1 Reference

Equation

$$
\begin{aligned}
\frac{d}{dt}u(t) &= (\mu + u^2 + u^3)(t) \\
\frac{d}{dt}\mu(t) &= \varepsilon
\end{aligned}
\tag{1.1}
$$

with B.C.

$$
\mu(0) = -\delta, \quad u(T) = \delta. \tag{1.2}
$$

where δ, ε, T are parameters.

(i) **Region 1**

- Ansatz and rescale in time

$$
u_-(t) = \varepsilon^{1/3} u_R(\tau(t) - \tau_0), \quad \mu(t) = \varepsilon t - \delta = \varepsilon^{2/3}(\tau - \tau_0).
$$

With

$$
\tau(t) = \varepsilon^{1/3} t, \quad \tau_0 = \varepsilon^{-2/3}\delta.
$$

After define $s := \tau - \tau_0$, $(u_R, s)^T$ solves

$$
\frac{d}{ds}u_R(s) = s + u_R(s)^2, \quad \frac{d}{ds}s = 1
$$

Which implies

$$
\frac{d}{dt}u_-(t) = \mu(t) + u_-(t)^2
$$

Yet another rescaling in time σ, defined via

$$
s = \psi(\sigma) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, \text{ for } \sigma \le -M \\ \Omega_0 - e^{-\sigma}, \text{ for } \sigma \ge M, \end{cases}
$$

and smooth interpolation in between, here Ω_0 is the blow-up time for $u_R(s)$. Note if $\varphi := \frac{d}{d\sigma}\psi(\sigma)$, then

$$
\varphi\frac{d}{ds} = \frac{d}{d\sigma}, \text{ and } \varepsilon^{-1/3}\varphi\frac{d}{dt} = \frac{d}{d\sigma}
$$

- Asymptotics for u_R and φ.

$$
\varphi(\sigma) = \begin{cases} (-\frac{3}{2}\sigma)^{-1/3}, \text{ as } \sigma \to -\infty \\ e^{-\sigma}, \text{ as } \sigma \to \infty. \end{cases}
$$

$$
u_R(\psi(\sigma)) \to \begin{cases} -(-\frac{3}{2}\sigma)^{1/3}, \text{ as } \sigma \to -\infty \\ e^{\sigma}, \text{ as } \sigma \to \infty. \end{cases}
$$

$$
2u_R\varphi(\sigma) to \begin{cases} -2 + \mathcal{O}((-\sigma)^{-3/2}), \text{ as } \sigma \to -\infty \\ 2 + \mathcal{O}(e^{-2\sigma}), \text{ as } \sigma \to \infty. \end{cases}
$$

- FP argument petrubation

$$u(t) = \varepsilon^{1/3}(u_R + v)(\sigma), \qquad \mu(t) = \varepsilon^{2/3}(s + \rho)(\sigma).$$

Equation for (v, ρ)

$$\frac{d}{d\sigma} v = 2(u_R \varphi) v + \varphi v^2 + \varphi \rho + \varepsilon^{1/3} \varphi (u_R + v)^3, \qquad \rho = 0.$$

- Gluing time
the gluing time σ_* is set to equal to $\log(\varepsilon^{-1/6}\delta)$, notice in terms of the original time t, this is at

$$s(\sigma_*) = \Omega_0 - \delta^{-1}\varepsilon^{1/6} = \tau - \tau_0 = \varepsilon^{1/3} t - \varepsilon^{-2/3}\delta \implies t = t_* := \varepsilon^{-1/3}[\Omega_0 + \varepsilon^{-2/3}\delta - \delta^{-1}\varepsilon^{1/6}]$$

We note then

$$u_-(t_*) = \varepsilon^{1/3}[(\Omega_0 - (\Omega_0 - \delta\varepsilon^{1/6}))^{-1} + \mathcal{O}(\varepsilon^{1/6})] = \varepsilon^{1/6}\delta^{-1} + \mathcal{O}(\varepsilon^{1/2})$$

- norms
We will stop at $\sigma = \sigma_*$, decide norm from the nonhomogeneous term $\varepsilon^{1/3}\varphi u_R^3$. We have for $0 \le \sigma \le \sigma_*$, that

$$\sup_{\sigma \le \sigma_*} \varepsilon^{1/3} \varphi u_R^3 \le \varepsilon^{1/3} e^{2\sigma_*} = \delta = \mathcal{O}_\varepsilon(1)$$

This is the nonhomogeneous term, so we just need to use the usual sup norm.

(ii) **Region 2**

- Ansatz and rescale

$$u_+(t) = (\delta\varepsilon^{-1/6} + t_* - t)^{-1}$$

This is designed so that $u_+(t_*) = \varepsilon^{1/6}\delta^{-1}$ and

$$|u_-(t) - u_+(t)| \simeq \mathcal{O}(\varepsilon^{2/3}|t - t_*| + \delta\varepsilon^{1/2}) = \mathcal{O}(\varepsilon^{1/2})$$

for t close enough to t_*.
Recall that we stop when $u(t = T) = \delta$, we need to consider the interval $[t_*, T]$. Note the B.C. gives

$$T \sim \delta\varepsilon^{-1/6} + t_* - \delta^{-1} = \varepsilon^{-1/3}\Omega_0 + \varepsilon^{-1}\delta - \delta^{-1}$$

We introduce the time ξ with the scaling

$$e^{-\xi} = u_+(t)^{-1}$$

- Asymptotics

$$u_+(t) = e^{\xi}$$

As for $\mu(t) = \varepsilon t - \delta$, we have

$$\begin{aligned}\mu &= \varepsilon(t_* + \delta\varepsilon^{-1/6} - u_+^{-1}) - \delta = \varepsilon^{2/3}\Omega_0 + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon u_+^{-1} \\ &= \Omega_0\varepsilon^{2/3} + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon e^{-\xi}\end{aligned}$$

- Gluing time

  We will glue at $\xi = \xi_*$, defined via

$$e^{\xi_*} = u_+(t_*) = \varepsilon^{1/6}\delta^{-1} \implies \xi_* = \log(\varepsilon^{1/6}\delta^{-1})$$

- FP argument with ansatz

$$u(t) = u_+(t) + w(t)$$

  By definition, $u_+(t)$ solves

$$\frac{d}{dt}u_+(t) = u_+(t)^2,$$

  Convert the equation in ξ time via

$$e^{-\xi}\frac{d}{dt} = \frac{d}{d\xi},$$

  so we get the equation for w in ξ variable

$$\frac{d}{d\xi}w = e^{-\xi}\mu + 2(e^{-\xi}u_+)w + e^{-\xi}w^2 + e^{-\xi}(u_+ + w)^3, \qquad \frac{d}{d\xi}\mu = \varepsilon e^{-\xi}$$

- norms

  We have

$$e^{-\xi}u_+^3 \sim e^{2\xi}$$

  also from definition of $u_+(t)$

$$\mu(t) = \Omega_0\varepsilon^{2/3} + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon e^{-\xi}$$

  Note $|e^{-\xi}| \leq e^{-\xi_*} \leq \delta\varepsilon^{-1/6}$, so that

$$\varepsilon e^{-2\xi} \leq \varepsilon\delta^2\varepsilon^{-1/3} \leq \delta^2\varepsilon^{2/3} \implies |\varepsilon e^{-\xi}\mu| = \mathcal{O}(\varepsilon^{2/3})$$

  which suggests the nonhomogeneous term is dominated by $e^{-\xi}u_+^3$ and hence a $e^{-2\xi}$ weight in the norm.

  In fact, due to the resonance of $e^{2\xi}$ with the linear part, we need to choose a slightly weaker norm, let $\eta \in]0,1[$, and our weight will be $e^{-(2-\eta)\xi}$. We check the nonhomogeneous term

$$\begin{aligned}
e^{-(2-\eta)\xi}e^{-\xi}\mu &= e^{-(3-\eta)\xi}(\Omega_0\varepsilon^{2/3} + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon e^{-\xi}) \\
&\leq e^{-(3-\eta)\xi}(\varepsilon^{2/3} + \varepsilon^{5/6} + \varepsilon e^{-\xi}) \leq \\
&\sim \delta^{3-\eta}\varepsilon^{\frac{\eta+1}{6}}
\end{aligned}$$

  We also check breifly the norm should work with the nonlinearity

  quadratic

$$\sup_{\xi\geq\xi_*} e^{-(2-\eta)\xi}|e^{-\xi}w^2| \leq \|w\| \sup |e^{-\xi}w| \leq \|w\| e^{-\xi}e^{(2-\eta)\xi} = \|w\| e^{(1-\eta)\xi}$$

  quadratic again

$$\sup_{\xi\geq\xi_*} e^{-(2-\eta)\xi}|e^{-\xi}u_+w^2| \leq \|w\| \sup |e^{-\xi}wu_+| \leq \|w\| e^{(2-\eta)\xi}$$

cubic

$$\sup_{\xi \geq \xi_*} |e^{-(2-\eta)\xi} e^{-\xi} w^3| \leq \|w\| \sup_{\xi \geq \xi_*} |e^{-\xi} w^2| \leq \|w\| e^{-\xi} e^{(4-2\eta)\xi}$$

Linear

$$\sup_{\xi \geq \xi_*} e^{-(2-\eta)\xi} |e^{-\xi} u_+^2 w| \leq \|w\| \sup |e^{\xi} u_+^2| \leq \|w\| e^{(1-\eta)\xi}$$

The Lipschitz constant will be of order $e^{-\xi} w \sim e^{(1-\eta)\xi}$, which is small on the relevant interval $\xi_* \leq \xi \leq 0$.