1 Introduction

Introduce something

2 Model problem for passage through the fold

The following problem will be studied using the gluing method instead of blow up.

$$\begin{aligned}\dot{u} &= \mu + u^2 + u^3 \\ \dot{\mu} &= \varepsilon\end{aligned} \tag{2.1}$$

with boundary condition

$$u(T) = \delta \text{ and } \mu(0) = -\delta, \tag{2.2}$$

where T is another parameter, the "time of flight" for the trajectory to shoot from $\mu = -\delta$ to $u = \delta$.

We first study the "blow up " problem, starting with rescale $u = \varepsilon^{1/3} u_1(\varepsilon^{1/3} t)$ and $\mu = \varepsilon^{2/3} \mu_1(\varepsilon^{1/3} t)$. We get the new equations (set $\tau = \varepsilon^{1/3} t$)

$$\begin{aligned}\partial_\tau u_1 &= \mu_1 + u_1^2 + (\varepsilon^{2/3} u_1^4) \\ \partial_\tau \mu_1 &= 1 + (\varepsilon^{1/3} u_1)\end{aligned} \tag{2.3}$$

The new boundary condition is

$$u_1(T) = \delta\varepsilon^{-1/3}, \mu_1(0) = -\delta\varepsilon^{-2/3} \tag{2.4}$$

Then if we set $s = \tau - \delta\varepsilon^{-2/3}$ and formally let $\varepsilon \to 0$, equation (2.3) has an explicit solution $u_1(\tau) = u_R(s)$ and $\mu_1(\tau) = s$. Where u_R is the unique solution to the riccati equation $\partial_s u_R = s + u_R^2$ with the specific asymptotics [reference].

$$u_R(s) = \begin{cases} (T_R - s)^{-1} + \mathcal{O}(T_R - s), \text{ as } s \to T_R \\ -(-s)^{1/2} - \frac{1}{4}(-s)^{-1} + \mathcal{O}(|s|^{-3/2}), \text{ as } s \to -\infty \end{cases} \tag{2.5}$$

From this and the boundary condition (2.4), we have the asymptotics for T:

$$T(\varepsilon) = \delta\varepsilon^{-1} + T_R\varepsilon^{-1/3} - \delta^{-1} + \mathcal{O}(\varepsilon^{2/3}) \tag{2.6}$$

Boundary condition $u_+(t = T) = \delta$, we derive the asymtotics for T,

$$T = T(\varepsilon) \sim \delta\varepsilon^{-1/6} + t_* - \delta^{-1} = \varepsilon^{-1/3}\Omega_0 + \varepsilon^{-1}\delta - \delta^{-1}.$$

Using the asymptotics for ψ and u_R, we calculate that

$$\varphi(\sigma) = \begin{cases} (-\frac{3}{2}\sigma)^{-1/3}, \text{ as } \sigma \to -\infty \\ e^{-\sigma}, \text{ as } \sigma \to \infty. \end{cases} \tag{2.7}$$

$$u_R(\psi(\sigma)) = \begin{cases} -(-\frac{3}{2}\sigma)^{1/3}, \text{ as } \sigma \to -\infty \\ e^{\sigma}, \text{ as } \sigma \to \infty. \end{cases} \tag{2.8}$$

$$a(\sigma) = \begin{cases} -2 + \mathcal{O}((-\sigma)^{-3/2}), \text{ as } \sigma \to -\infty \\ 2 + \mathcal{O}(e^{-2\sigma}), \text{ as } \sigma \to \infty. \end{cases} \tag{2.9}$$

so the linear operator $\mathcal{L} = \frac{d}{d\sigma} - A(\sigma)$, where $A(\sigma) \to A_{\pm} = \mathrm{diag}(\pm 2, 0)$ as $\sigma \to \pm\infty$. We need to find the right function spaces.

3 summary for set up

Equation

$$\begin{aligned} &\frac{d}{dt}u(t) = (\mu + u^2 + u^3)(t) \\ &\frac{d}{dt}\mu(t) = \varepsilon \end{aligned} \tag{3.1}$$

with B.C.

$$\mu(0) = -\delta, \quad u(T) = \delta. \tag{3.2}$$

where δ, ε, T are parameters.

3.1 The Riccati solution

This is taken from [Krupa, Szmolyan].

Consider the riccati equation

$$\frac{d}{dt}u(t) = t + u(t)^2 \tag{3.3}$$

(3.3) is known to have a unique solution (here we denote by u_R) with the following asymptotics:

$$u_R(t) = (\Omega_0 - t)^{-1} + \mathcal{O}(|\Omega_0 - t|)$$

as $t \to \Omega_0^-$ and

$$u_R(t) = -\sqrt{-t} + \mathcal{O}(|t|^{-1})$$

as $t \to -\infty$.

Here the constant Ω_0 is the smallest positive zero of a certain combination of Bessel functions of the first kind.

3.2 The t to σ time rescaling

step 1: Define ψ as

$$\psi = \varepsilon^{1/3}(t - \varepsilon^{-1}\delta)$$

step 2: Take $M > 0$ large, define σ as

$$\psi = \psi(\sigma) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, \text{ for } \sigma \le -M \\ \Omega_0 - e^{-\sigma}, \text{ for } \sigma \ge M, \end{cases}$$

and smooth interpolation in between so that $\psi(0) = 0$, here Ω_0 is the blow-up time for $u_R(s)$, the unique solution to the ricatti equation that satisfy the asymptotics.

We also define $\varphi(\sigma) := \frac{d}{d\sigma}\psi(\sigma)$.

For convenience let the map $t \mapsto \sigma$ be denoted as ρ.

3.3 Region I

In σ variable, we divide the real line into two segments. In different regions we will have different ansatz. Region I is defined by $\{\sigma : \sigma < 0\}$. Which corresponds to the original time t as $\{t : t < \varepsilon^{-1}\delta\}$.

3.3.1 Important times

- $t = 0$
- $t = t^*$, the (left) gluing time which corresponds to when $\sigma = \varepsilon^{-1/4} =: \sigma^*$.

3.3.2 ansatz in region I

The ansatz in region I takes the form

$$u_I(t) = \chi_s(\rho(t))u_s(t) + \chi_l(\rho(t))u_l(t) + W_s(t) + W_l(t)$$

Where

- $u_s(t)$ denotes the "singular" branch that forms the slow manifold (critical manifold?) of the original system. It is defined via the relation

  $$u_s(t) = h(\mu(t))$$

  for some smooth function h which solves

  $$0 = \mu(t) + h(\mu(t))^2 + h(\mu(t))^3. \tag{3.4}$$

  It has the following asymptotics:

  $$u_s(t) = -\sqrt{\delta - \varepsilon t} + \mathcal{O}(|\delta - \varepsilon t|). \tag{3.5}$$

  The equivalent in σ variable is

  $$u_s(\sigma) = -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathcal{O}(|\varepsilon\sigma|^{2/3}) \tag{3.6}$$

- $u_l(t)$ is defined by rescaling u_R and restrict it for $t < \varepsilon^{-1}\delta$. Specifically:

  $$u_l(t) = \varepsilon^{1/3}u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta)) \tag{3.7}$$

  It solves the equation

  $$\frac{d}{dt}u_l(t) = \mu(t) + u_l^2(t) \tag{3.8}$$

- The cutoff functions χ_s and χ_l are functions of σ directly, and they satisfy (for $\sigma \leq 0$)

  $$\chi_s(\sigma) = \begin{cases} 1, & \sigma \leq \sigma^* - 1 \\ 0, & \sigma \geq \sigma^* + 1. \end{cases} \tag{3.9}$$

  and

  $$\chi_l(\sigma) = \begin{cases} 0, & \sigma \leq \sigma^* - 1 \\ 1, & \sigma \geq \sigma^* + 1. \end{cases} \tag{3.10}$$

- norms

  From notes:
  $$W_\ell \approx \varepsilon^{2/3-\alpha}\langle\sigma\rangle^{2/3}$$
  and
  $$W_s \approx \varepsilon^{1-\alpha}\langle\varepsilon\sigma\rangle^{-2/3}$$

3.3.3 equation for ansatz in region I

3.4 Region II

Region II is defined by $\{\sigma : \sigma > 0\}$. Which corresponds to the original time t as $\{t : t > \varepsilon^{-1}\delta\}$.

3.4.1 Important times

- $T_\infty = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_0$, is the blow up time to the riccati solution, which corresponds to $\sigma = \infty$.
- $T = T_\infty - \delta^{-1}$, is the right boundary point
- $t_* = T_\infty - \varepsilon^{-1/6}\delta^{-1}$, is the (right) gluing time.

3.4.2 ansatz in region II

The ansatz in region II takes the form

$$u_{II}(t) = \chi_r(t)u_r(t) + \chi_b(t)u_b(t) + W_r(t) + W_b(t)$$

Where

- u_r has the same formula as u_l, except it is restricted on $t > \varepsilon^{-1}\delta$.
  $$u_r(t) = \varepsilon^{1/3}u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta)), \tag{3.11}$$
  it satisfies
  $$\frac{d}{dt}u_r(t) = \mu(t) + u_r^2(t). \tag{3.12}$$
- u_b is a "blow up" layer that is defined as follows:
  $$u_b(t) = (u_r(t_*)^{-1} + t_* - t)^{-1}, \tag{3.13}$$
  it satisfies
  $$\frac{d}{dt}u_b = u_b^2. \tag{3.14}$$

- The cutoff χ_b and χ_r are functions of t and it is true that $1 = \chi_b + \chi_r$, and they satisfy

$$\chi_r(t) = \begin{cases} 1, & t \leq t_* - 1 \\ 0, & t \geq t_* + 1. \end{cases} \tag{3.15}$$

  and

$$\chi_b(t) = \begin{cases} 0, & t \leq t_* - 1 \\ 1, & t \geq t_* + 1. \end{cases} \tag{3.16}$$

- We introduce the time ξ with the scaling

$$e^{-\xi} = u_b(t)^{-1}$$

Asymptotics

$$u_b(t) = e^{\xi}$$

As for $\mu(t) = \varepsilon t - \delta$, we have

$$\begin{aligned}\mu &= \varepsilon(t_* + \delta\varepsilon^{-1/6} - u_+^{-1}) - \delta = \varepsilon^{2/3}\Omega_0 + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon u_+^{-1} \\ &= \Omega_0\varepsilon^{2/3} + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon e^{-\xi}\end{aligned}$$

- Gluing time

the gluing time σ_* is set to equal to $\log(\varepsilon^{-1/6}\delta)$, notice in terms of the original time t, this is at

$$s(\sigma_*) = \Omega_0 - \delta^{-1}\varepsilon^{1/6} = \tau - \tau_0 = \varepsilon^{1/3}t - \varepsilon^{-2/3}\delta \implies t = t_* := \varepsilon^{-1/3}[\Omega_0 + \varepsilon^{-2/3}\delta - \delta^{-1}\varepsilon^{1/6}]$$

We note then

$$u_* := u_-(t_*) = \varepsilon^{1/3}[(\Omega_0 - (\Omega_0 - \delta^{-1}\varepsilon^{1/6}))^{-1} + \mathcal{O}(\varepsilon^{1/6})] = \varepsilon^{1/6}\delta + \mathcal{O}(\varepsilon^{1/2})$$

- since $u_b(t)$ solves

$$\frac{d}{dt}u_b(t) = u_b(t)^2,$$

Convert the equation in ξ time via

$$e^{-\xi}\frac{d}{dt} = \frac{d}{d\xi},$$

- norms see subsection on linear equation.

3.4.3 Distribute equations for ansatz in region II

Substituting u_{II} into (2.1) gives

$$\begin{aligned}\chi_r'u_r + u_r'\chi_r + \chi_b'u_b + u_b'\chi_b + W_r' + W_b' &= \mu + (\chi_r u_r + W_r + \chi_b u_b + W_b)^2 \\ &\quad + (\chi_r u_r + W_r + \chi_b u_b + W_b)^3\end{aligned}$$

to get the appropriate distribution of terms, we first simplify:

use $\chi_r + \chi_b = 1$, we have

$$\chi_r'u_r + \chi_b'u_b = \chi_r'(u_r - u_b)$$

by (3.12) and (3.14), we have

$$u_r'\chi_r = \chi_r(\mu + u_r^2), \quad u_b'\chi_b = \chi_b u_b^2$$

we expand the quardratic terms first, this gives

$$(\chi_r u_r + W_r)^2 + (\chi_b u_b + W_b)^2 + 2(\chi_r u_r + W_r)(\chi_b u_b + W_b)$$

We move $\chi_r'(u_r - u_b)$, $\chi_r u_r'$ and $\chi_b u_b'$ to the right handside, without the cubic terms, the right hand side becomes

$$\begin{aligned}
&\chi_r'(u_r - u_b) + (\chi_r^2 - \chi_r)u_r^2 + (\chi_b^2 - \chi_b)u_b^2 + 2\chi_r\chi_b u_r u_b \\
&+ 2\chi_r u_r W_r + 2\chi_b u_b W_b + 2\chi_r u_r W_b + 2\chi_b u_b W_r \\
&+ W_r^2 + W_b^2 + 2W_r W_b.
\end{aligned}$$

Since $\chi_r^2 - \chi_r = \chi_r(\chi_r - 1) = -\chi_r\chi_b$, $\chi_b^2 - \chi_b = \chi_b(\chi_b - 1) = -\chi_b\chi_r$, we have

$$(\chi_r^2 - \chi_r)u_r^2 + (\chi_b^2 - \chi_b)u_b^2 + 2\chi_r\chi_b u_r u_b = -\chi_b\chi_r(u_r - u_b)^2.$$

For $2\chi_r u_r W_r + 2\chi_b u_b W_b + 2\chi_r u_r W_b + 2\chi_b u_b W_r$, it can be simplified as

$$2(\chi_r u_r + \chi_b u_b)W_r = 2(u_r + \chi_b(u_b - u_r))W_r, \quad 2(\chi_r u_r + \chi_b u_b)W_b = 2(u_b + \chi_r(u_r - u_b))W_b.$$

Lastly, we note

$$2W_r W_b = 2\chi_r W_r W_b + 2\chi_b W_r W_b$$

Hence, up to quadratic terms, the original equation becomes

$$\begin{aligned}
W_r' + W_b' = &\chi_b\mu + \chi_r'(u_r - u_b) - \chi_b\chi_r(u_r - u_b)^2 + \\
&+ 2u_r W_r + 2\chi_b(u_b - u_r)W_r + \\
&+ 2u_b W_b + 2\chi_r(u_r - u_b)W_b + \\
&+ W_r^2 + W_b^2 + 2\chi_r W_b W_r + 2\chi_b W_b W_r
\end{aligned}$$

To simplify the cubic term so that it become natural to distribute the terms, first we have

$$\begin{aligned}
&(\chi_r u_r + W_r + \chi_b u_b + W_b)^3 \\
&= (\chi_r u_r + \chi_b u_b)^3 + (W_r + W_b)^3 + \\
&+ 3(\chi_r u_r + \chi_b u_b)(W_r + W_b)^2 + 3(\chi_r u_r + \chi_b u_b)^2(W_r + W_b)
\end{aligned}$$

For $(W_r + W_b)^3$ and $3(\chi_r u_r + \chi_b u_b)(W_r + W_b)^2$, we already distributed the quadratic term $(W_r + W_b)^2$ as $W_r^2 + W_b^2 + 2\chi_r W_b W_r + 2\chi_b W_b W_r$. The linear term $3(\chi_r u_r + \chi_b u_b)^2(W_r + W_b)$ can be ditrubuted as follows:

$$\begin{aligned}
3(\chi_r u_r + \chi_b u_b)^2(W_r + W_b) = &3[(\chi_r u_r)^2 + 2(\chi_r\chi_b)(u_r u_b)]W_r + \\
&+ 3(\chi_r u_r)^2 W_b + \\
&+ 3[(\chi_b u_b)^2 + 2(\chi_r\chi_b)(u_r u_b)]W_b + \\
&+ 3(\chi_b u_b)^2 W_r
\end{aligned}$$

The pure residual term $(\chi_r u_r + \chi_b u_b)^3$ equals

$$(\chi_r u_r)^3 + (\chi_b u_b)^3 + 3(\chi_r^2\chi_b)u_r^2 u_b + 3(\chi_b^2\chi_r)u_b^2 u_r$$

As we shall see, the distribution of the mixed terms is flexible. Hence we have

3.4.4 Equation for W_r

$$
\begin{aligned}
W_r' - 2u_r W_r = &\frac{\chi_r'}{2}(u_r - u_b) - \frac{\chi_b \chi_r}{2}(u_r - u_b)^2 \\
&+ 3[(\chi_r u_r)^2 + 2(\chi_r \chi_b)(u_r u_b)]W_r \\
&+ [3(\chi_r u_r)^2 + \chi_r(u_r - u_b)]W_b \\
&+ (\chi_r u_r)^3 + 3(\chi_r^2 \chi_b)u_r^2 u_b \\
&+ [1 + (W_r + W_b) + 3(\chi_r u_r + \chi_b u_b)](W_r^2 + 2\chi_r W_b W_r)
\end{aligned}
\tag{3.17}
$$

3.4.5 Equation for W_b

$$
\begin{aligned}
W_b' - 2u_b W_b = &\chi_b \mu + \frac{\chi_r'}{2}(u_r - u_b) - \frac{\chi_b \chi_r}{2}(u_r - u_b)^2 \\
&+ 3[(\chi_b u_b)^2 + 2(\chi_r \chi_b)(u_r u_b)]W_b \\
&+ [3(\chi_b u_b)^2 + \chi_b(u_b - u_r)]W_r \\
&+ (\chi_b u_b)^3 + 3(\chi_b^2 \chi_r)u_b^2 u_r \\
&+ [1 + (W_r + W_b) + 3(\chi_r u_r + \chi_b u_b)](W_b^2 + 2\chi_b W_b W_r)
\end{aligned}
\tag{3.18}
$$

3.4.6 Linear Equation and Norms

Denote the right hand side of (3.17) as R_r, use the time-scaling map between t and σ, the entire equation becomes

$$\left(\frac{d}{d\sigma} - 2\varphi u_R\right)\tilde{W}_r = \varepsilon^{-1/3}\varphi\tilde{R}_r \tag{3.19}$$

Here $\tilde{W}_r(\sigma) = W_r(\rho^{-1}(\sigma)) = W_r(t)$, and $\tilde{R}_r$ is similarly defined. We will abuse notation and drop the title below.

Similarly, to rescale equation (3.18), recall the time variable ξ is defined by $u_b(t) = e^{\xi}$, hence we obtain

$$\left(\frac{d}{d\xi} - 2\right)\tilde{W}_b = e^{-\xi}\tilde{R}_b \tag{3.20}$$

By the asmpyotic properties of φ and u_R, it is true that

$$2\varphi(\sigma)u_R(\sigma) \to 2 \text{ as } \sigma \to \infty.$$

Our goal now is to solve (3.19) on the interval $\sigma \in (0, \rho(T))$ and solve (3.20) on $\xi \in (\ln(u_b(\varepsilon^{-1}\delta)), \ln(u_b(T)))$ using a fixed point argument.

To do so, we introduce the function spaces below:

$$
\begin{aligned}
\mathcal{C}_{Wr} &= \left\{ u(\sigma) : \sup_{\sigma \geq 0} \left| \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} u(\sigma) \right| < \infty \right\} \\
&= \left\{ u(t) : \sup_{t \geq \varepsilon^{-1}\delta} \left| (T_\infty(\varepsilon) - t)^{2-\alpha} u(t) \right| < \infty \right\} \\
\mathcal{C}_{Wb} &= \left\{ u(\xi) : \sup_{e^{\xi} \geq u_b(\varepsilon^{-1}\delta)} \left| e^{(\alpha-2)\xi} u(\xi) \right| < \infty \right\} \\
&= \left\{ u(t) : \sup_{t \geq \varepsilon^{-1}\delta} \left| (T_\infty(\varepsilon) - t)^{2-\alpha} u(t) \right| < \infty \right\}
\end{aligned}
$$

Theorem 3.1. *For $W_r \in \mathcal{C}_{Wr}, W_b \in \mathcal{C}_{Wb}$, it is true that $\varepsilon^{-1/3}\varphi R_r \in \mathcal{C}_{Wr}$ and $e^{-\xi} R_b \in \mathcal{C}_{Wb}$. Specifically*

$$
\|\varepsilon^{-1/3}\varphi R_r\| = \mathcal{O}(\varepsilon^{5\alpha/6}) \tag{3.21}
$$

$$
\|e^{-\xi} R_b\| = \mathcal{O}(\varepsilon^{?}) \tag{3.22}
$$

Proof. We collect the estimates needed to prove this theorem:

For $\varepsilon^{-1}\delta \leq t \leq t_*$:

$$
\begin{aligned}
|u_r| \lesssim |u_b| &\lesssim (T_\infty - t)^{-1} \\
W_b &\lesssim (T_\infty - t_*)^{\alpha-2}
\end{aligned}
$$

For $t_* - 1 \leq t \leq T$:

$$
\begin{aligned}
W_r &\lesssim (T_\infty - T)^{\alpha-2} \\
|u_b(t) - u_r(t)| &\lesssim \varepsilon^{1/3}
\end{aligned}
$$

For $|t - t_*| \leq 1$:

$$
|u_b(t) - u_r(t)| \lesssim \varepsilon^{2/3}
$$

To be consistent, we convert these norms back to the t variable and establish the corresponding estimate. $\square$

4 Gluing

Ansatz

$$U(t) = \chi_-(t)u_-(t) + \chi_+(t)u_+(t) + W_-(t) + W_+(t;\beta),$$

where $W_+(t;\beta) = W_+(t) + \beta w_+^k$, $w_+^k = \chi_{\{t<\varepsilon^{-1}\delta\}}u_+^2$. Also recall $\mu(t) = \varepsilon t - \delta$.

"insert picture of χ_-, χ_+."

The support of χ_+ is $(t_* - 1, \infty)$ and the support of χ_- is $(-\infty, t_* + 1)$.

- Plug in the anstaz

  $$\begin{aligned}&\chi_-' u_- + \chi_- u_-' + \chi_+' u_+ + \chi_+ u_+' + W_-'(t) + W_+'(t) = \\ &= \mu + (\chi_- u_- + W_- + \chi_+ u_+ + W_+)^2 + (\chi_- u_- + W_- + \chi_+ u_+ + W_+)^3.\end{aligned}$$

  where $' = \frac{d}{dt}$.

  Useful identities

  $$\chi_- + \chi_+ = 1, \quad \frac{d}{dt}(\chi_- + \chi_+)(t) = 0,$$

  and

  $$\frac{d}{dt}u_- = \mu + u_-^2, \quad \frac{d}{dt}u_+ = u_+^2.$$

  Equation after cancellation:

  $$\begin{aligned}-\chi_+'(u_- - u_+) + W_-' + W_+' = {}& \chi_+\mu + \chi_-\chi_+(u_+ - u_-)u_- + 2\chi_- u_- W_- + W_-^2 \\ &+ \chi_-\chi_+(u_- - u_+)u_+ + 2\chi_+ u_+ W_+ + W_+^2 \\ &+ 2\chi_+ u_+ W_- + 2\chi_- u_- W_+ + 2W_- W_+ \\ &+ (\chi_- u_- + W_- + \chi_+ u_+ + W_+)^3\end{aligned}$$

- Distribute terms in the $-$ side

  $$\begin{aligned}W_-' - 2u_- W_- = {}& 2\chi_-(u_- - u_+)W_+ + (\chi_-\chi_+)u_-(u_- - u_+) + \frac{1}{2}\chi_-'(u_+ - u_-) + 2\chi_- W_- W_+ \\ &+ W_-^2 + (\chi_- u_- + W_-)^3 \\ &+ 3\chi_-^2\chi_+ u_-^2 u_+ + 6\chi_-\chi_+ u_+ u_- W_- + 6\chi_- u_- W_- W_+ + 3\chi_-(W_- + W_+)W_- W_+ \\ &+ 3(\chi_+ u_+)W_-^2 + 3(\chi_- u_-)^2 W_+\end{aligned}$$

- Distribute terms in the $+$ side

  $$\begin{aligned}W_+' - 2u_+ W_+ = {}& \chi_+\mu + 2\chi_+(u_+ - u_-)W_- + (\chi_-\chi_+)u_+(u_- - u_+) + \frac{1}{2}\chi_+'(u_- - u_+) + 2\chi_+ W_- W_+ \\ &+ W_+^2 + (\chi_+ u_+ + W_+)^3 \\ &+ 3\chi_+^2\chi_- u_+^2 u_- + 6\chi_+\chi_- u_+ u_- W_+ + 6\chi_+ u_+ W_+ W_- + 3\chi_+(W_- + W_+)W_- W_+ \\ &+ 3(\chi_- u_-)W_+^2 + 3(\chi_+ u_+)^2 W_-\end{aligned}$$

4.1 Linear equation.

Now the equation in W_- and W_+ can be written in the following form

$$W_-' - 2u_- W_- = \mathcal{R}_-,$$
$$W_+' - 2u_+ W_+ = \mathcal{R}_+$$

with $\mathcal{R}_\pm$ defined as in the distribution of terms.

First fix

$$\eta \in (1,2), \nu = 2 - \eta \in (0,1).$$

To be able to solve the linear equation, we first introduce the following weighted spaces, for the $-$ side we have

$$\mathcal{C}_v = \{u(t) \in \mathcal{C}(0,T) \mid \sup |v(t)u(t)| < \infty\}$$

where the weight $v(t)$ is defined as follows:

$$v(t) = \begin{cases} \delta^{-\frac{1}{4}}\varepsilon^{\frac{1}{3}(\nu-1)}(T_\infty - t)^\nu, \text{ for } t > \varepsilon^{-1}\delta \\ [\delta^{\frac{1}{4}}\varepsilon^{1/3} + \delta^{-\frac{1}{4}}(\delta - \varepsilon t)]^{-1}, \text{ for } t < \varepsilon^{-1}\delta \end{cases}$$

We can similarly define $\mathcal{C}_V$, with the other weight $V(t)$ defined as

$$V(t) = \begin{cases} \delta^{-\frac{1}{4}}\varepsilon^{\frac{1}{3}(\nu-1)}(T_\infty - t)^{\nu+1}, \text{ for } t > \varepsilon^{-1}\delta \\ [\delta^{\frac{1}{4}}\varepsilon^{\frac{2}{3}} + \delta^{-\frac{1}{4}}(\delta - \varepsilon t)^{\frac{3}{2}}]^{-1}, \text{ for } t < \varepsilon^{-1}\delta \end{cases}$$

and for the $+$ side we have:

$$\mathcal{C}_\eta(0,T) = \{u(t) \in \mathcal{C}(0,T) \mid \sup_{t\in(0,T)} |(T_\infty - t)^\eta u(t)| < \infty\}$$

(i) Time scale for W_-

The scaling of time is as follows:

$$s = \varepsilon^{\frac{1}{3}}(t - \varepsilon^{-1}\delta),$$

$$s = \psi(\sigma) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, \text{ for } \sigma \le -M \\ \Omega_0 - e^{-\sigma}, \text{ for } \sigma \ge M, \end{cases}$$

At $t = -\infty$, we have

$$s_{-\infty} = -\infty$$
$$\sigma_{-\infty} = -\infty$$

At $t = 0$, we have

$$s_0 := \varepsilon^{\frac{1}{3}}(0 - \varepsilon^{-1}\delta) = -\varepsilon^{-\frac{2}{3}}\delta$$
$$\sigma_0 := -\frac{2}{3}\delta^{\frac{3}{2}}\varepsilon^{-1}$$

At $t = t_*$, we have

$$s_* := \varepsilon^{\frac{1}{3}}(t_* - \varepsilon^{-1}\delta) = \Omega_0 - \delta^{-1}\varepsilon^{\frac{1}{6}}$$
$$\sigma_* := -\log(\delta^{-1}\varepsilon^{\frac{1}{6}})$$

At $t = T$, we have

$$s_T := \varepsilon^{\frac{1}{3}}(T - \varepsilon^{-1}\delta) = \Omega_0 - \delta^{-1}\varepsilon^{\frac{1}{3}}$$
$$\sigma_T := -\log(\delta^{-1}\varepsilon^{\frac{1}{3}})$$

At $t = T_\infty$, we have

$$s_\infty = \Omega_0$$
$$\sigma_\infty = \infty$$

Hence, as $\varepsilon \to 0$, we see that $\sigma_0 \to -\infty$ and $\sigma_T \to \infty$.

Therefore, rescale

$$\frac{d}{dt}W_- - 2u_-W_- = \mathcal{R}_- \text{ for } t \in (0, T)$$

into

$$\frac{d}{d\sigma}W_- - a(\sigma)W_- = \varepsilon^{-\frac{1}{3}}\varphi\mathcal{R}_- \text{ for } \sigma \in \left(-\frac{2}{3}\delta^{\frac{3}{2}}\varepsilon^{-1}, -\log(\delta^{-1}\varepsilon^{\frac{1}{3}})\right)$$

Recall $a(\sigma) \to \pm 2$ as $\sigma \to \pm\infty$.

(ii) Time scale for W_+.

It is scaled as thus

$$\xi = \log(u_+(t))$$

At $t = -\infty$, note $u_+ \to 0$ as $t \to -\infty$, then

$$\xi_{-\infty} := \log(0) = -\infty$$

At $t = 0$, we have

$$\xi_0 := \log(u_+(0)) \sim -\log T_\infty = -\log\left(\varepsilon^{-1}\delta + \varepsilon^{-\frac{1}{3}}\Omega_0\right) = -\log\left(\varepsilon^{-1}(\delta + \varepsilon^{\frac{2}{3}}\Omega_0)\right) \sim \log(\varepsilon)$$

At $t = T$, we have

$$\xi_T := \log(u_+(T)) \sim \log\left(\delta + \mathcal{O}(\varepsilon^{\frac{1}{2}})\right) \sim \log\delta$$

At $t = T_\infty$, we have

$$\xi_\infty := \log(u_+(T_\infty)) \sim -\log\left(\mathcal{O}(\varepsilon^{\frac{1}{2}})\right)$$

Therefore, rescale

$$\frac{d}{dt}W_+ - 2u_- W_+ = \mathcal{R}_+ \text{ for } t \in (0,T)$$

into

$$\frac{d}{d\xi}W_+ - 2W_+ = e^{-\xi}\mathcal{R}_+ \text{ for } \xi \in (\log(\varepsilon), \log(\delta))$$

then we can show the Fredholm properties of the linear operators as follows:

Theorem 4.1. *For $t \in (0,T)$, the linear operator on the $-$ side*

$$\frac{d}{dt} - 2u_-(t) : \mathcal{C}_v(0,T) \to \mathcal{C}_V(0,T)$$

and the linear operator on the $+$ side

$$\frac{d}{dt} - 2u_+(t) : \mathcal{C}_\eta(0,T) \to \mathcal{C}_{\eta+1}(0,T)$$

are Fredholm, and their indices are -1, 1, respectively..

Proof. For the W_- equation, recall we had the scalings $s = \varepsilon^{\frac{1}{3}}(t - \varepsilon^{-1}\delta)$, $\psi(\sigma) = s$, $\varphi = \partial_\sigma \psi$. Hence the equation in the σ-variable takes the form

$$\frac{d}{d\sigma}\tilde{W}_- - a(\sigma)\tilde{W}_- = \varepsilon^{-1/3}\varphi\tilde{\mathcal{R}}_-.$$

Where $\tilde{W}_-(\sigma) = W_-(\varepsilon^{-\frac{1}{3}}\psi(\sigma) + \varepsilon^{-1}\delta) = W_-(t)$, and similarly for $\tilde{\mathcal{R}}_-$. Now recall $a(\sigma) \to \pm 2$ as $\sigma \to \pm\infty$. In these variables, the weight satisfies

$$v(\sigma) \sim \begin{cases} \varepsilon^{-\frac{1}{3}} e^{-\nu\sigma}, \text{ for } \sigma > 0 \\ \varepsilon^{-\frac{2}{3}}[(-\sigma)^{\frac{2}{3}} + 1]^{-1}, \text{ for } \sigma < 0. \end{cases}$$

and

$$V(\sigma) \sim \begin{cases} \varepsilon^{-\frac{2}{3}} e^{-(\nu+1)\sigma}, \text{ for } \sigma > 0 \\ [\varepsilon|\sigma| + \varepsilon^{\frac{2}{3}}]^{-1} \text{ for } \sigma < 0. \end{cases}$$

Then for $\nu \neq 2$, the linear operators $\frac{d}{d\sigma} - a(\sigma)$ is Fredholm on the weighted spaces. Since $0 < \nu < 1$ and w has algebraic decay for $\sigma < 0$, we conclude that the Fredholm index is...

For the W_+ equation, we used the rescaling $u_+(t) = e^\xi$, and in the ξ equation, the $+$ side equation becomes

$$\frac{d}{d\xi}\tilde{W}_+ - 2\tilde{W}_+ = e^{-\xi}\mathcal{R}_+$$

the weight for W_+ is just $u_+(t)^\eta = e^{\eta\xi}$ and because of $1 < \eta < 2$, we see the linear opertator $\frac{d}{d\xi} - 2$ is Fredholm on this weighted function space.

To find the Fredholm index of this operator. □

4.2 Fixed point arguments-set up for W_r

We use the ansatz $U = W_r + u_r$ for $t \in (\varepsilon^{-1}\delta, T)$. Then W_r satisfies

$$\left(\frac{d}{dt} - 2u_r\right) W_r = W_r^2 + (u_r + W_r)^3 := R_r, \tag{4.1}$$

When we change variable from t to σ, we obtain, for $\sigma \in (0, \sigma_T)$,

$$\left(\frac{d}{d\sigma} - a(\sigma)\right) \tilde{W}_r(\sigma) = \varepsilon^{-1/3}\varphi \tilde{R}_r(\sigma).$$

Where $a(\sigma) = 2u_R(\psi(\sigma))\varphi(\sigma)$ satisfy

$$|a(\sigma) - 2| \leq Ce^{-2\sigma},$$

for some constant C, as $\sigma \to \infty$.

Invertibility of the linear operator The operator $\frac{d}{d\sigma} - a(\sigma) : C_{Wr} \to C_{Wr}$ will be invertible if we can find bounded solution to the equation

$$\left(\frac{d}{d\sigma} - a(\sigma)\right) u = f.$$

Variation of constants gives the formula

$$u(\sigma) = \exp\left(\int_\tau^\sigma a(\rho)d\rho\right) u(\tau) + \int_\tau^\sigma \exp\left(\int_s^\sigma a(\rho)d\rho\right) f(s)ds \tag{4.2}$$

If we are looking for bounded solution u on $C_{Wr}(0, \infty)$. Which implies $u(\tau) \leq \varepsilon^{(2-\alpha)/3} \exp\left((2-\alpha)\tau\right)$, so letting $\tau \to \infty$ gives the formula

$$u(\sigma) = \int_\infty^\sigma \exp\left(\int_s^\sigma a(\rho)d\rho\right) f(s)ds.$$

Using the convergence $|a(\sigma) - 2| \leq e^{-2\sigma}$ we discover that

$$\|u\|_{Wr} \leq C\|f\|_{Wr}$$

for some constant C. Moreover the homogeneous solution DOES NOT belong to the space $C_{Wr}(0, \infty)$, we get Fredholm -1!

However, we are solving on the finite interval $\sigma \in (0, \sigma_T)$, to get bounded inverse on this space, we use the solution formula, putting $\tau = \sigma_T$ in (4.2):

$$u(\sigma) = \exp\left(\int_{\sigma_T}^\sigma a(\rho)d\rho\right) u(\sigma_T) + \int_{\sigma_T}^\sigma \exp\left(\int_s^\sigma a(\rho)d\rho\right) f(s)ds$$

We see that

$$\begin{aligned} \|e^{\int_{\sigma_T}^\sigma a(\rho)d\rho} u(\sigma_T)\| &\lesssim \left|\varepsilon^{\frac{\alpha-2}{3}} e^{(\alpha-2)\sigma} e^{\int (a-2)} e^{2(\sigma-\sigma_T)} u(\sigma_T)\right| \\ &\lesssim |u(\sigma_T)| \end{aligned}$$

and

$$\left\|\int_{\sigma_T}^{\sigma} e^{\int_s^\sigma a(\rho)d\rho} f(s)ds\right\| \lesssim \left|\varepsilon^{\frac{\alpha-2}{3}} e^{(\alpha-2)\sigma}\left(\int_{\sigma_T}^{\sigma} e^{\int_s^\sigma a(\rho)d\rho} f(s)ds\right)\right|_\infty$$
$$\leq \left|e^{\alpha\sigma}\int_{\sigma_T}^{\sigma} e^{\int_s^\sigma (a(\rho)-2)d\rho} e^{-\alpha s}ds\right|_\infty \|f\|$$
$$\lesssim \left|\frac{1}{\alpha}\left(e^{\alpha(\sigma-\sigma_T)}-1\right)\right|_\infty \|f\| \lesssim \|f\|.$$

For some constant C independent of ε, not including the homogeneous part.

The parameter $u(\sigma_T)$ is choosen so that $|u(\sigma_T)| \leq \delta$.

We define the operator

$$\mathcal{L}u = \frac{d}{d\sigma}u - a(\sigma)u.$$

Nonlinear estimates Recall that

$$R_r(W_r) = W_r^2 + (u_r + W_r)^3$$

It is easier to estimate the nonlinear terms in the original $t-$variables, we estimate

$$\|R_r(W_r)(t)\| = \|\varepsilon^{-1/3}\varphi R_r(\sigma)\|_{C_{Wr}} = \sup_{\varepsilon^{-1}\delta\leq t\leq T} |(T_\infty - t)^{3-\alpha}(W_r^2 + (u_r + W_r)^3)|$$

for $W_r \in C_{Wr}$. But the latter implies that $|W_r|_\infty \leq (T_\infty - t)^{\alpha-2}\|W_r\|$, also we have that $|u_r| \lesssim (T_\infty - t)^{-1}$

$$\|R_r(W_r)\| \lesssim \left|(T_\infty - t)^{3-\alpha}(W_r^2 + u_r^3 + W_r^3)\right|_\infty$$
$$\lesssim (T_\infty - T)^{-\alpha} + (T_\infty - T)^{-(1-\alpha)}\|W_r\|^2 + (T_\infty - T)^{2\alpha-3}\|W_r\|^3$$
$$\lesssim \delta^\alpha + \delta^{1-\alpha}\|W_r\|^2 + \delta^{3-2\alpha}\|W_r\|^3$$

This implies that R_r maps a ball of radius $\delta^{\alpha/2}$ in C_{Wr} into itself, provided that δ is small enough. Indeed, we see if $\|W_r\| \leq \delta^{\alpha/2}$, then

$$\|R_r(W_r)\| \lesssim \delta^\alpha + \delta^{1-\alpha}\delta^\alpha + \delta^{3-\alpha/2} \lesssim \delta^\alpha \leq \delta^{\alpha/2}.$$

Denote $h(W_1, W_2) = (W_1 + W_2 + (u_r + W_1)^2 + (u_r + W_2)^2 + (u_r + W_1)(u_r + W_2))$, then

$$\|R_r(W_1) - R_r(W_2)\| = \sup |(T_\infty - t)^{3-\alpha}(W_1 - W_2)h(W_1, W_2)|$$
$$\lesssim |(T_\infty - t)h(W_1, W_2)|_\infty \|W_1 - W_2\|,$$

since $|W_{1,2}| \lesssim (T_\infty - t)^{\alpha-2}$ and $|u_r| \leq (T_\infty - t)^{-1}$, we have $|h(W_1, W_2)| \leq |W_1 + W_2| + O((T_\infty - t)^{-2})$, hence

$$|(T_\infty - t)h(W_1, W_2)|_\infty \lesssim (T_\infty - t)^{\alpha-1} \leq \delta^{1-\alpha}$$

Which shows $R_r(W)$ is Lipshitz in W with small ($\delta^{1-\alpha}$) Lipschitz constants.

Equation (4.1) is re-written in the form

$$W_r(\sigma) = \exp\left(\int_{\sigma_T}^{\sigma} a(\rho)d\rho\right) W_T + \int_{\sigma_T}^{\sigma} \exp\left(\int_s^\sigma a(\rho)d\rho\right)\varepsilon^{-1/3}\varphi R_r(W_r(s), \varepsilon)ds := \mathcal{T}(W_r, W_T, \varepsilon),$$

Previous estimates show that $\|\mathcal{T}(0,0,\varepsilon)\| \leq C\|u_r^3\| \leq C\delta^{1-\alpha}$, and $\mathcal{T}(W_r, W_T, \varepsilon)$ is Lipschitz in W_r, W_T with order $\delta^{1-\alpha}$, and for $\|W_r\| \lesssim \delta^{\alpha/2}$, $\|\mathcal{T}(W_r, W_T, \varepsilon)\| \lesssim \delta^{\alpha/2}$. Provided that W_T is chosen small enough ($\mathcal{O}(\delta)$), as indicated by the following estimate.

$$\left\|\exp\left(\int_{\sigma_T}^{\sigma} a\right) W_T\right\| = \sup_{\sigma \leq \sigma_T} |\varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} e^{2(\sigma-\sigma_T)} e^{\int_{\sigma_T}^{\sigma}(a-2)} W_T| \lesssim |\varepsilon^{\alpha/3} e^{\alpha\sigma} W_T| \leq |W_T|$$

4.3 Estimates for $W_r(0)$

Using the fixed point formula, we can write down equation for $W_r(0)$:

$$W_r(0) = \exp\left(\int_{\sigma_T}^0 a(\rho)d\rho\right) W_T + \int_{\sigma_T}^0 \exp\left(\int_s^0 a(\rho)d\rho\right) \varepsilon^{-1/3}\varphi R_r(W_r(s), \varepsilon)ds$$

then using the fact that

$$\begin{aligned} |\varepsilon^{-1/3}\varphi R_r(W_r(s))| &\leq \varepsilon^{(2-\alpha)/3}e^{(2-\alpha)s}\|\varepsilon^{-1/3}\varphi R_r(W_r(s))\| \\ &\lesssim \varepsilon^{(2-\alpha)/3}e^{(2-\alpha)s}\left(\delta^{1-\alpha}\|W_r^2\| + \delta^\alpha\right) \end{aligned}$$

we estimate

$$\begin{aligned} \left|e^{2\sigma_T}\left(W_r(0) - \exp\left(\int_{\sigma_T}^0 a(\rho)d\rho\right) W_T\right)\right| &\leq \int_0^{\sigma_T} e^{\int_s^0 (a-2)} e^{-2(s-\sigma_T)}|\varepsilon^{-1/3}\varphi R_r(W_r(s,\varepsilon))|ds \\ &\lesssim \int_0^{\sigma_T} e^{-2(s-\sigma_T)}\varepsilon^{(2-\alpha)/3}e^{(2-\alpha)s}(\delta^\alpha + \delta^{1-\alpha}\|W_r^2\|)ds \\ &\lesssim \varepsilon^{(2-\alpha)/3}\int_0^{\sigma_T} e^{-\alpha s}(\delta^\alpha + \delta^{1-\alpha}\|W_r^2\|)ds \\ &\lesssim \varepsilon^{-\alpha/3} \end{aligned}$$

4.4 Fixed point argument-set up for W_l