Here is what I was doing with the exist time:

The Ansatz chosen was

$$u_*(t;u_0) = \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t-\varepsilon^{-1}\delta);u_0)$$

where $u_R(\cdot;u_0)$ was the solution to the Ricatti equation with initial condition $u_R(0;u_0) = u_0$.

Which has the following asymptotics:

$$u_R(s;u_0) = (\Omega_\infty(u_0) - s)^{-1} + (\Omega_\infty(u_0) - s) r(\Omega_\infty(u_0) - s; u_0)$$

with

$$r(\Omega_\infty(u_0) - s; u_0) = -\frac{\Omega_\infty}{3} + \mathcal{O}(|\Omega_\infty(u_0) - s|)$$

Then we take a solution of the form $u_A(t) = u_*(t) + w_r(t)$, plug it into the equation

$$\frac{d}{dt}u(t) = \mu + u^2 + f(u,\mu;\varepsilon)$$

to get the solution satisfied by w_r, which is

$$\frac{d}{dt}w_r - 2u_* w_r = w_r^2 + f(u_* + w_r, \mu; \varepsilon) \tag{0.1}$$

After the rescaling this becomes

$$\left(\frac{d}{d\sigma} - a(\sigma)\right) W_r = \varepsilon^{-1/3}\varphi\left(W_r^2 + f(u_* + W_r, \mu; \varepsilon)\right) \tag{0.2}$$

for $\sigma \in (\sigma_m, \sigma_T)$.

where we have $|a(\sigma) - 2| \le Ce^{-2\sigma}$ as $\sigma \to \infty$.

We have proposed the function space

$$C_r = \{w(\sigma) : \sup |\varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} w| < \infty\}$$

And concluded that we need boundary condition at σ_T to complete the fixed point argument to solve equation (0.2).

Since we at the start required to have $u(T) = \delta$ as the definition for the exit time T. The simple choice we took was to use T such that

$$u_*(T) = 0$$

and hence

$$w_r(T) = 0 \text{ or } W_r(\sigma_T) = 0$$

as the needed boundary condition. The fixed point argument can be completed, as shown in section "Region A" of the entire writeup.

But from $u_*(T) = \delta$ we can compute the asymptotics of T to be

$$T = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty - \delta^{-1} + \mathcal{O}(\varepsilon^{2/3}),$$

so that in the end we have

$$\mu(T) = \varepsilon T - \delta = \varepsilon^{2/3}\Omega_\infty - \varepsilon\delta^{-1} + \mathcal{O}(\varepsilon^{5/3})$$

which I think is right, as this agrees with Lemma 2.10 in Krupa Szmolyan.

The problems is this expansion did not use any information about our correction w_r at $t = T$ (it is 0 by our assumption), so we cannot to expect to get the $\varepsilon \log(\varepsilon)$ just by using the asymptotics of the Riccati solution.