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Modulational instability, solitons and beam propagation in spatially nonlocal nonlinear media

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Abstract
We present an overview of recent advances in the understanding of optical beams in nonlinear media with a spatially nonlocal nonlinear response. We discuss the impact of nonlocality on the modulational instability of plane waves, the collapse of finite-size beams, and the formation and interaction of spatial solitons.

Keywords: modulational instabilities, solitons, collapse theory, nonlocal media

1. Introduction

A soliton is a localized wave that propagates without change through a nonlinear medium. Such a localized wave forms when the dispersion or diffraction associated with the finite size of the wave is balanced by the nonlinear change of the properties of the medium induced by the wave itself. Solitons are universal in nature and have been identified in such physical systems as fluids, plasmas, solids, matter waves and classical field theory. Spatial optical solitons—self-trapped light beams—have been proposed as building blocks in future ultra-fast all-optical devices. Spatial solitons can be used to create reconfigurable optical circuits that guide other light signals. Circuits with complex functionality and all-optical switching or processing can then be achieved through the evolution and interaction of one or more solitons [1]. This concept has now been verified in several optical materials [2, 3] and a number of new soliton effects have emerged through these studies, such as fusion, fission and formation of bound states. Due to the development of materials with stronger nonlinearities the optical power needed to create such virtual circuits has been reduced to the milliwatt and even microwatt level, bringing the concept nearer to practical implementation.

The soliton concept is an integral part of studies of the coherent excitations of Bose–Einstein condensates (BECs) [4]. Such BECs inherently have a spatially nonlocal nonlinear response due to the finite range of the inter-particle interaction potential. Spatial nonlocality, which is already an established concept in plasma physics [5–7], means that the response of the medium at a particular point is not determined solely by the wave intensity at that point (as in local media), but also depends on the wave intensity in its vicinity. The nonlocal nature often results from a transport process, such as atom diffusion [8], heat transfer [9, 10] or drift of electric charges [11]. It can also be induced by a long-range molecular interaction as in nematic liquid crystals which exhibit orientational nonlocal nonlinearity [12, 13]. Nonlocality is thus a generic feature of a large number of nonlinear systems. It has also recently become important in optics [14–16]. Although nonlocality can have a considerable impact on many nonlinear phenomena, studies of nonlocal nonlinear effects are still in their infancy.

In this paper we will discuss the role of spatial nonlocality in phenomena associated with propagation of optical beams in nonlinear media. These include modulational instability of plane waves, structural stability of localized beams and interaction of solitons.

2. Model

We consider an optical beam propagating along the z-axis of a nonlinear medium with the scalar electric field $E(\vec{r}, z) = \psi(\vec{r}, z)\exp(\mathrm{i}Kz - \mathrm{i}\Omega t) + \text{c.c.}$ Here $\vec{r}$ spans a D-dimensional transverse coordinate space, K is the wavenumber, Ω is the optical frequency, and $\psi(\vec{r}, z)$ is the slowly varying amplitude. We assume that the refractive index change $N(I)$ induced by the beam with intensity $I(\vec{r}, z) = |\psi(\vec{r}, z)|^2$ can be described by the phenomenological nonlocal model

$$N(I) = s\int R(\vec{\xi} - \vec{r})I(\vec{\xi}, z)\,\mathrm{d}\vec{\xi}, \tag{1}$$

where the integral $\int \mathrm{d}\vec{\xi}$ is over all transverse dimensions and $s = 1$ ($s = -1$) corresponds to a focusing (defocusing) nonlinearity. The response function $R(\vec{r})$, which is assumed to be real, localized and symmetric (i.e. $R(\vec{r}) = R(r)$, where $r = |\vec{r}|$), satisfies the normalization condition $\int R(\vec{r})\,\mathrm{d}\vec{r} = 1$. This model of nonlinearity leads to the following nonlocal nonlinear Schrödinger (NLS) equation governing the evolution of the beam

$$\mathrm{i}\partial_z\psi + \tfrac{1}{2}\nabla_\perp^2\psi + N(I)\psi = 0. \tag{2}$$

The width of the response function $R(r)$ determines the degree of nonlocality. For a singular response, $R(r) = \delta(r)$ (see figure 1(a)), the refractive index change becomes a local function of the light intensity, $N(I) = sI(\vec{r}, z)$, i.e. the refractive index change at a given point is solely determined by the light intensity at that very point, and equation (2) simplifies to the standard NLS equation

$$\mathrm{i}\partial_z\psi + \tfrac{1}{2}\nabla_\perp^2\psi + s\psi|\psi|^2 = 0. \tag{3}$$

With increasing width of $R(r)$ the light intensity in the vicinity of the point $\vec{r}$ also contributes to the index change at that point (see figure 1). While equation (1) is a phenomenological model, it nevertheless adequately describes several physical situations where the nonlocal nonlinear response is due to various transport effects, such as heat conduction or diffusion of molecules or atoms.

Before proceeding further it is worthwhile mentioning two important physical situations where the convolution term in equation (2) can be represented in a simplified form, which allows for an extensive analytical treatment of the resulting equation. First is the so-called weak nonlocality limit when the width of the response function is much less than the spatial extent of the beam (see figure 1(b)). In this case one can formally expand $I(\vec{r})$ in a Taylor series and retain only the first significant terms, which gives the following simple form of the nonlinearity

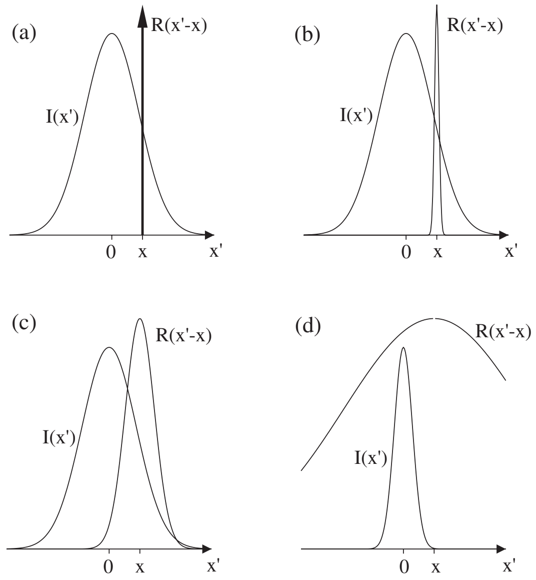

Figure 1. Various degrees of nonlocality, as given by the width of the nonlocal response function $R(x)$ and the intensity profile of the beam $I(x)$. Shown are (a) local, (b) weakly nonlocal, (c) general (nonlocal) and (d) highly nonlocal responses.

$$N(I) = s(I + \gamma\nabla_\perp^2 I), \qquad \gamma = \frac{1}{2D}\int r^2 R(r)\,\mathrm{d}\vec{r}, \tag{4}$$

where γ is a measure of the strength of the nonlocality. Here the nonlocal contribution to the Kerr-type local nonlinearity is reflected by the presence of the Laplacian of the wave intensity. A nonlinearity in this particular form appears in fact naturally in the theory of nonlinear effects in plasma [7]. It has been shown recently that the one-dimensional version of equation (2) with nonlinearity (4) supports propagation of stable bright and dark solitons [17]. Another limiting case, the so-called highly nonlocal limit, refers to the situation when the nonlocal response function is much wider than the beam itself (see figure 1(d)). In this case

$$N(I) = sR(r)P, \qquad P = \int I\,\mathrm{d}\vec{r}, \tag{5}$$

where P is the total power of the beam. Interestingly, in this case the propagation equation becomes local and linear. It describes the evolution of an optical beam trapped in an effective waveguide structure with the profile given by the nonlocal response function. This highly nonlocal limit was first explored by Snyder and Mitchell [14] in the context of the ‘accessible solitons’.

Although it is quite apparent in several physical situations that the nonlinear response in general is nonlocal (as in the case of thermal lensing), the nonlocal contribution to the refractive index change is often neglected [18, 19]. This is justified if the spatial scale of the beam remains large compared to the characteristic response length of the medium (given by the width of the response function). However, for very narrow beams or beams with fine spatial features (such as dark and bright solitons) the nonlocality can be of crucial importance and should be taken into account in the ensuing theoretical model.

3. Modulational instability

Modulational instability (MI) constitutes one of the most fundamental effects associated with wave propagation in nonlinear media. It signifies the exponential growth of a weak perturbation of the wave as it propagates. The gain leads to amplification of sidebands, which break up the otherwise uniform wave and generate fine localized structures (filamentation). Thus, it may act as a precursor for the formation of bright solitons. Conversely, the existence of stable dark solitons requires the absence of MI in the constant intensity background. The phenomenon of MI has been identified and studied in various physical systems, such as fluids [20], plasma [21], nonlinear optics [22], discrete nonlinear systems [23], waveguide arrays, and Fermi-resonant interfaces [24]. It has been shown that MI is strongly affected by mechanisms such as saturation of the nonlinearity [25], coherence properties of optical beams [26], and linear and nonlinear gratings [27]. The model (2) permits plane wave solutions of the form

$$\psi(\vec{r}, z) = \sqrt{\rho_0}\exp(\mathrm{i}\vec{k}_0 \cdot \vec{r} - \mathrm{i}\beta z), \qquad (6)$$

where $\rho_0 > 0$ is the wave intensity. The parameters satisfy the same dispersion relation $\beta = k_0^2/2 - s\rho_0$ as for the standard local NLS equation, because the response function is normalized. We now consider perturbed plane wave solutions in the form

$$\psi(\vec{r}, z) = \left[\sqrt{\rho_0} + a_1(\vec{\xi}, z)\right]\exp(\mathrm{i}\vec{k}_0 \cdot \vec{r} - \mathrm{i}\beta z), \qquad (7)$$

where

$$a_1(\vec{\xi}, z) = \int \tilde{a}_1(\vec{k})\exp(\mathrm{i}\vec{k}\cdot\vec{\xi} + \lambda z)\,\mathrm{d}\vec{k}, \qquad \vec{\xi} = \vec{r} - \vec{k}_0 z, \quad (8)$$

is the complex amplitude of the small perturbation referred to a coordinate frame moving with the group velocity $\vec{k}_0$. One can show that λ satisfies the equation [28]

$$\lambda^2 = -k^2\rho_0\left[\frac{k^2}{4\rho_0} - s\tilde{R}(k)\right], \qquad (9)$$

where $k = |\vec{k}|$ denotes the spatial frequency and $\tilde{R}(k)$ is the Fourier spectrum of $R(r)$. The sign of λ^2 determines the stability of the solution. For $\lambda^2 > 0$ the perturbation grows exponentially during propagation with the growth rate or gain given by Re $\{\lambda\}$, indicating MI. Equation (9) shows that the stability properties of the plane wave solutions are completely determined by the spectral properties of the nonlocal response function. A detailed analysis of all the possible scenarios is given in [29]. Here we summarize the main points.

The Fourier spectrum of typical response functions, such as Gaussians, Lorentzians and exponentials, is positive definite. Therefore, for a defocusing nonlinearity ($s = -1$), $\lambda^2 < 0$ and the plane wave solutions are always stable. In a focusing medium ($s = +1$) we always have $\lambda^2 > 0$, in a certain wavenumber band symmetrically centred about the origin, where k is sufficiently small. This means that the system will always exhibit a long wave MI in the focusing case, independent of the precise details of the response function. Nonlocality tends to suppress the instability, decreasing the growth rate and the width of the instability band. However, nonlocality can never eliminate the instability completely. This effect is illustrated in figure 2(a), where we show the instability growth rate in the case of a one-dimensional Gaussian response function $R(x) = \exp(-x^2/\sigma^2)/(\sqrt{\pi}\sigma)$.

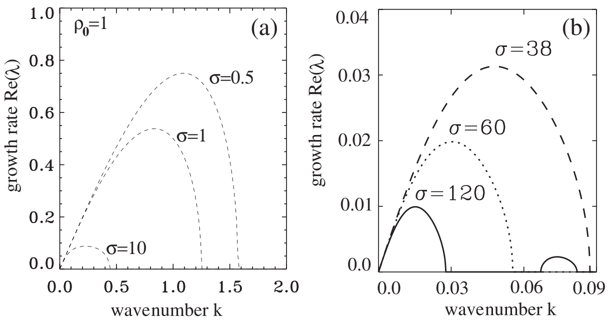

Figure 2. MI gain profile in self-focusing nonlocal media with a Gaussian response function for $\rho_0 = 1$ (a) and with a generalized Lorentzian response function for $n = 2$ and $\rho_0 = 1.25$ (b).

A drastically different behaviour is observed for response functions whose spectrum is not sign definite. Let us consider the generalized Lorentzian [30]

$$R(x) = \frac{\pi}{n\sigma}\frac{1}{\sin(\frac{\pi}{2n})}\frac{1}{1 + (\frac{x}{\sigma})^{2n}}, \qquad n = 2, 3, \ldots \qquad (10)$$

which for large n approximates the rectangular function. The spectrum of this function, which for $n = 2$ is $\tilde{R}(k) = \exp(-\sigma|k|/\sqrt{2})[\cos(\sigma|k|/\sqrt{2}) + \sin(\sigma|k|/\sqrt{2})]$ exhibits periodic changes of its sign. For focusing nonlinearity this may lead to the appearance of higher order instability bands in sufficiently nonlocal media (see figure 2(b) for $\sigma = 120$). Even more dramatic is the behaviour of the plane wave solution in a defocusing medium, where this type of nonlocality may promote a high frequency instability of the otherwise stable wave (see figure 3(a)). In figure 3(b) we show the development of the instability of the plane wave. The response function (10) has recently been discussed in the context of MI of partially coherent waves, where it represents the degree of coherence of the plane wave. Interestingly, it also promoted MI in that case [30]. It is worth mentioning that the spectrum of (10) for any value of the exponent $n > 2$ contains an oscillatory factor responsible for the formation of higher order gain bands in a way analogous to the case $n = 2$.

The above-mentioned stability properties of the plane wave may seem surprising in view of the fact that the nonlocality acts to smooth out any sharp modulations of the wavefront. This is a generic property of the nonlocality independent of the particular functional representation. One would therefore naively expect identical stability properties for all physically reasonable response functions. However, one can also look at the action of the nonlocality from a different perspective. In the Fourier domain the nonlocality acts as a filter with variable transmission determined by the form of the spectrum of the response function. For many nonlocal models, such as that with a Gaussian response

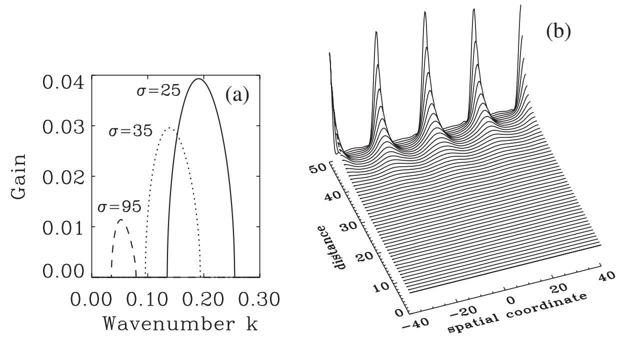

Figure 3. MI in self-defocusing nonlocal media with response function (10). (a) MI gain for $n = 2$ and $\rho_0 = 1$. (b) Development of instability for $n = 6$ and $\rho_0 = 3$.

function, the transmission characteristic of the filter has the form of a well-behaved, sign-definite function. However, in cases, such as those modelled by (10), this filter not only modulates the amplitude of the spectral components of the signal beam (perturbation to the plane wave), but also inverts the phase of selected components (bands). As this inversion is equivalent to a change of the sign of the nonlinearity (say, from defocusing to focusing), it leads, for instance, to amplification of certain harmonics in a defocusing nonlinear media. Recently Peccianti *et al* [31] conducted the first experimental studies of MI in a nonlocal medium. They used nematic liquid crystals, which are known to exhibit orientational nonlocal nonlinearity with a sign-definite exponential response. The effect of suppression of MI was clearly demonstrated.

4. Beam collapse

Collapse is a phenomenon well-known in the theory of wave propagation in nonlinear focusing media. It refers to the situation when strong self-focusing of a beam leads to a catastrophic increase (blow-up) of its intensity in a finite time or after a finite propagation distance [32–34]. Collapse has been observed in plasma waves [35], electromagnetic waves or laser beams [36], BECs [37], and even capillary–gravity waves on deep water [38].

Strictly speaking, the existence of collapse is an artefact of the model equation and signals the limit of its applicability. In the vicinity of the collapse regime some additional (unaccounted for) physical processes will come into play and stop the blow-up. Nevertheless, 'collapse-like' (or quasi-collapse) dynamics can still occur in real physical systems when nonlinearity leads to strong energy localization. In fact, recent experiments with ultra cold gases provided clear signatures of the collapse-like dynamics of atomic condensates [37, 39].

In the past there have been a few attempts to determine the role of nonlocality in the development of the collapse of finite-size beams. Turitsyn was the first to prove analytically the arrest of collapse for a specific choice of the nonlocal nonlinear response [40]. Recently Perez-Garcia *et al* [41] discussed the collapse suppression in the case of weak nonlocality, in a direct application for a BEC. The analysis of the collapse conditions in the case of a general BEC nonlocal model is difficult and has so far only been done numerically [42].

Here we present an analytical approach to beam collapse in nonlocal media, which is based on the technique introduced in [43], but which extends and generalizes the results to be valid for much more general response functions. We consider the general case of *symmetric, but otherwise arbitrarily shaped, non-singular response functions* and prove rigorously that a collapse cannot occur.

For localized or periodic solutions equations (1) and (2) conserve the power (in optics) or number of atoms (for BEC) P and the Hamiltonian H,

$$P = \int I \, \mathrm{d}\vec{r}, \qquad H = \tfrac{1}{2}\|\nabla_\perp \psi\|_2^2 - \tfrac{1}{2}\int N I \, \mathrm{d}\vec{r}, \quad (11)$$

where $\|\nabla_\perp\psi\|_2^2 \equiv \int |\nabla_\perp\psi|^2 \, \mathrm{d}\vec{r}$. In the local limit when the response function is a delta-function, the nonlinear response has the form $N(I) = sI$, as in local optical Kerr media described by the conventional NLS equation and in BECs described by the standard Gross–Pitaevskii equation. In this local limit multidimensional beams with a power higher than a certain critical value will experience unbounded self-focusing and *collapse* after a finite propagation distance.

It can be easily shown that in the two extreme limits of a weakly and highly nonlocal nonlinear response the collapse is prevented [7, 44]. Here we consider the general case of *symmetric, but otherwise arbitrarily shaped, non-singular response functions*. Introducing the D-dimensional Fourier transform and its inverse, it is straightforward to show that for $N(I)$ given by equation (1), when $s = 1$ the following relations hold [43]

$$\begin{aligned} |\tilde{I}(\vec{k})| &= \left| \int I(\vec{r}) e^{i\vec{k}\cdot\vec{r}} \, \mathrm{d}\vec{r} \right| \leqslant P, \\ \int N I \, \mathrm{d}\vec{r} &= \frac{1}{(2\pi)^D} \int \tilde{R}(\vec{k}) |\tilde{I}(\vec{k})|^2 \, \mathrm{d}\vec{k}. \end{aligned} \quad (12)$$

For any response functions for which the spectrum $\tilde{R}(k)$ is absolute integrable, we then have

$$\left| \int N I \, \mathrm{d}\vec{r} \right| \leqslant P^2 R_0, \qquad R_0 \equiv \frac{1}{(2\pi)^D} \int |\tilde{R}(k)| \, \mathrm{d}\vec{k}, \quad (13)$$

and hence we get $|H| \geqslant \|\nabla\psi\|_2^2/2 - R_0 P^2/2$. This inequality shows that the gradient norm $\|\nabla_\perp\psi\|_2^2$ is bounded from above by the conserved quantity $2|H| + R_0 P^2$ for all symmetric response functions under the only requirement that their spectrum is absolute integrable. It represents a rigorous proof that a collapse with the wave-amplitude locally going to infinity cannot occur in BECs or Kerr media with a nonlocal nonlinear response for any physically reasonable response functions, such as Gaussians, exponentials, sech-shaped, and even generalized Lorentzian response functions of the type (10).

This result represents a considerable generalization of the proof presented in [43], which required the spectrum of the response function to be *strictly positive definite*.

We emphasize that although the nonlocality prevents collapse, it does not remove the collapse-like dynamics of high-power beams. On the contrary, as long as the beam width is much larger than the extent of the nonlocality (width of the response function), the beam will naturally contract as in the

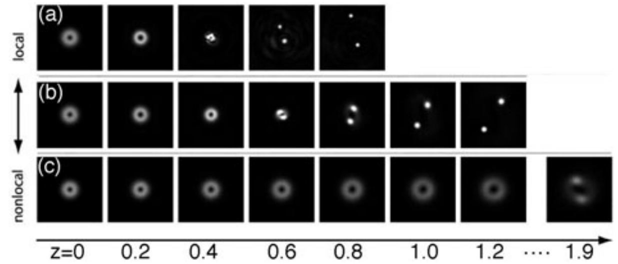

Figure 4. Propagation of a vortex beam (charge 1) in a self-focusing nonlocal Kerr-like medium. The nonlocality parameter $\sigma = 0$ (a), $\sigma = 1$ (b), $\sigma = 10$ (c).

local case. However the collapse will be arrested when the width becomes comparable with that of the response function. Finally, during the stationary phase of the beam propagation, its width will be comparable with the extent of the nonlocality.

The stabilizing effect of the nonlocality can be further illustrated by considering the propagation of a vortex beam in a self-focusing medium. Such a beam is characterized by a bright ring with a helical phase front with so-called charge, defined as a closed loop contour integral of the wave phase modulo 2π. A typical example is the Gaussian–Laguerre beam

$$\psi(\vec{r}) = r\exp[-(r/r_0)^2]\exp(\mathrm{i}\phi), \tag{14}$$

where r and ϕ are the radial and angular coordinates, respectively. This beam represents a vortex of charge one. Beams of such structure have been considered as candidates for vortex-type solitons in nonlinear self-focusing media [45]. However, it is well known that vortex beams cannot form stable stationary structures and disintegrate rather quickly when launched in self-focusing local Kerr nonlinear media [46]. One can expect to achieve a stabilization of the vortex beam by utilizing the nonlocal character of the nonlinearity. If the extent of the nonlocality is comparable with the size of the vortex beam then the resulting refractive index change will take the form of a broad circular waveguide, which could trap the vortex beam ensuring its stable propagation. In figure (4) we illustrate the effect of the nonlocality on propagation of a vortex beam in the form of equation (14). We numerically integrated equation (2) assuming a Gaussian nonlocal response $R(x) = \exp(-x^2/\sigma^2)/(\sqrt{\pi}\sigma)$. The plots in each row show the light intensity distribution at various propagation distances. The upper row represents local Kerr medium ($\sigma = 0$). It is clearly seen that the vortex beam experiences strong focusing and quickly breaks up into two fragments. As the nonlocality engages ($\sigma = 1$, middle row) the vortex instability is restrained and break-up occurs only after longer propagation. The distance of stable propagation significantly increases when nonlocality becomes larger ($\sigma = 10$, bottom row).

5. Nonlocal structure of parametric solitons

Unlike Kerr solitons, the formation of solitons in quadratic nonlinear (or $\chi^{(2)}$) materials does not involve a change of the refractive index [47]. Thus the underlying physics of quadratic solitons is often obscured by the mathematical model. Only recently Assanto and Stegeman used the cascading phase shift and parametric gain to give an intuitive interpretation of self-focusing, defocusing and soliton formation in $\chi^{(2)}$ materials [48]. Nevertheless several features of quadratic solitons are still without a physical interpretation, such as certain structural properties and the formation of bound states. We will show below that quadratic solitons can be described by nonlocal models. Such models provide simple physical explanations of these properties and many more, building on a simple waveguide analogy [49–51].

Consider a fundamental wave (FW) and its second harmonic (SH) propagating along the z-direction in a $\chi^{(2)}$ crystal under conditions for type I phase-matching. The normalized dynamical equations for the slowly varying envelopes $E_{1,2}(x,z)$ are then [52]

$$\begin{aligned} \mathrm{i}\partial_z E_1 + d_1\partial_x^2 E_1 + E_1^* E_2\exp(-\mathrm{i}\beta z) &= 0, \\ \mathrm{i}\partial_z E_2 + d_2\partial_x^2 E_2 + E_1^2\exp(\mathrm{i}\beta z) &= 0. \end{aligned} \tag{15}$$

In the spatial domain $d_1 \approx 2d_2$, $d_{1,2} > 0$ and x represents a transverse spatial direction. In the temporal domain $d_{1,2}$ is arbitrary and x represents time. β is the normalized phase-mismatch. Physical insight into equations (15) may be obtained from the cascading limit, in which the phase-mismatch is large, $\beta^{-1} \to 0$. Writing $E_2 = e_2\exp(\mathrm{i}\beta z)$ and assuming slow variation of $e_2(x,z)$ gives the NLS equation $\mathrm{i}\partial_z E_1 + d_1\partial_x^2 E_1 + \beta^{-1}|E_1|^2 E_1 = 0$, with $e_2 = E_1^2/\beta$. However, this model wrongly predicts several features that are known not to exist in equations (15) and even for stationary solutions it is inaccurate, since the term $\partial_x^2 E_2$ is neglected [51]. To obtain a more accurate model we assume a slow variation of the SH field $e_2(x,z)$ in the propagation direction only (i.e. only $\partial_z e_2$ is neglected), which leads to the *nonlocal equation for the FW*

$$\begin{aligned} \mathrm{i}\partial_z E_1 + d_1\partial_x^2 E_1 + \beta^{-1}N(E_1^2)E_1^* &= 0, \\ N(E_1^2) = \int_{-\infty}^{\infty} R(x-\xi)E_1^2(\xi,z)\,\mathrm{d}\xi&, \end{aligned} \tag{16}$$

with $E_2 = \beta^{-1}N\exp(\mathrm{i}\beta z)$. Equations (16) show that the interaction between the FW and SH is equivalent to the FW propagating in a medium with a nonlocal nonlinearity. In the Fourier domain the response function is a Lorentzian $\tilde{R}(k) = 1/(1 + s\sigma^2k^2)$, where $\sigma = |d_2/\beta|^{1/2}$ represents the degree of nonlocality and $s = \mathrm{sign}(d_2\beta)$. Both equations (15) and (16) are trivially extended to more transverse dimensions.

For $s = +1$, where the $\chi^{(2)}$-system (15) has a family of bright (for $d_1 > 0$) and dark (for $d_1 < 0$) soliton solutions [53], $\tilde{R}(k)$ is positive definite and localized, giving $R(x) = (2\sigma)^{-1}\exp(-|x|/\sigma)$. It is possible to show, e.g., that the nonlocal model (16) does not allow collapse in any physical dimension [51], a known property of the $\chi^{(2)}$ system (15) not captured by the cascading limit NLS equation. The cascading limit $\beta^{-1} \to 0$ is now seen to correspond to the local limit $\sigma \to 0$, in which the response function becomes a delta function, $R(x) \to \delta(x)$. With the nonlocal analogy one can further assign simple physically intuitive models to the weakly nonlocal limit $\sigma \ll 1$ and the strongly nonlocal limit $\sigma \gg 1$.

For $s = -1$, $\tilde{R}(k)$ has poles on the real axis and the response function becomes oscillatory with the Cauchy principal value $R(x) = (2\sigma)^{-1}\sin(|x|/\sigma)$. In this case the propagation of solitons has a close analogy with the evolution of a particle in a nonlinear oscillatory potential. In fact, it is possible to show that the oscillatory response function explains the fact that dark and bright quadratic solitons radiate linear waves [53].

Equations (16) show the important novel result that, in contrast to the conventional nonlocal NLS equation treated in detail in this work, the nonlocal response of the $\chi^{(2)}$ system depends on the square of the FW, not its intensity. Thus, the phase of the FW enters into the picture and one cannot directly transfer the known dynamical properties of plane waves and solitons, such as stability. The general model (16) and its weakly and strongly nonlocal limits thus represents novel equations, whose properties potentially allow us to understand yet unexplained dynamical properties of quadratic solitons.

In contrast, the stationary properties of nonlocal solitons, such as the dependence of their profiles on material parameters, directly apply to quadratic solitons. One can show that the structure of the nonlocal equation governing the stationary fields resulting from (16) is identical to the conventional nonlocal model for stationary solitons and thus it has the same weakly and strongly nonlocal limits with the same exact bright and dark soliton solutions. We recently showed that the nonlocal model elegantly explains the structural properties of both bright and dark solitons and their bound states and that it provides good approximate quadratic soliton solutions in large regimes of the parameter space [51].

6. Interaction of dark nonlocal solitons

To this point, we have concentrated on the properties of individual beams in nonlocal nonlinear media. Because of the specific nature of the nonlocality, which results in spatial advancing of the nonlinearity far beyond the actual spatial location of the beam, it is natural to expect strong influence of the nonlocality on interaction of well separated localized waves and solitons. For instance, in the case of two nearby optical beams, each of them will induce a refractive index change extending into the region of the other one, thereby affecting its trajectory. One can show that in a self-focusing medium nonlocality always provides an attractive force between interacting bright solitons. This effect has been recently demonstrated for the interaction of bright solitons formed in a liquid crystal [54]. It has been shown that even out-of-phase bright spatial solitons, which in a local medium always repel, experienced strong attraction that can only be overcome by a sufficiently large initial divergence of the soliton trajectories [54]. As a consequence of the nonlocality induced attraction, bound states of out-of-phase bright solitons can be formed [55]. In this section we will describe a novel phenomenon of the attraction of dark solitons [56] in nonlocal nonlinear media with a self-defocusing nonlinearity [57].

We will concentrate on the interaction of one-dimensional solitons. Without loss of generality, we consider the exponential response function $R(x) = (2\sigma)^{-1}\exp(-|x|/\sigma)$. Following the discussion in the preceding section about the equivalence between stationary nonlocal and parametric solitons, and the results of [51, 53], the nonlocal NLS equation with such a response function predicts the existence of single fundamental dark soliton solutions with nonmonotonic tails above a certain critical value of σ. As a result bound states involving two or more solitons can be formed.

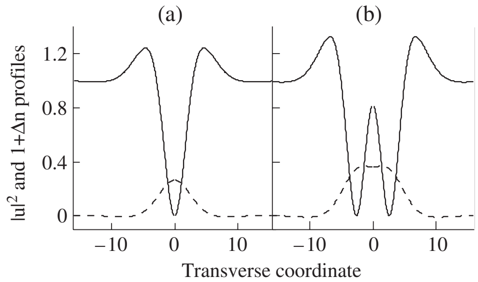

Figure 5. Intensity profile of numerically found stationary dark nonlocal solitons (solid curves). (a) Single dark soliton and (b) a bound state of two dark solitons for $\lambda = 1$ and $\sigma = 4$. The dashed curves indicate the soliton-induced refractive index change.

In figure 5 we show examples of numerically found dark soliton solutions and their bound states. The dashed curve illustrates the soliton-induced waveguide structure which guides the soliton. These solitons appear to be very robust.

The ability of dark solitons to form bound states and their subsequent stability is a direct consequence of the *nonlocality-induced long range attraction of solitons*. This effect can be qualitatively explained using the self-guiding concept. In a local defocusing medium the refractive index change corresponding to two distant dark solitons has the form of two waveguides separated by a region of lower refractive index (a potential barrier). In the presence of nonlocality, the effect of the convolution term in equation (2) is to decrease the index difference between these two separate waveguides (lower the barrier) thereby allowing light to penetrate the area between solitons. This, consequently, manifests as soliton attraction. The attraction of solitons can be clearly demonstrated by simulating numerically the interaction dynamics of two nearby solitons. The results of these simulations are summarized in figures 6(a)–(c). They clearly demonstrate that as the nonlocality parameter σ becomes comparable with the separation of the solitons they strongly attract and trap each other and subsequently propagate together as a bound state exhibiting transverse oscillations.

7. Conclusions

In conclusion, this paper has discussed the properties of optical beams propagating in nonlocal nonlinear Kerr-like media. We have shown that nonlocality leads to a variety of novel phenomena. In particular, it modifies the stability of plane waves, depending on whether the spectrum of the nonlocal response function is positive definite or not. If the spectrum is positive definite then nonlocality always tends to suppress MI, but can never eliminate it completely. If the spectrum is not positive definite then nonlocality may lead to the appearance of higher order instability bands for focusing media. For defocusing media the effect of nonlocality can be even more dramatic as it may actually initiate instability of the plane waves, leading to the formation of localized structures.

In local Kerr media multi-dimensional beams are unstable and collapse if they have a power above a certain critical value. We have shown that a nonlocal nonlinearity always will arrest a collapse and allow for the formation of stable multi-dimensional solitons. Furthermore, we have demonstrated that quadratic solitons formed in the process of second harmonic

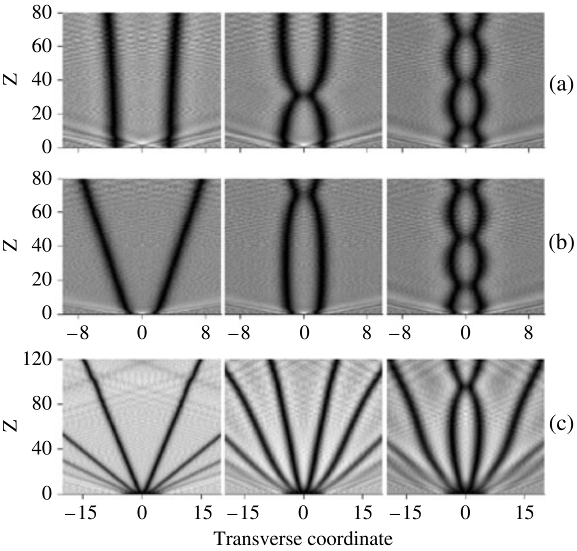

Figure 6. Attraction of dark nonlocal solitons formed either by two closely spaced phase jumps ((a), (b)) or a dark notch (gap) in the initial cw background intensity (c). In (a) the phase jump is π and the degree of nonlocality is $\sigma = 2$, while the initial soliton separation is $x_0 = 5.5, 4, 2.5$ from left to right. In (b) the phase jump is 0.95π and $x_0 = 2.5$, while $\sigma = 0.1, 1, 2$. In (c) the width of the intensity gap is 7.5, while $\sigma = 0.1, 3, 6$.

generation are equivalent to solitons of a nonlocal medium with an exponential response function. Finally, we have shown that nonlocality induces the attraction of normally repelling dark solitons and allows for the formation of stable bound states.

In this paper we have specifically concentrated on systems with spatially nonlocal Kerr responses. However, in many physical systems (BECs [44] as well as pulse propagation in non-isotropic media), the Kerr response is composed of a nonlocal (delayed) response supplemented by a local contribution within a certain ratio. In the modelling of ultrashort pulse propagation nonlocality applies to the temporal response. In that case the causality principle requires the nonlocality to be represented by non-symmetric response functions, and hence the results presented in this paper are not applicable to this situation. For more details on temporal aspects of nonlocality see [58–60] and the references therein.

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