Weighted Sobolev spaces and nonhomogeneous elliptic problems in the half-space

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À mes parents Irma et Max

À ma femme Laurence

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Friedrich Nietzsche, Ainsi parlait Zarathoustra.

[1]Dernières paroles attribuées au Vieux de la Montagne (XI^e siècle).

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Introduction

Le sujet de cette thèse est l'étude de certains problèmes aux limites elliptiques linéaires intervenant en mécanique et en physique, dans la géométrie particulière du demi-espace. De nombreux problèmes de la physique mathématique peuvent être modélisés par des équations aux dérivées partielles dans un tel domaine. C'est le cas en particulier de l'équation des ondes, de l'équation de Helmholtz pour la pression acoustique, des équations de Maxwell pour le champ électromagnétique et des équations de Navier-Stokes en mécanique des fluides. L'étude de ces problèmes célèbres passe par la résolution d'équations portant sur des opérateurs différentiels linéaires de base. Au premier rang de ceux-ci, on trouve l'opérateur de Laplace. En effet, l'équation de Laplace

$$\Delta u = f \quad \text{dans } \Omega,$$

avec une condition aux limites de type Dirichlet

$$u = g \quad \text{sur } \Gamma,$$

ou de type Neumann

$$\frac{\partial u}{\partial \boldsymbol{n}} = g \quad \text{sur } \Gamma,$$

a fait l'objet de nombreuses études tant en domaines bornés qu'en domaines non bornés. Une différence essentielle entre le borné et le non borné tient dans la description du comportement à l'infini des données du problème et de ses solutions éventuelles pour le second cas. Le premier domaine non borné est naturellement l'espace tout entier $\mathbb{R}^N$. Viennent ensuite les problèmes posés dans un domaine extérieur, c'est-à-dire où Ω est le complémentaire d'un compact de $\mathbb{R}^N$. Dans ce cas, la frontière est elle-même compacte et on peut utiliser une partition de l'unité pour ramener le problème posé à la somme de deux problèmes, l'un en domaine borné et l'autre dans $\mathbb{R}^N$. Avec le demi-espace, une nouvelle difficulté apparaît du fait que la frontière est non compacte. On se trouve ainsi conduit à définir des espaces de traces permettant de décrire le comportement à l'infini de leurs éléments. Cependant, la géométrie du demi-espace offre la particularité d'un prolongement des fonctions harmoniques à tout l'espace par le principe de réflexion établi par H.A. Schwarz. Cela permet donc là aussi d'utiliser les résultats précédemment établis dans tout l'espace. Pour aborder ces problèmes en domaines non bornés, plusieurs auteurs ont eu recours aux espaces $\hat{H}_0^{1,p}(\mathbb{R}^N)$ obtenus par complétion de l'espace $\mathcal{D}(\mathbb{R}^N)$ par rapport à la norme $\|\nabla.\|_{L^p(\mathbb{R}^N)}$, ou certains de leurs raffinements, ou encore aux espaces homogènes $D^{1,p}(\mathbb{R}^N)$ des fonctions $L^p_{loc}(\mathbb{R}^N)$ à gradient dans $L^p(\mathbb{R}^N)$. Les résultats obtenus dans ces cadres fonctionnels sont toutefois comparables et présentent l'inconvénient de ne rien dire du comportement à l'infini des données et des solutions. Une autre approche est celle des espaces de Sobolev avec poids. Elle présente l'avantage de donner des informations non seulement sur toutes les dérivées, mais aussi sur les fonctions elles-mêmes. C'est ce cadre fonctionnel que nous adopterons ici.

Outre la description du comportement à l'infini des fonctions en jeu, il y a une raison plus profonde qui impose l'adoption d'un cadre fonctionnel autre que celui des espaces de Sobolev classiques utilisés en domaine borné. En effet, l'inégalité de Poincaré, fondamentale pour la résolution de tels problèmes dans le cas borné, n'est plus satisfaite dans ces géométries pour certains opérateurs classiques (voir l'introduction du chapitre 2 pour l'opérateur biharmonique). Par contre dans les espaces de Sobolev avec poids, on retrouve des inégalités de type Poincaré comme conséquences naturelles d'une inégalité de Hardy ou d'une inégalité de Hardy généralisée. Ces inégalités de Poincaré sont quant à elles au cœur de la méthode variationnelle pour résoudre des problèmes aux limites faisant intervenir des opérateurs elliptiques. Nous utiliserons une classe d'espaces de Sobolev avec des poids logarithmiques (voir [5] et le chapitre 1), qui étendent ceux introduits par B. Hanouzet (voir [33] et la remarque 1.2.1) et qui permettent d'exclure moins de valeurs critiques que ces derniers.

Le demi-espace et l'espace entier sont donc les deux géométries entre lesquelles nous ferons de fréquents allers et retours. Pour revenir à l'équation de Laplace dans ces deux domaines, ou plus exactement tout d'abord à l'équation de Poisson dans $\mathbb{R}^N$, nous nous sommes basés sur les résultats d'isomorphismes établis par C. Amrouche, V. Girault et J. Giroire dans [5] et [6]. Concernant le demi-espace, on trouve des résultats partiels dans des espaces de Sobolev avec poids, avec notamment les travaux de V.G. Maz'ya, B.A. Plamenevskiĭ et L.I. Stupyalis (voir [38]) qui traitent du problème de Stokes et trouvent un champ de vitesses dans $W_0^{1,2}(\mathbb{R}^3_+)$ ou $W_1^{2,2}(\mathbb{R}^3_+)$. Pour le même problème, N. Tanaka (voir [43]) obtient un champ de vitesses dans $W_0^{m+2,2}(\mathbb{R}^3_+)$ avec $m > 0$. Toujours dans le cas hilbertien, T.Z. Boulmezaoud (voir [20]) a obtenu des résultats généraux, mais qui excluent cependant la dimension deux à cause des valeurs critiques inhérentes aux poids qu'il utilise (en fait les espaces de B. Hanouzet). Ces résultats ont été ensuite généralisés en théorie L^p par C. Amrouche et S. Nečasová (voir [7, 8]) pour les dimensions $N \geqslant 2$, avec la résolution de certains cas critiques au moyen de cette classe d'espaces avec des facteurs logarithmiques dans les poids.

Nous venons de parler du système de Stokes et cela nous ramène au sujet de cette thèse, après ce bref et très lacunaire état des lieux ... Notre objectif est de poursuivre ce travail de généralisation pour d'autres opérateurs elliptiques. Par généralisation, nous entendons un travail se développant suivant trois axes. Le premier est de fournir des résultats en théorie L^p pour $1 < p < \infty$, le second est de réduire au mieux les valeurs critiques — sachant que certains résultats peuvent encore être raffinés par l'emploi d'espaces adéquats, ce que nous n'avons pas exploité — et le dernier est d'envisager des conditions aux limites singulières et de chercher des solutions très faibles correspondantes. Notons d'ailleurs que ce dernier axe est devenu peu à peu prépondérant dans ce travail et illustre toute la richesse du raisonnement par dualité mis en œuvre dans ces questions.

Nous avons ainsi abordé dans un premier temps le problème biharmonique, pour utiliser ensuite ces résultats dans le problème de Stokes. Là encore, bien sûr, le terrain n'était pas vierge ! Citons, simplement pour la géométrie qui nous intéresse, le travail de R. Farwig et H. Sohr (voir [28]) pour le système de Stokes. Ces auteurs travaillent dans des espaces homogènes, mais utilisent un schéma de démonstration dont nous nous sommes en partie inspirés pour les solutions généralisées du problème de Stokes, en passant par la résolution d'un problème biharmonique. Les preuves que nous donnons divergent ensuite notablement du fait des cadres fonctionnels et des outils utilisés. Citons à nouveau un travail de T.Z. Boulmezaoud (voir [21]) pour ces deux problèmes. Cet auteur commence par résoudre le problème de Stokes par une méthode qui ne lui permet pas d'obtenir des solutions généralisées et résout ensuite le problème biharmonique. Nous avons cependant utilisé une caractérisation très intéressante qu'il donne du noyau de l'opérateur de Stokes, que nous obtenons de manière différente en partant de celui de l'opérateur biharmonique qui présente l'avantage de s'exprimer beaucoup plus naturellement. Il semblerait donc que nos objectifs de généralisation aient été atteints dans les trois axes évoqués précédemment.

Le découpage en différents chapitres de cette thèse retrace davantage l'ordre chronologique du travail de recherche qu'il ne laisse paraître une unité logique dans les questions abordées. En effet, on peut observer que certaines questions similaires reviennent dans différents chapitres et auraient pu avantageusement être traitées ensemble dans une partie autonome. Il s'agit en particulier des lemmes donnant un sens à des traces de fonctions dans des cas singuliers. On pourra toujours arguer que cela présente l'avantage pédagogique de la motivation des questions au fil de leur apparition naturelle ... Une raison plus prosaïque, mais aussi plus véridique, est que ce découpage provient de la rédaction initiale sous la forme d'articles autonomes pour différentes revues ! En fait la partie délicate de ce travail de synthèse a plutôt été de donner une apparence d'unité à l'ensemble en évitant les redondances. Nous avons en outre conservé la langue anglaise dans laquelle ont été rédigés les articles, non par conviction politique ou philosophique, mais pour une raison plus sordide de temps qui passe toujours trop vite.

Le premier chapitre est naturellement dévolu aux notations, aux définitions et propriétés des espaces fonctionnels et aux résultats fondamentaux sur lesquels nous nous appuyons dans la suite. Il s'agit principalement des résultats sur les problèmes de Dirichlet et de Neumann pour laplacien dans le demi-espace, ainsi que du lemme de traces. Tous ces résultats sont donnés sans démonstration et nous renvoyons à la bibliographie pour les détails.

Dans le second chapitre, nous abordons le problème biharmonique. Nous commençons par donner des résultats d'isomorphismes dans tout l'espace pour l'opérateur biharmonique, en utilisant ceux établis pour le Laplacien dans [5] et [6]. Ensuite, nous passons au problème biharmonique dans le demi-espace avec des conditions aux limites portant sur u et $\partial_N u$, où u est la solution cherchée. Après la caractérisation générale du noyau, nous étudions le problème homogène, pour lequel nous donnons des solutions généralisées et un résultat de régularité, puis nous passons enfin au problème non homogène pour lequel nous fournissons un résultat portant sur les solutions généralisées. Ce chapitre a fait l'objet d'une publication dans la revue Jounal of Differential Equations (voir [9]).

Le troisième chapitre porte sur les solutions fortes et un résultat de régularité pour le même problème. Nous revenons ensuite au problème homogène, pour lequel nous étudions le cas de conditions aux limites singulières et très singulières. Nous obtenons deux résultats par des techniques de dualité, fournissant ainsi des solutions faibles et très faibles de ce problème. Nous envisageons pour finir le cas d'autres conditions aux limites. Ce chapitre a fait l'objet d'un article à paraître dans le numéro de décembre de la revue Communication in Pure and Applied Analysis (voir [11]). Ces deux derniers chapitres ont aussi fait l'objet d'une note aux Comptes Rendus de l'Académie des Sciences (voir [10]).

Au quatrième chapitre, nous commençons l'étude du problème de Stokes avec des conditions de Dirichlet. Nous envisageons tout le spectre des régularités, pour fournir des solutions généralisées, fortes et très faibles, ainsi qu'un résultat de régularité. Nous nous sommes cependant limités ici aux poids basiques pour les comportements à l'infini, afin de dégager l'essentiel de la méthode. Ce chapitre provient lui aussi d'un article rédigé en collaboration avec S. Nečasová et à paraître dans la revue Jounal of Differential Equations (voir [12]).

Nous reprenons au cinquième chapitre le travail du précédent, mais pour tout le spectre des poids et donc des comportements à l'infini possibles. Surgissent alors naturellement la question du noyau dans sa généralité et les conditions d'orthogonalité pour les données. La méthode utilisée précédemment pour les solutions généralisées s'applique encore quand les poids considérés sont négatifs et nous récupérons les poids positifs par dualité. Les résultats donnés ici englobent ainsi ceux du précédent chapitre.

Au sixième chapitre, qui est aussi le fruit d'une collaboration avec S. Nečasová, nous nous intéressons toujours au problème de Stokes, mais avec des conditions de Navier (ou conditions de glissement) sur l'hyperplan frontière. Nous adaptons la méthode utilisée pour les conditions de Dirichlet à ce cas. Les objectifs et le plan sont les mêmes qu'au quatrième chapitre.

Le septième et dernier chapitre envisage un système de Stokes généralisé avec un terme d'élasticité et différents paramètres, système que l'on rencontre dans la littérature sur le sujet (voir H. Beirão da Veiga, [17]). Les conditions aux limites sont là aussi de type Navier. Après une étude préliminaire du problème correspondant dans tout l'espace, qui suit de près celle du problème classique effectuée par F. Alliot et C. Amrouche (voir [3]), nous montrons que la méthode du précédent chapitre s'applique tout à fait à ce cas dans le demi-espace.

Nous ne serions pas tout à fait complets si nous ne parlions pas des « oublis » de ce travail et des perspectives qu'il nous ouvre encore. Pour les questions laissées de côté, nous avons déjà cité la possibilité de combler les vides laissés par certaines valeurs critiques en introduisant des espaces judicieux (voir la référence [6]). Il y a aussi l'approche par les solutions fondamentales, avec par exemple en domaine borné l'article fondateur de L. Cattabriga (voir [24]) et dans le demi-espace les travaux de S. Ukai (voir [44]), puis plus récemment, ceux de M. Cannone, F. Planchon et M. Schonbek (voir [23]), avec laquelle il conviendrait d'établir un pont. Pour le travail en cours, il reste à généraliser aux poids quelconques le problème de Stokes avec des conditions de Navier, ce qui ne doit pas poser de problème. Quant aux perspectives, il faut voir comment utiliser ces résultats pour le problème non linéaire de Navier-Stokes. Une approche possible est de commencer par s'intéresser aux équations d'Oseen dans le demi-espace, sachant que des résultats dans tout l'espace et en domaine extérieur ont déjà été obtenus par C. Amrouche, H. Bouzit et U. Razafison. Il reste aussi tout le champ des problèmes d'évolution dans cette géométrie.

Chapitre 1

Functional framework and known results

1.1 Notations

For any real number $p > 1$, we always take p' to be the Hölder conjugate of p, *i.e.*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let Ω be an open set of $\mathbb{R}^N$, $N \geqslant 2$. Writing a typical point $x \in \mathbb{R}^N$ as $x = (x', x_N)$, where $x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, we will especially look on the upper half-space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N;\ x_N > 0\}$. We let $\overline{\mathbb{R}^N_+}$ denote the closure of $\mathbb{R}^N_+$ in $\mathbb{R}^N$ and let $\Gamma = \{x \in \mathbb{R}^N;\ x_N = 0\} \equiv \mathbb{R}^{N-1}$ denote its boundary. Let $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$ denote the Euclidean norm of x, we will use two basic weights

$$\varrho = (1 + |x|^2)^{1/2} \quad \text{and} \quad \lg \varrho = \ln(2 + |x|^2).$$

We denote by ∂_i the partial derivative $\frac{\partial}{\partial x_i}$, similarly $\partial_i^2 = \partial_i \circ \partial_i = \frac{\partial^2}{\partial x_i^2}$, $\partial_{ij}^2 = \partial_i \circ \partial_j = \frac{\partial^2}{\partial x_i \partial x_j}$, ... More generally, if $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{N}^N$ is a multi-index, then

$$\partial^\lambda = \partial_1^{\lambda_1} \cdots \partial_N^{\lambda_N} = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \cdots \partial x_N^{\lambda_N}} \quad \text{, where } |\lambda| = \lambda_1 + \cdots + \lambda_N.$$

In the sequel, for any integer q, we will use the following polynomial spaces:
— $\mathcal{P}_q$ is the space of polynomials of degree smaller than or equal to q;
— $\mathcal{P}_q^\Delta$ is the subspace of harmonic polynomials of $\mathcal{P}_q$;
— $\mathcal{P}_q^{\Delta^2}$ is the subspace of biharmonic polynomials of $\mathcal{P}_q$;
— $\mathcal{A}_q^\Delta$ is the subspace of polynomials of $\mathcal{P}_q^\Delta$, odd with respect to x_N, or equivalently, which satisfy the condition $\varphi(x', 0) = 0$;
— $\mathcal{N}_q^\Delta$ is the subspace of polynomials of $\mathcal{P}_q^\Delta$, even with respect to x_N, or equivalently, which satisfy the condition $\partial_N \varphi(x', 0) = 0$;

with the convention that these spaces are reduced to $\{0\}$ if $q < 0$.
For any real number s, we denote by $[s]$ the integer part of s.
Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X, *i.e.*

$$B' \perp X = \{f \in B';\ \forall v \in X,\ \langle f, v\rangle = 0\} = (B/X)'.$$

Lastly, if $k \in \mathbb{Z}$, we will constantly use the notation $\{1, \dots, k\}$ for the set of the first k positive integers, with the convention that this set is empty if k is nonpositive.

1.2 Weighted Sobolev spaces

For any nonnegative integer m, real numbers $p > 1$, α and β, we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha,\beta}(\Omega) = \Big\{ & u \in \mathcal{D}'(\Omega);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta-1}\,\partial^\lambda u \in L^p(\Omega); \\ & k+1 \leqslant |\lambda| \leqslant m,\ \varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta}\,\partial^\lambda u \in L^p(\Omega) \Big\}, \end{aligned} \tag{1.2.1}$$

where

$$k = \begin{cases} -1 & \text{if } \frac{N}{p} + \alpha \notin \{1, \dots, m\}, \\ m - \frac{N}{p} - \alpha & \text{if } \frac{N}{p} + \alpha \in \{1, \dots, m\}. \end{cases}$$

Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$\begin{aligned} \|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(& \sum_{0 \leqslant |\lambda| \leqslant k} \|\varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta-1}\,\partial^\lambda u\|^p_{L^p(\Omega)} \\ & + \sum_{k+1 \leqslant |\lambda| \leqslant m} \|\varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta}\,\partial^\lambda u\|^p_{L^p(\Omega)} \Big)^{1/p}. \end{aligned}$$

We also define the semi-norm:

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{|\lambda| = m} \|\varrho^{\alpha}\,(\lg \varrho)^{\beta}\,\partial^\lambda u\|^p_{L^p(\Omega)} \Big)^{1/p}.$$

The weights in definition (1.2.1) are chosen so that the corresponding space satisfies two fundamental properties. On the one hand, $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. On the other hand, the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ (see [7], Theorem 1.1): if

$$\frac{N}{p} + \alpha \notin \{1, \dots, m\} \quad \text{or} \quad (\beta - 1)p \neq -1, \tag{1.2.2}$$

then the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/\mathcal{P}_{q^*}$ a norm which is equivalent to the quotient norm,

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+), \quad \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/\mathcal{P}_{q^*}} \leqslant C\,|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}, \tag{1.2.3}$$

with $q^* = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. Now, we define the space

$$\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}};$$

which will be characterized in Lemma 1.3.1 as the subspace of functions with null traces in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. From that, we can introduce the space $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$ as the dual space of $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. In addition, under the assumption (1.2.2), $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ is a norm on $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ which is equivalent to the full norm,

$$\forall u \in \overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+), \quad \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)} \leqslant C\,|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}. \tag{1.2.4}$$

We will now recall some properties of the weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. We have the algebraic and topological imbeddings:

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \hookrightarrow W^{m-1,p}_{\alpha-1,\beta}(\mathbb{R}^N_+) \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta}(\mathbb{R}^N_+) \;\text{ if }\; \frac{N}{p} + \alpha \notin \{1,\dots,m\}.$$

When $\frac{N}{p} + \alpha = j \in \{1,\dots,m\}$, then we have:

$$W^{m,p}_{\alpha,\beta} \hookrightarrow \cdots \hookrightarrow W^{m-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{m-j,p}_{\alpha-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta-1}.$$

Note that in the first case, for any $\gamma \in \mathbb{R}$ such that $\frac{N}{p} + \alpha - \gamma \notin \{1,\dots,m\}$ and $m \in \mathbb{N}$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \longmapsto \varrho^\gamma u \in W^{m,p}_{\alpha-\gamma,\beta}(\mathbb{R}^N_+)$$

is an isomorphism. In both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \longmapsto \partial^\lambda u \in W^{m-|\lambda|,p}_{\alpha,\beta}(\mathbb{R}^N_+)$$

is continuous. Finally, it can be readily checked that the highest degree q of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is given by

$$q = \begin{cases} m - \left(\frac{N}{p} + \alpha\right) - 1, & \text{if } \begin{cases} \frac{N}{p} + \alpha \in \{1,\dots,m\} \;\text{ and }\; (\beta-1)p \geqslant -1, \\ \text{or} \\ \frac{N}{p} + \alpha \in \{j \in \mathbb{Z};\ j \leqslant 0\} \;\text{ and }\; \beta p \geqslant -1, \end{cases} \\ \left[m - \left(\frac{N}{p} + \alpha\right)\right], & \text{otherwise.} \end{cases} \tag{1.2.5}$$

Remark 1.2.1. In the case $\beta = 0$, we simply denote the space $W^{m,\,p}_{\alpha,0}(\Omega)$ by $W^{m,\,p}_{\alpha}(\Omega)$. In [33], Hanouzet introduced a class of weighted Sobolev spaces without logarithmic factors, with the same notation. We recall his definition under the notation $H^{m,\,p}_{\alpha}(\Omega)$:

$$H^{m,\,p}_{\alpha}(\Omega) = \Big\{u \in \mathcal{D}'(\Omega);\ 0 \leqslant |\lambda| \leqslant m,\ \varrho^{\alpha-m+|\lambda|}\,\partial^{\lambda} u \in L^p(\Omega)\Big\}.$$

It is clear that if $\frac{N}{p} + \alpha \notin \{1, \ldots, m\}$, we have $W^{m,\,p}_{\alpha}(\Omega) = H^{m,\,p}_{\alpha}(\Omega)$. The fundamental difference between these two families of spaces is that the assumption (1.2.2) and thus the Poincaré-type inequality (1.2.3), hold for any value of $(N,\, p,\, \alpha)$ in $W^{m,\,p}_{\alpha}(\Omega)$, but not in $H^{m,\,p}_{\alpha}(\Omega)$ if $\frac{N}{p} + \alpha \in \{1, \ldots, m\}$. $\diamondsuit$

1.3 The spaces of traces

In order to define the traces of functions of $W^{m,\,p}_{\alpha}(\mathbb{R}^N_+)$ (here we don't consider the case $\beta \neq 0$), for any $\sigma \in\,]0, 1[$, we introduce the space:

$$\begin{aligned} W^{\sigma,\,p}_{0}(\mathbb{R}^N) = \Big\{ & u \in \mathcal{D}'(\mathbb{R}^N);\ w^{-\sigma} u \in L^p(\mathbb{R}^N) \ \text{ and} \\ & \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < \infty \Big\}, \end{aligned} \tag{1.3.1}$$

where $w = \varrho$ if $N/p \neq \sigma$ and $w = \varrho\,(\lg \varrho)^{1/\sigma}$ if $N/p = \sigma$. It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{\sigma,\,p}_{0}(\mathbb{R}^N)} = \left(\Big\|\frac{u}{w^{\sigma}}\Big\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

Similarly, for any real number $\alpha \in \mathbb{R}$, we define the space:

$$\begin{aligned} W^{\sigma,\,p}_{\alpha}(\mathbb{R}^N) = \Big\{ & u \in \mathcal{D}'(\mathbb{R}^N);\ w^{\alpha-\sigma} u \in L^p(\mathbb{R}^N), \\ & \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\varrho^{\alpha}(x)\,u(x) - \varrho^{\alpha}(y)\,u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < \infty \Big\}, \end{aligned}$$

where $w = \varrho$ if $N/p + \alpha \neq \sigma$ and $w = \varrho\,(\lg \varrho)^{1/(\sigma-\alpha)}$ if $N/p + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$\begin{aligned} & W^{s,\,p}_{\alpha}(\mathbb{R}^N) = \Big\{u \in \mathcal{D}'(\mathbb{R}^N);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha-s+|\lambda|}\,(\lg \varrho)^{-1}\,\partial^{\lambda} u \in L^p(\mathbb{R}^N); \\ & k+1 \leqslant |\lambda| \leqslant [s]-1,\ \varrho^{\alpha-s+|\lambda|}\,\partial^{\lambda} u \in L^p(\mathbb{R}^N);\ |\lambda| = [s],\ \partial^{\lambda} u \in W^{\sigma,\,p}_{\alpha}(\mathbb{R}^N)\Big\}, \end{aligned}$$

where $k = s - N/p - \alpha$ if $N/p + \alpha \in \{\sigma, \ldots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. It is a reflexive Banach space equipped with the norm:

$$\|u\|_{W^{s,p}_\alpha(\mathbb{R}^N)} = \Big(\sum_{0 \leqslant |\lambda| \leqslant k} \|\varrho^{\alpha - s + |\lambda|} (\lg \varrho)^{-1} \partial^\lambda u\|^p_{L^p(\mathbb{R}^N)} + \sum_{k+1 \leqslant |\lambda| \leqslant [s]-1} \|\varrho^{\alpha - s + |\lambda|} \partial^\lambda u\|^p_{L^p(\mathbb{R}^N)} \Big)^{1/p} + \sum_{|\lambda| = [s]} \|\partial^\lambda u\|_{W^{\sigma,p}_\alpha(\mathbb{R}^N)}.$$

We can similarly define, for any real number β, the space:

$$W^{s,p}_{\alpha,\beta}(\mathbb{R}^N) = \Big\{ v \in \mathcal{D}'(\mathbb{R}^N);\ (\lg \varrho)^\beta\, v \in W^{s,p}_\alpha(\mathbb{R}^N) \Big\}.$$

We can prove some properties of the weighted Sobolev spaces $W^{s,p}_{\alpha,\beta}(\mathbb{R}^N)$. We have the algebraic and topological imbeddings in the case where $N/p + \alpha \notin \{\sigma, \ldots, \sigma + [s] - 1\}$:

$$W^{s,p}_{\alpha,\beta}(\mathbb{R}^N) \hookrightarrow W^{s-1,p}_{\alpha-1,\beta}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{\sigma,p}_{\alpha-[s],\beta}(\mathbb{R}^N),$$
$$W^{s,p}_{\alpha,\beta}(\mathbb{R}^N) \hookrightarrow W^{[s],p}_{\alpha+[s]-s,\beta}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-s,\beta}(\mathbb{R}^N).$$

When $N/p + \alpha = j \in \{\sigma, \ldots, \sigma + [s] - 1\}$, then we have:

$$W^{s,p}_{\alpha,\beta} \hookrightarrow \cdots \hookrightarrow W^{s-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{s-j,p}_{\alpha-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{\sigma,p}_{\alpha-[s],\beta-1},$$
$$W^{s,p}_{\alpha,\beta} \hookrightarrow W^{[s],p}_{\alpha+[s]-s,\beta} \hookrightarrow \cdots \hookrightarrow W^{[s]-j+1,p}_{\alpha-\sigma-j+1,\beta} \hookrightarrow W^{[s]-j,p}_{\alpha-\sigma-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-s,\beta-1}.$$

If u is a function on $\mathbb{R}^N_+$, we denote its trace of order j on the hyperplane Γ by:

$$\forall j \in \mathbb{N}, \quad \gamma_j u : x' \in \mathbb{R}^{N-1} \longmapsto \partial^j_N u(x', 0).$$

Let us recall the following traces lemma due to Hanouzet (see [33]) and extended by Amrouche-Nečasová (see [7]) to this class of weighted Sobolev spaces:

Lemma 1.3.1. *For any integer $m \geqslant 1$ and real number α, the mapping*

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) :\ \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1}),$$

can be extended to a linear continuous mapping, still denoted by γ,

$$\gamma :\ W^{m,p}_\alpha(\mathbb{R}^N_+) \longrightarrow \prod_{j=0}^{m-1} W^{m-j-1/p,p}_\alpha(\mathbb{R}^{N-1}).$$

Moreover γ is surjective and $\mathrm{Ker}\gamma = \mathring{W}^{m,p}_\alpha(\mathbb{R}^N_+)$.

1.4 The Laplace equation in $\mathbb{R}^N_+$

We shall now recall the fundamental results of the Laplace equation in the half-space, with nonhomogeneous Dirichlet or Neumann boundary conditions. These results have been proved by Boulmezaoud (see [20]) in the particular case where $p = 2$ for $N \geqslant 3$, then generalized by Amrouche-Nečasová (see [7]) and Amrouche (see [8]) in L^p theory for $N \geqslant 2$, with solutions of some critical cases by means of logarithmic factors in the weight. Let us also quote the partial results of Maz'ya-Plamenevskiĭ-Stupyalis (see [38]) for the Stokes system in $\mathbb{R}^3_+$ with the velocity obtained in $W^{1,\,2}_0(\mathbb{R}^3_+)$ or $W^{2,\,2}_1(\mathbb{R}^3_+)$, and those of Tanaka (see [43]) for the same problem and the velocity vector field in $W^{m+2,\,2}_0(\mathbb{R}^3_+)$ with $m > 0$.

Let us first recall the main result of the Dirichlet problem

$$(P_D)\ \begin{cases} \Delta u = f & \text{in } \mathbb{R}^N_+, \\ u = g & \text{on } \Gamma, \end{cases}$$

with a different behaviour at infinity according to ℓ.

Theorem 1.4.1 (Amrouche-Nečasová [7]). *Let $\ell \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell\}. \tag{1.4.1}$$

For any $f \in W^{-1,\,p}_{\ell}(\mathbb{R}^N_+)$ and $g \in W^{1-1/p,\,p}_{\ell}(\Gamma)$ satisfying the compatibility condition

$$\begin{aligned} &\forall \varphi \in \mathcal{A}^{\Delta}_{[1+\ell-N/p']}, \\ &\langle f, \varphi \rangle_{W^{-1,\,p}_{\ell}(\mathbb{R}^N_+) \times \mathring{W}^{1,\,p'}_{-\ell}(\mathbb{R}^N_+)} = \langle g, \partial_N \varphi \rangle_{W^{1-1/p,\,p}_{\ell}(\Gamma) \times W^{-1/p',\,p'}_{-\ell}(\Gamma)}, \end{aligned} \tag{1.4.2}$$

problem (P_D) admits a solution $u \in W^{1,\,p}_{\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$, and there exists a constant C such that

$$\inf_{q \in \mathcal{A}^{\Delta}_{[1-\ell-N/p]}} \|u + q\|_{W^{1,\,p}_{\ell}(\mathbb{R}^N_+)} \ \leqslant\ C \left(\|f\|_{W^{-1,\,p}_{\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{1-1/p,\,p}_{\ell}(\Gamma)} \right).$$

The second recall deals with this problem for more regular data.

Theorem 1.4.2 (Amrouche-Nečasová [7]). *Let $\ell \in \mathbb{Z}$ and $m \geqslant 1$ be two integers such that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell + 1\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell - m\}. \tag{1.4.3}$$

For any $f \in W^{m-1,\,p}_{m+\ell}(\mathbb{R}^N_+)$ and $g \in W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$, satisfying the compatibility condition (1.4.2), problem (P_D) has a solution $u \in W^{m+1,\,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$, and there exists a constant C such that

$$\inf_{q \in \mathcal{A}^{\Delta}_{[1-\ell-N/p]}} \|u + q\|_{W^{m+1,\,p}_{m+\ell}(\mathbb{R}^N_+)} \ \leqslant\ C \left(\|f\|_{W^{m-1,\,p}_{m+\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)} \right).$$

Concerning the Neumann problem

$$(P_N)\quad \begin{cases} \Delta u = f & \text{in } \mathbb{R}^N_+,\\ \partial_N u = g & \text{on } \Gamma, \end{cases}$$

let us first recall the result for the weakest data.

Theorem 1.4.3 (Amrouche [8]). *Let $\ell \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell+1\}. \tag{1.4.4}$$

For any $f \in W^{0,\,p}_{\ell}(\mathbb{R}^N_+)$ and $g \in W^{-1/p,\,p}_{\ell-1}(\Gamma)$ satisfying the compatibility condition

$$\begin{aligned} &\forall \varphi \in \mathcal{N}^{\Delta}_{[\ell-N/p']},\\ &\langle f, \varphi\rangle_{W^{0,\,p}_{\ell}(\mathbb{R}^N_+)\times W^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)} + \langle g, \varphi\rangle_{W^{-1/p,\,p}_{\ell-1}(\Gamma)\times W^{1-1/p',\,p'}_{-\ell+1}(\Gamma)} = 0, \end{aligned} \tag{1.4.5}$$

problem (P_N) admits a solution $u \in W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{N}^{\Delta}_{[2-\ell-N/p]}$, and there exists a constant C such that

$$\inf_{q\in\mathcal{N}^{\Delta}_{[2-\ell-N/p]}} \|u+q\|_{W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)} \;\leqslant\; C\left(\|f\|_{W^{0,\,p}_{\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{-1/p,\,p}_{\ell-1}(\Gamma)}\right).$$

As for the Dirichlet problem, we can prove the following result:

Theorem 1.4.4. *Let $\ell \in \mathbb{Z}$ and $m \geqslant 0$ be two integers such that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell-m\}. \tag{1.4.6}$$

For any $f \in W^{m,\,p}_{m+\ell}(\mathbb{R}^N_+)$ and $g \in W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$ satisfying the compatibility condition (1.4.5), problem (P_N) has a solution $u \in W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{N}^{\Delta}_{[2-\ell-N/p]}$, and there exists a constant C such that

$$\inf_{q\in\mathcal{N}^{\Delta}_{[2-\ell-N/p]}} \|u+q\|_{W^{m+1,\,p}_{m+\ell-1}(\mathbb{R}^N_+)} \;\leqslant\; C\left(\|f\|_{W^{m-1,\,p}_{m+\ell-1}(\mathbb{R}^N_+)} + \|g\|_{W^{m-1/p,\,p}_{m+\ell-1}(\Gamma)}\right).$$

Chapitre 2

Generalized solutions to the biharmonic problem in $\mathbb{R}^N$ & $\mathbb{R}^N_+$

2.1 Introduction

The purpose of this chapter is the resolution of the biharmonic problem with nonhomogeneous boundary conditions

$$(P)\quad \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma = \mathbb{R}^{N-1}, \\ \partial_N u = g_1 & \text{on } \Gamma. \end{cases}$$

Since this problem is posed in the half-space, it is important to specify the behaviour at infinity for the data and solutions. We have chosen to impose such conditions by setting our problem in weighted Sobolev spaces, where the growth or decay of functions at infinity are expressed by means of weights. These weighted Sobolev spaces provide a correct functional setting for unbounded domains, in particular because the functions in these spaces satisfy an optimal weighted Poincaré-type inequality. The weights chosen here behave at infinity as powers to $|x|$. The reason of this choice is given by the behaviour at infinity of the fundamental solution E_N to the biharmonic operator in $\mathbb{R}^N$. Let us recall for instance that

$$E_3(x) = c_3\,|x|, \quad E_4(x) = c_4 \ln|x|, \quad E_5(x) = \frac{c_5}{|x|},$$

and in particular if $f \in \mathcal{D}(\mathbb{R}^N)$, the convolution $E_N * f$ behaves at infinity as E_N. In this work, we shall consider more general data f; and the solutions will have a behaviour at infinity which will naturally depend on the one of data in $\mathbb{R}^N_+$ and on the boundary.

Let us throw light on this functional framework in the L^2 case. If we consider Problem (P) with homogeneous boundary conditions, *i.e.* $g_0 = g_1 = 0$, we can give the following variational formulation: For any given $f \in V'$, find $u \in V$ such that

$$\forall v \in V, \quad \int_{\mathbb{R}^N_+} \Delta u \, \Delta v \, dx = \langle f, v \rangle_{V' \times V} \, .$$

Which is the appropriate space V to use the Lax-Milgram's lemma? We must have firstly, for any $v \in V$, $\Delta v \in L^2(\mathbb{R}^N_+)$ and secondly, the coercivity condition for the bilinear form: $(u, v) \longmapsto \int_{\mathbb{R}^N_+} \Delta u \, \Delta v \, dx$.

According to 1.2.4, we have:

$$\forall v \in \overset{\circ}{W}{}^{2,2}_{0}(\mathbb{R}^N_+), \quad \|v\|_{W^{2,2}_0(\mathbb{R}^N_+)} \leqslant C \, \|\nabla^2 v\|_{L^2(\mathbb{R}^N_+)^{N^2}} .$$

Moreover,

$$\forall v \in \overset{\circ}{W}{}^{2,2}_{0}(\mathbb{R}^N_+), \quad \|\nabla^2 v\|_{L^2(\mathbb{R}^N_+)^{N^2}} = \|\Delta v\|_{L^2(\mathbb{R}^N_+)},$$

hence the coercivity of the form. Consequently, Problem (P) with $g_0 = g_1 = 0$ is well-posed on $V = \overset{\circ}{W}{}^{2,2}_{0}(\mathbb{R}^N_+)$. Which are the appropriate spaces of traces for the complete problem? Thanks to Lemma 1.3.1,

$$u \in W^{2,2}_0(\mathbb{R}^N_+) \ \Rightarrow \ (\gamma_0 u, \, \gamma_1 u) \in W^{3/2,2}_0(\mathbb{R}^{N-1}) \times W^{1/2,2}_0(\mathbb{R}^{N-1}),$$

consequently we must take $(g_0, \, g_1) \in W^{3/2,2}_0(\mathbb{R}^{N-1}) \times W^{1/2,2}_0(\mathbb{R}^{N-1})$ in the problem with nonhomogeneous boundary conditions.

Remark 2.1.1. If we consider the problem for the operator $I + \Delta^2$:

$$(\mathcal{Q}) \begin{cases} u + \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma, \\ \partial_N u = g_1 & \text{on } \Gamma, \end{cases}$$

we have the following variational formulation with $g_0 = g_1 = 0$: For any given $f \in V'$, find $u \in V$ such that $\forall v \in V$,

$$\int_{\mathbb{R}^N_+} u \, v \, dx + \int_{\mathbb{R}^N_+} \Delta u \, \Delta v \, dx = \langle f, v \rangle_{V' \times V} \, .$$

This form satisfies naturally the coercivity condition on $V = H^2_0(\mathbb{R}^N_+)$, where $H^2_0(\mathbb{R}^N_+)$ denotes here the classical Sobolev space of functions $v \in H^2(\mathbb{R}^N_+)$ such that $v = \partial_N v = 0$ on Γ. For the nonhomogeneous problem, we must take $(g_0, \, g_1) \in H^{3/2}(\mathbb{R}^{N-1}) \times H^{1/2}(\mathbb{R}^{N-1})$. ♢

Our analysis is based on the isomorphism properties of the biharmonic operator in the whole space and the resolution of the Dirichlet and Neumann problems for the Laplacian in the half-space. This last one is itself based on the isomorphism properties of the Laplace operator in the whole space and also on the reflection principle inherent in the half-space. Note here the double difficulty arising from the unboundedness of the domain in any direction and from the unboundedness of the boundary itself.

2.2 The biharmonic operator in $\mathbb{R}^N$

In this section, we shall give some isomorphism results relative to the biharmonic operator in the whole space. We shall rest on these for our investigation in the half-space. At first, we characterize the kernel

$$K = \left\{v \in W_\ell^{2,p}(\mathbb{R}^N);\ \Delta^2 v = 0 \ \text{ in } \mathbb{R}^N\right\}.$$

Lemma 2.2.1. *Let $\ell \in \mathbb{Z}$.*

(i) *If $\frac{N}{p} \notin \{1, \dots, -\ell\}$, then $K = \mathcal{P}^{\Delta^2}_{[2-\ell-N/p]}$.*

(ii) *If $\frac{N}{p} \in \{1, \dots, -\ell\}$, then $K = \mathcal{P}^{\Delta^2}_{1-\ell-N/p}$.*

Proof. Let $u \in K$. As we know that $\Delta^2 u = 0$ and moreover $u \in W_\ell^{2,p}(\mathbb{R}^N) \subset \mathcal{S}'(\mathbb{R}^N)$, the space of tempered distributions, we can deduce that u is a polynomial on $\mathbb{R}^N$. But according to (1.2.5), we know that the highest degree q of the polynomials contained in $W_\ell^{2,p}(\mathbb{R}^N)$ is given by:

$$q = \begin{cases} 1-\ell-N/p & \text{if } \frac{N}{p}+\ell \in \{j \in \mathbb{Z};\ j \leqslant 0\}, \\ [2-\ell-N/p] & \text{otherwise.} \end{cases}$$

We can thus see the conditions of the statement appear precisely. □

More generally, for any integer $m \in \mathbb{N}$, we define the kernel

$$K^m = \left\{v \in W_{m+\ell}^{m+2,p}(\mathbb{R}^N);\ \Delta^2 v = 0 \ \text{ in } \mathbb{R}^N\right\}.$$

The same arguments lead us to a result which includes the precedent, corresponding then to case $m = 0$.

Lemma 2.2.2. *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that*

(i) *$\frac{N}{p} \notin \{1, \dots, -\ell - m\}$, then $K^m = \mathcal{P}^{\Delta^2}_{[2-\ell-N/p]}$.*

(ii) *$\frac{N}{p} \in \{1, \dots, -\ell - m\}$, then $K^m = \mathcal{P}^{\Delta^2}_{1-\ell-N/p}$.*

We can now formulate the first result of isomorphism in $\mathbb{R}^N$:

Theorem 2.2.3. *Let $\ell \in \mathbb{Z}$. Under hypothesis (1.4.1), the following operator is an isomorphism:*

$$\Delta^2 : W_\ell^{2,p}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W_\ell^{-2,p}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}.$$

Proof. Let us recall (see [5]) that under assumption (1.4.1), the operator

$$\Delta : W^{2,p}_{\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta}_{[2-\ell-N/p]} \longrightarrow W^{0,p}_{\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta}_{[\ell-N/p']} \tag{2.2.1}$$

is an isomorphism. By duality, we can deduce that it is the same for the operator

$$\Delta : W^{0,p}_{\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta}_{[-\ell-N/p]} \longrightarrow W^{-2,p}_{\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta}_{[2+\ell-N/p']}. \tag{2.2.2}$$

If we suppose now that $\ell - N/p' < 0$, we can compose isomorphisms (2.2.1) and (2.2.2) to deduce that the operator

$$\Delta^2 : W^{2,p}_{\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{-2,p}_{\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta}_{[2+\ell-N/p']} \tag{2.2.3}$$

is an isomorphism. By duality, we can deduce that the operator

$$\Delta^2 : W^{2,p}_{\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta}_{[2-\ell-N/p]} \longrightarrow W^{-2,p}_{\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']} \tag{2.2.4}$$

is an isomorphism provided that we have $-\ell - N/p < 0$.

To combine (2.2.3) and (2.2.4), it remains to be noted that if $\ell - N/p' < 0$, then we have $\mathcal{P}^{\Delta^2}_{[2+\ell-N/p']} = \mathcal{P}^{\Delta}_{[2+\ell-N/p']} = \mathcal{P}_{[2+\ell-N/p']}$; and symmetrically, if $-\ell - N/p < 0$, we have $\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} = \mathcal{P}^{\Delta}_{[2-\ell-N/p]} = \mathcal{P}_{[2-\ell-N/p]}$. Moreover, if we note that the reunion of those two cases covers all integers $\ell \in \mathbb{Z}$, we can deduce that for any $\ell \in \mathbb{Z}$ satisfying (1.4.1), the operator

$$\Delta^2 : W^{2,p}_{\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{-2,p}_{\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']} \tag{2.2.5}$$

is an isomorphism. □

We can establish now a result for more regular data, with two preliminary lemmas.

Lemma 2.2.4. *Let $m \geqslant 1$ and $\ell \leqslant -2$ be two integers such that*

$$\frac{N}{p} \notin \{1, \dots, -\ell - m\}, \tag{2.2.6}$$

then the following operator is an isomorphism:

$$\Delta^2 : W^{m+2,p}_{m+\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{m-2,p}_{m+\ell}(\mathbb{R}^N).$$

Proof. We use here another isomorphism result (see [6]). Let $m \geqslant 1$ and $\ell \leqslant -1$ be two integers. Under hypothesis (2.2.6), the Laplace operator

$$\Delta : W^{m+1,p}_{m+\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta}_{[1-\ell-N/p]} \longrightarrow W^{m-1,p}_{m+\ell}(\mathbb{R}^N), \tag{2.2.7}$$

is an isomorphism. Then, replacing m by $m - 1$ and ℓ by $\ell + 1$, we can obtain that for $m \geqslant 2$ and $\ell \leqslant -2$, under hypothesis (2.2.6), the operator

$$\Delta : W^{m,p}_{m+\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta}_{[-\ell-N/p]} \longrightarrow W^{m-2,p}_{m+\ell}(\mathbb{R}^N), \tag{2.2.8}$$

is an isomorphism. Moreover (see [5]), for $\ell \leqslant -2$, the operator

$$\Delta : W^{1,p}_{1+\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta}_{[-\ell-N/p]} \longrightarrow W^{-1,p}_{1+\ell}(\mathbb{R}^N) \quad \text{if } N/p \notin \{1, \dots, -\ell-1\}, \tag{2.2.9}$$

is an isomorphism. Then, combining (2.2.8) and (2.2.9), we can deduce that for $m \geqslant 1$ and $\ell \leqslant -2$, under hypothesis (2.2.6), the operator

$$\Delta : W^{m,p}_{m+\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta}_{[-\ell-N/p]} \longrightarrow W^{m-2,p}_{m+\ell}(\mathbb{R}^N), \tag{2.2.10}$$

is an isomorphism. Replacing now m by $m+1$ and ℓ by $\ell-1$ in (2.2.7), we obtain that for $m \geqslant 0$ and $\ell \leqslant 0$, under hypothesis (2.2.6), the operator

$$\Delta : W^{m+2,p}_{m+\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta}_{[2-\ell-N/p]} \longrightarrow W^{m,p}_{m+\ell}(\mathbb{R}^N), \tag{2.2.11}$$

is an isomorphism. The lemma follows from the composition of isomorphisms (2.2.10) and (2.2.11). □

Lemma 2.2.5. *Let $m \geqslant 1$ an integer such that*

$$\frac{N}{p'} \neq 1 \quad \textit{or} \quad m = 1,$$

then the biharmonic operator

$$\Delta^2 : W^{m+2,p}_{m-1}(\mathbb{R}^N)/\mathcal{P}_{[3-N/p]} \longrightarrow W^{m-2,p}_{m-1}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']}$$

is an isomorphism.

Proof. Let us note that it suffices to prove that the operator is surjective. Here again, we compose two Laplace operators. We have the following isomorphism (see [5]): for $m \in \mathbb{N}$,

$$\Delta : W^{1+m,p}_{m}(\mathbb{R}^N)/\mathcal{P}_{[1-N/p]} \longrightarrow W^{-1+m,p}_{m}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']} \quad \text{if } N/p' \neq 1 \quad \text{or} \quad m = 0. \tag{2.2.12}$$

Replacing m by $m-1$, we obtain that for $m \geqslant 1$, the operator

$$\Delta : W^{m,p}_{m-1}(\mathbb{R}^N)/\mathcal{P}_{[1-N/p]} \longrightarrow W^{m-2,p}_{m-1}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']} \quad \text{if } N/p' \neq 1 \quad \text{or} \quad m = 1, \tag{2.2.13}$$

is an isomorphism. Composing with (2.2.11), for $\ell = -1$, we obtain the result. □

We can now give a global result for the biharmonic operator.

Theorem 2.2.6.

(i) *Let $\ell \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \dots, \ell + 1\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \dots, -\ell - 1\},$$

then the biharmonic operator

$$\Delta^2 : W^{3,\,p}_{\ell+1}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{-1,\,p}_{\ell+1}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}$$

is an isomorphism.

(ii) *Let $\ell \in \mathbb{Z}$ and $m \geqslant 2$ be two integers such that*

$$\frac{N}{p'} \notin \{1, \dots, \ell + 2\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \dots, -\ell - m\},$$

then the biharmonic operator

$$\Delta^2 : W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}$$

is an isomorphism.

Proof. For $\ell \leqslant -1$, it's clear that lemmas 2.2.4 and 2.2.5 exactly cover points (i) and (ii). It remains to establish the theorem for $\ell \geqslant 0$.

According to [5], for $\ell \geqslant 0$, the following operator is an isomorphism:

$$\begin{aligned} \Delta : W^{1,\,p}_{\ell+1}(\mathbb{R}^N) \longrightarrow W^{-1,\,p}_{\ell+1}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta}_{[2+\ell-N/p']} \\ \text{if } N/p' \notin \{1, \dots, \ell + 1\}. \end{aligned} \tag{2.2.14}$$

For $m \geqslant 1$ and $\ell \geqslant 1$, we also have the isomorphism:

$$\begin{aligned} \Delta : W^{m+1,\,p}_{m+\ell}(\mathbb{R}^N) \longrightarrow W^{m-1,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta}_{[1+\ell-N/p']} \\ \text{if } N/p' \notin \{1, \dots, \ell + 1\}. \end{aligned} \tag{2.2.15}$$

Replacing m by $m - 1$ and ℓ by $\ell + 1$, we deduce for $m \geqslant 2$ and $\ell \geqslant 0$, the isomorphism:

$$\begin{aligned} \Delta : W^{m,\,p}_{m+\ell}(\mathbb{R}^N) \longrightarrow W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta}_{[2+\ell-N/p']} \\ \text{if } N/p' \notin \{1, \dots, \ell + 2\}. \end{aligned} \tag{2.2.16}$$

Replacing m by $m + 1$ and ℓ by $\ell - 1$ in (2.2.15), we obtain for $m \geqslant 1$ and $\ell \geqslant 2$, the isomorphism:

$$\begin{aligned} \Delta : W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N) \longrightarrow W^{m,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta}_{[\ell-N/p']} \\ \text{if } N/p' \notin \{1, \dots, \ell\}. \end{aligned} \tag{2.2.17}$$

And now replacing m by $m+1$ in (2.2.12), we obtain for $m \geqslant 1$, the isomorphism:

$$\begin{aligned}\Delta : W^{m+2,\,p}_{m+1}(\mathbb{R}^N)/\mathcal{P}_{[1-N/p]} \longrightarrow\ & W^{m,\,p}_{m+1}(\mathbb{R}^N) \perp \mathcal{P}_{[1-N/p']} \\ & \text{if } N/p' \neq 1.\end{aligned} \tag{2.2.18}$$

Finally, if we return to (2.2.7) with $\ell = -1$ and $m+1$ instead of m, we have for $m \geqslant 1$, the isomorphism:

$$\Delta : W^{m+2,\,p}_{m}(\mathbb{R}^N)/\mathcal{P}_{[2-N/p]} \longrightarrow\ W^{m,\,p}_{m}(\mathbb{R}^N). \tag{2.2.19}$$

Then, combining (2.2.17), (2.2.18) and (2.2.19), we obtain for $m \geqslant 1$ and $\ell \geqslant 0$, the isomorphism:

$$\begin{aligned}\Delta : W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N)/\mathcal{P}_{[2-\ell-N/p]} \longrightarrow\ & W^{m,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta}_{[\ell-N/p']} \\ & \text{if } N/p' \notin \{1,\dots,\ell\}.\end{aligned} \tag{2.2.20}$$

It remains to justify the orthogonality condition to compose (2.2.20) with (2.2.14) or (2.2.16), which will give us respectively the isomorphisms of points (i) and (ii).

Let $f \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}$ with $m \geqslant 1$, then we have $f \perp \mathcal{P}^{\Delta}_{[2+\ell-N/p']}$ and according to (2.2.14) or (2.2.16), there exists $u \in W^{m,\,p}_{m+\ell}(\mathbb{R}^N)$ such that $\Delta u = f$. We will show that $u \perp \mathcal{P}^{\Delta}_{[\ell-N/p']}$. Let $\psi \in \mathcal{P}^{\Delta}_{[\ell-N/p']}$, we know that there exists $\varphi \in \mathcal{P}_{[2+\ell-N/p']}$ such that $\psi = \Delta\varphi$, *i.e.* $\varphi \in \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}$.

(a) Case $m = 1$: $u \in W^{1,\,p}_{\ell+1}(\mathbb{R}^N)$, $f \in W^{-1,\,p}_{\ell+1}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}$.

Let us note that $\psi \in W^{0,\,p'}_{-\ell}(\mathbb{R}^N)$ and $\varphi \in W^{2,\,p'}_{-\ell}(\mathbb{R}^N)$, since $\frac{N}{p'} \notin \{1,\dots,\ell\}$. We also have the imbedding $W^{2,\,p'}_{-\ell}(\mathbb{R}^N) \hookrightarrow W^{1,\,p'}_{-\ell-1}(\mathbb{R}^N)$, since $\frac{N}{p'} \neq \ell+1$. Then, we have $\psi = \Delta\varphi \in W^{-1,\,p'}_{-\ell-1}(\mathbb{R}^N)$. This implies

$$\begin{aligned}\langle u, \psi\rangle_{W^{1,\,p}_{\ell+1}(\mathbb{R}^N)\times W^{-1,\,p'}_{-\ell-1}(\mathbb{R}^N)} &= \langle u, \Delta\varphi\rangle_{W^{1,\,p}_{\ell+1}(\mathbb{R}^N)\times W^{-1,\,p'}_{-\ell-1}(\mathbb{R}^N)} \\ &= \langle \Delta u, \varphi\rangle_{W^{-1,\,p}_{\ell+1}(\mathbb{R}^N)\times W^{1,\,p'}_{-\ell-1}(\mathbb{R}^N)} \\ &= \langle f, \varphi\rangle_{W^{-1,\,p}_{\ell+1}(\mathbb{R}^N)\times W^{1,\,p'}_{-\ell-1}(\mathbb{R}^N)} \\ &= 0.\end{aligned}$$

(b) Case $m \geqslant 2$: $u \in W^{m,\,p}_{m+\ell}(\mathbb{R}^N)$, $f \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}$.

Since $\ell \geqslant 0$, we have $\frac{N}{p} + m + \ell \notin \{1,\dots,m\}$, therefore we can deduce the chain of imbeddings $W^{m,\,p}_{m+\ell}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{1,\,p}_{\ell+1}(\mathbb{R}^N)$. Moreover $\frac{N}{p'} \neq \ell+2$, then we also have $W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\ell+2}(\mathbb{R}^N) \hookrightarrow W^{-1,\,p}_{\ell+1}(\mathbb{R}^N)$. After that, we repeat the reasoning of case $m = 1$.

Then, we have $u \in W^{m,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta}_{[\ell-N/p']}$, and (2.2.20) shows us that there exists $z \in W^{m+2,\,p}_{m}(\mathbb{R}^N)$ such that $\Delta z = u$. Thus it follows that the operator

$$\Delta^2 : W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}$$

is an isomorphism. $\square$

2.3 Generalized solutions in $\mathbb{R}^N_+$

In this section, we shall deal with Problem (P) in the half-space.

For any $q \in \mathbb{Z}$, we introduce the space $\mathcal{B}_q$ as a subspace of $\mathcal{P}_q^{\Delta^2}$:

$$\mathcal{B}_q = \left\{ u \in \mathcal{P}_q^{\Delta^2};\ u = \partial_N u = 0 \text{ on } \Gamma \right\}.$$

We shall establish the main theorem of this chapter:

Theorem 2.3.1. *Let $\ell \in \mathbb{Z}$ and assume that*

$$\frac{N}{p'} \notin \{1, \dots, \ell\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \dots, -\ell\}. \tag{2.3.1}$$

For any $f \in W_\ell^{-2,p}(\mathbb{R}^N_+)$, $g_0 \in W_\ell^{2-1/p,p}(\Gamma)$ and $g_1 \in W_\ell^{1-1/p,p}(\Gamma)$ satisfying the compatibility condition

$$\begin{aligned} &\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']}, \\ &\qquad \langle f, \varphi \rangle_{W_\ell^{-2,p}(\mathbb{R}^N_+) \times \mathring{W}_{-\ell}^{2,p'}(\mathbb{R}^N_+)} + \langle g_1, \Delta\varphi \rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi \rangle_\Gamma = 0, \end{aligned} \tag{2.3.2}$$

problem (P) admits a solution $u \in W_\ell^{2,p}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, and there exists a constant C such that

$$\inf_{q \in \mathcal{B}_{[2-\ell-N/p]}} \|u + q\|_{W_\ell^{2,p}(\mathbb{R}^N_+)} \leqslant C \left(\|f\|_{W_\ell^{-2,p}(\mathbb{R}^N_+)} + \|g_0\|_{W_\ell^{2-1/p,p}(\Gamma)} + \|g_1\|_{W_\ell^{1-1/p,p}(\Gamma)} \right).$$

NB: (a) $\langle g_1, \Delta\varphi \rangle_\Gamma$ denotes the duality bracket $\langle g_1, \Delta\varphi \rangle_{W_\ell^{1-1/p,p}(\Gamma) \times W_{-\ell}^{-1/p',p'}(\Gamma)}$, and $\langle g_0, \partial_N \Delta\varphi \rangle_\Gamma$ the duality bracket $\langle g_0, \partial_N \Delta\varphi \rangle_{W_\ell^{2-1/p,p}(\Gamma) \times W_{-\ell}^{-1-1/p',p'}(\Gamma)}$.

(b) With hypothesis (2.3.1) on critical values, we find hypothesis (1.4.1) of Theorems 1.4.1 and 2.2.3.

2.3.1 Characterization of the kernel

Let us denote by $\mathcal{K}$ the kernel of the operator

$$(\Delta^2, \gamma_0, \gamma_1) : W_\ell^{2,p}(\mathbb{R}^N_+) \longrightarrow W_\ell^{-2,p}(\mathbb{R}^N_+) \times W_\ell^{2-1/p,p}(\Gamma) \times W_\ell^{1-1/p,p}(\Gamma),$$

i.e.

$$\mathcal{K} = \left\{ u \in W_\ell^{2,p}(\mathbb{R}^N_+);\ \Delta^2 u = 0 \text{ in } \mathbb{R}^N_+,\ u = \partial_N u = 0 \text{ on } \Gamma \right\}.$$

The following characterization uses the reflection principle (see Farwig [27]).

Lemma 2.3.2. *Let $\ell \in \mathbb{Z}$.*

(i) *If* $\frac{N}{p} \notin \{1, \ldots, -\ell\}$, *then* $\mathcal{K} = \mathcal{B}_{[2-\ell-N/p]}$.

(ii) *If* $\frac{N}{p} \in \{1, \ldots, -\ell\}$, *then* $\mathcal{K} = \mathcal{B}_{1-\ell-N/p}$.

Proof. Given $u \in \mathcal{K}$, we set

$$\tilde{u}(x', x_N) \;=\; \begin{cases} u(x', x_N) & \text{if } x_N \geqslant 0, \\ (-u - 2x_N\partial_N u - x_N^2 \Delta u)(x', -x_N) & \text{if } x_N < 0. \end{cases} \tag{2.3.3}$$

Then we have $\tilde{u} \in \mathcal{S}'(\mathbb{R}^N)$ and we show that $\Delta^2 \tilde{u} = 0$ in $\mathbb{R}^N$. We can deduce that $\tilde{u}$, and consequently u, is a polynomial. Furthermore, $u \in W^{2,p}_\ell(\mathbb{R}^N_+)$ implies that its maximum degree is the same as in Lemma 2.2.1. $\square$

More generally, for any $m \in \mathbb{N}$, we denote by $\mathcal{K}^m$ the kernel of the operator

$$(\Delta^2, \gamma_0, \gamma_1) : W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+) \longrightarrow W^{m-2,p}_{m+\ell}(\mathbb{R}^N_+) \times W^{m+2-1/p,p}_{m+\ell}(\Gamma) \times W^{m+1-1/p,p}_{m+\ell}(\Gamma),$$

i.e.

$$\mathcal{K}^m = \left\{ u \in W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+);\ \Delta^2 u = 0 \text{ in } \mathbb{R}^N_+,\ u = \partial_N u = 0 \text{ on } \Gamma \right\}.$$

Identical arguments lead us to the following result:

Lemma 2.3.3. *Let* $\ell \in \mathbb{Z}$ *and* $m \in \mathbb{N}$.

(i) *If* $\frac{N}{p} \notin \{1, \ldots, -\ell - m\}$, *then* $\mathcal{K}^m = \mathcal{B}_{[2-\ell-N/p]}$.

(ii) *If* $\frac{N}{p} \in \{1, \ldots, -\ell - m\}$, *then* $\mathcal{K}^m = \mathcal{B}_{1-\ell-N/p}$.

We now introduce the two operators Π_D and Π_N, defined by:

$$\forall r \in \mathcal{A}^\Delta_k, \quad \Pi_D r = \frac{1}{2} \int_0^{x_N} t\, r(x', t)\, dt,$$
$$\forall s \in \mathcal{N}^\Delta_k, \quad \Pi_N s = \frac{1}{2}\, x_N \int_0^{x_N} s(x', t)\, dt.$$

So we obtain the second characterization of $\mathcal{K}^m$:

Lemma 2.3.4. *Let* $\ell \in \mathbb{Z}$ *and* $m \in \mathbb{N}$. *Under hypothesis (2.2.6), we have*

$$\mathcal{K}^m = \mathcal{B}_{[2-\ell-N/p]} = \Pi_D \mathcal{A}^\Delta_{[-\ell-N/p]} \oplus \Pi_N \mathcal{N}^\Delta_{[-\ell-N/p]}. \tag{2.3.4}$$

Proof. A direct calculation with these operators yields the following formulas:

$$\forall r \in \mathcal{A}^\Delta_k, \quad \begin{cases} \Delta \Pi_D r = r & \text{in } \mathbb{R}^N_+, \\ \partial_N \Pi_D r = \dfrac{1}{2} x_N r & \text{in } \mathbb{R}^N_+, \\ \Pi_D r = \partial_N \Pi_D r = 0 & \text{on } \Gamma, \end{cases} \tag{2.3.5}$$

and

$$\forall s \in \mathcal{N}_k^\Delta, \quad \begin{cases} \Delta \Pi_N s = s & \text{in } \mathbb{R}_+^N, \\ \partial_N \Pi_N s = \dfrac{1}{2}\left(x_N s + \displaystyle\int_0^{x_N} s(x', t)\, dt\right) & \text{in } \mathbb{R}_+^N, \\ \Pi_N s = \partial_N \Pi_N s = 0 & \text{on } \Gamma. \end{cases} \tag{2.3.6}$$

Moreover, for any $r \in \mathcal{A}_k^\Delta$ and $s \in \mathcal{N}_k^\Delta$, $\Pi_D r \in \mathcal{P}_{k+2}$ and $\Pi_N s \in \mathcal{P}_{k+2}$. Thus, if $r \in \mathcal{A}^\Delta_{[-\ell-N/p]}$ and $s \in \mathcal{N}^\Delta_{[-\ell-N/p]}$, we can deduce that $\Pi_D r \in \mathcal{B}_{[2-\ell-N/p]}$ and $\Pi_N s \in \mathcal{B}_{[2-\ell-N/p]}$.
Conversely, if we consider $u \in \mathcal{B}_{[2-\ell-N/p]}$, then we have $\Delta u \in \mathcal{P}^\Delta_{[-\ell-N/p]}$. Since $\mathcal{P}^\Delta_{[-\ell-N/p]} = \mathcal{A}^\Delta_{[-\ell-N/p]} \oplus \mathcal{N}^\Delta_{[-\ell-N/p]}$, there exists $(r, s) \in \mathcal{A}^\Delta_{[-\ell-N/p]} \times \mathcal{N}^\Delta_{[-\ell-N/p]}$ such that $\Delta u = r + s$ in $\mathbb{R}^N_+$. According to formulas (2.3.5) and (2.3.6), the function $z = u - \Pi_D r - \Pi_N s$ satisfies: $\Delta z = 0$ in $\mathbb{R}^N_+$ and $z = \partial_N z = 0$ on Γ. The function z belonging to $\mathcal{A}^\Delta_{[2-\ell-N/p]} \cap \mathcal{N}^\Delta_{[2-\ell-N/p]} = \{0\}$, then $u = \Pi_D r + \Pi_N s$. Furthermore, the sum (2.3.4) is direct, because if $(r, s) \in \mathcal{A}^\Delta_{[-\ell-N/p]} \times \mathcal{N}^\Delta_{[-\ell-N/p]}$ such that $\Pi_D r = \Pi_N s = u$, then $\Delta u = r = s$. That implies $\Delta u = 0$ in $\mathbb{R}^N_+$ with $u = \partial_N u = 0$ on Γ, hence $u = 0$ in $\mathbb{R}^N_+$. □

The following proposition clarifies the kernel $\mathcal{B}_{[2-\ell-N/p]}$ in the simplest cases.

Proposition 2.3.5. *Let $\ell \in \mathbb{Z}$ such that $\frac{N}{p} \notin \{1, \ldots, -\ell\}$.*

(i) *If $-\ell - N/p < 0$, then $\mathcal{B}_{[2-\ell-N/p]} = \{0\}$.*

(ii) *If $0 < -\ell - N/p < 1$, then $\mathcal{B}_{[2-\ell-N/p]} = \mathbb{R}\, x_N^2$.*

Proof. If $-\ell - N/p < 0$, then we have $\mathcal{B}_{[2-\ell-N/p]} \subset \mathcal{P}_1$. Now, if $\varphi \in \mathcal{P}_1$ with $\varphi = \partial_N \varphi = 0$ on Γ, we necessarily have $\varphi = 0$. If $0 < -\ell - N/p < 1$, then $\mathcal{B}_{[2-\ell-N/p]} = \mathcal{B}_2 = \left\{\varphi \in \mathcal{P}_2^{\Delta^2};\ \varphi = \partial_N \varphi = 0 \text{ on } \Gamma\right\}$. Now, if $\varphi \in \mathcal{P}_2$ with $\varphi = \partial_N \varphi = 0$ on Γ, a direct calculation shows that $\varphi(x) = c\, x_N^2$, where $c \in \mathbb{R}$. □

Remark 2.3.6. This proposition yields an answer to important particular cases:

(i) If $\ell \geqslant 0$ or ($\ell = -1$ and $N/p > 1$), then $\mathcal{B}_{[2-\ell-N/p]} = \{0\}$.

(ii) If $\ell = -1$ and $N/p < 1$, then $\mathcal{B}_{[3-N/p]} = \mathcal{B}_2 = \mathbb{R}\, x_N^2$. ♢

2.3.2 The compatibility condition

We shall now show the necessity of condition (2.3.2) in Theorem 2.3.1.

Lemma 2.3.7. *Let $\ell \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell\}. \tag{2.3.7}$$

Let $f \in W^{-2,p}_\ell(\mathbb{R}^N_+)$, $g_0 \in W^{2-1/p,p}_\ell(\Gamma)$ and $g_1 \in W^{1-1/p,p}_\ell(\Gamma)$. If problem (P) admits a solution in $W^{2,p}_\ell(\mathbb{R}^N_+)$, then we have the compatibility condition:

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']},\ \langle f, \varphi\rangle_{W^{-2,p}_\ell(\mathbb{R}^N_+)\times \mathring{W}^{2,p'}_{-\ell}(\mathbb{R}^N_+)} + \langle g_1, \Delta\varphi\rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi\rangle_\Gamma = 0,$$

where $\langle g_1, \Delta\varphi\rangle_\Gamma$ denotes the duality bracket $\langle g_1, \Delta\varphi\rangle_{W^{1-1/p,p}_\ell(\Gamma)\times W^{-1/p',p'}_{-\ell}(\Gamma)}$ and $\langle g_0, \partial_N\Delta\varphi\rangle_\Gamma$ the duality bracket $\langle g_0, \partial_N\Delta\varphi\rangle_{W^{2-1/p,p}_\ell(\Gamma)\times W^{-1-1/p',p'}_{-\ell}(\Gamma)}$.

Remark 2.3.8. By Proposition 2.3.5, if $\ell - N/p' < 0$ and particularly if $\ell \leqslant 0$, we have $\mathcal{B}_{[2+\ell-N/p']} = \{0\}$. Thus there is no compatibility condition in these cases. $\diamondsuit$

Proof. So we assume that $\ell \geqslant 1$. The first point is to justify the dualities in the spaces of traces. Noting that under hypothesis (2.3.7), for any $\varphi \in \mathcal{B}_{[2+\ell-N/p']}$, we have $\varphi \in W^{3,p'}_{-\ell+1}(\mathbb{R}^N_+)$ and also $\varphi \in W^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)$, we can deduce that $\Delta\varphi|_\Gamma \in W^{1-1/p',p'}_{-\ell+1}(\Gamma)$ and $\partial_N\Delta\varphi|_\Gamma \in W^{1-1/p',p'}_{-\ell+2}(\Gamma)$. It remains to verify the imbeddings

$$W^{1-1/p',p'}_{-\ell+1}(\Gamma) \hookrightarrow W^{-1/p',p'}_{-\ell}(\Gamma), \tag{2.3.8}$$
$$W^{1-1/p',p'}_{-\ell+2}(\Gamma) \hookrightarrow W^{-1-1/p',p'}_{-\ell}(\Gamma). \tag{2.3.9}$$

(i) To show (2.3.8), we break down this imbedding into

$$W^{1-1/p',p'}_{-\ell+1}(\mathbb{R}^{N-1}) \hookrightarrow W^{0,p'}_{-\ell+1/p'}(\mathbb{R}^{N-1}), \tag{2.3.10}$$
$$W^{0,p'}_{-\ell+1/p'}(\mathbb{R}^{N-1}) \hookrightarrow W^{-1/p',p'}_{-\ell}(\mathbb{R}^{N-1}), \tag{2.3.11}$$

where (2.3.11) is equivalent by duality to

$$W^{1/p',p}_{\ell}(\mathbb{R}^{N-1}) \hookrightarrow W^{0,p}_{\ell-1/p'}(\mathbb{R}^{N-1}). \tag{2.3.12}$$

Observe that (2.3.10) is satisfied if and only if $\frac{N-1}{p'} - \ell + 1 \neq 1 - \frac{1}{p'}$, *i.e.* $\frac{N}{p'} \neq \ell$, which is included in (2.3.7). Likewise (2.3.12) holds if and only if $\frac{N-1}{p} + \ell \neq \frac{1}{p'}$, *i.e.* $\frac{N}{p} \neq -\ell + 1$, which cannot happen for $\ell \geqslant 1$.

(ii) Similarly, the imbedding (2.3.9) is equivalent to

$$W^{1-1/p',p'}_{-\ell+2}(\mathbb{R}^{N-1}) \hookrightarrow W^{0,p'}_{-\ell+1+1/p'}(\mathbb{R}^{N-1}) \tag{2.3.13}$$
$$W^{1+1/p',p}_{\ell}(\mathbb{R}^{N-1}) \hookrightarrow W^{0,p}_{\ell-1-1/p'}(\mathbb{R}^{N-1}). \tag{2.3.14}$$

The imbedding (2.3.13) holds if and only if $\frac{N}{p'} \neq \ell - 1$, which is included in (2.3.7). The imbedding (2.3.14) holds if and only if $\frac{N}{p} \notin \{-\ell+1,\ -\ell+2\}$. Since $\ell \geqslant 1$, it suffices that $\frac{N}{p} \neq 1$ for $\ell = 1$. Assume that $\ell = 1$ and $\frac{N}{p} = 1$, then we have $\frac{N}{p'} = N - 1$ and thus $\mathcal{B}_{[2+\ell-N/p']} = \mathcal{B}_{[4-N]}$. If $N \geqslant 3$, there is no compatibility condition because $\mathcal{B}_{[4-N]} = \{0\}$. If $N = 2$, then we have $p = p' = 2$ and $\frac{N}{p'} = 1$, but that is excluded by (2.3.7).

Now it is clear that for any $u \in \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ we have

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']}, \ \int_{\mathbb{R}^N_+} \varphi\, \Delta^2 u\, dx = \int_\Gamma u\, \Delta\partial_N \varphi\, dx' - \int_\Gamma \partial_N u\, \Delta\varphi\, dx'.$$

Let $u \in W^{2,p}_\ell(\mathbb{R}^N_+)$ and $\varphi \in \mathcal{B}_{[2+\ell-N/p']}$. Thanks to the density of $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $W^{2,p}_\ell(\mathbb{R}^N_+)$, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ such that $u_k \to u$ in $W^{2,p}_\ell(\mathbb{R}^N_+)$. Therefore $\Delta^2 u_k \to \Delta^2 u$ in $W^{-2,p}_\ell(\mathbb{R}^N_+)$, $u_k \to u$ in $W^{2-1/p,p}_\ell(\Gamma)$ and $\partial_N u_k \to \partial_N u$ in $W^{1-1/p,p}_\ell(\Gamma)$. Writing the previous formula for any u_k, we obtain by passing to the limit as $k \to \infty$

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']}, \ \big\langle \Delta^2 u, \varphi \big\rangle_{W^{-2,p}_\ell(\mathbb{R}^N_+)\times \mathring{W}^{2,p'}_{-\ell}(\mathbb{R}^N_+)} = \langle u, \partial_N\Delta\varphi\rangle_\Gamma - \langle \partial_N u, \Delta\varphi\rangle_\Gamma\,.$$

This proves the necessity of condition (2.3.2). □

2.3.3 The homogeneous problem

Here we consider the homogeneous problem in $\mathbb{R}^N_+$, *i.e.* $f = 0$, with standard boundary conditions. Let the problem

$$(P^0)\ \begin{cases} \Delta^2 u = 0 & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma, \\ \partial_N u = g_1 & \text{on } \Gamma, \end{cases}$$

with $g_0 \in W^{2-1/p,p}_\ell(\Gamma)$ and $g_1 \in W^{1-1/p,p}_\ell(\Gamma)$.

Lemma 2.3.9. *Let $\ell \in \mathbb{Z}$. Under hypothesis (2.3.1), for any $g_0 \in W^{2-1/p,p}_\ell(\Gamma)$ and $g_1 \in W^{1-1/p,p}_\ell(\Gamma)$ satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']}\ , \quad \langle g_1, \Delta\varphi\rangle_\Gamma - \langle g_0, \partial_N\Delta\varphi\rangle_\Gamma = 0, \tag{2.3.15}$$

problem (P^0) admits a solution $u \in W^{2,p}_\ell(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, with the estimate

$$\inf_{q\in\mathcal{B}_{[2-\ell-N/p]}} \|u+q\|_{W^{2,p}_\ell(\mathbb{R}^N_+)} \leqslant C\left(\|g_0\|_{W^{2-1/p,p}_\ell(\Gamma)} + \|g_1\|_{W^{1-1/p,p}_\ell(\Gamma)}\right).$$

Proof. Firstly, thanks to Lemma 2.3.4, note that condition (2.3.15) is equivalent to both conditions

$$\forall r \in \mathcal{A}^\Delta_{[\ell-N/p']}, \qquad \langle g_0, \partial_N r\rangle_\Gamma = 0, \tag{2.3.16}$$

$$\forall s \in \mathcal{N}^\Delta_{[\ell-N/p']}, \qquad \langle g_1, s\rangle_\Gamma = 0. \tag{2.3.17}$$

Consider the Dirichlet problem:

$$(R^0)\ \begin{cases} \Delta\vartheta = 0 & \text{in } \mathbb{R}^N_+, \\ \vartheta = g_0 & \text{on } \Gamma. \end{cases}$$

Since $g_0 \in W^{2-1/p,\,p}_{\ell}(\Gamma) = W^{1+1-1/p,\,p}_{1+(\ell-1)}(\Gamma)$, Theorem 1.4.2 holds with $m = 1$ and $\ell - 1$ instead of ℓ. Then hypothesis (1.4.3) becomes $\frac{N}{p'} \notin \{1, \dots, \ell\}$ and $\frac{N}{p} \notin \{1, \dots, -\ell\}$. Moreover compatibility condition (1.4.2) corresponds precisely to (2.3.16). We can deduce that problem (R^0) admits a solution $\vartheta \in W^{2,\,p}_{\ell}(\mathbb{R}^N_+)$.

Consider now the Neumann problem:

$$(S^0)\ \begin{cases} \Delta\zeta = 0 & \text{in } \mathbb{R}^N_+, \\ \partial_N\zeta = g_1 & \text{on } \Gamma. \end{cases}$$

Theorem 1.4.4 holds with $m = 0$. Moreover compatibility condition (1.4.5) corresponds precisely to (2.3.17). We can deduce that problem (S^0) admits a solution $\zeta \in W^{2,\,p}_{\ell}(\mathbb{R}^N_+)$. So we can readily verify that the function defined by

$$u = x_N\, \partial_N(\zeta - \vartheta) + \vartheta \tag{2.3.18}$$

is a solution to (P^0). However we must show that $u \in W^{2,\,p}_{\ell}(\mathbb{R}^N_+)$. For this, we remark that u satisfies

$$(T)\ \begin{cases} \Delta u = 2\,\partial_N^2(\zeta - \vartheta) & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma, \end{cases}$$

with $2\,\partial_N^2(\zeta - \vartheta) \in W^{0,\,p}_{\ell}(\mathbb{R}^N_+)$ and $g_0 \in W^{2-1/p,\,p}_{\ell}(\Gamma)$.

(i) If $\frac{N}{p} \neq -\ell + 1$, then we have the imbedding $W^{2,\,p}_{\ell}(\mathbb{R}^N_+) \hookrightarrow W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$. By (2.3.18), we deduce that $u \in W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$. Furthermore we have the following Green formula:

$$\forall r \in \mathcal{A}^{\Delta}_{[\ell-N/p']}, \quad \langle \Delta u, r\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} = \langle u, \partial_N r\rangle_{W^{1-1/p,\,p}_{\ell-1}(\Gamma)\times W^{-1/p',\,p'}_{-\ell+1}(\Gamma)}\,,$$

i.e.

$$\forall r \in \mathcal{A}^{\Delta}_{[\ell-N/p']}, \quad \big\langle 2\,\partial_N^2(\zeta - \vartheta), r\big\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} = \langle g_0, \partial_N r\rangle_{\Gamma}\,.$$

Thus the compatibility condition of problem $(\mathcal{T})$ is satisfied and thanks to Theorem 1.4.4, it admits a solution $y \in W^{2,\,p}_{\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta}_{[2-\ell-N/p]}$. So the function $z = u - y \in W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$ and satisfies

$$\begin{cases} \Delta z = 0 & \text{in } \mathbb{R}^N_+, \\ z = 0 & \text{on } \Gamma. \end{cases}$$

We can deduce that $z \in \mathcal{A}^{\Delta}_{[2-\ell-N/p]}$, *i.e.* $u = y + r$ with $r \in \mathcal{A}^{\Delta}_{[2-\ell-N/p]} \subset W^{2,p}_{\ell}(\mathbb{R}^N_+)$, which shows that $u \in W^{2,p}_{\ell}(\mathbb{R}^N_+)$.

(ii) If $\frac{N}{p} = -\ell + 1$, the previous imbedding does not hold. Then we only have $W^{2,p}_{\ell}(\mathbb{R}^N_+) \hookrightarrow W^{1,p}_{\ell-1,-1}(\mathbb{R}^N_+)$, with the introduction of a logarithmic weight in the second space. By (2.3.18), we can deduce that $u \in W^{1,p}_{\ell-1,-1}(\mathbb{R}^N_+)$. Furthermore we have $\ell - \frac{N}{p'} < 0$, thus there is no compatibility condition for (T) which admits consequently a solution $y \in W^{2,p}_{\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta}_1 = \mathbb{R}\, x_N$ which is included in $W^{2,p}_{\ell}(\mathbb{R}^N_+)$. The end of the proof is similar to the previous case. □

We can now extend this result to more regular data.

Lemma 2.3.10. *Let $\ell \in \mathbb{Z}$ and $m \geqslant 1$. Under hypothesis (1.4.6), for any $g_0 \in W^{m+2-1/p,p}_{m+\ell}(\Gamma)$ and $g_1 \in W^{m+1-1/p,p}_{m+\ell}(\Gamma)$, satisfying the compatibility condition (2.3.15), problem (P^0) admits a solution $u \in W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$ with the estimate*

$$\inf_{q \in \mathcal{B}_{[2-\ell-N/p]}} \|u + q\|_{W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+)} \leqslant C \left(\|g_0\|_{W^{m+2-1/p,p}_{m+\ell}(\Gamma)} + \|g_1\|_{W^{m+1-1/p,p}_{m+\ell}(\Gamma)} \right).$$

Proof. We strictly resume the proof of Lemma 2.3.9. In this case, we note that $g_0 \in W^{(m+1)+1-1/p,p}_{(m+1)+(\ell-1)}(\Gamma)$ and $g_1 \in W^{m+1-1/p,p}_{m+\ell}(\Gamma)$, then we use Theorems 1.4.2 and 1.4.4. To show that $u \in W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+)$, we must distinguish two cases. If $\frac{N}{p} \neq -\ell - m + 1$, then we have the imbedding $W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+) \hookrightarrow W^{m+1,p}_{m+\ell-1}(\mathbb{R}^N_+)$. If $\frac{N}{p} = -\ell - m + 1$, then we have $W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+) \hookrightarrow W^{m+1,p}_{m+\ell-1,-1}(\mathbb{R}^N_+)$. In the second case, we must remark that $\ell - \frac{N}{p'} < 0$, so there is again no compatibility condition for (T). □

Note that we have the chain of imbeddings $W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+) \hookrightarrow W^{m+1,p}_{m+\ell-1}(\mathbb{R}^N_+) \hookrightarrow \cdots \hookrightarrow W^{2,p}_{\ell}(\mathbb{R}^N_+)$ if and only if $\frac{N}{p} \notin \{-\ell-m+1, \dots, -\ell\}$, and then Lemma 2.3.10 is a regularity result with respect to Lemma 2.3.9.

2.3.4 Existence of a solution to this problem

We come back to the general problem (P) and Theorem 2.3.1. By Lemma 1.3.1, there exists a lifting function $u_g \in W^{2,p}_{\ell}(\mathbb{R}^N_+)$ of (g_0, g_1), *i.e.* $u_g = g_0$ on Γ and $\partial_N u_g = g_1$ on Γ, such that

$$\|u_g\|_{W^{2,p}_{\ell}(\mathbb{R}^N_+)} \leqslant C \left(\|g_0\|_{W^{2-1/p,p}_{\ell}(\Gamma)} + \|g_1\|_{W^{1-1/p,p}_{\ell}(\Gamma)} \right).$$

Set $h = f - \Delta^2 u_g \in W^{-2,p}_{\ell}(\mathbb{R}^N_+)$ and $v = u - u_g$, then problem (P) is equivalent to the following with homogeneous boundary conditions:

$$\Delta^2 v = h \quad \text{in } \mathbb{R}^N_+, \quad v = \partial_N v = 0 \quad \text{on } \Gamma.$$

Then, the compatibility condition (2.3.2) for Problem (P) becomes:

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']}\ , \quad \langle h, \varphi \rangle_{W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+)} = 0. \tag{2.3.19}$$

So, we can consider now the lifted problem

$$(P^\star)\ \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \Gamma, \\ \partial_N u = 0 & \text{on } \Gamma, \end{cases}$$

where $f \in W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)$ and $f \perp \mathcal{B}_{[2+\ell-N/p']}$.

Give at first a characterization of $W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)$:

Lemma 2.3.11. *For any $f \in W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)$, there exists $\mathbb{F} = (F_{ij})_{1\leqslant i,\,j\leqslant N} \in {W^{0,\,p}_{\ell}(\mathbb{R}^N_+)}^{N^2}$ such that*

$$f = \operatorname{div}\operatorname{div}\mathbb{F} = \sum_{i,\,j=1}^{N} \partial^2_{ij} F_{ij},$$

with the estimate

$$\sum_{i,\,j=1}^{N} \|F_{ij}\|_{W^{0,\,p}_{\ell}(\mathbb{R}^N_+)} \ \leqslant\ C\, \|f\|_{W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)}.$$

Proof. We know by Hardy's inequality (1.2.4) that the norm and the semi-norm in $\overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+)$ are equivalent, *i.e.* there exists a constant C such that

$$\forall u \in \overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+),$$
$$\|\nabla^2 u\|_{{W^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)}^{N^2}} \leqslant \|u\|_{\overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+)} \leqslant\ C\, \|\nabla^2 u\|_{{W^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)}^{N^2}}.$$

Let

$$\begin{array}{rrcl} T: & \overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+) & \longrightarrow & {W^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)}^{N^2} \\ & u & \longmapsto & \nabla^2 u. \end{array}$$

By the previous inequalities, T is a linear continuous injective mapping. We set $\Xi = T\left(\overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+)\right)$, equipped with the norm of ${W^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)}^{N^2}$, and $S = T^{-1} : \Xi \longrightarrow \overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+)$. The mapping $\mathbb{H} \in \Xi \longmapsto \langle f, S\mathbb{H}\rangle_{W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+)}$ is a linear functional on Ξ. Thanks to Hahn-Banach theorem, we can extend it to a linear functional on $W^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)^{N^2}$ denoted by Φ. Thanks to Riesz representation theorem, we know that there exists $\mathbb{F} = (F_{ij}) \in W^{0,\,p}_{\ell}(\mathbb{R}^N_+)^{N^2}$ such that

$$\forall \mathbb{H} = (h_{ij}) \in W^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)^{N^2}, \quad \langle \Phi, \mathbb{H} \rangle = \int_{\mathbb{R}^N_+} F_{ij}\, h_{ij}\, dx,$$

with Einstein convention of sumation on repeated indices. Particularly, if $\mathbb{H} \in \Xi$, we have

$$\langle f, S\mathbb{H} \rangle = \int_{\mathbb{R}^N_+} F_{ij}\, h_{ij}\, dx,$$

i.e.

$$\forall u \in \overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+), \quad \langle f, u \rangle = \int_{\mathbb{R}^N_+} F_{ij}\, \partial^2_{ij} u\, dx.$$

We can deduce that

$$\forall u \in \mathcal{D}(\mathbb{R}^N_+), \quad \langle f, u \rangle = \left\langle \partial^2_{ij} F_{ij}, u \right\rangle,$$

i.e. $f = \operatorname{div}\operatorname{div}\mathbb{F} = \partial^2_{ij} F_{ij}$. □

Now we can establish a first isomorphism result in the half-space:

Proposition 2.3.12. *Let $\ell \in \mathbb{Z}$. Under hypothesis (2.3.1), with $2+\ell-N/p' < 0$ or $2-\ell-N/p < 0$, the biharmonic operator*

$$\Delta^2 : \overset{\circ}{W}{}^{2,\,p}_{\ell}(\mathbb{R}^N_+)/\mathcal{B}_{[2-\ell-N/p]} \longrightarrow W^{-2,\,p}_{\ell}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2+\ell-N/p']}$$

is an isomorphism.

Proof. Let us first assume that $2+\ell-N/p' < 0$. Let $f \in W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)$. Then by Lemma 2.3.11, we can write $f = \partial^2_{ij} F_{ij}$ with $(F_{ij})_{1\leqslant i,\,j\leqslant N} \in W^{0,\,p}_{\ell}(\mathbb{R}^N_+)^{N^2}$. If we extend F_{ij} to $\mathbb{R}^N$ by 0, we obtain $(\tilde{F}_{ij})_{1\leqslant i,\,j\leqslant N} \in W^{0,\,p}_{\ell}(\mathbb{R}^N)^{N^2}$, and thus $\tilde{f} = \partial^2_{ij}\tilde{F}_{ij} \in W^{-2,\,p}_{\ell}(\mathbb{R}^N)$ as extension of f such that $\|\tilde{f}\|_{W^{-2,\,p}_{\ell}(\mathbb{R}^N)} \leqslant C\, \|f\|_{W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)}$. By Theorem 2.2.3, there exists $\tilde{z} \in W^{2,\,p}_{\ell}(\mathbb{R}^N)$ such that $\tilde{f} = \Delta^2 \tilde{z}$ in $\mathbb{R}^N$ and writing $z = \tilde{z}|_{\mathbb{R}^N_+}$, we have $f = \Delta^2 z$ in $\mathbb{R}^N_+$, with $z \in W^{2,\,p}_{\ell}(\mathbb{R}^N_+)$, $z|_\Gamma \in W^{2-1/p,\,p}_{\ell}(\Gamma)$ and $\partial_N z|_\Gamma \in W^{1-1/p,\,p}_{\ell}(\Gamma)$. Since $\mathcal{B}_{[2+\ell-N/p']} = \{0\}$, there is no compatibility condition for Lemma 2.3.9 which asserts the existence of a solution $v \in W^{2,\,p}_{\ell}(\mathbb{R}^N_+)$ to the homogeneous problem

$$\Delta^2 v = 0 \;\text{ in } \mathbb{R}^N_+, \quad v = z \;\text{ and }\; \partial_N v = \partial_N z \;\text{ on } \Gamma. \tag{2.3.20}$$

The function $u = z - v$ answers to problem $(P^\star)$ in this case.

So we have shown that if $2 + \ell - N/p' < 0$, the operator

$$\Delta^2 : \overset{\circ}{W}{}^{2,p}_{\ell}(\mathbb{R}^N_+)/\mathcal{B}_{[2-\ell-N/p]} \longrightarrow W^{-2,p}_{\ell}(\mathbb{R}^N_+)$$

is an isomorphism. Thus by duality we obtain the isomorphism

$$\Delta^2 : \overset{\circ}{W}{}^{2,p}_{\ell}(\mathbb{R}^N_+) \longrightarrow W^{-2,p}_{\ell}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2+\ell-N/p']},$$

if $2 - \ell - N/p < 0$. □

It remains to solve $(P^\star)$ if

$$2 + \ell - N/p' \geqslant 0 \quad \text{and} \quad 2 - \ell - N/p \geqslant 0. \tag{2.3.21}$$

It suffices to check the cases $\ell \in \{-1, 0, 1\}$, outside which condition (2.3.21) does not hold. For that, we establish a preliminary proposition:

Proposition 2.3.13. *Let $\ell \in \{-1, 0\}$ such that $N/p \neq 1$ if $\ell = -1$. For any $f \in W^{0,p}_{\ell}(\mathbb{R}^N_+)$, there exists $z \in W^{4,p}_{\ell}(\mathbb{R}^N_+)$ such that $\Delta^2 z = f$.*

Proof. Under these hypotheses, consider the extension $\tilde{f}$ of f to $\mathbb{R}^N$ by 0, so $\tilde{f} \in W^{0,p}_{\ell}(\mathbb{R}^N)$. Show at first that there exists $\tilde{z} \in W^{4,p}_{\ell}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$.

(a) If $\ell = -1$, then $\tilde{f} \in W^{0,p}_{-1}(\mathbb{R}^N)$ and we have $N/p \neq 1$. Thus Lemma 2.2.4 of isomorphism in $\mathbb{R}^N$ holds with $m = 2$ and $\ell = -3$, hence the existence of $\tilde{z} \in W^{4,p}_{-1}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$.

(b) If $\ell = 0$, then $\tilde{f} \in L^p(\mathbb{R}^N)$. Here again Lemma 2.2.4 holds with $m = 2$ and $\ell = -2$, hence the existence of $\tilde{z} \in W^{4,p}_{0}(\mathbb{R}^N)$ such that $\Delta^2 \tilde{z} = \tilde{f}$.

Then we come back to the restriction $z = \tilde{z}|_{\mathbb{R}^N_+}$ for which we naturally have $\Delta^2 z = f$ in $\mathbb{R}^N_+$. □

Now, we can fill the gap of Proposition 2.3.12:

Proposition 2.3.14. *Let $\ell \in \{-1, 0, 1\}$ such that*

$$\frac{N}{p'} \neq 1 \;\; \textit{if} \;\; \ell = 1 \qquad \textit{and} \qquad \frac{N}{p} \neq 1 \;\; \textit{if} \;\; \ell = -1.$$

Then the biharmonic operator

$$\Delta^2 : \overset{\circ}{W}{}^{2,p}_{\ell}(\mathbb{R}^N_+)/\mathcal{B}_{[2-\ell-N/p]} \longrightarrow W^{-2,p}_{\ell}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2+\ell-N/p']}$$

is an isomorphism.

Proof. At first we will use Lemma 2.3.11 and Proposition 2.3.13 to solve $(\mathcal{P}^\star)$ for $\ell \in \{-1, 0\}$.

Let $f \in W^{-2,p}_\ell(\mathbb{R}^N_+)$ with $\ell \in \{-1, 0\}$ verifying (2.3.1). By Lemma 2.3.11, there exists $\mathbb{F} = (F_{ij})_{1\leqslant i,j\leqslant N} \in W^{0,p}_\ell(\mathbb{R}^N_+)^{N^2}$ such that $f = \operatorname{div}\operatorname{div}\mathbb{F}$. It suffices to apply Proposition 2.3.13 to all the components F_{ij} of $\mathbb{F}$ to find $\mathbb{U} = (U_{ij})_{1\leqslant i,j\leqslant N} \in W^{4,p}_\ell(\mathbb{R}^N_+)^{N^2}$ such that $\Delta^2\mathbb{U} = \mathbb{F}$ in $\mathbb{R}^N_+$. Setting $z = \operatorname{div}\operatorname{div}\mathbb{U}$, we obtain $z \in W^{2,p}_\ell(\mathbb{R}^N_+)$ such that $\Delta^2 z = f$ in $\mathbb{R}^N_+$ because the operators div and Δ commute. Thus we have $z|_\Gamma \in W^{2-1/p,p}_\ell(\Gamma)$ and $\partial_N z|_\Gamma \in W^{1-1/p,p}_\ell(\Gamma)$, and Lemma 2.3.9 asserts the existence of a solution $v \in W^{2,p}_\ell(\mathbb{R}^N_+)$ to problem (2.3.20), since we have still $\mathcal{B}_{[2+\ell-N/p']} = \{0\}$ (see Remark 2.3.8). Then the function $u = z - v$ answers again to problem $(\mathcal{P}^\star)$ for $\ell \in \{-1, 0\}$.

Finally to solve the case $\ell = 1$, we proceed by duality from the case $\ell = -1$. We have the isomorphism

$$\Delta^2 : \overset{\circ}{W}{}^{2,p}_{-1}(\mathbb{R}^N_+)/\mathcal{B}_{[3-N/p]} \longrightarrow W^{-2,p}_{-1}(\mathbb{R}^N_+) \ \text{ if } \frac{N}{p} \neq 1,$$

hence, by duality, the operator

$$\Delta^2 : \overset{\circ}{W}{}^{2,p}_{1}(\mathbb{R}^N_+) \longrightarrow W^{-2,p}_{1}(\mathbb{R}^N_+) \perp \mathcal{B}_{[3-N/p']} \ \text{ if } \frac{N}{p'} \neq 1,$$

is also an isomorphism. □

Remark 2.3.15. It is also possible to solve directly the case $\ell = 1$. The first step is to extend Proposition 2.3.13 to $\ell = 1$ with $N/p' \neq 1$. Here we consider the extension $\tilde{f} \in W^{0,p}_1(\mathbb{R}^N)$ of $f \in W^{0,p}_1(\mathbb{R}^N_+)$ defined by:

$$\tilde{f}(x', x_N) = \begin{cases} f(x', x_N) & \text{if } x_N > 0, \\ 0 & \text{if } x_N = 0, \\ -f(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Then we use Lemma 2.2.5 with $m = 2$, which asserts the existence of a function $\tilde{z} \in W^{4,p}_1(\mathbb{R}^N)$ such that $\Delta^2\tilde{z} = \tilde{f}$ in $\mathbb{R}^N$, if $N/p' \neq 1$ and $\tilde{f} \perp \mathcal{P}^\Delta_{[1-N/p']}$. There are two cases: either $N/p' > 1$, then $\mathcal{P}^\Delta_{[1-N/p']} = \{0\}$ and there is no condition on $\tilde{f}$; or $N/p' < 1$, then $\mathcal{P}^\Delta_{[1-N/p']} = \mathcal{P}_0$ and we must have $\tilde{f} \perp \mathcal{P}_0$. But $N/p' < 1$ implies that $W^{0,p}_1(\mathbb{R}^N) \hookrightarrow L^1(\mathbb{R}^N)$ and we have

$$\int_{\mathbb{R}^N} \tilde{f}\,dx = 0,$$

as a straightforward consequence of this extension of f. That exactly means that $\tilde{f} \perp \mathcal{P}_0$. Thus $z = \tilde{z}|_{\mathbb{R}^N_+} \in W^{4,p}_\ell(\mathbb{R}^N_+)$ satisfies $\Delta^2 z = f$ in $\mathbb{R}^N_+$.

The second step is to resume the proof of Proposition 2.3.14 for $\ell = 1$ with $N/p' \neq 1$. If $N/p' > 1$, we have still $\mathcal{B}_{[3-N/p']} = \{0\}$, so the same reasoning holds; if $N/p' < 1$, we know that $\mathcal{B}_{[3-N/p']} = \mathbb{R}\, x_N^2$ and Lemma 2.3.9 requires the following compatibility condition for problem (2.3.20):

$$\forall \varphi \in \mathbb{R}\, x_N^2, \quad \langle \partial_N z, \Delta\varphi \rangle_\Gamma - \langle z, \partial_N \Delta\varphi \rangle_\Gamma = 0,$$

which boils down to

$$\langle \partial_N z, 1 \rangle_\Gamma = 0. \tag{2.3.22}$$

But remember that f must satisfy the orthogonality condition for $(P^\star)$, *i.e.* $\langle f, x_N^2 \rangle_{W_1^{-2,p}(\mathbb{R}^N_+) \times \mathring{W}_{-1}^{2,p'}(\mathbb{R}^N_+)} = 0$ and moreover we have $f = \Delta^2 z$ in $\mathbb{R}^N_+$; thus $\langle \Delta^2 z, x_N^2 \rangle_{W_1^{-2,p}(\mathbb{R}^N_+) \times \mathring{W}_{-1}^{2,p'}(\mathbb{R}^N_+)} = 0$. It suffices to write the Green formula

$$\left\langle \Delta^2 z, x_N^2 \right\rangle_{W_1^{-2,p}(\mathbb{R}^N_+) \times \mathring{W}_{-1}^{2,p'}(\mathbb{R}^N_+)} = -\left\langle \partial_N z, \Delta x_N^2 \right\rangle_\Gamma = -2\, \langle \partial_N z, 1 \rangle_\Gamma\,,$$

to see that (2.3.22) holds. $\diamondsuit$

To finish the proof of Theorem 2.3.1, it remains to combine Propositions 2.3.12 and 2.3.14, which provides the isomorphism

$$\Delta^2 : \mathring{W}_\ell^{2,p}(\mathbb{R}^N_+)/\mathcal{B}_{[2-\ell-N/p]} \longrightarrow W_\ell^{-2,p}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2+\ell-N/p']},$$

for any $\ell \in \mathbb{Z}$ verifying (2.3.1). This answers globally to problem $(P^\star)$ and thus to general problem (P) by means of the lifting function mentioned above.

Chapitre 3

Strong and very weak solutions to the biharmonic problem in $\mathbb{R}^N_+$

3.1 Introduction

In the previous chapter, we established the existence of generalized solutions to problem (P), *i.e.* solutions which belong to weighted Sobolev spaces of type $W^{2,p}_{\ell}(\mathbb{R}^N_+)$. Here, we are interested both in the existence of more regular solutions, as for instance strong solutions which belong to spaces of type $W^{4,p}_{\ell+2}(\mathbb{R}^N_+)$, and singular solutions which belong to $W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$ in the case $f = 0$ with singular boundary conditions. We also establish the existence of solutions which belong to intermediate spaces as for example $W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$. To finish this study, in the last section, we shall consider the biharmonic equation with different kinds of boundary conditions.

3.2 Weak solutions, strong solutions, regularity

The purpose of this section is the study of solutions to the nonhomogeneous problem (P) for more regular data. We shall now establish a global result which extends Theorem 2.3.1 to different types of data.

3.2.1 A global result in the nonhomogeneous case

Theorem 3.2.1. *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$ and assume that*

$$\frac{N}{p'} \notin \{1, \dots, \ell + \min\{m, 2\}\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \dots, -\ell - m\}. \tag{3.2.1}$$

For any $f \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+)$, $g_0 \in W^{m+2-1/p,\,p}_{m+\ell}(\Gamma)$ *and* $g_1 \in W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$ *satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']},\qquad \langle f, \varphi \rangle_{W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+)} + \langle g_1, \Delta\varphi\rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi\rangle_\Gamma = 0, \tag{3.2.2}$$

problem (P) *admits a solution* $u \in W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N_+)$, *unique up to an element of* $\mathcal{B}_{[2-\ell-N/p]}$, *with the estimate*

$$\inf_{q \in \mathcal{B}_{[2-\ell-N/p]}} \|u+q\|_{W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N_+)} \leqslant C \left(\|f\|_{W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+)} + \|g_0\|_{W^{m+2-1/p,\,p}_{m+\ell}(\Gamma)} + \|g_1\|_{W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)} \right).$$

Proof. Note at first that if $m = 0$, we find Theorem 2.3.1. The kernel has been globally characterized by (2.3.4). Let us recall that this kernel is reduced to $\{0\}$ if $\ell \geqslant 0$ and symmetrically the compatibility condition (3.2.2) vanishes if $\ell \leqslant 0$. Moreover under hypothesis (3.2.1), the imbeddings $W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+) \hookrightarrow W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)$, $W^{m+2-1/p,\,p}_{m+\ell}(\Gamma) \hookrightarrow W^{2-1/p,\,p}_{\ell}(\Gamma)$ and $W^{m+1-1/p,\,p}_{m+\ell}(\Gamma) \hookrightarrow W^{1-1/p,\,p}_{\ell}(\Gamma)$ hold for all $\ell \geqslant 1$, hence the necessity of (3.2.2) for any $m \in \mathbb{N}$. So it suffices to show the existence of a solution. By Lemma 1.3.1, we can consider the problem with homogeneous boundary conditions

$$(P^\star)\ \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \Gamma, \\ \partial_N u = 0 & \text{on } \Gamma, \end{cases}$$

with $f \perp \mathcal{B}_{[2+\ell-N/p']}$. This orthogonality condition naturally corresponds to the compatibility condition (3.2.2). We dealt with these questions in Chapter 2.

Let us now give the plan of the proof of the existence for $m \geqslant 1$:

(i) If $\ell \leqslant -2$, we establish globally the existence of a solution.

(ii) If $\ell \geqslant -1$ and $m = 1$, we show that by a direct construction.

(iii) If $\ell \geqslant -1$ and $m \geqslant 1$, we show that by induction on m from the previous case ($m = 1$), thanks to a regularity argument.

(i) Assume that $\ell \leqslant -2$. Then hypothesis (3.2.1) is reduced to (2.2.6). Let $f \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+)$. Let us first suppose $m \geqslant 2$. We know that there exists a continuous linear extension operator from $W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+)$ to $W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N)$ and thus $\tilde{f} \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N)$ which extends f to $\mathbb{R}^N$. Then we use Theorem 2.2.6 to obtain $\tilde{z} \in W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N)$ such that $\tilde{f} = \Delta^2 \tilde{z}$ in $\mathbb{R}^N$ and thus $f = \Delta^2 z$ in $\mathbb{R}^N_+$, with

$z = \tilde{z}|_{\mathbb{R}^N_+} \in W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N_+)$. Then Proposition 2.3.10 asserts the existence of a solution $v \in W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N_+)$ to the homogeneous problem

$$\Delta^2 v = 0 \quad \text{in } \mathbb{R}^N_+, \qquad v = z \quad \text{and} \quad \partial_N v = \partial_N z \quad \text{on } \Gamma,$$

with $z|_\Gamma \in W^{m+2-1/p,\,p}_{m+\ell}(\Gamma)$ and $\partial_N z|_\Gamma \in W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$. We remark again that $\mathcal{B}_{[2+\ell-N/p']} = \{0\}$ because $\ell \leqslant 0$, thus there is no compatibility condition. Then the function $u = z - v$ answer to problem $(P^\star)$ in this case.

Let us now consider the case $m = 1$, *i.e.* $f \in W^{-1,\,p}_{\ell+1}(\mathbb{R}^N_+)$. As we did for the distributions of $W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)$ in Lemma 2.3.11, we can show that there exists $\mathbf{F} = (F_i)_{1\leqslant i\leqslant N} \in W^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)^N$ such that $f = \operatorname{div}\mathbf{F} = \sum_{i=1}^N \partial_i F_i$, with the estimate $\sum_{i=1}^N \|F_i\|_{W^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)} \leqslant C\,\|f\|_{W^{-1,\,p}_{\ell+1}(\mathbb{R}^N_+)}$. Let us denote by $\tilde{\mathbf{F}} \in W^{0,\,p}_{\ell+1}(\mathbb{R}^N)^N$ the extension by 0 of $\mathbf{F}$ to $\mathbb{R}^N$. Since $\frac{N}{p} \notin \{1, \dots, -\ell-1\}$, by Theorem 2.2.6, there exists $\tilde{\Psi} \in W^{4,\,p}_{\ell+1}(\mathbb{R}^N)^N$ such that $\tilde{\mathbf{F}} = \Delta^2\tilde{\Psi}$ in $\mathbb{R}^N$. Setting $\tilde{\psi} = \operatorname{div}\tilde{\Psi}$ and $\psi = \tilde{\psi}|_{\mathbb{R}^N_+}$, so we have $\psi \in W^{3,\,p}_{\ell+1}(\mathbb{R}^N_+)$ and by Proposition 2.3.10, there exists $v \in W^{3,\,p}_{\ell+1}(\mathbb{R}^N_+)$ such that

$$\Delta^2 v = 0 \quad \text{in } \mathbb{R}^N_+, \quad v = \psi \quad \text{and} \quad \partial_N v = \partial_N \psi \quad \text{on } \Gamma.$$

The function $u = \psi - v \in W^{3,\,p}_{\ell+1}(\mathbb{R}^N_+)$ is a solution to Problem $(P^\star)$ in this case.

(ii) Assume that $\ell \geqslant -1$ and $m = 1$. Note that the distribution $f \in W^{-1,\,p}_{\ell+1}(\mathbb{R}^N_+)$ defines the linear functional L on $\mathcal{A}^{\Delta}_{[\ell+2-N/p']}$ by

$$L:\ r \longmapsto \langle f, r\rangle_{W^{-1,\,p}_{\ell+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-\ell-1}(\mathbb{R}^N_+)},$$

and introduce the inner product Φ on $\mathcal{A}^{\Delta}_{[\ell+2-N/p']} \times \mathcal{A}^{\Delta}_{[\ell+2-N/p']}$ defined by

$$\Phi:\ (\mu, r) \longmapsto \int_\Gamma \varrho'^{-2\ell-1-N/p+N/p'}\,\partial_N\mu\,\partial_N r\,\mathrm{d}x'.$$

Note that

$$r \in \mathcal{A}^{\Delta}_{[\ell+2-N/p']} \;\Rightarrow\; \varrho'^{-\ell-1+1/p'}\,\partial_N r \in L^{p'}(\Gamma),$$

and

$$\mu \in \mathcal{A}^{\Delta}_{[\ell+2-N/p']} = \mathcal{A}^{\Delta}_{[\ell+2-N/p'+N/p-N/p]} \;\Rightarrow\; \varrho'^{-\ell-1-N/p+N/p'+1/p}\,\partial_N\mu \in L^{p}(\Gamma).$$

Thus, thanks to Hölder inequality, Φ is well-defined. Then, there exists a unique $\mu \in \mathcal{A}^{\Delta}_{[\ell+2-N/p']}$ such that

$$\forall r \in \mathcal{A}^{\Delta}_{[\ell+2-N/p']},\ L(r) = \Phi(\mu, r),$$

i.e.

$$\forall r \in \mathcal{A}^{\Delta}_{[\ell+2-N/p']}, \quad \langle f, r\rangle_{W^{-1,p}_{\ell+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell-1}(\mathbb{R}^N_+)} = \int_\Gamma \varrho'^{-2\ell-1-N/p+N/p'}\,\partial_N\mu\,\partial_N r\,\mathrm{d}x'. \tag{3.2.3}$$

Let us set $\xi_0 = \varrho'^{-2\ell-1-N/p+N/p'}\,\partial_N\mu$, then we have $\xi_0 \in W^{1-1/p,\,p}_{\ell+1}(\Gamma)$ and (3.2.3) becomes

$$\forall r \in \mathcal{A}^{\Delta}_{[\ell+2-N/p']}, \quad \langle f, r\rangle_{W^{-1,p}_{\ell+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell-1}(\mathbb{R}^N_+)} = \langle \xi_0, \partial_N r\rangle_{W^{1-1/p,\,p}_{\ell+1}(\Gamma)\times W^{-1/p',\,p'}_{-\ell-1}(\Gamma)}. \tag{3.2.4}$$

That is precisely the compatibility condition of the Dirichlet problem

$$(Q)\quad \begin{cases} \Delta\xi = f & \text{in } \mathbb{R}^N_+,\\ \xi = \xi_0 & \text{on } \Gamma, \end{cases}$$

Thus, by Theorem 1.4.1 (replacing ℓ by $\ell+1$), problem (Q) admits a solution $\xi \in W^{1,p}_{\ell+1}(\mathbb{R}^N_+)$ under hypothesis (3.2.1). Here we shall use the characterization (2.3.4) of Lemma 2.3.4:

$$\mathcal{B}_{[2+\ell-N/p']} = \Pi_D\mathcal{A}^{\Delta}_{[\ell-N/p']} \oplus \Pi_N\mathcal{N}^{\Delta}_{[\ell-N/p']}.$$

Since $f \perp \mathcal{B}_{[2+\ell-N/p']}$, we have

$$\forall r \in \mathcal{A}^{\Delta}_{[\ell-N/p']}, \quad \langle \Delta\xi, \Pi_D r\rangle_{W^{-1,p}_{\ell+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell-1}(\mathbb{R}^N_+)} = \langle f, \Pi_D r\rangle = 0.$$

By a Green formula, we can deduce that

$$\forall r \in \mathcal{A}^{\Delta}_{[\ell-N/p']}, \quad \langle \xi, \Delta\Pi_D r\rangle_{W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)} = 0,$$

because $W^{1,p}_{\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)$ unless $\frac{N}{p} = -\ell$ or $\frac{N}{p'} = \ell$. The second possibility is excluded by (3.2.1), and since $\ell \geqslant -1$, the only problematic case is $\ell = -1$. But then $[\ell - N/p'] < 0$ and the condition vanishes. Thus, we have

$$\forall r \in \mathcal{A}^{\Delta}_{[\ell-N/p']}, \quad \langle \xi, r\rangle_{W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)} = 0,$$

which is the compatibility condition for the Dirichlet problem

$$(R^\star)\quad \begin{cases} \Delta\vartheta = \xi & \text{in } \mathbb{R}^N_+,\\ \vartheta = 0 & \text{on } \Gamma. \end{cases}$$

Thus, by Theorem 1.4.2 (with $m = 2$ and replacing ℓ by $\ell-1$), problem $(R^\star)$ admits a solution $\vartheta \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$ under hypothesis (3.2.1). Similarly we have

$$\forall s \in \mathcal{N}^{\Delta}_{[\ell-N/p']}, \quad \langle \Delta\xi, \Pi_N s\rangle_{W^{-1,p}_{\ell+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell-1}(\mathbb{R}^N_+)} = \langle f, \Pi_N s\rangle = 0,$$

therefore as previously, we have

$$\forall s \in \mathcal{N}^{\Delta}_{[\ell-N/p']}, \quad \langle \xi, s\rangle_{W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)} = 0,$$

which is the compatibility condition for Neumann problem

$$(S^\star)\ \begin{cases} \Delta\zeta = \xi & \text{in } \mathbb{R}^N_+,\\ \partial_N \zeta = 0 & \text{on } \Gamma. \end{cases}$$

As for problem $(R^\star)$, we can show that $(S^\star)$ admits a solution $\zeta \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$ under hypothesis (3.2.1) according to Theorem 1.4.4 with $m = 1$. Then the function defined by

$$u = x_N\, \partial_N(\zeta - \vartheta) + \vartheta \tag{3.2.5}$$

is a solution to $(P^\star)$. It remains to show that $u \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$.

If $\frac{N}{p} \neq -\ell$, then we have the imbedding $W^{3,p}_{\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{2,p}_{\ell}(\mathbb{R}^N_+)$ therefore $u \in W^{2,p}_{\ell}(\mathbb{R}^N_+)$. Moreover u satisfies the system

$$(T^\star)\ \begin{cases} \Delta u = 2\,\partial_N^2(\zeta - \vartheta) + \xi & \text{in } \mathbb{R}^N_+,\\ u = 0 & \text{on } \Gamma, \end{cases}$$

with $2\,\partial^2_N(\zeta-\vartheta) + \xi \in W^{1,p}_{\ell+1}(\mathbb{R}^N_+)$. As for $(R^\star)$, we know that problem $(T^\star)$ has a solution $y \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$. We can deduce that $u - y \in \mathcal{A}^{\Delta}_{[2-\ell-N/p]} \subset W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$, *i.e.* $u \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$.

If $\frac{N}{p} = -\ell$, then we have necessary $\ell = -1$ and moreover the imbedding $W^{3,p}_{0}(\mathbb{R}^N_+) \hookrightarrow W^{2,p}_{-1,-1}(\mathbb{R}^N_+)$ with $\ell - \frac{N}{p'} = \ell + \frac{N}{p} - N = -N < 0$, therefore no compatibility condition for $(T^\star)$. So we can still deduce that $u \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$.

(iii) Assume that $\ell \geqslant -1$ and $m \geqslant 1$. Consider $f \in W^{m-2,p}_{m+\ell}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2+\ell-N/p']}$. Remark at first that we have the imbedding

$$W^{m-2,p}_{m+\ell}(\mathbb{R}^N_+) \hookrightarrow W^{-1,p}_{\ell+1}(\mathbb{R}^N_+) \quad \text{if} \quad \frac{N}{p'} \neq \ell + 2 \ \text{ or } \ m = 1.$$

Then, thanks to the previous step, there exists a solution $u \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$ to problem $(P^\star)$. Let us prove by induction that, under hypothesis (3.2.1),

$$f \in W^{m-2,p}_{m+\ell}(\mathbb{R}^N_+) \ \Rightarrow\ u \in W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+). \tag{3.2.6}$$

For $m = 1$, (3.2.6) is true. Assume that (3.2.6) is true for $1, 2, \ldots, m$ and suppose that $f \in W^{m-1,p}_{m+1+\ell}(\mathbb{R}^N_+)$. Let us prove that $u \in W^{m+3,p}_{m+1+\ell}(\mathbb{R}^N_+)$. Let us first observe that $W^{m-1,p}_{m+1+\ell}(\mathbb{R}^N_+) \hookrightarrow W^{m-2,p}_{m+\ell}(\mathbb{R}^N_+)$, hence u belongs to $W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+)$ thanks to the induction hypothesis. Now, for any $i \in \{1, \ldots, N-1\}$,

$$\Delta(\varrho\,\partial_i u) = \varrho\,\Delta\partial_i u + \frac{2}{\varrho}\,x.\nabla\partial_i u + \left(\frac{N-1}{\varrho} + \frac{1}{\varrho^3}\right)\partial_i u.$$

Then let us set $v_i = \frac{2}{\varrho}\, x.\nabla\partial_i u + \left(\frac{N-1}{\varrho} + \frac{1}{\varrho^3}\right) \partial_i u$. We can remark that $v_i \in W^{m,p}_{m+\ell}(\mathbb{R}^N_+)$ and moreover we can write

$$\Delta^2(\varrho\,\partial_i u) = \Delta(\varrho\,\Delta\partial_i u) + \Delta v_i,$$

with $\Delta v_i \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+)$. It remains to see the first term, *i.e.*

$$\Delta(\varrho\,\partial_i\Delta u) = \varrho\,\partial_i f + \frac{2}{\varrho}\, x.\nabla\partial_i\Delta u + \left(\frac{N-1}{\varrho} + \frac{1}{\varrho^3}\right) \partial_i\Delta u.$$

We can see that $\Delta(\varrho\,\partial_i\Delta u) \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+)$, hence $\Delta^2(\varrho\,\partial_i u) \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+)$. Let us set $z_i = \varrho\,\partial_i u$ and $f_i = \Delta^2 z_i \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+)$. A priori, we only have $z_i \in W^{m+1,\,p}_{m-1+\ell}(\mathbb{R}^N_+)$. However, we know that $\gamma_0 u = \gamma_1 u = 0$, then we can deduce that

$$\gamma_0 z_i = (\varrho\,\partial_i u)|_\Gamma = 0 \quad\text{and}\quad \gamma_1 z_i = (\partial_N\varrho\,\partial_i u + \varrho\,\partial^2_{iN} u)|_\Gamma = 0, \quad \text{since } i \neq N.$$

Therefore

$$\Delta^2 z_i = f_i \;\text{ in } \mathbb{R}^N_+, \qquad z_i = \partial_N z_i = 0 \;\text{ on } \Gamma,$$

with $f_i \in W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N_+) \hookrightarrow W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)$ under hypothesis (3.2.1). Moreover, thanks to the Green formula, we have for any $\varphi \in \mathcal{B}_{[2+\ell-N/p']}$:

$$\left\langle \Delta^2 z_i, \varphi \right\rangle_{W^{-2,\,p}_{\ell}(\mathbb{R}^N_+) \times \mathring{W}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+)} = \left\langle z_i, \Delta^2\varphi \right\rangle_{\mathring{W}^{2,\,p}_{\ell}(\mathbb{R}^N_+) \times W^{-2,\,p'}_{-\ell}(\mathbb{R}^N_+)} = 0.$$

So the orthogonality condition $f_i \perp \mathcal{B}_{[2+\ell-N/p']}$ is satisfied for the problem

$$(Q_i)\;\begin{cases} \Delta^2\zeta_i = f_i & \text{in } \mathbb{R}^N_+,\\ \zeta_i = 0 & \text{on } \Gamma,\\ \partial_N\zeta_i = 0 & \text{on } \Gamma,\end{cases}$$

which admits, by the induction hypothesis, a solution $\zeta_i \in W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$. Thus $z_i - \zeta_i \in \mathcal{B}_{[2-\ell-N/p]}$, hence we can deduce that $z_i \in W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N_+)$. Since $z_i = \varrho\,\partial_i u$, that implies

$$\forall i \in \{1,\dots,N-1\}, \quad \partial_i u \in W^{m+2,\,p}_{m+1+\ell}(\mathbb{R}^N_+) \tag{3.2.7}$$

and consequently, for any $(i,j,k) \in \{1,\dots,N\}^2 \times \{1,\dots,N-1\}$,

$$\partial^3_{ijk}(\partial_N u) = \partial^3_{ijN}(\partial_k u) \;\in W^{m-1,\,p}_{m+1+\ell}(\mathbb{R}^N_+). \tag{3.2.8}$$

Furthermore, (3.2.7) gives us

$$\forall (i,j) \in \{1,\dots,N\} \times \{1,\dots,N-1\}, \quad \partial^2_i\partial^2_j u \in W^{m-1,\,p}_{m+1+\ell}(\mathbb{R}^N_+),$$

which implies

$$\partial_N^4 u = f - \sum_{\substack{i,j=1 \\ (i,j)\neq(N,N)}}^{N} \partial_i^2\partial_j^2 u \ \in W^{m-1,\,p}_{m+1+\ell}(\mathbb{R}^N_+). \tag{3.2.9}$$

Then combining (3.2.8) and (3.2.9), we obtain that $\nabla^3(\partial_N u) \in W^{m-1,\,p}_{m+1+\ell}(\mathbb{R}^N_+)^{N^3}$ and knowing that $\partial_N u \in W^{m+1,\,p}_{m+\ell}(\mathbb{R}^N_+)$, we can deduce that $\partial_N u \in W^{m+2,\,p}_{m+1+\ell}(\mathbb{R}^N_+)$, because $m \geqslant 1$. Adding this last point to (3.2.7), we have $\nabla u \in W^{m+2,\,p}_{m+1+\ell}(\mathbb{R}^N_+)^N$, hence we can conclude that $u \in W^{m+3,\,p}_{m+1+\ell}(\mathbb{R}^N_+)$. □

3.2.2 Panorama of basic cases

The purpose of this part is to extract the basic cases included in Theorem 3.2.1. We give them for the lifted problem $(P^\star)$. There is no orthogonality condition in these cases because $\ell \in \{-2,-1,0\}$, hence $\mathcal{B}_{[2+\ell-N/p']} = \{0\}$. For $m \geqslant 3$, we introduce the notation

$$\overset{\star}{W}{}^{m,\,p}_{\ell}(\mathbb{R}^N_+) = \left\{u \in W^{m,\,p}_{\ell}(\mathbb{R}^N_+);\ u = \partial_N u = 0 \text{ on } \Gamma\right\}.$$

Corollary 3.2.2. *The following biharmonic operators are isomorphisms:*

(i) *For* $\ell = 0$

$$\begin{array}{llll} \Delta^2: \ \overset{\circ}{W}{}^{2,\,p}_{0}(\mathbb{R}^N_+) & \longrightarrow & W^{-2,\,p}_{0}(\mathbb{R}^N_+). & \\ \Delta^2: \ \overset{\star}{W}{}^{3,\,p}_{1}(\mathbb{R}^N_+) & \longrightarrow & W^{-1,\,p}_{1}(\mathbb{R}^N_+), & \textit{if } N/p' \neq 1. \\ \Delta^2: \ \overset{\star}{W}{}^{4,\,p}_{2}(\mathbb{R}^N_+) & \longrightarrow & W^{0,\,p}_{2}(\mathbb{R}^N_+), & \textit{if } N/p' \notin \{1,2\}. \end{array}$$

(ii) *For* $\ell = -1$

$$\begin{array}{llll} \Delta^2: \ \overset{\circ}{W}{}^{2,\,p}_{-1}(\mathbb{R}^N_+)/\mathcal{B}_{[3-N/p]} & \longrightarrow & W^{-2,\,p}_{-1}(\mathbb{R}^N_+), & \textit{if } N/p \neq 1. \\ \Delta^2: \ \overset{\star}{W}{}^{3,\,p}_{0}(\mathbb{R}^N_+)/\mathcal{B}_{[3-N/p]} & \longrightarrow & W^{-1,\,p}_{0}(\mathbb{R}^N_+). & \\ \Delta^2: \ \overset{\star}{W}{}^{4,\,p}_{1}(\mathbb{R}^N_+)/\mathcal{B}_{[3-N/p]} & \longrightarrow & W^{0,\,p}_{1}(\mathbb{R}^N_+), & \textit{if } N/p' \neq 1. \end{array}$$

(iii) *For* $\ell = -2$

$$\begin{array}{llll} \Delta^2: \ \overset{\circ}{W}{}^{2,\,p}_{-2}(\mathbb{R}^N_+)/\mathcal{B}_{[4-N/p]} & \longrightarrow & W^{-2,\,p}_{-2}(\mathbb{R}^N_+), & \textit{if } N/p \notin \{1,2\}. \\ \Delta^2: \ \overset{\star}{W}{}^{3,\,p}_{-1}(\mathbb{R}^N_+)/\mathcal{B}_{[4-N/p]} & \longrightarrow & W^{-1,\,p}_{-1}(\mathbb{R}^N_+), & \textit{if } N/p \neq 1. \\ \Delta^2: \ \overset{\star}{W}{}^{4,\,p}_{0}(\mathbb{R}^N_+)/\mathcal{B}_{[4-N/p]} & \longrightarrow & L^p(\mathbb{R}^N_+). & \end{array}$$

Remark 3.2.3. Note that we have without any critical value, the isomorphism $\Delta^2 : \overset{\star}{W}{}^{3,p}_0(\mathbb{R}^N_+)/\mathcal{B}_{[3-N/p]} \longrightarrow W^{-1,p}_0(\mathbb{R}^N_+)$. On the other hand, we have the isomorphism $\Delta^2 : W^{3,p}_0(\mathbb{R}^N_+) \cap \overset{\circ}{W}{}^{2,p}_{-1}(\mathbb{R}^N_+)/\mathcal{B}_{[3-N/p]} \longrightarrow W^{-1,p}_0(\mathbb{R}^N_+)$ only if $N/p \neq 1$, which is necessary for the imbedding $W^{3,p}_0(\mathbb{R}^N_+) \hookrightarrow W^{2,p}_{-1}(\mathbb{R}^N_+)$. Hence the specificity of spaces $\overset{\star}{W}{}^{m,p}_\ell(\mathbb{R}^N_+)$. ♢

3.2.3 What is new?

Recall the Boulmezaoud theorem on the biharmonic problem (see [21]), using the spaces $H^{m,p}_\alpha(\mathbb{R}^N_+)$ instead of $W^{m,p}_\alpha(\mathbb{R}^N_+)$ (see Remark 1.2.1):

Theorem (Boulmezaoud [21]). *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$ and assume that*

$$\frac{N}{2} \notin \{1, \ldots, |\ell| + 2\}.$$

For any $f \in H^{m-1,2}_{m+\ell+1}(\mathbb{R}^N_+)$, $g_0 \in H^{m+5/2,2}_{m+\ell+1}(\Gamma)$ and $g_1 \in H^{m+3/2,2}_{m+\ell+1}(\Gamma)$ satisfying the compatibility condition (3.2.2), problem (P) has a solution $u \in H^{m+3,2}_{m+\ell+1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/2]}$ and this solution continuously depends on the data with respect to the quotient norm.

The most important point is about the regularity of data. In Theorem 3.2.1, we can take $f \in W^{-2,2}_\ell(\mathbb{R}^N_+)$, $g_0 \in W^{3/2,2}_\ell(\Gamma)$ and $g_1 \in W^{1/2,2}_\ell(\Gamma)$, whereas the lower level in Boulmezaoud theorem is for $f \in H^{-1,2}_{\ell+1}(\mathbb{R}^N_+)$, $g_0 \in H^{5/2,2}_{\ell+1}(\Gamma)$ and $g_1 \in H^{3/2,2}_{\ell+1}(\Gamma)$. The second point is about critical values which appear for all the even dimensions. Particularly for the dimensions $N = 2$ or $N = 4$, the Boulmezaoud theorem unfortunately does not give any answer to problem (P), whereas we can see in Corollary 3.2.2 that Theorem 3.2.1 gives solutions with f in $W^{-2,2}_0(\mathbb{R}^N_+)$, $W^{-1,2}_0(\mathbb{R}^N_+)$ or $L^2(\mathbb{R}^N_+)$ ... The last point concerns the underlying functional setting of our work, which is that of Lebesgue spaces $L^p(\Omega)$, with $1 < p < \infty$.

3.3 Singular boundary conditions

The purpose of the second part of this chapter is now to find some solutions to the homogeneous problem $(\mathcal{P}^0)$ for singular boundary conditions. We suggest an answer to this question through Theorems 3.3.4 and 3.3.5.

3.3.1 Extension of traces

In this section, we establish the existence of traces in special cases we shall use for the study of singular boundary conditions.

For any $\ell \in \mathbb{Z}$, we introduce the spaces

$$Y^p_\ell(\mathbb{R}^N_+) = \left\{ v \in W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+);\ \Delta^2 v \in W^{0,\,p}_{\ell+2}(\mathbb{R}^N_+) \right\},$$
$$Y^p_{\ell,\,1}(\mathbb{R}^N_+) = \left\{ v \in W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+);\ \Delta^2 v \in W^{0,\,p}_{\ell+2,\,1}(\mathbb{R}^N_+) \right\}.$$

They are reflexive Banach spaces equipped with their natural norms:

$$\|v\|_{Y^p_\ell(\mathbb{R}^N_+)} = \|v\|_{W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)} + \|\Delta^2 v\|_{W^{0,\,p}_{\ell+2}(\mathbb{R}^N_+)},$$
$$\|v\|_{Y^p_{\ell,\,1}(\mathbb{R}^N_+)} = \|v\|_{W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)} + \|\Delta^2 v\|_{W^{0,\,p}_{\ell+2,\,1}(\mathbb{R}^N_+)}.$$

Lemma 3.3.1. *Let $\ell \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell - 2\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell + 2\}, \tag{3.3.1}$$

then the space $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^p_\ell(\mathbb{R}^N_+)$ and in $Y^p_{\ell,\,1}(\mathbb{R}^N_+)$.

Proof. (i) We use an extension of the Riesz representation theorem to weighted Sobolev spaces: Given $T \in \big(Y^p_\ell(\mathbb{R}^N_+)\big)'$, there exists a unique pair $(u_1,\, u_2) \in \big(L^{p'}(\mathbb{R}^N_+)\big)^2$ such that

$$\forall \varphi \in Y^p_\ell(\mathbb{R}^N_+), \quad \langle T, \varphi \rangle = \int_{\mathbb{R}^N_+} u_1\, \varrho^{\ell-2}\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_+} u_2\, \varrho^{\ell+2}\, \Delta^2 \varphi \,\mathrm{d}x. \tag{3.3.2}$$

Let us suppose that $T = 0$ on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$, thus on $\mathcal{D}(\mathbb{R}^N_+)$. Then we can deduce from (3.3.2) that

$$\varrho^{\ell-2}\, u_1 + \Delta^2\left(\varrho^{\ell+2}\, u_2\right) = 0 \quad \text{in } \mathbb{R}^N_+. \tag{3.3.3}$$

We set $v_1 = \varrho^{\ell-2}\, u_1$ and $v_2 = \varrho^{\ell+2}\, u_2$, and we respectively denote by $\tilde{v}_1$ and $\tilde{v}_2$ the extensions by 0 of v_1 and v_2 to $\mathbb{R}^N$. We have for any $\varphi \in \mathcal{D}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \tilde{v}_1\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} \tilde{v}_2\, \Delta^2 \varphi \,\mathrm{d}x = \int_{\mathbb{R}^N_+} v_1\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_+} v_2\, \Delta^2 \varphi \,\mathrm{d}x = 0, \tag{3.3.4}$$

according to the assumption on T, since $\varphi|_{\mathbb{R}^N_+} \in \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$. Therefore we can deduce that $\tilde{v}_1 + \Delta^2 \tilde{v}_2 = 0$ in $\mathbb{R}^N$. We know that $\tilde{v}_1 \in W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N)$, then we also have $\Delta^2 \tilde{v}_2 \in W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N)$. Moreover, we have the following Green formula: for any $\varphi \in \mathcal{D}(\mathbb{R}^N)$,

$$\left\langle \Delta^2 \tilde{v}_2, \varphi \right\rangle_{W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N) \times W^{0,\,p}_{\ell-2}(\mathbb{R}^N)} = \left\langle \tilde{v}_2, \Delta^2 \varphi \right\rangle_{W^{0,\,p'}_{-\ell-2}(\mathbb{R}^N) \times W^{0,\,p}_{\ell+2}(\mathbb{R}^N)}, \tag{3.3.5}$$

and we know that $\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \subset W^{4,\,p}_{\ell+2}(\mathbb{R}^N) \hookrightarrow W^{0,\,p}_{\ell-2}(\mathbb{R}^N)$ under the hypothesis $\frac{N}{p} \notin \{1, \ldots, -\ell+2\}$. Since $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{4,\,p}_{\ell+2}(\mathbb{R}^N)$, we can deduce that

(3.3.5) holds for any $\varphi \in \mathcal{P}^{\Delta^2}_{[2-\ell-N/p]}$ and consequently that $\Delta^2\tilde{v}_2 \perp \mathcal{P}^{\Delta^2}_{[2-\ell-N/p]}$. Thanks to Theorem 2.2.6, with $m = 2$, $-\ell$ instead of ℓ and exchanging p and p', we can deduce that under hypothesis (3.3.1), we have $\tilde{v}_2 \in W^{4,p'}_{-\ell+2}(\mathbb{R}^N)$. Since $\tilde{v}_2$ is an extension by 0, it follows that $v_2 \in \mathring{W}^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Now, thanks to the density of $\mathcal{D}(\mathbb{R}^N_+)$ in $\mathring{W}^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)$, we have the following Green formula:

$$\begin{aligned}\forall \varphi \in Y^p_\ell(\mathbb{R}^N_+),\ \forall w \in \mathring{W}^{4,p'}_{-\ell+2}(\mathbb{R}^N_+) \cap W^{0,p'}_{-\ell-2}(\mathbb{R}^N_+),&\\ \int_{\mathbb{R}^N_+} \varphi\, \Delta^2 w \,\mathrm{d}x = \int_{\mathbb{R}^N_+} w\, \Delta^2 \varphi \,\mathrm{d}x.&\end{aligned} \tag{3.3.6}$$

Then it suffices to come back to (3.3.2) and to use (3.3.6) with $w = v_2$ which belongs to $\mathring{W}^{4,p'}_{-\ell+2}(\mathbb{R}^N_+) \cap W^{0,p'}_{-\ell-2}(\mathbb{R}^N_+)$, to obtain for any $\varphi \in Y^p_\ell(\mathbb{R}^N_+)$:

$$\langle T, \varphi\rangle = \int_{\mathbb{R}^N_+} v_1\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_+} v_2\, \Delta^2\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N_+} \big(v_1 + \Delta^2 v_2\big)\, \varphi \,\mathrm{d}x = 0,$$

according to (3.3.3). Then we have proved that $T = 0$ on $Y^p_\ell(\mathbb{R}^N_+)$, and the Hahn-Banach theorem assures us that $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^p_\ell(\mathbb{R}^N_+)$.

(ii) Likewise, we can prove the density of $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $Y^p_{\ell,1}(\mathbb{R}^N_+)$. The differences only concern the logarithmic factors in the weights.

Given $T \in \big(Y^p_{\ell,1}(\mathbb{R}^N_+)\big)'$, there exists a unique pair $(u_1,\, u_2) \in \big(L^{p'}(\mathbb{R}^N_+)\big)^2$ such that

$$\forall \varphi \in Y^p_{\ell,1}(\mathbb{R}^N_+), \quad \langle T, \varphi\rangle = \int_{\mathbb{R}^N_+} u_1\, \varrho^{\ell-2}\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_+} u_2\, \varrho^{\ell+2}\, \lg \varrho\, \Delta^2\varphi \,\mathrm{d}x. \tag{3.3.7}$$

Let us suppose that $T = 0$ on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$, then we have

$$\varrho^{\ell-2}\, u_1 + \Delta^2 \left(\varrho^{\ell+2} \lg \varrho\, u_2\right) = 0 \quad \text{ in } \mathbb{R}^N_+. \tag{3.3.8}$$

We set $v_1 = \varrho^{\ell-2}\, u_1$ and $v_2 = \varrho^{\ell+2} \lg \varrho\, u_2$, and we respectively denote by $\tilde{v}_1$ and $\tilde{v}_2$ the extensions by 0 of v_1 and v_2 to $\mathbb{R}^N$. We have the analog of identity (3.3.4) for any $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Therefore we can deduce that $\tilde{v}_1 + \Delta^2\tilde{v}_2 = 0$ in $\mathbb{R}^N$. We know that $\tilde{v}_1 \in W^{0,p'}_{-\ell+2}(\mathbb{R}^N)$, then we also have $\Delta^2\tilde{v}_2 \in W^{0,p'}_{-\ell+2}(\mathbb{R}^N)$, whence the analog of Green formula (3.3.5) where the duality of the right side is replaced by $W^{0,p'}_{-\ell-2,-1}(\mathbb{R}^N) \times W^{0,p}_{\ell+2,1}(\mathbb{R}^N)$. Since $\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \subset W^{4,p}_{\ell+2,1}(\mathbb{R}^N) \hookrightarrow W^{0,p}_{\ell-2}(\mathbb{R}^N)$ if $\frac{N}{p} \notin \{1,\ldots,-\ell+2\}$, we can deduce by density that this formula holds for any $\varphi \in \mathcal{P}^{\Delta^2}_{[2-\ell-N/p]}$ and consequently that $\Delta^2\tilde{v}_2 \perp \mathcal{P}^{\Delta^2}_{[2-\ell-N/p]}$. Thanks to Theorem 2.2.6, we can deduce that under hypothesis (3.3.1), we have $\tilde{v}_2 \in W^{4,p'}_{-\ell+2}(\mathbb{R}^N)$. It follows that $v_2 \in \mathring{W}^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)$. We also have the analog of Green formula (3.3.6) for any $w \in \mathring{W}^{4,p'}_{-\ell+2}(\mathbb{R}^N_+) \cap W^{0,p'}_{-\ell-2,-1}(\mathbb{R}^N_+)$. The end of the proof is quite similar to the previous case. $\square$

Thanks to this density lemma, we can prove the following result of traces:

Lemma 3.3.2. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.3.1), the mapping*

$$(\gamma_0,\, \gamma_1) : \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \mathcal{D}(\mathbb{R}^{N-1})^2,$$

can be extended to a linear continuous mapping

$$(\gamma_0,\, \gamma_1) : Y^p_{\ell,\,1}(\mathbb{R}^N_+) \longrightarrow W^{-1/p,\,p}_{\ell-2}(\Gamma) \times W^{-1-1/p,\,p}_{\ell-2}(\Gamma),$$

and we have the following Green formula:

$$\begin{aligned}
&\forall v \in Y^p_{\ell,\,1}(\mathbb{R}^N_+),\ \forall \varphi \in \overset{\star}{W}{}^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+),\\
&\big\langle \Delta^2 v, \varphi\big\rangle_{W^{0,\,p}_{\ell+2,\,1}(\mathbb{R}^N_+)\times W^{0,\,p'}_{-\ell-2,\,-1}(\mathbb{R}^N_+)} - \big\langle v, \Delta^2\varphi\big\rangle_{W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)\times W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N_+)}\\
&= \langle v, \partial_N\Delta\varphi\rangle_{W^{-1/p,\,p}_{\ell-2}(\Gamma)\times W^{1/p,\,p'}_{-\ell+2}(\Gamma)} - \langle \partial_N v, \Delta\varphi\rangle_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)\times W^{1+1/p,\,p'}_{-\ell+2}(\Gamma)}\,.
\end{aligned} \tag{3.3.9}$$

Proof. Let us first remark that for any $\varphi \in \overset{\star}{W}{}^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)$, we have

$$\Delta\varphi = \partial^2_N\varphi \ \text{ and } \ \partial_N\Delta\varphi = \partial^3_N\varphi \ \text{ on } \Gamma.$$

Moreover, we always have the imbedding $W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,\,p'}_{-\ell-2,\,-1}(\mathbb{R}^N_+)$. So we can write the following Green formula:

$$\begin{aligned}
&\forall v \in \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big),\ \forall \varphi \in \overset{\star}{W}{}^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+),\\
&\quad\int_{\mathbb{R}^N_+} \varphi\,\Delta^2 v\,\mathrm{d}x - \int_{\mathbb{R}^N_+} v\,\Delta^2\varphi\,\mathrm{d}x = \int_\Gamma v\,\partial_N\Delta\varphi\,\mathrm{d}x' - \int_\Gamma \partial_N v\,\Delta\varphi\,\mathrm{d}x'.
\end{aligned} \tag{3.3.10}$$

In particular, if $\varphi \in W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)$ and such that $\varphi = \partial_N\varphi = \partial^2_N\varphi = 0$ on Γ, we have

$$\left|\int_\Gamma v\,\partial_N\Delta\varphi\,\mathrm{d}x'\right| \leqslant\ \|v\|_{Y^p_{\ell,\,1}(\mathbb{R}^N_+)}\ \|\varphi\|_{W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)}.$$

For all $g \in W^{1-1/p',\,p'}_{-\ell+2}(\Gamma)$, thanks to Lemma 1.3.1, there exists a lifting function $\varphi_0 \in W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)$ such that $\varphi_0 = \partial_N\varphi_0 = \partial^2_N\varphi_0 = 0$ on Γ and $\partial^3_N\varphi_0 = g$ on Γ, satisfying moreover

$$\|\varphi_0\|_{W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)} \leqslant C\ \|g\|_{W^{1-1/p',\,p'}_{-\ell+2}(\Gamma)},$$

where C is a constant not depending on φ_0 and g. Then we have

$$\left|\int_\Gamma v\,g\,\mathrm{d}x'\right| \ \leqslant\ \|v\|_{Y^p_{\ell,\,1}(\mathbb{R}^N_+)}\ \|\varphi_0\|_{W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)}\ \leqslant\ C\ \|v\|_{Y^p_{\ell,\,1}(\mathbb{R}^N_+)}\ \|g\|_{W^{1-1/p',\,p'}_{-\ell+2}(\Gamma)}.$$

Thus

$$\|\gamma_0 v\|_{W^{-1/p,\,p}_{\ell-2}(\Gamma)}\ \leqslant\ C\ \|v\|_{Y^p_{\ell,\,1}(\mathbb{R}^N_+)}.$$

Therefore, the linear mapping $\gamma_0 : v \longmapsto v|_\Gamma$ defined on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is continuous for the norm of $Y^p_{\ell,1}(\mathbb{R}^N_+)$. Since $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^p_{\ell,1}(\mathbb{R}^N_+)$, γ_0 can be extended by continuity to a mapping $\gamma_0 \in \mathcal{L}\big(Y^p_{\ell,1}(\mathbb{R}^N_+);\ W^{-1/p,p}_{\ell-2}(\Gamma)\big)$.

To define the trace γ_1 on $Y^p_{\ell,1}(\mathbb{R}^N_+)$, we consider $\varphi \in W^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)$ such that $\varphi = \partial_N\varphi = \partial^3_N\varphi = 0$ on Γ. In this case, we have

$$\left|\int_\Gamma \partial_N v\, \Delta\varphi\, \mathrm{d}x'\right| \leqslant\ \|v\|_{Y^p_{\ell,1}(\mathbb{R}^N_+)}\ \|\varphi\|_{W^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)}.$$

For all $g \in W^{2-1/p',p'}_{-\ell+2}(\Gamma)$, thanks to Lemma 1.3.1, there exists a lifting function $\varphi_0 \in W^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)$ such that $\varphi_0 = \partial_N\varphi_0 = \partial^3_N\varphi_0 = 0$ on Γ and $\partial^2_N\varphi_0 = g$ on Γ, satisfying moreover

$$\|\varphi_0\|_{W^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)} \leqslant C\ \|g\|_{W^{2-1/p',p'}_{-\ell+2}(\Gamma)},$$

where C is a constant independent of φ_0 and g. Once again, the linear mapping $\gamma_1 : v \longmapsto \partial_N v|_\Gamma$, defined on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is continuous for the norm of $Y^p_{\ell,1}(\mathbb{R}^N_+)$, and it can be extended by continuity to a mapping $\gamma_1 \in \mathcal{L}\big(Y^p_{\ell,1}(\mathbb{R}^N_+);\ W^{-1-1/p,p}_{\ell-2}(\Gamma)\big)$.

To conclude this proof, we can deduce the formula (3.3.9) from (3.3.10) by density of $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $Y^p_{\ell,1}(\mathbb{R}^N_+)$. □

Remark 3.3.3. Note that the logarithmic factors are unnecessary in the case where $\frac{N}{p'} \notin \{\ell-1,\, \ell,\, \ell+1,\, \ell+2\}$, because the imbedding $W^{4,p'}_{-\ell+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell-2}(\mathbb{R}^N_+)$ holds. So we can replace the space $Y^p_{\ell,1}(\mathbb{R}^N_+)$ by $Y^p_\ell(\mathbb{R}^N_+)$ in the lemma, with a Green formula without logarithmic factors, *i.e.* where the first term of the left side is replaced by $\langle\Delta^2 v, \varphi\rangle_{W^{0,p}_{\ell+2}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell-2}(\mathbb{R}^N_+)}$. ♢

3.3.2 Very weak solutions

We now come back to the homogeneous problem, and we consider here singular boundary conditions. Let $g_0 \in W^{-1/p,p}_{\ell-2}(\Gamma)$ and $g_1 \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$, we search $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$ solution to the problem

$$(P^0)\ \begin{cases} \Delta^2 u = 0 & \text{in } \mathbb{R}^N_+,\\ u = g_0 & \text{on } \Gamma,\\ \partial_N u = g_1 & \text{on } \Gamma. \end{cases}$$

Let us first remark that if $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$ verifies (P^0) under hypothesis (3.3.1), then it belongs to $Y^p_{\ell,1}(\mathbb{R}^N_+)$ and thanks to Lemma 3.3.2, $\gamma_0 u \in W^{-1/p,p}_{\ell-2}(\Gamma)$ and $\gamma_1 u \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$, which gives a sense to (P^0).

Theorem 3.3.4. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.3.1), for any $g_0 \in W^{-1/p,\,p}_{\ell-2}(\Gamma)$ and $g_1 \in W^{-1-1/p,\,p}_{\ell-2}(\Gamma)$ satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']}\ , \quad \langle g_1, \Delta\varphi\rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi\rangle_\Gamma = 0, \tag{3.3.11}$$

problem (P^0) has a solution $u \in W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, with the estimate

$$\inf_{q\in\mathcal{B}_{[2-\ell-N/p]}} \|u+q\|_{W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)} \ \leqslant\ C\left(\|g_0\|_{W^{-1/p,\,p}_{\ell-2}(\Gamma)} + \|g_1\|_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)}\right).$$

Proof. Let $\mathcal{K}^{-2}$ denote the kernel of the operator associated to this problem. We can observe that problem (P^0) is equivalent to the formulation:

$$(Q)\ \begin{cases} \text{Find } u \in Y^p_{\ell,1}(\mathbb{R}^N_+)/\mathcal{K}^{-2} \text{ such that for any } v \in \overset{\star}{W}{}^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+), \\ \langle u, \Delta^2 v\rangle_{W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)\times W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N_+)} \ =\ \langle g_1, \Delta v\rangle_\Gamma - \langle g_0, \partial_N \Delta v\rangle_\Gamma\,, \end{cases}$$

where we have used the Green formula (3.3.9) of Lemma 3.3.2.

Now, let us solve problem (Q). For any $f \in W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2-\ell-N/p]}$, according to Theorem 3.2.1, with $m=2$, $-\ell$ instead of ℓ and exchanging p and p', the problem

$$(P^\star)\ \begin{cases} \Delta^2 v = f & \text{in } \mathbb{R}^N_+, \\ v = 0 & \text{on } \Gamma, \\ \partial_N v = 0 & \text{on } \Gamma, \end{cases}$$

admits a unique solution $v \in W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{B}_{[2+\ell-N/p']}$, under hypothesis (3.3.1). Moreover, v satisfies the estimate

$$\|v\|_{W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{B}_{[2+\ell-N/p']}} \ \leqslant\ C\ \|f\|_{W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N_+)},$$

where C denotes as usual a generic constant not depending on v and f. Consider the linear form $T\ :\ f \longmapsto \langle g_1, \Delta v\rangle_\Gamma - \langle g_0, \partial_N \Delta v\rangle_\Gamma$ defined on $W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2-\ell-N/p]}$. We have for any $q \in \mathcal{B}_{[2+\ell-N/p']}$,

$$\begin{aligned} |T(f)| &= \Big|\ \langle g_1, \Delta(v+q)\rangle_\Gamma - \langle g_0, \partial_N\Delta(v+q)\rangle_\Gamma\ \Big| \\ &\leqslant C\ \|v+q\|_{W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)}\ \Big(\|g_0\|_{W^{-1/p,\,p}_{\ell-2}(\Gamma)} + \|g_1\|_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)}\Big). \end{aligned}$$

Thus

$$\begin{aligned} |T(f)| &\leqslant C\ \|v\|_{W^{4,\,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{B}_{[2+\ell-N/p']}}\ \Big(\|g_0\|_{W^{-1/p,\,p}_{\ell-2}(\Gamma)} + \|g_1\|_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)}\Big) \\ &\leqslant C\ \|f\|_{W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N_+)}\ \Big(\|g_0\|_{W^{-1/p,\,p}_{\ell-2}(\Gamma)} + \|g_1\|_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)}\Big). \end{aligned}$$

Hence T is continuous on $W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2-\ell-N/p]}$, and according to Riesz representation theorem, there exists a unique $u \in W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)/\mathcal{B}_{[2-\ell-N/p]}$ such that $T(f) = \langle u, f\rangle_{W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)\times W^{0,\,p'}_{-\ell+2}(\mathbb{R}^N_+)}$. This means that u is a solution to problem (Q) and $\mathcal{K}^{-2} = \mathcal{B}_{[2-\ell-N/p]}$. □

3.3.3 Intermediate boundary conditions

To be comprehensive and also for the Stokes system, it remains to solve (P^0) for the data $g_0 \in W^{1-1/p,\,p}_{\ell-1}(\Gamma)$ and $g_1 \in W^{-1/p,\,p}_{\ell-1}(\Gamma)$. So, we fill the gap between generalized solutions of Theorem 2.3.1 and very weak solutions of Theorem 3.3.4. We could also call these solutions "very weak" and the ones of Theorem 3.3.4, "very very weak"...

Theorem 3.3.5. *Let $\ell \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell - 1\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell + 1\}. \tag{3.3.12}$$

For any $g_0 \in W^{1-1/p,\,p}_{\ell-1}(\Gamma)$ and $g_1 \in W^{-1/p,\,p}_{\ell-1}(\Gamma)$ satisfying the compatibility condition (3.3.11), problem (P^0) has a solution $u \in W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, with the estimate

$$\inf_{q \in \mathcal{B}_{[2-\ell-N/p]}} \|u + q\|_{W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)} \leqslant C \left(\|g_0\|_{W^{1-1/p,\,p}_{\ell-1}(\Gamma)} + \|g_1\|_{W^{-1/p,\,p}_{\ell-1}(\Gamma)} \right).$$

Remark 3.3.6. We can give a very quick proof of this result by interpolation between the previous case and the regular case, *i.e.* $g_0 \in W^{2-1/p,\,p}_{\ell}(\Gamma)$ and $g_1 \in W^{1-1/p,\,p}_{\ell}(\Gamma)$. But the problem with this reasoning is that we must combine the critical values of hypotheses (2.3.1) and (3.3.1), and then we obtain two supplementary values with respect to (3.3.12). Thus we shall give a direct proof similar to the singular case, with however some new arguments. $\diamondsuit$

For any $\ell \in \mathbb{Z}$, we introduce the space

$$Y^{1,\,p}_{\ell,\,1}(\mathbb{R}^N_+) = \left\{ v \in W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+);\ \Delta^2 v \in W^{0,\,p}_{\ell+2,\,1}(\mathbb{R}^N_+) \right\}.$$

It's a reflexive Banach space equipped with it's natural norm:

$$\|v\|_{Y^{1,\,p}_{\ell,\,1}(\mathbb{R}^N_+)} = \|v\|_{W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)} + \|\Delta^2 v\|_{W^{0,\,p}_{\ell+2,\,1}(\mathbb{R}^N_+)}.$$

We also define the subspace of $Y^{1,\,p}_{\ell,\,1}(\mathbb{R}^N_+)$

$$\mathring{Y}^{1,\,p}_{\ell,\,1}(\mathbb{R}^N_+) = \left\{ v \in \mathring{W}^{1,\,p}_{\ell-1}(\mathbb{R}^N_+);\ \Delta^2 v \in W^{0,\,p}_{\ell+2,\,1}(\mathbb{R}^N_+) \right\}.$$

Lemma 3.3.7. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.3.12), the space $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^{1,\,p}_{\ell,\,1}(\mathbb{R}^N_+)$.*

Proof. Let P be an extension operator mapping $W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$ into $W^{1,\,p}_{\ell-1}(\mathbb{R}^N)$. For any continuous linear form $T \in \big(Y^{1,\,p}_{\ell,\,1}(\mathbb{R}^N_+)\big)'$, there exists a unique pair $(u_1,\, u_2) \in W^{-1,\,p'}_{-\ell+1}(\mathbb{R}^N) \times W^{0,\,p'}_{-\ell-2,\,-1}(\mathbb{R}^N_+)$ such that for any $v \in Y^{1,\,p}_{\ell,\,1}(\mathbb{R}^N_+)$,

$$\langle T, v \rangle = \langle u_1, Pv \rangle_{W^{-1,\,p'}_{-\ell+1}(\mathbb{R}^N) \times W^{1,\,p}_{\ell-1}(\mathbb{R}^N)} + \int_{\mathbb{R}^N_+} u_2\, \Delta^2 v \,\mathrm{d}x.$$

Moreover, since T depends only on v and not on the restriction of Pv to $\mathbb{R}^N_-$, the support of u_1 is contained in $\overline{\mathbb{R}^N_+}$.

Thanks to the Hahn-Banach theorem, it suffices to show that any T which vanishes on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is actually zero on $Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)$. Indeed for any $\Psi \in \mathcal{D}(\mathbb{R}^N)$, we have

$$\langle u_1, \Psi\rangle + \big\langle \tilde{u}_2, \Delta^2\Psi\big\rangle = \langle u_1, P\psi\rangle + \int_{\mathbb{R}^N_+} u_2\,\Delta^2\psi\,\mathrm{d}x = 0,$$

where $\tilde{u}_2$ is the extension by 0 of u_2 to $\mathbb{R}^N$ and $\psi = \Psi|_{\mathbb{R}^N_+}$. It follows that

$$\Delta^2\tilde{u}_2 = -u_1 \text{ in } \mathbb{R}^N.$$

Thus we have $\Delta^2\tilde{u}_2 \in W^{-1,p'}_{-\ell+1}(\mathbb{R}^N)$. Since $\tilde{u}_2 \in W^{0,p'}_{-\ell-2,-1}(\mathbb{R}^N_+)$, we also have $\Delta^2\tilde{u}_2 \perp \mathcal{P}^{\Delta^2}_{[2-\ell-N/p]}$. Now, thanks to Theorem 2.2.6, we can deduce that under hypothesis (3.3.12), we have $\tilde{u}_2 \in W^{3,p'}_{-\ell+1}(\mathbb{R}^N)$. It follows that $u_2 \in \mathring{W}^{3,p'}_{-\ell+1}(\mathbb{R}^N_+)$. By density of $\mathcal{D}(\mathbb{R}^N_+)$ in $\mathring{W}^{3,p'}_{-\ell+1}(\mathbb{R}^N_+)$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^N_+)$ such that $\varphi_k \to u_2$ in $\mathring{W}^{3,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Thus for any $v \in Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)$, we have

$$\begin{aligned}\langle T, v\rangle &= \lim_{k\to\infty}\left\{\big\langle -\Delta^2\tilde{\varphi}_k, Pv\big\rangle + \int_{\mathbb{R}^N_+}\varphi_k\,\Delta^2 v\,\mathrm{d}x\right\}\\ &= \lim_{k\to\infty}\left\{-\int_{\mathbb{R}^N_+} v\,\Delta^2\varphi_k\,\mathrm{d}x + \int_{\mathbb{R}^N_+}\varphi_k\,\Delta^2 v\,\mathrm{d}x\right\}\\ &= 0.\end{aligned}$$

Thus T is identically zero. □

Lemma 3.3.8. *Let $\ell \in \mathbb{Z}$. Under hypothesis (3.3.12), the mapping*

$$\gamma_1 : \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \mathcal{D}(\mathbb{R}^{N-1}),$$

can be extended to a linear continuous mapping

$$\gamma_1 : Y^{1,p}_{\ell,1}(\mathbb{R}^N_+) \longrightarrow W^{-1/p,p}_{\ell-1}(\Gamma),$$

and we have the following Green formula:

$$\begin{aligned}&\forall v \in \mathring{Y}^{1,p}_{\ell,1}(\mathbb{R}^N_+),\ \forall\varphi \in \overset{\star}{W}{}^{3,p'}_{-\ell+1}(\mathbb{R}^N_+),\\ &\qquad\big\langle \Delta^2 v, \varphi\big\rangle_{W^{0,p}_{\ell+2,1}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell-2,-1}(\mathbb{R}^N_+)} - \big\langle v, \Delta^2\varphi\big\rangle_{\mathring{W}^{1,p}_{\ell-1}(\mathbb{R}^N_+)\times W^{-1,p'}_{-\ell+1}(\mathbb{R}^N_+)}\\ &\qquad\qquad = -\left\langle \partial_N v, \Delta\varphi\right\rangle_{W^{-1/p,p}_{\ell-1}(\Gamma)\times W^{1/p,p'}_{-\ell+1}(\Gamma)}.\end{aligned} \tag{3.3.13}$$

Proof. Since we always have the imbedding $W^{3,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell-2,-1}(\mathbb{R}^N_+)$, we can write the following Green formula:

$$\forall v \in \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big),\ \forall \varphi \in \overset{\star}{W}{}^{3,p'}_{-\ell+1}(\mathbb{R}^N_+),$$
$$\int_{\mathbb{R}^N_+} \varphi\, \Delta^2 v \,\mathrm{d}x + \int_{\mathbb{R}^N_+} \nabla v.\nabla\Delta\varphi \,\mathrm{d}x = -\int_\Gamma \partial_N v\, \Delta\varphi \,\mathrm{d}x'. \tag{3.3.14}$$

This implies that

$$\left|\int_\Gamma \partial_N v\, \Delta\varphi \,\mathrm{d}x'\right| \leqslant \|v\|_{Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)}\, \|\varphi\|_{W^{3,p'}_{-\ell+1}(\mathbb{R}^N_+)}.$$

For all $g \in W^{1-1/p',p'}_{-\ell+1}(\Gamma)$, thanks to Lemma 1.3.1, there exists a lifting function $\varphi_0 \in W^{3,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\varphi_0 = \partial_N\varphi_0 = 0$ on Γ and $\partial^2_N\varphi_0 = g$ on Γ, satisfying moreover

$$\|\varphi_0\|_{W^{3,p'}_{-\ell+1}(\mathbb{R}^N_+)} \leqslant C\, \|g\|_{W^{1-1/p',p'}_{-\ell+1}(\Gamma)},$$

where C is a constant not depending on φ_0 and g. Then we have

$$\left|\int_\Gamma g\, \partial_N v \,\mathrm{d}x'\right| \leqslant \|v\|_{Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)}\, \|\varphi_0\|_{W^{3,p'}_{-\ell+1}(\mathbb{R}^N_+)} \leqslant C\, \|v\|_{Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)}\, \|g\|_{W^{1-1/p',p'}_{-\ell+1}(\Gamma)}.$$

Therefore

$$\|\gamma_1 v\|_{W^{-1/p,p}_{\ell-1}(\Gamma)} \leqslant C\, \|v\|_{Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)}.$$

Thus the linear mapping $\gamma_1 : v \longmapsto \partial_N v|_\Gamma$ defined on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is continuous for the norm of $Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)$. Since $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)$, γ_1 can be extended by continuity to a mapping $\gamma_1 \in \mathcal{L}\big(Y^{1,p}_{\ell,1}(\mathbb{R}^N_+);\ W^{-1/p,p}_{\ell-1}(\Gamma)\big)$.

By density of $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)$, we can generalize the formula (3.3.14) to any $v \in Y^{1,p}_{\ell,1}(\mathbb{R}^N_+)$. Furthermore, thanks to the of $\mathcal{D}(\mathbb{R}^N_+)$ in $\overset{\circ}{W}{}^{1,p}_{\ell-1}(\mathbb{R}^N_+)$, we have for any $v \in \overset{\circ}{W}{}^{1,p}_{\ell-1}(\mathbb{R}^N_+)$ and $\varphi \in W^{3,p'}_{-\ell+1}(\mathbb{R}^N_+)$,

$$\langle \nabla v, \nabla\Delta\varphi\rangle_{W^{0,p}_{\ell-1}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell+1}(\mathbb{R}^N_+)} = \big\langle v, \Delta^2\varphi\big\rangle_{\overset{\circ}{W}{}^{1,p}_{\ell-1}(\mathbb{R}^N_+)\times W^{-1,p'}_{-\ell+1}(\mathbb{R}^N_+)}.$$

So we obtain the Green formula (3.3.13). □

Proof of Theorem 3.3.5. The first step is to reduce to zero the boundary condition on u in Problem (P^0). Let us consider the problem

$$(R^0)\ \begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^N_+, \\ w = g_0 & \text{on } \Gamma. \end{cases}$$

Thanks to (2.3.4), we know that $\mathcal{B}_{[2+\ell-N/p']} = \Pi_D \mathcal{A}^{\Delta}_{[\ell-N/p']} \oplus \Pi_N \mathcal{N}^{\Delta}_{[\ell-N/p']}$, thus the compatibility condition (3.3.11) on (P^0) implies

$$\forall r \in \mathcal{A}^{\Delta}_{[\ell-N/p']}, \quad \langle g_0, \partial_N r \rangle_{W^{1-1/p,\,p}_{\ell-1}(\Gamma)\times W^{-1/p',\,p'}_{-\ell+1}(\Gamma)} = 0,$$

which is the compatibility condition on (R^0). Thus, according to Theorem 1.4.1, problem (R^0) admits a solution $w \in W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$ under hypothesis (3.3.12). It follows that $w \in Y^{1,\,p}_{\ell,\,1}(\mathbb{R}^N_+)$, and thus $\gamma_1 w = h_1 \in W^{-1/p,\,p}_{\ell-1}(\Gamma)$. Let us set $v = u - w$, then problem (P^0) is equivalent to the following

$$\Delta^2 v = 0 \ \text{ in } \mathbb{R}^N_+, \quad v = 0 \ \text{ and } \ \partial_N v = g_1 - h_1 \ \text{ on } \Gamma. \tag{3.3.15}$$

Let $\mathcal{K}^{-1}$ denote the kernel of the operator associated to this problem. We can observe that Problem (3.3.15) is equivalent to the formulation:

$$(Q) \left\{ \begin{array}{l} \text{Find } v \in \overset{\circ}{Y}{}^{1,\,p}_{\ell,\,1}(\mathbb{R}^N_+)/\mathcal{K}^{-1} \text{ such that for any } \varphi \in \overset{\star}{W}{}^{3,\,p'}_{-\ell+1}(\mathbb{R}^N_+), \\ \quad \langle v, \Delta^2\varphi \rangle_{\overset{\circ}{W}{}^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)\times W^{-1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} \ = \ \langle g_1 - h_1, \Delta\varphi \rangle_{\Gamma}\,, \end{array}\right.$$

where we have used the Green formula (3.3.13) of Lemma 3.3.8.

Now, let us solve Problem (Q). For any $f \in W^{-1,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2-\ell-N/p]}$, according to Theorem 3.2.1, with $m = 1$, $-\ell$ instead of ℓ and exchanging p and p', the problem

$$(P^\star) \begin{cases} \Delta^2 z = f & \text{in } \mathbb{R}^N_+, \\ z = 0 & \text{on } \Gamma, \\ \partial_N z = 0 & \text{on } \Gamma, \end{cases}$$

admits a unique solution $z \in W^{3,\,p'}_{-\ell+1}(\mathbb{R}^N_+)/\mathcal{B}_{[2+\ell-N/p']}$, under hypothesis (3.3.12). Moreover, v satisfies the estimate

$$\|z\|_{W^{3,\,p'}_{-\ell+1}(\mathbb{R}^N_+)/\mathcal{B}_{[2+\ell-N/p']}} \ \leqslant \ C\ \|f\|_{W^{-1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}.$$

Consider the linear form $T\ :\ \ f \longmapsto \langle g_1 - h_1, \Delta z \rangle_\Gamma$. We can show that it is continuous on $W^{-1,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2-\ell-N/p]}$. Then, according to the Riesz representation theorem, there exists a unique $v \in \overset{\circ}{W}{}^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)/\mathcal{B}_{[2-\ell-N/p]}$ such that $T(f) = \langle v, f \rangle_{\overset{\circ}{W}{}^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)\times W^{-1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}$. This means that v is a solution to Problem (Q) and $\mathcal{K}^{-1} = \mathcal{B}_{[2-\ell-N/p]}$. $\square$

3.4 Other boundary conditions

The last part of this chapter is devoted to the biharmonic equation with other kinds of boundary conditions. These results will be useful in the next chapters concerning the Stokes problem with different types of boundary conditions.

I. Conditions on u and Δu

The biharmonic equation with boundary conditions on u and Δu

$$(Q)\quad \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma, \\ \Delta u = g_1 & \text{on } \Gamma. \end{cases}$$

Theorem 3.4.1. *Let $\ell \in \mathbb{Z}$. Under hypothesis (2.3.1), for any $f \in W^{-1,\,p}_{\ell}(\mathbb{R}^N_+)$, $g_0 \in W^{3-1/p,\,p}_{\ell}(\Gamma)$ and $g_1 \in W^{1-1/p,\,p}_{\ell}(\Gamma)$ satisfying the compatibility condition*

$$\begin{aligned} &\forall \varphi \in \mathcal{A}^{\Delta^2}_{[1+\ell-N/p']}, \\ &\qquad \langle f, \varphi \rangle_{W^{-1,\,p}_{\ell}(\mathbb{R}^N_+) \times \mathring{W}^{1,\,p'}_{-\ell}(\mathbb{R}^N_+)} - \langle g_1, \partial_N \varphi \rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi \rangle_\Gamma = 0, \end{aligned} \tag{3.4.1}$$

problem (Q) admits a solution $u \in W^{3,\,p}_{\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta^2}_{[3-\ell-N/p]}$, with the estimate

$$\begin{aligned} \inf_{q \in \mathcal{A}^{\Delta^2}_{[3-\ell-N/p]}} \|u+q\|_{W^{3,\,p}_{\ell}(\mathbb{R}^N_+)} \leqslant \\ C \left(\|f\|_{W^{-1,\,p}_{\ell}(\mathbb{R}^N_+)} + \|g_0\|_{W^{3-1/p,\,p}_{\ell}(\Gamma)} + \|g_1\|_{W^{1-1/p,\,p}_{\ell}(\Gamma)} \right). \end{aligned}$$

The kernel

We must characterize the kernel of the operator

$$(\Delta^2, \gamma_0, \gamma_0\Delta) : W^{3,\,p}_{\ell}(\mathbb{R}^N_+) \longrightarrow W^{-1,\,p}_{\ell}(\mathbb{R}^N_+) \times W^{3-1/p,\,p}_{\ell}(\Gamma) \times W^{1-1/p,\,p}_{\ell}(\Gamma).$$

Since $\frac{N}{p} \notin \{1, \ldots, -\ell\}$, we know that $\mathcal{P}_{[3-\ell-N/p]} \subset W^{3,\,p}_{\ell}(\mathbb{R}^N_+)$. Let u be a function of this kernel and set

$$\tilde{u}(x', x_N) = \begin{cases} u(x', x_N) & \text{if } x_N \geqslant 0, \\ -u(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Thus we have $\tilde{u} \in \mathcal{S}'(\mathbb{R}^N)$ and we show that $\Delta^2 \tilde{u} = 0$ in $\mathbb{R}^N$. We can deduce that $\tilde{u}$, and consequently u, is a polynomial. By identification in the half-space $x_N < 0$, we obtain that u is odd with respect to x_N. Conversely it is clear that any polynomial u odd with respect to x_N verifies $u = \Delta u = 0$ on Γ. Furthermore $u \in W^{3,\,p}_{\ell}(\mathbb{R}^N_+)$ implies that its degree is at the most $[3-\ell-N/p]$. So we have characterized this kernel as the space of biharmonic polynomials, odd with respect to x_N, of degree smaller than or equal to $[3-\ell-N/p]$. We denote it by $\mathcal{A}^{\Delta^2}_{[3-\ell-N/p]}$.

The compatibility condition

Problem (Q) admits a solution u in $W_\ell^{3,p}(\mathbb{R}_+^N)$ only if the condition (3.4.1) is satisfied, where $\langle g_1, \partial_N\varphi\rangle_\Gamma$ and $\langle g_0, \partial_N\Delta\varphi\rangle_\Gamma$ respectively denote the duality brackets $\langle g_1, \partial_N\varphi\rangle_{W_\ell^{1-1/p,p}(\Gamma)\times W_{-\ell}^{-1/p',p'}(\Gamma)}$ and $\langle g_0, \partial_N\Delta\varphi\rangle_{W_\ell^{3-1/p,p}(\Gamma)\times W_{-\ell}^{-2-1/p',p'}(\Gamma)}$.

Note that if $\ell \leqslant 0$, then $\mathcal{A}^{\Delta^2}_{[1+\ell-N/p']} = \{0\}$ and thus there is no compatibility condition. Let us now remark that if $\varphi \in \mathcal{A}^{\Delta^2}_{[1+\ell-N/p']}$, then $\varphi \in W^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$ and thus $\partial_N\varphi|_\Gamma \in W^{1-1/p',p'}_{-\ell+1}(\Gamma) \hookrightarrow W^{-1/p',p'}_{-\ell}(\Gamma)$. But we also have $\varphi \in W^{4,p'}_{-\ell+3}(\mathbb{R}^N_+)$ and thus $\partial_N\Delta\varphi \in W^{1-1/p',p'}_{-\ell+3}(\Gamma) \hookrightarrow W^{-2-1/p',p'}_{-\ell}(\Gamma)$. This gives a sense to (3.4.1). As in the previous chapter, for the generalized solutions, in Lemma 2.3.7, we can verify that these imbeddings hold under hypothesis (2.3.1) for $\ell \geqslant 1$ and we can prove in a similar fashion the necessity of condition (3.4.1).

Proof of Theorem 3.4.1

Let us first consider the Dirichlet problem

$$\Delta v = f \quad \text{in } \mathbb{R}^N_+, \qquad v = g_1 \quad \text{on } \Gamma,$$

which admits a solution $v \in W^{1,p}_\ell(\mathbb{R}^N_+)$, if the following compatibility condition is satisfied (see Theorem 1.4.1):

$$\forall \vartheta \in \mathcal{A}^{\Delta}_{[1+\ell-N/p']}, \quad \langle f, \vartheta\rangle_{W^{-1,p}_\ell(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell}(\mathbb{R}^N_+)} - \langle g_1, \partial_N\vartheta\rangle_\Gamma = 0. \tag{3.4.2}$$

Then, we must solve the second Dirichlet problem

$$\Delta u = v \quad \text{in } \mathbb{R}^N_+, \qquad u = g_0 \quad \text{on } \Gamma,$$

which admits a solution $u \in W^{3,p}_\ell(\mathbb{R}^N_+)$, if the following compatibility condition is satisfied (see Theorem 1.4.2):

$$\forall \psi \in \mathcal{A}^{\Delta}_{[-1+\ell-N/p']}, \quad \langle v, \psi\rangle_{W^{1,p}_\ell(\mathbb{R}^N_+)\times W^{-1,p'}_{-\ell}(\mathbb{R}^N_+)} - \langle g_0, \partial_N\psi\rangle_\Gamma = 0. \tag{3.4.3}$$

Now, let us show that the compatibility condition (3.4.1) of problem (Q) implies the conditions (3.4.2) and (3.4.3). Condition (3.4.1) must be satisfied for any $\varphi \in \mathcal{A}^{\Delta^2}_{[1+\ell-N/p']}$, thus for any $\vartheta \in \mathcal{A}^{\Delta}_{[1+\ell-N/p']}$, and then it is reduced to

$$\langle f, \vartheta\rangle_{W^{-1,p}_\ell(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell}(\mathbb{R}^N_+)} - \langle g_1, \partial_N\vartheta\rangle_\Gamma = 0,$$

i.e. precisely the condition (3.4.2). Now, note that by (3.4.1), v satisfies

$$\forall \varphi \in \mathcal{A}^{\Delta^2}_{[1+\ell-N/p']},$$
$$\langle \Delta v, \varphi\rangle_{W^{-1,p}_\ell(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell}(\mathbb{R}^N_+)} - \langle v, \partial_N\varphi\rangle_\Gamma - \langle g_0, \partial_N\Delta\varphi\rangle_\Gamma = 0.$$

It remains to write for such a φ, the Green formula

$$\langle \Delta v, \varphi \rangle_{W_\ell^{-1,p}(\mathbb{R}_+^N)\times \overset{\circ}{W}{}_{-\ell}^{1,p'}(\mathbb{R}_+^N)} = \langle v, \Delta\varphi \rangle_{W_\ell^{1,p}(\mathbb{R}_+^N)\times W_{-\ell}^{-1,p'}(\mathbb{R}_+^N)} + \langle v, \partial_N \varphi \rangle_\Gamma ,$$

to deduce the condition

$$\forall \varphi \in \mathcal{A}^{\Delta^2}_{[1+\ell-N/p']}, \quad \langle v, \Delta\varphi \rangle_{W_\ell^{1,p}(\mathbb{R}_+^N)\times W_{-\ell}^{-1,p'}(\mathbb{R}_+^N)} - \langle g_0, \partial_N \Delta\varphi \rangle_\Gamma = 0.$$

If we finally remark that any $\psi \in \mathcal{A}^{\Delta}_{[-1+\ell-N/p']}$ can be written $\psi = \Delta\varphi$ with $\varphi \in \mathcal{A}^{\Delta^2}_{[1+\ell-N/p']}$, we exactly find the condition (3.4.3).

Remark 3.4.2. Problem (Q) is ill-posed for $f \in W_\ell^{-2,p}(\mathbb{R}_+^N)$. However, if $f = 0$, we can consider less regular boundary conditions g_0 and g_1. $\diamondsuit$

Regularity of solutions to Problem (Q)

To complete Theorem 3.4.1, we can give a result for different types of data.

Theorem 3.4.3. *Let $\ell \in \mathbb{Z}$ and $m \geqslant 1$ be two integers and assume that*

$$\frac{N}{p'} \notin \{1, \dots, \ell+1\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \dots, -\ell-m\}. \tag{3.4.4}$$

For any $f \in W_{m+\ell}^{m-1,p}(\mathbb{R}_+^N)$, $g_0 \in W_{m+\ell}^{m+3-1/p,p}(\Gamma)$ and $g_1 \in W_{m+\ell}^{m+1-1/p,p}(\Gamma)$ satisfying the compatibility condition (3.4.1), problem (Q) admits a solution $u \in W_{m+\ell}^{m+3,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{A}^{\Delta^2}_{[3-\ell-N/p]}$, with the estimate

$$\inf_{q \in \mathcal{A}^{\Delta^2}_{[3-\ell-N/p]}} \|u+q\|_{W_{m+\ell}^{m+3,p}(\mathbb{R}_+^N)} \leqslant C\left(\|f\|_{W_{m+\ell}^{m-1,p}(\mathbb{R}_+^N)} + \|g_0\|_{W_{m+\ell}^{m+3-1/p,p}(\Gamma)} + \|g_1\|_{W_{m+\ell}^{m+1-1/p,p}(\Gamma)} \right).$$

It can be readily checked that the kernel is unchanged under the hypothesis $\frac{N}{p} \notin \{1, \dots, -\ell-m\}$. We also keep the compatibility condition (3.4.1) and the proof of the existence of a solution is similar to that employed for Theorem 3.4.1 by means of Theorem 1.4.2 for the two Dirichlet problems.

II. Conditions on $\partial_N u$ and $\partial_N \Delta u$

The biharmonic equation with boundary conditions on $\partial_N u$ and $\partial_N \Delta u$

$$(R) \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}_+^N, \\ \partial_N u = g_0 & \text{on } \Gamma, \\ \partial_N \Delta u = g_1 & \text{on } \Gamma. \end{cases}$$

Theorem 3.4.4. *Let $\ell \in \mathbb{Z}$ and assume that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell+1\}. \tag{3.4.5}$$

For any $f \in W_{\ell}^{0,\,p}(\mathbb{R}_+^N)$, $g_0 \in W_{\ell-1}^{2-1/p,\,p}(\Gamma)$ and $g_1 \in W_{\ell-1}^{-1/p,\,p}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{N}_{[\ell-N/p']}^{\Delta^2}, \quad \langle f, \varphi \rangle_{W_{\ell}^{0,\,p}(\mathbb{R}_+^N) \times W_{-\ell}^{0,\,p'}(\mathbb{R}_+^N)} + \langle g_1, \varphi \rangle_\Gamma + \langle g_0, \Delta\varphi \rangle_\Gamma = 0, \tag{3.4.6}$$

problem (R) admits a solution $u \in W_{\ell-1}^{3,\,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{N}_{[4-\ell-N/p]}^{\Delta^2}$, with the estimate

$$\inf_{q \in \mathcal{N}_{[4-\ell-N/p]}^{\Delta^2}} \|u+q\|_{W_{\ell-1}^{3,\,p}(\mathbb{R}_+^N)} \leqslant C\left(\|f\|_{W_{\ell}^{0,\,p}(\mathbb{R}_+^N)} + \|g_0\|_{W_{\ell-1}^{2-1/p,\,p}(\Gamma)} + \|g_1\|_{W_{\ell-1}^{-1/p,\,p}(\Gamma)}\right).$$

The kernel

We must characterize the kernel of the operator

$$(\Delta^2, \gamma_1, \gamma_1\Delta) : W_{\ell-1}^{3,\,p}(\mathbb{R}_+^N) \longrightarrow W_{\ell-1}^{-1,\,p}(\mathbb{R}_+^N) \times W_{\ell-1}^{2-1/p,\,p}(\Gamma) \times W_{\ell-1}^{-1/p,\,p}(\Gamma).$$

Let u be a function of this kernel and set

$$\tilde{u}(x', x_N) = \begin{cases} u(x', x_N) & \text{si } x_N \geqslant 0, \\ u(x', -x_N) & \text{si } x_N < 0. \end{cases}$$

Here again, $\tilde{u} \in \mathcal{S}'(\mathbb{R}^N)$ and we show that $\Delta^2 \tilde{u} = 0$ in $\mathbb{R}^N$. We can deduce that $\tilde{u}$, and consequently u, is a polynomial. By identification in the half-space $x_N < 0$, we obtain that u is even with respect to x_N. Conversely it is clear that any polynomial u even with respect to x_N verifies $\partial_N u = \partial_N \Delta u = 0$ on Γ. Furthermore $u \in W_{\ell-1}^{3,\,p}(\mathbb{R}_+^N)$ implies that its degree is at the most $[4-\ell-N/p]$. So we have characterized this kernel as the space of biharmonic polynomials, even with respect to x_N, of degree smaller than or equal to $[4-\ell-N/p]$. We denote it by $\mathcal{N}_{[4-\ell-N/p]}^{\Delta^2}$.

The compatibility condition

Problem (R) admits a solution u in $W_{\ell-1}^{3,\,p}(\mathbb{R}_+^N)$ only if the compatibility condition (3.4.6) is satisfied, where $\langle g_1, \varphi \rangle_\Gamma$ and $\langle g_0, \Delta\varphi \rangle_\Gamma$ respectively denote the duality brackets $\langle g_1, \varphi \rangle_{W_{\ell-1}^{-1/p,\,p}(\Gamma) \times W_{-\ell+1}^{1-1/p',\,p'}(\Gamma)}$ and $\langle g_0, \Delta\varphi \rangle_{W_{\ell-1}^{2-1/p,\,p}(\Gamma) \times W_{-\ell+1}^{-1-1/p',\,p'}(\Gamma)}$.

The arguments are exactly the same as in the other cases.

Proof of Theorem 3.4.4

We solve this case in the same way that the precedent, but this time by two successive Neumann problems.

$$\Delta v = f \ \text{ in } \mathbb{R}^N_+, \quad \partial_N v = g_1 \ \text{ on } \Gamma,$$

which admits a solution $v \in W^{1,p}_{\ell-1}(\mathbb{R}^N_+)$, if the following compatibility condition is satisfied (see Theorem 1.4.3):

$$\forall \vartheta \in \mathcal{N}^{\Delta}_{[\ell-N/p']}, \quad \langle f, \vartheta \rangle_{W^{0,p}_{\ell}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell}(\mathbb{R}^N_+)} + \langle g_1, \vartheta \rangle_{\Gamma} = 0. \tag{3.4.7}$$

Then

$$\Delta u = v \ \text{ in } \mathbb{R}^N_+, \quad \partial_N u = g_0 \ \text{ on } \Gamma,$$

which admits a solution $u \in W^{3,p}_{\ell-1}(\mathbb{R}^N_+)$, if the following compatibility condition is satisfied (see Theorem 1.4.4):

$$\forall \psi \in \mathcal{N}^{\Delta}_{[-2+\ell-N/p']}, \quad \langle v, \psi \rangle_{W^{1,p}_{\ell-1}(\mathbb{R}^N_+)\times W^{-1,p'}_{-\ell+1}(\mathbb{R}^N_+)} + \langle g_0, \psi \rangle_{\Gamma} = 0. \tag{3.4.8}$$

Here again the compatibility condition (3.4.6) of problem (R) implies the conditions (3.4.7) and (3.4.8). On the one hand condition (3.4.6) must be satisfied for any $\vartheta \in \mathcal{N}^{\Delta}_{[\ell-N/p']}$, and that gives (3.4.7).

On the other hand if we introduce the equations $\Delta v = f$ in $\mathbb{R}^N_+$ and $\partial_N v = g_1$ on Γ, in condition (3.4.6); with the Green formula

$$\langle \Delta v, \varphi \rangle_{W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)\times W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)} = \langle v, \Delta\varphi \rangle_{W^{1,p}_{\ell-1}(\mathbb{R}^N_+)\times W^{-1,p'}_{-\ell+1}(\mathbb{R}^N_+)} - \langle \partial_N v, \varphi \rangle_{\Gamma},$$

and the remark that any $\psi \in \mathcal{N}^{\Delta}_{[-2+\ell-N/p']}$ can be written $\psi = \Delta\varphi$ with $\varphi \in \mathcal{N}^{\Delta^2}_{[\ell-N/p']}$, then we obtain (3.4.8).

Regularity of solutions to Problem (R)

To complete Theorem 3.4.4, we can give a result for different types of data.

Theorem 3.4.5. *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$. Under hypothesis (1.4.6), for any $f \in W^{m,p}_{m+\ell}(\mathbb{R}^N_+)$, $g_0 \in W^{m+3-1/p,p}_{m+\ell}(\Gamma)$ and $g_1 \in W^{m+1-1/p,p}_{m+\ell}(\Gamma)$ satisfying the compatibility condition (3.4.6), problem (R) admits a solution $u \in W^{m+4,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{N}^{\Delta^2}_{[4-\ell-N/p]}$, with the estimate*

$$\inf_{q\in\mathcal{N}^{\Delta^2}_{[4-\ell-N/p]}} \|u+q\|_{W^{m+4,p}_{m+\ell}(\mathbb{R}^N_+)} \leqslant C\left(\|f\|_{W^{m,p}_{m+\ell}(\mathbb{R}^N_+)} + \|g_0\|_{W^{m+3-1/p,p}_{m+\ell}(\Gamma)} + \|g_1\|_{W^{m+1-1/p,p}_{m+\ell}(\Gamma)}\right).$$

It can be readily checked that the kernel is unchanged under the hypothesis $\frac{N}{p} \notin \{1,\ldots,-\ell-m\}$. We also keep the compatibility condition (3.4.6) and the proof for the existence is similar to that employed in Theorem 3.4.4 by means of Theorem 1.4.4 for the two Neumann problems.

III. Conditions on Δu and $\partial_N \Delta u$

Consider now the problem

$$\Delta^2 u = f \ \text{ in } \mathbb{R}^N_+, \quad \Delta u = g_0 \ \text{ and } \ \partial_N \Delta u = g_1 \ \text{ on } \Gamma.$$

Let us first note that these boundary conditions do not satisfy the complementing condition by Agmon-Douglis-Nirenberg (see [2]). Thus this problem is ill-posed. Indeed, if we set $v = \Delta u$, we obtain

$$\Delta v = f \ \text{ in } \mathbb{R}^N_+, \quad v = g_0 \ \text{ and } \ \partial_N v = g_1 \ \text{ on } \Gamma,$$

i.e. a Laplace equation with both Dirichlet and Neumann boundary conditions.

Chapitre 4

The Stokes system with Dirichlet boundary conditions

4.1 Introduction

The purpose of this chapter is the resolution of the Stokes system with nonhomogeneous Dirichlet boundary conditions. In the sequel, we will denote it by (S_D) (for Stokes system with Dirichlet conditions):

$$(S_D)\ \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi &= \boldsymbol{f} \quad \text{in } \mathbb{R}^N_+, \\ \operatorname{div} \boldsymbol{u} &= h \quad \text{in } \mathbb{R}^N_+, \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \Gamma = \mathbb{R}^{N-1}, \end{cases}$$

with data and solutions which live in weighted Sobolev spaces, expressing at the same time their regularity and their behavior at infinity. We will naturally base on the previously established results on the harmonic and biharmonic operators. We will also concentrate on the basic weights because they are the most usual and they avoid the question of the kernel for this operator and symmetricaly the compatibility condition for the data. In the next chapter, we will complete these results for the other types of weights in this class of spaces.

Among the first works on the Stokes problem in the half-space, we can cite Cattabriga. In [24], he appeals to the potential theory to explicitly get the velocity and pressure fields. For the homogeneous problem ($\boldsymbol{f} = \boldsymbol{0}$ and $h = 0$), for instance, he shows that if $\boldsymbol{g} \in \boldsymbol{L}^p(\Gamma)$ and the semi-norm $|\boldsymbol{g}|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)} < \infty$, then $\nabla \boldsymbol{u} \in \boldsymbol{L}^p(\mathbb{R}^N_+)$ and $\pi \in L^p(\mathbb{R}^N_+)$.

Similar results are given by Farwig-Sohr (see [28]) and Galdi (see [30]), who also have chosen the setting of homogeneous Sobolev spaces. On the other hand, Maz'ya-Plamenevskiĭ-Stupyalis (see [38]), work within the suitable setting of weighted Sobolev spaces and consider different sorts of boundary conditions. However, their results are limited to the dimension 3 and to the Hilbertian framework in which they give generalized and strong solutions. This is also the case of Boulmezaoud (see [21]), who only gives strong solutions. Otherwise, always in dimension 3, by Fourier analysis techniques, Tanaka considers the case of very regular data, corresponding to velocities which belong to $\boldsymbol{W}_2^{m+3,\,2}(\mathbb{R}^3_+)$, with $m \geqslant 0$ (see [43]).

Let us also quote, for the evolution Stokes or Navier-Stokes problems, Fujigaki-Miyakawa (see [29]), who are interested in the behaviour in $t \to +\infty$; Borchers-Miyakawa (see [19]) and Kozono (see [35]), for the L^N-Decay property; Ukai (see [44]), for the L^p-L^q estimates and Giga (see [31]), for the estimates in Hardy spaces.

4.2 The Stokes system in the whole space

Here again, this study requires to extend some problems in the half-space to the whole space. On the Stokes problem in $\mathbb{R}^N$,

$$(S): \qquad -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \mathbb{R}^N,$$

let us recall the fundamental results on which we are based in the sequel. First, for any $k \in \mathbb{Z}$, we introduce the space

$$\mathcal{S}_k = \left\{ (\boldsymbol{\lambda},\, \mu) \in \boldsymbol{\mathcal{P}}_k \times \mathcal{P}^{\Delta}_{k-1};\ \operatorname{div} \boldsymbol{\lambda} = 0,\ -\Delta \boldsymbol{\lambda} + \nabla \mu = \boldsymbol{0} \right\}.$$

Concerning the generalized solutions, we have the following result:

Theorem 4.2.1 (Alliot-Amrouche [3]). *Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \ldots, \ell\} \qquad \textit{and} \qquad N/p \notin \{1, \ldots, -\ell\}.$$

For any $(\boldsymbol{f},\, g) \in \left(\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^N) \times W^{0,\,p}_{\ell}(\mathbb{R}^N)\right) \perp \mathcal{S}_{[1+\ell-N/p']}$, problem (S) admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^N) \times W^{0,\,p}_{\ell}(\mathbb{R}^N)$, unique up to an element of $\mathcal{S}_{[1-\ell-N/p]}$, with the estimate

$$\inf_{(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}_{[1-\ell-N/p]}} \left(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^N)} + \|\pi + \mu\|_{W^{0,\,p}_{\ell}(\mathbb{R}^N)} \right) \leqslant C \left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^N)} + \|g\|_{W^{0,\,p}_{\ell}(\mathbb{R}^N)} \right).$$

We also have the following result for more regular data:

Theorem 4.2.2 (Alliot-Amrouche [3]). *Let $\ell \in \mathbb{Z}$ and $m \geqslant 1$ be two integers and assume that*

$$N/p' \notin \{1, \ldots, \ell + 1\} \qquad \textit{and} \qquad N/p \notin \{1, \ldots, -\ell - m\}.$$

For any $(\boldsymbol{f},\, g) \in \big(\boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^N) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^N)\big) \perp \mathcal{S}_{[1+\ell-N/p']}$, *problem* (S) *admits a solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^N) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^N)$, *unique up to an element of* $\mathcal{S}_{[1-\ell-N/p]}$, *with the estimate*

$$\inf_{(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}_{[1-\ell-N/p]}} \Big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^N)} + \|\pi+\mu\|_{W^{m,\,p}_{m+\ell}(\mathbb{R}^N)} \Big) \leqslant C \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^N)} + \|g\|_{W^{m,\,p}_{m+\ell}(\mathbb{R}^N)} \Big).$$

Note that if we suppose $\ell = 0$, then $\mathcal{S}_{[1-N/p']} = \boldsymbol{\mathcal{P}}_{[1-N/p']} \times \{0\}$ and the orthogonality condition $(\boldsymbol{f},\, g) \perp \mathcal{S}_{[1-N/p']}$ is equivalent to $\boldsymbol{f} \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}$.

4.3 Singular boundary conditions

The way we will take to solve the Stokes system is based on the existence of very weak solutions to homogeneous problems with singular boundary conditions. The first one is the biharmonic problem: find $u \in W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$ solution to the problem

$$(P): \qquad \Delta^2 u = 0 \ \text{ in } \mathbb{R}^N_+, \quad u = g_0 \ \text{ and } \ \partial_N u = g_1 \ \text{ on } \Gamma,$$

where $g_0 \in W^{1-1/p,\,p}_{\ell-1}(\Gamma)$ and $g_1 \in W^{-1/p,\,p}_{\ell-1}(\Gamma)$ are given. This question has been solved in the previous chapter by Theorem 3.3.5 (the intermediate boundary conditions).

Remark 4.3.1. In this chapter, we will particularly interested in the case $\ell = 1$. In this case, the compatibility condition (3.3.11) of Theorem 3.3.5 concerns the polynomials in $\mathcal{B}_{[3-N/p']}$. For the arguments to come about the compatibility conditions, let us recall that if $N > p'$, then $\mathcal{B}_{[3-N/p']} = \{0\}$ and if $N \leqslant p'$, then $\mathcal{B}_{[3-N/p']} = \mathcal{B}_2 = \mathbb{R}\, x_N^2$. ♢

We will also need a result of this type about the Neumann problem for the Laplacian: find $u \in W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)$ satisfying the problem

$$(Q): \qquad \Delta u = 0 \ \text{ in } \mathbb{R}^N_+, \quad \partial_N u = g \ \text{ on } \Gamma,$$

where $g \in W^{-1-1/p,\,p}_{\ell-2}(\Gamma)$.

Theorem 4.3.2 (Amrouche [8])**.** *Let* $\ell \in \mathbb{Z}$ *and assume that*

$$N/p' \notin \{1, \ldots, \ell-2\} \quad \text{and} \quad N/p \notin \{1, \ldots, -\ell+2\}. \tag{4.3.1}$$

For any $g \in W^{-1-1/p,\,p}_{\ell-2}(\Gamma)$ *satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{N}^{\Delta}_{[\ell-N/p']}, \quad \langle g, \varphi \rangle_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)\times W^{2-1/p',\,p'}_{-\ell+2}(\Gamma)} = 0, \tag{4.3.2}$$

problem (Q) *admits a solution* $u \in W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)$*, unique up to an element of* $\mathcal{N}^{\Delta}_{[2-\ell-N/p]}$*, with the estimate*

$$\inf_{q\in\mathcal{N}^{\Delta}_{[2-\ell-N/p]}} \|u+q\|_{W^{0,\,p}_{\ell-2}(\mathbb{R}^N_+)} \;\leqslant\; C\,\|g\|_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)}.$$

With the same arguments as for Theorem 4.3.2, we can prove an intermediate result for this problem:

Theorem 4.3.3. *Let* $\ell \in \mathbb{Z}$ *and assume that*

$$N/p' \notin \{1,\ldots,\ell-1\} \quad \text{and} \quad N/p \notin \{1,\ldots,-\ell+1\}. \tag{4.3.3}$$

For any $g \in W^{-1/p,\,p}_{\ell-1}(\Gamma)$ *satisfying the compatibility condition (4.3.2), problem* (Q) *admits a solution* $u \in W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$*, unique up to an element of* $\mathcal{N}^{\Delta}_{[2-\ell-N/p]}$*, with the estimate*

$$\inf_{q\in\mathcal{N}^{\Delta}_{[2-\ell-N/p]}} \|u+q\|_{W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)} \;\leqslant\; C\,\|g\|_{W^{-1/p,\,p}_{\ell-1}(\Gamma)}.$$

Now, we will establish a similar result about the Dirichlet problem for the Laplacian with very singular boundary conditions: find $u \in W^{-1,\,p}_{\ell-2}(\mathbb{R}^N_+)$ satisfying the problem

$$(R): \qquad \Delta u = 0 \;\text{ in } \mathbb{R}^N_+, \quad u = g \;\text{ on } \Gamma,$$

where $g \in W^{-1-1/p,\,p}_{\ell-2}(\Gamma)$.

Theorem 4.3.4. *Let* $\ell \in \mathbb{Z}$*. Under hypothesis (4.3.1), for any* $g \in W^{-1-1/p,\,p}_{\ell-2}(\Gamma)$ *satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{A}^{\Delta}_{[1+\ell-N/p']}, \quad \langle g, \partial_N\varphi\rangle_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)\times W^{2-1/p',\,p'}_{-\ell+2}(\Gamma)} = 0, \tag{4.3.4}$$

problem (R) *admits a solution* $u \in W^{-1,\,p}_{\ell-2}(\mathbb{R}^N_+)$*, unique up to an element of* $\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$*, with the estimate*

$$\inf_{q\in\mathcal{A}^{\Delta}_{[1-\ell-N/p]}} \|u+q\|_{W^{-1,\,p}_{\ell-2}(\mathbb{R}^N_+)} \;\leqslant\; C\,\|g\|_{W^{-1-1/p,\,p}_{\ell-2}(\Gamma)}.$$

Firstly, we must give a meaning to traces for a special class of distributions. We introduce the spaces

$$Y_\ell(\mathbb{R}^N_+) = \left\{v \in W^{-1,\,p}_{\ell-2}(\mathbb{R}^N_+);\; \Delta v \in W^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)\right\},$$
$$Y_{\ell,\,1}(\mathbb{R}^N_+) = \left\{v \in W^{-1,\,p}_{\ell-2}(\mathbb{R}^N_+);\; \Delta v \in W^{0,\,p}_{\ell+1,\,1}(\mathbb{R}^N_+)\right\}.$$

They are reflexive Banach spaces equipped with their natural norms:

$$\|v\|_{Y_\ell(\mathbb{R}^N_+)} = \|v\|_{W^{-1,\,p}_{\ell-2}(\mathbb{R}^N_+)} + \|\Delta v\|_{W^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)},$$
$$\|v\|_{Y_{\ell,\,1}(\mathbb{R}^N_+)} = \|v\|_{W^{-1,\,p}_{\ell-2}(\mathbb{R}^N_+)} + \|\Delta v\|_{W^{0,\,p}_{\ell+1,\,1}(\mathbb{R}^N_+)}.$$

Lemma 4.3.5. *Let $\ell \in \mathbb{Z}$. Under hypothesis (4.3.1), the space $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y_\ell(\mathbb{R}^N_+)$ and in $Y_{\ell,1}(\mathbb{R}^N_+)$.*

Proof. For every continuous linear form $T \in \big(Y_\ell(\mathbb{R}^N_+)\big)'$, there exists a unique pair $(f, g) \in \overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \times W^{0,p'}_{-\ell-1}(\mathbb{R}^N_+)$, such that

$$\forall v \in Y_\ell(\mathbb{R}^N_+), \quad \langle T, v\rangle = \langle f, v\rangle_{\overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \int_{\mathbb{R}^N_+} g\,\Delta v\,\mathrm{d}x. \tag{4.3.5}$$

Thanks to the Hahn-Banach theorem, it suffices to show that any T which vanishes on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is actually zero on $Y_\ell(\mathbb{R}^N_+)$. Let us suppose that $T = 0$ on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$, thus on $\mathcal{D}(\mathbb{R}^N_+)$. Then we can deduce from (4.3.5) that

$$f + \Delta g = 0 \quad \text{in } \mathbb{R}^N_+,$$

hence we have $\Delta g \in \overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Let $\tilde{f} \in W^{1,p'}_{-\ell+2}(\mathbb{R}^N)$ and $\tilde{g} \in W^{0,p'}_{-\ell-1}(\mathbb{R}^N)$ be respectively the extensions by 0 of f and g to $\mathbb{R}^N$. Thanks to (4.3.5), it is clear that $\tilde{f} + \Delta\tilde{g} = 0$ in $\mathbb{R}^N$, and thus $\Delta\tilde{g} \in W^{1,p'}_{-\ell+2}(\mathbb{R}^N)$. Now, thanks to the isomorphism results for the Laplace operator in $\mathbb{R}^N$ (see [6]), we can deduce that $\tilde{g} \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N)$, under hypothesis (4.3.1). Since $\tilde{g}$ is an extension by 0, it follows that $g \in \overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Then, by density of $\mathcal{D}(\mathbb{R}^N_+)$ in $\overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^N_+)$ such that $\varphi_k \to g$ in $\overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Thus, for any $v \in Y_\ell(\mathbb{R}^N_+)$, we have

$$\begin{aligned}
\langle T, v\rangle &= \langle -\Delta g, v\rangle_{\overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \langle g, \Delta v\rangle_{\overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-3,p}_{\ell-2}(\mathbb{R}^N_+)} \\
&= \lim_{k\to\infty}\Big\{\langle -\Delta\varphi_k, v\rangle_{\overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \langle \varphi_k, \Delta v\rangle_{\overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-3,p}_{\ell-2}(\mathbb{R}^N_+)}\Big\} \\
&= \lim_{k\to\infty}\Big\{-\int_{\mathbb{R}^N_+}\varphi_k\,\Delta v\,\mathrm{d}x + \int_{\mathbb{R}^N_+}\varphi_k\,\Delta v\,\mathrm{d}x\Big\} \\
&= 0,
\end{aligned}$$

i.e. T is identically zero.

For the density of $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $Y_{\ell,1}(\mathbb{R}^N_+)$, the only difference in the proof concerns the logarithmic factors in the weights, with $g \in W^{0,p'}_{-\ell-1,-1}(\mathbb{R}^N_+)$. □

Thanks to this density lemma, we can prove the following result of traces:

Lemma 4.3.6. *Let $\ell \in \mathbb{Z}$. Under hypothesis (4.3.1), the trace mapping $\gamma_0 : \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \mathcal{D}(\mathbb{R}^{N-1})$, can be extended to a linear continuous mapping*

$$\gamma_0 : Y_\ell(\mathbb{R}^N_+) \longrightarrow W^{-1-1/p,p}_{\ell-2}(\Gamma) \quad \textit{if } N/p' \notin \{\ell-1,\,\ell,\,\ell+1\},$$
$$\big(\textit{resp. } \gamma_0 : Y_{\ell,1}(\mathbb{R}^N_+) \longrightarrow W^{-1-1/p,p}_{\ell-2}(\Gamma) \quad \textit{if } N/p' \in \{\ell-1,\,\ell,\,\ell+1\}\big).$$

Moreover, we have the following Green formula

$$\begin{aligned} &\forall v \in Y_\ell(\mathbb{R}^N_+),\ \forall \varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \text{ such that } \varphi = \Delta\varphi = 0 \text{ on } \Gamma,\\ &\langle \Delta v, \varphi\rangle_{W^{0,p}_{\ell+1}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell-1}(\mathbb{R}^N_+)} - \langle v, \Delta\varphi\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}\\ &\qquad = \langle v, \partial_N\varphi\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} \end{aligned} \tag{4.3.6}$$

$\big($*resp. the Green formula for* $v \in Y_{\ell,1}(\mathbb{R}^N_+)$, *where the first term of the left-hand side is replaced by* $\langle \Delta v, \varphi\rangle_{W^{0,p}_{\ell+1,1}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell-1,-1}(\mathbb{R}^N_+)}\big)$.

Proof. Firstly, let us remark that for any $\varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$, the boundary condition $\varphi = \Delta\varphi = 0$ on Γ is equivalent to $\varphi = \partial_N^2\varphi = 0$ on Γ. Moreover, if $N/p' \notin \{\ell-1,\, \ell,\, \ell+1\}$, we have the imbedding $W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell-1}(\mathbb{R}^N_+)$. So we can write the following Green formula:

$$\begin{aligned} &\forall v \in \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big),\ \forall \varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \text{ such that } \varphi = \Delta\varphi = 0 \text{ on } \Gamma,\\ &\int_{\mathbb{R}^N_+} \varphi\,\Delta v\,\mathrm{d}x - \int_{\mathbb{R}^N_+} v\,\Delta\varphi\,\mathrm{d}x = \int_\Gamma v\,\partial_N\varphi\,\mathrm{d}x'. \end{aligned} \tag{4.3.7}$$

Since $\Delta\varphi = 0$ on Γ, we have the identity

$$\int_{\mathbb{R}^N_+} v\,\Delta\varphi\,\mathrm{d}x = \langle v, \Delta\varphi\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}\,.$$

This implies

$$\Big|\langle v, \partial_N\varphi\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}\Big| \leqslant \|v\|_{Y_\ell(\mathbb{R}^N_+)}\ \|\varphi\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)}.$$

By Lemma 1.3.1, for any $\mu \in W^{2-1/p',p'}_{-\ell+2}(\Gamma)$, there exists a lifting function $\varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$ such that $\varphi = 0,\ \partial_N\varphi = \mu$ and $\partial_N^2\varphi = 0$ on Γ, satisfying

$$\|\varphi\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)} \leqslant C\ \|\mu\|_{W^{2-1/p',p'}_{-\ell+2}(\Gamma)},$$

where C is a constant not depending on φ and μ. Then we can deduce that

$$\|\gamma_0 v\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)} \ \leqslant\ C\ \|v\|_{Y_\ell(\mathbb{R}^N_+)}.$$

Thus the linear mapping $\gamma_0 : v \longmapsto v|_\Gamma$ defined on $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is continuous for the norm of $Y_\ell(\mathbb{R}^N_+)$. Since $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y_\ell(\mathbb{R}^N_+)$, γ_0 can be extended by continuity to a mapping still called $\gamma_0 \in \mathcal{L}\big(Y_\ell(\mathbb{R}^N_+);\ W^{-1-1/p,p}_{\ell-2}(\Gamma)\big)$. Moreover, we also can deduce the formula (4.3.6) from (4.3.7) by density of $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $Y_\ell(\mathbb{R}^N_+)$. To finish, note that if $N/p' \in \{\ell-1,\, \ell,\, \ell+1\}$, we only have the imbedding $W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell-1,-1}(\mathbb{R}^N_+)$, hence the necessity to introduce the space $Y_{\ell,1}(\mathbb{R}^N_+)$ and the corresponding Green formula with logarithmic factors for these three critical values. $\square$

Proof of Theorem 4.3.4. We can observe that solve problem (R) is equivalent to find $u \in Y_\ell(\mathbb{R}^N_+)$ if $N/p' \notin \{\ell-1,\, \ell,\, \ell+1\}$ $\big($resp. $u \in Y_{\ell,1}(\mathbb{R}^N_+)$ if $N/p' \in \{\ell-1,\, \ell,\, \ell+1\}\big)$, satisfying

$$\begin{aligned}&\forall v \in\ W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \text{ such that } v = \Delta v = 0 \text{ on } \Gamma,\\ &\langle u, \Delta v\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)} \ =\ -\langle g, \partial_N v\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}\,.\end{aligned} \tag{4.3.8}$$

Indeed the direct implication is straightforward. Conversely, if u satisfies (4.3.8) then we have for any $\varphi \in \mathcal{D}(\mathbb{R}^N_+)$,

$$\langle \Delta u, \varphi\rangle_{W^{-3,p}_{\ell-2}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)} = \langle u, \Delta\varphi\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)} = 0,$$

thus $\Delta u = 0$ in $\mathbb{R}^N_+$. Moreover, by the Green formula (4.3.6), we have

$$\begin{aligned}&\forall v \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \text{ such that } v = \Delta v = 0 \text{ on } \Gamma,\\ &\langle g, \partial_N v\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = \langle u, \partial_N v\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}\,.\end{aligned}$$

By Lemma 1.3.1, for any $\mu \in W^{2-1/p',p'}_{-\ell+2}(\Gamma)$, there exists $v \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$ such that $v = 0,\ \partial_N v = \mu,\ \partial_N^2 v = 0$ on Γ. Consequently,

$$\langle u - g, \mu\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = 0,$$

i.e. $u - g = 0$ on Γ. Thus u satisfies (R).

Furthermore, for any $f \in \overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{A}^{\Delta}_{[1-\ell-N/p]}$, according to Theorem 1.4.2, we know that there exists a unique $v \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{A}^{\Delta}_{[1+\ell-N/p']}$ such that

$$\Delta v = f \ \text{ in } \mathbb{R}^N_+, \quad v = 0 \ \text{ on } \Gamma,$$

with the estimate

$$\|v\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{A}^{\Delta}_{[1+\ell-N/p']}} \ \leqslant\ C\ \|f\|_{W^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)},$$

where C denotes a generic constant not depending on v and f. Now, let us consider the linear form $T:\ f \longmapsto -\langle g, \partial_N v\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}$ defined on $\overset{\circ}{W}{}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{A}^{\Delta}_{[1-\ell-N/p]}$. Thanks to (4.3.4), we have for any $q \in \mathcal{A}^{\Delta}_{[1+\ell-N/p']}$,

$$\begin{aligned}|Tf| =&\ \left|\langle g, \partial_N(v+q)\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}\right|\\ \leqslant&\ C\ \|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}\ \|v+q\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)}\\ \leqslant&\ C\ \|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}\ \|v\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{A}^{\Delta}_{[3-N/p']}}\\ \leqslant&\ C\ \|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}\ \|f\|_{W^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}.\end{aligned}$$

Thus we have shown that T is continuous on $\mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{A}^{\Delta}_{[1-\ell-N/p]}$ and then, according to Riesz representation theorem, there exists a unique $u \in W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)/\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$ such that $Tf = \langle u, f\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}$. So we have (4.3.8) and u is the unique solution to problem (R). □

Similarly to the Neumann problem, we can give an intermediate result:

Theorem 4.3.7. *Let $\ell \in \mathbb{Z}$. Under hypothesis (4.3.3), for any $g \in W^{-1/p,p}_{\ell-1}(\Gamma)$ satisfying the compatibility condition (4.3.4), problem (R) admits a solution $u \in W^{0,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$, with the estimate*

$$\inf_{q\in\mathcal{A}^{\Delta}_{[1-\ell-N/p]}} \|u+q\|_{W^{0,p}_{\ell-1}(\mathbb{R}^N_+)} \leqslant C\,\|g\|_{W^{-1/p,p}_{\ell-1}(\Gamma)}.$$

4.4 Generalized solutions to the Stokes system

We will establish a first result about the generalized solutions to (S_D) in the homogeneous case. The following proposition is quite natural and we can find similar results in the literature although not expressed in weighted Sobolev spaces (see e.g. Farwig-Sohr [28], Galdi [30], Cattabriga [24]). Moreover, we take up some ideas in [28] and we considerably simplify the proof.

Proposition 4.4.1. *For any $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}_0(\Gamma)$, the Stokes problem*

$$-\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{0} \qquad \text{in } \mathbb{R}^N_+, \tag{4.4.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \mathbb{R}^N_+, \tag{4.4.2}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma, \tag{4.4.3}$$

has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^N_+)\times L^p(\mathbb{R}^N_+)$, with the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}_0(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \leqslant C\,\|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}_0(\Gamma)}. \tag{4.4.4}$$

Proof. (i) Firstly, we will show that system (4.4.1)–(4.4.3) can be reduced to three problems on the fundamental operators Δ^2 and Δ.

Applying the operator div to the first equation (4.4.1), we obtain

$$\Delta\pi = 0 \quad \text{in } \mathbb{R}^N_+. \tag{4.4.5}$$

Now, applying the operator Δ to the same equation (4.4.1), we deduce

$$\Delta^2\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+. \tag{4.4.6}$$

From the boundary condition (4.4.3), we take out

$$u_N = g_N \quad \text{on } \Gamma, \tag{4.4.7}$$

and moreover $\operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ, where $\operatorname{div}' \boldsymbol{u}' = \sum_{i=1}^{N-1} \partial_i u_i$.

Since $\operatorname{div} \boldsymbol{u} = 0$ in $\mathbb{R}^N_+$, we also have $\operatorname{div} \boldsymbol{u} = 0$ on Γ; then we can write $\partial_N u_N + \operatorname{div}' \boldsymbol{u}' = 0$ on Γ, hence

$$\partial_N u_N = -\operatorname{div}' \boldsymbol{g}' \quad \text{on } \Gamma. \tag{4.4.8}$$

Combining (4.4.6), (4.4.7) and (4.4.8), we get the following biharmonic problem

$$(P): \qquad \Delta^2 u_N = 0 \ \text{ in } \mathbb{R}^N_+, \quad u_N = g_N \ \text{ and } \ \partial_N u_N = -\operatorname{div}' \boldsymbol{g}' \ \text{ on } \Gamma.$$

Then, combining (4.4.5) with the trace on Γ of the N^{th} component in the equations (4.4.1), we obtain the following Neumann problem

$$(Q): \qquad \Delta \pi = 0 \ \text{ in } \mathbb{R}^N_+, \quad \partial_N \pi = \Delta u_N \ \text{ on } \Gamma.$$

Lastly, if we consider the $N-1$ first components of the equations (4.4.1) and (4.4.3), we can write the following Dirichlet problem

$$(R): \qquad \Delta \boldsymbol{u}' = \nabla' \pi \ \text{ in } \mathbb{R}^N_+, \quad \boldsymbol{u}' = \boldsymbol{g}' \ \text{ on } \Gamma.$$

(ii) Now, we will solve these three problems.

`Step 1:` Problem (P). Since $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/p,p}(\Gamma)$, we have $g_N \in W_0^{1-1/p,p}(\Gamma)$ and $\operatorname{div}' \boldsymbol{g}' \in W_0^{-1/p,p}(\Gamma)$. So (P) is an homogeneous biharmonic problem with singular boundary conditions, and we can apply Theorem 3.3.5 provided the compatibility condition (3.3.11) is fulfilled. If $1 - N/p' < 0$, then $\mathcal{B}_{[3-N/p']} = \{0\}$ and the condition vanishes. If $1 - N/p' \geqslant 0$, then $\mathcal{B}_{[3-N/p']} = \mathbb{R}\, x_N^2$ and this condition is equivalent to

$$\langle \operatorname{div}' \boldsymbol{g}', 1 \rangle_{W_0^{-1/p,p}(\Gamma) \times W_0^{1/p,p'}(\Gamma)} = 0. \tag{4.4.9}$$

Since $\mathcal{D}(\mathbb{R}^{N-1})$ is dense in $W_0^{1/p,p'}(\Gamma)$, we know that there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^{N-1})$ such that $\varphi_k \to 1$ in $W_0^{1/p,p'}(\Gamma)$, hence we can deduce

$$\langle \operatorname{div}' \boldsymbol{g}', 1 \rangle_{W_0^{-1/p,p}(\Gamma) \times W_0^{1/p,p'}(\Gamma)} \;=\; -\lim_{k\to\infty} \int_{\mathbb{R}^{N-1}} \boldsymbol{g}' \,.\nabla \varphi_k \,\mathrm{d}x' \;=\; 0.$$

Thus the orthogonality condition is fulfilled and problem (P) has a unique solution $u_N \in W_0^{1,p}(\mathbb{R}^N_+)$, satisfying

$$\begin{aligned} \|u_N\|_{W_0^{1,p}(\mathbb{R}^N_+)} &\leqslant C \left(\|g_N\|_{W_0^{1-1/p,p}(\Gamma)} + \|\operatorname{div}' \boldsymbol{g}'\|_{W_0^{-1/p,p}(\Gamma)} \right) \\ &\leqslant C \, \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}. \end{aligned} \tag{4.4.10}$$

`Step 2:` Problem (Q). Since $\Delta^2 u_N = 0$ in $\mathbb{R}^N_+$, we have $\Delta u_N \in Y_2(\mathbb{R}^N_+)$ and also $\Delta u_N \in Y_{2,1}(\mathbb{R}^N_+)$, hence $\Delta u_N|_\Gamma \in W_0^{-1-1/p,p}(\Gamma)$ by Lemma 4.3.6. Then we can apply Theorem 4.3.2, provided the compatibility condition (4.3.2) is fulfilled, *i.e.*

$$\forall \varphi \in \mathcal{N}^\Delta_{[2-N/p']}, \quad \langle \Delta u_N, \varphi \rangle_{W_0^{-1-1/p,p}(\Gamma) \times W_0^{2-1/p',p'}(\Gamma)} = 0.$$

Knowing that $\mathcal{N}^{\Delta}_{[2-N/p']} \subset \mathcal{P}_1$, an argument similar to that of the condition (4.4.9) in `Step 1` gives us this relation. We can conclude that problem (Q) has a unique solution $\pi \in L^p(\mathbb{R}^N_+)$, satisfying

$$\begin{aligned} \|\pi\|_{L^p(\mathbb{R}^N_+)} &\leqslant C\ \|\Delta u_N\|_{W_0^{-1-1/p,\,p}(\Gamma)} \\ &\leqslant C\ \|\Delta u_N\|_{Y_2(\mathbb{R}^N_+)} \ = \ C\ \|\Delta u_N\|_{W_0^{-1,\,p}(\mathbb{R}^N_+)} \\ &\leqslant C\ \|u_N\|_{W_0^{1,\,p}(\mathbb{R}^N_+)} \ \leqslant \ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)}. \end{aligned} \tag{4.4.11}$$

`Step 3`: Problem (R). By `Step 2`, we have $\nabla'\pi \in W_0^{-1,\,p}(\mathbb{R}^N_+)^{N-1}$ and moreover $\boldsymbol{g}' \in W_0^{1-1/p,\,p}(\Gamma)^{N-1}$. Since $\mathcal{A}^{\Delta}_{[1-N/p']} = \{0\}$, according to Theorem 1.4.1, we know that problem (R) has a unique solution $\boldsymbol{u}' \in W_0^{1,\,p}(\mathbb{R}^N_+)^{N-1}$, satisfying

$$\begin{aligned} \|\boldsymbol{u}'\|_{W_0^{1,\,p}(\mathbb{R}^N_+)^{N-1}} &\leqslant C\ \left(\|\nabla'\pi\|_{W_0^{-1,\,p}(\mathbb{R}^N_+)^{N-1}} + \|\boldsymbol{g}'\|_{W_0^{1-1/p,\,p}(\Gamma)^{N-1}}\right) \\ &\leqslant C\ \left(\|\pi\|_{L^p(\mathbb{R}^N_+)} + \|\boldsymbol{g}'\|_{W_0^{1-1/p,\,p}(\Gamma)^{N-1}}\right) \\ &\leqslant C\ \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)}. \end{aligned} \tag{4.4.12}$$

(iii) In order, we have found u_N, π and $\boldsymbol{u}'$, which satisfy (4.4.3) and partially satisfy (4.4.1), *i.e.*

$$-\Delta\boldsymbol{u}' + \nabla'\pi = \mathbf{0} \quad \text{in } \mathbb{R}^N_+.$$

It remains to show they satisfy (4.4.2) and the N^{th} component of (4.4.1), *i.e.*

$$-\Delta u_N + \partial_N\pi = 0 \quad \text{in } \mathbb{R}^N_+.$$

Consider such a pair $(\boldsymbol{u},\,\pi)$ satisfying problems (P), (Q) and (R). From the first equations of (P) and (Q), we obtain

$$\Delta(\Delta u_N - \partial_N\pi) = \Delta^2 u_N = 0 \quad \text{in } \mathbb{R}^N_+.$$

Thanks to the boundary condition of (Q), we can deduce that the distribution $\Delta u_N - \partial_N\pi \in W_0^{-1,p}(\mathbb{R}^N_+)$ satisfies the following Dirichlet problem

$$\Delta(\Delta u_N - \partial_N\pi) = 0 \ \text{ in } \mathbb{R}^N_+, \quad \Delta u_N - \partial_N\pi = 0 \ \text{ on } \Gamma.$$

Then, according to Theorem 4.3.4, we necessarily have $\Delta u_N - \partial_N\pi = 0$ in $\mathbb{R}^N_+$. Thus $(u,\,\pi)$ completely satisfies (4.4.1).

Now, applying the operator div to (4.4.1), we get $-\Delta\,\mathrm{div}\,\boldsymbol{u} + \Delta\pi = 0$ in $\mathbb{R}^N_+$, and by the main equation of (Q), *i.e.* (4.4.5), we obtain $\Delta\,\mathrm{div}\,\boldsymbol{u} = 0$ in $\mathbb{R}^N_+$. Moreover, from the boundary condition in (R), we get $\mathrm{div}'\,\boldsymbol{u}' = \mathrm{div}'\,\boldsymbol{g}'$ on Γ. Then, with the boundary condition in (P), we can write

$$\mathrm{div}\,\boldsymbol{u} = \mathrm{div}'\,\boldsymbol{u}' + \partial_N u_N = \mathrm{div}'\,\boldsymbol{g}' - \mathrm{div}'\,\boldsymbol{g}' = 0 \quad \text{on } \Gamma.$$

So, we have

$$\Delta \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^N_+, \qquad \operatorname{div} \boldsymbol{u} = 0 \quad \text{on } \Gamma,$$

with $\operatorname{div} \boldsymbol{u} \in L^p(\mathbb{R}^N_+)$ and then by Theorem 4.3.7, we can deduce that $\operatorname{div} \boldsymbol{u} = 0$ in $\mathbb{R}^N_+$, *i.e.* (4.4.2) is satisfied.

(iv) Finally, let us remark that the uniqueness of $(\boldsymbol{u}, \pi)$ is a consequence of the uniqueness of the solutions to problems (P), (Q) and (R). Moreover, the estimate (4.4.4) is a consequence of the estimates (4.4.10), (4.4.11) and (4.4.12). □

Now, we can solve the complete problem (S_D). For this, we will show that it can be reduced to an homogeneous problem, solved by Proposition 4.4.1.

Theorem 4.4.2. *For any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)$, $h \in L^p(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/p,p}(\Gamma)$, problem (S_D) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, and there exists a constant C such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)} + \|h\|_{L^p(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}\right). \tag{4.4.13}$$

Proof. Firstly, let us write $\boldsymbol{f} = \operatorname{div} \mathbb{F}$, where $\mathbb{F} = (\mathbf{F}_i)_{1 \leqslant i \leqslant N} \in \boldsymbol{L}^p(\mathbb{R}^N_+)^N$, with the estimate

$$\|\mathbb{F}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)^N} \leqslant C \, \|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)};$$

and let us respectively denote by $\tilde{\mathbb{F}} = (\tilde{\mathbf{F}}_i)_{1 \leqslant i \leqslant N} \in \boldsymbol{L}^p(\mathbb{R}^N)^N$ and $\tilde{h} \in L^p(\mathbb{R}^N)$ the extensions by 0 of $\mathbb{F}$ and h to $\mathbb{R}^N$. By Theorem 4.2.1, we know that there exists $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ solution to the problem

$$(\tilde{S}): \qquad -\Delta \tilde{\boldsymbol{u}} + \nabla \tilde{\pi} = \operatorname{div} \tilde{\mathbb{F}} \quad \text{and} \quad \operatorname{div} \tilde{\boldsymbol{u}} = \tilde{h} \quad \text{in } \mathbb{R}^N,$$

provided the condition $\operatorname{div} \tilde{\mathbb{F}} \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}$ is fulfilled. If $1 - N/p' < 0$, we obviously have $\boldsymbol{\mathcal{P}}_{[1-N/p']} = \{\mathbf{0}\}$, thus the condition vanishes. If $1 - N/p' \geqslant 0$, then we have $\boldsymbol{\mathcal{P}}_{[1-N/p']} = \mathbb{R}^N$ and this condition is equivalent to

$$\forall i = 1, \ldots, N, \quad \left\langle \operatorname{div} \tilde{\mathbf{F}}_i, 1 \right\rangle_{W_0^{-1,p}(\mathbb{R}^N) \times W_0^{1,p'}(\mathbb{R}^N)} = 0.$$

This is exactly the same argument as for the condition (4.4.9) in the previous proof. Thus the orthogonality condition is fulfilled, hence the existence of $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ solution to problem $(\tilde{S})$, satisfying

$$\begin{aligned} \|\tilde{\boldsymbol{u}}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^N)} + \|\tilde{\pi}\|_{L^p(\mathbb{R}^N)} &\leqslant C\left(\|\operatorname{div} \tilde{\mathbb{F}}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N)} + \|\tilde{h}\|_{L^p(\mathbb{R}^N)}\right) \\ &\leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)} + \|h\|_{L^p(\mathbb{R}^N_+)}\right). \end{aligned} \tag{4.4.14}$$

Consequently, we can reduce the system (S_D) to the homogeneous problem

$$(S^\sharp): \qquad -\Delta \boldsymbol{v} + \nabla \vartheta = \boldsymbol{0} \ \text{ and } \ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \mathbb{R}^N_+, \quad \boldsymbol{v} = \boldsymbol{g}^\sharp \ \text{ on } \Gamma,$$

where we have set $\boldsymbol{g}^\sharp = \boldsymbol{g} - \tilde{\boldsymbol{u}}|_\Gamma \in \boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)$. Now, thanks to Proposition 4.4.1, we know that $(S^\sharp)$ admits a unique solution $(\boldsymbol{v},\, \vartheta) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, satisfying

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+)} + \|\vartheta\|_{L^p(\mathbb{R}^N_+)} &\leqslant C\, \|\boldsymbol{g}^\sharp\|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)} \\ &\leqslant C \left(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^N_+)} + \|h\|_{L^p(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,\,p}(\Gamma)} \right). \end{aligned} \tag{4.4.15}$$

Then, $(\boldsymbol{u},\, \pi) = (\boldsymbol{v} + \tilde{\boldsymbol{u}}|_{\mathbb{R}^N_+},\, \vartheta + \tilde{\pi}|_{\mathbb{R}^N_+}) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ is solution to (S_D) and the estimate (4.4.13) is a consequence of the estimates (4.4.14) and (4.4.15). Finally, the uniqueness of the solution to (S_D) is a straightforward consequence of Proposition 4.4.1. □

Remark 4.4.3. In a forthcoming work, we will show that under hypotheses of Theorem 4.4.2 and if besides $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,\,q}(\mathbb{R}^N_+)$, $h \in L^q(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/q,\,q}(\Gamma)$, for any real number $q > 1$, then the solution $(\boldsymbol{u},\, \pi)$ given by Theorem 4.4.2 verifies moreover $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_0^{1,\,q}(\mathbb{R}^N_+) \times L^q(\mathbb{R}^N_+)$. ♢

4.5 Strong solutions & regularity for the Stokes system

In this section, we are interested in the existence of strong solutions (and then to regular solutions, see Corollaries 4.5.5 and 4.5.7), *i.e.* of solutions $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{2,\,p}_{\ell+1}(\mathbb{R}^N_+) \times W^{1,\,p}_{\ell+1}(\mathbb{R}^N_+)$. Here, we limit ourselves to the two cases $\ell = 0$ or $\ell = -1$. Note that in the case $\ell = 0$, we have $W_1^{2,\,p}(\mathbb{R}^N_+) \hookrightarrow W_0^{1,\,p}(\mathbb{R}^N_+)$ and $W_1^{1,\,p}(\mathbb{R}^N_+) \hookrightarrow L^p(\mathbb{R}^N_+)$. The proposition and theorem which follow show that the generalized solution of Theorem 4.4.2, with a stronger hypothesis on the data, is in fact a strong solution.

Proposition 4.5.1. *Assume that $\frac{N}{p'} \neq 1$. For any $\boldsymbol{g} \in \boldsymbol{W}_1^{2-1/p,\,p}(\Gamma)$, the Stokes problem (4.4.1)–(4.4.3) has a unique solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+) \times W_1^{1,\,p}(\mathbb{R}^N_+)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+)} + \|\pi\|_{W_1^{1,\,p}(\mathbb{R}^N_+)} \ \leqslant\ C\, \|\boldsymbol{g}\|_{\boldsymbol{W}_1^{2-1/p,\,p}(\Gamma)}.$$

Proof. The arguments for the estimate are unchanged with respect to the proof of Proposition 4.4.1. For the surjectivity and the uniqueness, note that we always have the imbedding $W_1^{2-1/p,\,p}(\Gamma) \hookrightarrow W_0^{1-1/p,\,p}(\Gamma)$. By Proposition 4.4.1, we can deduce that problem (4.4.1)–(4.4.3) admits a unique solution $(\boldsymbol{u},\, \pi) \in$

$\boldsymbol{W}_0^{1,p}(\mathbb{R}_+^N) \times L^p(\mathbb{R}_+^N)$, satisfying the estimate (4.4.4). Then, it suffices to go back to the proof of Proposition 4.4.1 and to use the established results about problems (P), (Q) and (R), to show that in fact $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$. In order, for problem (P), according to Lemma 2.3.9, we find $u_N \in W_1^{2,p}(\mathbb{R}_+^N)$; for problem (Q), thanks to Theorem 4.3.3, we find $\pi \in W_1^{1,p}(\mathbb{R}_+^N)$; for problem (R), according to Theorem 1.4.2, we find $\boldsymbol{u}' \in W_1^{2,p}(\mathbb{R}_+^N)^{N-1}$. Note that for these three results, the condition $N/p' \neq 1$ is always necessary. □

Now, we can study the strong solutions for the complete problem (S_D). As for the generalized solutions, we will show that it is equivalent to an homogeneous problem, solved by Proposition 4.5.1. The following theorem was established in the case $N = 3$, $p = 2$, by Maz'ya-Plamenevskiĭ-Stupyalis (see [38]).

Theorem 4.5.2. *Assume that* $\frac{N}{p'} \neq 1$. *For any* $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)$, $h \in W_1^{1,p}(\mathbb{R}_+^N)$ *and* $\boldsymbol{g} \in \boldsymbol{W}_1^{2-1/p,p}(\Gamma)$, *problem* (S_D) *admits a unique solution* $(\boldsymbol{u}, \pi)$ *which belongs to* $\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_1^{1,p}(\mathbb{R}_+^N)} \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)} + \|h\|_{W_1^{1,p}(\mathbb{R}_+^N)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_1^{2-1/p,p}(\Gamma)}\right).$$

Proof. Here again, the arguments for the estimate are unchanged with respect to the proof of Theorem 4.4.2. For the surjectivity and the uniqueness, note that the imbedding $W_1^{0,p}(\mathbb{R}_+^N) \hookrightarrow W_0^{-1,p}(\mathbb{R}_+^N)$ holds if $N/p' \neq 1$. Moreover, we have $W_1^{1,p}(\mathbb{R}_+^N) \hookrightarrow L^p(\mathbb{R}_+^N)$ and $W_1^{2-1/p,p}(\Gamma) \hookrightarrow W_0^{1-1/p,p}(\Gamma)$. Thus, thanks to Theorem 4.4.2, we know that problem (S_D) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}_+^N) \times L^p(\mathbb{R}_+^N)$, satisfying the estimate (4.4.13). To show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$, we want to find an extension $\tilde{\boldsymbol{f}}$ of $\boldsymbol{f}$ to $\mathbb{R}^N$, such that the orthogonality condition for the extended problem to the whole space $(\tilde{S})$ holds. To this end, we still can write $\boldsymbol{f} = \operatorname{div} \mathbb{F}$. Indeed, if $N/p' \neq 1$, for any $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)$, the Dirichlet problem

$$\Delta \boldsymbol{w} = \boldsymbol{f} \ \text{ in } \mathbb{R}_+^N, \quad \boldsymbol{w} = \boldsymbol{0} \ \text{ on } \Gamma,$$

admits a unique solution $\boldsymbol{w} \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N)$, according to Theorem 1.4.2. So, if we consider $\mathbb{F} = \nabla \boldsymbol{w} \in \boldsymbol{W}_1^{1,p}(\mathbb{R}_+^N)^N$, we have $\boldsymbol{f} = \operatorname{div} \mathbb{F}$. Now, it suffices to go back to the proof of Theorem 4.4.2. Here again, we know that there exists a continuous linear extension operator from $W_1^{1,p}(\mathbb{R}_+^N)$ to $W_1^{1,p}(\mathbb{R}^N)$, so we get $\tilde{\boldsymbol{f}} = \operatorname{div} \tilde{\mathbb{F}} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^N)$ and $\tilde{h} \in W_1^{1,p}(\mathbb{R}^N)$, hence the extended problem $(\tilde{S})$, which has, by Theorem 4.2.2, a solution $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^N) \times W_1^{1,p}(\mathbb{R}^N)$. Then, we obtain the equivalent problem $(S^\sharp)$ with $\boldsymbol{g}^\sharp \in \boldsymbol{W}_1^{2-1/p,p}(\Gamma)$ and this problem is solved by Proposition 4.5.1. □

Remark 4.5.3. To give a variant to this proof, we also can consider the extension $\tilde{\boldsymbol{f}} \in \boldsymbol{W}_1^{0,\,p}(\mathbb{R}^N)$ of $\boldsymbol{f}$ to $\mathbb{R}^N$ defined by:

$$\tilde{\boldsymbol{f}}(x',\, x_N) = \begin{cases} \boldsymbol{f}(x',\, x_N) & \text{if } x_N > 0, \\ -\boldsymbol{f}(x',\, -x_N) & \text{if } x_N < 0, \end{cases}$$

and $\tilde{h} \in W_1^{1,\,p}(\mathbb{R}^N)$ an extension of h to $\mathbb{R}^N$. Then by Theorem 4.2.2, there exists $(\tilde{\boldsymbol{u}},\, \tilde{\pi})$ solution to the problem

$$(\tilde{S}): \qquad -\Delta\tilde{\boldsymbol{u}} + \nabla\tilde{\pi} = \tilde{\boldsymbol{f}} \quad \text{and} \quad \operatorname{div}\tilde{\boldsymbol{u}} = \tilde{h} \quad \text{in } \mathbb{R}^N,$$

provided the orthogonality condition $\tilde{\boldsymbol{f}} \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}$ is fulfilled. Here again, if $1 - N/p' < 0$ this condition vanishes and if $1 - N/p' > 0$, we have

$$\forall i = 1, \ldots, N, \qquad \int_{\mathbb{R}^N} \tilde{\boldsymbol{f}}_i(x',\, x_N)\, \mathrm{d}x \;=\; 0.$$

Thus the orthogonality condition holds. The rest of the proof is identical. ♢

Remark 4.5.4. Similarly to Remark 4.4.3, we could show that under hypotheses of Theorem 4.5.2 and if moreover $\boldsymbol{f} \in \boldsymbol{W}_1^{0,\,q}(\mathbb{R}_+^N)$, $h \in W_1^{1,\,q}(\mathbb{R}_+^N)$ and $\boldsymbol{g} \in \boldsymbol{W}_1^{2-1/q,\,q}(\Gamma)$, with an arbitrary real number $q > 1$, then the solution $(\boldsymbol{u},\, \pi)$ given by Theorem 4.4.2 verify, besides, $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_1^{2,\,q}(\mathbb{R}_+^N) \times W_1^{1,\,q}(\mathbb{R}_+^N)$. ♢

We will now establish a global regularity result of solutions to the Stokes system (S_D), which includes the case of strong solutions and which rests on Theorem 4.4.2 and a regularity argument.

Corollary 4.5.5. *Let $m \in \mathbb{N}$ and assume that $\frac{N}{p'} \neq 1$ if $m \geqslant 1$. For any $\boldsymbol{f} \in \boldsymbol{W}_m^{m-1,\,p}(\mathbb{R}_+^N)$, $h \in W_m^{m,\,p}(\mathbb{R}_+^N)$ and $\boldsymbol{g} \in \boldsymbol{W}_m^{m+1-1/p,\,p}(\Gamma)$, problem (S_D) admits a unique solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_m^{m+1,\,p}(\mathbb{R}_+^N) \times W_m^{m,\,p}(\mathbb{R}_+^N)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_m^{m+1,\,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_m^{m,\,p}(\mathbb{R}_+^N)} \;\leqslant\; C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_m^{m-1,\,p}(\mathbb{R}_+^N)} + \|h\|_{W_m^{m,\,p}(\mathbb{R}_+^N)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_m^{m+1-1/p,\,p}(\Gamma)}\right).$$

Proof. Since we have $W_m^{m-1,\,p}(\mathbb{R}_+^N) \hookrightarrow W_0^{-1,\,p}(\mathbb{R}_+^N)$, $W_m^{m,\,p}(\mathbb{R}_+^N) \hookrightarrow L^p(\mathbb{R}_+^N)$ and $W_m^{m+1-1/p,\,p}(\Gamma) \hookrightarrow W_0^{1-1/p,\,p}(\Gamma)$, thanks to Theorem 4.4.2, we know that problem (S_D) admits a unique solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}_+^N) \times L^p(\mathbb{R}_+^N)$. We will show by induction that

$$\begin{aligned} (\boldsymbol{f},\, h,\, \boldsymbol{g}) \in \boldsymbol{W}_m^{m-1,\,p}(\mathbb{R}_+^N) \times W_m^{m,\,p}(\mathbb{R}_+^N) \times \boldsymbol{W}_m^{m+1-1/p,\,p}(\Gamma) \\ \Rightarrow (\boldsymbol{u},\, \pi) \in \boldsymbol{W}_m^{m+1,\,p}(\mathbb{R}_+^N) \times W_m^{m,\,p}(\mathbb{R}_+^N). \end{aligned} \tag{4.5.1}$$

For $m = 0$, (4.5.1) is true. Assume that (4.5.1) is true for $0, 1, \ldots, m$ and suppose that $(\boldsymbol{f},\, h,\, \boldsymbol{g}) \in \boldsymbol{W}_{m+1}^{m,\,p}(\mathbb{R}_+^N) \times W_{m+1}^{m+1,\,p}(\mathbb{R}_+^N) \times \boldsymbol{W}_{m+1}^{m+2-1/p,\,p}(\Gamma)$. Let us prove that $(\boldsymbol{u},\,\pi)\in \boldsymbol{W}^{m+2,\,p}_{m+1}(\mathbb{R}^N_+)\times W^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)$. Since $W^{m,\,p}_{m+1}(\mathbb{R}^N_+)\hookrightarrow W^{m-1,\,p}_{m}(\mathbb{R}^N_+)$, $W^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)\hookrightarrow W^{m,\,p}_{m}(\mathbb{R}^N_+)$ and $W^{m+2-1/p,\,p}_{m+1}(\Gamma)\hookrightarrow W^{m+1-1/p,\,p}_{m}(\Gamma)$, we know that $(\boldsymbol{u},\,\pi)\in \boldsymbol{W}^{m+1,\,p}_{m}(\mathbb{R}^N_+)\times W^{m,\,p}_{m}(\mathbb{R}^N_+)$ thanks to the induction hypothesis. Now, for any $i\in\{1,\dots,N-1\}$, we have

$$\begin{aligned}-\Delta(\varrho\,\partial_i\boldsymbol{u})+\nabla(\varrho\,\partial_i\pi)&\\ =\ \varrho\,\partial_i\boldsymbol{f}+\frac{2}{\varrho}\,x.\nabla\partial_i\boldsymbol{u}+\left(\frac{N-1}{\varrho}+\frac{1}{\varrho^3}\right)\partial_i\boldsymbol{u}+\frac{1}{\varrho}\,x\,\partial_i\pi.&\end{aligned}$$

Thus, $-\Delta(\varrho\,\partial_i\boldsymbol{u})+\nabla(\varrho\,\partial_i\pi)\in \boldsymbol{W}^{m-1,\,p}_{m}(\mathbb{R}^N_+)$. Moreover,

$$\operatorname{div}(\varrho\,\partial_i\boldsymbol{u})=\frac{1}{\varrho}\,x\,\partial_i\boldsymbol{u}+\varrho\,\partial_i h.$$

Thus, $\operatorname{div}(\varrho\,\partial_i\boldsymbol{u})\in W^{m,\,p}_{m}(\mathbb{R}^N_+)$. We also have $\gamma_0(\varrho\,\partial_i\boldsymbol{u})=\varrho'\,\partial_i\gamma_0\boldsymbol{u}=\varrho'\,\partial_i\boldsymbol{g}\in \boldsymbol{W}^{m+1-1/p,\,p}_{m}(\Gamma)$. So, by induction hypothesis, we can deduce that

$$\forall i\in\{1,\dots,N-1\},\quad (\partial_i\boldsymbol{u},\,\partial_i\pi)\in \boldsymbol{W}^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)\times W^{m,\,p}_{m+1}(\mathbb{R}^N_+).$$

It remains to prove that $(\partial_N\boldsymbol{u},\,\partial_N\pi)\in \boldsymbol{W}^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)\times W^{m,\,p}_{m+1}(\mathbb{R}^N_+)$. For that, let us observe that for any $i\in\{1,\dots,N-1\}$, we have

$$\begin{array}{lll}\partial_i\partial_N\boldsymbol{u} & =\partial_N\partial_i\boldsymbol{u} & \in \boldsymbol{W}^{m,\,p}_{m+1}(\mathbb{R}^N_+),\\ \partial^2_N u_i & =-\Delta' u_i+\partial_i\pi-f_i & \in W^{m,\,p}_{m+1}(\mathbb{R}^N_+),\\ \partial^2_N u_N & =\partial_N h-\partial_N\operatorname{div}'\boldsymbol{u}' & \in W^{m,\,p}_{m+1}(\mathbb{R}^N_+),\\ \partial_N\pi & =f_N+\Delta u_N & \in W^{m,\,p}_{m+1}(\mathbb{R}^N_+).\end{array}$$

Hence, $\nabla(\partial_N\boldsymbol{u})\in \boldsymbol{W}^{m,\,p}_{m+1}(\mathbb{R}^N_+)^N$ and knowing that $\partial_N\boldsymbol{u}\in \boldsymbol{W}^{m,\,p}_{m}(\mathbb{R}^N_+)$, we can deduce that $\partial_N\boldsymbol{u}\in \boldsymbol{W}^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)$, according to definition (1.2.1). Consequently, we have $\nabla\boldsymbol{u}\in \boldsymbol{W}^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)^N$. Likewise, we have $\nabla\pi\in \boldsymbol{W}^{m,\,p}_{m+1}(\mathbb{R}^N_+)$. Finally, we can conclude that $(\boldsymbol{u},\,\pi)\in \boldsymbol{W}^{m+2,\,p}_{m+1}(\mathbb{R}^N_+)\times W^{m+1,\,p}_{m+1}(\mathbb{R}^N_+)$. □

Now, we examine the basic case $\ell=-1$, corresponding to $f\in \boldsymbol{L}^p(\mathbb{R}^N_+)$. More precisely, we have the following result, corresponding to Theorem 4.5.2:

Theorem 4.5.6. *For any $\boldsymbol{f}\in \boldsymbol{L}^p(\mathbb{R}^N_+)$, $h\in W^{1,\,p}_0(\mathbb{R}^N_+)$ and $\boldsymbol{g}\in \boldsymbol{W}^{2-1/p,\,p}_0(\Gamma)$, problem (S_D) admits a solution $(\boldsymbol{u},\,\pi)\in \boldsymbol{W}^{2,\,p}_0(\mathbb{R}^N_+)\times W^{1,\,p}_0(\mathbb{R}^N_+)$, unique if $N>p$, unique up to an element of $(\mathbb{R}\,x_N)^{N-1}\times\{0\}\times\mathbb{R}$ if $N\leqslant p$, with the following estimate if $N\leqslant p$ (eliminate $(\boldsymbol{\lambda},\,\mu)$ if $N>p$):*

$$\begin{aligned}\inf_{(\boldsymbol{\lambda},\,\mu)\in(\mathbb{R}\,x_N)^{N-1}\times\{0\}\times\mathbb{R}}\Big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{2,\,p}_0(\mathbb{R}^N_+)}+\|\pi+\mu\|_{W^{1,\,p}_0(\mathbb{R}^N_+)}\Big)\ \leqslant&\\ C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)}+\|h\|_{W^{1,\,p}_0(\mathbb{R}^N_+)}+\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,\,p}_0(\Gamma)}\Big)\,.&\end{aligned}$$

Proof. The idea is to go back to the proof of Theorem 4.4.2 and we will throw light on the modifications. In contrast to Theorem 4.5.2, the extension $\tilde{\boldsymbol{f}}$ of $\boldsymbol{f}$ is of no importance because there is no orthogonality condition for the extended problem $(\tilde{S})$ (see Theorem 4.2.2). Then, we get the reduced problem $(S^\sharp)$. Now, to solve $(S^\sharp)$, this is the proof of Proposition 4.4.1. Problem (P) yields a unique $u_N \in W_0^{2,p}(\mathbb{R}^N_+)$, problem (Q) gives $\pi \in W_0^{1,p}(\mathbb{R}^N_+)$ unique up to an element of $\mathcal{N}^\Delta_{[1-N/p]}$; and (R) yields $\boldsymbol{u}' \in W_0^{2,p}(\mathbb{R}^N_+)^{N-1}$ unique up to an element of $(\mathcal{A}^\Delta_{[2-N/p]})^{N-1}$. The point (iii) of the proof is identical for all N and p (the kernels of the two Dirichlet problems are always reduced to zero). The last point concerns the kernel of the operator associated to this problem. If $N > p$, it is clearly reduced to zero and if $N \leqslant p$, we have $\mathcal{A}^\Delta_{[2-N/p]} = \mathbb{R}\, x_N$ and $\mathcal{N}^\Delta_{[1-N/p]} = \mathcal{P}_{[1-N/p]} = \mathbb{R}$. □

Thanks to the corresponding imbeddings, we can give a regularity result with the same proof as Corollary 4.5.5.

Corollary 4.5.7. *Let $m \in \mathbb{N}$. For any $\boldsymbol{f} \in \boldsymbol{W}^{m,p}_m(\mathbb{R}^N_+)$, $h \in W^{m+1,p}_m(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}^{m+2-1/p,p}_m(\Gamma)$, problem (S_D) admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+2,p}_m(\mathbb{R}^N_+) \times W^{m+1,p}_m(\mathbb{R}^N_+)$, unique if $N > p$, unique up to an element of $(\mathbb{R}\, x_N)^{N-1} \times \{0\} \times \mathbb{R}$ if $N \leqslant p$, with the following estimate if $N \leqslant p$ (eliminate $(\boldsymbol{\lambda}, \mu)$ if $N > p$):*

$$\inf_{(\boldsymbol{\lambda},\mu)\in(\mathbb{R}\,x_N)^{N-1}\times\{0\}\times\mathbb{R}} \left(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{m+2,p}_m(\mathbb{R}^N_+)} + \|\pi+\mu\|_{W^{m+1,p}_m(\mathbb{R}^N_+)}\right) \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{m,p}_m(\mathbb{R}^N_+)} + \|h\|_{W^{m+1,p}_m(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{m+2-1/p,p}_m(\Gamma)}\right).$$

4.6 Very weak solutions to the Stokes system

The aim of this section is to study the Stokes problem with singular data on the boundary. At first, we must give a meaning to singular data for the Stokes problem in the half-space. More precisely, we want to show that a boundary condition of the form $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)$ is meaningful. In mind of this paper, we limit ourselves to the two cases $\ell = 0$ or $\ell = 1$, *i.e.* to $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_{-1}(\Gamma)$ corresponding to a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,p}_{-1}(\mathbb{R}^N_+)$, or $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_0(\Gamma)$ corresponding to $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\mathbb{R}^N_+) \times W^{-1,p}_0(\mathbb{R}^N_+)$. In that way, for every $\ell \in \mathbb{Z}$, we introduce the space

$$\boldsymbol{M}_\ell(\mathbb{R}^N_+) = \left\{\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+);\ \boldsymbol{u} = \boldsymbol{0} \ \text{ and } \ \operatorname{div}\boldsymbol{u} = 0 \ \text{ on } \Gamma\right\}.$$

Lemma 4.6.1. *For any $\ell \in \mathbb{Z}$, we have the identity*

$$\boldsymbol{M}_\ell(\mathbb{R}^N_+) = \left\{\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+);\ \boldsymbol{u} = \boldsymbol{0} \ \text{ and } \ \partial_N u_N = 0 \ \text{ on } \Gamma\right\} \tag{4.6.1}$$

and the range space of the linear mapping $\gamma_1 : \boldsymbol{M}_\ell(\mathbb{R}^N_+) \to \boldsymbol{W}^{1/p,\,p'}_{-\ell+1}(\Gamma)$, *that is the trace of the normal derivative, is*

$$\boldsymbol{Z}_\ell(\Gamma) = \Big\{ \boldsymbol{w} \in \boldsymbol{W}^{1/p,\,p'}_{-\ell+1}(\Gamma);\ w_N = 0 \ \text{ on } \Gamma \Big\}. \tag{4.6.2}$$

Proof. Let $\boldsymbol{u} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\boldsymbol{u} = \boldsymbol{0}$ on Γ. Then $\operatorname{div} \boldsymbol{u} = \partial_N u_N$ on Γ and the identity (4.6.1) holds.

Moreover, it is clear that $\mathcal{I}m\ \gamma_1 \subset \boldsymbol{Z}_\ell(\Gamma)$. Conversely, given $\boldsymbol{w} \in \boldsymbol{Z}_\ell(\Gamma)$, by Lemma 1.3.1, there exists $\boldsymbol{u} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\boldsymbol{u} = \boldsymbol{0}$ and $\partial_N \boldsymbol{u} = \boldsymbol{w}$ on Γ. Since $w_N = 0$ on Γ, we have $\boldsymbol{u} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+)$ and $\boldsymbol{w} \in \mathcal{I}m\ \gamma_1$. $\square$

For any open subset Ω of $\mathbb{R}$, we also define the space

$$\boldsymbol{W}^{1,\,p'}_{-\ell}(\operatorname{div};\,\Omega) = \left[\begin{array}{ll} \big\{\boldsymbol{v} \in \boldsymbol{W}^{1,\,p'}_{-\ell}(\Omega);\ \operatorname{div} \boldsymbol{v} \in W^{1,\,p'}_{-\ell+1}(\Omega)\big\} & \text{if } \frac{N}{p'} \neq \ell, \\ \big\{\boldsymbol{v} \in \boldsymbol{W}^{1,\,p'}_{-\ell,\,-1}(\Omega);\ \operatorname{div} \boldsymbol{v} \in W^{1,\,p'}_{-\ell+1}(\Omega)\big\} & \text{if } \frac{N}{p'} = \ell; \end{array}\right.$$

which is a reflexive Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,\,p'}_{-\ell}(\operatorname{div};\,\Omega)} = \left[\begin{array}{ll} \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,\,p'}_{-\ell}(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{W^{1,\,p'}_{-\ell+1}(\Omega)} & \text{if } \frac{N}{p'} \neq \ell, \\ \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,\,p'}_{-\ell,\,-1}(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{W^{1,\,p'}_{-\ell+1}(\Omega)} & \text{if } \frac{N}{p'} = \ell; \end{array}\right.$$

and the following subspace of $\boldsymbol{W}^{1,\,p'}_{-\ell}(\operatorname{div};\,\mathbb{R}^N_+)$

$$\boldsymbol{X}_\ell(\mathbb{R}^N_+) = \left[\begin{array}{ll} \big\{\boldsymbol{v} \in \mathring{\boldsymbol{W}}^{1,\,p'}_{-\ell}(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} \in \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\big\} & \text{if } \frac{N}{p'} \neq \ell, \\ \big\{\boldsymbol{v} \in \mathring{\boldsymbol{W}}^{1,\,p'}_{-\ell,\,-1}(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} \in \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\big\} & \text{if } \frac{N}{p'} = \ell. \end{array}\right.$$

Before continuing, let us give the reason of this slightly complicated definition. This is the necessity of the imbedding $\boldsymbol{M}_\ell(\mathbb{R}^N_+) \hookrightarrow \boldsymbol{X}_\ell(\mathbb{R}^N_+)$; well, if $N/p' = \ell$, we do not have $W^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{1,\,p'}_{-\ell}(\mathbb{R}^N_+)$, but only $W^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{1,\,p'}_{-\ell,\,-1}(\mathbb{R}^N_+)$.

Lemma 4.6.2. *For any $\ell \in \mathbb{Z}$, the space $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ is dense in $\boldsymbol{X}_\ell(\mathbb{R}^N_+)$.*

Proof. Let $\boldsymbol{v} \in \boldsymbol{X}_\ell(\mathbb{R}^N_+)$ and $\tilde{\boldsymbol{v}}$ the extension by $\boldsymbol{0}$ of $\boldsymbol{v}$ to $\mathbb{R}^N$, then we have $\tilde{\boldsymbol{v}} \in \boldsymbol{W}^{1,\,p'}_{-\ell}(\operatorname{div};\,\mathbb{R}^N)$.

We begin to apply the cut off functions ϕ_k, defined on $\mathbb{R}^N$ for any $k \in \mathbb{N}$, by

$$\phi_k(x) = \begin{cases} \phi\left(\dfrac{k}{\ln|x|}\right), & \text{if } |x| > 1, \\ 1, & \text{otherwise,} \end{cases}$$

where $\phi \in \mathcal{C}^\infty([0,\,\infty[)$ is such that

$$\phi(t) = 0,\ \text{if } t \in [0,\,1]; \quad 0 \leqslant \phi(t) \leqslant 1,\ \text{if } t \in [1,\,2]; \quad \phi(t) = 1,\ \text{if } t \geqslant 2.$$

Let us recall Lemma 7.1 in [5], which is the essential argument for the sequel:

For all $x \in \mathbb{R}^N$, such that $|x| \in [e^{\frac{k}{2}}, e^k]$ with $k \geqslant 2$, and for all $\mu \in \mathbb{N}^N$, we have the estimate

$$|\partial^\mu \phi_k(x)| \leqslant \frac{c_\mu}{\varrho^{|\mu|} \lg \varrho}, \tag{4.6.3}$$

where c_μ is a constant independent of k.

We can deduce that

$$\phi_k \, \tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}_k \underset{k\to\infty}{\longrightarrow} \tilde{\boldsymbol{v}} \quad \text{in } \boldsymbol{W}^{1,p'}_{-\ell}(\mathbb{R}^N)$$

and

$$\operatorname{div}(\phi_k \, \tilde{\boldsymbol{v}}) = \phi_k \operatorname{div} \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{v}} \cdot \nabla \phi_k \underset{k\to\infty}{\longrightarrow} \operatorname{div} \tilde{\boldsymbol{v}} \quad \text{in } W^{1,p'}_{-\ell+1}(\mathbb{R}^N).$$

Let us notice that the estimate (4.6.3) is optimal to show the convergence to zero of the term $\boldsymbol{v} \cdot \nabla \phi_k$ in $W^{1,p'}_{-\ell+1}(\mathbb{R}^N)$.

Now, for any real number $\theta > 0$ and $x \in \mathbb{R}^N$, we set $\tilde{\boldsymbol{v}}_{k,\theta}(x) = \tilde{\boldsymbol{v}}_k(x - \theta \, e_N)$. Then $\tilde{\boldsymbol{v}}_{k,\theta} \in \boldsymbol{W}^{1,p'}_{-\ell}(\mathrm{div};\, \mathbb{R}^N)$ and $\operatorname{supp} \tilde{\boldsymbol{v}}_{k,\theta}$ is compact in $\mathbb{R}^N_+$, moreover

$$\lim_{\theta\to 0} \tilde{\boldsymbol{v}}_{k,\theta} = \tilde{\boldsymbol{v}}_k \quad \text{in } \boldsymbol{W}^{1,p'}_{-\ell}(\mathrm{div};\, \mathbb{R}^N).$$

Consequently, for any real number $\varepsilon > 0$ small enough, $\rho_\varepsilon * \tilde{\boldsymbol{v}}_{k,\theta} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ and

$$\lim_{\varepsilon\to 0} \lim_{\theta\to 0} \lim_{k\to\infty} \rho_\varepsilon * \tilde{\boldsymbol{v}}_{k,\theta} = \tilde{\boldsymbol{v}} \quad \text{in } \boldsymbol{W}^{1,p'}_{-\ell}(\mathrm{div};\, \mathbb{R}^N),$$

where ρ_ε is a mollifier. □

Let $\boldsymbol{X}'_\ell(\mathbb{R}^N_+)$ be the dual space of $\boldsymbol{X}_\ell(\mathbb{R}^N_+)$, we introduce the spaces:

$$\boldsymbol{T}_\ell(\mathbb{R}^N_+) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+);\; \Delta \boldsymbol{v} \in \boldsymbol{X}'_\ell(\mathbb{R}^N_+) \right\},$$
$$\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+) = \left\{ \boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+);\; \operatorname{div} \boldsymbol{v} = 0 \;\text{ in } \mathbb{R}^N_+ \right\},$$

which are reflexive Banach spaces for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{T}_\ell(\mathbb{R}^N_+)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+)} + \|\Delta \boldsymbol{v}\|_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)},$$

where $\|\cdot\|_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)}$ denotes the dual norm of the space $\boldsymbol{X}'_\ell(\mathbb{R}^N_+)$.

Lemma 4.6.3. *Let $\ell \in \mathbb{Z}$. Under hypothesis (4.3.3), the space $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $\boldsymbol{T}_\ell(\mathbb{R}^N_+)$.*

Proof. For every continuous linear form $\boldsymbol{z} \in \big(\boldsymbol{T}_\ell(\mathbb{R}^N_+)\big)'$, there exists a unique pair $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N_+) \times \boldsymbol{X}_\ell(\mathbb{R}^N_+)$, such that

$$\forall \boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+), \quad \langle \boldsymbol{z}, \boldsymbol{v} \rangle = \int_{\mathbb{R}^N_+} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x + \langle \Delta \boldsymbol{v}, \boldsymbol{g} \rangle_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+) \times \boldsymbol{X}_\ell(\mathbb{R}^N_+)}\,. \tag{4.6.4}$$

Thanks to the Hahn-Banach theorem, it suffices to show that any $\boldsymbol{z}$ which vanishes on $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is actually zero on $\boldsymbol{T}_\ell(\mathbb{R}^N_+)$. Let us suppose that $\boldsymbol{z} = \boldsymbol{0}$ on $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$, thus on $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$. Then we can deduce from (4.6.4) that

$$\boldsymbol{f} + \Delta \boldsymbol{g} = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+,$$

hence we have $\Delta \boldsymbol{g} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N_+)$, $\boldsymbol{g} \in \mathring{\boldsymbol{W}}^{1,p'}_{-\ell}(\mathbb{R}^N_+)$ and $\operatorname{div} \boldsymbol{g} \in \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Let $\tilde{\boldsymbol{f}} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N)$ and $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{1,p'}_{-\ell}(\mathbb{R}^N)$ be respectively the extensions by $\boldsymbol{0}$ of $\boldsymbol{f}$ and $\boldsymbol{g}$ to $\mathbb{R}^N$. From (4.6.4), we get $\tilde{\boldsymbol{f}} + \Delta \tilde{\boldsymbol{g}} = \boldsymbol{0}$ in $\mathbb{R}^N$, and thus $\Delta \tilde{\boldsymbol{g}} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N)$. Now, according to the isomorphism results for Δ in $\mathbb{R}^N$ (see [6]), we can deduce that $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N)$, under hypothesis (4.3.3). Since $\tilde{\boldsymbol{g}}$ is an extension by $\boldsymbol{0}$, it follows that $\boldsymbol{g} \in \mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Then, by density of $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ in $\mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$, there exists a sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ such that $\boldsymbol{\varphi}_k \to \boldsymbol{g}$ in $\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Thus, for any $\boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+)$, we have

$$\begin{aligned}
\langle \boldsymbol{z}, \boldsymbol{v} \rangle &= -\int_{\mathbb{R}^N_+} \boldsymbol{v} \cdot \Delta \boldsymbol{g} \,\mathrm{d}x + \langle \Delta \boldsymbol{v}, \boldsymbol{g} \rangle_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+) \times \boldsymbol{X}_\ell(\mathbb{R}^N_+)} \\
&= \lim_{k\to\infty} \left\{ -\int_{\mathbb{R}^N_+} \boldsymbol{v} \cdot \Delta \boldsymbol{\varphi}_k \,\mathrm{d}x + \langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}_k \rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^N_+) \times \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)} \right\} \\
&= 0,
\end{aligned}$$

i.e. $\boldsymbol{z}$ is identically zero. □

We also can show that, under hypothesis (4.3.3), $\Big\{ \boldsymbol{v} \in \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big);\ \operatorname{div} \boldsymbol{v} = 0 \Big\}$ is dense in $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$. To study the traces of functions which belong to $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$, we set

$$\boldsymbol{W}^{0,p}_\ell(\mathrm{div};\, \mathbb{R}^N_+) = \left[\begin{array}{ll} \big\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} \in W^{0,p}_\ell(\mathbb{R}^N_+) \big\} & \text{if } \frac{N}{p'} \neq \ell, \\ \big\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} \in W^{0,p}_{\ell,1}(\mathbb{R}^N_+) \big\} & \text{if } \frac{N}{p'} = \ell; \end{array} \right.$$

and their normal trace are described in the following lemma:

Lemma 4.6.4. *Let $\ell \in \mathbb{Z}$. The linear mapping*

$$\begin{aligned}
\gamma_{e_N} : \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big) &\longrightarrow \mathcal{D}(\mathbb{R}^{N-1}) \\
\boldsymbol{v} &\longmapsto v_N|_\Gamma,
\end{aligned}$$

that is the normal trace, can be extended to a linear continuous mapping

$$\gamma_{e_N} : \boldsymbol{W}^{0,p}_\ell(\mathrm{div};\, \mathbb{R}^N_+) \longrightarrow W^{-1/p,p}_{\ell-1}(\Gamma),$$

Moreover, we have the Green formula:

$$\begin{aligned}
&\forall \boldsymbol{v} \in \boldsymbol{W}^{0,p}_\ell(\mathrm{div};\, \mathbb{R}^N_+), \quad \forall \varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+), \\
&\qquad \int_{\mathbb{R}^N_+} \boldsymbol{v} \cdot \nabla \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_+} \varphi \operatorname{div} \boldsymbol{v} \,\mathrm{d}x = -\left\langle v_N, \varphi \right\rangle_{W^{-1/p,p}_{\ell-1}(\Gamma) \times W^{1/p,p'}_{-\ell+1}(\Gamma)}.
\end{aligned} \tag{4.6.5}$$

Proof. Let us remember that the assumption $N/p' \neq \ell$ is also necessary to have the imbedding $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell}(\mathbb{R}^N_+)$, which is underlying in the second term of the Green formula. Now, if $N/p' = \ell$, this imbedding fails, but in that case we have $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell,-1}(\mathbb{R}^N_+)$. That is the reason for the definition of the space $\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\, \mathbb{R}^N_+)$.

Here again, we can show by truncation and regularization that $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in both spaces $\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\, \mathbb{R}^N_+)$ and $\boldsymbol{W}^{0,p}_{\ell,1}(\mathrm{div};\, \mathbb{R}^N_+)$ as in Lemma 4.6.2. Note that the estimate (4.6.3) is optimal for the second space.

(i) Assume that $N/p' \neq \ell$. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ and $\varphi \in \mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$, then formula (4.6.5) obviously holds. Since $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$ and the mapping

$$\begin{aligned} \gamma_0 : W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) &\longrightarrow W^{1/p,p'}_{-\ell+1}(\Gamma) \\ \varphi &\longmapsto \varphi|_{\Gamma} \end{aligned}$$

is continuous, formula (4.6.5) holds for every $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ and $\varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$. By Lemma 1.3.1, for every $\mu \in W^{1/p,p'}_{-\ell+1}(\Gamma)$, there exists $\varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\varphi = \mu$ on Γ, with $\|\varphi\|_{W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)} \leqslant C\, \|\mu\|_{W^{1/p,p'}_{-\ell+1}(\Gamma)}$. Consequently,

$$\Big| \langle v_N, \mu \rangle_{W^{-1/p,p}_{\ell-1}(\Gamma)\times W^{1/p,p'}_{-\ell+1}(\Gamma)} \Big| \leqslant C\, \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)}\, \|\mu\|_{W^{1/p,p'}_{-\ell+1}(\Gamma)}.$$

Thus

$$\|v_N\|_{W^{-1/p,p}_{\ell-1}(\Gamma)} \leqslant C\, \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)}.$$

Hence we can deduce that the linear mapping γ_{e_N} is continuous for the norm of $\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\, \mathbb{R}^N_+)$. Since $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\, \mathbb{R}^N_+)$, the mapping γ_{e_N} can be extended by continuity to $\gamma_{e_N} \in \mathcal{L}\left(\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\, \mathbb{R}^N_+);\ W^{-1/p,p}_{\ell-1}(\Gamma)\right)$ and formula (4.6.5) holds for all $\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\, \mathbb{R}^N_+)$ and $\varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$.

(ii) The same arguments hold if $N/p' = \ell$. □

It follows that the functions $\boldsymbol{v}$ from $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$ are such their normal trace v_N belongs to $W^{-1/p,p}_{\ell-1}(\Gamma)$. Furthermore, for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ we have the following Green formula:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{M}_{\ell}(\mathbb{R}^N_+), \quad \int_{\mathbb{R}^N_+} \Delta \boldsymbol{v} \cdot \boldsymbol{\varphi}\, \mathrm{d}x = \int_{\mathbb{R}^N_+} \boldsymbol{v} \cdot \Delta \boldsymbol{\varphi}\, \mathrm{d}x + \int_{\Gamma} \boldsymbol{v} \cdot \partial_N \boldsymbol{\varphi}\, \mathrm{d}x'.$$

Let us now observe that the dual space $\boldsymbol{Z}'_{\ell}(\Gamma)$ of $\boldsymbol{Z}_{\ell}(\Gamma)$ can be identified with the space

$$\left\{ \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma);\ g_N = 0 \text{ on } \Gamma \right\},$$

and moreover that $\partial_N \boldsymbol{\varphi}$ sweeps $\boldsymbol{Z}_{\ell}(\Gamma)$ when $\boldsymbol{\varphi}$ sweeps $\boldsymbol{M}_{\ell}(\mathbb{R}^N_+)$. Thus, thanks to the density of $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ in $\boldsymbol{T}_{\ell}(\mathbb{R}^N_+)$, we can prove that the tangential trace of functions from $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$ belongs to $W^{-1/p,p}_{\ell-1}(\Gamma)$. So, their complete trace belongs to $\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)$ and we have

$$\begin{aligned} &\forall \boldsymbol{\varphi} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+), \quad \forall \boldsymbol{v} \in \boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+),\\ &\langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{X}'_\ell \times \boldsymbol{X}_\ell} = \langle \boldsymbol{v}, \Delta\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,p}_{\ell-1}\times \boldsymbol{W}^{0,p'}_{-\ell+1}} + \langle \boldsymbol{v}, \partial_N \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1/p,p}_{\ell-1}\times \boldsymbol{W}^{1/p,p'}_{-\ell+1}}. \end{aligned} \tag{4.6.6}$$

We now can solve the homogeneous Stokes problem with singular boundary conditions. We will give separately the results for $\ell = 0$ and $\ell = 1$. The proofs are quite similar and we will just detail the first case. The following proposition and corollary yield the existence of very weak solutions when the data are singular, so extending Proposition 4.4.1. Note that $W^{1,p}_0(\mathbb{R}^N_+) \hookrightarrow W^{0,p}_{-1}(\mathbb{R}^N_+)$ and $W^{1-1/p,p}_0(\Gamma) \hookrightarrow W^{-1/p,p}_{-1}(\Gamma)$ if $N \neq p$.

Proposition 4.6.5. *Assume that $\frac{N}{p} \neq 1$. For any $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_{-1}(\Gamma)$ such that $g_N = 0$, the Stokes problem (4.4.1)–(4.4.3) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,p}_{-1}(\mathbb{R}^N_+)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+)} + \|\pi\|_{W^{-1,p}_{-1}(\mathbb{R}^N_+)} \leqslant C\, \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}_{-1}(\Gamma)}.$$

Proof. (i) We will first show that if the pair $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,p}_{-1}(\mathbb{R}^N_+)$ satisfies (4.4.1) and (4.4.2), then we have $\boldsymbol{u} \in \boldsymbol{T}_{0,\sigma}(\mathbb{R}^N_+)$ and thus the boundary condition (4.4.3) makes sense. With this aim, thanks to Lemma 4.6.2, observe that if $\pi \in W^{-1,p}_{-1}(\mathbb{R}^N_+)$, then we have $\nabla\pi \in \boldsymbol{X}'_0(\mathbb{R}^N_+)$ and

$$\|\nabla \pi\|_{\boldsymbol{X}'_0(\mathbb{R}^N_+)} \leqslant C\, \|\pi\|_{W^{-1,p}_{-1}(\mathbb{R}^N_+)},$$

So, we have $\Delta\boldsymbol{u} \in \boldsymbol{X}'_0(\mathbb{R}^N_+)$ and the trace $\gamma_0\boldsymbol{u} \in \boldsymbol{W}^{-1/p,p}_{-1}(\Gamma)$.

(ii) Let us show that the problem (4.4.1)–(4.4.3) with $g_N = 0$ is equivalent to the variational formulation: Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,p}_{-1}(\mathbb{R}^N_+)$ such that

$$\begin{aligned} &\forall \boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}^N_+), \quad \forall \vartheta \in W^{1,p'}_1(\mathbb{R}^N_+),\\ &\langle \boldsymbol{u}, -\Delta\boldsymbol{v} + \nabla\vartheta\rangle_{\boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,p'}_{1}(\mathbb{R}^N_+)} - \langle \pi, \operatorname{div}\boldsymbol{v}\rangle_{W^{-1,p}_{-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_1(\mathbb{R}^N_+)}\\ &\qquad\qquad = \langle \boldsymbol{g}, \partial_N\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,p}_{-1}(\Gamma)\times \boldsymbol{W}^{1/p,p'}_1(\Gamma)}. \end{aligned} \tag{4.6.7}$$

(a) Let $(\boldsymbol{u}, \pi)$ be a solution to (4.4.1)–(4.4.3) with $g_N = 0$; then the Green formula (4.6.6) yields for all $\boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}^N_+)$,

$$\begin{aligned} \langle -\Delta\boldsymbol{u} + \nabla\pi, \boldsymbol{v}\rangle_{\boldsymbol{X}'_0\times\boldsymbol{X}_0} &= -\langle \boldsymbol{u}, \Delta\boldsymbol{v}\rangle_{\boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,p'}_{1}(\mathbb{R}^N_+)} -\\ &- \langle \boldsymbol{g}, \partial_N\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,p}_{-1}(\Gamma)\times \boldsymbol{W}^{1/p,p'}_1(\Gamma)} - \langle \pi, \operatorname{div}\boldsymbol{v}\rangle_{W^{-1,p}_{-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_1(\mathbb{R}^N_+)} = 0. \end{aligned}$$

Moreover, using the density of the functions of $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ with divergence zero in $\boldsymbol{T}_{0,\,\sigma}(\mathbb{R}^N_+)$, we obtain for all $\vartheta \in W^{1,\,p'}_1(\mathbb{R}^N_+)$,

$$\begin{aligned}\langle \boldsymbol{u}, \nabla\vartheta\rangle_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{1}(\mathbb{R}^N_+)} = &- \langle \operatorname{div}\boldsymbol{u}, \vartheta\rangle_{L^p(\mathbb{R}^N_+)\times L^{p'}(\mathbb{R}^N_+)} -\\ &- \langle u_N, \vartheta\rangle_{W^{-1/p,\,p}_{-1}(\Gamma)\times W^{1/p,\,p'}_{1}(\Gamma)} = 0.\end{aligned}$$

So we show that $(\boldsymbol{u},\, \pi)$ satisfies the variational formulation (4.6.7).

(b) Conversely, if $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$ satisfies the variational formulation (4.6.7), then taking $\boldsymbol{v} = \boldsymbol{0}$, we have for any $\vartheta \in \mathcal{D}(\mathbb{R}^N_+)$,

$$\langle \boldsymbol{u}, \nabla\vartheta\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} = \langle -\operatorname{div}\boldsymbol{u}, \vartheta\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} = 0,$$

hence $\operatorname{div}\boldsymbol{u} = 0$ in $\mathbb{R}^N_+$. We can deduce that $\boldsymbol{u} \in \boldsymbol{W}^{0,\,p}_{-1}(\mathrm{div};\, \mathbb{R}^N_+)$ and thus $u_N|_\Gamma \in W^{-1/p,\,p}_{-1}(\Gamma)$. Then, we can write for any $\vartheta \in W^{1,\,p'}_1(\mathbb{R}^N_+)$,

$$\langle \boldsymbol{u}, \nabla\vartheta\rangle_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{1}(\mathbb{R}^N_+)} = \langle u_N, \vartheta\rangle_{W^{-1/p,\,p}_{-1}(\Gamma)\times W^{1/p,\,p'}_{1}(\Gamma)} = 0.$$

Therefore, by the traces lemma (Lemma 1.3.1), we have for any $\varphi \in \mathcal{D}(\Gamma)$, $\langle u_N, \varphi\rangle_{\mathcal{D}'(\Gamma)\times\mathcal{D}(\Gamma)} = 0$, hence $u_N = 0$ on Γ. In addition, taking $\vartheta = 0$ in (4.6.7), we have for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$,

$$\langle \boldsymbol{u}, -\Delta\boldsymbol{v}\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} - \langle \pi, \operatorname{div}\boldsymbol{v}\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} = 0,$$

thus $\langle -\Delta\boldsymbol{u} + \nabla\pi, \boldsymbol{v}\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} = 0$, *i.e.* $-\Delta\boldsymbol{u} + \nabla\pi = 0$ in $\mathbb{R}^N_+$. We deduce that $\boldsymbol{u} \in \boldsymbol{T}_{0,\,\sigma}(\mathbb{R}^N_+)$ and taking $\vartheta = 0$ in (4.6.7), we finally get for any $\boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}^N_+)$,

$$\langle \boldsymbol{u}, \partial_N\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{1}(\Gamma)} = \langle \boldsymbol{g}, \partial_N\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{1}(\Gamma)},$$

where $\partial_N\boldsymbol{v}$ sweeps $\boldsymbol{Z}_0(\Gamma)$; hence $\boldsymbol{u}' = \boldsymbol{g}'$ on Γ. So, we have shown that $(\boldsymbol{u},\, \pi)$ is a solution to problem (4.4.1)–(4.4.3).

(iii) Let us solve problem (4.6.7). According to Theorem 4.5.2, we know that if $\frac{N}{p} \neq 1$, for all $\boldsymbol{f} \in \boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+)$ and $\varphi \in \overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+)$, there exists a unique $(\boldsymbol{v},\, \vartheta) \in \boldsymbol{M}_0(\mathbb{R}^N_+) \times W^{1,\,p'}_1(\mathbb{R}^N_+)$ solution to

$$-\Delta\boldsymbol{v} + \nabla\vartheta = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = \varphi \quad \text{in } \mathbb{R}^N_+, \qquad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma,$$

with the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{2,\,p'}_1(\mathbb{R}^N_+)} + \|\vartheta\|_{W^{1,\,p'}_1(\mathbb{R}^N_+)} \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+)} + \|\varphi\|_{W^{1,\,p'}_1(\mathbb{R}^N_+)}\right).$$

Then

$$\begin{aligned}\left|\langle \boldsymbol{g}, \partial_N\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{1}(\Gamma)}\right| &\leqslant C\, \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)} \|\boldsymbol{v}\|_{W^{2,\,p'}_1(\mathbb{R}^N_+)}\\ &\leqslant C\, \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{-1}}\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p'}_1} + \|\varphi\|_{W^{1,\,p'}_1}\right).\end{aligned}$$

In other words, we can say that the linear mapping

$$T:\ (\boldsymbol{f},\,\varphi)\longmapsto\langle\boldsymbol{g},\,\partial_N\boldsymbol{v}\rangle$$

is continuous on $\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+)\times\overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+)$, and according to the Riesz representation theorem, there exists a unique $(\boldsymbol{u},\,\pi)\in\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$ which is the dual space of $\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+)\times\overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+)$, such that

$$\begin{aligned}&\forall(\boldsymbol{f},\,\varphi)\in\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+)\times\overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+),\\&T(\boldsymbol{f},\,\varphi)=\langle\boldsymbol{u},\,\boldsymbol{f}\rangle_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+)}+\langle\pi,\,-\varphi\rangle_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)\times\overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+)}\,,\end{aligned}$$

i.e. the pair $(\boldsymbol{u},\,\pi)$ satisfies (4.6.7). □

We now can drop the hypothesis $g_N=0$.

Theorem 4.6.6. *Assume that* $\frac{N}{p}\neq 1$. *For any* $\boldsymbol{g}\in\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)$, *the Stokes problem (4.4.1)–(4.4.3) has a unique solution* $(\boldsymbol{u},\,\pi)\in\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)}+\|\pi\|_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)}\ \leqslant\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)}.$$

Proof. According to Theorem 4.3.3, we know that if $\frac{N}{p}\neq 1$, then there exists $\psi\in W^{1,\,p}_{-1}(\mathbb{R}^N_+)$ unique up to an element of $\mathcal{N}^{\Delta}_{[2-N/p]}$ solution to the following Neumann problem:

$$\Delta\psi=0\ \text{ in }\mathbb{R}^N_+,\quad \partial_N\psi=g_N\ \text{ on }\Gamma.$$

Let us set $\boldsymbol{w}=\nabla\psi$ and $\boldsymbol{g}^*=\boldsymbol{g}-\gamma_0\boldsymbol{w}$. Then $\boldsymbol{w}\in\boldsymbol{T}_{0,\,\sigma}(\mathbb{R}^N_+)$ and

$$\|\boldsymbol{w}\|_{\boldsymbol{T}_0(\mathbb{R}^N_+)}=\|\boldsymbol{w}\|_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)}\leqslant C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{-1}(\Gamma)}.$$

Furthermore, $\boldsymbol{g}^*$ satisfies the hypotheses of Proposition 4.6.5, hence the existence of a unique pair $(\boldsymbol{z},\,\pi)$ which satisfies

$$-\Delta\boldsymbol{z}+\nabla\pi=\boldsymbol{0}\ \text{ and }\ \operatorname{div}\boldsymbol{z}=0\ \text{ in }\mathbb{R}^N_+,\quad \boldsymbol{z}=\boldsymbol{g}^*\ \text{ on }\Gamma.$$

Then the pair $(\boldsymbol{z}+\boldsymbol{w},\,\pi)$ is the required solution. The uniqueness of this solution is a straightforward consequence of Proposition 4.6.5. □

Here is the corresponding results for the case $\ell=1$.

Proposition 4.6.7. *For any* $\boldsymbol{g}\in\boldsymbol{W}^{-1/p,\,p}_0(\Gamma)$ *such that* $g_N=0$, *and* $\boldsymbol{g}'\perp\mathbb{R}^{N-1}$ *if* $N\leqslant p'$, *the Stokes problem (4.4.1)–(4.4.3) has a unique solution* $(\boldsymbol{u},\,\pi)\in\boldsymbol{L}^p(\mathbb{R}^N_+)\times W^{-1,\,p}_0(\mathbb{R}^N_+)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)}+\|\pi\|_{W^{-1,\,p}_0(\mathbb{R}^N_+)}\ \leqslant\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_0(\Gamma)}.$$

Proof. The two differences from the weight $\ell = 0$ are the absence of critical value (the reason is that here, the dual problem solved by Theorem 4.5.6 has no critical value), and the orthogonality condition in the case $N \leqslant p'$ (which corresponds by duality to the non-zero kernel in Theorem 4.5.6 if $N \leqslant p$). The rest of the proof is similar. $\square$

Theorem 4.6.8. *For any $\boldsymbol{g} \in \boldsymbol{W}_0^{-1/p,\,p}(\Gamma)$ such that $\boldsymbol{g} \perp \mathbb{R}^N$ if $N \leqslant p'$, the Stokes problem (4.4.1)–(4.4.3) has a unique solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{L}^p(\mathbb{R}^N_+) \times W_0^{-1,\,p}(\mathbb{R}^N_+)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)} + \|\pi\|_{W_0^{-1,\,p}(\mathbb{R}^N_+)} \ \leqslant\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{-1/p,\,p}(\Gamma)}.$$

Remark 4.6.9. Let $p > 1$ be a real number. If $p < N$ and $r = Np/(N-p)$, then we have $W_0^{1-1/p,\,p}(\Gamma) \hookrightarrow W_0^{-1/r,\,r}(\Gamma)$. Indeed, by Theorem 1.4.4, for every $g \in W_0^{1-1/p,\,p}(\Gamma)$, there exists $u \in W_0^{2,\,p}(\mathbb{R}^N_+)$ such that

$$\Delta u = 0 \ \text{ in } \mathbb{R}^N_+, \quad \partial_N u = g \ \text{ on } \Gamma.$$

On the other hand, since the imbedding $W_0^{2,\,p}(\mathbb{R}^N_+) \hookrightarrow W_0^{1,\,r}(\mathbb{R}^N_+)$ holds, we deduce that $\boldsymbol{v} = \nabla u \in \boldsymbol{L}^r(\mathbb{R}^N_+)$ and $\operatorname{div} \boldsymbol{v} = 0 \in W_1^{0,\,r}(\mathbb{R}^N_+)$, *i.e.* $\boldsymbol{v} \in \boldsymbol{W}_1^{0,\,p}(\mathrm{div};\, \mathbb{R}^N_+)$. Moreover, as $r' \neq N$, according to Lemma 4.6.4, we get $\gamma_{e_N}\boldsymbol{v} = \partial_N u|_\Gamma = g \in W_0^{-1/r,\,r}(\Gamma)$. Consequently, if $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/p,\,p}(\Gamma) \hookrightarrow \boldsymbol{W}_0^{-1/r,\,r}(\Gamma)$, Proposition 4.4.1 and Theorem 4.6.8 respectively yield the unique solutions $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ and $(\boldsymbol{v},\, \vartheta) \in \boldsymbol{L}^r(\mathbb{R}^N_+) \times W_0^{-1,\,r}(\mathbb{R}^N_+)$, which are identical thanks to the Sobolev imbeddings $W_0^{1,\,p}(\mathbb{R}^N_+) \hookrightarrow L^r(\mathbb{R}^N_+)$ and $L^p(\mathbb{R}^N_+) \hookrightarrow W_0^{-1,\,r}(\mathbb{R}^N_+)$. $\diamondsuit$

Chapitre 5

Behaviour at infinity in the Stokes system

5.1 Introduction

This chapter is the continuation of the previous one in which we only dealt with the basic weights. Here, we are interested in a whole scale of weights. This lead us to be interested in the kernel of the operator associated to this problem and symmetricaly in the compatibility condition for the data. The main results of Chapter 4 will be naturally included in this one, but we will not discuss again these particular cases. We will also base on the previously established results on the harmonic and biharmonic operators.

5.2 Characterization of the kernel for the Stokes operator

In this section, we will give two characterizations of this kernel, and we will observe that it does not depend on the regularity according to the Sobolev imbeddings. Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$ and let us denote by $\mathcal{K}_\ell^{m,\,p}$ the kernel of the Stokes operator, *i.e.*

$$\begin{aligned}\mathcal{K}_\ell^{m,\,p} = \big\{(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_{m+\ell}^{m+1,\,p}(\mathbb{R}^N_+) \times W_{m+\ell}^{m,\,p}(\mathbb{R}^N_+);\\ -\,\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{0} \text{ and } \operatorname{div}\boldsymbol{u} = 0 \text{ in } \mathbb{R}^N_+,\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma\big\}\end{aligned}$$

and for any $k \in \mathbb{Z}$, introduce the following polynomial space

$$\begin{aligned}\mathcal{S}_k^D = \big\{(\boldsymbol{\lambda},\,\mu) \in \boldsymbol{\mathcal{P}}_k^{\Delta^2} \times \mathcal{P}_{k-1}^{\Delta};\\ -\,\Delta\boldsymbol{\lambda} + \nabla\mu = \boldsymbol{0} \text{ and } \operatorname{div}\boldsymbol{\lambda} = 0 \text{ in } \mathbb{R}^N_+,\ \boldsymbol{\lambda} = \boldsymbol{0} \text{ on } \Gamma\big\}.\end{aligned}$$

The first characterization uses the reflection principle. As preliminary result, we will show how to get a reflection principle for the Stokes system from those of harmonic and biharmonic functions. Let us notice that R. Farwig gives these formulas in [27], but without the method to get them. Let us especially quote R.J. Duffin, who first established in [26] the continuation formula of biharmonic functions in the three dimensional case and then analogous formulas for the Stokes flow equations. Lastly, A. Huber extended in [34] this principle to polyharmonic functions.

Lemma 5.2.1. *Let* $(\boldsymbol{u}, \pi) \in \boldsymbol{\mathcal{D}}'(\mathbb{R}^N_+) \times \mathcal{D}'(\mathbb{R}^N_+)$ *satisfying*

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} \quad \textit{and} \quad \operatorname{div} \boldsymbol{u} = 0 \qquad \textit{in } \mathbb{R}^N_+, \tag{$\aleph$}$$

then $(\boldsymbol{u}, \pi) \in \boldsymbol{\mathcal{C}}^\infty(\mathbb{R}^N_+) \times \mathcal{C}^\infty(\mathbb{R}^N_+)$*. In addition, if* $\boldsymbol{u} = \boldsymbol{0}$ *on* Γ*, then there exists an extension* $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{\mathcal{D}}'(\mathbb{R}^N) \times \mathcal{D}'(\mathbb{R}^N)$ *of* $(\boldsymbol{u}, \pi)$ *satisfying*

$$-\Delta \tilde{\boldsymbol{u}} + \nabla \tilde{\pi} = \boldsymbol{0} \quad \textit{and} \quad \operatorname{div} \tilde{\boldsymbol{u}} = 0 \qquad \textit{in } \mathbb{R}^N, \tag{$\beth$}$$

which is given by

$$(\star) \quad \begin{cases} \tilde{\boldsymbol{u}}'(x', x_N) &= \big(-\boldsymbol{u}' + 2\, x_N\, \nabla' u_N + x_N^2\, \nabla' \pi \big)(x', -x_N), \\ \tilde{u}_N(x', x_N) &= \big(-u_N - 2\, x_N\, \partial_N u_N - x_N^2\, \partial_N \pi \big)(x', -x_N), \\ \tilde{\pi}(x', x_N) &= \big(\pi - 2\, x_N\, \partial_N \pi - 4\, \partial_N u_N \big)(x', -x_N), \end{cases}$$

for any $(x', x_N) \in \mathbb{R}^N_-$*. Moreover, this extension is unique.*

Proof. (1) Applying the divergence operator to the first equation in ($\aleph$), we obtain $\Delta \pi = 0$ in $\mathbb{R}^N_+$. Since $\pi \in \mathcal{D}'(\mathbb{R}^N_+)$, we can deduce that $\pi \in \mathcal{C}^\infty(\mathbb{R}^N_+)$ by Weyl's lemma (see e.g. Dautray-Lions [25], vol. 2, p 327, Proposition 1).

Likewise, applying the harmonic operator to the first equation in ($\aleph$), we get $\Delta^2 \boldsymbol{u} = \boldsymbol{0}$ in $\mathbb{R}^N_+$. Since $\boldsymbol{u} \in \boldsymbol{\mathcal{D}}'(\mathbb{R}^N_+)$, we still deduce (using two times the same Proposition) that $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}^\infty(\mathbb{R}^N_+)$.

(2) For the uniqueness, let us consider $(\tilde{\boldsymbol{u}}_1, \tilde{\pi}_1)$ and $(\tilde{\boldsymbol{u}}_2, \tilde{\pi}_2)$ in $\boldsymbol{\mathcal{D}}'(\mathbb{R}^N) \times \mathcal{D}'(\mathbb{R}^N)$, which both extend $(\boldsymbol{u}, \pi)$ and satisfy ($\beth$). Let us set $\boldsymbol{U} = \tilde{\boldsymbol{u}}_2 - \tilde{\boldsymbol{u}}_1$ and $\Pi = \tilde{\pi}_2 - \tilde{\pi}_1$. Then $\Delta^2 \boldsymbol{U} = \boldsymbol{0}$ in $\mathbb{R}^N$ and thanks to the Proposition quoted before, we can also deduce that $\boldsymbol{U}$ is analytic in $\mathbb{R}^N$. Since $\boldsymbol{U} = \boldsymbol{0}$ in $\mathbb{R}^N_+$, the continuation analytic principle implies that in fact $\boldsymbol{U} = \boldsymbol{0}$ in $\mathbb{R}^N$. The same argument holds for Π.

(3) Now, let us assume the existence of an extension $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{\mathcal{D}}'(\mathbb{R}^N) \times \mathcal{D}'(\mathbb{R}^N)$ of $(\boldsymbol{u}, \pi)$ satisfying ($\beth$), and let us show the formulas ($\star$).

(i) With the same arguments as in point (1), we can see that $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{\mathcal{C}}^\infty(\mathbb{R}^N) \times \mathcal{C}^\infty(\mathbb{R}^N)$. In addition, we have $\tilde{\boldsymbol{u}} = \boldsymbol{0}$ and $\operatorname{div} \tilde{\boldsymbol{u}} = 0$ on Γ, thus $\partial_N \tilde{u}_N = 0$ on Γ. So, u_N satisfies the following biharmonic problem

$$\Delta^2 u_N = 0 \ \text{ in } \mathbb{R}^N_+ \quad \text{ and } \quad u_N = \partial_N u_N = 0 \ \text{ on } \Gamma.$$

From the continuation formula of biharmonic functions (2.3.3) and replacing Δu_N by $\partial_N \pi$, we immediately get the formula ($\star$) for u_N.

(ii) Let us set $\tilde{r} = 2\,\tilde{u}_N - x_N\,\tilde{\pi}$ in $\mathbb{R}^N$ and $r = \tilde{r}|_{\mathbb{R}^N_+}$. Then, it is easy to see that $\Delta\tilde{r} = 0$ in $\mathbb{R}^N$ and $\tilde{r} = 0$ on Γ, hence

$$\Delta r = 0 \ \text{ in } \mathbb{R}^N_+, \quad r = 0 \ \text{ on } \Gamma,$$

Thus, by the Schwarz reflection principle, we necessary have

$$\tilde{r}(x', x_N) = -r(x', -x_N) \ \text{ if } x_N < 0.$$

Since $\tilde{r} = 2\,\tilde{u}_N - x_N\,\tilde{\pi}$ in $\mathbb{R}^N_-$, we can write, for any $x_N < 0$,

$$\begin{aligned} x_N\,\tilde{\pi}(x', x_N) &= \big[2\,(-u_N - 2\,x_N\,\partial_N u_N - x_N^2\,\partial_N \pi) + (2\,u_N + x_N\,\pi)\big](x', -x_N), \\ &= \big(-4\,x_N\,\partial_N u_N - 2\,x_N^2\,\partial_N\pi + x_N\,\pi\big)(x', -x_N). \end{aligned}$$

Hence, dividing by x_N, we get the formula ($\star$) for π.

(iii) Lastly, we also must have $\Delta\tilde{\boldsymbol{u}}' = \nabla'\tilde{\pi}$ in $\mathbb{R}^N_-$. Thus, for any $x_N > 0$,

$$\begin{aligned} \Delta\tilde{\boldsymbol{u}}'(x', -x_N) &= \nabla'\tilde{\pi}(x', -x_N), \\ &= \big(\nabla'\pi + 2\,x_N\,\partial_N\nabla'\pi - 4\,\partial_N\nabla' u_N\big)(x', x_N), \\ &= \Delta\big(-\boldsymbol{u}' - 2\,x_N\,\nabla' u_N + x_N^2\,\nabla'\pi\big)(x', x_N). \end{aligned}$$

Let us introduce the function $\tilde{\boldsymbol{u}}'_*(x', x_N) = \tilde{\boldsymbol{u}}'(x', -x_N)$ in $\mathbb{R}^N_+$. Then, we can express the previous equality by

$$\Delta\big(\tilde{\boldsymbol{u}}'_* + \boldsymbol{u}' + 2\,x_N\,\nabla' u_N - x_N^2\,\nabla'\pi\big) = \boldsymbol{0} \ \text{ in } \mathbb{R}^N_+.$$

Moreover, we have $\big(\tilde{\boldsymbol{u}}'_* + \boldsymbol{u}' + 2\,x_N\,\nabla' u_N - x_N^2\,\nabla'\pi\big)(x', 0) = \boldsymbol{0}$; and

$$\begin{aligned} &\partial_N\big(\tilde{\boldsymbol{u}}'_* + \boldsymbol{u}' + 2\,x_N\,\nabla' u_N - x_N^2\,\nabla'\pi\big)(x', 0) \\ &\qquad = \big(-\partial_N\tilde{\boldsymbol{u}}' + \partial_N\boldsymbol{u}' + 2\,\nabla' u_N\big)(x', 0) = \boldsymbol{0}. \end{aligned}$$

Thus, $\tilde{\boldsymbol{u}}'_* = -\boldsymbol{u}' - 2\,x_N\,\nabla' u_N + x_N^2\,\nabla'\pi$ in $\mathbb{R}^N_+$. That is, for any $x_N > 0$,

$$\tilde{\boldsymbol{u}}'(x', -x_N) = \big(-\boldsymbol{u}' - 2\,x_N\,\nabla' u_N + x_N^2\,\nabla'\pi\big)(x', x_N).$$

So, replacing x_N by $-x_N$, we get the formula ($\star$) for $\boldsymbol{u}'$.

(4) Conversely, to show that the extension $(\tilde{\boldsymbol{u}}, \tilde{\pi})$ defined by the formulas ($\star$) belongs to $\boldsymbol{\mathcal{C}}^\infty(\mathbb{R}^N) \times \mathcal{C}^\infty(\mathbb{R}^N)$ and satisfies ($\sqsupset$), we refer to the proof by Duffin in [26]. It is very easy to see that in $\mathbb{R}^N_-$ and the only serious difficulty is the argument at the boundary. □

Now, we can give the first characterization of the Stokes kernel.

Lemma 5.2.2. *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$.*

(i) *If $N/p \notin \{1, \dots, -\ell - m\}$, then $\mathcal{K}^{m,p}_\ell = \mathcal{S}^D_{[1-\ell-N/p]}$.*

(ii) *If* $N/p \in \{1, \dots, -\ell - m\}$*, then* $\mathcal{K}_\ell^{m,p} = \mathcal{S}^D_{-\ell-N/p}$.

Proof. Let $(\boldsymbol{u},\, \pi) \in \mathcal{K}_\ell^{m,p}$. Using a weak formulation for the extension given by formulas $(\star)$, we can show that in fact $\tilde{\pi}$ and $\tilde{\boldsymbol{u}}$ are respectively harmonic and biharmonic tempered distributions in $\mathbb{R}^N$, thus polynomials. Moreover, according to (1.2.5), the highest degree of the polynomials contained in $W^{m+1,p}_{m+\ell}(\mathbb{R}^N_+)$, is given by

$$q = \begin{cases} -\ell - \frac{N}{p} & \text{if } \frac{N}{p} + \ell \in \{j \in \mathbb{Z};\ j \leqslant 0\},\\ \left[1 - \ell - \frac{N}{p}\right] & \text{otherwise,} \end{cases}$$

i.e. precisely the conditions of the statement. □

We can be more specific about polynomials which build up this kernel. The idea of this characterization is due to T.Z. Boulmezaoud (see [21]). We give it with a completely different proof, based on the kernels of the Dirichlet and Neumann problems for the Laplacian and the one of the biharmonic problem with Dirichlet boundary conditions in the half-space.

Concerning the kernel of the biharmonic operator $(\Delta^2, \gamma_0, \gamma_1)$ in $W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+)$, we showed in Lemma 2.3.4 that it is characterized for any $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$, under hypothesis $N/p \notin \{1, \dots, -\ell - m\}$, by:

$$\mathcal{B}_{[2-\ell-N/p]} = \Pi_D \mathcal{A}^\Delta_{[-\ell-N/p]} \oplus \Pi_N \mathcal{N}^\Delta_{[-\ell-N/p]}.$$

Moreover, thanks to the study of the very weak solutions for the singular boundary conditions in Section 3.3, we extended these results to the two supplementary cases $m \in \{-2,\, -1\}$.

We now can give the second characterization of the Stokes kernel in $\mathbb{R}^N_+$:

Lemma 5.2.3. *Let* $\ell \in \mathbb{Z}$, $m \in \mathbb{N}$ *and assume that* $N/p \notin \{1, \dots, -\ell-m\}$*. Then* $(\boldsymbol{u},\, \pi) \in \mathcal{K}_\ell^{m,p} = \mathcal{S}^D_{[1-\ell-N/p]}$ *if and only if there exists* $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^\Delta_{[1-\ell-N/p]}$ *such that*

$$\boldsymbol{u} = \boldsymbol{\varphi} - \nabla\big(\Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_N \varphi_N\big), \tag{5.2.1}$$

$$\pi = -\operatorname{div} \boldsymbol{\varphi}. \tag{5.2.2}$$

Proof. Given $(\boldsymbol{u},\, \pi) \in \mathcal{K}_\ell^{m,p}$, then we also have $\operatorname{div} \boldsymbol{u} = 0$ on Γ and thus $\partial_N u_N = 0$ on Γ. Moreover $\Delta \pi = 0$ in $\mathbb{R}^N_+$ and thus $\Delta^2 u_N = 0$ in $\mathbb{R}^N_+$. So we get the following biharmonic problem

$$\Delta^2 u_N = 0 \ \text{ in } \mathbb{R}^N_+ \quad \text{ and } \quad u_N = \partial_N u_N = 0 \ \text{ on } \Gamma.$$

Hence $u_N \in \mathcal{B}_{[1-\ell-N/p]}$ and there exists $(r,\, s) \in \mathcal{A}^\Delta_{[-1-\ell-N/p]} \times \mathcal{N}^\Delta_{[-1-\ell-N/p]}$ such that $u_N = \Pi_D r + \Pi_N s$.

According to the properties of the operators Π_D and Π_N, (2.3.5) and (2.3.6), we can deduce that $\partial_N \pi = \Delta u_N = r + s$ in $\mathbb{R}^N_+$ and thus π satisfies

$$\Delta \pi = 0 \ \text{ in } \mathbb{R}^N_+ \quad \text{ and } \quad \partial_N \pi = s \ \text{ on } \Gamma.$$

Then, there exists $\psi \in \mathcal{N}^{\Delta}_{[-\ell-N/p]}$ (see Theorems 1.4.3 and 1.4.4), such that

$$\pi = \psi + Ks \quad \text{in } \mathbb{R}^N_+, \tag{5.2.3}$$

where $Ks(x) = \int_0^{x_N} s(x',\, t)\,\mathrm{d}t$.

So, we have $\Delta u_N = r + s = \partial_N \pi = \partial_N \psi + s$ in $\mathbb{R}^N_+$, thus $r = \partial_N \psi$. Hence,

$$u_N = \Pi_D \partial_N \psi + \Pi_N s \quad \text{in } \mathbb{R}^N_+. \tag{5.2.4}$$

From (5.2.3), we get for every $i \in \{1, \dots, N-1\}$,

$$\begin{aligned} \Delta u_i = \partial_i \pi = \partial_i \psi + \partial_i K s \ &\in\ \mathcal{N}^{\Delta}_{[-1-\ell-N/p]} \oplus \mathcal{A}^{\Delta}_{[-1-\ell-N/p]} \\ &= \Delta \Pi_N \partial_i \psi + \Delta \Pi_D \partial_i K s. \end{aligned}$$

Then, $w_i = u_i - \Pi_N \partial_i \psi - \Pi_D \partial_i K s$ satisfies

$$\Delta w_i = 0 \quad \text{in } \mathbb{R}^N_+ \qquad \text{and} \qquad w_i = 0 \quad \text{on } \Gamma.$$

Hence the existence of $\varphi_i \in \mathcal{A}^{\Delta}_{[1-\ell-N/p]}$ (see Theorems 1.4.1 and 1.4.2), such that $w_i = \varphi_i$, *i.e.*

$$u_i = \Pi_N \partial_i \psi + \Pi_D \partial_i K s + \varphi_i.$$

Thereby, writing $\boldsymbol{\varphi}' = (\varphi_1, \dots, \varphi_{N-1})$, we get

$$\begin{aligned} \operatorname{div}' \boldsymbol{u}' &= \Pi_N \Delta' \psi + \Pi_D \Delta' K s + \operatorname{div}' \boldsymbol{\varphi}' \\ &= -\Pi_N \partial_N^2 \psi - \Pi_D \partial_N^2 K s + \operatorname{div}' \boldsymbol{\varphi}' \\ &= -\frac{1}{2}\, x_N\, \partial_N \psi - \frac{1}{2}\, (x_N\, \partial_N K s - K s) + \operatorname{div}' \boldsymbol{\varphi}' \\ &= -\frac{1}{2}\, x_N\, \partial_N \psi - \frac{1}{2}\, (x_N\, s - K s) + \operatorname{div}' \boldsymbol{\varphi}'. \end{aligned}$$

In addition, by (5.2.4), we have

$$\begin{aligned} \partial_N u_N &= \partial_N \Pi_D \partial_N \psi + \partial_N \Pi_N s \\ &= \frac{1}{2}\, x_N\, \partial_N \psi + \frac{1}{2} \left(x_N\, s + \int_0^{x_N} s(x',\, t)\,\mathrm{d}t \right) \\ &= \frac{1}{2}\, x_N\, \partial_N \psi + \frac{1}{2}\, (x_N\, s + K s)\,. \end{aligned}$$

Since $\operatorname{div} \boldsymbol{u} = 0$, we can deduce that $\operatorname{div}' \boldsymbol{\varphi}' = -Ks$ and thus (5.2.3) can be rewritten as $\pi = \psi - \operatorname{div}' \boldsymbol{\varphi}'$. Now, if we set $\varphi_N(x) = -\int_0^{x_N} \psi(x',\, t)\,\mathrm{d}t$, then we have $\psi = -\partial_N \varphi_N$ and $\varphi_N \in \mathcal{A}^{\Delta}_{[1-\ell-N/p]}$. So, we obtain $\pi = -\operatorname{div} \boldsymbol{\varphi}$, *i.e.* (5.2.2), with $\boldsymbol{\varphi} = (\boldsymbol{\varphi}',\, \varphi_N) \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1-\ell-N/p]}$.

Coming back to the velocity field, we get for every $i \in \{1, \dots, N-1\}$,

$$u_i = \varphi_i - \partial_i \Pi_N \partial_N \varphi_N - \partial_i \Pi_D \operatorname{div}' \boldsymbol{\varphi}'. \tag{5.2.5}$$

Likewise, for the normal component, (5.2.4) yields

$$\begin{aligned} u_N &= -\Pi_D \partial_N^2 \varphi_N + \Pi_N \partial_N K s \\ &= \frac{1}{2}\big(\varphi_N - x_N\,\partial_N \varphi_N\big) + \frac{1}{2}\,x_N\,K s \\ &= \varphi_N - \frac{1}{2}\,x_N\,\partial_N\varphi_N - \frac{1}{2}\,\varphi_N - \frac{1}{2}\,x_N\ \mathrm{div}'\,\boldsymbol{\varphi}' \\ &= \varphi_N - \partial_N \Pi_N \partial_N \varphi_N - \partial_N \Pi_D\,\mathrm{div}'\,\boldsymbol{\varphi}'. \end{aligned}$$

So, combining this with (5.2.5), we get $\boldsymbol{u} = \boldsymbol{\varphi} - \nabla\big(\Pi_N\partial_N\varphi_N + \Pi_D\,\mathrm{div}'\,\boldsymbol{\varphi}'\big)$, *i.e.* the statement (5.2.1).

Conversely, we can readily verify that such a pair $(\boldsymbol{u},\,\pi)$ belongs to $\mathcal{K}_\ell^{m,\,p}$. □

5.3 Generalized solutions

In this section, we will establish the central result on the generalized solutions to the Stokes system in the half-space, with Theorem 5.3.2. We will be interested in the existence of a solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}_+^N) \times W_\ell^{0,\,p}(\mathbb{R}_+^N)$ to (S_D), for data $\boldsymbol{f} \in \boldsymbol{W}_\ell^{-1,\,p}(\mathbb{R}_+^N)$, $h \in W_\ell^{0,\,p}(\mathbb{R}_+^N)$ and $\boldsymbol{g} \in \boldsymbol{W}_\ell^{1-1/p,\,p}(\Gamma)$. To avoid troubles with the compatibility conditions, we will start with the study of the negative weights. For this, as for the weight $\ell = 0$ in Chapter 4, we will adapt a method used by Farwig-Sohr in [28]. Then, we get back the positive weights by a duality argument, and the compatibility condition naturally comes from the kernel of the dual case.

First, we will establish the result for the homogeneous problem in the case of negative weights:

Lemma 5.3.1. *Let ℓ be a negative integer and assume that $N/p \notin \{1, \ldots, -\ell\}$. For any $\boldsymbol{g} \in \boldsymbol{W}_\ell^{1-1/p,\,p}(\Gamma)$, the homogeneous Stokes problem*

$$-\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{0} \qquad \text{in } \mathbb{R}_+^N, \tag{5.3.1}$$

$$\mathrm{div}\,\boldsymbol{u} = 0 \qquad \text{in } \mathbb{R}_+^N, \tag{5.3.2}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma, \tag{5.3.3}$$

has a solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}_+^N) \times W_\ell^{0,\,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathcal{S}^D_{[1-\ell-N/p]}$, with the estimate

$$\inf_{(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}^D_{[1-\ell-N/p]}} \Big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}_+^N)} + \|\pi+\mu\|_{W_\ell^{0,\,p}(\mathbb{R}_+^N)}\Big) \;\leqslant\; C\,\|\boldsymbol{g}\|_{\boldsymbol{W}_\ell^{1-1/p,\,p}(\Gamma)}.$$

Proof. The operator associated to this problem is clearly continuous, moreover its kernel is known. If we show that it is surjective, then the final estimate will be a straightforward consequence of the Banach Theorem. So, we only must prove the existence of a solution $(\boldsymbol{u}, \pi)$. The first point of this proof is strictly identical to the one of the proof of Theorem 4.4.1, but we recall it for the convenience of the reader. However, the arguments in the sequel are slightly different and we can see here the importance of the assumption that ℓ is negative. In spite of the required adaptations, this reasoning appears as a universal method for the solution to the Stokes system in the half-space, even for the other types of boundary conditions, as we will see in the next chapter.

(i) Firstly, we will show that system (5.3.1)–(5.3.3) can be reduced to a set of three problems on the fundamental operators Δ^2 and Δ.

Applying the operator div to the first equation (5.3.1), we obtain

$$\Delta\pi = 0 \quad \text{in } \mathbb{R}^N_+. \tag{5.3.4}$$

Now, applying the operator Δ to the same equation (5.3.1), we deduce

$$\Delta^2\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+. \tag{5.3.5}$$

From the boundary condition (5.3.3), we take out

$$u_N = g_N \quad \text{on } \Gamma, \tag{5.3.6}$$

and moreover $\operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ, where $\operatorname{div}' \boldsymbol{u}' = \sum_{i=1}^{N-1} \partial_i u_i$.

Since $\operatorname{div} \boldsymbol{u} = 0$ in $\mathbb{R}^N_+$, we also have $\operatorname{div} \boldsymbol{u} = 0$ on Γ; then we can write $\partial_N u_N + \operatorname{div}' \boldsymbol{u}' = 0$ on Γ, hence

$$\partial_N u_N = -\operatorname{div}' \boldsymbol{g}' \quad \text{on } \Gamma. \tag{5.3.7}$$

Combining (5.3.5), (5.3.6) and (5.3.7), we get the following biharmonic problem

$$(P): \qquad \Delta^2 u_N = 0 \ \text{ in } \mathbb{R}^N_+, \quad u_N = g_N \ \text{ and } \ \partial_N u_N = -\operatorname{div}' \boldsymbol{g}' \ \text{ on } \Gamma.$$

Then, combining (5.3.4) with the trace on Γ of the N^{th} component in the equations (5.3.1), we obtain the following Neumann problem

$$(Q): \qquad \Delta\pi = 0 \ \text{ in } \mathbb{R}^N_+, \quad \partial_N \pi = \Delta u_N \ \text{ on } \Gamma.$$

Lastly, if we consider the $N-1$ first components of the equations (5.3.1) and (5.3.3), we can write the following Dirichlet problem

$$(R): \qquad \Delta\boldsymbol{u}' = \nabla'\pi \ \text{ in } \mathbb{R}^N_+, \quad \boldsymbol{u}' = \boldsymbol{g}' \ \text{ on } \Gamma.$$

(ii) Next, we will solve these three problems.

`Step 1:` Problem (P). Since $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}_\ell(\Gamma)$, we have $g_N \in W^{1-1/p,p}_\ell(\Gamma)$ and $\operatorname{div}' \boldsymbol{g}' \in W^{-1/p,p}_\ell(\Gamma)$, so (P) is an homogeneous biharmonic problem with singular boundary conditions. Since $\ell < 0$, according to Theorem 3.3.5, we know that problem (P) has a solution $u_N \in W_\ell^{1,p}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[1-\ell-N/p]}$.

`Step 2:` Problem (Q). Since $\Delta^2 u_N = 0$ in $\mathbb{R}^N_+$, according to an appropriate trace result with Lemma 4.3.6, we can deduce that $\Delta u_N|_\Gamma \in W_\ell^{-1-1/p,p}(\Gamma)$. As $\ell < 0$, according to Theorem 4.3.2, we know that problem (Q) has a solution $\pi \in W_\ell^{0,p}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{N}^\Delta_{[-\ell-N/p]}$.

`Step 3:` Problem (R). Thanks to the previous result, we can deduce that $\nabla'\pi \in W_\ell^{-1,p}(\mathbb{R}^N_+)^{N-1}$ and moreover $\boldsymbol{g}' \in W_\ell^{1-1/p,p}(\Gamma)^{N-1}$. Since $\ell < 0$, according to Theorem 1.4.1, we know that problem (R) has a solution $\boldsymbol{u}' \in W_\ell^{1,p}(\mathbb{R}^N_+)^{N-1}$, unique up to an element of $(\mathcal{A}^\Delta_{[1-\ell-N/p]})^{N-1}$.

(iii) In order, we have found u_N, π and $\boldsymbol{u}'$, non-unique, which satisfy (5.3.3) and partially satisfy (5.3.1), more precisely such that

$$-\Delta \boldsymbol{u}' + \nabla'\pi = \mathbf{0} \quad \text{in } \mathbb{R}^N_+.$$

It remains to show we can choose them satisfying (5.3.2) and the N^{th} component of (5.3.1), *i.e.*

$$-\Delta u_N + \partial_N \pi = 0 \quad \text{in } \mathbb{R}^N_+.$$

Consider such a pair $(\boldsymbol{u}, \pi)$ satisfying problems (P), (Q) and (R). From the first equations of (P) and (Q), we obtain

$$\Delta(\Delta u_N - \partial_N \pi) = \Delta^2 u_N = 0 \quad \text{in } \mathbb{R}^N_+.$$

Thanks to the boundary condition of (Q), we can deduce that the distribution $\Delta u_N - \partial_N \pi \in W_\ell^{-1,p}(\mathbb{R}^N_+)$ satisfies the Dirichlet problem

$$\Delta(\Delta u_N - \partial_N \pi) = 0 \ \text{ in } \mathbb{R}^N_+, \quad \Delta u_N - \partial_N \pi = 0 \ \text{ on } \Gamma.$$

Then, according to Theorem 4.3.4, we have $\Delta u_N - \partial_N \pi = \mu \in \mathcal{A}^\Delta_{[-1-\ell-N/p]}$. Moreover, we can write $\mu = \Delta \Pi_D \mu$, with $\Pi_D \mu = q \in \mathcal{B}_{[1-\ell-N/p]}$. So, setting $u_N^* = u_N - q$, this time we get $\Delta u_N^* - \partial_N \pi = 0$ in $\mathbb{R}^N_+$, and besides u_N^* is still solution to problem (P).

Note that π is unchanged with u_N^*, because $\Delta q = \mu = 0$ on Γ.

Thus, if we set $\boldsymbol{u}^* = (\boldsymbol{u}', u_N^*)$, the pair $(\boldsymbol{u}^*, \pi)$ completely satisfies (5.3.1).

Next, as $\Delta \pi = 0$ in $\mathbb{R}^N_+$, we also have $\Delta \operatorname{div} \boldsymbol{u}^* = 0$ in $\mathbb{R}^N_+$. Moreover, from the boundary condition in (R), we obtain $\operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ. Then, with the boundary condition in (P), we can write

$$\operatorname{div} \boldsymbol{u}^* = \operatorname{div}' \boldsymbol{u}' + \partial_N u_N^* = \operatorname{div}' \boldsymbol{g}' - \operatorname{div}' \boldsymbol{g}' = 0 \quad \text{on } \Gamma.$$

So, we have $\operatorname{div} \boldsymbol{u}^* \in W_\ell^{0,p}(\mathbb{R}^N_+)$, which satisfies the Dirichlet problem

$$\Delta \operatorname{div} \boldsymbol{u}^* = 0 \ \text{ in } \mathbb{R}^N_+, \quad \operatorname{div} \boldsymbol{u}^* = 0 \ \text{ on } \Gamma.$$

Then, according to Theorem 4.3.7, we have $\operatorname{div}\boldsymbol{u}^* = \nu \in \mathcal{A}^{\Delta}_{[-\ell-N/p]}$. If we take for instance $r(x) = \int_0^{x_1} \nu(t,\, x_2, \ldots, x_N)\,\mathrm{d}t$, we have $\nu = \partial_1 r$ and thus $\nu = \operatorname{div}\boldsymbol{r}$, with $\boldsymbol{r} = (r,\, 0, \ldots, 0)$. Setting $\boldsymbol{u}^\dagger = \boldsymbol{u}^* - \boldsymbol{r}$, we get $\operatorname{div}\boldsymbol{u}^\dagger = 0$ in $\mathbb{R}^N_+$ and, as $r \in \mathcal{A}^{\Delta}_{[1-\ell-N/p]}$, we still have $u_1^\dagger = u_1 - r$ solution to the first component of the equations (5.3.1) and (5.3.3). Consequently, the pair $(\boldsymbol{u}^\dagger,\, \pi)$ now completely satisfies the problem (5.3.1)–(5.3.3). $\square$

Now, we can give the general result:

Theorem 5.3.2. *Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \ldots, \ell\} \quad \text{and} \quad N/p \notin \{1, \ldots, -\ell\}. \tag{5.3.8}$$

For any $\boldsymbol{f} \in \boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^N_+)$, $h \in W^{0,\,p}_{\ell}(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,\,p}_{\ell}(\Gamma)$, satisfying the compatibility condition

$$\begin{aligned}\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1+\ell-N/p']}, \qquad & \langle \boldsymbol{f} - \nabla h,\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^N_+)\times \mathring{\boldsymbol{W}}^{1,\,p'}_{-\ell}(\mathbb{R}^N_+)} + \\ & + \langle \operatorname{div}\boldsymbol{f},\, \Pi_D \operatorname{div}'\boldsymbol{\varphi}' + \Pi_N \partial_N \varphi_N\rangle_{W^{-2,\,p}_{\ell}(\mathbb{R}^N_+)\times \mathring{W}^{2,\,p'}_{-\ell}(\mathbb{R}^N_+)} + \\ & + \langle \boldsymbol{g},\, \partial_N \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{1-1/p,\,p}_{\ell}(\Gamma)\times \boldsymbol{W}^{-1/p',\,p'}_{-\ell}(\Gamma)} = 0,\end{aligned} \tag{5.3.9}$$

problem (S_D) admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^N_+) \times W^{0,\,p}_{\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{S}^D_{[1-\ell-N/p]}$, and there exists a constant C such that

$$\inf_{(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}^D_{[1-\ell-N/p]}} \Big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^N_+)} + \|\pi + \mu\|_{W^{0,\,p}_{\ell}(\mathbb{R}^N_+)}\Big) \leqslant$$
$$C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^N_+)} + \|h\|_{W^{0,\,p}_{\ell}(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,\,p}_{\ell}(\Gamma)}\right).$$

Proof. (i) First, we still assume that $\ell < 0$.

We write $\boldsymbol{f} = \operatorname{div}\mathbb{F}$, where $\mathbb{F} = (\mathbf{F}_i)_{1\leqslant i\leqslant N} \in \boldsymbol{W}^{0,\,p}_{\ell}(\mathbb{R}^N_+)^N$, with the estimate

$$\|\mathbb{F}\|_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathbb{R}^N_+)^N} \leqslant C\, \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^N_+)}.$$

Let us respectively denote by $\tilde{\mathbb{F}} = (\tilde{\mathbf{F}}_i)_{1\leqslant i\leqslant N} \in \boldsymbol{W}^{0,\,p}_{\ell}(\mathbb{R}^N)^N$ and $\tilde{h} \in W^{0,\,p}_{\ell}(\mathbb{R}^N)$ the extensions by 0 of $\mathbb{F}$ and h to $\mathbb{R}^N$. By Theorem 4.2.1, we know that there exists $(\tilde{\boldsymbol{u}},\, \tilde{\pi}) \in \boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^N) \times W^{0,\,p}_{\ell}(\mathbb{R}^N)$ solution to the problem

$$(\tilde{S}): \qquad -\Delta\tilde{\boldsymbol{u}} + \nabla\tilde{\pi} = \operatorname{div}\tilde{\mathbb{F}} \;\text{ and }\; \operatorname{div}\tilde{\boldsymbol{u}} = \tilde{h} \quad \text{in } \mathbb{R}^N.$$

Consequently, we can reduce the system (S_D) to the homogeneous problem

$$(S^\sharp): \qquad -\Delta\boldsymbol{v} + \nabla\vartheta = \boldsymbol{0} \;\text{ and }\; \operatorname{div}\boldsymbol{v} = 0 \;\text{ in } \mathbb{R}^N_+, \quad \boldsymbol{v} = \boldsymbol{g}^\sharp \;\text{ on } \Gamma,$$

where we have set $\boldsymbol{g}^\sharp = \boldsymbol{g} - \tilde{\boldsymbol{u}}|_\Gamma \in \boldsymbol{W}_\ell^{1-1/p,p}(\Gamma)$. Next, thanks to Lemma 5.3.1, we know that $(S^\sharp)$ admits a solution $(\boldsymbol{v},\, \vartheta) \in \boldsymbol{W}_\ell^{1,p}(\mathbb{R}_+^N) \times W_\ell^{0,p}(\mathbb{R}_+^N)$. Then, $(\boldsymbol{u},\, \pi) = (\boldsymbol{v} + \tilde{\boldsymbol{u}}|_{\mathbb{R}_+^N},\, \vartheta + \tilde{\pi}|_{\mathbb{R}_+^N}) \in \boldsymbol{W}_\ell^{1,p}(\mathbb{R}_+^N) \times W_\ell^{0,p}(\mathbb{R}_+^N)$ is solution to (S_D).

(ii) We now assume that $\ell > 0$.

We will reason by duality from the case $\ell < 0$. So, we have established that, under hypothesis (5.3.8), the Stokes operator

$$\begin{aligned} T :\ \big(\, \mathring{\boldsymbol{W}}_\ell^{1,p}(\mathbb{R}_+^N) \times W_\ell^{0,p}(\mathbb{R}_+^N)\big)/\mathcal{S}^D_{[1-\ell-N/p]} &\longrightarrow \boldsymbol{W}_\ell^{-1,p}(\mathbb{R}_+^N) \times W_\ell^{0,p}(\mathbb{R}_+^N) \\ (\boldsymbol{u},\, \pi) &\longmapsto (-\Delta \boldsymbol{u} + \nabla \pi,\, -\operatorname{div} \boldsymbol{u}) \end{aligned}$$

is an isomorphism for any integer $\ell < 0$ and real number $p > 1$. Thus, replacing p by p' and $-\ell$ by ℓ, we deduce that its adjoint operator

$$T^* : \mathring{\boldsymbol{W}}_\ell^{1,p}(\mathbb{R}_+^N) \times W_\ell^{0,p}(\mathbb{R}_+^N) \longrightarrow \big(\boldsymbol{W}_\ell^{-1,p}(\mathbb{R}_+^N) \times W_\ell^{0,p}(\mathbb{R}_+^N)\big) \perp \mathcal{S}^D_{[1+\ell-N/p']}$$

is an isomorphism for any integer $\ell > 0$ and real number $p > 1$, always under hypothesis (5.3.8). Moreover, by a density argument, we can readily show that

$$T^*(\boldsymbol{v},\, \vartheta) = (-\Delta \boldsymbol{v} + \nabla \vartheta,\, -\operatorname{div} \boldsymbol{v}).$$

So, we have proved that for any $\ell > 0$, problem (S_D) with $\boldsymbol{g} = \boldsymbol{0}$ admits a unique solution provided $(\boldsymbol{f},\, h) \perp \mathcal{S}^D_{[1+\ell-N/p']}$.

Now, it remains to show that the general problem (S_D) can be reduced to the particular case with $\boldsymbol{g} = \boldsymbol{0}$, by means of a lifting function; and then that the orthogonality condition on the lifted problem is equivalent to the compatibility condition (5.3.9).

First, by Lemma 1.3.1, there exists a lifting function $\boldsymbol{u}_g \in \boldsymbol{W}_\ell^{1,p}(\mathbb{R}_+^N)$ of $\boldsymbol{g}$, *i.e.* $\boldsymbol{u}_g = \boldsymbol{g}$ on Γ, such that

$$\|\boldsymbol{u}_g\|_{\boldsymbol{W}_\ell^{1,p}(\mathbb{R}_+^N)} \ \leqslant\ C\, \|\boldsymbol{g}\|_{\boldsymbol{W}_\ell^{1-1/p,p}(\Gamma)}.$$

Set $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_g$, then problem (S_D) is equivalent to the following, with homogeneous boundary conditions:

$$(S^\star)\ \begin{cases} -\Delta \boldsymbol{v} + \nabla \pi &= \boldsymbol{f} + \Delta \boldsymbol{u}_g \quad \text{in } \mathbb{R}_+^N, \\ \operatorname{div} \boldsymbol{v} &= h - \operatorname{div} \boldsymbol{u}_g \quad \text{in } \mathbb{R}_+^N, \\ \boldsymbol{v} &= \boldsymbol{0} \quad \text{on } \Gamma. \end{cases}$$

So, provided $(\boldsymbol{f} + \Delta \boldsymbol{u}_g,\, -h + \operatorname{div} \boldsymbol{u}_g) \perp \mathcal{S}^D_{[1+\ell-N/p']}$, we know that $(S^\star)$ admits a unique solution. This condition is written in the following way:

$$\forall (\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^D_{[1+\ell-N/p']}, \quad \langle \boldsymbol{f},\, \boldsymbol{\lambda}\rangle + \langle \Delta \boldsymbol{u}_g,\, \boldsymbol{\lambda}\rangle - \langle h,\, \mu\rangle + \langle \operatorname{div} \boldsymbol{u}_g,\, \mu\rangle = 0.$$

Moreover, we have the Green formula

$$\begin{aligned}\langle \Delta \boldsymbol{u}_g,\, \boldsymbol{\lambda}\rangle_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^N_+)\times \mathring{\boldsymbol{W}}^{1,\,p'}_{-\ell}(\mathbb{R}^N_+)} &= \int_{\mathbb{R}^N_+} \boldsymbol{u}_g \cdot \Delta \boldsymbol{\lambda}\,\mathrm{d}x + \langle \boldsymbol{g},\, \partial_N \boldsymbol{\lambda}\rangle_\Gamma\,,\\ &= \int_{\mathbb{R}^N_+} \boldsymbol{u}_g \cdot \Delta \boldsymbol{\lambda}\,\mathrm{d}x + \langle \boldsymbol{g}',\, \partial_N \boldsymbol{\lambda}'\rangle_\Gamma\,,\end{aligned}$$

because $\partial_N \boldsymbol{\lambda}_N = 0$ on Γ, according to the definition of the kernel. Next, we have another Green formula

$$\langle \operatorname{div} \boldsymbol{u}_g,\, \mu\rangle_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)} = -\int_{\mathbb{R}^N_+} \boldsymbol{u}_g \cdot \nabla \mu\,\mathrm{d}x - \langle g_N,\, \mu\rangle_\Gamma\,.$$

Finally, since $-\Delta\boldsymbol{\lambda} + \nabla \mu = \boldsymbol{0}$, we have

$$\int_{\mathbb{R}^N_+} \boldsymbol{u}_g \cdot \Delta \boldsymbol{\lambda}\,\mathrm{d}x - \int_{\mathbb{R}^N_+} \boldsymbol{u}_g \cdot \nabla \mu\,\mathrm{d}x = 0,$$

then we get a first formulation for this compatibility condition:

$$\forall (\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^D_{[1+\ell-N/p']}, \quad \langle \boldsymbol{f},\, \boldsymbol{\lambda}\rangle - \langle h,\, \mu\rangle + \langle \boldsymbol{g}',\, \partial_N \boldsymbol{\lambda}'\rangle_\Gamma - \langle g_N,\, \mu\rangle_\Gamma = 0.$$

Now, according to the characterization (5.2.1)–(5.2.2), we can replace each pair $(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^D_{[1+\ell-N/p']}$ by $\big(\boldsymbol{\varphi} - \nabla(\Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_N \varphi_N),\, -\operatorname{div} \boldsymbol{\varphi}\big)$, where $\boldsymbol{\varphi}$ belongs to $\boldsymbol{\mathcal{A}}^\Delta_{[1+\ell-N/p']}$. Then we have

$$\begin{aligned}\langle \boldsymbol{f},\, \boldsymbol{\lambda}\rangle_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^N_+)\times \mathring{\boldsymbol{W}}^{1,\,p'}_{-\ell}(\mathbb{R}^N_+)} &= \langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle - \langle \boldsymbol{f},\, \nabla(\Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_N \varphi_N)\rangle\,,\\ &= \langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle + \langle \operatorname{div} \boldsymbol{f},\, \Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_N \varphi_N\rangle\,,\end{aligned}$$

because $(\Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_N \varphi_N)|_\Gamma = 0$. Likewise,

$$\begin{aligned}\langle h,\, \mu\rangle_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)} &= \langle h,\, -\operatorname{div} \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)}\\ &= \langle \nabla h,\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^N_+)\times \mathring{\boldsymbol{W}}^{1,\,p'}_{-\ell}(\mathbb{R}^N_+)}\,.\end{aligned}$$

Moreover, we can remark that on the one hand $\mu = -\partial_N \varphi_N$ on Γ and on the other hand, according to (2.3.5) and (2.3.6), we have $\partial_N \boldsymbol{\lambda}' = \partial_N \boldsymbol{\varphi}'$ on Γ, hence the equivalent formulation:

$$\begin{aligned}&\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^\Delta_{[1+\ell-N/p']},\\ &\qquad \langle \boldsymbol{f} - \nabla h,\, \boldsymbol{\varphi}\rangle + \langle \operatorname{div} \boldsymbol{f},\, \Pi_D \operatorname{div}' \boldsymbol{\varphi}' + \Pi_N \partial_N \varphi_N\rangle + \langle \boldsymbol{g},\, \partial_N \boldsymbol{\varphi}\rangle_\Gamma = 0,\end{aligned}$$

i.e. the compatibility condition (5.3.9). □

5.4 Strong solutions and regularity

In this section, we are interested in the existence of strong solutions, *i.e.* of solutions $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{2,\,p}_{\ell+1}(\mathbb{R}^N_+) \times W^{1,\,p}_{\ell+1}(\mathbb{R}^N_+)$; and more generaly, in the regularity of solutions to the Stokes system (S_D) according to the data.

Theorem 5.4.1. *Let $\ell \in \mathbb{Z}$ and $m \geqslant 1$ be two integers and assume that*

$$N/p' \notin \{1, \dots, \ell+1\} \quad \text{and} \quad N/p \notin \{1, \dots, -\ell-m\}. \tag{5.4.1}$$

For any $\boldsymbol{f} \in \boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^N_+)$, $h \in W^{m,\,p}_{m+\ell}(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$, satisfying the compatibility condition (5.3.9), problem (S_D) admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^N_+) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{S}^D_{[1-\ell-N/p]}$, and there exists a constant C such that

$$\inf_{(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^D_{[1-\ell-N/p]}} \left(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^N_+)} + \|\pi+\mu\|_{W^{m,\,p}_{m+\ell}(\mathbb{R}^N_+)} \right) \leqslant$$
$$C \left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^N_+)} + \|h\|_{W^{m,\,p}_{m+\ell}(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{m+1-1/p,\,p}_{m+\ell}(\Gamma)} \right).$$

We have already proved this result for $\ell = 0$ and $\ell = -1$ in the previous chapter (see Corollaries 4.5.5 and 4.5.7). We will use similar arguments for the other negative weights, with the aim of minimizing the set of critical values, thanks to the known results on the harmonic and biharmonic operators in the half-space. Then, for the positive weights, we will use a regularity argument to avoid the compatibility conditions which would naturally appear in the auxiliary problems with the previous method.

At first, we adapt Lemma 5.3.1 and its proof for more regular data.

Lemma 5.4.2. *Let $\ell \leqslant -2$ and $m \geqslant 1$ be two integers and assume that*

$$N/p \notin \{1, \dots, -\ell-m\}. \tag{5.4.2}$$

For any $\boldsymbol{g} \in \boldsymbol{W}^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$, the Stokes problem (5.3.1)–(5.3.3) has a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^N_+) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{S}^D_{[1-\ell-N/p]}$, with the corresponding estimate.

Proof. Point (i) is clearly unchanged with respect to the proof of Lemma 5.3.1. Since $\boldsymbol{g} \in \boldsymbol{W}^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$, according to Lemma 2.3.10, we know that under hypothesis (5.4.2), problem (P) has a solution $u_N \in W^{m+1,\,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[1-\ell-N/p]}$. Hence $\Delta u_N|_\Gamma \in W^{m-1-1/p,\,p}_{m+\ell}(\Gamma)$, and then under hypothesis (5.4.2), problem (Q) has a solution $\pi \in W^{m,\,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{N}^\Delta_{[-\ell-N/p]}$ (see Theorem 4.3.3, for $m=1$; and Theorem 1.4.4, for $m \geqslant 2$). Hence $\nabla'\pi \in W^{m-1,\,p}_{m+\ell}(\mathbb{R}^N_+)^{N-1}$ and then under hypothesis (5.4.2), problem (R) has a solution $\boldsymbol{u}' \in W^{m+1,\,p}_{m+\ell}(\mathbb{R}^N_+)^{N-1}$, unique up to an element of $(\mathcal{A}^\Delta_{[1-\ell-N/p]})^{N-1}$ (see Theorem 1.4.2). Likewise, point (iii) is unchanged with respect to the proof of Lemma 5.3.1. □

Proof of Theorem 5.4.1. (i) Assume that $\ell \leqslant -2$. The proof is quite similar to the one of Theorem 5.3.2. Here again, the only question is the surjectivity of the Stokes operator for such data. For that, we must simply replace Theorem 4.2.1 by Theorem 4.2.2 and Lemma 5.3.1 by Lemma 5.4.2 in the proof of the existence of a solution for negative weights in Theorem 5.3.2.

(ii) Assume that $\ell > 0$. We simply extend the regularity argument used in Chapter 4 (see Corollaries 4.5.5 and 4.5.7) for the cases $\ell = 0$ and $\ell = -1$. Now, the hypothesis (5.4.1) is reduced to

$$N/p' \notin \{1, \dots, \ell + 1\}. \tag{5.4.3}$$

Since $N/p' \neq \ell+1$, we have the imbedding $W^{m-1,p}_{m+\ell}(\mathbb{R}^N_+) \hookrightarrow W^{-1,p}_{\ell}(\mathbb{R}^N_+)$, moreover, $W^{m,p}_{m+\ell}(\mathbb{R}^N_+) \hookrightarrow W^{0,p}_{\ell}(\mathbb{R}^N_+)$ and $W^{m+1-1/p,p}_{m+\ell}(\Gamma) \hookrightarrow W^{1-1/p,p}_{\ell}(\Gamma)$ hold. So, thanks to Theorem 5.3.2, we know that problem (S_D) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_{\ell}(\mathbb{R}^N_+) \times W^{0,p}_{\ell}(\mathbb{R}^N_+)$. We will show by induction that

$$\begin{aligned}(\boldsymbol{f}, h, \boldsymbol{g}) \in \boldsymbol{W}^{m-1,p}_{m+\ell}(\mathbb{R}^N_+) \times W^{m,p}_{m+\ell}(\mathbb{R}^N_+) \times \boldsymbol{W}^{m+1-1/p,p}_{m+\ell}(\Gamma)&\\ \Rightarrow (\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+1,p}_{m+\ell}(\mathbb{R}^N_+) \times W^{m,p}_{m+\ell}(\mathbb{R}^N_+).&\end{aligned} \tag{5.4.4}$$

For $m = 0$, (5.4.4) is true. Now assume that (5.4.4) is true for $0, 1, \dots, m$ and suppose that $(\boldsymbol{f}, h, \boldsymbol{g}) \in \boldsymbol{W}^{m,p}_{m+\ell+1}(\mathbb{R}^N_+) \times W^{m+1,p}_{m+\ell+1}(\mathbb{R}^N_+) \times \boldsymbol{W}^{m+2-1/p,p}_{m+\ell+1}(\Gamma)$. Let us prove that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+2,p}_{m+\ell+1}(\mathbb{R}^N_+) \times W^{m+1,p}_{m+\ell+1}(\mathbb{R}^N_+)$. Since we also have the imbeddings $W^{m,p}_{m+\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{m-1,p}_{m+\ell}(\mathbb{R}^N_+)$, $W^{m+1,p}_{m+\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{m,p}_{m+\ell}(\mathbb{R}^N_+)$ and $W^{m+2-1/p,p}_{m+\ell+1}(\Gamma) \hookrightarrow W^{m+1-1/p,p}_{m+\ell}(\Gamma)$, according to the induction hypothesis, we can deduce that the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{m+1,p}_{m+\ell}(\mathbb{R}^N_+) \times W^{m,p}_{m+\ell}(\mathbb{R}^N_+)$. Now, for any $i \in \{1, \dots, N-1\}$, we have

$$\begin{aligned}&-\Delta(\varrho\, \partial_i \boldsymbol{u}) + \nabla(\varrho\, \partial_i \pi)\\ &\qquad = \varrho\, \partial_i \boldsymbol{f} + \frac{2}{\varrho}\, x.\nabla \partial_i \boldsymbol{u} + \left(\frac{N-1}{\varrho} + \frac{1}{\varrho^3}\right) \partial_i \boldsymbol{u} + \frac{1}{\varrho}\, x\, \partial_i \pi.\end{aligned}$$

Thus, $-\Delta(\varrho\, \partial_i \boldsymbol{u}) + \nabla(\varrho\, \partial_i \pi) \in \boldsymbol{W}^{m-1,p}_{m+\ell}(\mathbb{R}^N_+)$. Moreover,

$$\operatorname{div}(\varrho\, \partial_i \boldsymbol{u}) = \frac{1}{\varrho}\, x\, \partial_i \boldsymbol{u} + \varrho\, \partial_i h.$$

Thus, $\operatorname{div}(\varrho\, \partial_i \boldsymbol{u}) \in W^{m,p}_{m+\ell}(\mathbb{R}^N_+)$. We also have $\gamma_0(\varrho\, \partial_i \boldsymbol{u}) = \varrho'\, \partial_i \gamma_0 \boldsymbol{u} = \varrho'\, \partial_i \boldsymbol{g} \in \boldsymbol{W}^{m+1-1/p,p}_{m+\ell}(\Gamma)$. So, by induction hypothesis, we can deduce that

$$\forall i \in \{1, \dots, N-1\}, \quad (\partial_i \boldsymbol{u}, \partial_i \pi) \in \boldsymbol{W}^{m+1,p}_{m+\ell+1}(\mathbb{R}^N_+) \times W^{m,p}_{m+\ell+1}(\mathbb{R}^N_+).$$

It remains to prove that $(\partial_N \boldsymbol{u}, \partial_N \pi) \in \boldsymbol{W}^{m+1,p}_{m+\ell+1}(\mathbb{R}^N_+) \times W^{m,p}_{m+\ell+1}(\mathbb{R}^N_+)$. For that, let us observe that for any $i \in \{1, \dots, N-1\}$, we have

$$\begin{array}{lll}\partial_i \partial_N \boldsymbol{u} & = \partial_N \partial_i \boldsymbol{u} & \in \boldsymbol{W}^{m,p}_{m+\ell+1}(\mathbb{R}^N_+),\\ \partial^2_N u_i & = -\Delta' u_i + \partial_i \pi - f_i & \in W^{m,p}_{m+\ell+1}(\mathbb{R}^N_+),\\ \partial^2_N u_N & = \partial_N h - \partial_N \operatorname{div}' \boldsymbol{u}' & \in W^{m,p}_{m+\ell+1}(\mathbb{R}^N_+),\\ \partial_N \pi & = f_N + \Delta u_N & \in W^{m,p}_{m+\ell+1}(\mathbb{R}^N_+).\end{array}$$

Hence, $\nabla(\partial_N \boldsymbol{u}) \in \boldsymbol{W}^{m,p}_{m+\ell+1}(\mathbb{R}^N_+)^N$ and knowing that $\partial_N \boldsymbol{u} \in \boldsymbol{W}^{m,p}_{m+\ell}(\mathbb{R}^N_+)$, we can deduce that $\partial_N \boldsymbol{u} \in \boldsymbol{W}^{m+1,p}_{m+\ell+1}(\mathbb{R}^N_+)$, according to definition (1.2.1). Consequently, we have $\nabla \boldsymbol{u} \in \boldsymbol{W}^{m+1,p}_{m+\ell+1}(\mathbb{R}^N_+)^N$. Likewise, $\nabla\pi \in \boldsymbol{W}^{m,p}_{m+\ell+1}(\mathbb{R}^N_+)$ and finally, we can conclude that $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{m+2,p}_{m+\ell+1}(\mathbb{R}^N_+) \times W^{m+1,p}_{m+\ell+1}(\mathbb{R}^N_+)$. □

5.5 Very weak solutions

The aim of this section is to study the homogeneous Stokes system (5.3.1)–(5.3.3) with singular data on the boundary. In Section 4.6, we gave a meaning to singular data for the Stokes problem in the half-space. More precisely, we showed that a boundary condition $\boldsymbol{u} = \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)$ is meaningful. Next, we solved problem (5.3.1)–(5.3.3) in the two cases $\ell = 0$ or $\ell = 1$. Here again, we will extend these results to the other weights, introducing the question of the kernel and, by duality, the compatibility condition in the proof. However, let us notice that we introduced and proved some preliminary definitions and properties in Section 4.6 with a view to the general case, *i.e.* for all $\ell \in \mathbb{Z}$. So we will directly use them in the proof of the following result which generalizes Theorems 4.6.6 and 4.6.8. Here again the reasoning is quite similar.

Theorem 5.5.1. *Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \dots, \ell - 1\} \quad \text{and} \quad N/p \notin \{1, \dots, -\ell + 1\}. \tag{5.5.1}$$

For any $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)$, satisfying the compatibility condition

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1+\ell-N/p']}, \qquad \langle \boldsymbol{g},\, \partial_N \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma) \times \boldsymbol{W}^{1/p,p'}_{-\ell+1}(\Gamma)} = 0, \tag{5.5.2}$$

problem (5.3.1)–(5.3.3) admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+) \times W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{S}^D_{[1-\ell-N/p]}$, and there exists a constant C such that

$$\inf_{(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^D_{[1-\ell-N/p]}} \Big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+)} + \|\pi + \mu\|_{W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)} \Big) \leqslant C\, \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)}.$$

Proof. `Step 1:` we assume that $g_N = 0$.

(i) Let us first show that if the pair $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+) \times W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)$ satisfies (5.3.1) and (5.3.2), then we have $\boldsymbol{u} \in \boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$ and thus the boundary condition (5.3.3) makes sense. With this aim, by means of the density of $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ in $\boldsymbol{X}_\ell(\mathbb{R}^N_+)$, observe that if $\pi \in W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)$, then we have $\nabla\pi \in \boldsymbol{X}'_\ell(\mathbb{R}^N_+)$ and

$$\|\nabla\pi\|_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)} \leqslant C\, \|\pi\|_{W^{-1,p}_{\ell-1}(\mathbb{R}^N_+)},$$

So, we have $\Delta \boldsymbol{u} \in \boldsymbol{X}'_\ell(\mathbb{R}^N_+)$ and the trace $\gamma_0 \boldsymbol{u} \in \boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)$.

(ii) Now, let us show that the problem (5.3.1)–(5.3.3) with $g_N = 0$ is equivalent to the variational formulation: Find $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)$ such that

$$\begin{aligned}
&\forall \boldsymbol{v} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+), \quad \forall \vartheta \in W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+),\\
&\langle \boldsymbol{u}, -\Delta \boldsymbol{v} + \nabla\vartheta\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} - \langle \pi, \operatorname{div} \boldsymbol{v}\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)} \\
&\qquad\qquad = \langle \boldsymbol{g}, \partial_N \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{-\ell+1}(\Gamma)}\,.
\end{aligned} \tag{5.5.3}$$

Indeed, let $(\boldsymbol{u},\, \pi)$ be a solution to (5.3.1)–(5.3.3) with $g_N = 0$; by means of the Green formula (4.6.6), we get for all $\boldsymbol{v} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+)$,

$$\begin{aligned}
\langle -\Delta \boldsymbol{u} + \nabla\pi, \boldsymbol{v}\rangle_{\boldsymbol{X}'_\ell\times \boldsymbol{X}_\ell} &= - \langle \boldsymbol{u}, \Delta \boldsymbol{v}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} - \\
&- \langle \boldsymbol{g}, \partial_N \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{-\ell+1}(\Gamma)} - \langle \pi, \operatorname{div} \boldsymbol{v}\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} = 0.
\end{aligned}$$

Moreover, using the density of the functions of $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ with divergence zero in $\boldsymbol{T}_{\ell,\,\sigma}(\mathbb{R}^N_+)$, we obtain for all $\vartheta \in W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$,

$$\begin{aligned}
\langle \boldsymbol{u}, \nabla\vartheta\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} &= - \langle \operatorname{div} \boldsymbol{u}, \vartheta\rangle_{W^{0,\,p}_{\ell}(\mathbb{R}^N_+)\times W^{0,\,p'}_{-\ell}(\mathbb{R}^N_+)} - \\
&- \langle u_N, \vartheta\rangle_{W^{-1/p,\,p}_{\ell-1}(\Gamma)\times W^{1/p,\,p'}_{-\ell+1}(\Gamma)} = 0.
\end{aligned}$$

So we show that $(\boldsymbol{u},\, \pi)$ satisfies the variational formulation (5.5.3). Conversely, we readily prove that if $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)$ satisfies the variational formulation (5.5.3), then $(\boldsymbol{u},\, \pi)$ is a solution to problem (5.3.1)–(5.3.3).

(iii) Next, let us solve problem (5.5.3). By Theorem 5.4.1, we know that under hypothesis (5.5.1), for all $(\boldsymbol{f},\, h) \in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \perp \mathcal{S}^D_{[1-\ell-N/p]}$, there exists a unique $(\boldsymbol{v},\, \vartheta) \in \boldsymbol{M}_\ell(\mathbb{R}^N_+) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)/\mathcal{S}^D_{[1+\ell-N/p']}$ solution to

$$-\Delta \boldsymbol{v} + \nabla\vartheta = \boldsymbol{f} \quad\text{and}\quad \operatorname{div} \boldsymbol{v} = h \quad\text{in } \mathbb{R}^N_+, \qquad \boldsymbol{v} = \boldsymbol{0} \quad\text{on } \Gamma,$$

with the estimate

$$\|(\boldsymbol{v},\, \vartheta)\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)/\mathcal{S}^D_{[1+\ell-N/p']}} \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} + \|h\|_{W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}\right).$$

Consider the linear form $T:\ (\boldsymbol{f},\, h) \longmapsto \langle \boldsymbol{g},\, \partial_N \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{-\ell+1}(\Gamma)}$ defined on $\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \perp \mathcal{S}^D_{[1-\ell-N/p]}$. According to (5.5.2), we have for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{A}}^{\Delta}_{[1+\ell-N/p']}$, or equivalently, for any $(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^D_{[1+\ell-N/p']}$,

$$\begin{aligned}
|T(\boldsymbol{f},\, h)| = &\ \left|\langle \boldsymbol{g}, \partial_N(\boldsymbol{v} + \boldsymbol{\varphi})\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1/p,\,p'}_{-\ell+1}(\Gamma)}\right| \\
\leqslant &\ C\, \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)}\, \|\boldsymbol{v} + \boldsymbol{\varphi}\|_{W^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}.
\end{aligned}$$

Thus

$$\begin{aligned}|T(\boldsymbol{f},\,h)| &\leqslant\; C\,\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)}\,\|(\boldsymbol{v},\,\vartheta)\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)/\mathcal{S}^D_{[1+\ell-N/p']}}\\ &\leqslant\; C\,\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)}\,\Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}+\|h\|_{W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}\Big)\,.\end{aligned}$$

In other words, T is continuous on $\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\perp\mathcal{S}^D_{[1-\ell-N/p]}$, and according to the Riesz representation theorem, we can deduce that there exists a unique $(\boldsymbol{u},\,\pi)\in\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)\times W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)/\mathcal{S}^D_{[1-\ell-N/p]}$ which is the dual space of $\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\perp\mathcal{S}^D_{[1-\ell-N/p]}$, such that

$$\begin{aligned}&\forall(\boldsymbol{f},\,h)\in\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+),\\ &\;T(\boldsymbol{f},\,h)=\langle\boldsymbol{u},\boldsymbol{f}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)\times\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}+\langle\pi,-h\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)\times\overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}\,,\end{aligned}$$

i.e. the pair $(\boldsymbol{u},\,\pi)$ satisfies (5.5.3) and the kernel of this operator is $\mathcal{S}^D_{[1-\ell-N/p]}$.

`Step 2:` we now can drop the hypothesis $g_N=0$.

For any $\boldsymbol{g}\in\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\perp\mathcal{N}^{\Delta}_{[\ell-N/p']}$, according to Theorem 4.3.3, we know that under hypothesis (5.5.1), there exists $\psi\in W^{1,\,p}_{\ell-1}(\mathbb{R}^N_+)$ unique up to an element of $\mathcal{N}^{\Delta}_{[2-\ell-N/p]}$, solution to the following Neumann problem:

$$\Delta\psi=0\ \text{ in }\mathbb{R}^N_+,\quad \partial_N\psi=g_N\ \text{ on }\Gamma.$$

Besides, we immediately notice that the orthogonality condition $\boldsymbol{g}\perp\mathcal{N}^{\Delta}_{[\ell-N/p']}$ is equivalent to the compatibility condition (5.5.2). Now, let us set $\boldsymbol{w}=\nabla\psi$ and $\boldsymbol{g}^*=\boldsymbol{g}-\gamma_0\boldsymbol{w}$. Then $\boldsymbol{w}\in\boldsymbol{T}_{\ell,\,\sigma}(\mathbb{R}^N_+)$, with the estimate

$$\|\boldsymbol{w}\|_{\boldsymbol{T}_\ell(\mathbb{R}^N_+)}=\|\boldsymbol{w}\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)}\leqslant C\,\|\boldsymbol{g}\|_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)}.$$

Furthermore, $\boldsymbol{g}^*$ is such that $\boldsymbol{g}^*_N=0$, hence the existence of a unique pair $(\boldsymbol{z},\,\pi)$ which satisfies

$$-\Delta\boldsymbol{z}+\nabla\pi=\boldsymbol{0}\ \text{ and }\ \operatorname{div}\boldsymbol{z}=0\ \text{ in }\mathbb{R}^N_+,\quad \boldsymbol{u}=\boldsymbol{g}^*\ \text{ on }\Gamma.$$

Then the pair $(\boldsymbol{z}+\boldsymbol{w},\,\pi)$ is a solution to problem (5.3.1)–(5.3.3). $\square$

Chapitre 6

The Stokes system with Navier boundary conditions

6.1 Introduction

The motion of a viscous incompressible fluid is described by the Navier-Stokes equations, which are non-linear. The Stokes system is a linear approximation of this model, available for slow motions. In the two previous chapters, we studied this system in a half-space with the classical Dirichlet boundary conditions, which correspond to an adhesion, or non-slip, condition of the fluid on the wall. But recent developments in microfluidic and nanofluidic technologies have renewed interest in the influence of surface roughness on the slip behavior of viscous fluids (see Priezjev and Troian, [41]). This issue have been subjected to discussion for over two centuries by many distinguished scientists who developed the foundations of fluid mechanics, including Bernoulli, Coulomb, Navier, Couette, Poisson, Stokes. There are two basic boundary conditions: Dirichlet boundary conditions (non-slip boundary conditions) and slip boundary conditions (Navier condition). It is intuitively clear that slip boundary conditions is much closer to the observed reality than non-slip boundary conditions whenever the rate of flow is sufficiently strong (turbulent regimes). However, there has been a common believe that even if the Navier slip conditions were correct, the corresponding slip length is likely to be small to influence the motion of macroscopic fluids. Recently, numerous experiments and simulations as well as theoretical studies have shown that the classical non-slip assumption can fail when the walls are sufficiently smooth. Strictly speaking, the slip length characterizing the contact between a fluid and a solid wall in relative motion is influenced by many different factors, among which the intrinsic affinity and commensurability between the liquid and solid molecular size as well as the macroscopic surface roughness caused by imperfections and tiny asperities play a significant role. The aim of this chapter is to investigate the Stokes problem with this type of slip boundary conditions in weighted Sobolev spaces. These type of boundary conditions was recently studied by Babin, Mahalov, Nicolaenko (see [14, 15, 16]) and by Bellout, Neustupa, Penel (see [18, 40]).

For the stokes problem in a domain Ω of $\mathbb{R}^N$,

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \Omega,$$

there are several possibilities of boundary conditions.

The classical homogeneous Dirichlet (non-slip) conditions:

$$\boldsymbol{u}|_{\partial\Omega} = \mathbf{0},$$

when $\partial\Omega$ is a fixed wall. This condition was suggested by Stokes in 1845.

The Navier (slip) conditions:

$$\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad (\mathcal{T} \cdot \boldsymbol{n})_\tau + k\, \boldsymbol{u} = \mathbf{0},$$

where $\mathcal{T}$ is the viscous stress tensor. For the incompressible isotropic fluid the viscous stress tensor has a form

$$\mathcal{T}_{ij}(u) = -\delta_{ij}\, \pi + 2\, \nu\, e_{ij},$$

where $e_{ij}(u) = \frac{1}{2}\, (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$.

Another boundary conditions (in the three dimensional case) we can find in the literature can be expressed by the equations

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} &= \mathbf{0}, \\ \boldsymbol{u} \cdot \boldsymbol{n} &= 0. \end{aligned}$$

In the half space the Navier conditions with $k = 0$ and previous boundary conditions have the same form and can be written in

$$u_N = 0, \quad \partial_N \boldsymbol{u}' = \mathbf{0} \quad \text{ on } \Gamma.$$

We would like to mention generalized impermeability boundary conditions, which we can find in the work of Bellout, Neustupa and Penel [18, 40].

$$\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathbf{curl}^2\, \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$$

on the fixed wall $\partial\Omega$.

In this chapter, we will consider the Stokes system with nonhomogeneous Navier boundary conditions. We will denote it by (S_N) (for Stokes system with Navier conditions):

$$(S_N) \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \mathbb{R}^N_+, \\ \qquad u_N = g_N \quad \text{and} \quad \partial_N \boldsymbol{u}' = \boldsymbol{g}' \quad \text{on } \Gamma, \end{cases}$$

6.2 Generalized solutions

We will first establish the result about the generalized solutions to (S_N) in the homogeneous case. The method is similar to the one employed for the Dirichlet conditions, but the auxiliary problems and the arguments for their resolution are appreciably different.

Homogeneous case

Here, we assume that $\boldsymbol{f} = \boldsymbol{0}$ and $h = 0$.

Proposition 6.2.1. *For any $g_N \in W_0^{1-1/p,\,p}(\Gamma)$ and $\boldsymbol{g}' \in W_0^{-1/p,\,p}(\Gamma)^{N-1}$ such that $\boldsymbol{g}' \perp \mathbb{R}^{N-1}$ if $N \leqslant p'$, the Stokes problem*

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} \qquad \text{in } \mathbb{R}^N_+, \tag{6.2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \mathbb{R}^N_+, \tag{6.2.2}$$

$$u_N = g_N \qquad \text{on } \Gamma, \tag{6.2.3}$$

$$\partial_N \boldsymbol{u}' = \boldsymbol{g}' \qquad \text{on } \Gamma, \tag{6.2.4}$$

has a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, unique if $N > p$, unique up to an element of $\mathbb{R}^{N-1} \times \{0\}^2$ if $N \leqslant p$, with the estimate

$$\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1} \times \{0\}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \leqslant C \left(\|g_N\|_{W_0^{1-1/p,\,p}(\Gamma)} + \|\boldsymbol{g}'\|_{W_0^{-1/p,\,p}(\Gamma)^{N-1}} \right)$$

if $N \leqslant p$, and the same without $\boldsymbol{\chi}$ if $N > p$.

Remark 6.2.2. Before giving the proof, let us notice that this problem is not standard. Indeed, we find the velocity field $\boldsymbol{u}$ in $\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+)$ with a boundary condition $\partial_N \boldsymbol{u}' = \boldsymbol{g}' \in W_0^{-1/p,\,p}(\Gamma)^{N-1}$ for its tangential components.

It is possible because $\Delta^2 \boldsymbol{u} = \boldsymbol{0}$ in $\mathbb{R}^N_+$ and thus $u_i \in Y_{1,\,1}^{1,\,p}(\mathbb{R}^N_+)$ (see page 50). Hence, by Lemma 3.3.8, the trace of $\partial_N u_i$ have a sense in $W_0^{-1/p,\,p}(\Gamma)$. ♢

Proof. (i) Firstly, we reduce system (6.2.1)–(6.2.4) to three problems on the fundamental operators Δ^2 and Δ.

According to (6.2.2) and applying the operators div and Δ to (6.2.1), we get both $\Delta\pi = 0$ and $\Delta^2 \boldsymbol{u} = \boldsymbol{0}$ in $\mathbb{R}^N_+$.

From the boundary condition (6.2.3), we take out

$$\forall i \in \{1,\, 2, \ldots,\, N-1\}, \qquad \partial_i^2 u_N = \partial_i^2 g_N \quad \text{on } \Gamma.$$

In addition, from (6.2.4), we take out

$$\partial_N^2 u_N = \partial_N(\partial_N u_N) = \partial_N(-\operatorname{div}' \boldsymbol{u}') = -\operatorname{div}' \boldsymbol{g}' \quad \text{on } \Gamma,$$

hence, the boundary condition

$$\Delta u_N = \Delta' g_N - \mathrm{div}' \boldsymbol{g}' \quad \text{on } \Gamma,$$

where $\Delta' = \sum_{j=1}^{N-1} \partial_j^2$. So, we get the following biharmonic problem

$$(B): \qquad \Delta^2 u_N = 0 \ \text{ in } \mathbb{R}^N_+, \quad u_N = g_N \ \text{ and } \ \Delta u_N = \Delta' g_N - \mathrm{div}' \boldsymbol{g}' \ \text{ on } \Gamma.$$

Moreover, we have two Neumann problems

$$(N1): \qquad \Delta \pi = 0 \ \text{ in } \mathbb{R}^N_+, \quad \partial_N \pi = \Delta u_N \ \text{ on } \Gamma,$$
$$(N2): \qquad \Delta \boldsymbol{u}' = \nabla' \pi \ \text{ in } \mathbb{R}^N_+, \quad \partial_N \boldsymbol{u}' = \boldsymbol{g}' \ \text{ on } \Gamma.$$

(ii) Now, we will solve these three problems.

`Step 1:` We deal with problem (B). Denoting $z_N = \Delta u_N$, we can split our problem in the following two Dirichlet problems:

$$\Delta z_N = 0 \ \text{ in } \mathbb{R}^N_+, \quad z_N = \Delta' g_N - \mathrm{div}' \boldsymbol{g}' \ \text{ on } \Gamma, \tag{6.2.5}$$
$$\Delta u_N = z_N \ \text{ in } \mathbb{R}^N_+, \quad u_N = g_N \ \text{ on } \Gamma. \tag{6.2.6}$$

Concerning (6.2.5), we notice that $\Delta' g_N - \mathrm{div}' \boldsymbol{g}' \in W_0^{-1-1/p,\,p}(\Gamma)$, then we can apply Theorem 4.3.4 with $\ell = 2$, provided condition (4.3.4) is satisfied, *i.e.* in the present case

$$\forall \varphi \in \mathcal{A}^{\Delta}_{[3-N/p']}, \quad \langle \Delta' g_N - \mathrm{div}' \boldsymbol{g}', \partial_N \varphi \rangle_{W_0^{-1-1/p,\,p}(\Gamma) \times W_0^{2-1/p',\,p'}(\Gamma)} = 0.$$

According to the degree of polynomials in $\mathcal{A}^{\Delta}_{[3-N/p']}$, this condition boils down to $\boldsymbol{g}' \perp (\mathcal{P}_{[1-N/p']})^{N-1}$, which is precisely the assumption of Proposition 6.2.1. Thus problem (6.2.5) has a unique solution $z_N \in W_0^{-1,\,p}(\mathbb{R}^N_+)$.

Concerning (6.2.6), we can apply Theorem 1.4.1 with $\ell = 0$ and without any condition since $\mathcal{A}^{\Delta}_{[1-N/p']} = \{0\}$. Thus problem (6.2.6) has a unique solution $u_N \in W_0^{1,\,p}(\mathbb{R}^N_+)$.

`Step 2:` We study now problem $(N1)$. Since $\Delta u_N \in W_0^{-1,\,p}(\mathbb{R}^N_+)$, it is necessary to check that the trace $\gamma_0 \Delta u_N$ has meaning. From definitions of $Y_\ell(\mathbb{R}^N_+)$ and $Y_{\ell,1}(\mathbb{R}^N_+)$ in Section 4.3, since $\Delta u_N \in W_0^{-1,\,p}(\Omega)$ and $\Delta^2 u_N = 0$, it follows that $\Delta u_N \in Y_2(\mathbb{R}^N_+)$ and $\Delta u_N \in Y_{2,1}(\mathbb{R}^N_+)$. Then, according to Lemma 4.3.6, we have $\Delta u_N \in W_0^{-1-1/p,\,p}(\Gamma)$. Now we can apply Theorem 4.3.2, provided the compatibility condition (4.3.2) is satisfied, *i.e.* in the present case

$$\forall \varphi \in \mathcal{N}^{\Delta}_{[2-N/p']}, \quad \langle \Delta u_N, \varphi \rangle_{W_0^{-1-1/p,\,p}(\Gamma) \times W_0^{2-1/p',\,p'}(\Gamma)} = 0.$$

But, according to the degree of polynomials in $\mathcal{N}^{\Delta}_{[2-N/p']}$, it is clear that in fact this condition vanishes. It implies the existence of a unique solution $\pi \in L^p(\mathbb{R}^N_+)$ to problem $(N1)$.

`Step 3`: Finally, we are dealing with problem $(N2)$. We split it in two parts:

$$\Delta \boldsymbol{v}' = \nabla' \pi \quad \text{in } \mathbb{R}^N_+, \qquad \partial_N \boldsymbol{v}' = \mathbf{0} \quad \text{on } \Gamma, \tag{6.2.7}$$

and

$$\Delta \boldsymbol{z}' = \mathbf{0} \quad \text{in } \mathbb{R}^N_+, \qquad \partial_N \boldsymbol{z}' = \boldsymbol{g}' \quad \text{on } \Gamma. \tag{6.2.8}$$

To solve (6.2.7), we introduce the auxiliary problem

$$\Delta w = \pi \quad \text{in } \mathbb{R}^N_+, \qquad \partial_N w = 0 \quad \text{on } \Gamma. \tag{6.2.9}$$

Since we have $\pi \in L^p(\mathbb{R}^N_+)$, we can apply Theorem 1.4.4 which yields a solution $w \in W^{2,p}_0(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{N}^{\Delta}_{[2-N/p]}$, to problem (6.2.9). Next, it suffices to put $\boldsymbol{v}' = \nabla' w$ to obtain a solution (non-unique) $\boldsymbol{v}' \in W^{1,p}_0(\mathbb{R}^N_+)^{N-1}$ to problem (6.2.7).

For problem (6.2.8), with $\boldsymbol{g}' \in W_0^{-1/p,p}(\Gamma)^{N-1}$, we must use Theorem 4.3.3. The compatibility condition is written in this case: $\boldsymbol{g}' \perp (\mathcal{P}_{[1-N/p']})^{N-1}$. Thus it is realized by the assumption of Proposition 6.2.1. So, this problem has a solution $\boldsymbol{z}' \in W^{1,p}_0(\mathbb{R}^N_+)^{N-1}$, unique up to an element of $(\mathcal{P}_{[1-N/p]})^{N-1}$.

Then, it is clear that the function $\boldsymbol{u}' = \boldsymbol{v}' + \boldsymbol{z}' \in W^{1,p}_0(\mathbb{R}^N_+)^{N-1}$ is solution to problem $(N2)$.

(iii) Conversely, it is necessary to show that from $u_N, \pi, \boldsymbol{u}'$, we get a solution $(\boldsymbol{u}, \pi)$ of the original problem (6.2.1)–(6.2.4).

From previous it is clear that

$$\begin{aligned} -\Delta \boldsymbol{u}' + \nabla' \pi &= 0 && \text{in } \mathbb{R}^N_+, \\ u_N &= g_N && \text{on } \Gamma, \\ \partial_N \boldsymbol{u}' &= \boldsymbol{g}' && \text{on } \Gamma. \end{aligned}$$

It remains to prove that

$$-\Delta u_N + \partial_N \pi = 0 \quad \text{in } \mathbb{R}^N_+ \tag{6.2.10}$$

and finally, the relation (6.2.2).

For (6.2.10), thanks to the first equations of (B) and $(N1)$, we get

$$\Delta(\Delta u_N - \partial_N \pi) = \Delta^2 u_N = 0 \quad \text{in } \mathbb{R}^N_+.$$

With the boundary condition of $(N1)$, it follows that $\Delta u_N - \partial_N \pi$ satisfies the problem

$$\Delta(\Delta u_N - \partial_N \pi) = 0 \quad \text{in } \mathbb{R}^N_+, \qquad \Delta u_N - \partial_N \pi = 0 \quad \text{on } \Gamma.$$

Since $\Delta u_N - \partial_N \pi \in W^{-1,p}_0(\mathbb{R}^N_+)$, Theorem 4.3.4 shows that we necessarily have $\Delta u_N - \partial_N \pi = 0$ in $\mathbb{R}^N_+$.

For (6.2.2), the boundary condition of $(N2)$ implies $\partial_N \operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ. Besides, from the boundary conditions of (B), we get $\partial_N^2 u_N = -\operatorname{div}' \boldsymbol{g}'$ on Γ. Then we have

$$\partial_N \operatorname{div} \boldsymbol{u} = \partial_N \operatorname{div}' \boldsymbol{u}' + \partial_N^2 u_N = \operatorname{div}' \boldsymbol{g}' - \operatorname{div}' \boldsymbol{g}' = 0 \quad \text{on } \Gamma.$$

So, $\operatorname{div} \boldsymbol{u}$ satisfies the problem

$$\Delta \operatorname{div} \boldsymbol{u} = 0 \ \text{ in } \mathbb{R}^N_+, \quad \partial_N \operatorname{div} \boldsymbol{u} = 0 \ \text{ on } \Gamma.$$

Since $\operatorname{div} \boldsymbol{u} \in L^p(\mathbb{R}^N_+)$, thanks to Theorem 4.3.2, we get $\operatorname{div} \boldsymbol{u} = 0$ in $\mathbb{R}^N_+$.

(iv) Concerning the uniqueness question, we have notice that u_N and π are unique. Let $\boldsymbol{u}' = (u_i)_{1 \leqslant i \leqslant N-1}$ and $\boldsymbol{u}'_* = (u_i^*)_{1 \leqslant i \leqslant N-1}$ be solutions to $(N2)$, then

$$\begin{array}{rcll} \Delta(u_i - u_i^*) & = & 0 & \text{in } \mathbb{R}^N_+, \\ \partial_N(u_i - u_i^*) & = & 0 & \text{on } \Gamma, \end{array}$$

where $u_i - u_i^* \in W_0^{1,p}(\mathbb{R}^N_+)$. Thus, according to Theorem 1.4.3, we can deduce that $u_i - u_i^* \in \mathcal{N}^{\Delta}_{[1-N/p]}$. It remains to remark that $\mathcal{N}^{\Delta}_{[1-N/p]} = \mathbb{R}$ if $N \leqslant p$, and $\mathcal{N}^{\Delta}_{[1-N/p]} = \{0\}$ if $N > p$.

Finally, the estimate of Proposition 6.2.1 is a straightforward consequence of the Banach Theorem. □

Nonhomogeneous case

Now, we can deal with the complete problem.

Theorem 6.2.3. *Assume that* $\frac{N}{p'} \neq 1$. *For any* $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^N_+)$, $h \in W_1^{1,p}(\mathbb{R}^N_+)$, $g_N \in W_0^{1-1/p,p}(\Gamma)$ *and* $\boldsymbol{g}' \in W_0^{-1/p,p}(\Gamma)^{N-1}$, *satisfying the following compatibility condition if* $N < p'$:

$$\forall i \in \{1, \dots, N-1\}, \quad \int_{\mathbb{R}^N_+} f_i \, \mathrm{d}x = \langle g_i, 1 \rangle_{W_0^{-1/p,p}(\Gamma) \times W_0^{1/p,p'}(\Gamma)}, \tag{6.2.11}$$

problem (S_N) *admits a solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, *unique if* $N > p$, *unique up to an element of* $\mathbb{R}^{N-1} \times \{0\}^2$ *if* $N \leqslant p$, *with the estimate*

$$\begin{aligned} &\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1} \times \{0\}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \\ &\quad \leqslant C \left(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^N_+)} + \|h\|_{W_1^{1,p}(\mathbb{R}^N_+)} + \|g_N\|_{W_0^{1-1/p,p}(\Gamma)} + \|\boldsymbol{g}'\|_{W_0^{-1/p,p}(\Gamma)^{N-1}} \right) \end{aligned}$$

if $N \leqslant p$, *and the same without* $\boldsymbol{\chi}$ *if* $N > p$.

Proof. We can give a proof quite similar to the one of the nonhomogeneous case for the Stokes system with Dirichlet boundary conditions, by extension of the data $\boldsymbol{f}$ and h to the whole space (see Section 4.4). But another way is to combine this result with the homogeneous case for the Stokes system with Navier boundary conditions. We will follow this one.

Firstly, we introduce the auxiliary problem

$$\begin{aligned} -\Delta \boldsymbol{z} + \nabla \eta &= \boldsymbol{f} \quad \text{in } \mathbb{R}^N_+, \\ \operatorname{div} \boldsymbol{z} &= h \quad \text{in } \mathbb{R}^N_+, \\ \boldsymbol{z} &= \boldsymbol{0} \quad \text{on } \Gamma. \end{aligned} \tag{6.2.12}$$

With the assumption $\frac{N}{p'} \neq 1$, according to Theorem 4.5.2, we know that problem (6.2.12) admits a unique solution $(\boldsymbol{z}, \eta) \in \boldsymbol{W}^{2,p}_1(\mathbb{R}^N_+) \times W^{1,p}_1(\mathbb{R}^N_+)$. Thus we can deduce that $\partial_N \boldsymbol{z}'|_\Gamma \in W^{1-1/p,p}_1(\Gamma)^{N-1}$. In addition, we can notice that we have the imbeddings $W^{2,p}_1(\mathbb{R}^N_+) \hookrightarrow W^{1,p}_0(\mathbb{R}^N_+)$ and $W^{1,p}_1(\mathbb{R}^N_+) \hookrightarrow L^p(\mathbb{R}^N_+)$ without condition, whereas we have $W^{1-1/p,p}_1(\Gamma) \hookrightarrow W^{-1/p,p}_0(\Gamma)$ only if $\frac{N}{p'} \neq 1$.

Indeed, we can break it down into

$$W^{1-1/p,p}_1(\Gamma) \hookrightarrow W^{0,p}_{1/p}(\Gamma) \quad \text{and} \quad W^{0,p}_{1/p}(\Gamma) \hookrightarrow W^{-1/p,p}_0(\Gamma).$$

The first one holds without condition and, by duality, the second one is equivalent to $W^{1/p,p'}_0(\Gamma) \hookrightarrow W^{0,p'}_{-1/p}(\Gamma)$, which holds if $\frac{N-1}{p'} \neq \frac{1}{p}$, *i.e.* $\frac{N}{p'} \neq 1$.

So, $(\boldsymbol{z}, \eta) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ and above all $\gamma_1 \boldsymbol{z}' \in W^{-1/p,p}_0(\Gamma)^{N-1}$, which allows us to consider the second auxiliary problem

$$\begin{aligned} -\Delta \boldsymbol{v} + \nabla \vartheta = \boldsymbol{0} \quad &\text{and} \quad \operatorname{div} \boldsymbol{v} = 0 && \text{in } \mathbb{R}^N_+, \\ v_N = g_N \quad &\text{and} \quad \partial_N \boldsymbol{v}' = \boldsymbol{g}' - \partial_N \boldsymbol{z}' && \text{on } \Gamma, \end{aligned} \tag{6.2.13}$$

where $\boldsymbol{g}' - \partial_N \boldsymbol{z}'|_\Gamma = \boldsymbol{g}' - \gamma_1 \boldsymbol{z}' \in W^{-1/p,p}_0(\Gamma)^{N-1}$. Then, we can apply Proposition 6.2.1, which yields $(\boldsymbol{v}, \vartheta) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ solution to (6.2.13), provided the orthogonality condition

$$\forall \boldsymbol{\varphi}' \in \mathbb{R}^{N-1}, \quad \langle \boldsymbol{g}' - \gamma_1 \boldsymbol{z}', \boldsymbol{\varphi}' \rangle_{W^{-1/p,p}_0(\Gamma)^{N-1} \times W^{1/p,p'}_0(\Gamma)^{N-1}} = 0 \tag{6.2.14}$$

is satisfied if $N < p'$. Now, we must write this condition by only means of data. It suffices to notice that we have for all $\boldsymbol{\varphi} \in \mathbb{R}^{N-1} \times \{0\}$,

$$\begin{aligned} \int_{\mathbb{R}^N_+} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x &= \int_{\mathbb{R}^N_+} (-\Delta \boldsymbol{z} + \nabla \eta) \cdot \boldsymbol{\varphi} \, \mathrm{d}x \\ &= \langle \gamma_1 \boldsymbol{z}', \boldsymbol{\varphi}' \rangle_{W^{-1/p,p}_0(\Gamma)^{N-1} \times W^{1/p,p'}_0(\Gamma)^{N-1}}, \end{aligned}$$

to deduce that the condition (6.2.14) is written

$$\forall \boldsymbol{\varphi}' \in \mathbb{R}^{N-1}, \quad \int_{\mathbb{R}^N_+} \boldsymbol{f}' \cdot \boldsymbol{\varphi}' \, \mathrm{d}x = \langle \boldsymbol{g}', \boldsymbol{\varphi}' \rangle_{W^{-1/p,p}_0(\Gamma)^{N-1} \times W^{1/p,p'}_0(\Gamma)^{N-1}},$$

that is, more simply, the compatibility condition (6.2.11).

Then, the pair $(\boldsymbol{u},\, \pi) = (\boldsymbol{v} + \boldsymbol{z},\, \vartheta + \eta)$ which belongs to $\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ is a solution to (S_N).

Finally, the uniqueness of solutions to (S_N) is a straightforward consequence of Proposition 6.2.1. □

Remark 6.2.4. With these boundary conditions, it is not reasonable to consider data $(\boldsymbol{f},\, h)$ which belong to $\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$. Indeed, with such data we get a solution to problem (6.2.12) in the space $W_0^{1,\,p}(\mathbb{R}^N_+)$ for the velocity field $\boldsymbol{z}$ and we cannot give a sense to the trace of $\partial_N \boldsymbol{z}'$ in that case. This limitation is not due to the method employed here, but we find the same situation as in the Neumann problem for the Laplacian in $\mathbb{R}^N_+$ (see Theorem 1.4.3). ♢

6.3 Strong solutions and regularity

In this section, we are interested in the existence of strong solutions, *i.e.* of solutions $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+) \times W_1^{1,\,p}(\mathbb{R}^N_+)$, and next to get a general regularity result. We start with the homogeneous problem.

Proposition 6.3.1. *Assume that $\frac{N}{p'} \neq 1$. For any $g_N \in W_1^{2-1/p,\,p}(\Gamma)$ and $\boldsymbol{g}' \in W_1^{1-1/p,\,p}(\Gamma)^{N-1}$ such that $\boldsymbol{g}' \perp \mathbb{R}^{N-1}$ if $N < p'$, problem (6.2.1)–(6.2.4) admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+) \times W_1^{1,\,p}(\mathbb{R}^N_+)$, unique if $N > p$, unique up to an element of $\mathbb{R}^{N-1} \times \{0\}^2$ if $N \leqslant p$, with the estimate*

$$\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1} \times \{0\}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+)} + \|\pi\|_{W_1^{1,\,p}(\mathbb{R}^N_+)} \leqslant C \left(\|g_N\|_{W_1^{2-1/p,\,p}(\Gamma)} + \|\boldsymbol{g}'\|_{W_1^{1-1/p,\,p}(\Gamma)^{N-1}} \right)$$

if $N \leqslant p$, and the same without $\boldsymbol{\chi}$ if $N > p$.

Proof. We have seen before that $W_1^{1-1/p,\,p}(\Gamma) \hookrightarrow W_0^{-1/p,\,p}(\Gamma)$ if $\frac{N}{p'} \neq 1$. Moreover, we have the imbedding $W_1^{2-1/p,\,p}(\Gamma) \hookrightarrow W_0^{1-1/p,\,p}(\Gamma)$ without condition.

Then, from Proposition 6.2.1, we can deduce that problem (6.2.1)–(6.2.4) admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$. Now, it suffices to go back to the proof of Proposition 6.2.1 and to use the established results about problems (B), $(N1)$ and $(N2)$, to show that in fact $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+) \times W_1^{1,\,p}(\mathbb{R}^N_+)$.

In order, for problem (B), according to Theorem 4.3.7, we find $z_N \in W_1^{0,\,p}(\mathbb{R}^N_+)$ solution to the first problem (6.2.5), with the assumption $\boldsymbol{g}' \perp (\mathcal{P}_{[1-N/p']})^{N-1}$; and according to Theorem 1.4.2, we find $u_N \in W_1^{2,\,p}(\mathbb{R}^N_+)$ solution to the second problem (6.2.6). For problem $(N1)$, thanks to Theorem 4.3.3, we find $\pi \in W_1^{1,\,p}(\mathbb{R}^N_+)$. For problem $(N2)$, according to Theorem 1.4.2, we find $\boldsymbol{u}' \in W_1^{2,\,p}(\mathbb{R}^N_+)^{N-1}$. Note that for all these results, the condition $N/p' \neq 1$ is always necessary. □

We now can give the result in the nonhomogeneous case.

Theorem 6.3.2. *Assume that* $\frac{N}{p'} \neq 1$. *For any* $\boldsymbol{f} \in \boldsymbol{W}_1^{0,\,p}(\mathbb{R}^N_+)$, $h \in W_1^{1,\,p}(\mathbb{R}^N_+)$, $g_N \in W_1^{2-1/p,\,p}(\Gamma)$ *and* $\boldsymbol{g}' \in W_1^{1-1/p,\,p}(\Gamma)^{N-1}$, *satisfying the compatibility condition (6.2.11) if* $N < p'$, *problem* (S_N) *has a solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+) \times W_1^{1,\,p}(\mathbb{R}^N_+)$, *unique if* $N > p$, *unique up to an element of* $\mathbb{R}^{N-1} \times \{0\}^2$ *if* $N \leqslant p$, *with the estimate*

$$\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1}\times\{0\}} \|\boldsymbol{u}+\boldsymbol{\chi}\|_{\boldsymbol{W}_1^{2,\,p}(\mathbb{R}^N_+)} + \|\pi\|_{W_1^{1,\,p}(\mathbb{R}^N_+)}$$
$$\leqslant\ C\ \left(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,\,p}(\mathbb{R}^N_+)} + \|h\|_{W_1^{1,\,p}(\mathbb{R}^N_+)} + \|g_N\|_{W_1^{2-1/p,\,p}(\Gamma)} + \|\boldsymbol{g}'\|_{W_1^{1-1/p,\,p}(\Gamma)^{N-1}}\right)$$

if $N \leqslant p$, *and the same without* $\boldsymbol{\chi}$ *if* $N > p$.

Proof. The proof of Theorem 6.2.3 work in this case. It suffices to take the strong result for Stokes system with Dirichlet boundary conditions, *i.e.* Theorem 4.5.2, to solve (6.2.12); and Proposition 6.3.1 to solve (6.2.13). □

Corollary 6.3.3. *Let* $m \geqslant 1$ *be an integer and assume that* $\frac{N}{p'} \neq 1$. *For any* $\boldsymbol{f} \in \boldsymbol{W}_m^{m-1,\,p}(\mathbb{R}^N_+)$, $h \in W_m^{m,\,p}(\mathbb{R}^N_+)$, $g_N \in W_m^{m+1-1/p,\,p}(\Gamma)$ *and* $\boldsymbol{g}' \in W_m^{m-1/p,\,p}(\Gamma)^{N-1}$, *satisfying the compatibility condition (6.2.11) if* $N < p'$, *problem* (S_N) *admits a solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_m^{m+1,\,p}(\mathbb{R}^N_+) \times W_m^{m,\,p}(\mathbb{R}^N_+)$, *unique if* $N > p$, *unique up to an element of* $\mathbb{R}^{N-1} \times \{0\}^2$ *if* $N \leqslant p$, *with the estimate*

$$\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1}\times\{0\}} \|\boldsymbol{u}+\boldsymbol{\chi}\|_{\boldsymbol{W}_m^{m+1,\,p}(\mathbb{R}^N_+)} + \|\pi\|_{W_m^{m,\,p}(\mathbb{R}^N_+)}\ \leqslant$$
$$C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_m^{m-1,\,p}(\mathbb{R}^N_+)} + \|h\|_{W_m^{m,\,p}(\mathbb{R}^N_+)} + \|g_N\|_{W_m^{m+1-1/p,\,p}(\Gamma)} + \|\boldsymbol{g}'\|_{W_m^{m-1/p,\,p}(\Gamma)^{N-1}}\right)$$

if $N \leqslant p$, *and the same without* $\boldsymbol{\chi}$ *if* $N > p$.

Proof. Here again, we can refer to the proof of the regularity result for Stokes system with Dirichlet boundary conditions, that is Corollary 4.5.5 on page 74. The only change in the proof is about traces of the tangential components of the velocity field. However, assuming that $\boldsymbol{g}' \in W_{m+1}^{m+1-1/p,\,p}(\Gamma)^{N-1}$, since $\gamma_0\varrho = \varrho'$ and $\gamma_1\varrho = 0$, then we have $\gamma_1(\varrho\,\partial_i\boldsymbol{u}') = \varrho'\,\partial_i\gamma_1\boldsymbol{u}' = \varrho'\,\partial_i\boldsymbol{g}' \in \boldsymbol{W}_m^{m-1/p,\,p}(\Gamma)$, which allows us to apply the induction hypothesis. □

Remark 6.3.4. Another way to prove Corollary 6.3.3, is to resume the method of Section 6.2, using the regularity results for harmonic, biharmonic and Stokes (with Dirichlet boundary conditions) problems in the half-space. ♢

6.4 Very weak solutions

The aim of this section is to study the homogeneous problem (6.2.1)–(6.2.4):

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} \quad &\text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^N_+, \\ u_N = g_N \quad &\text{and} \quad \partial_N \boldsymbol{u}' = \boldsymbol{g}' \quad \text{on } \Gamma, \end{aligned}$$

with singular data on the boundary, that is more precisely with

$$\boldsymbol{g}' \in W^{-1-1/p,\,p}_{-1}(\Gamma)^{N-1} \quad \text{and} \quad g_N \in W^{-1/p,\,p}_{-1}(\Gamma).$$

The outline of the reasoning is the same as in the case of Dirichlet conditions (Section 4.6), but we must write an adapted Green formula for the variational formulation and thus find the good spaces with appropriate densities. We will give the complete proof, points already seen at Section 4.6 excepted. Here again, we will establish these preliminary definitions and properties with a view to the general case, *i.e.* for all $\ell \in \mathbb{Z}$.

For every $\ell \in \mathbb{Z}$, we introduce

$$\boldsymbol{M}_\ell(\mathbb{R}^N_+) = \left\{ \boldsymbol{u} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+);\ u_N = 0,\ \partial_N \boldsymbol{u}' = \boldsymbol{0} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 \ \text{ on } \Gamma \right\},$$

as a subspace of $\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$, equipped with the inherited norm. We also define the space

$$\boldsymbol{X}_\ell(\mathbb{R}^N_+) = \left[\begin{array}{ll} \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,\,p'}_{-\ell-1}(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} \in W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \right\} & \text{if } \frac{N}{p'} \notin \{\ell,\, \ell+1\}, \\ \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} \in W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \right\} & \text{if } \frac{N}{p'} \in \{\ell,\, \ell+1\}; \end{array} \right.$$

which is a reflexive Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{X}_\ell(\mathbb{R}^N_+)} = \left[\begin{array}{ll} \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p'}_{-\ell-1}(\mathbb{R}^N_+)} + \|\operatorname{div} \boldsymbol{v}\|_{W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} & \text{if } \frac{N}{p'} \notin \{\ell,\, \ell+1\}, \\ \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^N_+)} + \|\operatorname{div} \boldsymbol{v}\|_{W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} & \text{if } \frac{N}{p'} \in \{\ell,\, \ell+1\}; \end{array} \right.$$

and the following subspace of $\boldsymbol{X}_\ell(\mathbb{R}^N_+)$

$$\mathring{\boldsymbol{X}}_\ell(\mathbb{R}^N_+) = \left[\begin{array}{ll} \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,\,p'}_{-\ell-1}(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} \in \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \right\} & \text{if } \frac{N}{p'} \notin \{\ell,\, \ell+1\}, \\ \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^N_+);\ \operatorname{div} \boldsymbol{v} \in \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \right\} & \text{if } \frac{N}{p'} \in \{\ell,\, \ell+1\}. \end{array} \right.$$

Finally, let us we denote by $\boldsymbol{X}'_\ell(\mathbb{R}^N_+)$ the dual space of $\mathring{\boldsymbol{X}}_\ell(\mathbb{R}^N_+)$.

Lemma 6.4.1. *For any $\ell \in \mathbb{Z}$, the space $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ is dense in $\mathring{\boldsymbol{X}}_\ell(\mathbb{R}^N_+)$.*

Proof. For every continuous linear form $J \in \boldsymbol{X}'_\ell(\mathbb{R}^N_+)$, there exists a unique pair $(\boldsymbol{f},\, g) \in \boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)$, such that

$$\forall \boldsymbol{v} \in \mathring{\boldsymbol{X}}_\ell(\mathbb{R}^N_+), \quad \langle J, \boldsymbol{v}\rangle = \int_{\mathbb{R}^N_+} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \langle g, \operatorname{div}\boldsymbol{v}\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}. \tag{6.4.1}$$

Thanks to Hahn-Banach theorem, it suffices to show that any J which vanishes on $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ is actually zero on $\mathring{\boldsymbol{X}}_\ell(\mathbb{R}^N_+)$. Let us suppose that $J = 0$ on $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$. Then we can deduce from (6.4.1) that

$$\boldsymbol{f} - \nabla g = 0 \quad \text{ in } \mathbb{R}^N_+,$$

hence we have $\nabla g \in \boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)$. Then, we can deduce that $g \in W^{1,\,p}_{\ell+1}(\mathbb{R}^N_+)$. Now, it is a standard density argument which allows us to see from (6.4.1) that J is identically zero. □

Next, we introduce the two spaces:

$$\boldsymbol{T}_\ell(\mathbb{R}^N_+) = \left\{\boldsymbol{v} \in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+);\ \Delta\boldsymbol{v} \in \boldsymbol{X}'_\ell(\mathbb{R}^N_+)\right\},$$
$$\boldsymbol{T}_{\ell,\,\sigma}(\mathbb{R}^N_+) = \left\{\boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+);\ \operatorname{div}\boldsymbol{v} = 0 \ \text{ in } \mathbb{R}^N_+\right\},$$

which are reflexive Banach spaces for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{T}_\ell(\mathbb{R}^N_+)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)} + \|\Delta\boldsymbol{v}\|_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)},$$

where $\|\cdot\|_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)}$ denotes the dual norm of the space $\boldsymbol{X}'_\ell(\mathbb{R}^N_+)$.

We now can give the essential lemma, both to give a sense to the traces in our singular problem, and for the duality reasoning on which is based the main result.

Lemma 6.4.2. *Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \dots, \ell-1\} \quad \text{and} \quad N/p \notin \{1, \dots, -\ell+1\}. \tag{6.4.2}$$

The linear mapping

$$(\gamma_{e_N},\, \gamma'_1) : \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \boldsymbol{\mathcal{D}}(\mathbb{R}^{N-1})$$
$$\boldsymbol{v} \longmapsto (v_N|_\Gamma,\, \partial_N \boldsymbol{v}'|_\Gamma),$$

can be extended to a linear continuous mapping

$$(\gamma_{e_N},\, \gamma'_1) : \boldsymbol{T}_{\ell,\,\sigma}(\mathbb{R}^N_+) \longrightarrow W^{-1/p,\,p}_{\ell-1}(\Gamma) \times \big(W^{-1-1/p,\,p}_{\ell-1}(\Gamma)\big)^{N-1}.$$

In addition, we have the Green formula:

$$\begin{aligned}
&\forall \boldsymbol{v} \in \boldsymbol{T}_{\ell,\,\sigma}(\mathbb{R}^N_+), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+),\\
&\langle \Delta\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)\times\mathring{\boldsymbol{X}}_\ell(\mathbb{R}^N_+)} = \langle \boldsymbol{v}, \Delta\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)\times\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} -\\
&- \langle \partial_N \boldsymbol{v}', \boldsymbol{\varphi}'\rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma)\times\boldsymbol{W}^{1+1/p,\,p'}_{-\ell+1}(\Gamma)} + \langle v_N, \partial_N\varphi_N\rangle_{W^{-1/p,\,p}_{\ell-1}(\Gamma)\times W^{1/p,\,p'}_{-\ell+1}(\Gamma)}.
\end{aligned} \tag{6.4.3}$$

Proof. (i) We start with the normal trace, that is the linear mapping

$$\begin{aligned}\gamma_{e_N} : \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big) &\longrightarrow \mathcal{D}(\mathbb{R}^{N-1})\\ \boldsymbol{v} &\longmapsto v_N|_\Gamma,\end{aligned}$$

Let us consider the space

$$\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+) = \left[\begin{array}{ll}\big\{\boldsymbol{v}\in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+);\ \operatorname{div}\boldsymbol{v}\in W^{0,\,p}_{\ell}(\mathbb{R}^N_+)\big\} & \text{if } \frac{N}{p'}\neq \ell,\\ \big\{\boldsymbol{v}\in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+);\ \operatorname{div}\boldsymbol{v}\in W^{0,\,p}_{\ell,\,1}(\mathbb{R}^N_+)\big\} & \text{if } \frac{N}{p'}= \ell;\end{array}\right.$$

which is a reflexive Banach space equipped with its natural norm

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)} = \left[\begin{array}{ll}\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)} + \|\operatorname{div}\boldsymbol{v}\|_{W^{0,\,p}_{\ell}(\mathbb{R}^N_+)} & \text{if } \frac{N}{p'}\neq \ell,\\ \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)} + \|\operatorname{div}\boldsymbol{v}\|_{W^{0,\,p}_{\ell,\,1}(\mathbb{R}^N_+)} & \text{if } \frac{N}{p'}= \ell.\end{array}\right.$$

As in Lemma 4.6.2, we can show by truncation and regularization that $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)$.

Moreover, by density of $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$, we have

$$\begin{aligned}&\forall \boldsymbol{v}\in \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big),\quad \forall\varphi\in W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+),\\ &\qquad\int_{\mathbb{R}^N_+}\boldsymbol{v}\cdot\nabla\varphi\,\mathrm{d}x\ +\int_{\mathbb{R}^N_+}\varphi\,\operatorname{div}\boldsymbol{v}\,\mathrm{d}x = -\int_\Gamma v_N\,\varphi\,\mathrm{d}x',\end{aligned}$$

hence

$$\left|\int_\Gamma v_N\,\varphi\,\mathrm{d}x'\right|\ \leqslant\ \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)}\ \|\varphi\|_{W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}.$$

Let $\mu\in W^{1/p,\,p'}_{-\ell+1}(\Gamma)$. By Lemma 1.3.1, there exists $\varphi\in W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\varphi=\mu$ on Γ, with $\|\varphi\|_{W^{1,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}\ \leqslant\ C\ \|\mu\|_{W^{1/p,\,p'}_{-\ell+1}(\Gamma)}$. Consequently,

$$\left|\int_\Gamma v_N\,\mu\,\mathrm{d}x'\right|\ \leqslant\ C\ \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)}\ \|\mu\|_{W^{1/p,\,p'}_{-\ell+1}(\Gamma)},$$

and thus

$$\|v_N\|_{W^{-1/p,\,p}_{\ell-1}(\Gamma)}\ \leqslant\ C\ \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)}. \tag{6.4.4}$$

Hence we can deduce that γ_{e_N} is continuous for the norm of $\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)$ and, since $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)$, the mapping γ_{e_N} can be extended by continuity to $\gamma_{e_N}\in\mathcal{L}\Big(\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+);\ W^{-1/p,\,p}_{\ell-1}(\Gamma)\Big)$. That obviously answers to the question of normal trace for the functions in $\boldsymbol{T}_{\ell,\,\sigma}(\mathbb{R}^N_+)$, since we have in this space the inequality $\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell}(\mathrm{div};\,\mathbb{R}^N_+)}\ \leqslant\ \|\boldsymbol{v}\|_{\boldsymbol{T}_{\ell}(\mathbb{R}^N_+)}$.

(ii) Next, we are interested in the trace of normal derivative of tangential components, that is the linear mapping

$$\begin{aligned}\gamma_1' : \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big) &\longrightarrow \mathcal{D}(\mathbb{R}^{N-1})^{N-1}\\ \boldsymbol{v} &\longmapsto \partial_N \boldsymbol{v}'|_\Gamma,\end{aligned}$$

Firstly, the density of $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ in $\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$, yields the following Green formula:

$$\begin{gathered}\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+),\\ \int_{\mathbb{R}^N_+} \Delta \boldsymbol{v} \cdot \boldsymbol{\varphi} \,\mathrm{d}x = \int_{\mathbb{R}^N_+} \boldsymbol{v} \cdot \Delta \boldsymbol{\varphi} \,\mathrm{d}x - \int_\Gamma \partial_N \boldsymbol{v}' \cdot \boldsymbol{\varphi}' \,\mathrm{d}x' + \int_\Gamma v_N \,\partial_N \varphi_N \,\mathrm{d}x'.\end{gathered} \tag{6.4.5}$$

According to (6.4.4), we can deduce the following estimate:

$$\left| \int_\Gamma \partial_N \boldsymbol{v}' \cdot \boldsymbol{\varphi}' \,\mathrm{d}x' \right| \;\leqslant\; C \;\|\boldsymbol{v}\|_{\boldsymbol{T}_\ell(\mathbb{R}^N_+)} \;\|\boldsymbol{\varphi}\|_{W^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}.$$

Let $\boldsymbol{\mu}' \in W^{1+1/p,\,p'}_{-\ell+1}(\Gamma)^{N-1}$. By Lemma 1.3.1, there exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\boldsymbol{\varphi} = (\boldsymbol{\mu}',\, 0)$ and $\partial_N \boldsymbol{\varphi} = (\boldsymbol{0},\, -\operatorname{div}' \boldsymbol{\mu}')$ on Γ — so we have $\operatorname{div} \boldsymbol{\varphi} = 0$ on Γ and thus $\boldsymbol{\varphi} \in \boldsymbol{M}_\ell(\mathbb{R}^N_+)$ —, with $\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} \;\leqslant\; C \;\|\boldsymbol{\mu}'\|_{W^{1+1/p,\,p'}_{-\ell+1}(\Gamma)}$. Consequently,

$$\left| \int_\Gamma \partial_N \boldsymbol{v}' \cdot \boldsymbol{\mu}' \,\mathrm{d}x' \right| \;\leqslant\; C \;\|\boldsymbol{v}\|_{\boldsymbol{T}_\ell(\mathbb{R}^N_+)} \;\|\boldsymbol{\mu}'\|_{W^{1+1/p,\,p'}_{-\ell+1}(\Gamma)},$$

and thus

$$\|\partial_N \boldsymbol{v}'\|_{W^{-1-1/p,\,p}_{\ell-1}(\Gamma)} \;\leqslant\; C \;\|\boldsymbol{v}\|_{\boldsymbol{T}_\ell(\mathbb{R}^N_+)}.$$

Hence we can deduce that the linear mapping γ_1' is continuous for the norm of $\boldsymbol{T}_{\ell,\,\sigma}(\mathbb{R}^N_+)$. In addition, we can show that the space $\Big\{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big);\; \operatorname{div} \boldsymbol{v} = 0\Big\}$ is dense in $\boldsymbol{T}_{\ell,\,\sigma}(\mathbb{R}^N_+)$. Therefore, the mapping γ_1' can be extended by continuity to $\gamma_1' \in \mathcal{L}\Big(\boldsymbol{T}_{\ell,\,\sigma}(\mathbb{R}^N_+);\; W^{-1-1/p,\,p}_{\ell-1}(\Gamma)\Big)$.

To finish this proof, we also can deduce the formula (6.4.3) from (6.4.5) by that last density. □

We now can solve the homogeneous problem (6.2.1)–(6.2.4) with singular data on the boundary. We will do it in two times. The first step is to consider $g_N = 0$, next we will remove this assumption. Here, we are only interested in the case $\ell = 0$ and then hypothesis (6.4.2) is reduced to $N \neq p$. Let us notice that with this hypothesis, we have the imbeddings $W^{1,\,p}_0(\mathbb{R}^N_+) \hookrightarrow W^{0,\,p}_{-1}(\mathbb{R}^N_+)$ and $W^{1-1/p,\,p}_0(\Gamma) \hookrightarrow W^{-1/p,\,p}_{-1}(\Gamma)$, in addition $W^{-1/p,\,p}_0(\Gamma) \hookrightarrow W^{-1-1/p,\,p}_{-1}(\Gamma)$ holds without condition. It allows us to link the very weak solutions of this section to the generalized solutions of Proposition 6.2.1.

Proposition 6.4.3. *Assume that* $\frac{N}{p} \neq 1$. *For any* $\boldsymbol{g}' \in W_{-1}^{-1-1/p,\,p}(\Gamma)^{N-1}$ *such that* $\boldsymbol{g}' \perp \mathbb{R}^{N-1}$ *if* $N \leqslant p'$, *the Stokes problem (6.2.1)–(6.2.4) with* $g_N = 0$ *admits a solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,\,p}(\mathbb{R}_+^N)$, *unique if* $N > p$, *unique up to an element of* $\mathbb{R}^{N-1} \times \{0\}^2$ *if* $N < p$, *with the estimate*

$$\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1} \times \{0\}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_{-1}^{-1,\,p}(\mathbb{R}_+^N)} \;\leqslant\; C\, \|\boldsymbol{g}'\|_{W_{-1}^{-1-1/p,\,p}(\Gamma)^{N-1}}$$

if $N < p$, *and the same without* $\boldsymbol{\chi}$ *if* $N > p$.

Proof. (i) Given a pair $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,\,p}(\mathbb{R}_+^N)$ which satisfies (6.2.1) and (6.2.2), then we have $\boldsymbol{u} \in \boldsymbol{T}_{0,\,\sigma}(\mathbb{R}_+^N)$ and thus the boundary conditions (6.2.3) and (6.2.4) makes sense. Indeed, observe that if $\pi \in W_{-1}^{-1,\,p}(\mathbb{R}_+^N)$, we can write for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}_+^N)$,

$$\langle \nabla \pi, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}_+^N) \times \boldsymbol{\mathcal{D}}(\mathbb{R}_+^N)} = -\, \langle \pi, \operatorname{div} \boldsymbol{\varphi} \rangle_{\mathcal{D}'(\mathbb{R}_+^N) \times \mathcal{D}(\mathbb{R}_+^N)} \,.$$

Consider the linear form:

$$J : \boldsymbol{\varphi} \longmapsto -\, \langle \pi, \operatorname{div} \boldsymbol{\varphi} \rangle_{W_{-1}^{-1,\,p}(\mathbb{R}_+^N) \times \mathring{W}_1^{1,\,p'}(\mathbb{R}_+^N)} \,,$$

defined on $\mathcal{D}(\mathbb{R}_+^N)$. Thanks to Lemma 6.4.1, we can extend J by density to $\boldsymbol{X}_0(\mathbb{R}_+^N)$; moreover, we have:

$$|J \boldsymbol{\varphi}| \;\leqslant\; \|\pi\|_{W_{-1}^{-1,\,p}(\mathbb{R}_+^N)} \, \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_0(\mathbb{R}_+^N)}.$$

Hence J is continuous on $\boldsymbol{X}_0(\mathbb{R}_+^N)$ and by the Riesz representation theorem, we can deduce that $\nabla \pi \in \boldsymbol{X}_0'(\mathbb{R}_+^N)$. In addition, we have the following Green formula:

$$\begin{aligned} &\forall \boldsymbol{\varphi} \in \boldsymbol{X}_0(\mathbb{R}_+^N), \\ &\langle \nabla \pi, \boldsymbol{\varphi} \rangle_{\boldsymbol{X}_0'(\mathbb{R}_+^N) \times \mathring{\boldsymbol{X}}_0(\mathbb{R}_+^N)} = -\, \langle \pi, \operatorname{div} \boldsymbol{\varphi} \rangle_{W_{-1}^{-1,\,p}(\mathbb{R}_+^N) \times \mathring{W}_1^{1,\,p'}(\mathbb{R}_+^N)} \,, \end{aligned} \tag{6.4.6}$$

with the estimate

$$\|\nabla \pi\|_{\boldsymbol{X}_0'(\mathbb{R}_+^N)} \;\leqslant\; \|\pi\|_{W_{-1}^{-1,\,p}(\mathbb{R}_+^N)}.$$

Since $\nabla \pi \in \boldsymbol{X}_0'(\mathbb{R}_+^N)$, we also have $\Delta \boldsymbol{u} \in \boldsymbol{X}_0'(\mathbb{R}_+^N)$, hence $\boldsymbol{u} \in \boldsymbol{T}_{0,\,\sigma}(\mathbb{R}_+^N)$, and thus we have both the trace $\gamma_0 u_N \in \boldsymbol{W}_{-1}^{-1/p,\,p}(\Gamma)$ and the trace $\gamma_1 \boldsymbol{u}' \in \boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)$.

(ii) Let us show that the problem (6.2.1)–(6.2.4) with $g_N = 0$ is equivalent to the variational formulation: Find $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,\,p}(\mathbb{R}_+^N)$ such that

$$\begin{aligned} &\forall \boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}_+^N), \quad \forall \vartheta \in W_1^{1,\,p'}(\mathbb{R}_+^N), \\ &\langle \boldsymbol{u}, -\Delta \boldsymbol{v} + \nabla \vartheta \rangle_{\boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N) \times \boldsymbol{W}_1^{0,\,p'}(\mathbb{R}_+^N)} - \langle \pi, \operatorname{div} \boldsymbol{v} \rangle_{W_{-1}^{-1,\,p}(\mathbb{R}_+^N) \times \mathring{W}_1^{1,p'}(\mathbb{R}_+^N)} \\ &\qquad\qquad = -\, \langle \boldsymbol{g}', \boldsymbol{v}' \rangle_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma) \times \boldsymbol{W}_1^{1+1/p,\,p'}(\Gamma)} \,. \end{aligned} \tag{6.4.7}$$

(a) Let $(\boldsymbol{u},\,\pi)$ be a solution to (6.2.1)–(6.2.4) with $g_N = 0$; then the Green formulas (6.4.3) and (6.4.6) yield for all $\boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}^N_+)$,

$$\begin{aligned}\langle -\Delta\boldsymbol{u} + \nabla\pi, \boldsymbol{v}\rangle_{\boldsymbol{X}_0'(\mathbb{R}^N_+)\times\mathring{\boldsymbol{X}}_0(\mathbb{R}^N_+)} &= -\,\langle \boldsymbol{u}, \Delta\boldsymbol{v}\rangle_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times\boldsymbol{W}^{0,\,p'}_{1}(\mathbb{R}^N_+)} + \\ + \langle \boldsymbol{g}', \boldsymbol{v}'\rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{-1}(\Gamma)\times\boldsymbol{W}^{1+1/p,\,p'}_{1}(\Gamma)} &- \langle \pi, \operatorname{div}\boldsymbol{v}\rangle_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)\times\mathring{W}^{1,\,p'}_{1}(\mathbb{R}^N_+)} = 0.\end{aligned}$$

Moreover, using the density of the functions of $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ with divergence zero in $\boldsymbol{T}_{0,\,\sigma}(\mathbb{R}^N_+)$, we obtain for all $\vartheta \in W^{1,\,p'}_1(\mathbb{R}^N_+)$,

$$\begin{aligned}\langle \boldsymbol{u}, \nabla\vartheta\rangle_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times\boldsymbol{W}^{0,\,p'}_{1}(\mathbb{R}^N_+)} &= -\,\langle \operatorname{div}\boldsymbol{u}, \vartheta\rangle_{L^p(\mathbb{R}^N_+)\times L^{p'}(\mathbb{R}^N_+)} - \\ &- \langle u_N, \vartheta\rangle_{W^{-1/p,\,p}_{-1}(\Gamma)\times W^{1/p,\,p'}_{1}(\Gamma)} = 0.\end{aligned}$$

So we show that $(\boldsymbol{u},\,\pi)$ satisfies the variational formulation (6.4.7).

(b) Conversely, if $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$ satisfies the variational formulation (6.4.7), then taking $\boldsymbol{v} = \boldsymbol{0}$, we have for any $\vartheta \in \mathcal{D}(\mathbb{R}^N_+)$,

$$\langle \boldsymbol{u}, \nabla\vartheta\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} = \langle -\operatorname{div}\boldsymbol{u}, \vartheta\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} = 0,$$

hence $\operatorname{div}\boldsymbol{u} = 0$ in $\mathbb{R}^N_+$. We can deduce that $\boldsymbol{u} \in \boldsymbol{W}^{0,p}_{-1}(\mathrm{div};\,\mathbb{R}^N_+)$ and thus $u_N|_\Gamma \in W^{-1/p,p}_{-1}(\Gamma)$. Then, we can write for any $\vartheta \in W^{1,\,p'}_1(\mathbb{R}^N_+)$,

$$\langle \boldsymbol{u}, \nabla\vartheta\rangle_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)\times\boldsymbol{W}^{0,\,p'}_{1}(\mathbb{R}^N_+)} = \langle u_N, \vartheta\rangle_{W^{-1/p,\,p}_{-1}(\Gamma)\times W^{1/p,\,p'}_{1}(\Gamma)} = 0.$$

Therefore, by the traces lemma (Lemma 1.3.1), we have for any $\varphi \in W^{1/p,\,p'}_1(\Gamma)$, $\langle u_N, \varphi\rangle_{W^{-1/p,\,p}_{-1}(\Gamma)\times W^{1/p,\,p'}_{1}(\Gamma)} = 0$, hence $u_N = 0$ on Γ. In addition, taking $\vartheta = 0$ in (6.4.7), we have for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$,

$$\langle \boldsymbol{u}, -\Delta\boldsymbol{v}\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} - \langle \pi, \operatorname{div}\boldsymbol{v}\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} = 0,$$

thus $\langle -\Delta\boldsymbol{u} + \nabla\pi, \boldsymbol{v}\rangle_{\mathcal{D}'(\mathbb{R}^N_+)\times\mathcal{D}(\mathbb{R}^N_+)} = 0$, *i.e.* $-\Delta\boldsymbol{u}+\nabla\pi = 0$ in $\mathbb{R}^N_+$. We deduce that $\boldsymbol{u} \in \boldsymbol{T}_{0,\,\sigma}(\mathbb{R}^N_+)$ and taking $\vartheta = 0$ in (6.4.7), we finally get for any $\boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}^N_+)$,

$$\langle \partial_N\boldsymbol{u}', \boldsymbol{v}'\rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{-1}(\Gamma)\times\boldsymbol{W}^{1+1/p,\,p'}_{1}(\Gamma)} = \langle \boldsymbol{g}', \boldsymbol{v}'\rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{-1}(\Gamma)\times\boldsymbol{W}^{1+1/p,\,p'}_{1}(\Gamma)}\,.$$

Moreover, as we saw at point (ii) in the proof of Lemma 6.4.2 on page 113, for all $\boldsymbol{\mu}' \in W^{1+1/p,\,p'}_1(\Gamma)^{N-1}$, there exists $\boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}^N_+)$ such that $\boldsymbol{v}' = \boldsymbol{\mu}'$ on Γ; consequently $\partial_N\boldsymbol{u}' = \boldsymbol{g}'$ on Γ. So, we have shown that $(\boldsymbol{u},\,\pi)$ is a solution to problem (6.2.1)–(6.2.4).

(iii) Let us solve problem (6.4.7). According to Theorem 6.3.2, we know that if $\frac{N}{p} \neq 1$, for all $\boldsymbol{f} \in \boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+) \perp \big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}\big)$ and $\varphi \in \mathring{W}^{1,\,p'}_1(\mathbb{R}^N_+)$, there exists a unique $(\boldsymbol{v},\, \vartheta) \in \big(\boldsymbol{M}_0(\mathbb{R}^N_+) \times W^{1,\,p'}_1(\mathbb{R}^N_+)\big)/\big((\mathcal{P}_{[1-N/p']})^{N-1} \times \{0\}^2\big)$ solution to

$$-\Delta \boldsymbol{v} + \nabla\vartheta = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} = \varphi \quad \text{in } \mathbb{R}^N_+,$$
$$v_N = 0 \quad \text{and} \quad \partial_N \boldsymbol{v}' = \boldsymbol{0} \quad \text{on } \Gamma,$$

with the estimate

$$\inf_{\boldsymbol{\chi} \in (\mathcal{P}_{[1-N/p']})^{N-1} \times \{0\}} \|\boldsymbol{v} + \boldsymbol{\chi}\|_{\boldsymbol{W}^{2,\,p'}_1(\mathbb{R}^N_+)} + \|\vartheta\|_{W^{1,\,p'}_1(\mathbb{R}^N_+)}$$
$$\leqslant C \left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+)} + \|\varphi\|_{W^{1,\,p'}_1(\mathbb{R}^N_+)} \right).$$

Now, consider the linear form

$$T: \ (\boldsymbol{f},\, \varphi) \longmapsto -\, \langle \boldsymbol{g}',\, \boldsymbol{v}' \rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{-1}(\Gamma) \times \boldsymbol{W}^{1+1/p,\,p'}_1(\Gamma)},$$

defined on $\big(\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+) \times \overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+)\big) \perp \big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}^2\big)$. Since $\boldsymbol{g}' \perp (\mathcal{P}_{[1-N/p']})^{N-1}$, we have for any $\boldsymbol{\chi}' \in (\mathcal{P}_{[1-N/p']})^{N-1}$,

$$|T(\boldsymbol{f},\, \varphi)| = \left| \langle \boldsymbol{g}',\, \boldsymbol{v}' + \boldsymbol{\chi}' \rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{-1}(\Gamma) \times \boldsymbol{W}^{1+1/p,\,p'}_1(\Gamma)} \right|$$
$$\leqslant \ C \ \|\boldsymbol{v}\|_{\boldsymbol{W}^{2,\,p'}_1(\mathbb{R}^N_+)/(\mathcal{P}_{[1-N/p']})^{N-1} \times \{0\}} \ \|\boldsymbol{g}'\|_{\boldsymbol{W}^{-1-1/p,\,p}_{-1}(\Gamma)}$$
$$\leqslant \ C \ \left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+)} + \|\varphi\|_{W^{1,\,p'}_1(\mathbb{R}^N_+)} \right) \ \|\boldsymbol{g}'\|_{\boldsymbol{W}^{-1-1/p,\,p}_{-1}(\Gamma)}.$$

Hence T is continuous on $\big(\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+) \times \overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+)\big) \perp \big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}^2\big)$, and according to the Riesz representation theorem, we know that there exists a unique $(\boldsymbol{u},\, \pi) \in \big(\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)\big)/\big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}^2\big)$ — which is the dual space of $\big(\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+) \times \overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+)\big) \perp \big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}^2\big)$ —, such that

$$\forall (\boldsymbol{f},\, \varphi) \in \big(\boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+) \times \overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+)\big) \perp \big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}^2\big),$$
$$T(\boldsymbol{f},\, \varphi) = \langle \boldsymbol{u}, \boldsymbol{f} \rangle_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times \boldsymbol{W}^{0,\,p'}_1(\mathbb{R}^N_+)} + \langle \pi,\, -\varphi \rangle_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+) \times \overset{\circ}{W}{}^{1,\,p'}_1(\mathbb{R}^N_+)},$$

i.e. the pair $(\boldsymbol{u},\, \pi)$ satisfies (6.4.7). □

We now can drop the hypothesis $g_N = 0$.

Theorem 6.4.4. *Assume that $\frac{N}{p} \neq 1$. For any $\boldsymbol{g}' \in W^{-1-1/p,\,p}_{-1}(\Gamma)^{N-1}$ such that $\boldsymbol{g}' \perp \mathbb{R}^{N-1}$ if $N \leqslant p'$, and $g_N \in W^{-1/p,\,p}_{-1}(\Gamma)$, the Stokes problem (6.2.1)–(6.2.4)* *admits a solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$*, unique if* $N > p$*, unique up to an element of* $\mathbb{R}^{N-1} \times \{0\}^2$ *if* $N < p$*, with the estimate*

$$\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1} \times \{0\}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)} + \|\pi\|_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)} \leqslant C \left(\|g_N\|_{W^{-1/p,\,p}_{-1}(\Gamma)} + \|\boldsymbol{g}'\|_{W^{-1-1/p,\,p}_{-1}(\Gamma)^{N-1}} \right)$$

if $N < p$*, and the same without* $\boldsymbol{\chi}$ *if* $N > p$*.*

Proof. According to Theorem 4.3.3, we know that if $\frac{N}{p} \neq 1$, then there exists $\psi \in W^{1,\,p}_{-1}(\mathbb{R}^N_+)$ unique up to an element of $\mathcal{N}^{\Delta}_{[2-N/p]}$ solution to the following Neumann problem:

$$\Delta\psi = 0 \ \text{ in } \mathbb{R}^N_+, \quad \partial_N \psi = g_N \ \text{ on } \Gamma.$$

Let us set $\boldsymbol{w} = \nabla\psi$ and $\boldsymbol{g}'_* = \boldsymbol{g}' - \partial_N \boldsymbol{w}'$ on Γ. Then we have $\boldsymbol{w} \in \boldsymbol{T}_{0,\,\sigma}(\mathbb{R}^N_+)$, hence $\boldsymbol{g}'_* \in W^{-1-1/p,\,p}_{-1}(\Gamma)^{N-1}$, with the estimate

$$\|\boldsymbol{w}\|_{\boldsymbol{T}_0(\mathbb{R}^N_+)} = \|\boldsymbol{w}\|_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)} \leqslant C\, \|g_N\|_{W^{-1/p,\,p}_{-1}(\Gamma)}.$$

Moreover, $\boldsymbol{g}'_*$ satisfies the orthogonality condition of Proposition 6.4.3, hence the existence of a pair $(\boldsymbol{z},\, \pi) \in \boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$ which satisfies

$$-\Delta \boldsymbol{z} + \nabla\pi = \boldsymbol{0} \ \text{ and } \ \operatorname{div} \boldsymbol{z} = 0 \ \text{ in } \mathbb{R}^N_+,$$
$$z_N = 0 \ \text{ and } \ \partial_N \boldsymbol{z}' = \boldsymbol{g}'_* \ \text{ on } \Gamma.$$

Then the pair $(\boldsymbol{u},\, \pi) = (\boldsymbol{z} + \boldsymbol{w},\, \pi)$ is the required solution. The uniqueness of this solution is a straightforward consequence of Proposition 6.4.3. □

Chapitre 7

A generalized Stokes system

7.1 Introduction

In this chapter we are interested in the study of systems of Stokes type

$$(S_N^e)\quad \begin{cases} -\nu\,\Delta \boldsymbol{u} - \mu\,\nabla\operatorname{div}\boldsymbol{u} + \nabla\pi &= \boldsymbol{f} \quad \text{in } \mathbb{R}^N_+,\\ \lambda\,\pi + \operatorname{div}\boldsymbol{u} &= h \quad \text{in } \mathbb{R}^N_+,\\ u_N &= g_N \quad \text{on } \Gamma,\\ \partial_N \boldsymbol{u}' &= \boldsymbol{g}' \quad \text{on } \Gamma, \end{cases}$$

where the constants ν, μ and λ satisfy the assumptions $\nu > 0$, $\lambda \geqslant 0$ and $\mu + \nu > 0$. First, we can remark that the elasticity term $-\mu\,\nabla\operatorname{div}\boldsymbol{u}$ in the first equation vanishes by using the second equation in order to substitute $\operatorname{div}\boldsymbol{u}$. However, the calculations made under the assumption $\mu \neq 0$ will be useful in studying some problems related to compressible fluids.

Naturally, it is also possible to see the classical Stokes system as the limit case $\lambda = 0$ of this generalized problem. This point of view can be interesting in numerical approximation (see H. Beirão da Veiga, [17]).

Since the previous chapter was dedicated to the classical Stokes system with Navier condition, in the present one, we will assume that $\lambda \neq 0$. Besides, we will use both the method elaborated in the previous chapters and the specificity of this system — particularly for the very weak solutions —.

7.2 The generalized Stokes system in $\mathbb{R}^N$

As usual, our method requires the extension of problems given in the half-space to the whole space. Then a necessary step is to consider the corresponding Stokes system in $\mathbb{R}^N$:

$$(S^e)\quad \begin{cases} -\nu\,\Delta \boldsymbol{u} - \mu\,\nabla\operatorname{div}\boldsymbol{u} + \nabla\pi &= \boldsymbol{f} \quad \text{in } \mathbb{R}^N,\\ \lambda\,\pi + \operatorname{div}\boldsymbol{u} &= h \quad \text{in } \mathbb{R}^N. \end{cases}$$

In this section, we adapt to this case, with minor modifications, the arguments used by Alliot-Amrouche in [3] for the classical Stokes system.

Let us denote by T the corresponding operator:

$$T : (\boldsymbol{u},\, \pi) \longmapsto (-\nu\, \Delta \boldsymbol{u} - \mu\, \nabla\, \mathrm{div}\, \boldsymbol{u} + \nabla \pi,\; -\lambda\, \pi - \mathrm{div}\, \boldsymbol{u}).$$

7.2.1 Existence and uniqueness results

We assume that $(\boldsymbol{f},\, h) = (\boldsymbol{0},\, 0)$ and we first consider the operator T defined on the space of tempered distributions $\boldsymbol{\mathcal{S}}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$. Using the second equation in order to substitute $-\lambda\pi$ for $\mathrm{div}\, \boldsymbol{u}$ in the first equation, we get

$$-\nu\, \Delta \boldsymbol{u} + (1 + \lambda\, \mu)\nabla \pi = \boldsymbol{0} \quad \text{ in } \mathbb{R}^N.$$

Applying the divergence operator to this equation, we obtain $\Delta \pi = 0$ in $\mathbb{R}^N$. Finally, applying the Laplacian to the same equation, we find $\Delta^2 \boldsymbol{u} = \boldsymbol{0}$ in $\mathbb{R}^N$. So, π and $\boldsymbol{u}$ are respectively tempered harmonic and biharmonic distributions, thus polynomials. Consequently, the kernel of T is quite similar to the kernel of the classical Stokes operator: for any $k \in \mathbb{Z}$, we introduce the space

$$\mathcal{S}_k^e = \left\{ (\boldsymbol{\chi},\, q) \in \boldsymbol{\mathcal{P}}_k^{\Delta^2} \times \mathcal{P}_{k-1}^{\Delta};\; \lambda\, q + \mathrm{div}\, \boldsymbol{\chi} = 0,\; -\nu\, \Delta \boldsymbol{\chi} - \mu\, \nabla\, \mathrm{div}\, \boldsymbol{\chi} + \nabla q = \boldsymbol{0} \right\},$$

and we have the following uniqueness result:

Lemma 7.2.1. *Let $\ell \in \mathbb{Z}$, $m \in \mathbb{N}$ and assume that $N/p \notin \{1, \ldots, -\ell - m\}$, then the kernel of T defined on $\boldsymbol{W}_{m+\ell}^{m+1,\, p}(\mathbb{R}_+^N) \times W_{m+\ell}^{m,\, p}(\mathbb{R}_+^N)$ is the space $\mathcal{S}_{[1-\ell-N/p]}^e$.*

Now, we are interested in the question of existence of solutions. Let $(\boldsymbol{u},\, \pi) \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N)$ be a pair solution to problem (S^e). The second equation of (S^e) allows us to substitute $h - \lambda\pi$ for $\mathrm{div}\, \boldsymbol{u}$ in the first equation, then taking the divergence of this one, we get

$$\big(1 + \lambda\, (\nu + \mu)\big)\, \Delta \pi = \mathrm{div}\, \boldsymbol{f} + (\nu + \mu)\, \Delta h. \tag{7.2.1}$$

Besides, for the velocity field, we have

$$\nu\, \Delta \boldsymbol{u} = (1 + \lambda \mu)\, \nabla \pi - \boldsymbol{f} - \mu\, \nabla h. \tag{7.2.2}$$

Thus, as for the classical Stokes system in $\mathbb{R}^N$, it suffices to solve these two Poisson's equations. Indeed, if $(\boldsymbol{v},\, \tau)$ verifies (7.2.1)–(7.2.2), then we get

$$-\nu\, \Delta \boldsymbol{v} - \mu\, \nabla(h - \lambda\, \tau) + \nabla \tau = \boldsymbol{f} \quad \text{ in } \boldsymbol{\mathcal{S}}'(\mathbb{R}^N), \tag{7.2.3}$$

$$\Delta\, \mathrm{div}\, \boldsymbol{v} = \Delta(h - \lambda\, \tau) \quad \text{ in } \mathcal{S}'(\mathbb{R}^N), \tag{7.2.4}$$

and thus, $\mathrm{div}\, \boldsymbol{v} - h + \lambda\, \tau = \varphi$, where φ is a harmonic polynomial. So we can use the following lemma proved in [3]:

Lemma 7.2.2. *For any $k \in \mathbb{N}$, $\mathcal{P}_k^{\Delta} = \mathrm{div}\, \big(\boldsymbol{\mathcal{P}}_{k+1}^{\Delta}\big)$.*

Therefore $\varphi = \mathrm{div}\, \boldsymbol{\chi}$, where $\boldsymbol{\chi} \in \boldsymbol{\mathcal{P}}_{k+1}^{\Delta}$ and the pair $(\boldsymbol{v} - \boldsymbol{\chi},\, \tau)$ satisfies the initial problem (S^e).

7.2.2 Generalized solutions

Theorem 7.2.3. *Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \ldots, \ell\} \quad \text{and} \quad N/p \notin \{1, \ldots, -\ell\}. \tag{7.2.5}$$

For any $(\boldsymbol{f},\, h) \in \big(\boldsymbol{W}_\ell^{-1,\,p}(\mathbb{R}^N) \times W_\ell^{0,\,p}(\mathbb{R}^N)\big) \perp \mathcal{S}^e_{[1+\ell-N/p']}$*, problem* (S^e) *admits a solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}^N) \times W_\ell^{0,\,p}(\mathbb{R}^N)$*, unique up to an element of* $\mathcal{S}^e_{[1-\ell-N/p]}$*, with the estimate*

$$\inf_{(\boldsymbol{\chi},\, q) \in \mathcal{S}^e_{[1-\ell-N/p]}} \Big(\|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}^N)} + \|\pi + q\|_{W_\ell^{0,\,p}(\mathbb{R}^N)} \Big) \leqslant C \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}_\ell^{-1,\,p}(\mathbb{R}^N)} + \|h\|_{W_\ell^{0,\,p}(\mathbb{R}^N)} \Big).$$

Proof. We proceed in three steps. First we solve the case $\ell = 0$, then we consider the negative weights to avoid troubles with the compatibility conditions and last, we obtain the solutions for positive weights by a duality argument.

(i) The Stokes operator

$$T: \ \big(\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^N) \times L^p(\mathbb{R}^N)\big)/\mathcal{S}^e_{[1-N/p]} \longrightarrow \big(\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^N) \times L^p(\mathbb{R}^N)\big) \perp \mathcal{S}^e_{[1-N/p']}$$

is an isomorphism.

The operator T is clearly continuous, moreover T is injective by Lemma 7.2.1, then by the Banach Theorem, it remains to show that it is surjective. Let us consider a pair $(\boldsymbol{f},\, h) \in \big(\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^N) \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}\big) \times L^p(\mathbb{R}^N)$, then $\operatorname{div} \boldsymbol{f}$ belongs to $W_0^{-2,\,p}(\mathbb{R}^N)$. Moreover, for any $\varphi \in \mathcal{P}_{[2-N/p']}$, we have

$$\langle \operatorname{div} \boldsymbol{f},\, \varphi \rangle_{W_0^{-2,\,p}(\mathbb{R}^N) \times W_0^{2,\,p'}(\mathbb{R}^N)} = \langle \boldsymbol{f},\, \nabla\varphi \rangle_{W_0^{-1,\,p}(\mathbb{R}^N) \times W_0^{1,\,p'}(\mathbb{R}^N)} = 0,$$

i.e. $\operatorname{div} \boldsymbol{f} \in W_0^{-2,\,p}(\mathbb{R}^N) \perp \mathcal{P}_{[2-N/p']}$ and the same argument holds for Δh. Then, according to the isomorphism[1] (2.2.2) with $\ell = 0$, there exists $\pi \in L^p(\mathbb{R}^N)$ solution to (7.2.1). Furthermore, for any $\psi \in W_0^{1,\,p'}(\mathbb{R}^N)$, $1 \leqslant i \leqslant N$,

$$\langle \partial_i \pi,\, \psi \rangle_{W_0^{-1,\,p}(\mathbb{R}^N) \times W_0^{1,\,p'}(\mathbb{R}^N)} = -\, \langle \pi,\, \partial_i \psi \rangle_{L^p(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)}\,.$$

That implies $\partial_i \pi \perp \mathbb{R}$ if $N/p' \leqslant 1$, and the same argument holds for $\partial_i h$. Thus, according to the isomorphism (2.2.12) with $m = 0$, there exists $\boldsymbol{u} \in W_0^{1,\,p}(\mathbb{R}^N)$ solution to (7.2.2). In addition, as we have seen above, $\operatorname{div} \boldsymbol{u} - h + \lambda\, \pi$ is a harmonic polynomial. Since it belongs to $L^p(\mathbb{R}^N)$, it is actually zero. So $(\boldsymbol{u},\, \pi)$ verifies $T(\boldsymbol{u},\, \pi) = (\boldsymbol{f},\, -h)$, which proves the surjectivity of T.

(ii) For any $\ell < 0$, the Stokes operator

$$T: \ \big(\boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}^N) \times W_\ell^{0,\,p}(\mathbb{R}^N)\big)/\mathcal{S}^e_{[1-\ell-N/p]} \longrightarrow \boldsymbol{W}_\ell^{-1,\,p}(\mathbb{R}^N) \times W_\ell^{0,\,p}(\mathbb{R}^N)$$

[1] Let us recall that the isomorphisms for the Laplacian are established in [5] and [6].

is an isomorphism.

It is the same reasoning to solve the two Poisson's equations (7.2.1) and (7.2.2), but using this time successively the isomorphisms (2.2.2), and (2.2.9) with ℓ instead of $1+\ell$. Then, modifying these solutions with a polynomial constructed by means of Lemma 7.2.2, we finally get a solution to (S^e).

(iii) For any $\ell > 0$, the adjoint operator of T,

$$T^* : \boldsymbol{W}_{\ell}^{1,p}(\mathbb{R}^N) \times W_{\ell}^{0,p}(\mathbb{R}^N) \longrightarrow \big(\boldsymbol{W}_{\ell}^{-1,p}(\mathbb{R}^N) \times W_{\ell}^{0,p}(\mathbb{R}^N)\big) \perp \mathcal{S}^e_{[1+\ell-N/p']}$$

is an isomorphism.

We get it by duality, replacing $-\ell$ by ℓ and p' by p. In addition, by a density argument, we show that

$$T^*(\boldsymbol{v},\, \vartheta) = (-\nu\, \Delta \boldsymbol{v} - \mu\, \nabla \operatorname{div} \boldsymbol{v} + \nabla\vartheta,\, -\lambda\, \vartheta - \operatorname{div} \boldsymbol{v}).$$

i.e. T is selfadjoint and the proof is complete. □

7.2.3 Regularity of solutions

Theorem 7.2.4. *Let $\ell \in \mathbb{Z}$ and $m \geqslant 1$ be two integers and assume that*

$$N/p' \notin \{1, \ldots, \ell+1\} \quad \text{and} \quad N/p \notin \{1, \ldots, -\ell-m\}. \tag{7.2.6}$$

For any $(\boldsymbol{f},\, h) \in \big(\boldsymbol{W}_{m+\ell}^{m-1,p}(\mathbb{R}^N) \times W_{m+\ell}^{m,p}(\mathbb{R}^N)\big) \perp \mathcal{S}^e_{[1+\ell-N/p']}$, problem (S^e) admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_{m+\ell}^{m+1,p}(\mathbb{R}^N) \times W_{m+\ell}^{m,p}(\mathbb{R}^N)$, unique up to an element of $\mathcal{S}^e_{[1-\ell-N/p]}$, with the estimate

$$\inf_{(\boldsymbol{\chi},\, q) \in \mathcal{S}^e_{[1-\ell-N/p]}} \Big(\|\boldsymbol{u}+\boldsymbol{\chi}\|_{\boldsymbol{W}_{m+\ell}^{m+1,p}(\mathbb{R}^N)} + \|\pi+q\|_{W_{m+\ell}^{m,p}(\mathbb{R}^N)} \Big) \leqslant C \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}_{m+\ell}^{m-1,p}(\mathbb{R}^N)} + \|g\|_{W_{m+\ell}^{m,p}(\mathbb{R}^N)} \Big).$$

Proof. For the negative weights, it is the same reasoning as for the generalized solutions, but using the regularity results for the Laplacian: (2.2.10) if $\ell \leqslant -2$, or (2.2.13) if $\ell = -1$, to solve (7.2.1); and (2.2.7) to solve (7.2.2). However, the case $N = p'$ for $\ell = -1$ and $m \geqslant 2$ is a critical value of the isomorphism (2.2.13), then it require the use of a critical result on the Laplace operator to solve (7.2.1). According to [6], the following Laplace operator is an isomorphism

$$\begin{aligned} \Delta : W_{m}^{1+m,p}(\mathbb{R}^N)/\mathcal{P}_{[1-N/p]} \longrightarrow X_{m}^{m-1,p}(\mathbb{R}^N) \perp \mathbb{R} \\ \text{if } N = p',\ m \geqslant 1, \end{aligned} \tag{7.2.7}$$

where the family of spaces X is defined as follows: for any $m \in \mathbb{Z}$, $\ell \in \mathbb{N}$,

$$\begin{aligned} X_{\ell}^{m+\ell,p}(\mathbb{R}^N) = \Big\{ u \in W_0^{m,p}(\mathbb{R}^N);\ \forall \lambda \in \mathbb{N}^N,\ 0 \leqslant |\lambda| \leqslant \ell, \\ x^{\lambda}\, u \in W_0^{m+|\lambda|,p}(\mathbb{R}^N);\ u \in W_{\mathrm{loc}}^{m+\ell,p}(\mathbb{R}^N) \Big\}, \end{aligned} \tag{7.2.8}$$

and its dual space is denoted by $X_{-\ell}^{-m-\ell,p'}(\mathbb{R}^N)$.

So, replacing m by $m-1$ in (7.2.7), we get the isomorphism

$$\Delta : W_{m-1}^{m,p}(\mathbb{R}^N)/\mathcal{P}_{[1-N/p]} \longrightarrow X_{m-1}^{m-2,p}(\mathbb{R}^N) \perp \mathbb{R} \quad \text{if } N = p',\ m \geqslant 2,$$

which precisely fills the gap of isomorphism (2.2.13) for this critical value.

In addition, we can show that $X_{m-1}^{m-2,p}(\mathbb{R}^N) = W_{m-1}^{m-2,p}(\mathbb{R}^N) \cap W_0^{-1,p}(\mathbb{R}^N)$. Since $\boldsymbol{f} \in \boldsymbol{W}_{m-1}^{m-1,p}(\mathbb{R}^N)$, we have $\operatorname{div} \boldsymbol{f} \in W_{m-1}^{m-2,p}(\mathbb{R}^N)$, and thanks to the imbedding $W_{m-1}^{m-1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, we also have $\operatorname{div} \boldsymbol{f} \in W_0^{-1,p}(\mathbb{R}^N)$, hence $\operatorname{div} \boldsymbol{f} \in X_{m-1}^{m-2,p}(\mathbb{R}^N)$. In the same way, we have $\Delta g \in X_{m-1}^{m-2,p}(\mathbb{R}^N)$, and thus we are able to solve (7.2.1). The rest of the proof is quite similar.

For $\ell \geqslant 0$, contrary to the generalized solutions, the duality reasoning fails, however we can use a regularity argument similar to the one of Section 5.4 for the classical system. We will develop it for the problem in the half-space. $\square$

7.3 Generalized solutions in $\mathbb{R}^N_+$

Where we come back to the half-space and to start, we are interested in the homogeneous problem.

Proposition 7.3.1. *For any $g_N \in W_0^{1-1/p,p}(\Gamma)$ and $\boldsymbol{g}' \in W_0^{-1/p,p}(\Gamma)^{N-1}$ such that $\boldsymbol{g}' \perp \mathbb{R}^{N-1}$ if $N \leqslant p'$, the Stokes problem*

$$-\nu\Delta \boldsymbol{u} - \mu\nabla \operatorname{div} \boldsymbol{u} + \nabla\pi = \boldsymbol{0} \qquad \text{in } \mathbb{R}^N_+, \tag{7.3.1}$$

$$\lambda\pi + \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \mathbb{R}^N_+, \tag{7.3.2}$$

$$u_N = g_N \qquad \text{on } \Gamma, \tag{7.3.3}$$

$$\partial_N \boldsymbol{u}' = \boldsymbol{g}' \qquad \text{on } \Gamma, \tag{7.3.4}$$

has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, unique if $N > p$, unique up to an element of $\mathbb{R}^{N-1} \times \{0\}^2$ if $N \leqslant p$, with the estimate

$$\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1}\times\{0\}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \leqslant C \left(\|g_N\|_{W_0^{1-1/p,p}(\Gamma)} + \|\boldsymbol{g}'\|_{W_0^{-1/p,p}(\Gamma)^{N-1}} \right)$$

if $N \leqslant p$, and the same without $\boldsymbol{\chi}$ if $N > p$.

Proof. First, let us notice a particular case, which is naturally included in this result, but which requires a particular treatment. Indeed, if $\lambda\,\mu = -1$, we simply get a Dirichlet problem for the Laplacian on the normal component of the velocity field u_N and a Neumann problem on its tangential components $\boldsymbol{u}'$. Then, applying Theorems 1.4.1 and 1.4.3, respectively for u_N and $\boldsymbol{u}'$, we find the orthogonality condition and the kernel of our statement. Moreover, we directly find the pressure from the velocity field thanks to the second equation. In the sequel of the proof, we will assume that $\lambda\,\mu \neq -1$.

(i) Reduction of system (7.3.1)–(7.3.4).

As for the question of the uniqueness in the whole space, we deduce from (7.3.1) and (7.3.2) that we have both $\Delta\pi = 0$ and $\Delta^2\boldsymbol{u} = \boldsymbol{0}$ in $\mathbb{R}^N_+$.

Then, we have $\Delta^2 u_N = 0$ in $\mathbb{R}^N_+$ and $u_N = g_N$ on Γ.

Now, let us extract another boundary condition on Δu_N from this system. From (7.3.2), we get

$$\lambda\,\partial_N\pi + \partial_N \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^N_+, \tag{7.3.5}$$

that we substitute in the N^{th} component of (7.3.1), to obtain

$$\lambda\,\nu\,\Delta u_N + (1+\lambda\,\mu)\,\partial_N \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^N_+.$$

We can deduce that

$$\begin{aligned}
\lambda\,\nu\,\Delta u_N + (1+\lambda\,\mu)\,(\operatorname{div}'\boldsymbol{g}' + \partial_N^2 u_N) &= 0 \quad \text{on } \Gamma,\\
\lambda\,\nu\,\Delta u_N + (1+\lambda\,\mu)\,(\operatorname{div}'\boldsymbol{g}' + \Delta u_N - \Delta' u_N) &= 0 \quad \text{on } \Gamma,\\
\big(1+\lambda\,(\mu+\nu)\big)\,\Delta u_N + (1+\lambda\,\mu)\,(\operatorname{div}'\boldsymbol{g}' - \Delta' g_N) &= 0 \quad \text{on } \Gamma,
\end{aligned}$$

hence,

$$\Delta u_N = \frac{1+\lambda\,\mu}{1+\lambda\,(\mu+\nu)}\,(\Delta' g_N - \operatorname{div}'\boldsymbol{g}') \quad \text{on } \Gamma.$$

About the pressure, looking again at the N^{th} component of (7.3.1), with (7.3.5), we have

$$\partial_N\pi = \nu\,\Delta u_N - \lambda\,\mu\,\partial_N\pi \quad \text{in } \mathbb{R}^N_+,$$

hence (since $\lambda\,\mu \neq -1$),

$$\partial_N\pi = \frac{\nu}{1+\lambda\,\mu}\,\Delta u_N \quad \text{on } \Gamma.$$

Finally, from (7.3.2), we also get

$$\lambda\,\nabla'\pi + \nabla' \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^N_+,$$

that we substitute in the tangential components of (7.3.1), to obtain

$$\Delta\boldsymbol{u}' = \frac{1+\lambda\,\mu}{\nu}\,\nabla'\pi \quad \text{in } \mathbb{R}^N_+.$$

Let us denote by κ_1 and κ_2 the two constants $\kappa_1 = \frac{1+\lambda\,\mu}{1+\lambda\,(\mu+\nu)}$ and $\kappa_2 = \frac{\nu}{1+\lambda\,\mu}$. So, we have found the following three problems

$$(B): \qquad \Delta^2 u_N = 0 \ \text{ in } \mathbb{R}^N_+, \quad u_N = g_N \ \text{ and } \ \Delta u_N = \kappa_1\,(\Delta' g_N - \operatorname{div}'\boldsymbol{g}') \ \text{ on } \Gamma.$$

$$(N1): \qquad \Delta \pi = 0 \ \text{ in } \mathbb{R}^N_+, \quad \partial_N \pi = \kappa_2\, \Delta u_N \ \text{ on } \Gamma,$$
$$(N2): \qquad \Delta \boldsymbol{u}' = \frac{1}{\kappa_2}\, \nabla' \pi \ \text{ in } \mathbb{R}^N_+, \quad \partial_N \boldsymbol{u}' = \boldsymbol{g}' \ \text{ on } \Gamma.$$

(ii) Solution of these three problems.

Since κ_1 and κ_2 are non-zero constants, it is clear that these problems have exactly the same form as those of Section 6.2 for the classical Stokes system. Therefore, we find $u_N \in W_0^{1,p}(\mathbb{R}^N_+)$, $\pi \in L^p(\mathbb{R}^N_+)$ and $\boldsymbol{u}' \in W_0^{1,p}(\mathbb{R}^N_+)^{N-1}$ with the same orthogonality condition on $\boldsymbol{g}'$, and the uniqueness of u_N and π.

(iii) Conversely, to show that solving (B), $(N1)$ and $(N2)$, we get a solution $(\boldsymbol{u}, \pi)$ to the original problem (7.3.1)–(7.3.4), we must make a few calculations.

The first equation of $(N2)$ is written

$$-\nu\, \Delta \boldsymbol{u}' + (1 + \lambda\, \mu)\, \nabla' \pi = 0 \quad \text{ in } \mathbb{R}^N_+. \tag{7.3.6}$$

Thanks to the first equations of (B) and $(N1)$, we get

$$\Delta\big(-\nu\, \Delta u_N + (1 + \lambda\, \mu)\, \partial_N \pi \big) = 0 \quad \text{ in } \mathbb{R}^N_+.$$

In addition, the boundary condition of $(N1)$ can be written

$$-\nu\, \Delta u_N + (1 + \lambda\, \mu)\, \partial_N \pi = 0 \quad \text{ on } \Gamma.$$

Since $-\nu\, \Delta u_N + (1 + \lambda\, \mu)\, \partial_N \pi \in W_0^{-1,p}(\mathbb{R}^N_+)$, according to Theorem 4.3.4, we necessarily have

$$-\nu\, \Delta u_N + (1 + \lambda\, \mu)\, \partial_N \pi = 0 \quad \text{ in } \mathbb{R}^N_+. \tag{7.3.7}$$

The boundary condition of $(N2)$ implies $\partial_N \operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ. Besides, the boundary conditions of (B) yield

$$\begin{aligned} \frac{1}{\kappa_1}\, \Delta u_N - \Delta' g_N &= -\operatorname{div}' \boldsymbol{g}' \quad \text{ on } \Gamma, \\ \partial_N^2 u_N + \lambda\, \kappa_2\, \Delta u_N &= -\operatorname{div}' \boldsymbol{g}' \quad \text{ on } \Gamma, \end{aligned}$$

hence, with the boundary condition of $(N1)$,

$$\partial_N^2 u_N + \lambda\, \partial_N \pi = -\operatorname{div}' \boldsymbol{g}' \quad \text{ on } \Gamma.$$

We can deduce

$$\begin{aligned} \partial_N \operatorname{div} \boldsymbol{u} &= \partial_N \operatorname{div}' \boldsymbol{u}' + \partial_N^2 u_N \quad \text{ on } \Gamma, \\ &= \operatorname{div}' \boldsymbol{g}' - \operatorname{div}' \boldsymbol{g}' - \lambda\, \partial_N \pi \quad \text{ on } \Gamma, \end{aligned}$$

that is

$$\partial_N(\lambda\, \pi + \operatorname{div} \boldsymbol{u}) = 0 \quad \text{ on } \Gamma.$$

Moreover, from (7.3.6) and (7.3.7), we obtain $\operatorname{div}(-\nu\,\Delta\boldsymbol{u}) = 0$ in $\mathbb{R}^N_+$, hence

$$\Delta(\lambda\,\pi + \operatorname{div}\boldsymbol{u}) = 0 \quad \text{in } \mathbb{R}^N_+.$$

Since $\lambda\,\pi + \operatorname{div}\boldsymbol{u} \in L^p(\mathbb{R}^N_+)$, by Theorem 4.3.2, we get $\lambda\,\pi + \operatorname{div}\boldsymbol{u} = 0$ in $\mathbb{R}^N_+$, that is the equation (7.3.2). Finally, substituting this last relation in (7.3.6) and (7.3.7), we find the first equation (7.3.1) of our system.

(iv) Concerning the uniqueness — up to the constants, if $N \leqslant p$ — for the tangential components of the velocity field $\boldsymbol{u}'$, we can still use the same argument as for the classical Stokes system (see Section 6.2). □

We now can deal with the nonhomogeneous problem.

Theorem 7.3.2. *Assume that $\frac{N}{p'} \neq 1$. For any $\boldsymbol{f} \in \boldsymbol{W}^{0,\,p}_1(\mathbb{R}^N_+)$, $h \in W^{1,\,p}_1(\mathbb{R}^N_+)$, $g_N \in W^{1-1/p,\,p}_0(\Gamma)$ and $\boldsymbol{g}' \in W^{-1/p,\,p}_0(\Gamma)^{N-1}$, satisfying the following compatibility condition if $N < p'$:*

$$\forall i \in \{1, \dots, N-1\}, \qquad \int_{\mathbb{R}^N_+} f_i \,\mathrm{d}x = \nu\,\langle g_i,\, 1\rangle_{W^{-1/p,\,p}_0(\Gamma)\times W^{1/p,\,p'}_0(\Gamma)}\,, \tag{7.3.8}$$

problem (S^{e}_N) admits a solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,\,p}_0(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, unique if $N > p$, unique up to an element of $\mathbb{R}^{N-1} \times \{0\}^2$ if $N \leqslant p$, with the estimate

$$\begin{aligned} &\inf_{\boldsymbol{\chi}\in\mathbb{R}^{N-1}\times\{0\}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}^{1,\,p}_0(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \\ &\quad\leqslant\; C\; \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p}_1(\mathbb{R}^N_+)} + \|h\|_{W^{1,\,p}_1(\mathbb{R}^N_+)} + \|g_N\|_{W^{1-1/p,\,p}_0(\Gamma)} + \|\boldsymbol{g}'\|_{W^{-1/p,\,p}_0(\Gamma)^{N-1}}\Big)\,. \end{aligned}$$

if $N \leqslant p$, and the same without $\boldsymbol{\chi}$ if $N > p$.

Proof. First, let us remark that the case $\lambda\,\mu = -1$ is naturally included in this result. Indeed, the condition (7.3.8) is necessary to solve the Neumann problem for the tangential components of the velocity field by means of Theorem 1.4.3.

In the general case, we introduce the Dirichlet problem

$$\Delta\boldsymbol{w} = \boldsymbol{f} \;\text{ in } \mathbb{R}^N_+, \quad \boldsymbol{w} = \boldsymbol{0} \;\text{ on } \Gamma.$$

According to Theorem 1.4.2, it admits a unique solution $\boldsymbol{w} \in \boldsymbol{W}^{2,\,p}_1(\mathbb{R}^N_+)$. So, if we consider $\mathbb{F} = (\mathbf{F}_i)_{1\leqslant i\leqslant N} = \nabla\boldsymbol{w} \in \boldsymbol{W}^{1,\,p}_1(\mathbb{R}^N_+)^N$, we have $\boldsymbol{f} = \operatorname{div}\mathbb{F}$. Knowing that there exists a continuous linear extension operator from $W^{1,\,p}_1(\mathbb{R}^N_+)$ to $W^{1,\,p}_1(\mathbb{R}^N)$, we get $\tilde{\boldsymbol{f}} = \operatorname{div}\tilde{\mathbb{F}} \in \boldsymbol{W}^{0,\,p}_1(\mathbb{R}^N)$, $\tilde{h} \in W^{1,\,p}_1(\mathbb{R}^N)$, and the extended problem

$$\left\{\begin{aligned} -\nu\,\Delta\tilde{\boldsymbol{z}} - \mu\,\nabla\operatorname{div}\tilde{\boldsymbol{z}} + \nabla\tilde{\eta} \;&= \tilde{\boldsymbol{f}} \quad &&\text{in } \mathbb{R}^N, \\ \lambda\,\tilde{\eta} + \operatorname{div}\tilde{\boldsymbol{z}} \;&= \tilde{h} &&\text{in } \mathbb{R}^N. \end{aligned}\right. \tag{7.3.9}$$

According to Theorem 7.2.4 with $\ell = 0$ and $m = 1$, under hypothesis $\frac{N}{p'} \neq 1$, problem (7.3.9) admits a solution $(\tilde{\boldsymbol{z}}, \tilde{\eta}) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^N) \times W_1^{1,p}(\mathbb{R}^N)$, provided the condition $\tilde{\boldsymbol{f}} \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}$ is fulfilled — indeed, $\mathcal{S}^e_{[1-N/p']} = \boldsymbol{\mathcal{P}}_{[1-N/p']} \times \{0\}$ —. But, thanks to the relation $\tilde{\boldsymbol{f}} = \operatorname{div} \tilde{\mathbb{F}}$, we can see that this condition is always fulfilled. Let us note $\boldsymbol{z} = \tilde{\boldsymbol{z}}|_{\mathbb{R}^N_+} \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^N_+)$ and $\eta = \tilde{\eta}|_{\mathbb{R}^N_+} \in W_1^{1,p}(\mathbb{R}^N_+)$, then we have $(\boldsymbol{z}, \eta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, $\gamma_0 z_N \in W_0^{1-1/p,p}(\Gamma)$ and $\gamma_1 \boldsymbol{z}' \in W_0^{-1/p,p}(\Gamma)^{N-1}$, and we can introduce the auxiliary problem

$$\begin{aligned} -\nu\,\Delta \boldsymbol{v} - \mu \nabla \operatorname{div} \boldsymbol{v} + \nabla \vartheta = \boldsymbol{0} \quad &\text{and} \quad \lambda\,\vartheta + \operatorname{div} \boldsymbol{v} = 0 &&\text{in } \mathbb{R}^N_+, \\ v_N = g_N - z_N \quad &\text{and} \quad \partial_N \boldsymbol{v}' = \boldsymbol{g}' - \partial_N \boldsymbol{z}' &&\text{on } \Gamma. \end{aligned} \tag{7.3.10}$$

where $g_N - z_N|_\Gamma \in W_0^{1-1/p,p}(\Gamma)$ and $\boldsymbol{g}' - \partial_N \boldsymbol{z}'|_\Gamma \in W_0^{-1/p,p}(\Gamma)^{N-1}$. Then, we can apply Proposition 7.3.1, which yields $(\boldsymbol{v}, \vartheta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ solution to problem (7.3.10), provided the orthogonality condition

$$\forall \boldsymbol{\varphi}' \in \mathbb{R}^{N-1}, \quad \langle \boldsymbol{g}' - \gamma_1 \boldsymbol{z}', \boldsymbol{\varphi}' \rangle_{W_0^{-1/p,p}(\Gamma)^{N-1} \times W_0^{1/p,p'}(\Gamma)^{N-1}} = 0 \tag{7.3.11}$$

is satisfied if $N < p'$. Now, to write this condition by only means of data of the initial problem, it suffices to notice that we have for all $\boldsymbol{\varphi} \in \mathbb{R}^{N-1} \times \{0\}$,

$$\begin{aligned} \int_{\mathbb{R}^N_+} \boldsymbol{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}x &= \int_{\mathbb{R}^N_+} (-\nu\,\Delta \boldsymbol{z} - \mu \nabla \operatorname{div} \boldsymbol{z} + \nabla \eta) \cdot \boldsymbol{\varphi}\,\mathrm{d}x \\ &= \nu\, \langle \gamma_1 \boldsymbol{z}', \boldsymbol{\varphi}' \rangle_{W_0^{-1/p,p}(\Gamma)^{N-1} \times W_0^{1/p,p'}(\Gamma)^{N-1}}, \end{aligned}$$

hence we deduce that the condition (7.3.11) is written

$$\forall \boldsymbol{\varphi}' \in \mathbb{R}^{N-1}, \quad \int_{\mathbb{R}^N_+} \boldsymbol{f}' \cdot \boldsymbol{\varphi}'\,\mathrm{d}x = \nu\, \langle \boldsymbol{g}', \boldsymbol{\varphi}' \rangle_{W_0^{-1/p,p}(\Gamma)^{N-1} \times W_0^{1/p,p'}(\Gamma)^{N-1}},$$

that is, the compatibility condition (7.3.8).

Then, the pair $(\boldsymbol{u}, \pi) = (\boldsymbol{v} + \boldsymbol{z}, \vartheta + \eta)$ which belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ is a solution to (S^e_N).

Finally, the uniqueness of solutions to (S^e_N) is a straightforward consequence of Proposition 7.3.1. □

7.4 Strong solutions and regularity

In this section, we are interested in the existence of strong solutions, *i.e.* of solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^N_+) \times W_1^{1,p}(\mathbb{R}^N_+)$, and next to get a general regularity result. We start with the homogeneous problem.

Proposition 7.4.1. *Assume that $\frac{N}{p'} \neq 1$. For any $g_N \in W_1^{2-1/p,p}(\Gamma)$ and $\boldsymbol{g}' \in W_1^{1-1/p,p}(\Gamma)^{N-1}$ such that $\boldsymbol{g}' \perp \mathbb{R}^{N-1}$ if $N < p'$, problem (7.3.1)–(7.3.4) admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$, unique if $N > p$, unique up to an element of $\mathbb{R}^{N-1} \times \{0\}^2$ if $N \leqslant p$, with the estimate*

$$\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1} \times \{0\}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_1^{1,p}(\mathbb{R}_+^N)} \leqslant C \left(\|g_N\|_{W_1^{2-1/p,p}(\Gamma)} + \|\boldsymbol{g}'\|_{W_1^{1-1/p,p}(\Gamma)^{N-1}} \right)$$

if $N \leqslant p$, and the same without $\boldsymbol{\chi}$ if $N > p$.

Proof. We have the imbeddings $W_1^{2-1/p,p}(\Gamma) \hookrightarrow W_0^{1-1/p,p}(\Gamma)$ and, since $\frac{N}{p'} \neq 1$, $W_1^{1-1/p,p}(\Gamma) \hookrightarrow W_0^{-1/p,p}(\Gamma)$.

Then, from Proposition 7.3.1, we can deduce that problem (7.3.1)–(7.3.4) admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}_+^N) \times L^p(\mathbb{R}_+^N)$. Now, it suffices to go back to the proof of Proposition 7.3.1 and to use the established results about problems (B), $(N1)$ and $(N2)$, to show that in fact $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$. □

We now can give the result in the nonhomogeneous case.

Theorem 7.4.2. *Assume that $\frac{N}{p'} \neq 1$. For any $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)$, $h \in W_1^{1,p}(\mathbb{R}_+^N)$, $g_N \in W_1^{2-1/p,p}(\Gamma)$ and $\boldsymbol{g}' \in W_1^{1-1/p,p}(\Gamma)^{N-1}$, satisfying the compatibility condition (7.3.8) if $N < p'$, problem (S_N^e) has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$, unique if $N > p$, unique up to an element of $\mathbb{R}^{N-1} \times \{0\}^2$ if $N \leqslant p$, with the estimate*

$$\inf_{\boldsymbol{\chi} \in \mathbb{R}^{N-1} \times \{0\}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_1^{1,p}(\mathbb{R}_+^N)} \leqslant C \left(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)} + \|h\|_{W_1^{1,p}(\mathbb{R}_+^N)} + \|g_N\|_{W_1^{2-1/p,p}(\Gamma)} + \|\boldsymbol{g}'\|_{W_1^{1-1/p,p}(\Gamma)^{N-1}} \right)$$

if $N \leqslant p$, and the same without $\boldsymbol{\chi}$ if $N > p$.

Proof. The proof of Theorem 7.3.2 work in this case. It suffices to take the strong result for generalized Stokes system in $\mathbb{R}^N$, *i.e.* Theorem 7.2.4, to solve (7.3.9); and Proposition 7.4.1 to solve (7.3.10). □

To finish, here is the corresponding regularity result.

Corollary 7.4.3. *Let $m \geqslant 1$ be an integer and assume that $\frac{N}{p'} \neq 1$. For any $\boldsymbol{f} \in \boldsymbol{W}_m^{m-1,p}(\mathbb{R}_+^N)$, $h \in W_m^{m,p}(\mathbb{R}_+^N)$, $g_N \in W_m^{m+1-1/p,p}(\Gamma)$ and $\boldsymbol{g}' \in W_m^{m-1/p,p}(\Gamma)^{N-1}$, satisfying the compatibility condition (7.3.8) if $N < p'$, problem (S_N^e) admits a* *solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_m^{m+1,\,p}(\mathbb{R}_+^N) \times W_m^{m,\,p}(\mathbb{R}_+^N)$*, unique if* $N > p$*, unique up to an element of* $\mathbb{R}^{N-1} \times \{0\}^2$ *if* $N \leqslant p$*, with the estimate*

$$\inf_{\boldsymbol{\chi}\in\mathbb{R}^{N-1}\times\{0\}} \|\boldsymbol{u}+\boldsymbol{\chi}\|_{\boldsymbol{W}_m^{m+1,\,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_m^{m,\,p}(\mathbb{R}_+^N)} \leqslant$$
$$C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_m^{m-1,\,p}(\mathbb{R}_+^N)} + \|h\|_{W_m^{m,\,p}(\mathbb{R}_+^N)} + \|g_N\|_{W_m^{m+1-1/p,\,p}(\Gamma)} + \|\boldsymbol{g}'\|_{W_m^{m-1/p,\,p}(\Gamma)^{N-1}}\right)$$

if $N \leqslant p$*, and the same without* $\boldsymbol{\chi}$ *if* $N > p$.

Proof. The simplest way is to resume the proof of Theorem 7.3.2. First, we solve the extended problem (7.3.9) by means of Theorem 7.2.4 for any integer $m \geqslant 1$; next, for the homogeneous problem (7.3.10), we can use the regularity results on the biharmonic and harmonic problems in the half-space to solve the auxiliary problems (B), $(N1)$ and $(N2)$, following the method employed in the proof of Proposition 7.3.1. □

7.5 Very weak solutions

Influence of the parameter λ, the problem from another point of view

As we remarked at the begining of the chapter, if $\lambda = 0$, we find the classical Stokes system which was the subject of the previous chapter. Now, if $\lambda \neq 0$, we can totally uncouple the velocity field from the pressure in the main equation.

First, in the whole space, the system (S^e) is clearly equivalent to

$$(S^e)\quad \begin{cases} -\nu\,\Delta\boldsymbol{u} - (\mu + \frac{1}{\lambda})\,\nabla\operatorname{div}\boldsymbol{u} = \boldsymbol{f} - \frac{1}{\lambda}\,\nabla h & \text{in } \mathbb{R}^N, \\ \pi = \frac{1}{\lambda}\,(h - \operatorname{div}\boldsymbol{u}) & \text{in } \mathbb{R}^N. \end{cases}$$

Denoting by A the operator $-\nu\,\Delta - (\mu + \frac{1}{\lambda})\,\nabla\operatorname{div}$, we can rewrite the main equation more simply

$$A\boldsymbol{u} = \mathbf{F} \quad \text{in } \mathbb{R}^N, \tag{7.5.1}$$

where $\mathbf{F} = \boldsymbol{f} - \frac{1}{\lambda}\,\nabla h$. Let us still notice that if $\lambda\,\mu = -1$, the operator A is nothing else but the Laplacian. Hence, solving (S^e) is equivalent to solve (7.5.1) — indeed, knowing the velocity field, we immediately get the pressure π —, moreover, the kernel of A is the velocity field's part, uncoupled from the pressure, in the kernel of T. So, we could express the results on system (S^e), that is Theorems 7.2.3 and 7.2.4, in terms adapted to equation (7.5.1).

Next, in the half-space, we also can formulate the problem (S_N^e) by means of equation (7.5.1) combined with the boundary conditions, *i.e.*

$$\begin{cases} A\boldsymbol{u} = \mathbf{F} & \text{in } \mathbb{R}_+^N, \\ u_N = g_N & \text{on } \Gamma, \\ \partial_N\boldsymbol{u}' = \boldsymbol{g}' & \text{on } \Gamma. \end{cases} \tag{7.5.2}$$

Here again, we could give a version adapted to problem (7.5.2) for all the results on (S_N^e). Conversely, in this section, we will use the form (7.5.2) to study the case of singular boundary conditions in the homogeneous problem. For that, we need some preliminary results.

Traces and Green formula

For any $\ell \in \mathbb{Z}$, let us introduce the spaces

$$\mathbf{U}_\ell(\mathbb{R}^N_+) = \left\{\boldsymbol{v} \in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+);\ A\boldsymbol{v} \in \boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)\right\},$$
$$\mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+) = \left\{\boldsymbol{v} \in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+);\ A\boldsymbol{v} \in \boldsymbol{W}^{0,\,p}_{\ell+1,\,1}(\mathbb{R}^N_+)\right\}.$$

They are reflexive Banach spaces equipped with their natural norms:

$$\|\boldsymbol{v}\|_{\mathbf{U}_\ell(\mathbb{R}^N_+)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)} + \|A\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)},$$
$$\|\boldsymbol{v}\|_{\mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)} + \|A\boldsymbol{v}\|_{\boldsymbol{W}^{0,\,p}_{\ell+1,\,1}(\mathbb{R}^N_+)}.$$

Lemma 7.5.1. *Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \ldots, \ell-1\} \quad \text{and} \quad N/p \notin \{1, \ldots, -\ell+1\}. \tag{7.5.3}$$

The space $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $\mathbf{U}_\ell(\mathbb{R}^N_+)$ and in $\mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+)$.

Proof. We give the proof for $\mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+)$, but it is similar for the space $\mathbf{U}_\ell(\mathbb{R}^N_+)$. For every continuous linear form $\boldsymbol{z} \in \big(\mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+)\big)'$, there exists a unique pair $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \times \boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^N_+)$, such that

$$\forall \boldsymbol{v} \in \mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+), \quad \langle \boldsymbol{z}, \boldsymbol{v}\rangle = \int_{\mathbb{R}^N_+} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\mathbb{R}^N_+} \boldsymbol{g}\cdot A\boldsymbol{v}\,\mathrm{d}x. \tag{7.5.4}$$

According to the Hahn-Banach theorem, it suffices to show that any $\boldsymbol{z}$ which vanishes on $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is actually zero on $\mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+)$. Let us suppose that $\boldsymbol{z} = \boldsymbol{0}$ on $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$, thus on $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$. Then we can deduce from (7.5.4) that

$$\boldsymbol{f} + A\boldsymbol{g} = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+,$$

hence we have $A\boldsymbol{g} \in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Let $\tilde{\boldsymbol{f}} \in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N)$ and $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^N)$ be respectively the extensions by $\boldsymbol{0}$ of $\boldsymbol{f}$ and $\boldsymbol{g}$ to $\mathbb{R}^N$. Thanks to (7.5.4), it is clear that $\tilde{\boldsymbol{f}} + A\tilde{\boldsymbol{g}} = \boldsymbol{0}$ in $\mathbb{R}^N$, and thus $A\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N)$. Hence, according to theorem 7.2.4 — for equation (7.5.1) —, we can deduce that $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N)$, under hypothesis (7.5.3). Since $\tilde{\boldsymbol{g}}$ is an extension by $\boldsymbol{0}$, it follows that we have $\boldsymbol{g} \in \mathring{\boldsymbol{W}}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Then, by density of $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ in $\mathring{\boldsymbol{W}}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$, there exists a sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ such that $\boldsymbol{\varphi}_k \to \boldsymbol{g}$ in $\mathring{\boldsymbol{W}}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Thus we have, for any $\boldsymbol{v} \in \mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+)$,

$$\begin{aligned}\langle \boldsymbol{z}, \boldsymbol{v}\rangle &= -\int_{\mathbb{R}^N_+} A\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\mathbb{R}^N_+} \boldsymbol{g}\cdot A\boldsymbol{v}\,\mathrm{d}x\\ &= \lim_{k\to\infty}\left\{-\int_{\mathbb{R}^N_+} A\boldsymbol{\varphi}_k\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\mathbb{R}^N_+} \boldsymbol{\varphi}_k\cdot A\boldsymbol{v}\,\mathrm{d}x\right\}\\ &= 0,\end{aligned}$$

i.e. $\boldsymbol{z}$ is identically zero. □

Thanks to this density lemma, we can prove the following result of traces:

Lemma 7.5.2. *Let* $\ell \in \mathbb{Z}$ *with (7.5.3).*

(i) If $N/p' \notin \{\ell,\, \ell+1\}$*, then the mapping*

$$\begin{aligned}(\gamma_{e_N},\, \gamma_1') : \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big) &\longrightarrow \boldsymbol{\mathcal{D}}(\mathbb{R}^{N-1})\\ \boldsymbol{v} &\longmapsto (v_N|_\Gamma,\, \partial_N \boldsymbol{v}'|_\Gamma),\end{aligned}$$

can be extended to a linear continuous mapping

$$(\gamma_{e_N},\, \gamma_1') : \mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+) \longrightarrow W^{-1/p,\,p}_{\ell-1}(\Gamma) \times \big(W^{-1-1/p,\,p}_{\ell-1}(\Gamma)\big)^{N-1}.$$

In addition, we have the Green formula

$$\begin{aligned}&\forall \boldsymbol{v} \in \mathbf{U}_\ell(\mathbb{R}^N_+),\ \forall \boldsymbol{\varphi} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \text{ such that } (\varphi_N,\, \partial_N\boldsymbol{\varphi}') = \mathbf{0} \text{ on } \Gamma,\\ &\langle A\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell-1}(\mathbb{R}^N_+)} - \langle \boldsymbol{v}, A\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} =\\ &\quad -\nu\,\langle v_N, \partial_N\varphi_N\rangle_{W^{-1/p,\,p}_{\ell-1}(\Gamma)\times W^{1-1/p',\,p'}_{-\ell+1}(\Gamma)} +\\ &\quad\quad +\nu\,\langle \partial_N\boldsymbol{v}', \boldsymbol{\varphi}'\rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma)} -\\ &\quad\quad\quad -(\mu+\frac{1}{\lambda})\,\langle v_N, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1/p,\,p}_{\ell-1}(\Gamma)\times W^{1-1/p',\,p'}_{-\ell+1}(\Gamma)}.\end{aligned}\tag{7.5.5}$$

(ii) If $N/p' \in \{\ell,\, \ell+1\}$*, the same result holds with* $\mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+)$ *instead of* $\mathbf{U}_\ell(\mathbb{R}^N_+)$ *and where* $\langle A\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell+1,\,1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^N_+)}$ *remplace the first term in the Green formula.*

Proof. (i) Case $N/p' \notin \{\ell,\, \ell+1\}$.

So, we have the imbedding $W^{2,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell-1,\,-1}(\mathbb{R}^N_+)$, hence the following Green formula:

$$\begin{aligned}&\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big),\ \forall \boldsymbol{\varphi} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \text{ such that } (\varphi_N,\, \partial_N\boldsymbol{\varphi}') = \mathbf{0} \text{ on } \Gamma,\\ &\int_{\mathbb{R}^N_+} \boldsymbol{\varphi}\cdot A\boldsymbol{v}\,\mathrm{d}x - \int_{\mathbb{R}^N_+} \boldsymbol{v}\cdot A\boldsymbol{\varphi}\,\mathrm{d}x\ =\\ &-\nu\int_\Gamma v_N\,\partial_N\varphi_N\,\mathrm{d}x' + \nu\int_\Gamma \partial_N\boldsymbol{v}'\cdot\boldsymbol{\varphi}'\,\mathrm{d}x' - \big(\mu+\frac{1}{\lambda}\big)\int_\Gamma v_N\,\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x'.\end{aligned}\tag{7.5.6}$$

In particular, if $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$ is such that $\boldsymbol{\varphi} = \mathbf{0}$ and $\partial_N \boldsymbol{\varphi}' = \mathbf{0}$ on Γ, we have

$$\left| \int_\Gamma v_N \, \partial_N \varphi_N \, \mathrm{d}x' \right| \;\leqslant\; \frac{\lambda}{1+\lambda(\mu+\nu)} \; \|\boldsymbol{v}\|_{\mathbf{U}_\ell(\mathbb{R}^N_+)} \; \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}.$$

Let $g \in W^{1-1/p',\,p'}_{-\ell+1}(\Gamma)$. By Lemma 1.3.1, there exists a lifting function $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\boldsymbol{\varphi} = \mathbf{0}$, $\partial_N \boldsymbol{\varphi}' = \mathbf{0}$ and $\partial_N \varphi_N = g$ on Γ, satisfying moreover

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} \;\leqslant\; C \; \|g\|_{W^{1-1/p',\,p'}_{-\ell+1}(\Gamma)},$$

where C is a constant not depending on $\boldsymbol{\varphi}$ and g. Then we can deduce that

$$\left| \int_\Gamma v_N \, g \, \mathrm{d}x' \right| \;\leqslant\; C \; \|\boldsymbol{v}\|_{\mathbf{U}_\ell(\mathbb{R}^N_+)} \; \|g\|_{W^{1-1/p',\,p'}_{-\ell+1}(\Gamma)},$$

and thus

$$\|v_N\|_{W^{-1/p,\,p}_{\ell-1}(\Gamma)} \;\leqslant\; C \; \|\boldsymbol{v}\|_{\mathbf{U}_\ell(\mathbb{R}^N_+)}.$$

Hence we can deduce that $\gamma_{e_N} : \boldsymbol{v} \longmapsto v_N|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is continuous for the norm of $\mathbf{U}_\ell(\mathbb{R}^N_+)$. Since $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $\mathbf{U}_\ell(\mathbb{R}^N_+)$, the mapping γ_{e_N} can be extended by continuity to $\gamma_{e_N} \in \mathcal{L}\big(\mathbf{U}_{\ell,1}(\mathbb{R}^N_+);\; W^{-1/p,\,p}_{\ell-1}(\Gamma)\big)$.

To define the trace γ'_1 on $\mathbf{U}_\ell(\mathbb{R}^N_+)$, we consider now $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\varphi_N = 0$, $\partial_N \boldsymbol{\varphi}' = \mathbf{0}$ and $\nu \partial_N \varphi_N + (\mu + \frac{1}{\lambda}) \operatorname{div} \boldsymbol{\varphi} = 0$ on Γ. Then, we have

$$\left| \int_\Gamma \partial_N \boldsymbol{v}' \cdot \boldsymbol{\varphi}' \, \mathrm{d}x' \right| \;\leqslant\; \frac{1}{\nu} \; \|\boldsymbol{v}\|_{\mathbf{U}_\ell(\mathbb{R}^N_+)} \; \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)}.$$

Let $\boldsymbol{g}' \in W^{2-1/p',\,p'}_{-\ell+1}(\Gamma)^{N-1}$. By Lemma 1.3.1, there exists a lifting function $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\boldsymbol{\varphi}' = \boldsymbol{g}'$, $\varphi_N = 0$, $\partial_N \boldsymbol{\varphi}' = \mathbf{0}$ and $\partial_N \varphi_N = -\kappa_1 \operatorname{div}' \boldsymbol{g}'$ on Γ, where $\kappa_1 = \frac{1+\lambda\,\mu}{1+\lambda\,(\mu+\nu)}$, satisfying moreover

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+)} \;\leqslant\; C \; \|\boldsymbol{g}\|_{W^{2-1/p',\,p'}_{-\ell+1}(\Gamma)},$$

where C is a constant not depending on $\boldsymbol{\varphi}$ and $\boldsymbol{g}'$. Then we can deduce that

$$\left| \int_\Gamma \partial_N \boldsymbol{v}' \cdot \boldsymbol{g}' \, \mathrm{d}x' \right| \;\leqslant\; C \; \|\boldsymbol{v}\|_{\mathbf{U}_\ell(\mathbb{R}^N_+)} \; \|\boldsymbol{g}'\|_{W^{2-1/p',\,p'}_{-\ell+1}(\Gamma)},$$

and thus

$$\|\boldsymbol{v}'\|_{W^{1-1/p,\,p}_{\ell-1}(\Gamma)} \;\leqslant\; C \; \|\boldsymbol{v}\|_{\mathbf{U}_\ell(\mathbb{R}^N_+)}.$$

Hence we can deduce that $\gamma'_1 : \boldsymbol{v} \longmapsto \partial_N \boldsymbol{v}'|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is continuous for the norm of $\mathbf{U}_\ell(\mathbb{R}^N_+)$. Since $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $\mathbf{U}_\ell(\mathbb{R}^N_+)$, the mapping γ'_1 can be extended by continuity to $\gamma'_1 \in \mathcal{L}\big(\mathbf{U}_\ell(\mathbb{R}^N_+);\; W^{-1-1/p,\,p}_{\ell-1}(\Gamma)^{N-1}\big)$.

To finish, we also can deduce the formula (7.5.5) from (7.5.6) by density of $\boldsymbol{\mathcal{D}}\big(\overline{\mathbb{R}^N_+}\big)$ in $\mathbf{U}_\ell(\mathbb{R}^N_+)$.

(ii) Case $N/p' \in \{\ell,\, \ell+1\}$.

Then the imbedding $W^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,\,p'}_{-\ell-1}(\mathbb{R}^N_+)$ fails, and we only have $W^{2,\,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^N_+)$. To avoid these two supplementary critical values with respect to hypothesis (7.5.3), the idea is to define the space $\mathbf{U}_{\ell,\,1}(\mathbb{R}^N_+)$ with a logarithmic factor in the weight to replace the first term in the Green formula (7.5.6) by the duality pairing $\langle A\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell+1,\,1}(\mathbb{R}^N_+)\times \boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^N_+)}$. The proof is identical. $\square$

Homogeneous problem with singular boundary conditions

Our purpose is now to solve the homogeneous problem (7.3.1)–(7.3.4):

$$\begin{aligned} -\nu\,\Delta\boldsymbol{u} - \mu\,\nabla\operatorname{div}\boldsymbol{u} + \nabla\pi = \boldsymbol{0} \quad &\text{and} \quad \lambda\,\pi + \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^N_+,\\ u_N = g_N \quad &\text{and} \quad \partial_N\boldsymbol{u}' = \boldsymbol{g}' \qquad\quad \text{on } \Gamma, \end{aligned}$$

with singular data on the boundary, that is more precisely with $g_N \in W^{-1/p,\,p}_{-1}(\Gamma)$ and $\boldsymbol{g}' \in W^{-1-1/p,\,p}_{-1}(\Gamma)^{N-1}$. Naturally, we will use the formulation

$$\begin{gathered} A\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+,\\ u_N = g_N \ \text{ and } \ \partial_N\boldsymbol{u}' = \boldsymbol{g}' \quad \text{on } \Gamma, \end{gathered} \tag{7.5.7}$$

that is the homogeneous version of problem (7.5.2), for system (7.3.1)–(7.3.4). As usual for the singular problems, the main tool is the Green formula (7.5.5), established in Lemma 7.5.2, which allows us to get the variational formulation of problem (7.5.2); then we can argue by duality to solve it.

Theorem 7.5.3. *Assume that* $\frac{N}{p} \neq 1$. *For any* $\boldsymbol{g}' \in W^{-1-1/p,\,p}_{-1}(\Gamma)^{N-1}$ *such that* $\boldsymbol{g}' \perp \mathbb{R}^{N-1}$ *if* $N \leqslant p'$, *and* $g_N \in W^{-1/p,\,p}_{-1}(\Gamma)$, *the Stokes problem (7.3.1)–(7.3.4) admits a solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+) \times W^{-1,\,p}_{-1}(\mathbb{R}^N_+)$, *unique if* $N > p$, *unique up to an element of* $\mathbb{R}^{N-1}\times\{0\}^2$ *if* $N < p$, *with the estimate*

$$\begin{aligned} \inf_{\boldsymbol{\chi}\in\mathbb{R}^{N-1}\times\{0\}} \|\boldsymbol{u}+\boldsymbol{\chi}\|_{\boldsymbol{W}^{0,\,p}_{-1}(\mathbb{R}^N_+)} &+ \|\pi\|_{W^{-1,\,p}_{-1}(\mathbb{R}^N_+)}\\ &\leqslant\ C\,\Big(\|g_N\|_{W^{-1/p,\,p}_{-1}(\Gamma)} + \|\boldsymbol{g}'\|_{W^{-1-1/p,\,p}_{-1}(\Gamma)^{N-1}}\Big) \end{aligned}$$

if $N < p$, *and the same without* $\boldsymbol{\chi}$ *if* $N > p$.

Proof. `Step 1:` We assume that $g_N = 0$.

(i) We can observe that problem (7.5.7) with $g_N = 0$ is equivalent to the following variational formulation: find $u \in \mathbf{U}_0(\mathbb{R}^N_+)$ — $\mathbf{U}_{0,\,1}(\mathbb{R}^N_+)$ if $\frac{N}{p'} = 1$ —

satisfying

$$\begin{aligned}&\forall \boldsymbol{v} \in \boldsymbol{W}_1^{2,\,p'}(\mathbb{R}^N_+) \text{ such that } (v_N,\, \partial_N \boldsymbol{v}') = \boldsymbol{0} \text{ on } \Gamma,\\ &\langle \boldsymbol{u}, A\boldsymbol{v}\rangle_{\boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}^N_+)\times \boldsymbol{W}_1^{0,\,p'}(\mathbb{R}^N_+)} = -\nu\ \langle \boldsymbol{g}', \boldsymbol{v}'\rangle_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)\times \boldsymbol{W}_1^{2-1/p',\,p'}(\Gamma)}\,.\end{aligned} \tag{7.5.8}$$

Indeed the direct implication is straightforward. Conversely, if $\boldsymbol{u}$ satisfies (7.5.8), then we have for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$,

$$\langle A\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_1^{0,\,p}(\mathbb{R}^N_+)\times \boldsymbol{W}_{-1}^{0,\,p'}(\mathbb{R}^N_+)} = \langle \boldsymbol{u}, A\boldsymbol{\varphi}\rangle_{\boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}^N_+)\times \boldsymbol{W}_1^{0,\,p'}(\mathbb{R}^N_+)} = 0,$$

thus $A\boldsymbol{u} = 0$ in $\mathbb{R}^N_+$. Moreover, by the Green formula (7.5.5), we have

$$\begin{aligned}&\forall \boldsymbol{v} \in \boldsymbol{W}_1^{2,\,p'}(\mathbb{R}^N_+) \text{ such that } (v_N,\, \partial_N \boldsymbol{v}') = \boldsymbol{0} \text{ on } \Gamma,\\ \nu\ \langle \boldsymbol{g}', \boldsymbol{v}'\rangle_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)\times \boldsymbol{W}_1^{2-1/p',\,p'}(\Gamma)} &= -\nu\ \langle u_N, \partial_N v_N\rangle_{W_{-1}^{-1/p,\,p}(\Gamma)\times W_1^{1-1/p',\,p'}(\Gamma)} +\\ &\quad + \nu\ \langle \partial_N \boldsymbol{u}', \boldsymbol{v}'\rangle_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)\times \boldsymbol{W}_1^{2-1/p',\,p'}(\Gamma)} -\\ &\quad -(\mu + \frac{1}{\lambda})\ \langle u_N, \operatorname{div} \boldsymbol{v}\rangle_{W_{-1}^{-1/p,\,p}(\Gamma)\times W_1^{1-1/p',\,p'}(\Gamma)}\,.\end{aligned}$$

By Lemma 1.3.1, for any $\zeta \in W_1^{1-1/p',\,p'}(\Gamma)$, there exists $\boldsymbol{v} \in \boldsymbol{W}_1^{2,\,p'}(\mathbb{R}^N_+)$ such that $\boldsymbol{v} = \boldsymbol{0}$, $\partial_N \boldsymbol{v}' = \boldsymbol{0}$ and $\partial_N v_N = \zeta$ on Γ. Consequently,

$$\langle u_N, \zeta\rangle_{W_{-1}^{-1/p,\,p}(\Gamma)\times W_1^{1-1/p',\,p'}(\Gamma)} = 0,$$

i.e. $u_N = 0$ on Γ. Likewise, for any $\boldsymbol{\zeta}' \in W_1^{2-1/p',p'}(\Gamma)^{N-1}$, there exists $\boldsymbol{v} \in \boldsymbol{W}_1^{2,\,p'}(\mathbb{R}^N_+)$ such that $\boldsymbol{v}' = \boldsymbol{\zeta}'$, $v_N = 0$ and $\partial_N \boldsymbol{v}' = \boldsymbol{0}$ on Γ. Consequently,

$$\langle \partial_N \boldsymbol{u}', \boldsymbol{\zeta}'\rangle_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)\times \boldsymbol{W}_1^{2-1/p',\,p'}(\Gamma)} = \langle \boldsymbol{g}', \boldsymbol{\zeta}'\rangle_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)\times \boldsymbol{W}_1^{2-1/p',\,p'}(\Gamma)}\,,$$

i.e. $\partial_N \boldsymbol{u}' = \boldsymbol{g}'$ on Γ.

(ii) Now, let us solve problem (7.5.8). According to Theorem 7.4.2 — adapted to problem (7.5.2) —, we know that if $\frac{N}{p} \neq 1$, for all $\boldsymbol{f} \in \boldsymbol{W}_1^{0,\,p'}(\mathbb{R}^N_+) \perp \big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}\big)$, there exists a unique $\boldsymbol{v} \in \boldsymbol{W}_1^{2,\,p'}(\mathbb{R}^N_+)/\big((\mathcal{P}_{[1-N/p']})^{N-1} \times \{0\}\big)$ solution to

$$\begin{aligned} A\boldsymbol{v} = \boldsymbol{f} \quad &\text{in } \mathbb{R}^N_+,\\ v_N = 0 \ \text{ and } \ \partial_N \boldsymbol{v}' = \boldsymbol{0} \quad &\text{on } \Gamma,\end{aligned}$$

with the estimate

$$\inf_{\boldsymbol{\chi} \in (\mathcal{P}_{[1-N/p']})^{N-1}\times\{0\}} \|\boldsymbol{v} + \boldsymbol{\chi}\|_{\boldsymbol{W}_1^{2,\,p'}(\mathbb{R}^N_+)} \ \leqslant\ C\ \|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,\,p'}(\mathbb{R}^N_+)}.$$

Now, consider the linear form

$$J: \ \boldsymbol{f} \longmapsto -\nu \left\langle \boldsymbol{g}', \, \boldsymbol{v}' \right\rangle_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)\times \boldsymbol{W}_{1}^{1+1/p,\,p'}(\Gamma)},$$

defined on $\boldsymbol{W}_{1}^{0,\,p'}(\mathbb{R}_+^N) \perp \big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}\big)$. Since $\boldsymbol{g}' \perp (\mathcal{P}_{[1-N/p']})^{N-1}$, we have for any $\boldsymbol{\chi}' \in (\mathcal{P}_{[1-N/p']})^{N-1}$,

$$\begin{aligned} |J\boldsymbol{f}| &= \Big|\langle \boldsymbol{g}', \, \boldsymbol{v}' + \boldsymbol{\chi}' \rangle_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)\times \boldsymbol{W}_{1}^{1+1/p,\,p'}(\Gamma)}\Big| \\ &\leqslant \ C \, \|\boldsymbol{v}\|_{\boldsymbol{W}_{1}^{2,\,p'}(\mathbb{R}_+^N)/(\mathcal{P}_{[1-N/p']})^{N-1}\times\{0\}} \, \|\boldsymbol{g}'\|_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)} \\ &\leqslant \ C \, \|\boldsymbol{f}\|_{\boldsymbol{W}_{1}^{0,\,p'}(\mathbb{R}_+^N)} \, \|\boldsymbol{g}'\|_{\boldsymbol{W}_{-1}^{-1-1/p,\,p}(\Gamma)}. \end{aligned}$$

Hence J is continuous on $\boldsymbol{W}_{1}^{0,\,p'}(\mathbb{R}_+^N) \perp \big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}\big)$, and thanks to the Riesz representation theorem, we can deduce that there exists a unique $\boldsymbol{u} \in \boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N)/\big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}\big)$, such that

$$\begin{aligned} \forall \boldsymbol{f} \in \boldsymbol{W}_{1}^{0,\,p'}(\mathbb{R}_+^N) \perp \big((\mathcal{P}_{[1-N/p]})^{N-1} \times \{0\}\big), \\ J\boldsymbol{f} = \langle \boldsymbol{u}, \boldsymbol{f} \rangle_{\boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N)\times \boldsymbol{W}_{1}^{0,\,p'}(\mathbb{R}_+^N)}, \end{aligned}$$

i.e. $\boldsymbol{u}$ satisfies (7.5.8).

`Step 2:` The general case (where we drop the hypothesis $g_N = 0$).

According to Theorem 4.3.3, we know that if $\frac{N}{p} \neq 1$, then there exists $\psi \in W_{-1}^{1,\,p}(\mathbb{R}_+^N)$ unique up to an element of $\mathcal{N}_{[2-N/p]}^{\Delta}$ solution to the following Neumann problem:

$$\Delta \psi = 0 \ \text{ in } \mathbb{R}_+^N, \quad \partial_N \psi = g_N \ \text{ on } \Gamma.$$

Let us set $\boldsymbol{w} = \nabla \psi$ and $\boldsymbol{g}'_* = \boldsymbol{g}' - \partial_N \boldsymbol{w}'$ on Γ. Then we have $A\boldsymbol{w} = \boldsymbol{0}$, hence $\boldsymbol{w} \in \mathbf{U}_0(\mathbb{R}_+^N)$ — $\mathbf{U}_{0,\,1}(\mathbb{R}_+^N)$ if $\frac{N}{p'} = 1$ — and $\boldsymbol{g}'_* \in W_{-1}^{-1-1/p,\,p}(\Gamma)^{N-1}$, with the estimate

$$\|\boldsymbol{w}\|_{\mathbf{U}_0(\mathbb{R}_+^N)} = \|\boldsymbol{w}\|_{\boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N)} \ \leqslant \ C \, \|g_N\|_{W_{-1}^{-1/p,\,p}(\Gamma)}.$$

Moreover, $\boldsymbol{g}'_*$ satisfies the orthogonality condition of Theorem 7.5.3, hence the existence of $\boldsymbol{z} \in \boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N)$ which satisfies

$$\begin{aligned} A\boldsymbol{z} = \boldsymbol{0} \ \text{ in } \mathbb{R}_+^N, \\ z_N = 0 \ \text{ and } \ \partial_N \boldsymbol{z}' = \boldsymbol{g}'_* \ \text{ on } \Gamma. \end{aligned}$$

Then $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{w} \in \boldsymbol{W}_{-1}^{0,\,p}(\mathbb{R}_+^N)$ is the required solution. □

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