Alternatives to the blow-up method in singular perturbation problems

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Abstract

We revisit the classical problem of determine the asymptotic expansion of the solution near the passage of a fold point in a singularly perturbed system, where the theory of normally hyperbolic invariant manifold cannot be directly applied, the standard remedy is the blow-up method first demonstrated by Krupa and Szmolyan. In this paper we show how one can use a functional analytic approach (thus less geometrically-flavored) to achieve the same goal.

1 Introduction

In this paper we study singularly perturbed ordinary differential equations (ODEs) of the form

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y; \varepsilon), \\ \dot{y} &= g(x, y; \varepsilon), \end{aligned} \qquad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m, \quad 0 < \varepsilon \ll 1, \tag{1.1}$$

where f and g are C^k functions for $k >= 3$.

The standard way of studying (1.1) is using methods from geometric singular perturbation theory. An brief overview of this theory involves treating system (1.1), which is referred to as the *slow-system*, together with its equivalent counterpart, the *fast-system*:

$$\begin{aligned} x' &= f(x, y; \varepsilon), \\ y' &= \varepsilon g(x, y; \varepsilon), \end{aligned} \tag{1.2}$$

where if τ denotes the (slow) time variable in system (1.1), then $t = \tau/\varepsilon$ is the (fast) time variable used in system (1.2). The dynamics can then be studied by setting $\varepsilon = 0$ in both systems to obtain the so-called *reduced problem*

$$\begin{aligned} 0 &= f(x, y, 0), \\ \dot{y} &= g(x, y, 0), \end{aligned} \tag{1.3}$$

and the *layer problem*

$$\begin{aligned} x' &= f(x, y, 0), \\ y' &= 0. \end{aligned} \tag{1.4}$$

The basic premise of the theory, which was first laid out by Fenichel, is that the dynamics of reduced problem (1.3) happens on the *critical manifold*

$$S := \{(x, y) \mid f(x, y; 0) = 0\},$$

one then focuses on a *normally hyperbolic submanifold* of equlibra $S_0 \subset S$ of the layer problem (1.4), which will perturb to a so-called "slow manifold", S_ε for $0 < \varepsilon \ll 1$, on which the dynamics of (1.1) is an ε-perturbation of the reduced problem (1.3). In addition to the existence of a slow manifold, we have the existence of stable and unstable invariant foliation along with base S_0, which also persists for $\varepsilon > 0$.

The above approach relies heavily on the notion of normal hyperbolicity, which may not be always satisfied in the problems to be studied. The most common case is the so-called *fold point*, where the critical manifold S loses its normal hyperbolicity near bifurcation points due to a zero eigenvalue in the Jacobian $\frac{\partial f}{\partial x}$.

To overcome these difficulties, Krupa and Szmolyan proposed the method of *blow-up* to extend the reach of geometric singular perturbation theory. Roughly speaking, it is a set of coordinate transformations which desingularizes the vector field near the fold point so that information can be gained by using standard tools in dynamical system.

Their example was the following extended system

$$\begin{aligned} u' &= \mu + u^2 + f(u, \mu; \varepsilon), \\ \mu' &= \varepsilon g(u, \mu; \varepsilon), \\ \varepsilon' &= 0 \end{aligned} \tag{1.5}$$

where (μ, u, ε) in a sufficiently small neighborhood $\mathcal{U}$ of the origin so that the critical manifold

$$S_0 = \{(\mu, u) \mid \mu + u^2 + f(u, \mu; 0) = 0\}$$

has only $(0, 0)$ as the fold point. Further, a generic condition on the nonlinearity f, g are assumed, so that they have the following expansions

$$f(u, \mu; \varepsilon) = \mathcal{O}(\varepsilon, u\mu, \mu^2, u^3), \qquad g(u, \mu; \varepsilon) = 1 + \mathcal{O}(u, \mu, \varepsilon), \tag{1.6}$$

for $(\mu, u, \varepsilon) \in \mathcal{U}$.

Away from the fold point $(0, 0, 0)$, there exist the attracting manifold M_a with a section S_a sketched in Figure 1. By standard Fenichel's theory, S_a perturbs to $S_{a,\varepsilon}$ until it reaches the fold point, thus a natural question is to track how a trajectory starting on the slow manifold $S_{a,\varepsilon}$ passes through the fold point.

Krupa and Szmolyan starts by setting two sections $\Delta_{in} = \{(-\delta, u) \mid u \in J,\ J \text{ is some small interval }\}$ and $\Delta_{out} = \{(\mu, \sqrt{\delta}) \mid \mu \in \mathbb{R})\}$ with $\delta > 0$ small but fixed, which are also shown in figure 1. To track the passage through the fold amounts to study the transition map

$$\pi : \Delta_{in} \to \Delta_{out},$$

Krupa and Szmolyan then proceed by defining the blow up transformation

$$u = \bar{r}\bar{u}, \quad \mu = \bar{r}^2\bar{\mu}, \quad \varepsilon = \bar{r}^3\bar{\varepsilon},$$

which "blows up" the vector field of the extended (μ, u, ε) system near $(0, 0, 0)$ into a vector field on the ball $B = S^2 \times [0, \varepsilon_0^{1/3}] \ni (\bar{u}, \bar{\mu}, \bar{\varepsilon}, \bar{r})$ for some $\varepsilon_0 > 0$ small, and further directional blow-ups to obtain charts K_1, K_2, K_3 which was used to describe the flows in regions near different parts of the manifold B. After a careful and thorough analysis, they were able to prove the following results:

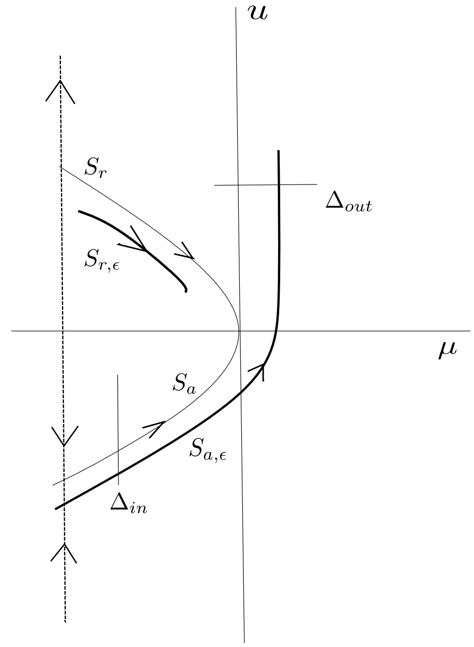

Figure 1: *critical manifold and slow manifolds, section near the fold*

Theorem 1.1. *For ε sufficiently small, the following statements are true:*

(i) The manifold $S_{a,\varepsilon}$ passes through Δ_{out} at a point $(h(\varepsilon), \sqrt{\delta})$ where $h(\varepsilon) = \mathcal{O}(\varepsilon^{2/3})$.

(ii) The transition map π is a contraction with rate $\mathcal{O}(e^{-c/\varepsilon})$, where c is a positive constant.

In this paper, we focus on the same problem (1.5) with an aim to recreate the result of Theorem 1.1 via a different, functional-analytical approach. Instead of proceeding with the geometrically-flavored blow-up approach, we directly prove a trajectory exists with the properties claimed in Theorem 1.1 by dividing the time of passage into appropriate parts and setting up a corresponding ansatz in each region, closing the arguments with carefully reformulating the existence question as a fixed-point argument.

Outline The reminder of the paper is organized as follows. In Section 2 we give a precise statement of our result, as well as an overview of our set ups. In Sections 3, 4 and 5 we construct the ansatz mentioned earlier and in Section 6 we show how to piece together all the parts to arrive at our main results.

Notation We use $A \lesssim B$ to indicate that there is a constant C such that $A \leq C \cdot B$, independent of the properties of A and B. We also use the brackets $\langle x \rangle$ to represent the expression $\sqrt{x^2+1}$.

2 Main Result

We first make a change of variable to transform equation (1.5) into a more convenient form in Section 2.1. Then we introduces the different ansatz used in Section 2.2 and 2.3. We then give a statement of our main result in Section 2.4 and in Section 2.5 we explain how we divide up the passage time into 3 different regions A, B, and C to prepare for the proof of this theorem.

2.1 Euler multiplier

Let τ denote the independent time variable in (1.5), since for u, μ, ε small, $g(u,\mu,\varepsilon) = 1 + \mathcal{O}(u,\mu,\varepsilon) > 0$, we can define a new time $t = t(\tau)$ by $\frac{dt}{d\tau} = g$, consequently, equation (1.5) is transformed into

$$\begin{aligned} &\frac{d}{dt}u = \mu + u^2 + \tilde{f}(u,\mu;\varepsilon), \\ &\frac{d}{dt}\mu = \varepsilon, \end{aligned} \tag{2.1}$$

where now $\tilde{f}$ satisfies the asymptotics

$$\tilde{f}(u,\mu,\varepsilon) = \mathcal{O}(\varepsilon, u\mu, \mu^2, \varepsilon u, \varepsilon\mu, \varepsilon u^2, u^3). \tag{2.2}$$

The critical manifold

$$\tilde{S}_0 = \{(\mu,u) \mid \mu + u^2 + \tilde{f}(u,\mu;0) = 0\},$$

still has $(0,0,0)$ as the only fold point, and our goal is to track how a trajectory of the flow of (2.1) passes through the fold. Following the set up of the sections Δ_{in} and Δ_{out} in Krupa-Szmolyan, we propose the following boundary conditions

$$\mu(0) = -\delta, \qquad u(T) = \delta, \tag{2.3}$$

where T is also an unknown variable which marks the "time of exit" as the trajectory hits the section Δ_{out}.

That is, we think the tracking of the passage of the fold as a shooting problem, if we can prove the existence of a solution (μ, u) to system (2.1) with the boundary condition (2.3), and give the expansion for the components μ, then we will prove the results in (1.5). Our contribution in this paper is that the method we used to prove the existence of such a solution is different than the blow-up approach.

In the rest of the paper, we shall drop the tilde to use $f(u,\mu,\varepsilon)$ as the nonlinearity and S_0 to denote the critical manifold.

2.2 The Riccati equation

First, we want to get an idea of what kinds of ansatz one should use. Consider (2.1) with $f(u,\mu;\varepsilon) = 0$, which can then be scaled such that $\varepsilon = 1, \mu = s$, and

$$\frac{d}{ds}u(s) = s + u(s)^2, \tag{2.4}$$

which is the Ricatti equation in its simplest form. We denote any solution to (2.4) as u_R, it is known to have a unique solution (denoted by $\bar{u}_R$) with the following asymptotic expansions:

$$\bar{u}_R(s) = \begin{cases} (\Omega_0 - s)^{-1} + \mathcal{O}(\Omega_0 - s), \text{ as } s \to \Omega_0, \\ -(-s)^{1/2} - \frac{1}{4}(-s)^{-1} + \mathcal{O}(|s|^{-3/2}), \text{ as } s \to -\infty. \end{cases} \tag{2.5}$$

Here the constant $\Omega_0 \approx 2.3381$ is the smallest positive zero of

$$J_{-1/3}(2z^{3/2}/3) + J_{1/3}(2z^{3/2}/3),$$

where $J_{\pm 1/3}$ are Bessel functions of the first kind. (See [Krupa, Szmolyan])

More generally, we consider family of solutions $u_R(s; u_0)$ of the Riccati equation (2.4) such that $u_R(0; u_0) = u_0$. That is, we take the initial condition u_0 as a parameter to the Riccati equation. For the special Riccati solution $\bar{u}_R$, we denote $\bar{u}_R(0)$ as $\bar{u}_0$. In fact, using simple phase plane analysis, we can extend the asymptotic results about the special Riccati solution $\bar{u}_R$, (2.5) to the u_0-parameter dependent family $u_R(s; u_0)$, as the following proposition states.

Proposition 2.1. *There exist $\eta > 0$ small, so that for all initial condition u_0 with $|u_0 - \bar{u}_0| < \eta$, there exist a number $\Omega_\infty = \Omega_\infty(u_0)$ which depends on u_0 smoothly, and a solution $u_R(s; u_0)$ of (2.4) on $[0, \Omega_\infty(u_0))$ with the following asymptotic expansion as $s \to \Omega_\infty(u_0)$:*

$$u_R(s; u_0) = \frac{1}{\Omega_\infty - s} + (\Omega_\infty - s) r(\Omega_\infty - s; u_0), \tag{2.6}$$

where the function r is smooth in both variables and satisfies

$$r(\Omega_\infty - s; u_0) = -\frac{\Omega_\infty}{3} + \mathcal{O}(\Omega_\infty - s), \tag{2.7}$$

as $s \to \Omega_\infty$.

we postpone the proof of this proposition to the appendix.

2.3 Critical manifold

Another piece of the ansatz comes from part of the critical manifold, we expect this because away from the fold, the critical manifold has an attracting branch S_a which implies trajectory has to stay close to it.

Recall the critical manifold for system (2.8) is the set of points (μ, u) in a small neighborhood of the origin which satisfies

$$\mu + u^2 + f(u, \mu; 0) = 0. \tag{2.8}$$

From (2.2) we see that $\mu = -u^2 + \mathcal{O}(u^3)$, by rescaling $\mu = -\mu_1^2$ with μ_1 positive and $u = \mu_1 u_1$ we obtain

$$u_1^2 = 1 + \mathcal{O}(\mu_1),$$

and hence two branches of solutions

$$u_1 = \pm\sqrt{1} + \mathcal{O}(\mu_1) \implies u = \pm\sqrt{-\mu} + \mathcal{O}(\mu).$$

We therefore define

$$\begin{aligned} u_-(\mu) &= -\sqrt{-\mu} + \mathcal{O}(|\mu|), \\ u_+(\mu) &= +\sqrt{-\mu} + \mathcal{O}(|\mu|). \end{aligned} \tag{2.9}$$

In particular, we set $u_s(t) = u_-(\mu(t))$, so that

$$0 = \mu(t) + u_s^2(t) + f(u_s(t), \mu; 0), \tag{2.10}$$

Due to the simple form of (2.1) and (2.3), we have that $\mu(t) = \varepsilon t - \delta$, hence $u_s(\mu)$ satisfies

$$u_s(t) = -\sqrt{\delta - \varepsilon t} + \mathcal{O}(|\delta - \varepsilon t|). \tag{2.11}$$

2.4 Main result - summary

We now state our main result. Recall $\mathcal{U}$ is a neighborhood small enough so that (1.6) holds for $(\mu, u, \varepsilon) \in \mathcal{U}$.

Theorem 2.2. *Let Ω_0 be the constant defined in (2.5). Fix $\delta > 0$ and α with $0 < \alpha < 3/4$. There exist $\varepsilon_0 > 0$, a constant $C = C(\delta, \alpha)$, such that for all $0 < \varepsilon < \varepsilon_0$ and u_i with $|u_i - u_-(-\delta)| \leq C\varepsilon^{1-\alpha/3}$, a solution of the rescaled system*

$$\begin{aligned} \frac{d}{dt}u &= \mu + u^2 + f(u, \mu; \varepsilon), \\ \frac{d}{dt}\mu &= \varepsilon. \end{aligned} \tag{2.12}$$

with the initial condition

$$\begin{aligned} u(0) &= u_i, \\ \mu(0) &= -\delta, \end{aligned} \tag{2.13}$$

exists for $t \in (0, T)$, where the end point T is the desired "exit time" in the sense that

$$u(T) = \delta. \tag{2.14}$$

Moreover, T has the following expansion in ε:

$$T = T(\varepsilon; u_i) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_0 + \mathcal{H}(\varepsilon; u_i), \tag{2.15}$$

with the term $\mathcal{H}(\varepsilon; u_i)$ satisfies

$$\mathcal{H}(\varepsilon; u_i) = \mathcal{O}(\varepsilon^{-\alpha/3}), \qquad Lip_{u_i}\mathcal{H}(\varepsilon; u_i) =>??? < . \tag{2.16}$$

In particular, since $\mu(t) = \varepsilon t - \delta$, we have the following expansion of the exit location $\mu(T)$ on the section Δ_{out} in ε:

$$\mu(T) = \varepsilon^{2/3}\Omega_0 + \varepsilon\mathcal{H}(\varepsilon; u_i) = \varepsilon^{2/3}\Omega_0 + \mathcal{O}(\varepsilon^{1-\alpha/3}) \tag{2.17}$$

To fully recover Theorem 1.1, we have the following corollary, which is proven using the standard Fenichel theory.

Corollary 2.3. *For $\alpha, \delta > 0$, there is a compact interval $K \subset (-\infty, u_+(-\delta))$, independent of ε so that for all $u_i \in K$ with $(u_i, \mu, \varepsilon) \in \mathcal{U}$, the same conclusion of Theorem 2.2 holds. In fact, there exist a constant c independent of ε, such that the Lipschitz constant of $\mathcal{H}(\varepsilon; u_i)$ in the expansion (2.15) satisfies*

$$Lip_{u_i}\mathcal{H}(\varepsilon; u_i) = \mathcal{O}(e^{-c/\varepsilon}), \tag{2.18}$$

for all $u_i \in K$.

The proof of Theorem 2.2 and 2.3 will take up the rest of the paper, we first prepare ourselves with the setup of the problem in the following sections.

2.5 Division of regions and the rescaling of time

Now that we can introduce the ansatz. We divide the time from $t = 0$ to $t = T$ into three "Regions" where we take a different ansatz on each region.

We first start with the exit time $t = T$ when the trajectory hits the section Δ_{out}. Also, we use T to mark the rightmost boundary of the region A, where our proposed solution takes the form

$$u_A(t; u_0) = u_*(t; u_0) + w_r(t; u_0),$$

here the function $u_* = u_*(t; u_0)$ is defined as:

$$u_*(t; u_0) := \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta); u_0), \tag{2.19}$$

where $u_R = u_R(s; u_0)$ is the family of solutions to the Riccati equation which were shown to exist in Proposition 2.1, it solves the initial value problem

$$\frac{d}{dt} u_*(t; u_0) = \mu(t) + u_*^2(t; u_0), \qquad u_*(\varepsilon^{-1}\delta; u_0) = \varepsilon^{\frac{1}{3}} u_0, \tag{2.20}$$

with ε and u_0 as parameters. The function w_r is a correction term whose properties will be given in later sections.

In region A, the ansatz $u_*(t)$ is merely a rescaled version of the Riccati solution $u_R(s; u_0)$, which follows the first half of the asymptotic expansion in Proposition 2.1. When $u_*(t)$ start to be controlled by the other part of the asymptotic expansion, we will need to adaptively change our ansatz or function space in order to "glue" the solutions, an intuitive, but rather arbitrary place to switch regions is at $t = \varepsilon^{-1}\delta$. This marks the start of region B, where our choice of solution is as follows:

$$u_B(t) = \bar{u}_*(t) + w_\ell(t).$$

Where $\bar{u}_*$ is the function

$$\bar{u}_*(t) = u_*(t; \bar{u}_0) = \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta); \bar{u}_0) = \varepsilon^{1/3} \bar{u}_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta)), \tag{2.21}$$

and $\bar{u}_*$ solves the equation

$$\frac{d}{dt} \bar{u}_*(t) = \mu(t) + \bar{u}_*^2(t), \tag{2.22}$$

so similarly $\bar{u}_*$ is a rescaled version of the special solution to the Riccati equation. Also similar to the situation in region A, w_ℓ is a correction term whose properties will be described later.

The next piece of the ansatz will be used to connect the piece u_B, which roughly follows the special Riccati solution $\bar{u}_R$, to the attracting branch S_a of the critical manifold S_0, a natural "gluing point" is at where the error between $\bar{u}_R$ and $u_s(t)$, is small. Calculation shows that this is at a point $t = t^*$ where we roughly have

$$(\delta - \varepsilon t^*) \approx -\varepsilon^{1/2}, \tag{2.23}$$

hence we choose t^* as a natural transition point from region B to the last region, region C, where it covers the rest of the passage time until at $t = 0$. The corresponding solution will take the form

$$u_C(t) = u_s(t) + w_s(t),$$

the $w_s(t)$ is yet another correction term whose properties will be discussed later.

In summary, region A, B and C are divided as follows:

$$\begin{aligned} &\text{Region A:} \quad (\varepsilon^{-1}\delta, T), \quad \text{solution in region A:} \quad u_A(t) = u_*(t) + w_r(t), \\ &\text{Region B:} \quad (t^*, \varepsilon^{-1}\delta), \quad \text{solution in region B:} \quad u_B(t) = \bar{u}_*(t) + w_\ell(t), \\ &\text{Region C:} \quad (0, t^*), \quad \text{solution in region C:} \quad u_A(t) = u_*(t) + w_s(t). \end{aligned} \tag{2.24}$$

Now we can briefly describe the strategy of our proof, we plug in the ansatz into Equation (2.1) to derive the equation for the ansatz w_r, w_ℓ and w_s, choose appropriate function space with norms and set up the equations for ansatz as fixed point argument on these function spaces. A main technical part of our proof consists of appropriately rescale the time $t \in (0,T)$ so that we gain hyperbolicity in the sense that the linearized operator at the ansatz become Fredholm in the new time scale. This is the key observation in our approach, comparable to the blow-up approach which also gains hyperbolicity via a carefully chosen change of variable. Having solved the correction term w_r, w_ℓ and w_s, we can then collect information about their asymptotic expansion to confirm the corresponding solution has the properties we need.

Therefore, we need to describe how we transform time t into time σ to gain hyperbolicity. This is demonstrated in the following steps.

(i) Step 1: Define ψ as
$$\psi = \varepsilon^{1/3}(t - \varepsilon^{-1}\delta),$$

(ii) Step 2: Fix $M > 0$ large, define σ as
$$\psi = \psi(\sigma; u_0) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, & \text{for } \sigma \le -M, \\ \Omega_\infty(u_0) - e^{-\sigma}, & \text{for } \sigma \ge M, \end{cases} \tag{2.25}$$

here Ω_∞ is the blow-up time for u_R found in Proposition 2.1.

Also, we denote $\varphi(\sigma) := \frac{d}{d\sigma}\psi(\sigma)$, which has the following asymptotic expansions:
$$\varphi(\sigma) = \begin{cases} (-\frac{3}{2}\sigma)^{-1/3}, & \text{for } \sigma \le -M, \\ e^{-\sigma}, & \text{for } \sigma \ge M, \end{cases} \tag{2.26}$$

(iii) Step 3: For $\sigma \in (-M, M)$, we define $\psi(\sigma)$ as the straight line connecting the two points $(M, \Omega_\infty - e^{-M})$ and $(-M, -(\frac{3}{2}M)^{2/3})$. As a result, if we define $\sigma_m = \sigma_m(u_0)$ as the value of σ such that $\psi(\sigma_m; u_0) = 0$, then we have
$$\frac{|\sigma_m - M|}{M} = \left| \frac{(\frac{3M}{2})^{2/3} - (\Omega_\infty - e^{-M})}{(\frac{3M}{2})^{2/3} - (\Omega_\infty + e^{-M})} - 1 \right| \le C M^{-2/3},$$

for some constant C independent of u_0.

Therefore we can write
$$\sigma_m = M + M_r, \qquad |M_r| \le C M^{1/3}. \tag{2.27}$$

After defining the time transformation, the original exit time $t = T$ will be transformed into $\sigma = \sigma_T$ under this change of variable, and similarly the boundary between region A and region B is transformed from $t = \varepsilon^{-1}\delta$ to $\sigma = \sigma_m$ and the boundary between region B and region C is transformed from $t = t^*$ to $\sigma = \sigma^*$, and $t = 0$ into $\sigma = \sigma_0$. We will see later that the $\sigma_0, \sigma^*, \sigma_m, \sigma_T$ satisfies:

$$\sigma_0 \approx -\delta^{3/2}\varepsilon^{-1}, \quad \sigma^* \approx -\varepsilon^{-1/4}, \quad \sigma_m = \mathcal{O}_\varepsilon(1), \quad \sigma_T \approx -\log(\varepsilon^{1/3}\delta^{-1})$$

In summary, the regions in time σ are as follows:

$$\begin{aligned} &\text{Region A:} \quad (\sigma_m, \sigma_T), \\ &\text{Region B:} \quad (\sigma^*, \sigma_m), \\ &\text{Region C:} \quad (\sigma_0, \sigma^*), \end{aligned} \tag{2.28}$$

and the following picture summarizes the relationship between the two time scales t and σ, as well as the corresponding regions divided.

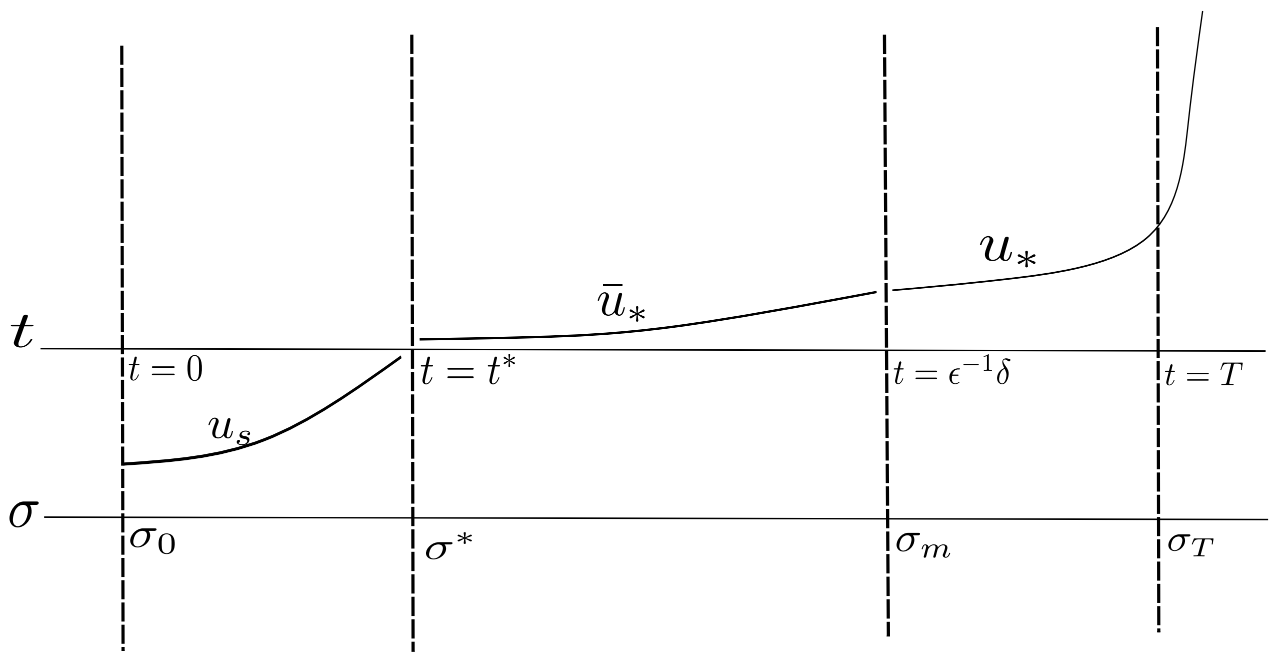

Figure 2: *time scales used to divide the regions*

3 Region A

Region A corresponds to the t-time interval $\{t : t > \varepsilon^{-1}\delta\}$. In this section we will give the precise form of the solution which solves system (2.1) on this region.

3.1 Solution in region A

Theorem 3.1. *Fix $\alpha > 0, \delta > 0, \eta > 0$ small enough, there exists $\varepsilon_A > 0$ and a constant $C = C(\alpha, \delta, \eta)$, such that for all $0 < \varepsilon < \varepsilon_A$, and all $|u_0 - \bar{u}_0| < \eta$, there exist a time $T = T(\varepsilon; u_0)$, such that a solution of* *the form*

$$u_A(t;u_0) = u_*(t;u_0) + w_r(t;u_0), \tag{3.1}$$

to the rescaled system (2.12) *exists on the time interval* $t \in (\varepsilon^{-1}\delta, T)$*, where*

$$u_*(t;u_0) = \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta); u_0), \tag{3.2}$$

and $u_R(\cdot;u_0)$ *is the family of solution to Riccati equation that were shown to exist in Proposition 2.1. Moreover, we have*

(1) $T = T(\varepsilon;u_0) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0) - \delta^{-1} + T_r$ *with* $T_r = \mathcal{O}(\varepsilon^{2/3}\delta^{-3})$ *where the constant* $\Omega_\infty(u_0)$ *is the same mentioned in Proposition 2.1,*

(2) $w_r(T;u_0) = 0$ *and* $u_*(T, u_0) = \delta$*,*

(3) w_r *is continuous with* $|w_r(t;u_0)| \le C|\varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0) - t|^{\alpha-2}$ *for* $t \in (\varepsilon^{-1}\delta, T)$*,*

(4) *the function* $\mathcal{F} : (\bar{u}_0 - \eta, \bar{u}_0 + \eta) \to \mathbb{R}$ *with* $\mathcal{F}(u_0) = w_r(\varepsilon^{-1}\delta; u_0)$ *is smooth, with Lipschitz constant satisfy*

$$|Lip\ \mathcal{F}| \le C\varepsilon^{2/3}.$$

We will prove this theorem in the following sections.

3.2 The exit time $T(u_0)$

Take $\eta > 0$ so that the function $u_R = u_R(\cdot;u_0)$ exists for $|u_0 - \bar{u}_0| < \eta$ as demonstrated in Proposition 2.1. The exit time T is then defined by the condition

$$\delta = u_*(T;u_0) = \varepsilon^{1/3} u_R(\varepsilon^{1/3}(T - \varepsilon^{-1}\delta); u_0),$$

since the expansion for u_R is given in (2.6) , if we define $\psi_T = \varepsilon^{1/3}(T - \varepsilon^{-1}\delta)$, then ψ_T satisfies

$$\frac{1}{\Omega_\infty - \psi_T} + (\Omega_\infty - \psi_T) r(\Omega_\infty - \psi_T) = \varepsilon^{-1/3}\delta,$$

from which we get the leading order expansion $\Omega_\infty - \psi_T = \mathcal{O}(\varepsilon^{1/3}\delta^{-1})$. A fixed point argument gives

$$\Omega_\infty - \psi_T = \varepsilon^{1/3}\delta^{-1} + \mathcal{O}(\varepsilon\delta^{-3}),$$

hence the expansion for $T = T(\varepsilon;u_0)$ is

$$T = T(\varepsilon;u_0) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0) - \delta^{-1} + T_r, \tag{3.3}$$

with $|T_r| \le C\varepsilon^{2/3}\delta^{-3}$, for some constant C independent of u_0, as $\varepsilon \to 0$.

For convenience, we define

$$T_\infty = T_\infty(\varepsilon;u_0) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0),$$

so that $T = T_\infty - \delta^{-1} + T_r$.

3.3 Equation for w_r and rescaling

We now plug in $u = u_* + w_r$ into equation (2.1), and derive the equation for w_r

$$w_r' - 2u_* w_r = w_r^2 + f(u_* + w_r, \mu; \varepsilon) := R_r(w_r; \varepsilon, u_0), \tag{3.4}$$

moreover, we enforce the boundary condition $u(T; u_0) = \delta$, hence this gives the boundary condition for w_r at $t = T$:

$$w_r(T; u_0) = 0, \tag{3.5}$$

therefore, we need to solve equation (3.4) on the interval $t \in (\varepsilon^{-1}\delta, T)$, with boundary condition (3.5).

Next, we rescale equation (3.4) into σ-time variable by using the t to σ-time rescaling in section 2.5, and obtain

$$\left(\frac{d}{d\sigma} - a(\sigma; \varepsilon, u_0)\right) W_r = \varepsilon^{-1/3} \varphi \mathcal{R}_r(W_r; \varepsilon, u_0), \tag{3.6}$$

where

- The term $a(\sigma; \varepsilon, u_0)$ is defined as and has asymptotic expansion

  $$a(\sigma; \varepsilon, u_0) := 2\varphi(\sigma) u_R(\psi(\sigma); u_0) = 2 + \mathcal{O}(e^{-2\sigma}) \text{ as } \sigma \to \infty,$$

  we remark that this convergence as $\sigma \to \infty$ is uniform in u_0 due to the definition of our time-rescaling.

- The function $W_r(\sigma)$ is the rescaled version of $w_r(t)$ in the σ-variable, $w_r(t) = w_r(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = W_r(\sigma)$. U_* is similarly the rescaled version of u_*, $U_*(\sigma; u_0) = u_*(t; u_0) = \varepsilon^{1/3} u_R(\psi(\sigma; u_0); u_0)$.

- The function $\mathcal{R}_r$ is a rescaled version of R_r such that

  $$\mathcal{R}_r(W_r; \varepsilon, u_0) = W_r^2 + f(U_* + W_r, \mu; \varepsilon),$$

To get the corresponding boundary condition of 3.5, we need to know the corresponding σ-time for the t-time interval $t \in (\varepsilon^{-1}\delta, T)$.

At $t = \varepsilon^{-1}\delta$, the corresponding σ time is at $\sigma = \sigma_m$, from its definition in section 2.5.

At $t = T$, we have $\varepsilon^{1/3}(T - \varepsilon^{-1}\delta) = \Omega_\infty - \varepsilon^{1/3}\delta^{-1} + \varepsilon^{1/3} T_r = \psi(\sigma_T) = \Omega_\infty - e^{-\sigma_T}$ from (3.3), hence, for ε small enough, we get that the corresponding σ-time to $t = T$ is

$$\sigma_T = \sigma_T(u_0) = -\log(\varepsilon^{1/3}(\delta^{-1} - T_r)) = -\log(\varepsilon^{1/3}\delta^{-1}) - \log(1 - \delta T_r), \tag{3.7}$$

then, the complete boundary value problem we wish to solve is

$$\begin{gathered} \frac{d}{d\sigma} W_r - a(\sigma; \varepsilon, u_0) W_r = \varepsilon^{-1/3} \varphi \mathcal{R}_r(W_r), \text{ for } \sigma \in (\sigma_m, \sigma_T), \\ W_r(\sigma_T) = 0. \end{gathered} \tag{3.8}$$

3.4 Linear equation and norms

Our goal now is to solve (3.8) on an appropriate function space, to do so we first slightly enlarge the time interval (σ_m, σ_T) where the boundary value problem is posed.

From the definition of σ_T (3.7), we see that

$$|\sigma_T - (-\log(\varepsilon^{1/3}\delta^{-1}))| \le |\log(1-\delta T_r)| \le C|\delta T_r| \le C\varepsilon^{2/3}\delta^{-2}, \tag{3.9}$$

for some constant C independent of those u_0 with $|u_0 - \bar{u}_0| < \eta$.

We now define $\sigma_{\inf}$ and $\sigma_{\sup}$ as follows:

$$\sigma_{\inf} = \inf_{|u_0-\bar{u}_0|<\eta} \sigma_m(u_0), \qquad \sigma_{\sup} = \sup_{|u_0-\bar{u}_0|<\eta} \sigma_T(u_0).$$

From the definition of σ_m in section 2.5 and (3.9) we have

$$M \approx \sigma_{\inf}, \quad \sigma_{\sup} \approx -\log(\varepsilon^{1/3}\delta^{-1}),$$

with the error independent of u_0. Therefore we introduce the following weighted function space:

$$\mathcal{C}_r = \left\{ w(\sigma) : \sup_{\sigma_{\sup} \ge \sigma \ge \sigma_{\inf}} \left| \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} w(\sigma) \right| < \infty \right\}.$$

We establish the invertibility of the linear operator A_r which acts on $w \in \mathcal{C}_r$ as

$$A_r w = \left(\frac{d}{d\sigma} w - a(\sigma; \varepsilon, u_0) w, w(\sigma_T) \right),$$

in the following

Proposition 3.2. *$A_r = A_r(u_0, \varepsilon) : \mathcal{D} \subset \mathcal{C}_r \to \mathcal{C}_r \times \mathbb{R}$ and is invertible, with its inverse depends smoothly and uniformly bounded in u_0, ε.*

Proof. Consider the conjugate operator of A_r, given by

$$\begin{aligned} \tilde{A}_r v &= \left(e^{(\alpha-2)\sigma} \left(\frac{d}{d\sigma} - a(\sigma; \varepsilon, u_0) \right) e^{(2-\alpha)\sigma} v, v(\sigma_T) \right) \\ &= \left(\frac{d}{d\sigma} v - (\alpha + a(\sigma; \varepsilon, u_0) - 2) v, v(\sigma_T) \right), \end{aligned}$$

for $v(\sigma) = \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} w(\sigma) \in \mathcal{C}(\sigma_{\inf}, \sigma_{\sup})$.

The associated conjugate equation of

$$A_r w = (f, w_T) \text{ with } f \in \mathcal{C}_r, w_T \in \mathbb{R},$$

is

$$\tilde{A}_r v = (\varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} f, v_T) \text{ with } \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} f \in C, v_T = \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma_T} w_T.$$

Since $\alpha > 0$ and $|a(\sigma; \varepsilon, u_0) - 2| = \mathcal{O}(e^{-2\sigma}) \to 0$ as $\sigma \to \infty$ uniformly in ε, u_0. We apply lemma 6.10 with $L = \sigma_T$ to conclude that there exists a constant C independent of ε, u_0 with

$$\|w\|_{\mathcal{C}_r} = |v|_\infty \le C(|\varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} f|_\infty + |v_T|) \le C(\|f\|_{\mathcal{C}_r} + |w_T|), \tag{3.10}$$

notice $|v_T| \le w_T$ by the asymptotic expansion of $\sigma_T = \mathcal{O}(-\log(\varepsilon^{1/3}))$. By the definition of $\sigma_{\inf}$ and $\sigma_{\sup}$, A_r is uniformly invertible in u_0 on $\mathcal{C}_r$, this finishes the proof of the proposition. ■

3.5 Nonlinear estimates

In this section we estimate the nonlinear term

$$\mathcal{R}_r(W_r) = W_r^2 + f(U_* + W_r, \mu; \varepsilon),$$

in the $\mathcal{C}_r$ norm to prove

Proposition 3.3. *If $W_r \in \mathcal{C}_r$, then $\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r) \in \mathcal{C}_r$, and*

$$\|\varepsilon^{-1/3}\varphi\mathcal{R}_r\| = \mathcal{O}(\delta^\alpha). \tag{3.11}$$

Proof. Proposition 2.6 shows

$$U_*(\sigma; u_0) = \varepsilon^{\frac{1}{3}}(e^\sigma + e^{-\sigma} r(e^{-\sigma}; u_0)) \text{ as } \sigma \to \infty,$$

therefore

$$|U_*(\sigma)| \lesssim \varepsilon^{\frac{1}{3}} e^\sigma \le \varepsilon^{1/3} e^{\sigma_{\sup}} = \mathcal{O}(\delta) \text{ for all } \sigma \ge \sigma_{\inf}.$$

From the definition of the time rescaling in section 2.5 we have

$$\mu = \varepsilon t - \delta = \varepsilon^{2/3}\psi(\sigma) \lesssim \varepsilon^{2/3},$$

for all $\sigma \ge \sigma_{\inf}$.

As $W_r \in \mathcal{C}_r$, we have

$$|W_r(\sigma)| \lesssim \varepsilon^{\frac{2-\alpha}{3}} e^{(2-\alpha)\sigma} \ll |U_*(\sigma)|, \text{ for } \sigma \in (\sigma_{\inf}, \sigma_{\sup}).$$

Using these facts, we have

$$\|\varepsilon^{-\frac{1}{3}}\varphi W_r^2\|_{\mathcal{C}_r} = \sup|\varepsilon^{-\frac{1}{3}}\varphi W_r| \lesssim \varepsilon^{\frac{1-\alpha}{3}} e^{(1-\alpha)\sigma} \lesssim \varepsilon^{\frac{1-\alpha}{3}} e^{(1-\alpha)\sigma_{\sup}} \lesssim \delta^{1-\alpha},$$

as $f(u, \mu; \varepsilon) = \mathcal{O}(\varepsilon(1 + u + \mu + u^2), u\mu, \mu^2, u^3)$ and since U_*, W_r, μ are all small in sup norm, we have

$$f(U_* + W_r, \mu; \varepsilon) = \mathcal{O}(\varepsilon, (U_* + W_r)\mu, \mu^2, (U_* + W_r)^3) = \mathcal{O}(\varepsilon, U_*\mu, \mu^2, U_*^3),$$

hence

$$\|\varepsilon^{-\frac{1}{3}}\varphi\varepsilon\|_{\mathcal{C}_r} = \sup|\varepsilon^{-\frac{1}{3}}\varphi \cdot \varepsilon \cdot \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma}| \lesssim \varepsilon^{\alpha/3} e^{(\alpha-3)\sigma_m} = \mathcal{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi U_*\mu\|_{\mathcal{C}_r} = \sup|\varepsilon^{-\frac{1}{3}}\varphi \cdot U_*\mu \cdot \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma}| \lesssim \varepsilon^{\alpha/3} e^{(\alpha-2)\sigma_m} = \mathcal{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi\mu^2\|_{\mathcal{C}_r} \lesssim \|\varepsilon^{-\frac{1}{3}}\varphi\varepsilon\|_{\mathcal{C}_r} \text{ since } \mu^2 = \mathcal{O}(\varepsilon^{4/3}) \ll \varepsilon,$$

and lastly

$$\|\varepsilon^{-\frac{1}{3}}\varphi U_*^3\|_{\mathcal{C}_r} \lesssim \varepsilon^{\frac{\alpha}{3}} e^{\alpha\sigma} \lesssim \varepsilon^{\frac{\alpha}{3}} e^{\alpha\sigma_{\sup}} = \mathcal{O}(\delta^\alpha).$$

Combining all the estimates we conclude that

$$\|\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r)\| = \max\{\mathcal{O}(\delta^\alpha), \mathcal{O}(\delta^{1-\alpha})\},$$

since we assumed $\alpha \ll 1$, it follows that $\|\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r)\| = \mathcal{O}(\delta^\alpha)$, this completes the proof. ■

3.6 Fixed point argument and the proof of Theorem 3.1

In this section we prove theorem 3.1 by setting up an appropriate fixed point argument.

Proof of theorem 3.1. Items (1) and (2) in the assertion of the theorem has been demonstrated in section 3.2 and 3.3. Items (3) is a direct consequence of the fact that $W_r \in \mathcal{C}_r$, to prove this, we first rewrite equation (3.6) and the boundary condition $W_r(\sigma_T) = w_T$ as

$$F_r(W_r, w_T; \varepsilon, u_0) = 0,$$

where $F_r : \mathcal{C}_r \times \mathbb{R} \to \mathcal{C}_r \times \mathbb{R}$ is defined as

$$\begin{aligned} F_r(W_r, w_T; \varepsilon, u_0) &= A_r W_r - \left(\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), w_T\right) \\ &= \left(\frac{d}{d\sigma}W_r - aW_r - \varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), W_r(\sigma_T) - w_T\right). \end{aligned}$$

Let $X = \mathcal{C}_r \times I$, where I is some small interval containing 0, we introduce the solution map $\mathcal{S} : X \to \mathcal{C}_r \times \mathbb{R}$ as follows:

$$\mathcal{S}(W_r, w_T; \varepsilon, u_0) = (W_r - A_r^{-1}F_r(W_r, w_T; \varepsilon, u_0), w_T),$$

From propositions 3.2 and 3.3, we conclude

- $\|\mathcal{S}(0,0;\varepsilon,u_0)\| = \|\left(-A_r^{-1}F_r(0,0;\varepsilon,u_0), 0\right)\| \leq \|A_r^{-1}\|\|F_r(0,0;\varepsilon,u_0)\| \lesssim |\|\varepsilon^{-1/3}\varphi\mathcal{R}_r(0)\| \lesssim \delta^\alpha$, uniformly in ε and u_0.
- $\mathcal{S}$ is a smooth map in W_r, w_T as well as the parameters ε, u_0.
- Since the derivative of $f(U_* + W_r, \mu; \varepsilon)$ with respect to W_r is $D_{W_r} f(U_* + W_r, \mu; \varepsilon) = \mathcal{O}(\mu, U_*^2)$, the linearization of $\mathcal{S}$ at $(0,0)$, $D_{(W_r, w^*)}\mathcal{S}(0,0)$ satisfies

  $$\|D_{(W_r,w^*)}\mathcal{S}(0,0)\|_{op} \lesssim \sup|\varepsilon^{-1/3}\varphi(\mu + U_*^2)| = \mathcal{O}(\delta),$$

  where $\|\cdot\|_{op}$ denotes the operator norm of the associated linear operator. Moreover, for $\|W_r\|$ small enough and $|w_T| \leq \delta_1$, we have $D_{(W_r,w_T)}\mathcal{S}(W_r, w_T; \varepsilon, u_0) = D_{(0,0)}\mathcal{S}(W_r, w_T; \varepsilon, u_0) + \mathcal{O}(\|W_r\|_{\mathcal{C}_r})$, which is uniformly small in ε and u_0.

Therefore, for (W_r, w_T) in a small ball of X, we can apply an iteration scheme and utilize the Banach fix point theorem to the existence of a fixed point, hence a solution to equation (3.6) exists. Moreover, this solution depends smoothly on the parameter ε, u_0. By picking $w_T = 0$, we have shown that a unique fixed point $W_r \in \mathcal{C}_r$ exists and solves equation (3.6).

Finally, to prove item (4) we need to estimate the Lipschitz constant for the map

$$\Psi : u_0 \mapsto w_r(\varepsilon^{-1}\delta; u_0) = W_r(\sigma_m; u_0),$$

which maps from a small interval I containing $\bar{u}_0$ to $\mathbb{R}$. We can write Ψ as the composition of two maps $\Psi = \Psi_1 \circ \Psi_2$ where $\Psi_2 : I \to \mathcal{C}_r$ is the map

$$u_0 \mapsto W_r(\sigma; u_0),$$

and $\Psi_1 : \mathcal{C}_r \to \mathbb{R}$ is the evaluation map

$$W_r(\sigma; u_0) \mapsto W_r(\sigma_m, u_0).$$

To estimate the number $\mathrm{Lip}_{u_0}\Psi$, we need to estimate the number $\mathrm{Lip}_{u_0}\Psi_2$ and Lip Ψ_1.

To estimate $\mathrm{Lip}_{u_0}\Psi_2$, it suffices to estimate the following two quantities

$$C_1 = \mathrm{Lip}_{W_r}\mathcal{S}, \text{ and } C_2 = \mathrm{Lip}_{u_0}\mathcal{S},$$

because W_r is the fixed point of the map $\mathcal{S}$, which implies

$$\mathrm{Lip}_{u_0}\Psi_2 \le C_2/(1 - C_1).$$

From the definition of $\mathcal{S}$, we see that

$$C_1 \le \mathrm{Lip}_{W_r}|\varepsilon^{-1/3}\varphi R_r(W_r)| \le \mathrm{Lip}_{W_r}|\varepsilon^{-1/3}\varphi W_r^2| \le \sup_{W_r \in \mathcal{C}_r} |\varepsilon^{-1/3}\varphi W_r| = \mathcal{O}(\delta^{1-\alpha}),$$

where the last line follows from proposition 3.3.

To estimate C_2. We notice that

$$C_2 \le \text{ Lip }_{u_0}|\varepsilon^{-1/3}\varphi U_*^3(\sigma; u_0)|.$$

However if $u_0, \tilde{u}_0$ are two different numbers near $\bar{u}_0$,

$$\|\varepsilon^{-1/3}\varphi[U_*^3(\sigma; u_0) - U_*^3(\sigma; \tilde{u}_0)]\|_{\mathcal{C}_r} \le \|\varepsilon^{-1/3}\varphi U_*^2\|_{\mathcal{C}_r} \sup|U_*(\sigma; u_0) - U_*(\sigma; \tilde{u}_0)|,$$

proposition 2.6 shows $U_*(\sigma; u_0) = \varepsilon^{1/3}(e^\sigma + e^{-\sigma}r(e^{-\sigma}; u_0))$ for σ large, hence

$$\partial_{u_0}U_*(\sigma; u_0) \le C\varepsilon^{1/3},$$

for some constant independent of u_0, on the other hand

$$\|\varepsilon^{-1/3}\varphi U_*^2\|_{\mathcal{C}_r} = \mathcal{O}(\varepsilon^{(\alpha-1)/3}),$$

so we conclude that

$$\|\varepsilon^{-1/3}\varphi[U_*^3(\sigma; u_0) - U_*^3(\sigma; \tilde{u}_0)]\|_{\mathcal{C}_r} \le C\varepsilon^{\alpha/3}|u_0 - \tilde{u}_0|,$$

or $C_2 = \mathcal{O}(\varepsilon^{\alpha/3})$. Hence

$$\mathrm{Lip}_{u_0}\Psi_2 = \mathcal{O}(\varepsilon^{\alpha/3}).$$

On the other hand, the evaluation map Ψ_1 is a linear map which satisfies

$$|\Psi_1(W) - \Psi_1(\widetilde{W})| = |W(\sigma_m) - \widetilde{W}(\sigma_m)| \leq \|W - \widetilde{W}\|_{\mathcal{C}_r} \varepsilon^{(2-\alpha)/3} e^{(2-\alpha)\sigma_m} \lesssim \varepsilon^{(2-\alpha)/3}\|W - \widetilde{W}\|,$$

for $W, \widetilde{W}$ in $\mathcal{C}_r$, from the definition of its norm, therefore

$$\mathrm{Lip}\ \Psi_1 = \mathcal{O}(\varepsilon^{(2-\alpha)/3}),$$

combine the two estimates we conclude that

$$\mathrm{Lip}_{u_0}\Psi \leq \left(\mathrm{Lip}_{u_0}\Psi_2\right)\left(\mathrm{Lip}\ \Psi_1\right) = \mathcal{O}(\varepsilon^{2/3}),$$

which completes the proof of Theorem 3.1. ■

4 Region B

Region B corresponds to the t-time interval $t^* < t < \varepsilon^{-1}\delta$. In this section, we give the precise form of the solution in this region and prove the needed properties.

4.1 Solution in region B

Theorem 4.1. *Fix $\delta > 0, \alpha > 0$ small enough, there exists $\varepsilon_B > 0$ and a constant $C = C(\delta, \alpha)$ such that for all $0 < \varepsilon < \varepsilon_B$, and w^* with $|w^*| \leq C\varepsilon^{1/2-\alpha/3}$, a solution u_B of the form*

$$u_B(t) = \bar{u}_*(t) + w_\ell(t), \tag{4.1}$$

to system (2.12) with the initial condition

$$u(t^*) = \bar{u}_*(t^*) + w^*, \qquad \mu(t^*) = \varepsilon t^* - \delta \tag{4.2}$$

exists on the time interval $t \in (t^*, \varepsilon^{-1}\delta)$, where $\bar{u}_*$ is the function defined as in (2.21). Moreover, we have that the correction function w_ℓ is continuous with

$$|w_\ell(t)| \leq C\varepsilon^{(2-\alpha)/3}|\varepsilon^{1/3}(t - \varepsilon^{-1}\delta) + 1|, \tag{4.3}$$

for $t \in (t^*, \varepsilon^{-1}\delta)$.

We will prove this theorem in the rest of this section.

4.2 Equation of W_ℓ and rescaling

As before, we plug the ansatz into (2.1) to derive the equation satisfied by w_ℓ.

$$w_\ell' - 2\bar{u}_* w_\ell = w_\ell^2 + f(\bar{u}_* + w_\ell, \mu; \varepsilon) := R_\ell(w_\ell, \mu; \varepsilon). \tag{4.4}$$

We want to solve this equation on $t \in (t^*, \varepsilon^{-1}\delta)$. Following previous steps, we next rescale the equation to the σ-time variable using the time rescaling map $\psi = \psi(\sigma; \bar{u}_0)$ and we obtain

$$\frac{d}{d\sigma} W_\ell - b(\sigma) W_\ell = \varepsilon^{-1/3} \varphi \mathcal{R}_\ell(W_\ell, \mu, \varepsilon), \tag{4.5}$$

where

- The equation is posed on $\sigma \in (\sigma^*, \sigma_m(\bar{u}_0))$ where $\sigma^* \approx -\varepsilon^{-1/4}$ and $\sigma_m(\bar{u}_0) := \bar{\sigma}_m$ follows the notation used in section 2.5.
- The term $b(\sigma)$ is defined and has asymptotic expansion:

  $$b(\sigma) := 2u_R(\psi(\sigma))\varphi(\sigma) = -2 + \mathcal{O}(|\sigma|^{-1}),$$

  as $\sigma \to -\infty$. Again, the convergence is uniform in ε.
- The function $W_\ell(\sigma)$ is the rescaled version of $w_\ell(t)$ in the σ-variable, $w_\ell(t) = w_\ell(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = W_\ell(\sigma)$. $\bar{U}_*$ is similarly the rescaled version of $\bar{u}_*$, $\bar{U}_*(\sigma) = \bar{u}_*(t) = \varepsilon^{1/3}\bar{u}_R(\psi(\sigma))$.
- The function $\mathcal{R}_\ell$ is a rescaled version of R_ℓ such that $\mathcal{R}_\ell(W_r; \varepsilon, u_0) = W_\ell^2 + f(\bar{U}_* + W_\ell, \mu; \varepsilon)$,

We also need to specify the boundary value at the left end point $\sigma = \sigma^*$, the complete system we want to solve is

$$\begin{aligned} \frac{d}{d\sigma} W_\ell - b(\sigma) W_\ell &= \varepsilon^{-1/3} \varphi \mathcal{R}_\ell(W_\ell), \;\; \sigma \in (\sigma^*, \bar{\sigma}_m), \\ W_\ell(\sigma^*) &= w^*. \end{aligned} \tag{4.6}$$

4.3 Linear equation and norms

Similarly, the proof of theorem 4.1 consists of solving (4.5) via a fixed point argument on the following function space:

$$\mathcal{C}_\ell = \left\{ w(\sigma) : \sup_{\sigma^* < \sigma < \bar{\sigma}_m} |\varepsilon^{(\alpha-2)/3} \langle \sigma \rangle^{-2/3} w(\sigma)| < \infty \right\}.$$

To begin with, let us define the operator A_ℓ by

$$A_\ell w = \left(\frac{d}{d\sigma} w - b(\sigma) w, w(\sigma^*) \right),$$

for $w \in \mathcal{D}(A_\ell) \subset \mathcal{C}_\ell$.

Proposition 4.2. *$A_\ell : \mathcal{D}(A_\ell) \subset C_{W_\ell} \to C_{W_\ell} \times \mathbb{R}$, and A_ℓ is bounded invertible with its inverse uniformly bounded in ε.*

Proof. Similar to the proof of proposition 3.2, let $v(\sigma) = \varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}w(\sigma)$, we consider the conjugate linear operator

$$\begin{aligned}\tilde{A}_\ell v &= \left(\langle\sigma\rangle^{-2/3}\left(\frac{d}{d\sigma} - b(\sigma;\varepsilon)\right)\langle\sigma\rangle^{2/3}v, v(\sigma^*)\right)\\ &= \left(\frac{d}{d\sigma}v - \tilde{b}(\sigma;\varepsilon)v, v(\sigma^*)\right),\end{aligned}$$

where $\tilde{b}$ satisfies

$$\tilde{b} = b(\sigma;\varepsilon) - \frac{2}{3}\langle\sigma\rangle^{-1} = -2 + \mathcal{O}(|\sigma|^{-1}) \to -2,$$

uniformly in ε as $\sigma \to -\infty$.

The associated conjugate equation of

$$A_\ell w = (f, w^*) \text{ with } f \in \mathcal{C}_\ell, \text{ and } w^* \in \mathbb{R},$$

is

$$\tilde{A}_\ell v = (\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}f, v^*) \text{ with } \varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}f \in C, v^* = \varepsilon^{(\alpha-2)/3}\langle\sigma^*\rangle^{-2/3}w^*.$$

Again we apply lemma 6.2 to conclude that there exist a constant C independent of ε such that

$$\|w\|_{\mathcal{C}_\ell} = |v|_\infty \le C(|\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}f|_\infty + |v^*|) = C(\|f\|_{\mathcal{C}_\ell} + \varepsilon^{\alpha/3-1/2}|w^*|), \tag{4.7}$$

which shows the claim, provided that $|w^*| = \mathcal{O}(\varepsilon^{1/2-\alpha/3})$. ■

4.4 Nonlinear estimates

For $\sigma \in (\sigma^*, \bar{\sigma}_m)$, we will estimate the nonlinear term

$$\mathcal{R}_\ell(W_\ell) = \varepsilon^{-1/3}\varphi(\sigma)\left[W_\ell^2 + f(\bar{U}_* + W_\ell, \mu; \varepsilon)\right],$$

in the $\mathcal{C}_\ell$ norm. As a result, we have

Proposition 4.3. *If $W_\ell \in \mathcal{C}_\ell$, then $\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell(\sigma)) \in \mathcal{C}_\ell$ and $\|\varepsilon^{-1/3}\varphi\mathcal{R}_\ell\|_{\mathcal{C}_\ell} = \mathcal{O}(\varepsilon^{\alpha/3})$.*

Proof. From the asymptotic expansion (2.5), we have that

$$\bar{U}_*(\sigma) = \varepsilon^{1/3}\bar{u}_R(\psi(\sigma)) \lesssim |\varepsilon\sigma|^{1/3} \le \varepsilon^{1/4}, \quad \varphi(\sigma) \lesssim \langle\sigma\rangle^{-1/3},$$

for $\sigma^* \le \sigma \le \bar{\sigma}_m$.

Also, for σ in this range, we have $\mu = \varepsilon t - \delta = \varepsilon^{2/3}\psi(\sigma)$ which from definition of the time rescaling in 2.5 that it satisfies

$$|\mu| \lesssim |\varepsilon\sigma|^{2/3} \le \varepsilon^{1/2}.$$

If $W_\ell \in \mathcal{C}_\ell$, then it is true that

$$|W_\ell(\sigma)| \lesssim \varepsilon^{(2-\alpha)/3}\langle\sigma\rangle^{2/3} \ll |\bar{U}_*(\sigma)|,$$

also recall $|\sigma^*| = \mathcal{O}(\varepsilon^{-1/4})$ and $|\bar{\sigma}_m| = \mathcal{O}(1)$, from these facts we have

$$\|\varepsilon^{-1/3}\varphi W_\ell^2\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar{\sigma}_m)} \varepsilon^{(1-\alpha)/3}\langle\sigma\rangle^{1/3} \leq \varepsilon^{(1-\alpha)/3}\langle\sigma^*\rangle^{1/3} = \mathcal{O}(\varepsilon^{(3-4\alpha)/12}),$$

as $f(u,\mu;\varepsilon) = \mathcal{O}(\varepsilon(1+u+\mu+u^2), u\mu, \mu^2, u^3)$, we have

$$f(\bar{U}_* + W_\ell, \mu; \varepsilon) = \mathcal{O}(\varepsilon, (\bar{U}_* + W_\ell)\mu, \mu^2, (\bar{U}_* + W_\ell)^3) = \mathcal{O}(\varepsilon, \bar{U}_*\mu, \mu^2, \bar{U}_*^3).$$

Therefore we have the following estimates:

$$\|\varepsilon^{-1/3}\varphi\varepsilon\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar{\sigma}_m)} \varepsilon^{-1/3}\varphi(\sigma)\varepsilon(\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}) \lesssim \varepsilon^{\alpha/3}\langle\sigma\rangle^{-1} = \mathcal{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-1/3}\varphi\bar{U}_*\mu\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar{\sigma}_m)} \varepsilon^{-1/3}\varphi|\varepsilon\sigma|^{2/3}|\varepsilon\sigma|^{1/3}\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3} = \mathcal{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-1/3}\varphi\mu^2\|_{\mathcal{C}_\ell} \lesssim \|\varepsilon^{-1/3}\varphi\varepsilon\|_{\mathcal{C}_\ell} \text{ since } \mu^2 \lesssim \varepsilon,$$

and

$$\|\varepsilon^{-1/3}\varphi\bar{U}_*^3\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar{\sigma}_m)} \varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}|\varepsilon\sigma|^{2/3} = \mathcal{O}(\varepsilon^{\alpha/3}).$$

Combining all the estimates we conclude that $\|\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell)\|_{\mathcal{C}_\ell} = \mathcal{O}(\varepsilon^{\alpha/3})$ if $W_\ell \in \mathcal{C}_\ell$. ■

4.5 Fixed point argument and the proof of Theorem 4.1

We are ready to prove theorem 4.1.

Proof of theorem 4.1. The proof consists of rewriting equation (3.8) as a fixed point equation. Using the linear operator A_ℓ, we define $F_\ell : \mathcal{C}_\ell \times \mathbb{R} \to \mathcal{C}_\ell \times \mathbb{R}$ as

$$F_\ell(W_\ell, w^*; \varepsilon) = A_\ell W_\ell - (\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell), w^*),$$

Let $X = \mathcal{C}_\ell \times (-\varepsilon^{1/2-\alpha/3}, \varepsilon^{1/2-\alpha/3})$, we introduce the map $\mathcal{S} : X \to \mathcal{C}_\ell \times \mathbb{R}$ as follows:

$$\mathcal{S}(W_\ell, w^*; \varepsilon) = (W_\ell - A_\ell^{-1}F_\ell(W_\ell, w^*; \varepsilon), w^*).$$

From propositions 4.2 and 4.3, we conclude

- $\|\mathcal{S}(0,0;\varepsilon)\| = \|\left(-A_\ell^{-1}F_\ell(0,0;\varepsilon), 0\right)\| \leq \|A_\ell^{-1}\|\|F_\ell(0,0;\varepsilon)\| \lesssim \|\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(0)\| \lesssim \varepsilon^{\alpha/3}$, uniformly in ε.
- $\mathcal{S}$ is a smooth map in W_ℓ, w^* as well as the parameters ε.

- Since the derivative of $f(\bar{U}_* + W_\ell, \mu; \varepsilon)$ with respect to W_ℓ is $D_{W_\ell} f(\bar{U}_* + W_\ell, \mu; \varepsilon) = \mathcal{O}(\mu, \bar{U}_*^2)$, the linearization of $\mathcal{S}$ at $(0,0)$, $D_{(W_\ell, w^*)}\mathcal{S}(0,0)$ satisfies

$$\|D_{(W_\ell,w^*)}\mathcal{S}(0,0)\|_{op} \lesssim \sup |\varepsilon^{-1/3}\varphi(\mu + \bar{U}_*^2)| = \mathcal{O}(\varepsilon^{1/4}),$$

where $\|\cdot\|_{op}$ denotes the operator norm of the associated linear operator. Moreover, for $\|W_\ell\|$ small enough and $|w^*| = \mathcal{O}(\varepsilon^{1/2-\alpha/3})$, we have $D_{(W_\ell,w^*)}\mathcal{S}(W_\ell, w^*; \varepsilon) = D_{(0,0)}\mathcal{S}(W_\ell, w^*; \varepsilon) + \mathcal{O}(\|W_\ell\|_{\mathcal{C}_\ell})$, which is uniformly small in ε.

Therefore, for (W_ℓ, w^*) in a small ball of X, we apply Banach's fixed point argument to the map $\mathcal{S}$ to obtain a solution of equation (4.6) with $W_\ell \in \mathcal{C}_\ell$ and $w^* = \mathcal{O}(\varepsilon^{1/2-\alpha/3})$. Scaling back from σ to t-time, we obtain claim (4.3). ■

5 Region C

This region corresponds to the t-time interval $0 < t < t^*$. We will give the precise form the solution in this region and prove the needed properties.

5.1 Solution in region C

Theorem 5.1. *Fix $\delta > 0, \alpha > 0$ small enough, there exists $\varepsilon_C > 0$ and a constant $C = C(\delta, \alpha)$ such that for all $0 < \varepsilon < \varepsilon_C$, and w_0 with $|w_0| \le C\varepsilon^{1-\alpha/3}$, a solution u_C of the form*

$$u_C(t) = u_s(t) + w_s(t), \tag{5.1}$$

to system (2.12) with the initial condition

$$u(0) = u_s(0) + w_0, \qquad \mu(0) = -\delta \tag{5.2}$$

exists on the time interval $t \in (0, t^*)$, where u_s is the function defined as in (2.11). Moreover, we have that the correction function w_s is continuous with

$$|w_s(t)| \le C\varepsilon^{1-\alpha/3}(\varepsilon t - \delta)^{-1}, \tag{5.3}$$

for $t \in (t^*, \varepsilon^{-1}\delta)$.

This theorem will be proved in the rest of this section.

5.2 Equation of W_s and rescaling

Once again, we plug u_C into (2.1) and use (2.10) to obtain the equation satisfied by w_s.

$$\frac{d}{dt}w_s - 2u_s w_s = w_s^2 + f(u_s + w_s, \mu; \varepsilon) - f(u_s, \mu; 0) - \frac{d}{dt}u_s := R_s(w_s), \tag{5.4}$$

which is posed on $t \in (0, t^*)$.

Rescaling to σ time, we obtain

$$\frac{d}{d\sigma}W_s - c(\sigma)W_s = \varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s), \tag{5.5}$$

where

- The equation is posed on $\sigma \in (\sigma_0, \sigma^*)$. where $\sigma^* \approx -\varepsilon^{-1/4}$ is the left most point in region B and $\sigma_0 = -\frac{2}{3}\delta^{3/2}\varepsilon^{-1}$.
- Like W_r and W_ℓ in region A and region B, the function $W_s(\sigma)$ is the rescaled version of $w_s(t)$ in the σ-variable, $w_s(t) = w_s(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = W_s(\sigma)$.

  Rescaling to σ variable leads to the asymptotic expansion of the rescaled version of $u_s(t)$:

$$U_s(\sigma) := u_s(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathcal{O}(|\varepsilon\sigma|^{2/3}). \tag{5.6}$$

- The function $\mathcal{R}_s$ is likewise a rescaled version of R_s such that

$$\mathcal{R}_s(W_s; \varepsilon) = W_s^2 + f(U_s + W_s, \mu; \varepsilon) - f(U_s, \mu; 0) - \varepsilon^{1/3}\varphi^{-1}\frac{d}{d\sigma}U_s(\sigma),$$

- The term $c(\sigma)$ is defined and has the asymptotic expansions:

$$c(\sigma) = 2\varepsilon^{-\frac{1}{3}}U_s(\sigma)\varphi(\sigma) = -2 + \mathcal{O}(\varepsilon^{1/3}|\sigma|^{1/3}),$$

  as $\sigma \to -\infty$.

Once again we need to specify the boundary value at the left end point $\sigma = \sigma_0$, the complete system we want to solve is

$$\begin{aligned} \frac{d}{d\sigma}W_s - c(\sigma)W_s &= \varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s),\ \sigma \in (\sigma_0, \sigma^*), \\ W_s(\sigma_0) &= w_0. \end{aligned} \tag{5.7}$$

5.3 Linear equation and norms

The proof of theorem 5.1 will be complete if we are able to solve (5.7) using a fixed point argument similar to region A and B. The function space on which we will solve the W_s equation via a fixed point argument is:

$$\mathcal{C}_s = \left\{ w(\sigma) : \sup_{\sigma_0 < \sigma < \sigma^*} |\varepsilon^{\frac{\alpha}{3}-1}\langle\varepsilon\sigma\rangle^{\frac{2}{3}}w(\sigma)| < \infty \right\},$$

And similarly we define the linear operator A_s which acts on $w \in \mathcal{D}(A_s) \subset \mathcal{C}_s$ by

$$A_s w = \left(\frac{d}{d\sigma}w - cw, w(\sigma_0)\right).$$

Proposition 5.2. *$A_s : \mathcal{D}(A_s) \subset \mathcal{C}_s \to \mathcal{C}_s \times \mathbb{R}$, and A_s is bounded invertible with its inverse uniformly bounded in ε.*

Proof. Unlike the case for linear operator A_r and A_ℓ, lemma 6.2 cannot be directly used for the operator A_s since from the asymptotic expansion of c we see that $c(\sigma)$ does not converge to -2 as $\sigma \to -\infty$, in fact, it diverges to $-\infty$ as $\sigma \to -\infty$. However, for $\sigma \in (\sigma_0, \sigma^*)$, we have

$$|c(\sigma) - (-2)| \lesssim |\varepsilon\sigma|^{1/3} \lesssim \delta^{1/2},$$

hence for δ small, A_s is a small perturbation of the operator

$$L_s : w \mapsto \left(\frac{d}{d\sigma}w + 2w, w(\sigma_0)\right).$$

To see the invertibility of L_s on the weighted space $\mathcal{C}_s$, let $v(\sigma) = \varepsilon^{\alpha/3-1}\langle\varepsilon\sigma\rangle^{2/3}w(\sigma)$ and consider the conjugate linear operator

$$\begin{aligned}\tilde{L}_s : v \mapsto &\left(\langle\varepsilon\sigma\rangle^{-2/3}\left(\frac{d}{d\sigma}+2\right)\langle\varepsilon\sigma\rangle^{2/3}v(\sigma), v(\sigma_0)\right)\\ &= \left(\left(\frac{d}{d\sigma}+2+\frac{2}{3}\varepsilon\langle\varepsilon\sigma\rangle^{-1}\right)v, v(\sigma^*)\right),\end{aligned}$$

which acts on $v \in \mathcal{C}(\sigma_0, \sigma^*)$.

Hence, the conjugate linear equation of

$$L_s w = (f, w_0),$$

is

$$\tilde{L}_s v = (\tilde{f}, v_0),$$

with $v_0 = \varepsilon^{\alpha/3-1}\langle\varepsilon\sigma_0\rangle^{2/3}w_0$ and $\tilde{f} = \varepsilon^{\alpha/3-1}\langle\varepsilon\sigma\rangle^{2/3}f$, which is a differential equation of the form

$$\left(\frac{d}{d\sigma}+2+\mathcal{O}(\varepsilon)\right)v = f, \qquad v(\sigma_0) = v_0.$$

Its linear part is yet another small perturbation of the linear operator $\frac{d}{d\sigma}+2$ on the uniform space $\mathcal{C}(\sigma_0, \sigma^*)$, integrating this equation yields

$$|v|_\infty \leq C(|v_0| + |f|_\infty),$$

for some constant C independent of ε. Equivalently, in terms of w we have

$$\|w\|_{\mathcal{C}_s} \leq C(|\delta\varepsilon^{\alpha/3-1}w_0| + \|f\|_{\mathcal{C}_s}), \tag{5.8}$$

which shows the invertibility of L_s and uniformity of the inverse in ε, provided that $w_0 = \mathcal{O}(\delta^{-1}\varepsilon^{1-\alpha/3})$. The same property hence holds for A_s as well. ■

5.4 Nonlinear estimates

For $\sigma \in (\sigma_0, \sigma^*)$, we estimate the nonlinear term $\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s(\sigma))$ in the $\mathcal{C}_s$ norm. Similar to proposition 3.3 and 4.3, we have

Proposition 5.3. *If $W_s \in \mathcal{C}_s$, then $\varepsilon^{-\frac{1}{3}}\varphi\mathcal{R}_s(W_s(\sigma)) \in \mathcal{C}_s$ and $\|\varepsilon^{-\frac{1}{3}}\varphi\mathcal{R}_s\|_{\mathcal{C}_s} = \mathcal{O}(\delta^{1/2})$.*

Proof. Recall that

$$\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s;\varepsilon) = \varepsilon^{-1/3}\varphi\left[W_s^2 + f(U_s+W_s,\mu;\varepsilon) - f(U_s,\mu;0)\right] - \frac{d}{d\sigma}U_s(\sigma),$$

From the definition of $\psi(\sigma)$ we have

$$|\varphi(\sigma)| \lesssim |\sigma|^{-1/3},$$

and $\mu = \varepsilon t - \delta = \varepsilon^{2/3}\psi(\sigma)$ satisfies

$$|\mu| \lesssim |\varepsilon\sigma|^{2/3}$$

From (5.6), we have that

$$U_s(\sigma) = -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathcal{O}(|\varepsilon\sigma|^{2/3}),$$
$$\frac{d}{d\sigma}U_s(\sigma) = -\frac{1}{2}\varepsilon(\varepsilon\sigma)^{-2/3} + \mathcal{O}(\varepsilon|\varepsilon\sigma|^{-1/3}),$$

and for $W_s \in \mathcal{C}_s$, it holds that

$$|W_s(\sigma)| \lesssim \varepsilon^{1-\alpha/3}\langle\varepsilon\sigma\rangle^{-2/3}.$$

Hence we have the following estimates:

$$\left\|\frac{d}{d\sigma}U_s(\sigma)\right\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{\alpha/3-1}\langle\sigma\rangle^{2/3}\varepsilon|\varepsilon\sigma|^{-2/3} = \mathcal{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-1/3}\varphi W_s^2(\sigma)\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}\varepsilon^{1-\alpha/3}\langle\varepsilon\sigma\rangle^{-2/3} = \mathcal{O}(\varepsilon^{\frac{1}{4}-\alpha/3}),$$

The term $\varepsilon^{-1/3}\varphi[f(U_s+W_s,\mu;\varepsilon) - f(U_s,\mu;0)]$ is estimated as follows, we decompose $f(U_s+W_s,\mu;\varepsilon) - f(U_s,\mu;0)$ into $f_1 + f_2$, where

$$f_1 = f(U_s+W_s,\mu;\varepsilon) - f(U_s+W_s,\mu;0),$$

and

$$f_2 = f(U_s+W_s,\mu;0) - f(U_s,\mu;0).$$

Since $f(u,\mu;\varepsilon) = \mathcal{O}(\varepsilon(1+u+\mu+u^2), u\mu, \mu^2, u^3)$, we have

$$f_1 = \mathcal{O}(\varepsilon(1+U_s+W_s+\mu+U_s^2)) = \mathcal{O}(\varepsilon),$$

and

$$f_2 = \mathcal{O}(W_s\mu + U_sW_s^2 + U_s^2W_s + W_s^3).$$

For f_1 we simply estimate

$$\|\varepsilon^{-1/3}f_1\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}\varphi(\varepsilon)\varepsilon^{\alpha/3-1}|\varepsilon\sigma|^{2/3} \lesssim \varepsilon^{\alpha/3}|\varepsilon\sigma_0|^{1/3} = \mathcal{O}(\varepsilon^{\alpha/3}).$$

For f_2 we have

$$\|\varepsilon^{-1/3}\varphi W_s\mu\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}|\mu| \lesssim |\varepsilon\sigma_0|^{1/3} = \mathcal{O}(\delta^{1/2}),$$
$$\|\varepsilon^{-1/3}\varphi U_s^2W_s(\sigma)\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}|\varepsilon\sigma|^{2/3} \lesssim |\varepsilon\sigma_0|^{1/3} = \mathcal{O}(\delta^{1/2}),$$
$$\|\varepsilon^{-1/3}\varphi U_sW_s^2(\sigma)\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}|\varepsilon\sigma|^{1/3}\varepsilon^{1-\alpha/3}|\varepsilon\sigma|^{-2/3} = \mathcal{O}(\varepsilon^{1/2-\alpha/3}),$$

and

$$\|\varepsilon^{-1/3}\varphi W_s^3(\sigma)\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}(\varepsilon^{1-\alpha/3}|\varepsilon\sigma|^{-2/3})^2 = \mathcal{O}(\varepsilon^{3/4-2\alpha/3}).$$

Hence $\|\varepsilon^{-1/3}f_2\|_{C_s} = \mathcal{O}(\delta^{1/2})$, combining all the estimates we conclude that

$$\|\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s)\| = \mathcal{O}(\delta^{1/2}),$$

which finishes the proof. ■

5.5 Fixed point argument and the proof of Theorem 5.1

Proof of theorem 5.1. The proof is almost identical to the proof of 4.1. Using the linear operator A_s, we define the nonlinear operator $F_s : \mathcal{C}_s \times \mathbb{R} \to \mathcal{C}_s \times \mathbb{R}$ as

$$F_s(W_s, w_0; \varepsilon) := A_s W_s - (\varepsilon^{-1/3}\varphi \mathcal{R}_s(W_s), w_0),$$

Let $X = \mathcal{C}_s \times (-\delta^{-1}\varepsilon^{1-\alpha/3}, \delta^{-1}\varepsilon^{1-\alpha/3})$, we introduce the map $\mathcal{S} : X \to \mathcal{C}_s \times \mathbb{R}$ as follows:

$$\mathcal{S}(W_s, w_0; \varepsilon) := (W_s - A_s^{-1} F_s(W_s, w_0; \varepsilon), w_0).$$

We conclude from proposition 5.2 and 5.3 that:

- $\|\mathcal{S}(0,0;\varepsilon)\| = \| \left(-A_s^{-1} F_s(0,0;\varepsilon), 0\right) \| \leq \|A_s^{-1}\| \|F_s(0,0;\varepsilon)\| \lesssim \|\varepsilon^{-1/3}\varphi\mathcal{R}_s(0)\| \lesssim \varepsilon^{\alpha/3}$, uniformly in ε.
- $\mathcal{S}$ is a smooth map in W_s, w_0 as well as the parameter ε.
- Since the derivative of $f(U_s + W_s, \mu; \varepsilon)$ with respect to W_s at $W_s = 0$ is $D_{W_s} f(U_s + W_s, \mu; \varepsilon) = \mathcal{O}(\mu, U_s^2)$, the linearization of $\mathcal{S}$ at $(0,0)$, $D_{(W_s,w_0)}\mathcal{S}(0,0)$ satisfies

  $$\|D_{(W_s,w_0)}\mathcal{S}(0,0)\|_{op} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} |\varepsilon^{-1/3}\varphi(\mu + U_s^2)| = \mathcal{O}(\delta^{1/2}),$$

  where $\|\cdot\|_{op}$ denotes the operator norm of the associated linear operator. Moreover, for $\|W_s\|$ small enough and $|w_0| = \mathcal{O}(\delta^{-1}\varepsilon^{1-\alpha/3})$, we have $D_{(W_s,w_0)}\mathcal{S}(W_s, w_0;\varepsilon) = D_{(W_s,w_0)}\mathcal{S}(0,0;\varepsilon) + \mathcal{O}(\|W_\ell\|_{\mathcal{C}_\ell})$, which is uniformly small in ε.

Therefore, for (W_s, w_0) in a small ball of X, we apply Banach's fixed point argument to the map $\mathcal{S}$ to obtain a solution of equation (5.7) with $W_s \in \mathcal{C}_s$ and $w_0 = \mathcal{O}(\varepsilon^{1-\alpha/3})$. Scaling back from σ to t-time, we obtain claim (5.3). ■

6 Gluing

So far we have showed solutions of the form u_A, u_B, and u_C exists on region A, B and C, respectively. To prove Theorem 2.2 then, we will need to patch together u_A, u_B and u_C on the whole time interval $t \in (0, T)$.

Starting with the left most point in region C, Theorem 5.1 shows (2.1) has a solution of the form

$$u_C(t) = u_s(t) + w_s(t; w_0),$$

exists, provided that we pick the initial condition $w_0 = \mathcal{O}(\delta^{-1}\varepsilon^{1-\alpha/3})$.

At t^*, the right end of region C, we see $u_C(t^*) = u_s(t^*) + w_s(t^*; w_0)$ has the following expansion:

$$u_s(t^*) = -\sqrt{\delta - \varepsilon t^*} + \mathcal{O}(|\delta - \varepsilon t^*|) = -\varepsilon^{1/4} + \mathcal{O}(\varepsilon^{1/2}),$$
$$w_s(t^*) \lesssim \varepsilon^{1-\alpha/3}(\varepsilon t^* - \delta)^{-1} = \mathcal{O}(\varepsilon^{1/2-\alpha/3}),$$

therefore

$$u_C(t^*) = -\varepsilon^{1/4} + \mathcal{O}(\varepsilon^{1/2-\alpha/3}).$$

Notice at t^*, $\bar{u}_*(t^*)$ has the following expansion in ε:

$$\bar{u}_*(t^*) = \varepsilon^{1/3}\bar{u}_R(\varepsilon^{1/3}(t^* - \varepsilon^{-1}\delta)) = -\varepsilon^{1/4} + \mathcal{O}(\varepsilon^{1/2}).$$

Hence, if we set $w^* = u_s(t^*) - \bar{u}_*(t^*) + w_s(t^*)$, then $w^* = \mathcal{O}(\varepsilon^{1/2-\alpha/3})$, we can apply Theorem 4.1 with the initial condition $u(t^*) = \bar{u}_*(t^*) + w^*$ to get a solution of (2.1) of the form

$$u_B(t) = \bar{u}_*(t) + w_\ell(t; w^*),$$

in region B. So that

$$u_C(t^*) = u_B(t^*). \tag{6.1}$$

Moving on, again from Theorem 4.1, we conclude that at the point $t = \varepsilon^{-1}\delta$, the right end of region B, the value of $u_B(\varepsilon^{-1}\delta) = \bar{u}_*(\varepsilon^{-1}\delta) + w_\ell(\varepsilon^{-1}\delta; w^*)$ has the following expansion in ε:

$$\begin{aligned}\bar{u}_*(\varepsilon^{-1}\delta) &= \varepsilon^{1/3}\bar{u}_R(0) = \varepsilon^{1/3}\bar{u}_0,\\ w_\ell(\varepsilon^{-1}\delta) &= \mathcal{O}(\varepsilon^{(2-\alpha)/3}),\end{aligned}$$

therefore

$$u_B(\varepsilon^{-1}\delta) = \varepsilon^{1/3}\bar{u}_0 + \mathcal{O}(\varepsilon^{(2-\alpha)/3}).$$

On the other hand, Theorem 3.1 proves a solution to (2.1) of the form $u_A(t; u_0) = u_*(t; u_0) + w_r(t; u_0)$ exists for $t \in (\varepsilon^{-1}\delta, T)$ with $u_A(T; u_0) = \delta$, provided that u_0 is close to $\bar{u}_0$. Moreover, at $t = \varepsilon^{-1}\delta$ the value $u_A(\varepsilon^{-1}\delta; u_0)$ has the following expansion in ε:

$$\begin{aligned}u_*(\varepsilon^{-1}\delta; u_0) &= \varepsilon^{1/3}u_R(0; u_0) = \varepsilon^{1/3}u_0,\\ w_r(\varepsilon^{-1}\delta; u_0) &= \mathcal{O}(\varepsilon^{(2-\alpha)/3}),\end{aligned}$$

therefore

$$u_A(\varepsilon^{-1}\delta; u_0) = u_*(\varepsilon^{-1}\delta; u_0) + w_r(\varepsilon^{-1}\delta; u_0) = \varepsilon^{1/3}u_0 + \mathcal{O}(\varepsilon^{(2-\alpha)/3}).$$

Therefore, the matching condition at $\varepsilon^{-1}\delta$ is

$$u_A(\varepsilon^{-1}\delta; u_0) = u_B(\varepsilon^{-1}\delta), \tag{6.2}$$

Using the expansions we obtained, this amounts to solve the equation

$$0 = \varepsilon^{1/3}(u_0 - \bar{u}_0) + w_r(\varepsilon^{-1}\delta; u_0) - w_\ell(\varepsilon^{-1}\delta), \tag{6.3}$$

in the variable u_0. Let $\phi(\varepsilon; u_0) := w_r(\varepsilon^{-1}\delta; u_0) - w_\ell(\varepsilon^{-1}\delta)$, we conclude from Theorem 3.1 and Theorem 4.1 that

$$\phi(\varepsilon; u_0) = \mathcal{O}(\varepsilon^{(2-\alpha)/3}),$$

uniformly in u_0 and

$$\mathrm{Lip}_{u_0}\phi(\varepsilon; u_0) = \mathcal{O}(\varepsilon^{2/3}).$$

Hence we divide the right hand side of (6.3) by $\varepsilon^{1/3}$, and apply the implicit function theorem around the point $(u_0 = \bar{u}_0, \varepsilon = 0)$ to conclude that there exist a branch $u_0 = u_0(\varepsilon)$ with $u_0(\varepsilon) = \bar{u}_0 + \mathcal{O}(\varepsilon^{(1-\alpha)/3})$, the matching condition (6.2) is satisfied.

Therefore, we have proved the following theorem, which implies Theorem 2.2.

Theorem 6.1. *Let $u_-(\mu)$ be the attracting branch of the critical manfold defined in (2.10) and $\bar{u}_0$ be the value of the function $\bar{u}_R(s)$ defined in (2.5) evaluated at $s=0$. Fix $\alpha > 0$ and $\delta > 0$ small enough, there exist $\varepsilon_0 > 0$, and a constant $C = C(\delta,\alpha)$, such that for all $0 < \varepsilon < \varepsilon_0$ and u_i with $|u_i - u_-(-\delta)| \le C\varepsilon^{1-\alpha/3}$, there exist a branch of initial conditions $u_0 = u_0(\varepsilon) = \bar{u}_0 + \mathcal{O}(\varepsilon^{(1-\alpha)/3})$, and a solution of the rescaled system (2.12) with the initial condition (2.13) of the following form*

$$u(t) = \begin{cases} u_A(t;u_0(\varepsilon)) = u_*(t;u_0(\varepsilon)) + w_r(t;u_0(\varepsilon)), \\ u_B(t) = \bar{u}_*(t) + w_\ell(t), \\ u_C(t) = u_s(t) + w_s(t), \end{cases} \tag{6.4}$$

and

$$\mu(t) = \varepsilon t - \delta, \tag{6.5}$$

exists on $t \in (0,T)$, where all of the correction functions w_r, w_ℓ and w_s satisfy the properties stated in Theorems 3.1, 4.1 and 5.1, respectively.

Proof of theorem 2.2 and 2.3. Theorem 2.2 follows from Theorem 6.1 almost directly, we just need to show the expansion (2.15) and (2.16).

Recall that $\Omega_\infty(u_0)$ depends smoothly on u_0 by Proposition 2.1, hence there is a constant C such that

$$|\Omega_\infty(u_0) - \Omega_\infty(\bar{u}_0)| \le C|u_0 - \bar{u}_0|,$$

for u_0 close to $\bar{u}_0$. Now we substitute the branch $u_0(\varepsilon)$ with $u_0(\varepsilon) = \bar{u}_0 + \mathcal{O}(\varepsilon^{(1-\alpha)/3})$ and use the expansion (3.3), we have

$$\begin{aligned} T &= \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0(\varepsilon)) + \delta^{-1} + \mathcal{O}(\varepsilon^{2/3}) \\ &= \varepsilon^{-1}\delta + \varepsilon^{-1/3}(\Omega_\infty(\bar{u}_0) + \mathcal{O}(\varepsilon^{(1-\alpha)/3})) + \delta^{-1} + \mathcal{O}(\varepsilon^{2/3}) \\ &= \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_0 + \mathcal{H}(\varepsilon), \end{aligned}$$

with $\mathcal{H}(\varepsilon) = \mathcal{O}(\varepsilon^{-\alpha/3})$.

To show the dependence of $\mathcal{H}$ on the initial condition u_i in (2.16) takes a little more work, to begin, we consider the following maps:

$$\begin{aligned} u_i &\mapsto u_C(t;u_i), \\ u_C(t;u_i) &\mapsto u_C(t^*;u_i), \\ u_C(t^*;u_i) := u^* &\mapsto u_B(t;u^*), \\ u_B(t;u^*) &\mapsto u_B(\varepsilon^{-1}\delta;u^*), \end{aligned}$$

which are all smooth either because they are evaluation maps or by the smooth dependence of solutions on the initial conditions.

Let $\mathcal{K}$ be the map $u_i \mapsto u_B(\varepsilon^{-1}\delta;u^*)$. It can be seen from (3.10),(4.7), (5.8) and the definition of the norms that

$$\begin{aligned} \|u_C - \tilde{u}_C\|_{\mathcal{C}_s} &\le C\varepsilon^{\alpha/3-1}|u_i - \tilde{u}_i|, \\ |u_C(t^*) - \tilde{u}_C(t^*)| &\le C\varepsilon^{1/2-\alpha/3}\|u_C - \tilde{u}_C\|_{\mathcal{C}_s}, \\ \|u_B - \tilde{u}_B\|_{\mathcal{C}_\ell} &\le \varepsilon^{\alpha/3-1/2}|u_C(t^*) - \tilde{u}_C(t^*)|, \\ |u_B(\varepsilon^{-1}\delta) - \tilde{u}_B(\varepsilon^{-1}\delta)| &\le C\varepsilon^{(2-\alpha)/3}\|u_B - \tilde{u}_B\|_{\mathcal{C}_\ell}, \end{aligned}$$

which implies

$$\mathrm{Lip}_{u_i}\mathcal{K} \le C\varepsilon^{-1/3} \tag{6.6}$$

Next, recall that the branch $u_0 = u_0(\varepsilon)$ was solved by the condition (6.3) which shows the map $u_B(\varepsilon^{-1}\delta) \mapsto u_0$ has Lipschitz constant $\le C\varepsilon^{-1/3}$, which shows the map

$$u_i \mapsto u_0 \mapsto \Omega_\infty(u_0) = \Omega_\infty(u_0(u_i)) \tag{6.7}$$

has Lipschitz constant $\le C\varepsilon^{-2/3}$.

Recall now that $\mu(T) = \varepsilon T - \delta = \varepsilon^{2/3}\Omega_\infty(u_0) + \mathcal{O}(1)$. Therefore we conclude the map

$$u_i \mapsto \mu(T) \tag{6.8}$$

has Lipschitz constant of order $\mathcal{O}(1)$.

But remember the fact that all u_i was chosen such that $|u_i - u_-(-\delta)| \le C\varepsilon^{1-\alpha/3}$.

Now we prove Corollary 2.3, recall $(\mu, u, \varepsilon) \in \mathcal{U}$ where $\mathcal{U}$ is a small neighborhood of the origin $(0,0,0)$. Take a compact interval K such that $(\mu, u, \varepsilon) \subset \mathcal{U}$ for all $u \in K$, consider a trajectory (μ, u) with initial condition $\mu(0) = -2\delta$ and $u(0) \in K$. By standard Fenichel theory, at $\varepsilon = 0$, there exist stable foliations with base the lower branch $u_-(\mu)$ of the critical manifold $S_0 = \{(\mu, u) : \mu + u^2 + f(\mu, u, 0) = 0\}$ for $\mu \in [-2\delta, -\delta]$. Further, for $0 < \varepsilon \ll 1$, the foliation persists and the part of the critical manifold $u_-(\mu)$ perturbs to a nearby slow manifold $u_-^\varepsilon(\mu)$. Moreover, there exist asymptotic rate $\gamma = \gamma(\delta) > 0$ and a constant $C = C(\delta)$, independent of ε such that the solution $u(t)$ starts at $\mu(0) = -2\delta$ and $u(0) \in K$ converges exponentially close to the slow manifold:

$$|u(t) - u_-^\varepsilon(\mu(t))| \le Ce^{-\gamma t}. \tag{6.9}$$

Let T_* be the time of flight for a solution which starts at $\mu = -2\delta$ at $t = 0$ to reach $\mu = -\delta$. Since $\mu' = \varepsilon$ we see directly that $T_* = \delta/\varepsilon$. From (6.9) we see that $|u(T_*) - u_-^\varepsilon(-\delta)| \le Ce^{-\gamma\cdot\delta/\varepsilon}$. But again by Fenichel theory, $u_-^\varepsilon(\mu)$ is $\mathcal{O}(\varepsilon)$ distance away from $u_-(\mu)$. This implies

$$|u(T_*) - u_-(-\delta)| = \mathcal{O}(\varepsilon),$$

which satisfies the assumption of Theorem 6.1 if we restart the equation with $u = u(T_*)$ and $\mu = -\delta$ at $t = 0$ and follow this trajectory, by Theorem 6.1 we conclude that the trajectory hit the section Δ_{out} at the point $(h(\varepsilon), \delta)$ with $h(\varepsilon)$ has the expansion (2.15).

Since the choice for the staring point $\mu = -2\delta$ was arbitrary, the same argument applies whenever one starts with $\mu \in [-2\delta, -\delta]$ and $u \in K$, this proves there is a rectangular region for which we have the dependence $u_i \mapsto \mu(T)$ is of order $\mathcal{O}(\varepsilon^{-c/\varepsilon})$. Which proves Theorem 2.2 and Corollary 2.3. ∎

Remark: Since $\alpha > 0$ could be chosen arbitrary small, expansion (2.15) suggests that the next order term after $\varepsilon^{2/3}\Omega_0$ might be of order $\varepsilon\log(\varepsilon)$. This is true, a well-known from the matched-asymptotics community and is affirmed by Krupa and Szmolyan using the blow-up methods.

Appendix

We first show how to extend the asymptotic expansion (2.5) to a more general family of solutions.

A A family of the solution to the Riccati equation

Proof of proposition 2.1. To get the dependence from u_0 to Ω_∞, we first add the equation $\frac{d}{ds}s = 1$ to equation (2.4) to get a autonomous $2-$dimensional system in the (s,u) plane. Consider a small neighborhood I containing $\bar{u}_0$ on the vertical u-axis, then $u_R(s;u_0)$ is the trajectory that starts at $u_0 \in I$. The map $P_1 : I \to \mathbb{R}$ defined by $P_1(p) = u(2;p)$ is smooth in p, as the blow up time for $\bar{u}_R(s;\bar{u}_0)$ is $\Omega_0 > 2$. Moreover, the image $P_1(I)$ is a finite interval on the vertical line $s = 2$ containing $\bar{u}_R(2;u_0)$ bounded away from 0, since the trajectory $u_R(s;\bar{u}_0)$ crosses the horizontal axis around $s = 1$ and the vector field goes upwards in the first quadrant of the (s,u)-plane.

Denote $\tilde{u}_0 := P_1(u_0)$ for brevity (technically, the interval $P_1(I)$ is a small section of the line $s = 2$, with a little abuse of notation, we identify $\tilde{u}_0$ with the second coordinate of the point $P_1(u_0)$). Again in the Riccati equation (2.4), we make a change of variable by setting $z(s) = 1/u(s)$, the equation z satisfies is:

$$\frac{d}{ds}z(s) = -z^2 s - 1.$$

Let $J = \{1/\tilde{u}_0 \mid \tilde{u}_0 \in P_1(I)\}$ and $z(s;1/\tilde{u}_0)$ is the trajectory which starts at $1/\tilde{u}_0$. We claim that $z(s;1/\bar{u}_0)$ reaches 0 at a finite time $\Omega_\infty = \Omega_\infty(1/\bar{u}_0)$. To see this, first notice there is no equilibrium for the two dimensional system $\frac{d}{ds}s = 1, \frac{d}{ds}z = -z^2 s - 1$. Then, on the boundary $s = 2$, the vector field takes the form $(1, -2z^2 - 1)$, which makes any trajectory starting at a point on J moving down towards the right. Moreover, the vector field $(1, -sz^2 - 1)$ always pointing down in the first quadrant of the (s,z) plane, so trajectories cannot go upwards. Lastly, the vector field crosses the horizontal axis non-tangentially, it identically equals $(1,-1)$ throughout the line $z = 0$, hence, any trajectory which starts at a point on J will cross $z = 0$ in finite time at a unique point $\Omega_\infty = \Omega_\infty(1/\tilde{u}_0)$. The dependence of Ω_∞ on $1/\tilde{u}_0$ is smooth by the smooth dependence on initial conditions.

We now define another map $P_2 : J \to \mathbb{R}$ by $P_2(1/\tilde{u}_0) = \Omega_\infty(1/\tilde{u}_0)$, we get a smooth map $P : I \to \mathbb{R}$ by the composition

$$P = P_2 \circ f \circ P_1,$$

where $f(z) = 1/z$ is the inversion map. Since each of the map in the composition is smooth, $P : u_0 \mapsto \Omega_\infty = \Omega_\infty(u_0)$ is smooth as well.

To get the asymptotic expansion, we set $\xi = \Omega_\infty - s$, then $\tilde{z}(\xi) = z(\Omega_\infty - \xi)$ solves

$$\frac{d}{d\xi}\tilde{z} = \tilde{z}^2(\Omega_\infty - \xi) + 1,$$

and $\tilde{z}(0) = 0$.

Hence we can assume the expansion for $\tilde{z}$ near $\xi = 0$ is of the form

$$\tilde{z} = \xi + z_2\xi^2 + z_3\xi^3 + \mathcal{O}(\xi^4),$$

for some constant z_2, z_3. Differentiating this expansion, use the equation $\tilde{z}$ solves and comparing coefficients, we find that $z_2 = 0, z_3 = \Omega_\infty/3$. Changing back from $\tilde{z}(\xi)$ to $z = z(s)$ with $s = \Omega_\infty - \xi$ and recall $z(s) = 1/u(s)$, we find that $u_R(s; u_0)$ has expansion (2.6) with remainder r satisfies (2.7).

■

Next we show the main perturbation lemma used to prove the invertibility of the linearized operators at the ansatz.

B Uniform invertibility of boundary value problems

Lemma 6.2. *Consider the following boundary value problems*

$$\dot{u}(\sigma) = a(\sigma)u + f(\sigma), \qquad u(L) = u_L, \tag{6.10a}$$

$$\dot{u}(\sigma) = b(\sigma)u + g(\sigma), \qquad u(-M) = u_M, \tag{6.10b}$$

where equation (6.10a) is posed on $\sigma \in [0, L]$ with $L > 0$ and (6.10b) is posed on $\sigma \in [-M, 0]$ with $M > 0$. Assume $a(\sigma), b(\sigma)$ are continuous functions such that

$$a(\sigma) \to a_+ > 0, \qquad \sigma \to \infty, \tag{6.11a}$$

$$b(\sigma) \to b_- < 0, \qquad \sigma \to -\infty, \tag{6.11b}$$

then (6.10) has a unique solutions u_a, u_b which satisfies

$$|u_a|_\infty \leq C_a(u_L + |f|_\infty), \tag{6.12a}$$

$$|u_b|_\infty \leq C_b(u_m + |g|_\infty), \tag{6.12b}$$

for some constants C_a, C_b independent of L and M.

Proof. We only prove the result for (6.10a) since the other case is similar.

Step I

To begin with, consider the asymptotic equation

$$\dot{u} = a_+ u + f(\sigma), \qquad u(L) = 0. \tag{6.13}$$

posed on $\sigma \in [0, L]$. (6.12a) holds for (6.13) since in this case we have

$$\begin{aligned}
u_a(\sigma) &= e^{a_+(\sigma - L)} u_L + \int_L^t e^{a_+(\sigma - s)} f(s) ds \\
&\leq 2|u_L| + \left| \int_L^t e^{a_+(\sigma - s)} ds \right| |f|_\infty \\
&\leq 2|u_L| + \frac{1}{a_+} \left| e^{t-L} - 1 \right| |f|\infty \\
&\leq 2(|u_L| + |f|_\infty).
\end{aligned}$$

Step II

Next, give $\eta > 0$ small enough and independent of L, there exist $\sigma_* \leq L$ such that $|a(\sigma) - a_+| < \eta$ for all $\sigma > \sigma*$. It is important to note here that one can choose σ_* independent of L as long as L is large enough. A Neumann series argument shows that in this case the operator

$$u \mapsto \left(\frac{d}{dt}u - a(t)u, u(L)\right)$$

is a $\eta-$perturbation of the asymptotic operator

$$u \mapsto \left(\frac{d}{dt}u - a_+u, u(L)\right),$$

which acts on the space of continuous functions $\mathcal{C}([\sigma_*, L])$ with uniform norm (with domain), hence (6.11a) holds with the sup norm taken on $[\sigma_*, L]$.

Step III

Finally, for $\sigma \in [0, \sigma_*]$, the solution is given by the following formula

$$u(\sigma) = \exp\left(\int_{\sigma_*}^{\sigma} a(\tau)d\tau\right) u(\sigma_*) + \int_{\sigma_*}^{\sigma} \exp\left(-\int_{\sigma}^{s} a(\tau)d\tau\right) f(s)ds$$

since $\sigma_* < \infty$ and does not depend on L, there exist a constant C_1 independent of L so that

$$\max\left\{\left|\exp\left(\int_{\sigma_*}^{\sigma} a(\tau)d\tau\right)\right|, \left|\int_{\sigma_*}^{\sigma} \exp\left(-\int_{\sigma}^{s} a(\tau)d\tau\right)\right|\right\} \leq C_1,$$

moreover, the value $u(\sigma_*)$ satisfies

$$u(\sigma_*) \leq \sup_{\sigma \in (\sigma_*, L)} |u(\sigma)| \leq C_2(u_L + |f|_\infty)$$

for some constant C_2 independent of L from the conclusion in step 2. Therefore on $[0, \sigma_*]$ the solution satisfies

$$\sup_{\sigma \in [0, \sigma_*]} |u(\sigma)| \leq C_1C_2(u_L + |f|_\infty) + C_1|f|_\infty \leq C(u_L + |f|_\infty)$$

where obviously C does not depend on L. Therefore we conclude that

$$\sup_{\sigma \in [0, L]} = |u|_\infty \leq C(u_L + |f|_\infty)$$

which is (6.12a). ∎

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