limsup_{v\to 0^+}\frac{f(v)}{vf'(v)}<\infty.$$

In addition suppose that, for all $u>0$,

$$(n+2)f(u)-(n-2)uf'(u)>0, \tag{6.4}$$

where $n\geqq 2$ *is an integer, and that*

$$\liminf_{u\to 0^+}\frac{f(u)}{u^p}=+\infty,\qquad p=\frac{n+2}{n-2}, \tag{6.5}$$

and

$$\liminf_{u\to +\infty}\frac{f(u)}{u}>0. \tag{6.6}$$

Inequality (6.5) *is vacuous if* $n=2$. *Then if* $u(r)$ *is a solution of* (6.1) *or* (6.2), $u(r)$ *has a first zero* $z(d)$, *i.e., there exists* $z=z(d)>a$ (*for equation* (6.1)) *or* $z=z(d)>0$ (*for equation* (6.2)) *such that* $u(z)=0$.

The following is a typical corollary of Theorem 6.3; we assume in Corollary 6.7 that $n\geqq 2$.

COROLLARY 6.7. *Assume that* $f(u)=\sum_{j=1}^{k}A_ju^{p_j}$ *for* $u\geqq 0$, *where* $1\leqq p_j\leqq (n+2)/(n-2)$ *and* $0<A_j$ *for all* j *and where at least one* p_j *satisfies* $p_j<(n+2)/(n-2)$. *Then if* $f(u)$ *is extended to be an odd function, every solution of* (6.1) *or* (6.2) *oscillates.*

We shall prove Theorem 6.3 by transforming (6.1) or (6.2) and then proving a series of lemmas about the transformed equation. If $u(r)=u(r,d)$ is a solution of (6.1) and $w(s)$ is given by

$$r^{\gamma}w(s)=u(r),\qquad \gamma=\frac{2-n}{2},\qquad s=\log r, \tag{6.8}$$

then $w(s)=w(s,d)$ satisfies

$$\begin{gathered} w''(s)-\left(\frac{n-2}{2}\right)^2 w+e^{((n+2)/2)s}f(e^{((2-n)/2)s}w)=0,\qquad s\geqq\alpha,\\ w(\alpha)=0,\qquad w'(\alpha)=\delta>0,\qquad \alpha\equiv\log a. \end{gathered} \tag{6.9}$$

If $u(r)$ satisfies (6.2), $n\geqq 3$ and $w(s)$ is given by (6.8), then w satisfies the differential equation in (6.9) for $s>-\infty$, but the boundary conditions are replaced by

$$\lim_{s\to-\infty} w(s)=0, \qquad \lim_{s\to-\infty} w'(s)=0. \tag{6.10}$$

In fact a calculation gives that

$$w'(s)=r^{n/2-1}\left[\frac{n-2}{2}u(r)+ru_r(r)\right], \tag{6.11}$$

so that, if $n\geqq 3$, $w'(s)>0$ for s large negative. If $n=2$, (6.11) shows (if $f(u)>0$ for $u>0$) that $w'(s)<0$ for s large negative and

$$\lim_{s\to-\infty} w(s)=d, \qquad \lim_{s\to-\infty} w'(s)=0. \tag{6.12}$$

(Notice that we have not considered the question of the domain of $u(r)$ carefully, but this will be treated below.)

If $f(u)$ is C^1 for $u\geqq 0$ and $f(u)>0$ for $u>0$, then for $n=2$ equation (6.9) implies that $w(s)$ is concave down as long as $w(s)>0$ and (6.11) shows that $w'(s)<0$ for s large negative. It follows that $w(s)$ has a first zero in this case, and we have

PROPOSITION 6.13. *Assume that f is C^1 and $f(u)>0$ for $u>0$. If $n=2$, any solution $u(r)$ of* (6.2) *has a first zero.*

To handle the other cases we begin with the following lemma.

LEMMA 6.14. *Assume that $w(s)$ satisfies the differential equation in* (6.9) *on some interval (α_1, α_2), where possibly $\alpha_1=-\infty$. Suppose that $w(s)>0$ for $\alpha_1<s<\alpha_2$, $\lim_{s\to\alpha_1} w(s)=0$ and $w'(s)>0$ for $\alpha_1<s<\beta$, where β is some number greater than α_1. In addition, suppose that $f:\mathbb{R}\to\mathbb{R}$ is C^1 and that f satisfies* (6.4) *for all $u>0$. Then either $w'(s)>0$ for all s in (α_1, α_2) or there exists $\tau\in(\alpha_1, \alpha_2)$ such that $w'(s)>0$ for $\alpha_1<s<\tau$ and $w'(s)<0$ for $\tau<s<\alpha_2$.*

Proof: If $w'(s)>0$ for all $s\in(\alpha_1, \alpha_2)$ we are done, so assume this is not the case and define $\tau=\inf\{s>\alpha_1: w'(s)=0\}$.

Our first claim is that there exists $\varepsilon>0$ such that $w'(s)<0$ for $\tau<s<\tau+\varepsilon$. We know that $w''(\tau)\leqq 0$; to prove the existence of $\varepsilon>0$ it suffices to prove that $w''(\tau)<0$. By way of contradiction, assume it is not, then $w''(\tau)=0$. If we define $g(s, v)$ by

$$g(s, v)=-\left(\frac{n-2}{2}\right)^2 v+e^{((n+2)/2)s}f(e^{((2-n)/2)s}v), \tag{6.15}$$

and $\xi(s)=e^{((2-n)/2)s}w(s)$, a calculation gives (recalling $w'(\tau)=0$)

$$w^{(3)}(\tau)=-\tfrac{1}{2}e^{((n+2)/2)\tau}[(n+2)f(\xi(\tau))-(n-2)f'(\xi(\tau))\xi(\tau)]. \tag{6.16}$$

Inequality (6.4) then implies that $w^{(3)}(\tau)<0$, and this implies that $w(s)>w(\tau)$ for $s<\tau$ and s near τ, which contradicts the choice of τ.

Now it is legitimate to define σ by

$$\sigma=\sup\{t>\tau\colon w'(s)<0 \text{ for } \tau<s<t\}.$$

If $\sigma=\alpha_2$, we are done, so assume $\sigma<\alpha_2$, in which case $w'(\sigma)=0$. Select σ_1, $\alpha_1<\sigma_1<\tau$, such that $w(\sigma_1)=w(\sigma)$ (the reader should graph $w(s)$ on $[\sigma_1,\sigma]$). If we multiply (6.9) by $w'(s)$ and integrate from σ_1 to σ we obtain

$$-\tfrac{1}{2}(w'(\sigma_1))^2+\int_{\sigma_1}^{\sigma} e^{((n+2)/2)s}f(\xi(s))w'(s)\,ds=0. \tag{6.17}$$

If we define $F(v)=\int_0^v f(t)\,dt$, a calculation gives (writing ξ for $\xi(s)$)

$$e^{((n+2)/2)s}f(\xi)w'(s)=\frac{d}{ds}(e^{ns}F(\xi))-e^{ns}\left[nF(\xi)-\frac{n-2}{s}f(\xi)\xi\right]. \tag{6.18}$$

Equation (6.18) yields

$$\begin{aligned}\int_{\sigma_1}^{\sigma} e^{((n+2)/2)s}f(\xi(s))w'(s)\,ds&=e^{ns}F(\xi)\Big|_{\sigma_1}^{\sigma}\\ &\quad-\int_{\sigma_1}^{\sigma} e^{ns}\left[nF(\xi)-\frac{n-2}{2}\,\xi f(\xi)\right]ds.\end{aligned} \tag{6.19}$$

Calculating we obtain (for $v>0$)

$$2\frac{d}{dv}\left[nF(v)-\frac{n-2}{2}vf(v)\right]=(n+2)f(v)-(n-2)vf'(v)>0;$$

hence $nF(v)-((n-2)/2)vf(v)$ is monotonic increasing for $v>0$ and positive for $v>0$. In particular, because

$$w(s)>w(\sigma)\equiv c, \qquad \sigma_1<s<\sigma,$$

we must have, for $\sigma_1<s<\sigma$,

$$\begin{aligned}&nF(\xi(s))-\frac{n-2}{2}\,\xi(s)f(\xi(s))\\ &\quad>nF(e^{((2-n)/2)s}c)-\frac{n-2}{2}(e^{((2-n)/2)s}c)f(e^{((2-n)/2)s}c)\equiv\bar{h}(s),\end{aligned} \tag{6.20}$$

and $\bar{h}(s)$ defined in (6.20) is positive for all s. Substituting (6.20) in (6.19), one obtains

$$\int_{\sigma_1}^{\sigma} e^{((n+2)/2)s}f(\xi)w'(s)\,ds<e^{ns}F(\xi)|_{\sigma_1}^{\sigma}-\int_{\sigma_1}^{\sigma} e^{ns}\bar{h}(s)\,ds. \tag{6.21}$$

A calculation yields

$$\frac{d}{ds}[e^{ns}F(e^{((2-n)/2)s}c)]=e^{ns}\bar{h}(s) \tag{6.22}$$

and using (6.22) in (6.21) one finds

$$\int_{\sigma_1}^{\sigma} e^{((n+2)/2)s}f(\xi(s))w'(s)\,ds<0. \tag{6.23}$$

Using (6.23) in (6.17), one obtains a contradiction.

Remark 6.24. If $f(u)=u^p$, $p=(n+2)/(n-2)$, the function $g(s,v)$ in (6.15) depends only on v. Therefore, the uniqueness theorem for the initial value problem for ordinary differential equations implies that if $w(s)$ is a solution of the differential equation in (6.9) and $w'(\tau)=0$ for some τ, then $w''(\tau)\neq 0$. If τ is as in Lemma 6.14, $w''(\tau)<0$. An examination shows that the rest of the proof of Lemma 6.14 is still valid, although some inequalities become equalities. Thus Lemma 6.14 is valid for $f(u)=u^p$, $p=(n+2)/(n-2)$.

We shall also need a simple identity.

LEMMA 6.25. *Assume that $w(s)$ is a C^2 solution of the differential equation in* (6.9) *for $s_1\leqq s\leqq s_2$. If $\xi\equiv\xi(s)\equiv e^{((2-n)/2)s}w(s)$ and $F(v)\equiv\int_0^v f(t)\,dt$, $w(s)$ satisfies*

$$\begin{aligned}\tfrac{1}{2}(w'(s))^2\Big|_{s_1}^{s_2}-\tfrac{1}{2}\left(\frac{n-2}{2}\right)^2 w(s)^2\Big|_{s_1}^{s_2}+e^{ns}F(\xi)\Big|_{s_1}^{s_2}\\ =\int_{s_1}^{s_2}e^{ns}\left[nF(\xi)-\frac{n-2}{2}\xi f(\xi)\right]ds.\end{aligned} \tag{6.26}$$

Proof: Multiply the defining equation for w by $w'(s)$, use (6.18) to substitute for $e^{((n+2)/2)s}f(\xi)w'(s)$, and integrate from s_1 to s_2.

Proof of Theorem 6.3: We shall suppose the theorem is false (so $u(r)>0$ for all r in its domain) and obtain a contradiction. If u satisfies (6.2), then u is decreasing (because $f(u)>0$ for $u>0$) and positive for all u in its domain. It follows easily then (from Proposition 2.35 and standard theorems for ordinary differential equations) that $u(r)$ is defined for all $r>0$. If u satisfies (6.1), one can see that $r^{n-1}u_r(r)$ is decreasing for $r\geqq a$, and one concludes that $u_r(r)\leqq (da^{n-1})/r^{n-1}$ for $r\geqq a$. It follows easily that $u(r)$ is defined for all $r\geqq a$ in this case.

Let $w(s)$ be defined by (6.8), where u satisfies (6.1) or (6.2). We can assume (because of Proposition 6.13) that $n\geqq 3$ if u satisfies (6.2). We are assuming that $w(s)$ is defined and positive on (α,∞), where $\alpha=-\infty$ or $\alpha=\log a$.

If we allow the value $+\infty$ for a limit, Lemma 6.14 implies that $\lim_{s\to+\infty}w(s)$ exists. Our first claim is that $\lim_{s\to\infty}w(s)=0$. If $n=2$, $w(s)$ satisfies

$$w''(s)+\left(e^{2s}\frac{f(w)}{w}\right)w=0. \tag{6.27}$$

If $\lim_{s\to\infty} w(s) > 0$, we can apply the Sturm comparison theorem to (6.27): condition (6.6) and the assumption that $f(w) > 0$ for $w > 0$ imply that

$$\lim_{s\to\infty} e^{2s}\frac{f(w(s))}{w(s)} = +\infty,$$

so $w(s)$ has a first zero, a contradiction.

Thus we can assume $n > 2$. If $\lim_{s\to\infty} w(s)$ is finite and positive, $\lim_{s\to\infty}\xi(s) = 0$ and we can write the differential equation for $w(s)$ as

$$w''(s) + w\left[-\left(\frac{n-s}{2}\right)^2 + (w^{4/(n-2)})\frac{f(\xi)}{\xi^p}\right] = 0, \qquad p = \frac{n+2}{n-2}. \tag{6.28}$$

Condition (6.5) and the Sturm comparison theorem then imply that $w(s)$ has a zero, a contradiction. It still remains to prove, however, that $\lim_{s\to\infty} w(s)$ is not $+\infty$. Suppose that $\lim_{s\to+\infty} w(s) = +\infty$. We have the identities (for $p \equiv (n+2)/(n-2)$)

$$e^{((n+2)/2)s}\frac{f(e^{((2-n)/2)s}w)}{w} = e^{2s}\frac{f(\xi)}{\xi} = w^{4/(n-2)}\frac{f(\xi)}{\xi^p}. \tag{6.29}$$

Conditions (6.5) and (6.6) imply that there exists a positive constant k such that

$$\frac{f(\xi)}{\xi} \geqq k \quad \text{if} \quad \xi \geqq 1, \tag{6.30}$$

and

$$\frac{f(\xi)}{\xi^p} \geqq k \quad \text{if} \quad 0 < \xi < 1. \tag{6.31}$$

Using these inequalities we conclude that

$$e^{((n+2)/2)s}\frac{f(e^{((2-n)/2)s}w(s))}{w(s)} \geqq k \min(e^{2s}, w(s)^{4/(n-2)}). \tag{6.32}$$

The right-hand side of (6.32) approaches infinity as $s \to \infty$, so the Sturm comparison theorem again implies that $w(s)$ has a zero, a contradiction.

Thus we have proved that $\lim_{s\to\infty} w(s) = 0$. Notice that for any $\varepsilon > 0$ we must have

$$\liminf_{s\to+\infty} e^{((n+2)/2)s}\frac{f(\xi(s))}{w(s)} < \left(\frac{n-2}{2}\right)^2 + \varepsilon.$$

Otherwise, a Sturm comparison argument applied to (6.9) would imply that $w(s)$ has a first zero. Multiplying both sides of the inequality by $w(s)$ one obtains

$$\liminf_{s\to+\infty} e^{((n+2)/2)s} f(\xi(s)) = 0. \tag{6.33}$$

It seems convenient to consider two subcases.

Case 1. Assume that

$$\limsup_{s\to+\infty} e^{((n+2)/2)s} f(\xi(s)) \leqq 1.$$

The assumptions on $f(v)$ in Theorem 6.3 imply that for all v with $0 < v < v_0$ we have

$$F(v) = \int_0^v f(t)\,dt \leqq v f(v). \tag{6.34}$$

If we apply (6.34) to $v = \xi(s) = \xi$ (for s large) we obtain

$$e^{ns} F(\xi) \leqq e^{ns} \xi f(\xi) = w(s)[e^{((n+2)/2)s} f(\xi(s))] \leqq 2w(s). \tag{6.35}$$

It follows that

$$\lim_{s\to+\infty} e^{ns} F(\xi(s)) = 0. \tag{6.36}$$

Because $\lim_{s\to\infty} w(s) = 0$, we also conclude that there exists a sequence $t_j \to +\infty$ with $w'(t_j) \to 0$.

If $\alpha = -\infty$ (so that u satisfies (6.2)), let $b_j \to -\infty$; otherwise choose $b_j = \log a = \alpha$. We now apply Lemma 6.25 with $s_1 = b_j$ and $s_2 = t_j$, and obtain

$$\begin{aligned} &\tfrac{1}{2}(w'(s))^2\Big|_{b_j}^{t_j} - \tfrac{1}{2}\left(\frac{n-2}{2}\right)^2 w(s)^2\Big|_{b_i}^{t_j} + e^{ns} F(\xi(d))\Big|_{b_i}^{t_j} \\ &= \int_{b_j}^{t_j} e^{ns}\left[nF(\xi) - \frac{n-2}{2}\,\xi f(\xi)\right] ds. \end{aligned} \tag{6.37}$$

If we take the lim sup as $j \to \infty$ of the left-hand side of (6.37), we obtain a nonpositive number (strictly negative if $\alpha > -\infty$). If we take the lim sup of the right-hand side of (6.37), we obtain a positive number, because (as we proved in Lemma 6.14)

$$nF(v) - \frac{n-2}{2}\, v f(v) > 0$$

for all $v > 0$. This is a contradiction.

Case 2. Assume that

$$\limsup_{s\to\infty} e^{((n+2)/2)s} f(\xi(s)) > 1.$$

The same argument as used in Case 1 will yield a contradiction if we can find a sequence $t_j \to +\infty$ such that

$$\lim_{j\to\infty} w'(t_j) = 0 \quad \text{and} \quad e^{((n+2)/2)t_j} f(\xi_j) < 1, \quad \xi_j \equiv \xi(t_j); \tag{6.38}$$

thus to complete the proof of Theorem 6.3 it suffices to find t_j satisfying (6.38). Define $h(s)$ by

$$h(s)=e^{((n+2)/2)s}f(\xi(s)).$$

By using (6.33) and the assumption of Case 2, we can find a sequence $\sigma_1<\tau_1<\sigma_2<\tau_2<\cdots$ such that $h(\sigma_j)>1$, $h(\tau_j)<1$ and $\lim_{j\to\infty}\sigma_j=\infty$. Let $t_j\in(\sigma_j,\sigma_{j+1})$ be a point where

$$h(t_j)=\min_{\sigma_j<s<\sigma_{j+1}} h(s), \tag{6.39}$$

so that $h(t_j)<1$ and $h'(t_j)=0$. The equation $h'(t_j)=0$ implies (setting $\xi_j=\xi(t_j)$)

$$-\frac{n+2}{2}f(\xi_j)w(t_j)+\frac{n-2}{2}f'(\xi_j)\xi_j w(t_j)=f'(\xi_j)\xi_j w'(t_j). \tag{6.40}$$

If j is taken so large that $\xi_j<v_0$ (v_0 as in the statement of Theorem 6.3), there exists a constant C such that

$$0\leqq\frac{f(\xi_j)}{f'(\xi_j)\xi_j}\leqq C. \tag{6.41}$$

If one divides both sides of (6.40) by $f'(\xi_j)\xi_j$ and uses the fact that $\lim_{j\to\infty}w(t_j)=0$, one obtains from (6.40) and (6.41) that

$$\lim_{j\to\infty}w'(t_j)=0.$$

Thus t_j satisfies (6.38), and the proof is complete.

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