1 Hyperbolic Gluing

Consider the 2-d system

$$\dot{u} = Au + f(u), \qquad u = (u^1(t), u^2(t))^T,$$

Where A is a constant coefficient hyperbolic matrix, with exactly 1 stable/unstable direction. And the dot is d/dt. f denotes higher order terms so $f(0) = 0$ and $Df(0) = 0$.

Fix some $\delta > 0$ small, we have a 1-d (local) stable and unstable manifold u_- and u_+, which can be locally straightened: $u_- : \{u^1 = 0\}$ and $u_+ : \{u^2 = 0\}$

so that $u_\mp \to (0, 0)^T$ as $t \to \pm\infty$ and $u^2_-(0) = \delta, u^1_+(0) = \delta$ (after some shift in time).

Fix $T > 0$ (not necessarily large), we want to solve the boundary value problem $u^1(T) = u^2(-T) = \delta$ by looking for a solution u to the system in the vicinity of the stable/unstable manifold $u_\pm$.

We need the following property: $u_\pm$ satisfies the estimate

$$\|u_-(t)\| \le \delta e^{-\gamma t} \text{ for } t \ge 0$$

and

$$\|u_+(t)\| \le \delta e^{\gamma t} \text{ for } t \le 0$$

for some constants $\gamma > 0$ and.

1.1 The Ansatz

Let $\chi_\pm(t)$ be a smooth partition of unity of the real line $(-\infty, \infty)$ such that

(i) $\chi_- + \chi_+ = 1$;

(ii) $\chi_- = 1$ for $t < -T$, $\chi_- = 0$ for $t > T$;

(iii) $\chi_+ = 0$ for $t < -T$, $\chi_+ = 1$ for $t > T$

Our ansatz would take the form (note the time shift)

$$u(t) = \chi_-(t)u_-(t+T) + \chi_+(t)u_+(t-T) + w_-(t+T) + w_+(t-T)$$

Here the corrector term $w = w_- + w_+$ is split into two parts w_- and w_+, which we consider on the halfline $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}^- = (-\infty, 0)$.

We then introduce exponentially weighted function space on $\mathbb{R}^+$ and $\mathbb{R}^-$. Let us fix an exponential weight $\eta > 0$ whose exact value will be determined later, so that for $t \in \mathbb{R}^+ = (0, \infty)$ we have

$$\|w_-\|_{C^1_\eta} := |e^{\eta t}(w_-(t) + \dot{w}_-(t))|_\infty < \infty$$

and for $t \in \mathbb{R}^- = (-\infty, 0)$ we have

$$\|w_+\|_{C^1_\eta} := |e^{-\eta t}(w_+(t) + \dot{w}_+(t))|_\infty < \infty$$

these $w_\pm$ are unshifted!

We need to determine equations in $w_\pm$ separately! Note the equation is to be solved for $|t| < T$, together with the boundary values $u^1(T) = u^2(-T) = \delta$.

(i) turns out $w_-(2T) + w_+(0) = (0, u^2(T))^T$ and $w_-(0) + w_+(-2T) = (u^1(-T), 0)^T$.

(ii) need $\dot{u} = Au + f(u)$, so we have on the left

$$\dot{u} = (\dot{\chi}_- u_- + \chi_- \dot{u}_-) + (\dot{\chi}_+ u_+ + \chi_+ \dot{u}_+) + \dot{w}_- + \dot{w}_+$$

which must equal to the right

$$Au + f(u) = A(\chi_- u_- + \chi_+ u_+ + w_- + w_+) + f(\chi_- u_- + \chi_+ u_+ + w_- + w_+).$$

Using the fact that $u_\pm$ are solutions to the ODE and $\chi_\pm$ are scalar-valued which can be pulled out in front of A, we simplify:

$$(\dot{w}_- + \dot{w}_+) - A(w_- + w_+) = \dot{\chi}_- u_- + \dot{\chi}_+ u_+ + f(\chi_- u_- + \chi_+ u_+ + w_- + w_+) - \chi_- f(u_-) - \chi_+ f(u_+)$$

We next split the above equation separately in w_- and w_+....

1.2 Spliting the error

Let us first adjust the linear part into

$$(\dot{w}_- + \dot{w}_+) - A(w_- + w_+) - (f'(u_+)w_+ + f'(u_-)w_-).$$

Then we first group the right hand as follows:

$$R := \underbrace{\dot{\chi}_- u_- + \dot{\chi}_+ u_+}_{:=R_0} + f(\chi_- u_- + \chi_+ u_+ + w_- + w_+) - \chi_- f(u_-) - \chi_+ f(u_+) - (f'(u_+)w_+ + f'(u_-)w_-).$$

Next, define the commutator (f' is shorthand for Df, also keep in mind the time shift $u_\pm(t \mp T)$ on u_j.

$$[f, \chi_\pm] = \sum_{j=\pm} \chi_j f(u_j) - f(\sum_{j=\pm} \chi_j u_j); \quad [f', \chi_\pm] = \sum_{j=\pm} \chi_j f'(u_j) - f'(\sum_{j=\pm} \chi_j u_j),$$

We then group $R - R_0$ as follows:

$$R - R_0 = f(\sum_j \chi_j u_j + w_j) - f(\sum_j \chi_j u_j) - \underbrace{[f, \chi_\pm]}_{:=-R_1} - \sum_j f'(u_j) w_j,$$

We next decompose $R - R_0 - R_1$ by first Taylor expand f around $\sum_j \chi_j u_j$

$$R - R_0 - R_1 = f'(\sum_j \chi_j u_j) \sum_j w_j - \sum_j f'(u_j) w_j + R_2$$

Here R_2 would be of the order $O(w^2)$ with $w = w_- + w_+$. We then have $R - R_0 - R_1 - R_2$ being decomposed again:

$$\begin{aligned}
R - \sum_{j=0}^{2} R_j &= f'(\sum_j \chi_j u_j) \sum_j w_j - \sum_j f'(u_j) w_j \\
&= \sum_j \chi_j f'(u_j) \sum_j w_j - \underbrace{[f', \chi_\pm] \sum_j w_j}_{:=-R_3 \sum w_j} - \sum_j f'(u_j) w_j \\
&= R_3 - (\chi_- f'(u_+) w_+ + \chi_+ f'(u_-) w_- + (\chi_- f'(u_-) w_+ + \chi_+ f'(u_+) w_-)
\end{aligned}$$

in the end we group $\chi_- f'(u_+)w_+$ and $\chi_- f'(u_-)w_+$ to be R_4, and the rest $\chi_+ f'(u_-)w_-$ with $\chi_+ f'(u_+)w_-$ to be R_5.

Thus we have split the error R into 6 parts, summarize:

$$\begin{aligned}
R_0 &= \dot{\chi}_- u_- + \dot{\chi}_+ u_+ \\
R_1 &= -[f, \chi_\pm] = [\chi_\pm, f] \\
R_2 &= f(\sum_j \chi_j u_j + w_j) - f(\sum_j \chi_j u_j) - f'(\sum_j \chi_j u_j) \sum w_j \\
R_3 &= -[f', \chi_\pm] = [\chi_\pm, f'] \\
R_4 &= -\chi_- f'(u_+)w_+ + \chi_- f'(u_-)w_+ \\
R_5 &= -\chi_+ f'(u_-)w_- + \chi_+ f'(u_+)w_-
\end{aligned}$$

1.3 Equation of the corrector and estimates

We first set up the equation for w_- and w_+, note these $w_\pm$ are shifted, we define $w_-T(\cdot) := w_-(\cdot + T)$ and $w_+T(\cdot) := w_+(\cdot - T)$, the equation we have will be equation for w_-^T and w_+^T, respectively, and the domain for both $w_\pm^T$ is $(-T, T)$.

equation for w_-^T:

$$\mathcal{L}_- w_-^T := \dot{w}_-^T - (A + f'(u_-^T) + R_3)w_-^T = \dot{\chi}_- u_-^T + \chi_-(R_1 + R_2) + R_4 := R_-(w_-^T; w_+^T)$$

and equation for w_+^T:

$$\mathcal{L}_+ w_+^T := \dot{w}_+^T - (A + f'(u_+^T) + R_3)w_+^T = \dot{\chi}_+ u_+^T + \chi_+(R_1 + R_2) + R_5 := R_+(w_+^T; w_-^T)$$

Want to solve w_-^T and w_+^T through a fixed point argument. Use the space $C^1_\eta(R_-)$ for w_+ and $C^1_\eta(R_+)$, consider $\mathcal{L}_\pm$ as an operator from C^1_η to C^0_η. The control for linear parts must be done using exponential dichotomy inherited from the hyperbolicity of the matrix A and the smallness of $u_\pm^T + R_3$ in $(-T, T)$.

(i) **Estimates for** R_0, R_1, R_3

These terms do not involve w, we shall show they are small in the η-weighted norm. The linear part will be controlled by using exponential dichtomy from the hyperbolicity of A and the fact that $f'(u_\pm^T) + R_3$ are uniformly small.

Let us focus on the equation for w_- first,

- note I have distributed R_0 into a χ_--part and a χ_+-part, for the equation for w_-^T, what needs to be estimated is just $\dot{\chi}_- u_-^T$.

  Since χ_- is constant outside of $|t| > T$, we need only consider $|t| < T$, but then $u_-^T(t) = u_-(t+T)$ will satisfy $\|u_-^T(t)\| \le \delta$ (sup norm). Hence if δ is sufficiently small, then in the weighted norm $\dot{\chi}_- u_-^T$ will be as small as needed.
- for the commutator term R_1, because of the $\chi_\pm$, the time interval that are relevant is $(-T, T)$ (outside of which $R_1 = 0$) But on these intervals again using $\|u_-^T(t)\| < \delta$. and $f(0) = 0$ to get R_1 and R_3 are as small as needed.

- similarly for R_3, using $f'(0) = 0$.

  Here I am not using any information about T being large, but just the smallness of $u^T_\pm$ on the time interval $(-T, T)$.

(ii) **Estimates for R_2**

Recall that for $|t| < T$, we have

$$R_2(t) = f(\sum_j \chi_j u_j + w_-(t-T) + w_+(t-T)) - f(\sum_j \chi_j u_j) - f'\left(\sum_j \chi_j u_j\right)\{w_-(t-T) + w_+(t-T)\}.$$

This is the remainder term which is of higher order in $w(t) = w_-(t+T) + w_+(t-T)$, since for $|t| < T$, we have $|u_\pm(t \mp T)| \leq \delta$ by set up, using Taylor's theorem, we have $|R_2(t)| = \mathcal{O}(|w|^2)$, which will be small if we are working in some small ball in the function space C^1_η for w.

(iii) **Estimates for R_4 and R_5**

We have set

$$R_4 = -\chi_- f'(u^T_+)w^T_+ + \chi_- f'(u^T_-)w^T_+$$

Again, due to the cut off χ_-, for $t < -T$, the decay at infinity of w_+ ensures $e^{\eta(t+T)}w_+(t-T)$ is exponentially small. And the focus is on $|t| < T$.

For these t in these range, we need to estimate sup norm of R_4 under the weight $e^{\eta(t+T)}$. If I just use $\|u_\pm\| \leq \delta$, I will end up with $\|e^{t+T}R_4(t)\| \leq g(\delta)e^{\eta(t+T)}e^{\eta(t-T)} \leq e^{2\eta t}g(\delta)$ for some function $g(\delta) = O(\delta^2)$. Note $t \in [-T, T]$ could make $e^{2\eta t}$ big. But if δ is sufficiently small I think this can be taken care of.

A similar argument applies to R_5.

from here we need to show the equation $\mathcal{L}_\pm w_\pm = R_\pm$ can be solved using an iteration argument, which amounts to show $\|R_\pm\|$ is small given $w_\pm$ small, and $\mathcal{L}^{-1}_\pm R_\pm$ is a contraction, say when we working on some balls in the function space.

conclusion for the "flying time" if T is given, then the size of the boundary condition will depend on T. (if T is large, then δ need to be sufficiently small), which is quite natural since the flying time goes to infinity as the trajectories get close to the invariant manifolds.

2 Non-hyperbolic Gluing

2.1 A decoupled system

Start with the following very simple 2-d system:

$$\begin{aligned}\dot{u} &= -u^2 \\ \dot{v} &= v\end{aligned} \tag{2.1}$$

This system decouples, of course. But we wish to demonstrate the method from a simple example.

Clearly, the u-axis $\{v = 0\}$ is invariant and the solution is explicitly parameterized by

$$U_-(t) = (\frac{1}{t + C_-}, 0)^T := (u_*(t), 0)^T$$

for some constant C_-. Likewise, the invariant v-axis $\{u = 0\}$ is parametrized by

$$U_+(t) = (0, C_+e^t)^T := (0, v_*)^T$$

for some constant C_+.

We want to solve a boundary value problem $U(t) = (u(t), v(t))^T$ such that (u, v) satisfies the ODE, with the boundary condition $u(1) = \delta$, $v(1) = \delta$. We follow the hyperbolic case, take an ansatz of the form

$$U(t) = \chi_-(t)U_-(t + T + 1) + \chi_+(t)U_+(t - T + 1) + w_-(t + T + 1) + w_+(t - T + 1).$$

Now, use the fact that U satisfy the system $\dot{U} = AU + F(U)$, where A is the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and the nonlinear term $F(U) = F(u, v) = (-u^2, 0)^T$, after some calculation we arrived at the following equations for $w_\pm$.

$$\dot{w}_- - Aw_- = -\dot{\chi}_-U_- + F(\chi_\pm U_\pm + w_\pm) - \chi_\pm F(U_\pm) \tag{2.2}$$

$$\dot{w}_+ - Aw_+ = -\dot{\chi}_+U_+ \tag{2.3}$$

Notice here $w_\pm$ are vector-valued, $w_\pm = (w_\pm^1, w_\pm^2)^T$. Let us focus on the equation for w_+ first, by the structure of A and U_+, the equation for the first component of w_+ is actually just

$$\dot{w}_+^1 = 0,$$

and the equation for the second component is

$$\dot{w}_+^2 = w_+^2 - \dot{\chi}_+v_*$$

Following the moral of the hyperbolic gluing, $w_+(\cdot)$ should be decaying at $-\infty$, hence we must have $w_+^1 \equiv 0$. And we may explicitly solve w_+^2, which is given by

$$w_+^2(t) = w_+^2(0)e^t + C_+e^t\left(\chi_+(T - 1) - \chi_+(t + T - 1)\right)$$

Of course in the general case we would not get such an explicit formula, but we can get the estimate that show w_+ lies in some exponentially weighted space on the interval $(-\infty, 0)$.

Next we focus on the equation for w_-, first we subtract both sides of the equation by the term $f'(U_-)w_-$, which equals $\begin{pmatrix} -2u_* & 0 \\ 0 & 0 \end{pmatrix} w_-$ to adjust the linear term.

The equation for w_- now reads

$$\dot{w}_- - \begin{pmatrix} -2u_* & 0 \\ 0 & 1 \end{pmatrix} w_- = -\dot{\chi}_- U_- + \begin{pmatrix} \chi_- u_*^2 - (\chi_- u_* + w_+^1 + w_-^1)^2 + 2u_* w_-^1 \\ 0 \end{pmatrix}$$

Again, the equation for the second component of w_- is just

$$\dot{w}_-^2 - w_-^2 = 0$$

we got $w_-(t) = Ae^t$ for some constant A, however, in order for $w_-(\cdot)$ to decay at $+\infty$, we must choose $A = 0$, thus we have $w_-^2 = 0$.

Therefore, we end up with the equation for the first component, which is

$$\dot{w}_-^1 - (-2u_*)w_-^1 = -\dot{\chi}_- u_* + (\chi_- u_*^2 - (\chi_- u_* + w_-^1)^2 + 2u_* w_-^1)$$

To solve it, we need to rescale in time.

Define the new time variable τ such that $dt/d\tau = (-2u_*(t + T + 1))^{-1}$. Put $\tilde{w}(\tau) = w_-^1(t(\tau))$, after multiplying the equation by $(-2u_*)^{-1}$. We have

$$\frac{d}{d\tau}\tilde{w} - \tilde{w} = (-2u_*)^{-1}\left(-\dot{\chi}_- u_* + (\chi_- - \chi_-^2)u_*^2 + 2(1-\chi_-)u_*\tilde{w} - (\tilde{w})^2\right) := (-2u_*)^{-1}R$$

We can now work with exponentially weighted space (in the variable τ, let us solve the above equation for $\tau \in [0, \infty)$ (corresponding to $t \in [1, \infty)$, since explicitly $t = \exp(\tau)$.)

Now if we assume $\tilde{w} \in C^1_\nu$ for some weight ν, due to the multiplication of the right hand side by $(-2u_*)^{-1} \sim t \sim \exp(\tau)$, we lose the localization from ν to $\nu - 1$. Which means we need to estimate the remainder R in the $C^1_{\nu-1}$ norm.

- Estimates for R

2.2 Equation with more general nonlinear terms

$$\begin{aligned} \dot{u} &= -u^2 + \mathcal{O}(uv, v^2, u^3) \\ \dot{v} &= v + \mathcal{O}(uv, u^2, v^2) \end{aligned} \tag{2.4}$$

The u^2 term might be a bit troublesome, since v is going to be exponentially localized, while u^2 decay at most algebraically

we want to do the same thing as in the hyperbolic case, now the problem is that due to non-hyperbolicity, we need to work in an appropritely re-scaled time (and accordingly choose the correct function space) to recover the Fredholm properties.

Now, by standard theory, equation (2.4) has a solution which asymptotically decays algebraically: $u_*(t) = \mathcal{O}(t^{-1})$ as $t \to \infty$.

The v equation determines uniquely the unstable manifold v_*, which decays exponentially in backward time $v_*(t) = \mathcal{O}(e^t)$ as $t \to -\infty$.

Our goal is to 2find an orbit near the origin by solve the following boundary value problem: $(u, v)(t)$ solves (2.4) and $u(-T) = \delta, v(T) = \delta$ for flying time T and small $\delta > 0$ given.

Again, u_* and v_* are shifted in time so that it satisfies $u_*(1) = \delta$ and $v_*(1) = \delta$, with the same asymptotics. Let us denote the curve $U_-(t) = (u_*(t), 0)$ and $U_+(t) = (0, v_*(t))$

Following the hyperbolic case, we take the ansatz to be of the form

$$U(t) = \chi_-(t)U_-(t + T + 1) + \chi_+(t)U_+(t - T + 1) + w_-(t + T + 1) + w_+(t - T + 1)$$

where $\chi_\pm$ is the same partition of unity.

The calculation to determine the equation for w_- and w_+ follows pretty much the same procedure, note the matrix A here is of the form $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, also note the linearization $Df(U_-)$ now will decay algebraically due to the non-hyperbolicity $\|Df(U_-(t))\| = \mathcal{O}(t^{-1}), t \to \infty$.

Let us again focus on the equation for w_-:

$$\mathcal{L}_- w_-^T := \dot{w}_-^T - (A + Df(U_-^T) + R_3)w_-^T = \dot{\chi}_- U_-^T + \chi_-(R_1 + R_2) + R_4 := R_-(w_-^T; w_+^T),$$

where the T super script means the time shift: $U_-^T(t) = U_-(t + T + 1)$.

let us abbreviate w_-^T by w, so we can use the notation $w(t) = (w^1(t), w^2(t)) \in \mathbb{R}^2$.

Recall that $R_1 = [\chi_\pm, f]$ and $R_3 = [\chi_\pm, f']$ are the commutator terms, and $R_2 = f(\sum_j \chi_j u_j + w_j) - f(\sum_j \chi_j u_j) - f'(\sum_j \chi_j u_j) \sum w_j$ is the higher order remainder, with $R_4 = (f'(u_-) - f'(u_+))\chi_- w_+$ is w_+ considered as an perturbation of w_-.

To solve the above equation, we need to choose the appropriate function spaces. Due to the form of the matrix A, the second component $w^2(t)$ should still be the usual exponentially weighted space.

To see what space should we choose for w^1, note the equation for w^1 takes the form

$$\dot{w}^1(t) - (D_u f^1(U_-^T) + R_3^1)w^1(t) + D_v f^2(U_-^T)w^2(t) = R_-^1(w_-, w_+)(t) \tag{2.5}$$

where $f(u, v) = (f^1(u, v), f^2(u, v))^T$ is the nonlinear term, with $f^1(u, v) = -u^2 + \mathcal{O}(uv, v^2, u^3)$ and $f^2(u, v) = \mathcal{O}(uv, u^2, v^2)$. The linearization is evaluated at $U_-^T(t) = (u_*(t), 0)$

We will do a rescale of time, since $u_*(t) \neq 0,\ D_u f^1(U_-^T) \neq 0$ for $t \in [-T, T]$, introduce a new variable τ such that

$$\frac{dt}{d\tau} = \frac{1}{D_u f^1(U_-(t))} = \frac{1}{D_u f^1(u_*(t), 0)}$$

This implicitly defines t as a function of τ, we write $t = t(\tau)$.

In the case $f^1(u, v) = -u^2$, we have $u_*(t) = 1/t$ and hence $t = \exp(\tau)$. Define $\tilde{w}^1(\tau) = w^1(t(\tau))$, we have $\frac{d}{d\tau}\tilde{w}^1 = \frac{dw^1}{dt}\frac{dt}{d\tau}$, multiply (2.5) by $(D_u f^1(u_*(t)))^{-1}$. We get the equation in the rescale time τ (we choose to suppress the superscript again):

$$\frac{d}{d\tau}\tilde{w}(\tau) - \left(1 + (D_u f^1)^{-1} R_3^1\right)\tilde{w}(\tau) + (D_u f^1)^{-1} D_v f^2 w^2(t(\tau)) = (D_u f^1)^{-1} R_-^1(t(\tau))$$

This suggest that we solve $\tilde{w}(\tau)$ in an exponentially weighted space, in terms of w, it will be an algebraically weighted space in the time variable t. Also, notice that the term $(D_u f^1)^{-1} R_-^1$ suggests that we loses 1 algebraic localization.

Let us choose an weight ν whose exact value will be specified later, and work with the exponentially localized space C_ν^1, with norm

$$\|\tilde{w}\|_{C_\nu^1} := |e^{\nu t}(\tilde{w}(t) + \dot{\tilde{w}}(t))|_\infty < \infty$$

Also note: there is a term involve the second component w^2, however, it lies in the exponentially weighted space, hence will not cause any problem when considered as a perturbation of the first component $\tilde{w}$.

2.3 Estimate the remainder

We solve the problem

$$\frac{d}{d\tau}\tilde{w} - \tilde{w} = (D_u f^1)^{-1}\left(\tilde{w} + D_v f^2 w^2 + R_-^1\right)$$

As an fixed point problem, from the space $X \subset C_{\nu+1}^1$ to the space $Y \subset C_\nu^1$, where X and Y are small-ε balls in the respect spaces.

(i) **Estimates for** $(D_u f^1)^{-1} D_v f^2 w^2(t(\tau))$

The second component $w^2(t)$ already lies in some exponentially weighted space (in the variable t!)

(ii) **Estimates for** $(D_u f^1)^{-1}\dot{\chi}_- U_-^T$

For this one I shall use the explicit construction of χ

(iii) **Estimates for** $(D_u f^1)^{-1} R_1$ **and** $(D_u f^1)^{-1} R_3$

This part is somewhat confusing.

(iv) **Estimates for** $(D_u f^1)^{-1} R_2$

This part we may determine the range of ν.