1 Introduction

Introduce something

2 Model problem for passage through the fold

The following problem will be studied using the gluing method instead of blow up.

$$\begin{aligned} \dot{u} &= \mu + u^2 + u^3 \\ \dot{\mu} &= \varepsilon \end{aligned} \tag{2.1}$$

with boundary condition

$$u(T) = \delta \text{ and } \mu(0) = -\delta, \tag{2.2}$$

where T is another parameter, the "time of flight" for the trajectory to shoot from $\mu = -\delta$ to $u = \delta$.

We first study the "blow up " problem, starting with rescale $u = \varepsilon^{1/3} u_1(\varepsilon^{1/3} t)$ and $\mu = \varepsilon^{2/3} \mu_1(\varepsilon^{1/3} t)$. We get the new equations (set $\tau = \varepsilon^{1/3} t$)

$$\begin{aligned} \partial_\tau u_1 &= \mu_1 + u_1^2 + (\varepsilon^{2/3} u_1^4) \\ \partial_\tau \mu_1 &= 1 + (\varepsilon^{1/3} u_1) \end{aligned} \tag{2.3}$$

The new boundary condition is

$$u_1(T) = \delta \varepsilon^{-1/3}, \mu_1(0) = -\delta \varepsilon^{-2/3} \tag{2.4}$$

Then if we set $s = \tau - \delta\varepsilon^{-2/3}$ and formally let $\varepsilon \to 0$, equation (2.3) has an explicit solution $u_1(\tau) = u_R(s)$ and $\mu_1(\tau) = s$. Where u_R is the unique solution to the riccati equation $\partial_s u_R = s + u_R^2$ with the specific asymptotics [reference].

$$u_R(s) = \begin{cases} (T_R - s)^{-1} + \mathcal{O}(T_R - s), \text{ as } s \to T_R \\ -(-s)^{1/2} - \frac{1}{4}(-s)^{-1} + \mathcal{O}(|s|^{-3/2}), \text{ as } s \to -\infty \end{cases} \tag{2.5}$$

From this and the boundary condition (2.4), we have the asymptotics for T:

$$T(\varepsilon) = \delta\varepsilon^{-1} + T_R \varepsilon^{-1/3} - \delta^{-1} + \mathcal{O}(\varepsilon^{2/3}) \tag{2.6}$$

Boundary condition $u_+(t = T) = \delta$, we derive the asymtotics for T,

$$T = T(\varepsilon) \sim \delta\varepsilon^{-1/6} + t_* - \delta^{-1} = \varepsilon^{-1/3}\Omega_0 + \varepsilon^{-1}\delta - \delta^{-1}.$$

Using the asymptotics for ψ and u_R, we calculate that

$$\varphi(\sigma) = \begin{cases} (-\frac{3}{2}\sigma)^{-1/3}, \text{ as } \sigma \to -\infty \\ e^{-\sigma}, \text{ as } \sigma \to \infty. \end{cases} \tag{2.7}$$

$$u_R(\psi(\sigma)) = \begin{cases} -(-\frac{3}{2}\sigma)^{1/3}, \text{ as } \sigma \to -\infty \\ e^{\sigma}, \text{ as } \sigma \to \infty. \end{cases} \tag{2.8}$$

$$a(\sigma) = \begin{cases} -2 + \mathcal{O}((-\sigma)^{-3/2}), \text{ as } \sigma \to -\infty, \\ 2 + \mathcal{O}(e^{-2\sigma}), \text{ as } \sigma \to \infty. \end{cases} \tag{2.9}$$

3 summary for set up

Equation

$$
\begin{aligned}
&\frac{d}{dt}u(t) = (\mu + u^2 + u^3)(t) \\
&\frac{d}{dt}\mu(t) = \varepsilon
\end{aligned}
\tag{3.1}
$$

with B.C.

$$
\mu(0) = -\delta, \qquad u(T) = \delta. \tag{3.2}
$$

where δ, ε, T are parameters.

3.1 The Riccati solution

This is taken from [Krupa, Szmolyan].

Consider the riccati equation

$$
\frac{d}{dt}u(t) = t + u(t)^2 \tag{3.3}
$$

(3.3) is known to have a unique solution (here we denote by u_R) with the following asymptotics:

$$
u_R(t) = (\Omega_0 - t)^{-1} + \mathcal{O}(|\Omega_0 - t|)
$$

as $t \to \Omega_0^-$ and

$$
u_R(t) = -\sqrt{-t} + \mathcal{O}(|t|^{-1})
$$

as $t \to -\infty$.

Here the constant Ω_0 is the smallest positive zero of a certain combination of Bessel functions of the first kind.

3.2 The t to σ time rescaling

step 1: Define ψ as

$$
\psi = \varepsilon^{1/3}(t - \varepsilon^{-1}\delta)
$$

step 2: Take $M > 0$ large, define σ as

$$
\psi = \psi(\sigma) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, \text{ for } \sigma \leq -M \\ \Omega_0 - e^{-\sigma}, \text{ for } \sigma \geq M, \end{cases}
$$

and smooth interpolation in between so that $\psi(0) = 0$, here Ω_0 is the blow-up time for $u_R(s)$, the unique solution to the ricatti equation that satisfy the asymptotics.

$$
\varepsilon^{-1/3}\varphi\frac{d}{dt} = \frac{d}{d\sigma}
$$

We also define $\varphi(\sigma) := \frac{d}{d\sigma}\psi(\sigma) = e^{-\sigma}$ for $\sigma \geq M$ and is equal to $(-\frac{2}{3}\sigma)^{-1/3}$.

For convenience let the map $t \mapsto \sigma$ be denoted as ρ.

3.3 Region I

In σ variable, we divide the real line into two segments. In different regions we will have different ansatz. Region I is defined by

$$\left\{\sigma : -\frac{2}{3}\delta^{\frac{3}{2}}\varepsilon^{-1} < \sigma < 0\right\}.$$

Which corresponds to the original time t as

$$\left\{t : 0 < t < \varepsilon^{-1}\delta\right\}.$$

3.3.1 Important times

- $t = 0$
- $t = t^*$, the (left) gluing time which corresponds to when $\sigma = -\varepsilon^{-1/4} =: \sigma^*$, this is determined when the corresponding remainder term in the asympototics of u_s and u_ℓ are equal (which happens when

$$|\varepsilon\sigma|^{2/3} = \varepsilon^{1/3}|\sigma|^{-2/3} \implies |\sigma| = \varepsilon^{-1/4},$$

see the asympototic formula of u_s and u_ℓ below)

3.3.2 ansatz in region I

The ansatz in region I takes the form

$$u_I(t) = \chi_s(\rho(t))u_s(t) + \chi_l(\rho(t))u_l(t) + W_s(t) + W_l(t)$$

Where

- $u_s(t)$ denotes the "singular" branch that forms the slow manifold (critical manifold?) of the original system. It is defined via the relation

$$u_s(t) = h(\mu(t))$$

for some smooth function h which solves

$$0 = \mu(t) + h(\mu(t))^2 + h(\mu(t))^3. \tag{3.4}$$

It has the following asymptotics:

$$u_s(t) = -\sqrt{\delta - \varepsilon t} + \mathcal{O}(|\delta - \varepsilon t|). \tag{3.5}$$

The equivalent in σ variable is

$$u_s(\sigma) = -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathcal{O}(|\varepsilon\sigma|^{2/3}) \tag{3.6}$$

- $u_l(t)$ is defined by rescaling u_R and restrict it for $t < \varepsilon^{-1}\delta$. Specifically:

$$u_l(t) = \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta)). \tag{3.7}$$

  It solves the equation

$$\frac{d}{dt} u_l(t) = \mu(t) + u_l^2(t), \tag{3.8}$$

  and has the asymptotics

$$u_\ell = -\sqrt{\delta - \varepsilon t} + \mathcal{O}(\varepsilon(\delta - \varepsilon t)^{-1}),$$

  or in equivalent σ variable

$$u_\ell(\sigma) = -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathcal{O}(\varepsilon^{1/3}|\sigma|^{-2/3}).$$

- The cutoff functions χ_s and χ_l are functions of σ directly, and they satisfy (for $\sigma \leq 0$)

$$\chi_s(\sigma) = \begin{cases} 1, & \sigma \leq \sigma^* - 1 \\ 0, & \sigma \geq \sigma^* + 1. \end{cases} \tag{3.9}$$

  and

$$\chi_l(\sigma) = \begin{cases} 0, & \sigma \leq \sigma^* - 1 \\ 1, & \sigma \geq \sigma^* + 1. \end{cases} \tag{3.10}$$

- Norms

  From notes:

$$W_\ell \approx \varepsilon^{(2-\alpha)/3} \langle \sigma \rangle^{2/3}$$

  and

$$W_s \approx \varepsilon^{1-\alpha/3} \langle \varepsilon\sigma \rangle^{-2/3}$$

  are the weights we proposed.

3.3.3 ditrubution of terms

$$\begin{aligned} W_s' + W_\ell' = &-\chi_s'(u_s - u_\ell) + \chi_s \mu + \chi_s u_s' \\ &+ (W_s + W_\ell + \chi_s u_s + \chi_\ell u_\ell)^2 + \\ &+ (W_s + W_\ell + \chi_s u_s + \chi_\ell u_\ell)^3 \end{aligned}$$

use the fact that

$$u_\ell' = \mu + u_\ell^2$$

and

$$\mu + u_s^2 + u_s^3 = 0,$$

we arrive at

$$\begin{aligned}W_s' + W_\ell' = &-\chi_s'(u_s - u_\ell) - \chi_s\chi_\ell(u_s - u_\ell)^2 + \chi_s u_s' + \\ &+ 2(\chi_s u_s + \chi_\ell u_\ell)(W_s + W_\ell) + \\ &+ (W_s + W_\ell)^2 + \\ &+ (W_s + W_\ell + \chi_s u_s + \chi_\ell u_\ell)^3 - \chi_s u_s^3\end{aligned}$$

Here we propose W_s solves the equation:

$$W_\ell' - 2u_\ell W_\ell = \chi_s'(u_s - u_\ell) - (\chi_s\chi_\ell)(u_s - u_\ell)^2 + W_\ell^2 + \cdots$$

But already W_ℓ^2 term gives some trouble:

Rescale to σ time, the above equation become

$$\frac{d}{d\sigma} W_\ell - a(\sigma) W_\ell = \varepsilon^{-1/3}\varphi W_\ell^2 + \cdots$$

where $a(\sigma) = 2u_R(\sigma)\varphi$, satisfies

$$a(\sigma) = -2 + \mathcal{O}(|\sigma|^{-3/2})$$

as $\sigma \to -\infty$. And $\varphi(\sigma) \to -(-3\sigma/2)^{-1/3}$ as $\sigma \to -\infty$.

So that in our proposed norm

$$\|\varepsilon^{-1/3}\varphi W_\ell^2\| \leq \varepsilon^{-1/3}|\sigma|^{-1/3}\varepsilon^{(2-\alpha)/3}|\sigma|^{2/3} \leq \varepsilon^{(1-\alpha)/3}|\sigma|^{1/3}$$

The problem is the range of σ, here there is no cutoff term that multiplies W_ℓ^2, so $|\sigma| \leq \varepsilon^{-1}$ by the definition of region I at the begining. Which makes the above term to be $\varepsilon^{-\alpha/3}$ large.

For other terms, u_ℓ^3 is fine, and the residual term $(u_s - u_\ell)\chi_s'$, $\chi_s\chi_\ell(u_s - u_\ell)^2$ behave good with the purposed norm, so I think the main issue is from the W_ℓ^2 term.

4 Ansatz without the cutoff functions

We will use ansatz without cut off functions.

- For $t \in (t^*, \varepsilon^{-1}\delta)$ (corresponds to $\sigma^* \leq \sigma \leq 0$), the ansatz takes the form $u = u_\ell + W_\ell$.
- For $t \in (0, t^*)$ (corresponds to $-\frac{2}{3}\delta^{3/2}\varepsilon^{-1} \leq \sigma \leq \sigma^*$), the ansatz takes the form $u = u_s + W_s$.

4.1 Equation of W_ℓ

$$\begin{aligned}W_\ell' - 2u_\ell W_\ell &= W_\ell^2 + (u_\ell + W_\ell)^3 \\ &= (3u_\ell^2)W_\ell + (1 + 3u_\ell)W_\ell^2 + W_\ell^3 + u_\ell^3\end{aligned} \tag{4.1}$$

We want to solve this equation on $t \in (t^*, \varepsilon^{-1}\delta)$.

4.1.1 Linear equation of W_ℓ

Rescale to σ variable:

$$\frac{d}{d\sigma}W_\ell - b(\sigma)W_\ell = \varepsilon^{-1/3}\varphi R_\ell(W_\ell) \tag{4.2}$$

Asymptotics for $b(\sigma)$:

$$b(\sigma) = 2\varepsilon^{-1/3}u_\ell(\psi(\sigma))\varphi(\sigma) = 2u_R(\psi(\sigma))\varphi(\sigma) = -2 + \mathcal{O}(|\sigma|^{-1})$$

as $\sigma \to -\infty$.

Function space:

$$C_{W_\ell} = \left\{u(\sigma) \mid \sup|\varepsilon^{\frac{\alpha-2}{3}}\langle\sigma\rangle^{-\frac{2}{3}}u(\sigma)| < \infty\right\}$$

Variation of constants gives the formula

$$W_\ell(\sigma) = \exp\left(\int_\tau^\sigma b(\rho)d\rho\right)W_\ell(\tau) + \int_\tau^\sigma \exp\left(\int_s^\sigma b(\rho)d\rho\right)\varepsilon^{-\frac{1}{3}}\varphi R_\ell(W_\ell)ds. \tag{4.3}$$

For $\sigma \in (\sigma^*, 0)$, we will estimate the term $\varepsilon^{-1/3}\varphi(\sigma)\left[(3u_\ell^2)W_\ell + (1+3u_\ell)W_\ell^2 + W_\ell^3 + u_\ell^3\right]$ under the integral given by the formula.

Theorem 4.1. $\sup_{\sigma^*\le\sigma\le 0}\varepsilon^{-1/3}\varphi R_\ell(W_\ell(\sigma)) = \mathcal{O}(\varepsilon^?)$.

Proof. We start with the term u_ℓ^3, we wish to show that

$$\varepsilon^{\frac{\alpha-2}{3}}\langle\sigma\rangle^{-\frac{2}{3}}\int_{\sigma^*}^\sigma \exp\left(\int_s^\sigma b(\rho)d\rho\right)\varepsilon^{-\frac{1}{3}}\varphi(s)u_\ell^3(s)ds.$$

is uniformly bounded in ε.

Recall the asymptotics $u_\ell(s) \lesssim (\varepsilon s)^{1/3} + \mathcal{O}(\varepsilon^{1/3}|\sigma|^{-2/3})$, $\varphi(s) \le s^{-1/3}$ and the asymptotics of $b(\sigma)$ above, we find out that it is sufficient to estimate instead

$$\varepsilon^{\frac{\alpha-2}{3}}\langle\sigma\rangle^{-\frac{2}{3}}\int_{\sigma^*}^\sigma \varepsilon^{2/3}e^{-2(\sigma-s)}s^{2/3}ds.$$

This integral is given by the following incomplete gamma functions, and is bounded by

$$\varepsilon^{\frac{\alpha}{3}}\left[1 - e^{-2(\sigma-\sigma^*)}(\sigma^*/\sigma)^{2/3} + \langle\sigma\rangle^{-2/3}e^{-2\sigma}\left(\Gamma(\frac{2}{3}, -2\sigma) - \Gamma(\frac{2}{3}, -2\sigma^*)\right)\right] = \mathcal{O}(\varepsilon^{\alpha/3}).$$

Next, we estimate the term

$$\varepsilon^{-1/3}\varphi(\sigma)W_\ell^3.$$

We need to show that

$$\varepsilon^{\frac{\alpha-2}{3}}\langle\sigma\rangle^{-\frac{2}{3}}\int_{\sigma^*}^\sigma \exp\left(\int_s^\sigma b(\rho)d\rho\right)\varepsilon^{-\frac{1}{3}}\varphi(s)W_\ell^3(s)ds$$

is uniformly bounded in ε given that $W_\ell \in C_{W_\ell}$.

Equivalently, it is sufficient to estimate

$$\varepsilon^{\frac{\alpha-2}{3}}\langle\sigma\rangle^{-\frac{2}{3}}\int_{\sigma^*}^\sigma e^{-2(\sigma-s)}\varepsilon^{-\frac{1}{3}}\varphi(s)[\varepsilon^{\frac{2-\alpha}{3}}\langle s\rangle^{\frac{2}{3}}]^3ds,$$

which turns out (?) to be bounded by

$$\varepsilon^{1-\frac{2\alpha}{3}}|\sigma| \leq \varepsilon^{1-\frac{2\alpha}{3}}|\sigma^*| = \mathcal{O}(\varepsilon^{\frac{9-8\alpha}{12}})$$

Similarly, we have for the quadratic term

$$\varepsilon^{-1/3}\varphi(\sigma)W_\ell^2 = \mathcal{O}(\varepsilon^{\frac{3-4\alpha}{12}}), \quad \varepsilon^{-1/3}\varphi(\sigma)u_\ell W_\ell^2 = \mathcal{O}(\varepsilon^{\frac{2-\alpha}{3}}).$$

And for the linear term

$$\varepsilon^{-1/3}\varphi(\sigma)u_\ell^2 W_\ell = \mathcal{O}(\varepsilon^{\frac{1}{4}}).$$

□

4.2 Equation of W_s

$$W_s' - 2u_s W_s = (3u_s^2)W_s + (1+3u_s)W_s^2 + W_s^3 - u_s' \tag{4.4}$$

We want to solve this equation on $t \in (0, t^*)$.

4.2.1 Linear equation of W_s

$$\frac{d}{d\sigma}W_s - c(\sigma)W_s = \varepsilon^{-1/3}\varphi R_s(W_s) \tag{4.5}$$

Asymptotics for $c(\sigma)$:

$$c(\sigma) = 2\varepsilon^{-\frac{1}{3}}u_s(\psi(\sigma))\varphi(\sigma) = -2 + \mathcal{O}(\varepsilon^{1/3}|\sigma|^{1/3})$$

as $\sigma \to -\infty$.

Function space:

$$C_{W_s} = \left\{ u(\sigma) \mid \sup|\varepsilon^{\frac{\alpha}{3}-1}\langle\varepsilon\sigma\rangle^{\frac{2}{3}}u(\sigma)| < \infty \right\}$$

Variation of constants gives the formula

$$W_s(\sigma) = \exp\left(\int_\tau^\sigma c(\rho)d\rho\right)W_s(\tau) + \int_\tau^\sigma \exp\left(\int_s^\sigma c(\rho)d\rho\right)\varepsilon^{-\frac{1}{3}}\varphi R_s(W_s)ds \tag{4.6}$$