1 Introduction

Notation: sometimes we use $A \lesssim B$ to indicate that there is a constant C such that $A \le C \cdot B$, if this constant C does not depend on certain parameters that A or B might depends on, we will be specific and write directly $A \le C \cdot B$.

2 Set up

Equation

$$\begin{aligned} \frac{d}{d\tau}u &= \mu + u^2 + f(u,\mu,\varepsilon), \\ \frac{d}{d\tau}\mu &= \varepsilon g(u,\mu,\varepsilon), \end{aligned} \tag{2.1}$$

where $f(u,\mu,\varepsilon) = \mathcal{O}(\varepsilon, u\mu, \mu^2, u^3)$ and $g(u,\mu,\varepsilon) = 1 + \mathcal{O}(u,\mu,\varepsilon)$.

Boundary condition

$$\mu(0) = -\delta, \quad u(T) = \delta, \tag{2.2}$$

where δ, ε are parameters, and T is dependent on ε and δ which will be solved as part of the equation.

2.1 Euler multiplier

Since for u, μ, ε small, $g(u,\mu,\varepsilon) = 1 + \mathcal{O}(u,\mu,\varepsilon)$, we can define a new time $t = t(\tau)$ by $\frac{dt}{d\tau} = g$, and rewrite (2.1) as

$$\begin{aligned} \frac{d}{dt}u &= \mu + u^2 + \tilde{f}(u,\mu,\varepsilon), \\ \frac{d}{dt}\mu &= \varepsilon, \end{aligned} \tag{2.3}$$

where now $\tilde{f}(u,\mu,\varepsilon) = \mathcal{O}(\varepsilon, u\mu, \mu^2, \varepsilon u, \varepsilon\mu, \varepsilon u^2, u^3)$. Moreover, by shifting τ apporpirtely, we retain the boundary condition

$$\mu(t=0) = -\delta, \quad u(\tilde{T} = t(T)) = \delta.$$

Therefore, we will use (2.3) for analysis in the rest of the paper.

2.2 The Riccati equation

Consider the Riccati equation

$$\frac{d}{ds}u(s) = s + u(s)^2 \tag{2.4}$$

We denote any solution to (2.4) as u_R, it is known to have a unique solution (denoted by $\bar{u}_R$) with the following asymptotics:

$$\bar{u}_R(s) = \begin{cases} (\Omega_0 - s)^{-1} + \mathcal{O}(\Omega_0 - s), \text{ as } s \to \Omega_0, \\ -(-s)^{1/2} - \frac{1}{4}(-s)^{-1} + \mathcal{O}(|s|^{-3/2}), \text{ as } s \to -\infty. \end{cases} \tag{2.5}$$

Here the constant $\Omega_0 \approx 2.3381$ is the smallest positive zero of

$$J_{-1/3}(2z^{3/2}/3) + J_{1/3}(2z^{3/2}/3),$$

where $J_{\pm 1/3}$ are Bessel functions of the first kind. (See [Krupa, Szmolyan])

More generally, we consider family of solutions $u_R(s; u_0)$ of the Riccati equation (2.4) such that $u_R(0; u_0) = u_0$. That is, we take the initial condition u_0 as a parameter to the Riccati equation. For the special Riccati solution $\bar{u}_R$, we denote $\bar{u}_R(0)$ as $\bar{u}_0$. We will use a family of such solutions to build our ansatz, whose properties are summerized in the following proposition:

Proposition 2.1. *There exist blow up time $\Omega_\infty = \Omega_\infty(u_0)$ that depends smoothly on u_0 for $|u_0 - \bar{u}_0| < \eta$, η small, with $\Omega_\infty(\bar{u}_0) = \Omega_0$, and the corresponding solution $u_R(s; u_0)$ is of the form*

$$u_R(s; u_0) = \frac{1}{\Omega_\infty - s} + (\Omega_\infty - s) r(\Omega_\infty - s; u_0), \tag{2.6}$$

where the function r is smooth in both variables and satisfies

$$r(\Omega_\infty - s; u_0) = -\frac{\Omega_\infty}{3} + \mathcal{O}(\Omega_\infty - s), \tag{2.7}$$

as $s \to \Omega_\infty$.

Proof. To get the dependence from u_0 to Ω_∞, we first add the equation $\frac{d}{ds}s = 1$ to equation (2.4) to get a autonomous 2−dimensional system in the (s, u) plane. Consider a small neighbourhood I containing $\bar{u}_0$ on the vertical u-axis, then $u_R(s; u_0)$ is the trajectory that starts at $u_0 \in I$. The map $P_1 : I \to \mathbb{R}$ defined by $P_1(p) = u(2; p)$ is smooth in p, as the blow up time for $\bar{u}_R(s; \bar{u}_0)$ is $\Omega_0 > 2$. Moreover, the image $P_1(I)$ is a finite interval on the vertical line $s = 2$ containing $\bar{u}_R(2; u_0)$ bounded away from 0, since the trajectory $u_R(s; \bar{u}_0)$ crosses the horizontal axis around $s = 1$ and the vector field goes upwards in the first quardrant of the (s, u)-plane.

Denote $\tilde{u}_0 := P_1(u_0)$ for brevity (technically, the interval $P_1(I)$ is a small section of the line $s = 2$, with a little abuse of notation, we identify $\tilde{u}_0$ with the second coordinate of the point $P_1(u_0)$). Again in the Riccati equation (2.4), we make a change of variable by setting $z(s) = 1/u(s)$, the equation z satisfies is:

$$\frac{d}{ds} z(s) = -z^2 s - 1.$$

Let $J = \{1/\tilde{u}_0 \mid \tilde{u}_0 \in P_1(I)\}$ and $z(s; 1/\tilde{u}_0)$ is the trajectory which starts at $1/\tilde{u}_0$. We claim that $z(s; 1/\bar{u}_0)$ reaches 0 at a finite time $\Omega_\infty = \Omega_\infty(1/\bar{u}_0)$. To see this, first notice there is no equilibrium for the two dimensional system $\frac{d}{ds}s = 1, \frac{d}{ds}z = -z^2 s - 1$. Then, on the boundary $s = 2$, the vector field takes the form $(1, -2z^2 - 1)$, which makes any trajectory starting at a point on J moving down towards the right. Moreover, the vector field $(1, -sz^2 - 1)$ always pointing down in the first qurdrant of the (s, z) plane, so trajectories cannot go upwards. Lastly, the vector field crosses the horizontal axis non-tangentially, it identically equals $(1, -1)$ throughout the line $z = 0$, hence, any trajectory which starts at a point on J will cross $z = 0$ in finite time at a unique point $\Omega_\infty = \Omega_\infty(1/\tilde{u}_0)$. The dependence of Ω_∞ on $1/\tilde{u}_0$ is smooth by the smooth dependence on initial conditions.

We now define another map $P_2 : J \to \mathbb{R}$ by $P_2(1/\tilde{u}_0) = \Omega_\infty(1/\tilde{u}_0)$, we get a smooth map $P : I \to \mathbb{R}$ by the composition

$$P = P_2 \circ f \circ P_1,$$

where $f(z) = 1/z$ is the inversion map. Since each of the map in the composition is smooth, $P : u_0 \mapsto \Omega_\infty = \Omega_\infty(u_0)$ is smooth as well.

To get the asymptotic expansion, we set $\xi = \Omega_\infty - s$, then $\tilde{z}(\xi) = z(\Omega_\infty - \xi)$ solves

$$\frac{d}{d\xi}\tilde{z} = \tilde{z}^2(\Omega_\infty - \xi) + 1,$$

and $\tilde{z}(0) = 0$.

Hence we can assume the expansion for $\tilde{z}$ near $\xi = 0$ is of the form

$$\tilde{z} = \xi + z_2\xi^2 + z_3\xi^3 + \mathcal{O}(\xi^4),$$

for some constant z_2, z_3. Differentiating this expansion, use the equation $\tilde{z}$ solves and comparing coefficients, we find that $z_2 = 0, z_3 = \Omega_\infty/3$. Changing back from $\tilde{z}(\xi)$ to $z = z(s)$ with $s = \Omega_\infty - \xi$ and recall $z(s) = 1/u(s)$, we find that $u_R(s; u_0)$ has expansion (2.6) with remainder r satisfies (2.7).

■

2.3 The t to σ time rescaling

The solution to (2.1) will be based on the solution of the Riccati equation (2.4). Due to the parameter ε, we need to rescale time t to obtain appropriate equations as $\varepsilon \to 0$.

We will rescale time t to time σ using the following steps.

- Step 1: Define ψ as
$$\psi = \varepsilon^{1/3}(t - \varepsilon^{-1}\delta)$$
- Step 2: Fix $M > 0$ large, define σ as
$$\psi = \psi(\sigma; u_0) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, \text{ for } \sigma \le -M \\ \Omega_\infty(u_0) - e^{-\sigma}, \text{ for } \sigma \ge M, \end{cases}$$
here Ω_∞ is the blow-up time for u_R found in proposition 2.1.
- Step 3: For $\sigma \in (-M, M)$, we define $\psi(\sigma)$ as the straight line connecting the two points $(M, \Omega_\infty - e^{-M})$ and $(-M, -(\frac{3}{2}M)^{2/3})$. As a result, if we define $\sigma_m = \sigma_m(u_0)$ as the value of σ such that $\psi(\sigma_m; u_0) = 0$, then we have
$$\frac{|\sigma_m - M|}{M} = \left| \frac{(\frac{3M}{2})^{2/3} - (\Omega_\infty - e^{-M})}{(\frac{3M}{2})^{2/3} - (\Omega_\infty + e^{-M})} - 1 \right| \le CM^{-2/3},$$
for some constant C independent of u_0.

  Therefore we can write
$$\sigma_m = M + M_r, \qquad |M_r| \le CM^{1/3} \tag{2.8}$$

We also denote $\varphi(\sigma) := \frac{d}{d\sigma}\psi(\sigma)$.

2.4 Region A

Region I corresponds to the t-time interval $\{t : t > \varepsilon^{-1}\delta\}$.

2.4.1 Ansatz in region A

The ansatz in region A takes the form

$$u_A(t) = u_* + w_r.$$

The function $u_* = u_*(t; u_0)$ is defined as:

$$u_*(t; u_0) := \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta); u_0), \tag{2.9}$$

where $u_R = u_R(s; u_0)$ is the family of solutions to the Riccati equation which were shown to exist in proposition 2.1, it solves the initial value problem

$$\frac{d}{dt} u_*(t; u_0) = \mu(t) + u_*^2(t; u_0), \qquad u_*(\varepsilon^{-1}\delta; u_0) = \varepsilon^{\frac{1}{3}} u_0 \tag{2.10}$$

with ε and u_0 as parameters. The function w_r is a correction term whose properties are summerized in the following theorem.

Theorem 2.2. *For all δ, α small enough, there exists η, ε_1, C, such that for all $0 < \varepsilon < \varepsilon_1$, and all $|u_0 - \bar{u}_0| < \eta$, there exist a time $T = T(\varepsilon; u_0)$ and a solution to (2.1) of the form*

$$u(t; u_0) = u_*(t; u_0) + w_r(t; u_0)$$

exists on the time interval $t \in (\varepsilon^{-1}\delta, T)$, such that w_r and T satisfies

(1) $T = T(\varepsilon; u_0) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0) - \delta^{-1} + T_r$ *with* $|T_r| \leq C\varepsilon^{2/3}\delta^{-3}$,

(2) $w_r(T; u_0) = 0$ *and* $u_*(T, u_0) = \delta$,

(3) $|w_r(\varepsilon^{-1}\delta; u_0)| \leq C\varepsilon^{\frac{2-\alpha}{3}}$,

(4) $\sup_t |(T_\infty - t)^{2-\alpha} w_r(t; u_0)| \leq C$,

(5) *The function $w_r(\varepsilon^{-1}\delta; \cdot)$ is smooth, with Lipschitz constant $|Lip\ w_r(\varepsilon^{-1}\delta; \cdot)| \leq C\varepsilon^{\frac{\alpha}{3}}$.*

We will prove this theorem in the following sections.

2.4.2 The exit time $T(u_0)$

The exit time T is defined by the boundary condition

$$\delta = u_*(T; u_0) = \varepsilon^{1/3} u_R(\varepsilon^{1/3}(T - \varepsilon^{-1}\delta); u_0),$$

since the expansion for u_R is given in (2.6) , if we define $s_T = \varepsilon^{1/3}(T - \varepsilon^{-1}\delta)$, then s_T satisfies

$$\frac{1}{\Omega_\infty - s_T} + (\Omega_\infty - s_T) r(\Omega_\infty - s_T) = \varepsilon^{-1/3}\delta,$$

from which we get the leading order expansion $\Omega_\infty - s_T = \mathcal{O}(\varepsilon^{1/3}\delta^{-1})$. A fixed point argument gives

$$\Omega_\infty - s_T = \varepsilon^{1/3}\delta^{-1} + \mathcal{O}(\varepsilon\delta^{-3}),$$

hence the expansion for $T = T(\varepsilon; u_0)$ is

$$T = T(\varepsilon; u_0) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0) - \delta^{-1} + T_r, \tag{2.11}$$

with $|T_r| \le C\varepsilon^{2/3}\delta^{-3}$, for some constant C independent of u_0, as $\varepsilon \to 0$.

For conveneience, we define

$$T_\infty = T_\infty(\varepsilon; u_0) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0),$$

so that $T = T_\infty - \delta^{-1} + T_r$.

2.4.3 Equation for w_r and rescaling

We now plug in the anstaz $u = u_* + w_r$ into equation (2.1), and derive the equation for w_r

$$\begin{aligned} w_r' - 2u_* w_r &= w_r^2 + (u_* + w_r)^3 \\ &= 3u_*^2 w_r + (1 + 3u_*)w_r^2 + w_r^3 + u_*^3 := R_r(w_r) = R_r(w_r; \varepsilon, u_0), \end{aligned} \tag{2.12}$$

moreover, we enforce the boundary condition $u(T; u_0) = \delta$, hence this gives the boundary condition for w_r at $t = T$:

$$w_r(T; u_0) = 0, \tag{2.13}$$

therefore, equation (2.12) is posed on the interval $t \in (\varepsilon^{-1}\delta, T)$, with boundary condition (2.13).

Next, we rescale the equation (2.12) into σ-time variable by using the t to σ-time rescaling in section 2.3, and obtain

$$\left(\frac{d}{d\sigma} - a(\sigma; \varepsilon, u_0)\right) W_r = \varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r; \varepsilon, u_0), \tag{2.14}$$

where

- The term $a(\sigma; \varepsilon, u_0)$ is defined as and has asymptotics

  $$a(\sigma; \varepsilon, u_0) := 2\varphi(\sigma)u_R(\psi(\sigma); u_0) = 2 + \mathcal{O}(e^{-2\sigma}) \text{ as } \sigma \to \infty,$$

  we remark that this convergence as $\sigma \to \infty$ is uniform in u_0 due to the definition of our time-rescaling.

- The function $W_r(\sigma)$ is the rescaled version of $w_r(t)$ in the σ-variable, $w_r(t) = w_r(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = W_r(\sigma)$. U_* is similarly the rescaled version of u_*, $U_*(\sigma; u_0) = u_*(t; u_0) = \varepsilon^{1/3}u_R(\psi(\sigma; u_0); u_0)$.

- The function $\mathcal{R}_r$ is a rescaled version of R_r such that $\mathcal{R}_r(W_r; \varepsilon, u_0) = 3U_*^2 W_r + (1 + 3U_*)W_r^2 + U_*^3$,

To get the corresponding boundary condition of 2.13, we need to know the corresponding σ-time for the t-time interval $t \in (\varepsilon^{-1}\delta, T)$.

At $t = \varepsilon^{-1}\delta$, the corresponding σ time is at $\sigma = \sigma_m$, from its definition in section 2.3.

At $t = T$, we have $\varepsilon^{1/3}(T - \varepsilon^{-1}\delta) = \Omega_\infty - \varepsilon^{1/3}\delta^{-1} + \varepsilon^{1/3}T_r = \psi(\sigma_T) = \Omega_\infty - e^{-\sigma_T}$ from (2.11), hence, for ε small enough, we get that the corresponding σ-time to $t = T$ is

$$\sigma_T = \sigma_T(u_0) = -\log(\varepsilon^{1/3}(\delta^{-1} - T_r)) = -\log(\varepsilon^{1/3}\delta^{-1}) - \log(1 - \delta T_r),$$

which implies that

$$|\sigma_T - (-\log(\varepsilon^{1/3}\delta^{-1}))| \leq |\log(1 - \delta T_r)| \leq C|\delta T_r| \leq C\varepsilon^{2/3}\delta^{-2}$$

for some constant C independent of u_0.

We now define $\sigma_{\inf}$ and $\sigma_{\sup}$ as follows:

$$\sigma_{\inf} = \inf_{|u_0 - \bar{u}_0| < \eta} \sigma_m(u_0), \qquad \sigma_{\sup} = \sup_{|u_0 - \bar{u}_0| < \eta} \sigma_T(u_0)$$

and pose the equation (2.14) on the time interval $\sigma \in (\sigma_{\inf}, \sigma_{\sup})$. From above we can see that

$$cM \leq \sigma_{\inf}, \quad \sigma_{\sup} \leq -C\log(\varepsilon^{1/3}\delta^{-1})$$

for some constant c, C with $c < 1, C > 1$, independent of u_0, ε. The point is that we have eliminated the u_0 dependence on the time interval $(\sigma_{\inf}, \sigma_{\sup})$.

2.4.4 Linear equation and norms

Our goal now is to solve (2.14) on the interval $\sigma \in (\sigma_{\inf}, \sigma_{\sup})$ using a fixed point argument. To do so, we introduce the function space below:

$$\mathcal{C}_r = \left\{ w(\sigma) : \sup_{\sigma_{\sup} \geq \sigma \geq \sigma_{\inf}} \left| \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} w(\sigma) \right| < \infty \right\}.$$

We establish the invertibility of the linear operator A_r which acts on $w \in C_r$ as

$$A_r w = \left(\frac{d}{d\sigma} w - a(\sigma; \varepsilon, u_0) w, w(\sigma_T) \right)$$

in the following

Proposition 2.3. *$A_r = A_r(u_0, \varepsilon) : \mathcal{D} \subset C_r \to C_r \times \mathbb{R}$ and is invertible, with its inverse smoothly depends and uniformly bounded in u_0, ε.*

Proof. Fix η small, then there is $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, we can choose $\sigma_* \in (\sigma_{\inf}, \sigma_{\sup})$ so that $|a(\sigma; \varepsilon, u_0) - 2| \leq \eta$ for $\sigma \geq \sigma_*$. As a result there is a constant C, independent of ε and u_0, such that $|\sigma_* - \sigma_{\inf}| \leq C$. In fact, we can choose $\sigma_* \leq \sigma_T$ since $|\sigma_T - \sigma_{\sup}| \leq |\sigma_T - (-\log(\varepsilon^{1/3}\delta^{-1})| \leq C\varepsilon^{2/3}\delta^{-2}$.

For $\sigma_{\sup} \geq \sigma \geq \sigma_*$, the linear operator A_r is a small perturbation of the invertible linear operator $w \to \left(\frac{dw}{d\sigma} - 2w, w(\sigma_T)\right)$. Indeed, for $f \in C_r$, the equation $\frac{d}{d\sigma}w - 2w = f$ has the solution

$$w(\sigma) = e^{2(\sigma-\sigma_T)}w(\sigma_T) + \int_{\sigma_T}^{\sigma} e^{2(\sigma-s)}f(s)ds,$$

moreover, it holds that $\|w\|_{C_r} \leq C(\delta^{\alpha-2}|w(\sigma_T)| + 2\alpha^{-1}\|f\|_{C_r})$ for some constant C independent of u_0, ε, which shows the inverse is independent of ε and u_0. If the prescribed value at $w(\sigma_T)$ is of order $\delta^{2-\alpha}$.

For $\sigma < \sigma_*$, we can directly write down the solution of $(d/d\sigma - a)w = f$ as

$$w(\sigma) = \exp\left(\int_{\sigma_*}^{\sigma} a(\tau)d\tau\right)w(\sigma_*) + \int_{\sigma_*}^{\sigma} \exp\left(-\int_{\sigma}^{s} a(\tau)d\tau\right)f(s)ds$$

Again, the inverse is bounded uniformly in ε, u_0 because of the bound $|\sigma_* - \sigma_{\inf}| \leq C$. To see this, note $w(\sigma_*)$ can be evaluated using the solution on the intveral $(\sigma_*, \sigma_{\sup})$, which satisfies $w(\sigma_*) \lesssim \varepsilon^{\frac{2-\alpha}{3}}e^{(2-\alpha)\sigma_*}$, so

$$\left\|\exp\left(\int_{\sigma_*}^{\sigma} a(\tau)d\tau\right)w(\sigma_*)\right\|_{C_r} \leq e^{(2-\alpha)(\sigma_*-\sigma)}\exp\left(\int_{\sigma_*}^{\sigma} a(\tau)d\tau\right) \lesssim e^{(2-\alpha)C},$$

and

$$\left\|\int_{\sigma_*}^{\sigma} \exp\left(-\int_{\sigma}^{s} a(\tau)d\tau\right)f(s)ds\right\|_{C_r} \leq \int_{\sigma_*}^{\sigma} \exp\left(-\int_{\sigma}^{s} a(\tau)d\tau\right)e^{(2-\alpha)(s-\sigma)}\|f\|_{C_r}ds \lesssim e^{(2-\alpha)C}\|f\|_{C_r}$$

Combining the case $\sigma < \sigma_*$ and $\sigma \geq \sigma_*$ together, we conclude that A_r is uniformly invertible in ε and u_0 on the space C_r. ■

2.4.5 Nonlinear estimates

In this section we estimate the nonlinear term R_r.

Proposition 2.4. *If $W_r \in \mathcal{C}_r$, then $\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r) \in \mathcal{C}_r$, and*

$$\|\varepsilon^{-1/3}\varphi\mathcal{R}_r\| = \mathcal{O}(\delta^{\alpha}) \tag{2.15}$$

Proof. Recall that $\mathcal{R}_r(W_r) = 3U_*^2W_r + (1+3U_*)W_r^2 + W_r^3 + U_*^3$.

Proposition 2.6 shows

$$U_*(\sigma; u_0) = \varepsilon^{\frac{1}{3}}(e^{\sigma} + e^{-\sigma}r(e^{-\sigma}; u_0)) \text{ as } \sigma \to \infty,$$

therefore $|u_*(\sigma)| \lesssim \varepsilon^{\frac{1}{3}}e^{\sigma}$ for all $\sigma \geq \sigma_{\inf}$.

As $W_r \in C_r$, we have

$$|W_r(\sigma)| \lesssim \varepsilon^{\frac{2-\alpha}{3}}e^{(2-\alpha)\sigma}.$$

Using these facts, we have

$$\|\varepsilon^{-\frac{1}{3}}\varphi U_*^3\|_{C_r} \lesssim \varepsilon^{\frac{\alpha}{3}}e^{\alpha\sigma} \lesssim \varepsilon^{\frac{\alpha}{3}}e^{\alpha\sigma_{\sup}} \lesssim \delta^{\alpha},$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi W_r^2\|_{C_r} = \sup|\varepsilon^{-\frac{1}{3}}\varphi W_r| \lesssim \varepsilon^{\frac{1-\alpha}{3}}e^{(1-\alpha)\sigma} \lesssim \varepsilon^{\frac{1-\alpha}{3}}e^{(1-\alpha)\sigma_{\sup}} \lesssim \delta^{1-\alpha},$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi W_r^3\|_{C_r} = \sup|\varepsilon^{-\frac{1}{3}}\varphi W_r^2| \lesssim \varepsilon^{\frac{3-2\alpha}{3}}e^{(3-2\alpha)\sigma} \lesssim \varepsilon^{\frac{3-2\alpha}{3}}e^{(3-2\alpha)\sigma_{\sup}} \lesssim \delta^{3-2\alpha},$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi U_*^2 W_r\|_{C_r} = \sup|\varepsilon^{-\frac{1}{3}}\varphi u_r^2| \lesssim \varepsilon^{\frac{1}{3}}e^{\sigma} \lesssim \varepsilon^{\frac{1}{3}}e^{\sigma_{\sup}} \lesssim \delta.$$

■

2.4.6 Fixed point argument and the proof of Theorem 2.2

In this section we prove theorem 2.2 by setting up an appropriate fixed point argument.

Proof of theorem 2.2. Items (1) and (2) in the assertion of the theorem has been demonstrated in section 2.4.2 and 2.4.3. Items (3) and (4) is a direct consequence of $W_r \in C_r$ and the definition for the norm of C_r, to prove this, we first rewrite equation (2.14) and the boundary condition $W_r(\sigma_T) = W_T$ as

$$F_r(W_r, W_T; \varepsilon, u_0) = 0$$

where $F_r : C_r \times \mathbb{R} \to C_r \times \mathbb{R}$ is defined as

$$\begin{aligned} F_r(W_r, W_T; \varepsilon, u_0) &= A_r W_r - \left(\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), W_T\right) \\ &= \left(\frac{d}{d\sigma}W_r - aW_r - \varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), W_r(\sigma_T) - W_T\right). \end{aligned}$$

Now we are ready to use a fixed point argument to solve the equation

$$A_r W_r = (\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), W_T).$$

Let $X = C_r \times (-\delta_1, \delta_1)$, where $\delta_1 = \mathcal{O}(\delta^{2-\alpha})$ is small, we introduce the solution map $\mathcal{S} : X \to C_r \times \mathbb{R}$ as follows:

$$\mathcal{S}(W_r, W_T; \varepsilon, u_0) = (W_r - A_r^{-1}F_r(W_r, W_T; \varepsilon, u_0), W_T)$$

From the propositions above, we conclude

- $\|\mathcal{S}(0, 0; \varepsilon, u_0)\| = \|\left(-A_r^{-1}F_r(0, 0; \varepsilon, u_0), 0\right)\| \le \|A_r^{-1}\|\|F_r(0, 0; \varepsilon, u_0)\| \lesssim \|\|\varepsilon^{-1/3}\varphi\mathcal{R}_r(0)\| \lesssim \delta^{\alpha}$, uniformly in ε and u_0.
- $\mathcal{S}$ is a smooth map in W_r, W_T as well as the parameters ε, u_0.
- The linearization of $\mathcal{S}$ at $(0, 0)$, $D_{(W_r, W_T)}\mathcal{S}(0, 0)$, is equal to $A_r^{-1}\varepsilon^{-1/3}\varphi(3u_r^2)$, whose norm satisfies

  $$\|A_r^{-1}\varepsilon^{-1/3}\varphi(3u_r^2)\| \lesssim \sup|\varepsilon^{-1/3}\varphi(3u_r^2)| = \mathcal{O}(\delta)$$

  Moreover, for $\|W_r\|$ small enough and $|W_T| \le \delta_1$, we have $D_{(W_r, W_T)}\mathcal{S}(W_r, W_T; \varepsilon, u_0) = A_r^{-1}(3u_r^2) + \mathcal{O}(\|W_r\|_{C_r})$, which is uniformly small in ε and u_0.

Therefore, for (W_r, W_T) in a small ball of X, we can apply an iteration scheme and utilize the Banach fix point theorem to the existence of a fixed point, hence a solution to equation (2.14) exists. Moreover, this solution depends smoothly on the parameter ε, u_0. By picking $W_T = 0$, we have shown that a unique fixed point $W_r \in C_r$ exists and solves equation (2.14).

Finally, to prove item (5) we need to estimate the Lipschitz constant for the map $u_0 \mapsto w_r(\varepsilon^{-1}\delta; u_0) = W_r(\sigma_m; u_0)$, It suffices to estimate the following two quantities

$$C_1 = \mathrm{Lip}_{W_r}\mathcal{S}, \text{ and } C_2 = \mathrm{Lip}_{u_0}\mathcal{S},$$

because W_r is the fixed point of the map $\mathcal{S}$, which implies

$$\mathrm{Lip}_{u_0} W_r(\sigma; u_0) \leq C_2/(1 - C_1).$$

From the definition of $\mathcal{S}$, we see that

$$C_1 \leq \mathrm{Lip}_{W_r}|\varepsilon^{-1/3}\varphi R_r(W_r)| \leq \mathrm{Lip}_{W_r}|\varepsilon^{-1/3}\varphi W_r^2| \leq \sup_{W_r \in C_r}|\varepsilon^{-1/3}\varphi W_r| = \mathcal{O}(\delta^{1-\alpha})$$

where the last line follows from proposition 2.4.

To estimate C_2. We notice that

$$C_2 \leq \text{ Lip }_{u_0}|\varepsilon^{-1/3}\varphi U_*^3(\sigma; u_0)|.$$

However,

$$\|\varepsilon^{-1/3}\varphi[U_*^3(\sigma; u_0) - U_*^3(\sigma; \tilde{u}_0)]\|_{C_r} \leq \|\varepsilon^{-1/3}\varphi U_*^2\|_{C_r} \sup|U_*(\sigma; u_0) - U_*(\sigma; \tilde{u}_0)|$$

proposition 2.6 shows $U_* = \varepsilon^{1/3}(e^\sigma + e^{-\sigma} r(e^{-\sigma}; u_0))$ for σ large, hence

$$\partial_{u_0} U_*(\sigma; u_0) \leq C\varepsilon^{1/3}$$

for some constant independent of u_0, on the other hand

$$\|\varepsilon^{-1/3}\varphi U_*^2\|_{C_r} = \mathcal{O}(\varepsilon^{(\alpha-1)/3}),$$

so we conclude that

$$\|\varepsilon^{-1/3}\varphi[U_*^3(\sigma; u_0) - U_*^3(\sigma; \tilde{u}_0)]\|_{C_r} \leq C\varepsilon^{\alpha/3}|u_0 - \tilde{u}_0|,$$

or $C_2 = \mathcal{O}(\varepsilon^{\alpha/3})$. Using the remarks above, this shows $\mathrm{Lip}_{u_0} W_r(\sigma; u_0) = \mathcal{O}(\varepsilon^{\alpha/3})$. Rescale back from σ to t-time, we have completed the proof of theorem 2.2. ■

2.5 Region B

Region B is defined as $t^* < t < \varepsilon^{-1}\delta$.

2.5.1 Ansatz in region B

The ansatz takes the form $u = \bar{u}_* + w_\ell$.

Where $\bar{u}_*$ is the function

$$\bar{u}_*(t) = u_*(t;\bar{u}_0) = \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t-\varepsilon^{-1}\delta);\bar{u}_0) = \varepsilon^{1/3}\bar{u}_R(\varepsilon^{1/3}(t-\varepsilon^{-1}\delta)). \tag{2.16}$$

$\bar{u}_*$ solves the equation

$$\frac{d}{dt}\bar{u}_*(t) = \mu(t) + \bar{u}_*^2(t), \tag{2.17}$$

that is, $\bar{u}_*$ is merely a rescaled version of the special solution to the Riccati equation.

Similarly to the situation in region A, w_ℓ is a correction term whose properties are summerized in the following

Theorem 2.5. *For all δ, α small enough, there exists ε_2, C, such that for all $0 < \varepsilon < \varepsilon_2$, and a solution to (2.3) of the form*

$$u_B(t) = \bar{u}_*(t) + w_\ell(t),$$

exists on the time interval $t \in (t^*, \varepsilon^{-1}\delta)$, such that w_ℓ satisfies

$$w_\ell(t) \le C\varepsilon^{(2-\alpha)/3}|\varepsilon^{1/3}(t-\varepsilon^{-1}\delta)+1|. \tag{2.18}$$

We prove this theorem in the rest of this section.

2.5.2 Equation of W_ℓ and rescaling

As before, we plug in the ansatz into (2.3) to derive the equation satisfied by w_ℓ.

$$\begin{aligned} w_\ell' - 2\bar{u}_* w_\ell &= w_\ell^2 + (\bar{u}_* + w_\ell)^3 \\ &= (3\bar{u}_*^2)w_\ell + (1+3\bar{u}_*)w_\ell^2 + w_\ell^3 + \bar{u}_*^3 := R_\ell(w_\ell). \end{aligned} \tag{2.19}$$

We want to solve this equation on $t \in (t^*, \varepsilon^{-1}\delta)$. Following previous steps, we next rescale the equation to the σ-time variable using the time rescaling map $\psi = \psi(\sigma;\bar{u}_0)$ and we obtain

$$\frac{d}{d\sigma}W_\ell - b(\sigma)W_\ell = \varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell) \tag{2.20}$$

Where

- The equation is posed on $\sigma \in (\sigma^*, \sigma_m(\bar{u}_0))$ where $\sigma^* \approx -\varepsilon^{-1/4}$ and $\sigma_m(\bar{u}_0) := \bar{\sigma}_m$ follows the notation used in section 2.3.

- The term $b(\sigma)$ is defined and has asymptotics:

$$b(\sigma) := 2u_R(\psi(\sigma))\varphi(\sigma) = -2 + \mathcal{O}(|\sigma|^{-1})$$

  as $\sigma \to -\infty$. Agian, the convergence is uniform in ε.

- The function $W_\ell(\sigma)$ is the rescaled version of $w_\ell(t)$ in the σ-variable, $w_\ell(t) = w_\ell(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = W_\ell(\sigma)$. $\bar{U}_*$ is similarly the rescaled version of $\bar{u}_*$, $\bar{U}_*(\sigma) = \bar{u}_*(t) = \varepsilon^{1/3}\bar{u}_R(\psi(\sigma))$.

- The function $\mathcal{R}_\ell$ is a rescaled version of R_ℓ such that $\mathcal{R}_\ell(W_r; \varepsilon, u_0) = 3U_*^2 W_r + (1 + 3U_*)W_r^2 + U_*^3$,

2.5.3 Linear equation and norms

Similarly, the proof of theorem 2.5 consists of solving (2.20) via a fixed point argument on the following function space:

$$C_\ell = \left\{ w(\sigma) : \sup_{\sigma^* < \sigma < \bar{\sigma}_m} |\varepsilon^{\frac{\alpha-2}{3}} \langle \sigma \rangle^{-\frac{2}{3}} w(\sigma)| < \infty \right\}.$$

To begin with, let us define the operator A_ℓ by

$$A_\ell w = \left(\frac{d}{d\sigma} w - b(\sigma) w, w(\sigma^*) \right),$$

for $w \in \mathcal{D}(A_\ell) \subset C_\ell$.

Proposition 2.6. *$A_\ell : \mathcal{D}(A_\ell) \subset C_{W_\ell} \to C_{W_\ell} \times \mathbb{R}$, and A_ℓ is bounded invertible with its inverse uniformly bounded in ε.*

Proof. Similar to the proof of proposition 2.3, we may find $\sigma^{**} \in (\sigma^*, \bar{\sigma}_m)$ so that $|b(\sigma) - (-2)| < \eta$ for any small η given provided $\sigma < \sigma^{**}$. Moreover, this σ^{**} can be chosen to be independent of ε as $\sigma^* = \mathcal{O}(\varepsilon^{-1/4})$ and $\bar{\sigma}_m = \mathcal{O}(1)$.

Then, for $\sigma \in (\sigma^*, \sigma^{**})$, A_ℓ is a perturbation of the invertible operator

$$w \mapsto \left(\frac{d}{d\sigma} w + 2w, w(\sigma^*) \right),$$

which can be seen as follows: for $f \in C_\ell$, consider the initial value problem

$$\frac{d}{d\sigma} w + 2w = f, \qquad w(\sigma^*) = w^*,$$

which has solution

$$w(\sigma) = e^{2(\sigma^* - \sigma)} w^* + \int_{\sigma^*}^{\sigma} e^{2(\tau - \sigma)} f(\tau) d\tau.$$

Notice that

$$\|e^{2(\sigma^* - \sigma)} w^*\|_{C_\ell} \leq |\langle \sigma^{**} \rangle|^{-2/3} |w^* \varepsilon^{(\alpha-2)/3}| \leq |w^* \varepsilon^{(\alpha-2)/3}|$$

and

$$\left\| \int_{\sigma^*}^{\sigma} e^{2(\tau - \sigma)} f(\tau) d\tau \right\|_{C_\ell} \leq$$

■

2.5.4 Nonlinear estimates

For $\sigma \in (\sigma^*, 0)$, we will estimate the nonlinear term $\varepsilon^{-1/3}\varphi(\sigma)\left[(3u_\ell^2)W_\ell + (1+3u_\ell)W_\ell^2 + W_\ell^3 + u_\ell^3\right]$ in the C_{W_ℓ} norm. As a result, we have

Proposition 2.7. *$\varepsilon^{-\frac{1}{3}}\varphi R_\ell(W_\ell(\sigma)) \in C_{W_\ell}$ and $\|\varepsilon^{-\frac{1}{3}}\varphi R_\ell\|_{C_{W_\ell}} = \mathcal{O}(\varepsilon^?)$.*

Proof. ∎

2.6 Region C

This region is the interval (in t-time) $0 < t < t^*$. Recall t^* is the (left) gluing time which corresponds to when $\sigma = \varepsilon^{-1/4} =: \sigma^*$.

2.6.1 Ansatz in region C

The ansatz takes the form

$$u = u_s + W_s.$$

Where $u_s(t)$ denotes the "singular" branch that forms the slow manifold (critical manifold?) of the original system. It is defined via the relation (approximately)

$$u_s(t) = h(\mu(t))$$

for some smooth function h which solves

$$0 = \mu(t) + h(\mu(t))^2 + h(\mu(t))^3. \tag{2.21}$$

It has the following asymptotics:

$$u_s(t) = -\sqrt{\delta - \varepsilon t} + \mathcal{O}(|\delta - \varepsilon t|). \tag{2.22}$$

The equivalent in σ variable is

$$u_s(\sigma) = -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathcal{O}(|\varepsilon\sigma|^{2/3}) \tag{2.23}$$

2.6.2 Equation of W_s and rescaling

The ansatz for $t \in (0, t^*)$ is of the form $u = u_s + W_s$, hence we obtain the equation for W_s.

$$W_s' - 2u_sW_s = (3u_s^2)W_s + (1+3u_s)W_s^2 + W_s^3 - u_s' \tag{2.24}$$

We want to solve this equation on $t \in (0, t^*)$.

2.6.3 Linear equation and norms

Under the same rescaling, the W_s equation in σ time is

$$\frac{d}{d\sigma}W_s - c(\sigma)W_s = \varepsilon^{-1/3}\varphi R_s(W_s), \tag{2.25}$$

for $\sigma \in (-(2/3)\delta^{3/2}\varepsilon^{-1}, \sigma^*)$.

Where $c(\sigma)$ is defined and has the asymptotics:

$$c(\sigma) = 2\varepsilon^{-\frac{1}{3}}u_s(\psi(\sigma))\varphi(\sigma) = -2 + \mathcal{O}(\varepsilon^{1/3}|\sigma|^{1/3}),$$

as $\sigma \to -\infty$.

The function space on which we will solve the W_s equation via a fixed point argument is:

$$C_{W_s} = \left\{ w(\sigma) \mid \sup |\varepsilon^{\frac{\alpha}{3}-1}\langle\varepsilon\sigma\rangle^{\frac{2}{3}}w(\sigma)| < \infty \right\}$$

Similarly, we define the linear operator A_s on $w \in C_{W_s}$ by

$$A_s w = \left(\frac{d}{d\sigma}w - cw, w(\sigma_0) \right)$$

Proposition 2.8. *$A_s : C_{W_s} \to C_{W_s} \times \mathbb{R}$, and A_s is bounded invertible with its inverse uniformly bounded in ε.*

Proof. small perturbation from the case $c \equiv 0$. ■

2.6.4 Nonlinear estimates

For $\sigma \in (\sigma_0, \sigma^*)$, we estimate the nonlinear term $\varepsilon^{-1/3}\varphi(\sigma)\left[(3u_\ell^2)W_\ell + (1+3u_\ell)W_\ell^2 + W_\ell^3 + u_\ell^3\right]$ in the C_{W_s} norm. Similar to the theorem for W_ℓ, we have

Proposition 2.9. *$\varepsilon^{-\frac{1}{3}}\varphi R_s(W_s(\sigma)) \in C_{W_s}$ and $\|\varepsilon^{-\frac{1}{3}}\varphi R_s\|_{C_{W_s}} = \mathcal{O}(\varepsilon^?)$.*

Proof. ■

3 Gluing

To be completed.