1 Hyperbolic Gluing

Consider the 2-d system

$$\dot{u} = Au + f(u), \qquad u = (u^1(t), u^2(t))^T,$$

Where A is a constant coefficient hyperbolic matrix, with exactly 1 stable/unstable direction. And the dot is d/dt. f denotes higher order terms so $f(0) = 0$ and $Df(0) = 0$.

Fix some $\delta > 0$ small, we have a 1-d (local) stable and unstable manifold u_- and u_+, which can be locally straightened: $u_- : \{u^1 = 0\}$ and $u_+ : \{u^2 = 0\}$

so that $u_\mp \to (0,0)^T$ as $t \to \pm\infty$ and $u_-^2(0) = \delta, u_+^1(0) = \delta$ (after some shift in time).

Fix $T > 0$ (not necessarily large), we want to solve the boundary value problem $u^1(T) = u^2(-T) = \delta$ by looking for a solution u to the system in the vicinity of the stable/unstable manifold $u_\pm$.

We need the following property: $u_\pm$ satisfies the estimate

$$\|u_-(t)\| \leq \delta e^{-\gamma t} \text{ for } t \geq 0$$

and

$$\|u_+(t)\| \leq \delta e^{\gamma t} \text{ for } t \leq 0$$

for some constants $\gamma > 0$ and.

1.1 The Ansatz

Let $\chi_\pm(t)$ be a smooth partition of unity of the real line $(-\infty, \infty)$ such that

(i) $\chi_- + \chi_+ = 1$;

(ii) $\chi_- = 1$ for $t < -1$, $\chi_- = 0$ for $t > 1$;

(iii) $\chi_+ = 0$ for $t < -1$, $\chi_+ = 1$ for $t > 1$

Our ansatz would take the form (note the time shift)

$$u(t) = u_-(t+T) + u_+(t-T) + w_-(t+T) + w_+(t-T)$$

Here the corrector term $w = w_- + w_+$ is split into two parts w_- and w_+, which we consider on the halfline $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}^- = (-\infty, 0)$.

We then introduce exponentially weighted function space on $\mathbb{R}^+$ and $\mathbb{R}^-$. Let us fix an exponential weight $\eta > 0$ whose exact value will be determined later, so that for $t \in \mathbb{R}^+ = (0, \infty)$ we have

$$\|w_-\|_{C^1_\eta} := |e^{\eta t}(w_-(t) + \dot{w}_-(t))|_\infty < \infty$$

and for $t \in \mathbb{R}^- = (-\infty, 0)$ we have

$$\|w_+\|_{C^1_\eta} := |e^{-\eta t}(w_+(t) + \dot{w}_+(t))|_\infty < \infty$$

these $w_\pm$ are unshifted!

We need to determine equations in $w_\pm$ separately! Note the equation is to be solved for $|t| < T$, together with the boundary values $u^1(T) = u^2(-T) = \delta$.

(i) turns out $w_-(2T) + w_+(0) = (0, u^2(T))^T$ and $w_-(0) + w_+(-2T) = (u^1(-T), 0)^T$.

(ii) need $\dot{u} = Au + f(u)$, so we have on the left

$$\dot{u} = (\dot{\chi}_- u_- + \chi_- \dot{u}_-) + (\dot{\chi}_+ u_+ + \chi_+ \dot{u}_+) + \dot{w}_- + \dot{w}_+$$

which must equal to the right

$$Au + f(u) = A(\chi_- u_- + \chi_+ u_+ + w_- + w_+) + f(\chi_- u_- + \chi_+ u_+ + w_- + w_+).$$

Using the fact that $u_\pm$ are solutions to the ODE and $\chi_\pm$ are scalar-valued which can be pulled out in front of A, we simplify:

$$(\dot{w}_- + \dot{w}_+) - A(w_- + w_+) = \dot{\chi}_- u_- + \dot{\chi}_+ u_+ + f(\chi_- u_- + \chi_+ u_+ + w_- + w_+) - \chi_- f(u_-) - \chi_+ f(u_+)$$

We next split the above equation separately in w_- and w_+....

1.2 Spliting the error

Let us first adjust the linear part into

$$(\dot{w}_- + \dot{w}_+) - A(w_- + w_+) - (f'(u_+)w_+ + f'(u_-)w_-).$$

Then we first group the right hand as follows:

$$R := \underbrace{\dot{\chi}_- u_- + \dot{\chi}_+ u_+}_{:= R_0} + f(\chi_- u_- + \chi_+ u_+ + w_- + w_+) - \chi_- f(u_-) - \chi_+ f(u_+) - (f'(u_+)w_+ + f'(u_-)w_-).$$

Next, define the commutator (f' is shorthand for Df, also keep in mind the time shift $u_\pm(t \mp T)$ on u_j.

$$[f, \chi_\pm] = \sum_{j=\pm} \chi_j f(u_j) - f(\sum_{j=\pm} \chi_j u_j); \quad [f', \chi_\pm] = \sum_{j=\pm} \chi_j f'(u_j) - f'(\sum_{j=\pm} \chi_j u_j),$$

We then group $R - R_0$ as follows:

$$R - R_0 = f(\sum_j \chi_j u_j + w_j) - f(\sum_j \chi_j u_j) - \underbrace{[f, \chi_\pm]}_{:= -R_1} - \sum_j f'(u_j) w_j,$$

We next decompose $R - R_0 - R_1$ by first Taylor expand f around $\sum_j \chi_j u_j$

$$R - R_0 - R_1 = f'(\sum_j \chi_j u_j) \sum_j w_j - \sum_j f'(u_j) w_j + R_2$$

Here R_2 would be of the order $O(w^2)$ with $w = w_- + w_+$. We then have $R - R_0 - R_1 - R_2$ being decomposed again:

$$\begin{aligned} R - \sum_{j=0}^{2} R_j &= f'(\sum_j \chi_j u_j) \sum_j w_j - \sum_j f'(u_j) w_j \\ &= \sum_j \chi_j f'(u_j) \sum_j w_j - \underbrace{[f', \chi_\pm] \sum_j w_j}_{:= -R_3 \sum w_j} - \sum_j f'(u_j) w_j \\ &= R_3 \sum_j w_j - (\chi_- f'(u_+) w_+ + \chi_+ f'(u_-) w_- + (\chi_- f'(u_-) w_+ + \chi_+ f'(u_+) w_-) \end{aligned}$$

in the end we expand the first term $R_3 \sum w_j$ and group $R_3 w_+, \chi_- f'(u_+)w_+$ and $\chi_- f'(u_-)w_+$ to be R_4, and the rest $R_3 w_-, \chi_+ f'(u_-)w_-$ with $\chi_+ f'(u_+)w_-$ to be R_5.

Thus we have split the error R into 6 parts, summarize:

$$
\begin{aligned}
R_0 &= \dot{\chi}_- u_- + \dot{\chi}_+ u_+ \\
R_1 &= -[f, \chi_\pm] = [\chi_\pm, f] \\
R_2 &= f(\sum_j \chi_j u_j + w_j) - f(\sum_j \chi_j u_j) - f'(\sum_j \chi_j u_j) \sum w_j \\
R_3 &= -[f', \chi_\pm] = [\chi_\pm, f'] \\
R_4 &= -\chi_- f'(u_+)w_+ + \chi_- f'(u_-)w_+ + R_3 w_+ \\
R_5 &= -\chi_+ f'(u_-)w_- + \chi_+ f'(u_+)w_- + R_3 w_-
\end{aligned}
$$

1.3 Equation of the corrector and estimates

We first set up the equation for w_- and w_+, note these $w_\pm$ are shifted, we define $w_-T(\cdot) := w_-(\cdot + T)$ and $w_+T(\cdot) := w_+(\cdot - T)$, the equation we have will be equation for w_-^T and w_+^T, respectively, and the domain for both $w_\pm^T$ is $(-T, T)$.

equation for w_-^T:

$$\mathcal{L}_- w_-^T := \dot{w}_-^T - (A + f'(u_-^T) + R_3)w_-^T = \dot{\chi}_- u_-^T + \chi_-(R_1 + R_2) + R_4 := R_-(w_-^T; w_+^T)$$

and equation for w_+^T:

$$\mathcal{L}_+ w_+^T := \dot{w}_+^T - (A + f'(u_+^T) + R_3)w_+^T = \dot{\chi}_+ u_+^T + \chi_+(R_1 + R_2) + R_5 := R_+(w_+^T; w_-^T)$$

Want to solve w_-^T and w_+^T through a fixed point argument. Use the space $C^1_\eta(R_-)$ for w_+ and $C^1_\eta(R_+)$, consider $\mathcal{L}_\pm$ as an operator from C^1_η to C^0_η. The control for linear parts must be done using exponential dichotomy inherited from the hyperbolicity of the matrix A and the smallness of $u_\pm^T + R_3$ in $(-T, T)$.

(i) **Estimates for R_0, R_1, R_3**

These terms do not involve w, we shall show they are small in the η-weighted norm. The linear part will be controlled by using exponential dichtomy from the hyperbolicity of A and the fact that $f'(u_\pm^T) + R_3$ are uniformly small.

Let us focus on the equation for w_- first,

- note I have distributed R_0 into a χ_--part and a χ_+-part, for the equation for w_-^T, what needs to be estimated is just $\dot{\chi}_- u_-^T$.

  Since χ_- is constant outside of $|t| > T$, we need only consider $|t| < T$, but then $u_-^T(t) = u_-(t+T)$ will satisfy $\|u_-^T(t)\| \leq \delta$ (sup norm). Hence if δ is sufficiently small, then in the weighted norm $\dot{\chi}_- u_-^T$ will be as small as needed.
- for the commutator term R_1, because of the $\chi_\pm$, the time interval that are relevant is $(-T, T)$ (outside of which $R_1 = 0$) But on these intervals again using $\|u_-^T(t)\| < \delta$. and $f(0) = 0$ to get R_1 and R_3 are as small as needed.

- similarly for R_3, using $f'(0) = 0$.

Here I am not using any information about T being large, but just the smallness of $u_\pm^T$ on the time interval $(-T, T)$.

(ii) **Estimates for R_2**

Recall that for $|t| < T$, we have

$$R_2(t) = f(\sum_j \chi_j u_j + w_-(t-T) + w_+(t-T)) - f(\sum_j \chi_j u_j) - f'\left(\sum_j \chi_j u_j\right)\{w_-(t-T) + w_+(t-T)\}.$$

This is the remainder term which is of higher order in $w(t) = w_-(t+T) + w_+(t-T)$, since for $|t| < T$, we have $|u_\pm(t \mp T)| \leq \delta$ by set up, using Taylor's theorem, we have $|R_2(t)| = \mathcal{O}(|w|^2)$, which will be small if we are working in some small ball in the function space C^1_η for w.

(iii) **Estimates for R_4 and R_5**

We have set

$$R_4 = -\chi_- f'(u_+^T)w_+^T + \chi_- f'(u_-^T)w_+^T$$

Again, due to the cut off χ_-, for $t < -T$, the decay at infinity of w_+ ensures $e^{\eta(t+T)}w_+(t-T)$ is exponentially small. And the focus is on $|t| < T$.

For these t in these range, we need to estimate sup norm of R_4 under the weight $e^{\eta(t+T)}$. If I just use $\|u_\pm\| \leq \delta$, I will end up with $\|e^{t+T}R_4(t)\| \leq g(\delta)e^{\eta(t+T)}e^{\eta(t-T)} \leq e^{2\eta t}g(\delta)$ for some function $g(\delta) = O(\delta^2)$. Note $t \in [-T, T]$ could make $e^{2\eta t}$ big. But if δ is sufficiently small I think this can be taken care of.

A similar argument applies to R_5.

from here we need to show the equation $\mathcal{L}_\pm w_\pm = R_\pm$ can be solved using an iteration argument, which amounts to show $\|R_\pm\|$ is small given $w_\pm$ small, and $\mathcal{L}_\pm^{-1}R_\pm$ is a contraction, say when we working on some balls in the function space.

conclusion for the "flying time" if T is given, then the size of the boundary condition will depend on T. (if T is large, then δ need to be sufficiently small), which is quite natural since the flying time goes to infinity as the trajectories get close to the invariant manifolds.

2 Non-hyperbolic Gluing

2.1 A decoupled system

Start with the following very simple 2-d system:

$$\begin{aligned} \dot{u} &= -u^2 \\ \dot{v} &= v \end{aligned} \tag{2.1}$$

This system decouples, of course. But we wish to demonstrate the method from a simple example. Clearly, the u-axis $\{v = 0\}$ is invariant and the solution is explicitly parameterized by

$$U_-(t) = (\frac{1}{t + C_-}, 0)^T := (u_*(t), 0)^T$$

for some constant C_-. Likewise, the invariant v-axis $\{u = 0\}$ is parametrized by

$$U_+(t) = (0, C_+ e^t)^T := (0, v_*)^T$$

for some constant C_+.

We want to solve a boundary value problem $U(t) = (u(t), v(t))^T$ such that (u, v) satisfies the ODE, with the boundary condition $u(1) = \delta$, $v(1) = \delta$. We follow the hyperbolic case, take an ansatz of the form

$$U(t) = \chi_-(t)U_-(t + T + 1) + \chi_+(t)U_+(t - T + 1) + w_-(t + T + 1) + w_+(t - T + 1).$$

Now, use the fact that U satisfy the system $\dot{U} = AU + F(U)$, where A is the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and the nonlinear term $F(U) = F(u, v) = (-u^2, 0)^T$, after some calculation we arrived at the following equations for $w_\pm$.

$$\dot{w}_- - Aw_- = -\dot{\chi}_- U_- + F(\chi_\pm U_\pm + w_\pm) - \chi_\pm F(U_\pm) \tag{2.2}$$

$$\dot{w}_+ - Aw_+ = -\dot{\chi}_+ U_+ \tag{2.3}$$

Notice here $w_\pm$ are vector-valued, $w_\pm = (w_\pm^1, w_\pm^2)^T$. Let us focus on the equation for w_+ first, by the structure of A and U_+, the equation for the first component of w_+ is actually just

$$\dot{w}_+^1 = 0,$$

and the equation for the second component is

$$\dot{w}_+^2 = w_+^2 - \dot{\chi}_+ v_*$$

Following the moral of the hyperbolic gluing, $w_+(\cdot)$ should be decaying at $-\infty$, hence we must have $w_+^1 \equiv 0$. And we may explicitly solve w_+^2, which is given by

$$w_+^2(t) = w_+^2(0)e^t + C_+ e^t \left(\chi_+(T - 1) - \chi_+(t + T - 1)\right)$$

Of course in the general case we would not get such an explicit formula, but we can get the estimate that show w_+ lies in some exponentially weighted space on the interval $(-\infty, 0)$.

Next we focus on the equation for w_-, first we subtract both sides of the equation by the term $f'(U_-)w_-$, which equals $\begin{pmatrix} -2u_* & 0 \\ 0 & 0 \end{pmatrix} w_-$ to adjust the linear term.

The equation for w_- now reads

$$\dot{w}_- - \begin{pmatrix} -2u_* & 0 \\ 0 & 1 \end{pmatrix} w_- = -\dot{\chi}_- U_- + \begin{pmatrix} \chi_- u_*^2 - (\chi_- u_* + w_+^1 + w_-^1)^2 + 2u_* w_-^1 \\ 0 \end{pmatrix}$$

Again, the equation for the second component of w_- is just

$$\dot{w}_-^2 - w_-^2 = 0$$

we got $w_-(t) = Ae^t$ for some constant A, however, in order for $w_-(\cdot)$ to decay at $+\infty$, we must choose $A = 0$, thus we have $w_-^2 = 0$.

Therefore, we end up with the equation for the first component, which is

$$\dot{w}_-^1 - (-2u_*)w_-^1 = -\dot{\chi}_- u_* + (\chi_- u_*^2 - (\chi_- u_* + w_-^1)^2 + 2u_* w_-^1)$$

To solve it, we need to rescale in time.

Define the new time variable τ such that $dt/d\tau = (-2u_*(t + T + 1))^{-1}$. Put $\tilde{w}(\tau) = w_-^1(t(\tau))$, after multiplying the equation by $(-2u_*)^{-1}$. We have

$$\frac{d}{d\tau}\tilde{w} - \tilde{w} = (-2u_*)^{-1}\left(-\dot{\chi}_- u_* + (\chi_- - \chi_-^2)u_*^2 + 2(1 - \chi_-)u_*\tilde{w} - (\tilde{w})^2\right) := (-2u_*)^{-1}R$$

We can now work with exponentially weighted space (in the variable τ, let us solve the above equation for $\tau \in [0, \infty)$ (corresponding to $t \in [1, \infty)$, since explicitly $t = \exp(\tau)$.)

Now if we assume $\tilde{w} \in C^1_\nu$ for some weight ν, due to the multiplication of the right hand side by $(-2u_*)^{-1} \sim t \sim \exp(\tau)$, we lose the localization from ν to $\nu - 1$. Which means we need to estimate the remainder R in the $C^1_{\nu-1}$ norm.

- Estimates for R

In fact, one can check that w_-^1 and w_+^2 are given explicitly by $w_-^1 = \chi_+ u_*$ and $w_+^2 = \chi_- v_*$.

2.2 Equation with more general nonlinear terms

$$\dot{u} = -u^2 + f(u, v) \tag{2.4}$$

$$\dot{v} = v + g(u, v), \tag{2.5}$$

where $f(u, v) = \mathcal{O}(uv, v^2, u^3)$ and $g(u, v) = \mathcal{O}(uv, u^2, v^2)$ as $|(u, v)| \to 0$.

We want to do the same thing as in the hyperbolic case, now the problem is that due to non-hyperbolicity, we need to work in an appropritely re-scaled time (and accordingly choose the correct function space) to recover the Fredholm properties.

Now, by standard theory, equation (2.4) has a solution which asymptotically decays algebraically: $u_*(t) = \mathcal{O}(t^{-1})$ as $t \to \infty$.

The v equation determines uniquely the unstable manifold v_*, which decays exponentially in backward time $v_*(t) = \mathcal{O}(e^t)$ as $t \to -\infty$.

Our goal is to find an orbit near the origin by solve the following boundary value problem: $(u, v)(t)$ solves (2.4) and $u(-T) = \delta, v(T) = \delta$ for flying time T and small $\delta > 0$ given.

2.2.1 Construction of the center manifold

For this example we consider the equation

$$\dot{u}(t) = -u^2(t) + u^3(t)$$

we want to construct the solution which decays to 0 with order t^{-1} as $t \to \infty$. However, plug in the ansatz $u = t^{-1} + v$ gives an equation

$$\dot{v}(t) = -\frac{2}{t}v(t) - v^2(t) + (v(t) + \frac{1}{t})^3$$

if we do the rescaling $\frac{d}{d\tau} = t\frac{d}{dt}$, let $' = \frac{d}{d\tau}$ and $w(\tau) = v(t(\tau))$, we have

$$w'(\tau) + 2w(\tau) = -e^{\tau}w^2(\tau) + e^{\tau}(e^{-\tau} + w(\tau))^3$$

Variation of constants/integration factor gives

$$w(\tau) = e^{2(\sigma-\tau)}w(\sigma) + e^{-2\tau}\int_{\sigma}^{\tau} e^{3s}(w^2(s) + (e^{-s} + w(s))^3)$$

This is problematic when choosing the right function space for w: we cannot have $w(\tau) = O(e^{-2\tau})$ as $\tau \to \infty$, because of the $e^{3s}(e^{-s})^3 = 1$ term in the integrand. This implies some logarithmic correction in the asymptotics of the center manifold we are constructing.

Instead, let us try the ansatz

$$u(t) = t^{-1} + t^{-2}\log(t) + v(t) := u_{\#}(t) + v(t)$$

we end up with the equation

$$\dot{v}(t) + 2u_{\#}(t)v(t) = -v^2 + 3\frac{1}{t^2}(\frac{\log t}{t^2} + v) + 3\frac{1}{t}(\frac{\log t}{t^2} + v)^2 + (\frac{\log t}{t^2} + v)^3 - \frac{\log^2 t}{t^4} := f(v, t)$$

we have $f(v,t) = \mathcal{O}(\frac{\log^2 t}{t^4} + (\frac{\log t}{t^3} + \frac{1}{t^2})v + v^2)$, as $v \to 0$, uniformly in t.

Use again integration factor, set $p(t) = \exp(\int 2u_{\#}(t)dt)$, we have

$$v(t) = p(t)^{-1}p(\sigma)v(\sigma) + p(t)^{-1}\int_{\sigma}^{t} p(s)f(v(s),s)ds$$

We have $u_{\#}(t) = \mathcal{O}(t^{-1})$ as $t \to \infty$, so $p(t) = \mathcal{O}(t^2)$.

Fix $\delta > 0$ small, we impose the decay $v(t) = \mathcal{O}(t^{\delta-3})$ as $t \to \infty$, then letting $\sigma \to \infty$, we have

$$v(t) = p(t)^{-1}\int_{\infty}^{t} p(s)f(v(s),s)ds := F(v,t)$$

Set the weight function $\langle t\rangle^{\gamma} = (\sqrt{t^2+1})^{\gamma}$, with $\gamma = 3 - \delta$.

Then the weighted space $L^{\infty}_{\gamma}(I) = \{v(x); \langle x\rangle^{\gamma}v(x) \in L^{\infty}(I)\}$ for $I = (a,\infty) \subset \mathbb{R}$. where a is some large number (precise value determined later). Let X be the closed ball around 0 of L^{∞}_{γ} with radius ε.

To complete the fixed point argument, we need to check

(i) $F(v,t)$ is a contraction on X.

(ii) $F(\cdot,t)$ maps X into X.

Contraction First, to see $F(v,t)$ is a contraction on X, let $v_1, v_2 \in X$. want to show we can find $c < 1$ so that $\|F(v_1,t) - F(v_2,t)\|_X \le c\|v_1 - v_2\|_X$.

Now

$$\|F(v_1,t) - F(v_2,t)\|_X = \sup_{t\ge a}\left|\frac{\langle t\rangle^{3-\delta}}{p(t)}\int_{\infty}^{t} p(s)[f(v_1(s),s) - f(v_2(s),s)]ds\right|$$

Since $f(v,t) = \mathcal{O}\left(\frac{\log^2 t}{t^4} + (\frac{\log t}{t^3} + \frac{1}{t^2})v + v^2\right)$. we have

$$|D_v f(v,t)| = 2v + \frac{3}{t^2} + \frac{9}{t}\left(v + \frac{\log t}{t^2}\right)^2 + \frac{6}{t}\left(\frac{\log t}{t^2} + v\right)$$

use mean value theorem, there is some $\tilde{v}$ with $|\tilde{v}|_{\infty} < \varepsilon$ such that

$$|f(v_1,s) - f(v_2,s)| \le \left|\int_{\theta=0}^{\theta=1} D_v f(v_2 + \theta(v_1 - v_2), t)d\theta\right| |v_1(s) - v_2(s)|$$

use the fact that $v_1, v_2 \in X$, denote $v_\theta(s) = v_2 + \theta(v_1 - v_2)(s)$, we have

$$|v_\theta(s)| \le \langle s\rangle^{\delta-3}\|v_\theta(s)\|_X \le \varepsilon\langle s\rangle^{\delta-3}$$

then for some constant C

$$\begin{aligned}|f(v_1,s) - f(v_2,s)| &\le C\left((1 + \frac{1}{s} + \frac{\log^2 s}{s^3})\varepsilon\langle s\rangle^{\delta-3} + \frac{1}{s^2} + \frac{\log s}{s^3}\right)|v_1(s) - v_2(s)| \\ &\le C\frac{1}{s^2}\|v_1 - v_s\|_X\langle s\rangle^{\delta-3}\end{aligned}$$

Thus

$$\int p(s)|f(v_1) - f(v_2)| \leq C\|v_1 - v_2\|_X \int_{s\geq t} \langle s\rangle^{\delta-3} ds$$

we can choose a so that $Cp(t)^{-1}\langle t\rangle^{3-\delta} \int_{s\geq t}\langle s\rangle^{\delta-3}ds < \frac{1}{2}$ for $t \geq a$. This shows F is a contraction.

Maps X into itself To see $F(v(s), s) \in X$ for $v(s) \in X$, we estimate

$$\sup_{t\geq a}\left|\frac{\langle t\rangle^{3-\delta}}{p(t)} \int_t^\infty p(s)f(v(s), s)ds\right|$$

since $|v(s)| \leq |v|_X \langle s\rangle^{\delta-3}$, the leading order of $f(v(s), s)$ comes from the $\log^2 s/s^4$ term, we compute that

$$\int_t^\infty |p(s)\frac{\log^2 s}{s^4}|ds \leq C\frac{\log t}{t}$$

Hence $\langle t\rangle^{3-\delta}p(t)^{-1}\frac{\log t}{t} = \mathcal{O}(t^{-\delta}\log t)$, which is bounded (barely!)

2.2.2 Functional framework

More generally, we will show the operator

$$f(\cdot) \mapsto p(t)^{-1}\int_\infty^t p(s)f(s)ds$$

is bounded from $L^p_\gamma(I)$ to $M^{1,p}_{\gamma-1}(I)$, where $I = (a, \infty)$. (will determine γ later)
where we have introduced the following function space:

$$M^{m,p}_\gamma(\Omega) = \{u(t) \in L^1_{loc}(\Omega); \quad \langle t\rangle^{\gamma+k}\partial_t^k u \in L^p(\Omega), \quad k \leq s, k \in \mathbb{Z}\}$$

where $p \in (1, \infty)$.
Set $u(t) = p(t)^{-1}\int_\infty^t p(s)f(s)ds$, we want to show

$$\langle t\rangle^{\gamma-1}u(t) \in L^p(I)$$

recall that $p(t) = \exp(\int 2u_\#(t)dt)$, $u_\#(t) = t^{-1} + \mathcal{O}(t^{-2}\log(t))$, and $p(t) = \mathcal{O}(t^2)$.
Consider the expression

$$u(t) = p(t)^{-1}\int_\infty^t p(s)f(s)ds.$$

Use $p(t) = \mathcal{O}(t^2)$ and $t \geq a$ we can instead estimate the L^p norm of the following expression

$$t^{-2}\int_\infty^t s^2 f(s)ds$$

We will use an exponential scaling $t = e^\tau$, set $\tilde{\gamma} = \gamma - (1 - p^{-1})$, and define $w(\tau) = e^{\tilde{\gamma}\tau}u(e^\tau)$.

We have, for $\tau \in (\log a, \infty) := I_a$

$$\|w(\tau)\|_{L^p(I_a)}^p = \int_{I_a} e^{p\tilde{\gamma}\tau} |u(e^\tau)|^p d\tau = \int_{I_a} e^{p(\gamma-1)\tau} |u(e^\tau)|^p e^\tau d\tau = \int_I (t^{\gamma-1}|u(t)|)^p dt = \|u\|_{L^P_{\gamma-1}(I)}^p$$

Set $g(\sigma) = e^{(\tilde{\gamma}+1)\sigma} f(e^\sigma)$, and $h(\cdot) = e^{(\tilde{\gamma}-2)(\cdot)}$, we note that

$$\|g\|_{L^p(I_a)} = \int_{I_a} e^{(\gamma+p^{-1})p\sigma} |f(e^\sigma)|^p d\sigma = \int_{I_a} |e^{\sigma\gamma} f(e^\sigma)|^p e^\sigma d\sigma = \|f\|_{L^p_\gamma(I)}.$$

On the other hand, use another scaling $s = e^\sigma$, we can write

$$\begin{aligned}
\int_{I_a} |w(\tau)|^p d\tau &\leq \int_{I_a} \left| e^{\tilde{\gamma}\tau} e^{-2\tau} \int_\infty^{\sigma=\tau} e^{2\sigma} f(e^\sigma) e^\sigma d\sigma \right|^p d\tau \\
&= \int_{I_a} \left| \int_\infty^\tau e^{(\tilde{\gamma}-2)(\tau-\sigma)} e^{(\tilde{\gamma}+1)\sigma} f(e^\sigma) d\sigma \right|^p d\tau \\
&= \|h * g\|_{L^p(I_a)}^p
\end{aligned}$$

Then we can apply Young's inequality and obtain

$$\|u\|_{L^P_{\gamma-1}(I)} = \|w(\tau)\|_{L^p(I_a)} \leq \|h * g\|_{L^p(I_a)} \leq \|h\|_{L^1(I_a)} \|g\|_{L^p(I_a)} \leq C\|f\|_{L^p_\gamma(I)}$$

for some constant C, provided $\tilde{\gamma} - 2 > 0$. Which translates to $\gamma > 3 - p^{-1}$.

The case for $p = \infty$ When $p = \infty$, the whole arguments are actually easier. In particular, we still have the same conclusion, that

$$f(\cdot) \mapsto p(\cdot)^{-1} \int_\infty^\cdot p(s) f(s) ds$$

is bounded from $L^\infty_\gamma(I)$ to $M^{1,\infty}_{\gamma-1}(I)$.

2.2.3 Nonhyperbolic gluing by explicit construction of the center manifold

We now go back to system (2.4) and (2.5), we want to actually construct the center and unstable manifold using the functional framework in the previous subsection and then follow the hyperbolic gluing part.

More precisely, we want to show that there exists a unique (up to time translation) $U_-(t) = (u_-(t), v_-(t))^T$ which decays to 0 as $t \to \infty$, and $U_+(t) = (u_+(t), v_+(t))^T$ which decays to 0 as $t \to -\infty$. We also wish to give the asymptotics at $\pm\infty$ directly for $U_\pm$.

Existence for U_- To set up the appropriate functional framwork, we will use an ansatz $u_-(t) = u_\#(t) + w(t)$, where $u_\#(t) = t^{-1} + bt^{-2}\log(t)$ for some prefactor b chosen later.

We then use variation of constants to rewrite the ODE system into the following system of integral equation

$$\begin{aligned}
w(t) &= \int_\infty^t p(s)p(t)^{-1} F(w(s), v_-(s), s) ds = \left(\frac{d}{dt} + 2u_\#(t)\right)^{-1} F(w, v_-, t) \\
v_-(t) &= \int_\infty^t e^{(t-s)} G(w(s), v_-(s), s) ds = \left(\frac{d}{dt} - 1\right)^{-1} G(w, v_-, t)
\end{aligned}$$

Where

$$F(w(s),s) = -w^2(s) + f(u_\#(s)+w(s), v_-(s)) - (bs^{-3} + b^2s^{-4}(\log(s))^2),$$

and

$$G(v_-(s),s) = g(u_\#(s)+w(s), v_-(s)).$$

By choosing b appropriately, we can eliminate the term with order $\mathcal{O}(s^{-3})$, leaving the inhomogeneity with order $s^{-4}(\log(s))^2$ only.

Let us solve this for $w \in M^{1,\infty}_{\gamma-1}(I_a)$ and $v_- \in W^{1,\infty}_2(I_a)$, where $W^{k,p}_\gamma$ is the weighted Sobolev space (Definition...)

Since the nonlinear function f, g are quadratic in leading order and $w \in M^{1,\infty}_{\gamma-1}, v_- \in W^{1,\infty}_2$, we have that $F(w(s), v_-(s), s) \in L^\infty_\gamma$ and $G(w(s), v_-(s), s) \in L^\infty_\gamma$ provided that $4 > \gamma > 3 - 2^{-1}$. (need $4 > \gamma$ since we want $t^{-4}\log(t)^2 \in L^\infty_\gamma$, need $\gamma > 3 - 2^{-1}$ since we want $(d/dt + 2u_\#)^{-1}$ maps L^∞_γ to $M^{1,\infty}_{\gamma-1}$ and be bounded).

On the other hand, due to exponential localization, the operator $(d/dt - 1)^{-1} : L^\infty_2 \to W^{1,\infty}_2$ is an isomorphism.

Then, by picking $\mathcal{X} = B_1 \times B_2$ where B_1, B_2 are some balls with small enough radius centered at 0 in $M^{1,2}_{\gamma-1}$ and $W^{1,2}_\gamma$, respectively (may need to pick a large enough). Set

$$W = (w, v_-)^T, \mathcal{L}^{-1} = \begin{pmatrix} (\frac{d}{dt} + 2u_\#)^{-1} & 0 \\ 0 & (\frac{d}{dt} - 1)^{-1} \end{pmatrix}, \mathcal{F}(W,t) = (F(W,t), G(W,t))^T$$

we can use a standard fixed point argument solve the equation $W = \mathcal{L}^{-1}\mathcal{F}(W)$ for $W \in \mathcal{X}$.

(Need to fix $v = v_\# + z$ to set up a more proper equation....)

Existence for U_+ The unstable manifold will have an exponential decay as $t \to -\infty$, we prove this using a similar functional analytic framework.

Solving shilnikov BV the complicated way Now we are solving the same BV with $F(U) = \mathcal{O}(|U|^2)$ as $|U| \to 0$.

Denote by $f^T(t) := f(t-T)$ the shifted version of a function f, our ansatz is still

$$U(t) = U_-^{-T}(t) + U_+^T(t) + R(t)$$

We get the equation

$$\frac{d}{dt}R(t) - AR(t) = F((U_-^{-T} + U_+^T + R)(t)) - F(U_-^{-T}(t)) - F(U_+^T(t)).$$

The right hand side is decomposed by

$$F((U_-^{-T} + U_+^T + R)(t)) - F(U_-^{-T}(t)) - F(U_+^T(t)) = M(t)R + C + Q(R),$$

where $M(t)$ is the linearization of F at $U_-^{-T} + U_+^T$

$$M(t) = [DF(U_-^{-T}(t) + U_+^T(t))]$$

where C denotes cross term

$$C = F(U_-^{-T} + U_+^T) - F(U_-^{-T}) - F(U_+^T),$$

and $Q(R)$ is the "adjusted" quadratic nonlinearity in R

$$Q(R) = F(U_-^{-T} + U_+^T + R) - F(U_-^{-T} + U_+^T) - DF(U_-^{-T} + U_+^T)R = \mathcal{O}(|R|^2).$$

We further decompose the linear part

$$M(t) = M_c(t) + DF(U_-^{-T}(t)) + DF(U_+^T(t)) := M_c(t) + M_-(t) + M_+(t)$$

(where $M_c = DF(U_-^{-T} + U_+^T) - DF(U_-^{-T}) - DF(U_+^T)$)

Let χ_- and χ_+ be the same p.o.u introduced above then put $R_+ = \chi_+ R$ and $R_- = \chi_- R$.

We will decompose $Q(R)$ as

$$Q_c(R_-, R_+) := [Q(R) - Q(R_-) - Q(R_+)], \quad Q(R) = Q_c(R_-, R_+) + Q(R_-) + Q(R_+)$$

We get the equation for R_- and R_+ respectively,

$$\frac{d}{dt}R_-(t) - AR_-(t) = \{[M_-(t) + \chi_-[M_+(t) + M_c(t)]\}R_- + \chi_-[(M_- + M_c)R_+ + C] + Q(R_-) + \chi_- Q_c(R_-, R_+),$$

and

$$\frac{d}{dt}R_+(t) - AR_+(t) = \{M_+(t) + \chi_+[M_-(t) + M_c(t)]\}R_+ + \chi_+[(M_+ + M_c)R_- + C] + Q(R_+) + \chi_+ Q_c(R_-, R_+)$$

The final modification is to apply the shift operator S^T to the R_- equation and S^{-T} to the R_+ equation.

To slightly simplify notations, denote

$$\boldsymbol{\rho}(t) = R_-^T(t) = S^T(\chi_- R)(t) = (\chi_- R)(t-T),$$

and

$$\boldsymbol{\sigma}(t) = R_+^{-T}(t) = S^{-T}(\chi_+ R)(t) = (\chi_+ R)(t+T).$$

also set $\mathcal{M}_-(t) = M_-(t) + \chi_-[M_+(t) + M_c(t)]$ and $\mathcal{M}_+(t) = M_+(t) + \chi_+[M_-(t) + M_c(t)]$

Then the equations are

$$\left[\frac{d}{dt} - A - \mathcal{M}_-^T(t)\right]\boldsymbol{\rho}(t) = (\chi_-^T[(M_-^T + M_c^T)\boldsymbol{\sigma}^{2T} + C^T] + Q(\boldsymbol{\rho}(t)) + \chi_-^T(t)Q_c(\boldsymbol{\rho}(t), \boldsymbol{\sigma}^{2T}(t)) \tag{2.6}$$

and

$$\left[\frac{d}{dt} - A - \mathcal{M}_+^{-T}(t)\right]\boldsymbol{\sigma}(t) = (\chi_+^{-T}[(M_+^{-T} + M_c^{-T})\boldsymbol{\rho}^{-2T} + C^{-T}] + Q(\boldsymbol{\sigma}(t)) + \chi_+^{-T}(t)Q_c(\boldsymbol{\rho}^{-2T}(t), \boldsymbol{\sigma}(t)) \tag{2.7}$$

To solve the boundary value problem we solve the above two equation on the halflines $[0,\infty)$ and $(-\infty, 0]$ respectively.

To be more specific, set the function space

$$X_\rho = [C_\delta^1((0,\infty))]^2, X_\sigma = [C_\delta^1((-\infty,0))]^2$$

with norm

$$\|u\|_{X_\rho} =: \|u\|_\rho = \sup_{t\in(0,\infty)} \|e^{\delta t}[u(t) + u'(t)]\|, \quad \|u\|_{X_\sigma} =: \|u\|_\sigma = \sup_{t\in(-\infty,0)} \|e^{-\delta t}[u(t) + u'(t)]\|$$

where δ denotes exponential weights in corresponding intervals.

For the $\boldsymbol{\rho}$ equation, we need the estimate in space X_ρ.

First the perturbation of the linear part

$$\mathcal{M}_-^T(t) = M_-(t-T) + \chi_-(t-T)[M_+(t-T) + M_c(t-T)].$$

By definition, $M_-(t) = DF(U_-^{-T}(t)) = DF(U_-(t+T))$, so $M_-(t-T) = DF(U_-(t))$. While $M_+(t-T) = DF(U_+(t-2T))$. And U_- has the bound $\|U_-\|_{\gamma,-} = \sup_{t\in(0,\infty)} |U_-(t)e^{\gamma t}| < \infty$, and U_+ has the bound $\|U_+\|_{\gamma,+} = \sup_{t\in(-\infty,0)} |U_+(t)e^{-\gamma t}| < \infty$.

So for $M_-^T(t)$ we have

$$e^{\delta t} M_-^T(t) \leq Ce^{\delta t}|U_-(t)| \leq C|U_-|_{\gamma,-}e^{(\delta-\gamma)t}$$

On the other hand

$$\chi_-(t-T)M_+(t-T) = \chi_-(t-T)DF(U_+(t-2T)) \leq |\chi_-(t-T)||U_+(t-2T)|,$$

since support of $\chi_-(t-T)$ lies in $t\in(-\infty, T+1)$, for $t\in(0,\infty)$, we are considering the function $U_+(t-2T)$ on the window $t\in(0,T+1)$, and consequently, the function $U_+(\cdot)$ on the window $(-2T, -T+1)$. This gives

$$e^{\delta t}\chi_-(t-T)M_+(t-T) \leq C \sup_{t\in(-2T,-T+1)} |e^{\delta t}U_+(t)| \leq C|U_+|_{\gamma,+} \sup_{t\in(-2T,-T+1)} e^{(\delta+\gamma)t} \leq Ce^{-(\delta+\gamma)T}|U_+|_{\gamma,+}$$

Lastly, recall $M_c(t) = DF(U_-^{-T} + U_+^T) - DF(U_-^{-T}) - DF(U_+^T)$, we have the pointwise estimate (the norm is the norm of M_c as a matrix)

$$\|M_c(t)\| \leq C|U_-^{-T}(t)||U_+^T(t)|$$

Hence

$$e^{\delta t}\chi_-(t-T)M_c(t) \leq C|U_-|_{\gamma,-}e^{(\delta-\gamma)t}|U_+|_{\gamma,+}e^{-\gamma T}.$$

We therefore conclude that

$$\|e^{\delta t}\mathcal{M}^T(t)\| \leq C(|U_-|_{\gamma,-}e^{(\delta-\gamma)t} + |U_+|_{\gamma,+}e^{-(\delta+\gamma)T} + e^{-\gamma T+(\delta-\gamma)t}|U_-|_{\gamma,-}|U_+|_{\gamma,+}) \leq C(|U_-|_{\gamma,-} + |U_+|_{\gamma,+})$$

as long as we choose $0 < \delta < \gamma$, we can take supremum over all $t \in (0, \infty)$, as the right hand side is independent of t.

In a very similar vein, we obtain the estimates for the residual term

$$\begin{aligned}\left|e^{\delta t}\chi_-^T[(M_-^T + M_c^T)\boldsymbol{\sigma}^{2T} + C^T](t)\right| &\leq |U_-|_{\gamma,-}e^{(\delta-\gamma)t}\left(\sup_{t\in(-2T,-T+1)}|\boldsymbol{\sigma}(t)(1+U_+(t))| + |U_+(t)|\right) \\ &\leq |U_-|_{\gamma,-}[|\boldsymbol{\sigma}|_\sigma(1+|U_+|_{\gamma,+}) + |U_+|_{\gamma,+}]\end{aligned}$$

and the quadratic terms

$$\left|e^{\delta t}Q(\boldsymbol{\rho}(t)) + \chi_-^T Q_c(\boldsymbol{\rho}, \boldsymbol{\sigma}^{2T})(t)\right| \leq |\boldsymbol{\rho}|_\rho^2 e^{-2\delta t} + |\boldsymbol{\rho}|_\rho \sup_{t\in(-2T,-T+1)}|\boldsymbol{\sigma}(t)| \leq |\boldsymbol{\rho}|_\rho(1 + |\boldsymbol{\sigma}|_\sigma)$$

In particular, this says the right hand of equation (2.6) lies in X_ρ. And the size is controlled by

$$|U_-|_{\gamma,-}, |U_+|_{\gamma,+}, |\boldsymbol{\rho}|_\rho, |\boldsymbol{\sigma}|_\sigma,$$

moreover, the linear part is a perturbation ($\mathcal{M}^T(t)$ is small in L^∞ norm) of the operator $\dfrac{d}{dt} - A$.

The exact same thing is true for equation (2.7). We can now set up a fixed point argument by starting with $U_-, U_+, \boldsymbol{\rho}, \boldsymbol{\sigma}$ small in their respective norms, and one conclude the solvability of the equations using exponential dichotomies of the operator $\dfrac{d}{dt} - A - \mathcal{M}^T(t)$ and $\dfrac{d}{dt} - A - \mathcal{M}^{-T}(t)$, which follows by robustness.

Remark: the solution is unique up to the choice of the "initial" condition $\boldsymbol{\rho}(0)$ and $\boldsymbol{\sigma}(0)$, which are just the boundary condition for the corrector term R at $\pm T$ (after being modified by the p.o.u) and the flying time T, we see that the above argument works for any $T \geq 0$. Note also that the decaying of $\boldsymbol{\rho}, \boldsymbol{\sigma}$ at $\infty, -\infty$ closes the problem.

Nonhyperbolic gluing We are studying the equation

$$\dot{U}(t) - AU(t) = F(U(t)) \tag{2.8}$$

for $t \in (-T, T)$, $U \in \mathbb{R}^2$.

Here $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and we assume the following for the nonlinearity F

Hypothesis (N) *Write $U = (u, v)$ and $F = (F_1, F_2) = (F_1(u, v), F_2(u, v))$, then we require*

- *F is smooth,*
- *$F(0, 0) = (0, 0)$, $DF(0, 0) = 0$,*
- *$\partial_u^2 F_1(u, v) < 0$.*

By an appropriate scale in time or the variable u, we can normalize so that $\partial_u^2 F_1(u, v) = -1$.

Let P_- be the projection to the center-stable direction and P_+ the projection to unstable direction, the boundary conditions are

$$P_- U(-T) = u_0, \text{ and } P_+ U(T) = v_0, \text{ where } u_0 > 0, v_0 \in \mathbb{R}. \tag{2.9}$$

Our main result is

Theorem 2.1. *Assume A and F are given as above, then fix any $T > 0$, there exist $\varepsilon > 0$ small, so that for u_0, v_0 with $u_0 > 0, |(u_0, v_0)| < \varepsilon$ there exists a unique solution $U = U(t; T, u_0, v_0)$ to equation (2.8) on $(-T, T)$ and satisfies the boundary condition (2.9).*

Before begin the proof, we outline our methods below.

Recall that, for (2.8), we have established the existence of the following special solutions, which exist on the half lines $(0, \infty)$ and $(-\infty, 0)$.

(Notation: $p \lesssim q$ means there is a constant C such that $p \leq Cq$. Similarly we define $p \gtrsim q$, and $p \simeq q$ if $p \gtrsim q$ and $p \lesssim q$.)

Theorem 2.2. *There exist unique solutions $U_-(t)$ and $U_+(t)$ of equation (2.8) with the following properties*

(i) $U_- = (u_-, v_-)^T$, with $u_-(t) \lesssim u_0(u_0 t + 1)^{-1}$ for $t \in [0, \infty)$ and $v_- = \mathcal{O}(|u_-|^2)$ as $|u_-| \to 0$. Given any $u_0 = u_-(0)$.

(ii) $U_+ = (u_+, v_+)^T$, with $v_+(t) \leq C v_0 e^t$ for $t \in (-\infty, 0]$ and $|u_+| = \mathcal{O}(|v_+|^2)$ as $|v_+| \to 0$, given $v_0 = v_+(0)$.

For a function $f(t)$, denote $f^T(t) := f(t - T)$, the right shift of f by T. We will take an ansatz of the form

$$U(t) = U_-^{-T}(t) + U_+^T(t) + W(t)$$

where W is a correction term.

Substitute this ansatz into equation (2.8), we obtain the equation for W,

$$\dot{W} - AW = F(U_-^{-T} + U_+^T + W) - F(U_-^{-T}) - F(U_+^T),$$

for which we write in the following form

$$\dot{W} - AW = M(t)W + R + Q(W), \tag{2.10}$$

where

$$M(t) = DF(U_-^{-T}(t) + U_+^T(t)),$$

R is "residual"

$$R(t) = F(U_-^{-T}(t) + U_+^T(t)) - F(U_-^{-T}(t)) - F(U_+^T(t)),$$

and Q is quadratic in W

$$Q(W) = F(U_-^{-T} + U_+^T + W) - F(U_-^{-T} + U_+^T) - M(t)W = \mathcal{O}(|W|^2).$$

We will proceed in the following steps:

(i) Decompose $W = W_- + W_+$, $W_\pm = \chi_\pm W$ and shift in time to get separate equations,

(ii) Establishing the Fredholm properties of the linear part in the algebraically localized spaces,

(iii) Estimate the nonlinear terms in these spaces as well.

Step 1 Introduce the partition of unity χ_- and χ_+ in $(-\infty, \infty)$, where

- support of $\chi_- \subset (-\infty, 1)$ and $\chi_- = 1$ on $(-\infty, -1)$,
- $\chi_+ = 1 - \chi_-$. support of $\chi_+ \subset (-1, \infty)$ and $\chi_+ = 1$ on $(1, \infty)$.

Now set $W_+ = \chi_+ W$ and $W_- = \chi_- W$. We first We further decompose the linear part $M(t)$

$$M(t) = M_c(t) + DF(U_-^{-T}(t)) + DF(U_+^T(t)) := M_c(t) + M_-(t) + M_+(t)$$

(where $M_c = DF(U_-^{-T} + U_+^T) - DF(U_-^{-T}) - DF(U_+^T)$), then we decompose the quadratic part $Q(W)$ as

$$Q_c(W_-, W_+) := [Q(W) - Q(W_-) - Q(W_+)], \quad Q(W) = Q(W_- + W_+) = Q_c(W_-, W_+) + Q(W_-) + Q(W_+).$$

Denote

$$\boldsymbol{\rho}(t) = W_-^T(t) = (\chi_- W)(t - T),$$

and

$$\boldsymbol{\sigma}(t) = W_+^{-T}(t) = (\chi_+ W)(t + T).$$

Note that supp $\boldsymbol{\rho} \subset (-\infty, T+1)$ and supp $\boldsymbol{\sigma} \subset (-T-1, \infty)$. We get the equations

$$\left[\frac{d}{dt} - A - \mathcal{M}_-^T(t)\right] \boldsymbol{\rho}(t) = (\chi_-^T[(M_-^T + M_c^T)\boldsymbol{\sigma}^{2T} + R^T] + Q(\boldsymbol{\rho}(t)) + \chi_-^T Q_c(\boldsymbol{\rho}(t), \boldsymbol{\sigma}^{2T}(t)), \tag{2.11}$$

and

$$\left[\frac{d}{dt} - A - \mathcal{M}_+^{-T}(t)\right] \boldsymbol{\sigma}(t) = (\chi_+^{-T}[(M_+^{-T} + M_c^{-T})\boldsymbol{\rho}^{-2T} + R^{-T}] + Q(\boldsymbol{\sigma}(t)) + \chi_+^{-T} Q_c(\boldsymbol{\rho}^{-2T}(t), \boldsymbol{\sigma}(t)). \quad (2.12)$$

Where

$$\mathcal{M}_-(t) = M_-(t) + \chi_-(t)[M_+(t) + M_c(t)]$$

and

$$\mathcal{M}_+(t) = M_+(t) + \chi_+(t)[M_-(t) + M_c(t)]$$

Now, we will solve equation (2.11) on the interval $(0, \infty)$ and solve (2.12) on the interval $(-\infty, 0)$ by setting up fixed point equations in appropriate function spaces.

Step 2 We write the linear operator $\partial_t - A - \mathcal{M}_-^T(t)$ in the form

$$\left(\partial_t - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + M_-^T(t)\right) + \chi_-^T[M_+ + M_c] := \mathcal{L}_- + \mathcal{P}_-$$

Introduce the following algebraically weighted space (Kondratiev spaces), which increases localization as one taking derivatives

$$M_\gamma^{m,p}(\Omega) = \{u(t) \in L^1_{loc}(\Omega); \quad \langle t\rangle^{\gamma+k}\partial_t^k u \in L^p(\Omega), \quad k \leq s, k \in \mathbb{Z}\}$$

where $p \in (1, \infty]$.

We have the following Fredholm properties:

Theorem 2.3. *$\mathcal{L}_- : \mathcal{D} = M_{\gamma-1}^{1,\infty}(0,\infty) \times W_{\tilde{\gamma}}^{1,\infty}(0,\infty) \subset L_{\gamma-1}^\infty(0,\infty) \times L_{\tilde{\gamma}}^\infty(0,\infty) \to L_\gamma^\infty(0,\infty) \times L_{\tilde{\gamma}}^\infty(0,\infty)$ is Fredholm with index 1. Here $\tilde{\gamma}$ satisfies $\tilde{\gamma} < \gamma < \tilde{\gamma} + 1$.*

Proof. Recall $M_-(t)$ is the linearization at the center-stable manifold $U_-(t)$, for which we can always normalize so that the nonlinear terms starts with $[F(u,v)]_1 = f(u,v) = -u^2 + h.o.t.$ Then we have the following

$$[\mathcal{M}_-]_{11}(t) = -2u_-(t) + \mathcal{O}(t^{-2}) = \frac{-2u_0}{u_0 t + 1} + \mathcal{O}(t^{-2})$$

Where $u_0 > 0$ is the initial condition $u_-(0)$.

The resolvent equation $\mathcal{L}_-(u,v)^T = (f,g)^T$ is decoupled into

$$\dot{u} + p(t)u = f \tag{2.13}$$

$$\dot{v} - v = g \tag{2.14}$$

with $p(t) = 2/(t + u_0^{-1}) + \mathcal{O}(t^{-2})$ for all $t \geq 0$. The equation is to be solved on the half line $[0, \infty)$.

Using the variation of constants formula, denote $I(t) = e^{\int_0^t p(s)ds}$ the integration factor of equation (2.13). We get

$$u(t) = I(t)^{-1} I(\tau) u(\tau) + I(t)^{-1} \int_\tau^t I(s) f(s) ds$$

from the expansion of $p(t)$, we get the following asymptotics for $I(t)$:

$$I(t) = e^{\mathcal{O}(t^{-1})}(t + u_0^{-1})^2 \text{ as } t \to \infty$$

To get the decaying solution, we want $u(\tau) \to 0$ as $\tau \to \infty$, hence the voc formula becomes

$$u(t) = -I(t)^{-1} \int_t^\infty I(s)f(s)ds$$

Since $f(t) \in L^\infty_\gamma(0,\infty)$ with norm $\|f\|_\gamma = |\langle t\rangle^\gamma f(t)|_\infty$ where $\langle t\rangle = \sqrt{t^2+1}$.

we have

$$\begin{aligned}|t^{\gamma-1}u(t)| &\le t^{\gamma-1}I(t)^{-1}\int_t^\infty |I(s)f(s)|ds \le t^{\gamma-1}I(t)^{-1}\int_t^\infty C(s+u_0^{-1})^2(s+1)^{-\gamma}ds \\ &\le Ct^{\gamma-1}I(t)^{-1}t^{3-\gamma} \simeq Ct^2 I(t)^{-1} \le C \text{ as } t \to \infty\end{aligned}$$

provided that $\gamma - 2 > 0$, or $\gamma - 1 > 1$.

So, if $\gamma > 2$, and given $f \in L^\infty_\gamma(0,\infty)$, we can solve to get a unique solution. Moreover, inspecting the homogeneous equation $\dot{u} + p(t)u = 0$, we see the solution is given by $u(t) \simeq I(t)^{-1} \simeq t^{-2}$ as $t \to \infty$. This provides a one-dimensional kernel in $M^{1,\infty}_{\gamma-1}$ for $\gamma < 3$.

Range is closed: let $f_n \in Rg(\partial_t - p(t)) \subset L^\infty_\gamma$ and $f_n \to f$ in L^∞_γ. This means $|\langle t\rangle^\gamma(f_n - f)|_\infty \to 0$ as $n \to \infty$. For $2 < \gamma < 3$, we get $u_n = (\partial_t - p(t))^{-1}f_n$, the unique decaying solution in $M^{1,\infty}_{\gamma-1} \subset L^\infty_{\gamma-1}$, then

$$\|\langle t\rangle^{\gamma-1}(u_n - u_m)(t)\|_\infty \le C\|f_n - f_m\|_{\gamma,\infty} \to 0.$$

So that $u_n \to u$ for some $u \in M^{1,\infty}_{\gamma-1}$ and we must have $\dot{u} - p(t)u = f$.

The other equation (2.14), is solved also by voc, with the explicit formula

$$v(t) = -\int_t^\infty e^{t-s}f(s)ds,$$

for $f(s) \simeq s^{-\gamma}$, we get

$$|v(t)| \le C\int_t^\infty (s-t)^{-1}s^{-\gamma}ds \le Ct^{-\gamma}$$

for any $\gamma > 0$. Moreover, the kernel consist of scalar multiplication of the function e^t, which is not in the space $L^\infty_\gamma(0,\infty)$ for any $\gamma > 0$. This shows the second component is an invertible operator from $W^{1,\infty}_\gamma \to L^\infty_\gamma$ for any $\gamma > 1$. Note that we do not lose any localization on the second component. □

Theorem 2.4. *The operator $\mathcal{L}_- + \mathcal{P}_- : \mathcal{D} \to L^\infty_\gamma(0,\infty) \times L^\infty_\gamma(0,\infty)$ is Fredholm with index 1. Moreover, given $f \in L^\infty_\gamma(0,\infty) \times L^\infty_\gamma(0,\infty)$, and $u_0 \times \mathbb{R}^2$, there exists a unique solution u to $\mathcal{L}_-u = f$, with the estimate*

$$\|u\|_{\mathcal{D}} \le C(\|f\|_\gamma + |u_0|)$$

for some constant C.

Proof. We have seen that the diagonal operator $\frac{d}{dt} - diag(1/t, 1)$ has a pseudo inverse on the space $\mathcal{D}$, we need to check the off-diagonal multiplication operators are small perturbations on the respected space. □

Next, we study the linearization at the unstable manifold, we write the linear operator $\dfrac{d}{dt} - A - \mathcal{M}_+^{-T}(t)$ in the form

$$\left(\frac{d}{dt} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} [\mathcal{M}_+]_{11} & [\mathcal{M}_+]_{12} \\ [\mathcal{M}_+]_{21} & [\mathcal{M}_+]_{22} \end{pmatrix} := \mathcal{L}_+ + \mathcal{P}_+$$

recall we had

$$\mathcal{M}_+(t) = M_+(t) + \chi_+[M_-(t) + M_c(t)], \text{ with } M_+(t) = DF(U_+^T(t)).$$

So that $\mathcal{M}_+^{-T}(t)$ has the asymptotics

$$\mathcal{M}_+^{-T}(t) = DF(U_+(t)) + \chi_+^{-T}[M_-^{-T} + M_c^{-T}] = \mathcal{O}(e^t)$$

as $t \to -\infty$. Since χ_+^{-T} has support on $(-1-T, \infty)$.

We wish to establish Fredholm property of the linear operator

$$\frac{d}{dt} - A - \mathcal{M}_+^{-T}(t)$$

on the space $\mathcal{D} = M_{\gamma-1}^{1,\infty}(-\infty, 0) \times W_{\tilde{\gamma}}^{1,\infty}(-\infty, 0) \to L_\gamma^\infty(-\infty, 0) \times L_{\tilde{\gamma}}^\infty(-\infty, 0)$.

Theorem 2.5. *The operator $\mathcal{L}_+$ is Fredholm from $\mathcal{D}$ to... with index* 1

Proof. TBD □

Step 3

(i) Residual term. Here we want to show the residual term is small in the right hand side is small in the space $L_\gamma^\infty(0,\infty) \times L_\gamma^\infty(0,\infty)$. If $\boldsymbol{\rho} \in M_{\gamma-1}^{1,\infty}(0,\infty) \times W_\gamma^{1,\infty}(0,\infty)$ and $\boldsymbol{\sigma} \in M_{\gamma-1}^{1,\infty}(-\infty,0) \times W_\gamma^{1,\infty}(-\infty,0)$. We need to estimate the following quantity

$$\sup_{t\in[0,\infty)} \left|\langle t\rangle^\gamma \chi_-^T[(M_-^T + M_c^T)\boldsymbol{\sigma}^{2T} + R^T](t)\right| = \sup_{t\in[0,T+1)} \left|\langle t\rangle^\gamma[(M_-^T + M_c^T)\boldsymbol{\sigma}^{2T} + R^T]\right|$$

(a) $\langle t\rangle^\gamma[M_-^T(t) + M_c^T(t)]\boldsymbol{\sigma}^{2T}(t)$

We have $\sup_{t\in(0,T+1)} |\boldsymbol{\sigma}^{2T}(t)| \le \sup_{t\in(-2T,-T+1)} |\boldsymbol{\sigma}(t)| \le |\boldsymbol{\sigma}||T-1|^{1-\gamma}$

(recall that the norm of $\boldsymbol{\sigma}$ satisfies $\sup_{t\in(-\infty,0)} |\langle t\rangle^{\gamma-1}\boldsymbol{\sigma}(t)| \le |\boldsymbol{\sigma}|$)

On the other hand

$$|\langle t\rangle^\gamma M_-^T(t)| \lesssim |\langle t\rangle^\gamma U_-(t)| \le |u_0(u_0 t+1)^{-1}\langle t\rangle^\gamma| \le C\langle t\rangle^{\gamma-1}$$

Hence $\sup_{t\in(0,\infty)} |\langle t\rangle^\gamma \chi_-^T(t) M_-^T(t)\boldsymbol{\sigma}^{2T}(t)| \le C|\boldsymbol{\sigma}|$

For the crossed part M_c recall

$$\|M_c(t)\| \le C|U_-^{-T}(t)||U_+^T(t)|, \text{ so that } \|M_c^T(t)\| \le C|U_-(t)||U_+^{2T}(t)|$$

Therefore we know that

$$\begin{aligned}\sup_{t\in[0,\infty)} |\langle t\rangle^\gamma \chi_-^T(t) M_c^T(t)\boldsymbol{\sigma}^{2T}(t)| &= \sup_{t\in[0,T+1)} |\langle t\rangle^\gamma M_c^T(t)\boldsymbol{\sigma}^{2T}(t)| \\ &\le \sup_{t\in[0,T+1)} |\langle t\rangle^\gamma U_-^{-T}(t) U_+^{2T}(t)\boldsymbol{\sigma}^{2T}(t)| \\ &= |\boldsymbol{\sigma}||U_+|_+\langle T+1\rangle^{\gamma-1}e^{-(T-1)}|T-1|^{1-\gamma} = \mathcal{O}(|\boldsymbol{\sigma}||U_+|_+)\end{aligned}$$

(b) For the "pure" residual part R, we need to estimate the quantity

$$\sup_{t\in[0,T+1)} \left|\langle t\rangle^\gamma R^T(t)\right| = \sup_{t\in[0,T+1)} \left|\langle t\rangle^\gamma [F(U_-(t)+U_+^{2T}(t)) - F(U_-(t)) - F(U_+^{2T}(t))]\right|$$

recall, since F is quadratic, the following estimate, which is similar to that of M_c, hold pointwise

$$|F(U_-(t)+U_+^{2T}(t)) - F(U_-(t)) - F(U_+^{2T}(t))| \leq C|U_-(t)||U_+^{2T}(t)|$$

consequently we have

$$\sup_{t\in[0,T+1)} \left|\langle t\rangle^\gamma R^T(t)\right| \leq |U_+|\langle T+1\rangle^{\gamma-1}e^{-(T-1)} = \mathcal{O}(|U_+|).$$

(ii) Lastly, one estimates the quadratic terms.

$$Q(\boldsymbol{\rho}(t)) + \chi_-^T(t)Q_c(\boldsymbol{\rho}(t), \boldsymbol{\sigma}^{2T}(t))$$

We want this to be small in $L_\gamma^\infty(0,\infty)\times L_\gamma^\infty(0,\infty)$, we have

$$|Q(\boldsymbol{\rho}(t))| \leq C|\boldsymbol{\rho}(t)|^2$$

recall

$$Q_c(R_-,R_+) := [Q(R) - Q(R_-) - Q(R_+)], \qquad Q(R) = Q_c(R_-,R_+) + Q(R_-) + Q(R_+)$$

hence the following holds

$$|Q_c(\boldsymbol{\rho}(t), \boldsymbol{\sigma}^{2T}(t))| \leq C|\boldsymbol{\rho}(t)||\boldsymbol{\sigma}^{2T}(t)|$$

Now $|\boldsymbol{\rho}(t)| \leq |\boldsymbol{\rho}|_{M_{\gamma-1}\times W_\gamma}\langle t\rangle^{1-\gamma}$, so $\langle t\rangle^\gamma Q(\boldsymbol{\rho}(t)) \leq |\boldsymbol{\rho}|\langle t\rangle^{2-\gamma}$.
Since $\gamma > 2$, $\langle t\rangle^{2-\gamma}$ is bounded (decays at infinity) on the half line $[0,\infty)$.
Finally, $\sup_{t\in[0,T+1)}\langle t\rangle^\gamma|\boldsymbol{\rho}(t)||\boldsymbol{\sigma}^{2T}(t)| \leq \langle T+1\rangle^{2-\gamma}|\boldsymbol{\rho}||\boldsymbol{\sigma}|$

Solving the system Similarly we obtain an estimate for the $\boldsymbol{\sigma}$ equations, We then proceed to set up an fixed point argument, utilizing the above estimates on the linear, residual, and quadratic part, and conclude a unique solution (up to the choice of initial value) exists in the space $M_{\gamma-1}^{1,\infty}(0,\infty)\times W_\gamma^{1,\infty}(0,\infty)$.

Of theorem 2.1. We have done the necessary estimates in ... □

Variation of constant to solve $\dot{u} - u = f$, we have

$$u(t) = e^{t-\tau}u(\tau) + \int_\tau^t e^{-s}f(s)ds$$

to find a bounded solution on $t\in(0,\infty)$, we let $\tau\to\infty$, since we want bounded u, linear term drops out, we get $u(t) = \int_\infty^t e^{-s}f(s)ds$.
Note we have no kernel, to see there isn't a cokernel, i.e. $\dim(\ker \frac{d}{dt}^* - 1)^\perp = 0$. Suppose $v \perp \ker \frac{d}{dt} + 1$, so that $\int_0^\infty e^{-t}v(t)dt = 0$ but v should have a sign... impossible.
$\int Auv = \int[\dot{u} - u]v = \int -\dot{v}u - uv = \int uA^*v$

3 Model problem for passage through the fold

We want to first apply the functional-analytic approach to the following problem:

$$\begin{aligned}\dot{u} &= \mu + u^2 + (u^4) \\ \dot{\mu} &= \varepsilon + (\varepsilon u)\end{aligned} \tag{3.1}$$

with boundary condition

$$u(T) = \delta \text{ and } \mu(0) = -\delta, \tag{3.2}$$

where T is another parameter, the "time of flight" for the trajectory to shoot from $\mu = -\delta$ to $u = \delta$.

We first study the "blow up " problem, starting with rescale $u = \varepsilon^{1/3}u_1(\varepsilon^{1/3}t)$ and $\mu = \varepsilon^{2/3}\mu_1(\varepsilon^{1/3}t)$. We get the new equations (set $\tau = \varepsilon^{1/3}t$)

$$\begin{aligned}\partial_\tau u_1 &= \mu_1 + u_1^2 + (\varepsilon^{2/3}u_1^4) \\ \partial_\tau \mu_1 &= 1 + (\varepsilon^{1/3}u_1)\end{aligned} \tag{3.3}$$

The new boundary condition is

$$u_1(T) = \delta\varepsilon^{-1/3}, \mu_1(0) = -\delta\varepsilon^{-2/3} \tag{3.4}$$

Then if we set $s = \tau - \delta\varepsilon^{-2/3}$ and formally let $\varepsilon \to 0$, equation (3.3) has an explicit solution $u_1(\tau) = u_R(s)$ and $\mu_1(\tau) = s$. Where u_R is the unique solution to the riccati equation $\partial_s u_R = s + u_R^2$ with the specific asymptotics [reference].

$$u_R(s) = \begin{cases} (T_R - s)^{-1} + \mathcal{O}(T_R - s), \text{ as } s \to T_R \\ -(-s)^{1/2} - \frac{1}{4}(-s)^{-1} + \mathcal{O}(|s|^{-3/2}), \text{ as } s \to -\infty \end{cases} \tag{3.5}$$

From this and the boundary condition (3.4), we have the asymptotics for T:

$$T(\varepsilon) = \delta\varepsilon^{-1} + T_R\varepsilon^{-1/3} - \delta^{-1} + \mathcal{O}(\varepsilon^{2/3}) \tag{3.6}$$

Towards solving the boundary value problem, we start by setting up a perturbative problem, for $\varepsilon > 0$ small. Insert the ansatz $u_1(\tau) = u_R(s) + v(\tau)$ and $\mu_1(\tau) = s + \rho(\tau)$ into (3.3). We find

$$\begin{aligned}\partial_\tau v &= 2u_R v + \rho + v^2 + \varepsilon^{2/3}(u_R + v)^4 \\ \partial_\tau \rho &= \varepsilon^{1/3}(u_R + v)\end{aligned} \tag{3.7}$$

We then do another rescaling to put the time interval into $(-\infty, \infty)$, we define a new time variable $\sigma \in (-\infty, \infty)$ which is related to the time s via a smooth map $\psi : (-\infty, \infty) \to (-\infty, T_R)$ with the properrites:

$$s = \psi(\sigma) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, \text{ for } \sigma \leq -M \\ T_R - e^{-\sigma}, \text{ for } \sigma \geq M, \end{cases} \tag{3.8}$$

and smooth interpolation in between.

With this new time variable, set $\varphi(\sigma) = \psi'(\sigma)$ and $a(\sigma) = 2\varphi(\sigma)u_R(\psi(\sigma))$, we obtain

$$\begin{aligned}\partial_\sigma v &= av + \varphi\rho + \varphi[v^2 + \varepsilon^{2/3}(u_R + v)^4] \\ \partial_\sigma \rho &= \varphi\varepsilon^{1/3}(u_R + v)\end{aligned} \tag{3.9}$$

Using the asymptotics for ψ and u_R, we calculate that

$$\varphi(\sigma) = \begin{cases}(-\frac{3}{2}\sigma)^{-1/3}, \text{ as } \sigma \to -\infty \\ e^{-\sigma}, \text{ as } \sigma \to \infty.\end{cases} \tag{3.10}$$

$$u_R(\psi(\sigma)) = \begin{cases}-(-\frac{3}{2}\sigma)^{1/3}, \text{ as } \sigma \to -\infty \\ e^{\sigma}, \text{ as } \sigma \to \infty.\end{cases} \tag{3.11}$$

$$a(\sigma) = \begin{cases}-2 + \mathcal{O}((-\sigma)^{-3/2}), \text{ as } \sigma \to -\infty \\ 2 + \mathcal{O}(e^{-2\sigma}), \text{ as } \sigma \to \infty.\end{cases} \tag{3.12}$$

so the linear operator $\mathcal{L} = \frac{d}{d\sigma} - A(\sigma)$, where $A(\sigma) \to A_\pm = \operatorname{diag}(\pm 2, 0)$ as $\sigma \to \pm\infty$. We need to find the right function spaces.

3.1 function spaces

Recall we want to solve the boundary value problem for the time variable $t \in (0, T)$, using the asymptotics we calculated, we have that

$$\sigma \in \left(-\frac{2}{3}\delta^{3/2}\varepsilon^{-1}, -\frac{1}{3}\log(\varepsilon) + \mathcal{O}(1)\right)$$

where the $\mathcal{O}(1)$ term start with $\log(\delta)$.

We need yet another rescaling in order to solve equation (3.9) on function space independent of ε.

Define $\tilde{v} = \varepsilon^{-1/3}v$ and $\tilde{\rho} = \varepsilon^{-1/3}\rho$. They satisfy

$$\begin{aligned}\partial_\sigma \tilde{v} &= a\tilde{v} + \varphi\tilde{\rho} + \varepsilon^{1/3}\varphi[\tilde{v}^2 + (u_R + \varepsilon^{1/3}\tilde{v})^4] \\ \partial_\sigma \tilde{\rho} &= \varphi(u_R + \varepsilon^{1/3}\tilde{v})\end{aligned} \tag{3.13}$$

Function space for $\sigma \leq 0$ Investigating the ρ equation and the term $\rho\varphi$ in the v equation, we need

$$X_- = \{(v, \rho)^T \mid v \in M^{1,\infty}_{-\frac{2}{3}}(\mathbb{R}_-), \rho \in M^{1,\infty}_{-1}(\mathbb{R}_-)\}$$

Function space for $\sigma \geq 0$ On the positive real axis, we set

$$X_+ = \{(v, \rho)^T \mid W_{-2}. \sup_{0 \leq \sigma \leq \infty} |\varepsilon^{-2/3}e^{-3\sigma}v(\sigma)| + \sup_{0 \leq \sigma \leq \infty} \varepsilon^{-1/3}|\sigma^{-1}\rho(\sigma)| < \infty\}$$

Now set $X = \{\chi_-(v, \rho) \in X_-, \chi_+(v, \rho) \in X_+\}$ where $\chi_\pm$ is a smooth partition of unity with $supp\chi_+ \subset (-1, \infty)$ and $\chi_-(x) = \chi_+(-x)$.

Fredholm property of the linear part The linear part is

$$\mathcal{L}(v,\rho)^T(\sigma) = \left[\frac{d}{d\sigma} - A(\sigma)\right](v,\rho)^T(\sigma)$$

here $A(\sigma) = \begin{pmatrix} a(\sigma) & \varphi(\sigma) \\ \varepsilon^{1/3}\varphi(\sigma) & 0 \end{pmatrix}$. Considered on $\mathcal{D}(\mathcal{L})$, into X.

We have

Theorem 3.1. *$\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset X \to X$ is Fredholm with index 1 and onto.*

Proof. Since $\varphi(\sigma) \to 0$ as $\sigma \to \pm\infty$, $\mathcal{L} = \frac{d}{d\sigma} - A(\sigma)$ is a compact perturbation of the operator $\mathcal{L}_\infty = \frac{d}{d\sigma} - A_\infty$ where $A_\infty(\sigma) = \mathrm{diag}(2,0)$ for $\sigma \geq 0$ and $A_\infty(\sigma) = \mathrm{diag}(-2,0)$ for $\sigma < 0$. We shall count the Fredholm index of $\mathcal{L}_\infty$ instead.

Since $\mathcal{L}_\infty$ is of the form $\begin{pmatrix} \frac{d}{d\sigma} - a_\infty & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$, where $a_\infty(\sigma) = 2$ for $\sigma \geq 0$ and equals -2 for $\sigma < 0$. Acting on the space $M_{\gamma-1}$

For the $v-$component, the index is given by the differences between the morse index [Stability Dynamical spectral, ...], for the $\sigma \geq 0$ direction, the weight is 3, the matrix eigenvalue is $2 < 3$, hence the morse index is 0; for the $\sigma \leq 0$ direction, the weight is 0, the matrix eigenvalue is $-2 < 0$, so the morse index is again 0. Thus the index for the $v-$ component operator is 0.

For the $\rho-$ component, by spliting the problem to the positive and negative half line, the Fredholm index can be computed as

$$i\left(\frac{d}{d\sigma}^{+}\right) + i\left(\frac{d}{d\sigma}^{-}\right) - 1$$

where $\frac{d}{d\sigma}^{\pm}$ means the operator acting on $\mathbb{R}^{\pm}$, respectively. Accroding to the [W,G,S] paper, their Fredholm index are both equal to 1, hence $i = 1$.

Thus the total index of $\mathcal{L} = 0 + 1 = 1$, as climed. The kernel of $\mathcal{L}$ is at most one-dimensional, it has to be 1, hence $\mathcal{L}$ is onto, this finishes the proof. □

3.2 Contraction mapping

Equation (3.9) is written as

$$\mathcal{L}(v,\rho)^T = \varphi(N_1, N_2)^T(v,\rho,\varepsilon)$$

with $N_1 = v^2 + \varepsilon^{2/3}(u_R + v)^4); N_2 = \varphi\varepsilon^{1/3}u_R$

we know at $\varepsilon = 0$, (v,ρ) is a solution, we want to continue this solution for $\varepsilon > 0$.

Since $\mathcal{L}$ is Fredholm with index 1, we use the bordering lemma to make an operator $\tilde{\mathcal{L}}$ which is invertible.

We define $\tilde{\mathcal{L}}(v,\rho)^T$ as the map $(v,\rho)^T \mapsto (\mathcal{L}(v,\rho)^T, \rho(0))$.

Nonlinear parts, we make sure that the term $v-$components

$$\varphi v^2, \varphi\varepsilon^{2/3}(u_R+v)^4,$$

$\rho-$components

$$\varepsilon^{1/3}\varphi u_R$$

are all compatible the the norms.

For $\sigma > 0$, we check

$$|\varepsilon^{-2/3}e^{-3\sigma}\varphi v^2| \le |\varphi v||v|_+ \le e^{-\sigma}\varepsilon^{2/3}e^{3\sigma}|v|_+ < C|v|_+,$$

and (using $(x+y)^4 \le C(x^4+y^4)$ for some C)

$$|\varepsilon^{-2/3}e^{-3\sigma}\varphi\varepsilon^{2/3}(u_R+v)^4| \le C|e^{-4\sigma}(u_R^4+v^4)| \le C + Ce^{-4\sigma}v^4 \le C + C\varepsilon^{8/3}e^{8\sigma}|v|_+$$

for $\sigma \le -(1/3)\log\varepsilon$.

For $\sigma < 0$, we have

$$|\varepsilon^{-1/3}\sigma^{-2/3}\varphi v^2| \le |\varphi v||v|_- \le |(-\frac{3}{2}\sigma)^{-1/3}\varepsilon^{1/3}\sigma^{2/3}| \le C|\varepsilon^{1/3}\sigma^{1/3}| < \infty$$

and

$$\begin{aligned}|\varepsilon^{-1/3}\sigma^{-2/3}\varphi\varepsilon^{2/3}(u_R^4+v^4)| &\le |\varepsilon^{1/3}\sigma^{-2/3}(-\frac{3}{2}\sigma)^{4/3}(-\frac{3}{2}\sigma)^{-1/3}| + |\varepsilon^{1/3}\sigma^{-2/3}\varphi v^4| \\ &\le |\varepsilon^{1/3}\sigma^{1/3}| + |\varepsilon^{1/3}\sigma^{-2/3}(-\frac{3}{2}\sigma)^{-1/3}\varepsilon^{4/3}\sigma^{8/3}| \\ &\le C + C|\varepsilon^{5/3}\sigma^{5/3}| < \infty\end{aligned}$$

for $0 \ge \sigma \ge -1/\varepsilon$.

The ρ components are easier to check, in fact, the choice for the norm is motivated by looking at the right hand side of the ρ equations first.

Small LIpshcitz constant

From the above estimates we see that the term φv^2 and φv^4 satisfies

for $\sigma > 0$,

$$|v_1^2 - v_2^2|_+ = |\varepsilon^{-2/3}e^{-3\sigma}\varphi(v_1^2 - v_2^2)| \le \varphi|v_1+v_2||v_1-v_2|_+ \le C|v_1-v_2|_+$$

and

$$|\varepsilon^{-2/3}e^{-3\sigma}\varphi(v_1^4 - v_2^4)| \le \varphi(\sum_{i+j=3} v_1^i v_2^j)|v_1-v_2|_+ \le C|v_1-v_2|_+$$

If v_1, v_2 are close to 0 in the weighted norm, C can be made less than 1, for example, say v_1, v_2 are such that $|v_1|_+, |v_2|_+ < \eta$, then $|v_{1,2}(\sigma)| \le \eta\varepsilon^{2/3}e^{3\sigma}$, so that $\varphi|v_1+v_2(\sigma)| \le 2\eta\varepsilon^{2/3}e^{2\sigma} \le 2\eta < 1$, for $\sigma \le -(1/3)\log\varepsilon$, if $\eta < 1/2$. To start the contraction mapping, introduce the cut off function $\chi_\varepsilon(\sigma)$ which has support in $(-\varepsilon^{-1}, -\frac{1}{3}\log(\varepsilon))$.

Define $\mathcal{F}(v,\rho,\varepsilon) = \varphi(N_1,N_2)^T\chi_\varepsilon$, then our equation is

$$\mathcal{L}(v,\rho)^T = \mathcal{F}(v,\rho,\varepsilon)$$

4 Reference

Equation

$$
\begin{aligned}
\frac{d}{dt}u(t) &= (\mu + u^2 + u^3)(t) \\
\frac{d}{dt}\mu(t) &= \varepsilon
\end{aligned}
\tag{4.1}
$$

with B.C.

$$
\mu(0) = -\delta, \quad u(T) = \delta. \tag{4.2}
$$

where δ, ε, T are parameters.

(i) **Region 1**

- Ansatz and rescale in time

$$
u_-(t) = \varepsilon^{1/3} u_R(\tau(t) - \tau_0), \quad \mu(t) = \varepsilon t - \delta = \varepsilon^{2/3}(\tau - \tau_0).
$$

With

$$
\tau(t) = \varepsilon^{1/3} t, \quad \tau_0 = \varepsilon^{-2/3}\delta.
$$

After define $s := \tau - \tau_0$, $(u_R, s)^T$ solves

$$
\frac{d}{ds}u_R(s) = s + u_R(s)^2, \quad \frac{d}{ds}s = 1
$$

Which implies

$$
\frac{d}{dt}u_-(t) = \mu(t) + u_-(t)^2
$$

Yet another rescaling in time σ, defined via

$$
s = \psi(\sigma) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, \text{ for } \sigma \leq -M \\ \Omega_0 - e^{-\sigma}, \text{ for } \sigma \geq M, \end{cases}
$$

and smooth interpolation in between, here Ω_0 is the blow-up time for $u_R(s)$. Note if $\varphi := \frac{d}{d\sigma}\psi(\sigma)$, then

$$
\varphi \frac{d}{ds} = \frac{d}{d\sigma}, \text{ and } \varepsilon^{-1/3}\varphi \frac{d}{dt} = \frac{d}{d\sigma}
$$

- Asymptotics for u_R and φ.

$$
\varphi(\sigma) = \begin{cases} (-\frac{3}{2}\sigma)^{-1/3}, \text{ as } \sigma \to -\infty \\ e^{-\sigma}, \text{ as } \sigma \to \infty. \end{cases}
$$

$$
u_R(\psi(\sigma)) \to \begin{cases} -(-\frac{3}{2}\sigma)^{1/3}, \text{ as } \sigma \to -\infty \\ e^{\sigma}, \text{ as } \sigma \to \infty. \end{cases}
$$

$$
2u_R\varphi(\sigma) \to \begin{cases} -2 + \mathcal{O}((-\sigma)^{-3/2}), \text{ as } \sigma \to -\infty \\ 2 + \mathcal{O}(e^{-2\sigma}), \text{ as } \sigma \to \infty. \end{cases}
$$

- FP argument petrubation

$$u(t) = \varepsilon^{1/3}(u_R + v)(\sigma), \quad \mu(t) = \varepsilon^{2/3}(s+\rho)(\sigma).$$

Equation for (v,ρ)

$$\frac{d}{d\sigma}v = 2(u_R\varphi)v + \varphi v^2 + \varphi\rho + \varepsilon^{1/3}\varphi(u_R+v)^3, \quad \rho = 0.$$

- Gluing time
the gluing time σ_* is set to equal to $\log(\varepsilon^{-1/6}\delta)$, notice in terms of the original time t, this is at

$$s(\sigma_*) = \Omega_0 - \delta^{-1}\varepsilon^{1/6} = \tau - \tau_0 = \varepsilon^{1/3}t - \varepsilon^{-2/3}\delta \implies t = t_* := \varepsilon^{-1/3}[\Omega_0 + \varepsilon^{-2/3}\delta - \delta^{-1}\varepsilon^{1/6}]$$

We note then

$$u_* := u_-(t_*) = \varepsilon^{1/3}[(\Omega_0 - (\Omega_0 - \delta^{-1}\varepsilon^{1/6}))^{-1} + \mathcal{O}(\varepsilon^{1/6})] = \varepsilon^{1/6}\delta + \mathcal{O}(\varepsilon^{1/2})$$

- norms
We will stop at $\sigma = \sigma_*$, decide norm from the nonhomogeneous term $\varepsilon^{1/3}\varphi u_R^3$. We have for $0 \le \sigma \le \sigma_*$, that

$$\sup_{\sigma\le\sigma_*} \varepsilon^{1/3}\varphi u_R^3 \le \varepsilon^{1/3}e^{2\sigma_*} = \delta = \mathcal{O}_\varepsilon(1)$$

This is the nonhomogeneous term, so we just need to use the usual sup norm.

(ii) **Region 2**

- Ansatz and rescale

$$u_+(t) = (u_*^{-1} + t_* - t)^{-1}$$

so that we have $u_+ = u_- = u_*$ at $t = t_*$. Moreover, we have

$$|u_-(t) - u_+(t)| \simeq \mathcal{O}(\varepsilon^{2/3}|t - t_*|) = \mathcal{O}(\varepsilon^{2/3})$$

for t close enough to t_*. (show using a Gronwall argument).
Recall that we stop when $u_+(t = T) = \delta$, we need to consider the interval $[t_*, T]$. This gives the asymtotics for T,

$$T \sim \delta\varepsilon^{-1/6} + t_* - \delta^{-1} = \varepsilon^{-1/3}\Omega_0 + \varepsilon^{-1}\delta - \delta^{-1}.$$

We introduce the time ξ with the scaling

$$e^{-\xi} = u_+(t)^{-1}$$

- Asymptotics

$$u_+(t) = e^{\xi}$$

As for $\mu(t) = \varepsilon t - \delta$, we have

$$\begin{aligned}\mu &= \varepsilon(t_* + \delta\varepsilon^{-1/6} - u_+^{-1}) - \delta = \varepsilon^{2/3}\Omega_0 + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon u_+^{-1} \\ &= \Omega_0\varepsilon^{2/3} + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon e^{-\xi}\end{aligned}$$

- Gluing time

  We will glue at $\xi = \xi_*$, defined via

$$e^{\xi_*} = u_+(t_*) = \varepsilon^{1/6}\delta^{-1} + \mathcal{O}(\varepsilon^{1/2}) \implies \xi_* = \log(\varepsilon^{1/6}\delta^{-1}) + ...$$

- FP argument with ansatz

$$u(t) = u_+(t) + w(t)$$

  By definition, $u_+(t)$ solves

$$\frac{d}{dt}u_+(t) = u_+(t)^2,$$

  Convert the equation in ξ time via

$$e^{-\xi}\frac{d}{dt} = \frac{d}{d\xi},$$

  so we get the equation for w in ξ variable

$$\frac{d}{d\xi}w = e^{-\xi}\mu + 2(e^{-\xi}u_+)w + e^{-\xi}w^2 + e^{-\xi}(u_+ + w)^3, \qquad \frac{d}{d\xi}\mu = \varepsilon e^{-\xi}$$

- norms

  solving directly for μ we have $\mu \sim e^{-\xi}$ so the nonhomogeneous term in the w equation have

$$e^{-\xi}u_+^3 \sim e^{2\xi}$$

  also

$$\mu(t) = \Omega_0\varepsilon^{2/3} + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon e^{-\xi}$$

  Note $|e^{-\xi}| \leq e^{-\xi_*} \leq \delta\varepsilon^{-1/6}$, so that

$$\varepsilon e^{-2\xi} \leq \varepsilon\delta^2\varepsilon^{-1/3} \leq \delta^2\varepsilon^{2/3} \implies |\varepsilon e^{-\xi}\mu| = \mathcal{O}(\varepsilon^{2/3})$$

  which suggests the nonhomogeneous term is dominated by $e^{-\xi}u_+^3$ and hence a $e^{-2\xi}$ weight in the norm.

  In fact, due to the resonance of $e^{2\xi}$ with the linear part, we need to choose a slightly weaker norm, let $\eta \in (0,1)$, and our weight will be $e^{-(2-\eta)\xi}$. We check the nonhomogeneous term

$$\begin{aligned} e^{-(2-\eta)\xi}e^{-\xi}\mu &= e^{-(3-\eta)\xi}(\Omega_0\varepsilon^{2/3} + (\delta - \delta^{-1})\varepsilon^{5/6} - \varepsilon e^{-\xi}) \\ &\leq e^{-(3-\eta)\xi}(\varepsilon^{2/3} + \varepsilon^{5/6} + \varepsilon e^{-\xi}) \leq \\ &\sim \delta^{3-\eta}\varepsilon^{\frac{\eta+1}{6}} \end{aligned}$$

  We also check breifly the norm should work with the nonlinearity

  quadratic

$$\sup_{0\geq\xi\geq\xi_*} e^{-(2-\eta)\xi}|e^{-\xi}w^2| \leq \|w\| \sup|e^{-\xi}w| \leq \|w\|e^{-\xi}e^{(2-\eta)\xi} = \|w\|e^{(1-\eta)\xi}$$

  quadratic again

$$\sup_{0\geq\xi\geq\xi_*} e^{-(2-\eta)\xi}|e^{-\xi}u_+w^2| \leq \|w\| \sup|e^{-\xi}wu_+| \leq \|w\|e^{(2-\eta)\xi}$$

cubic

$$\sup_{0\geq\xi\geq\xi_*} |e^{-(2-\eta)\xi}e^{-\xi}w^3| \leq \|w\| \sup_{\xi\geq\xi_*} |e^{-\xi}w^2| \leq \|w\|e^{-\xi}e^{(4-2\eta)\xi}$$

Linear

$$\sup_{0\geq\xi\geq\xi_*} e^{-(2-\eta)\xi}|e^{-\xi}u_+^2 w| \leq \|w\| \sup |e^{\xi}u_+^2| \leq \|w\|e^{(1-\eta)\xi}$$

The Lipschitz constant will be of order $e^{-\xi}w \sim e^{(1-\eta)\xi}$, which is small on the relevant interval $\xi_* \leq \xi \leq 0$.

- Time scale, starts at $t = 0$:

$$\begin{aligned}
t_{mid} &= \varepsilon^{-1}\delta && \text{middle point of the minus side} \\
t_* &= T_\infty - \varepsilon^{-1/6}\delta^{-1} && \text{gluing time} \\
T &= T_\infty - \delta^{-1} && \text{right boundary} \\
T_\infty &= \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_0 && \text{Ricatti blow up time}
\end{aligned}$$

(iii) **Gluing** Ansatz

$$U(t) = \chi_-(t)u_-(t) + \chi_+(t)u_+(t) + W_-(t) + W_+(t)$$

with $W_-(t) = \varepsilon^{1/3}w_-(t)$. Also recall $\mu(t) = \varepsilon t - \delta$.

"insert picture of χ_-, χ_+."

The support of χ_+ is $(t_* - 1, \infty)$ and the support of χ_- is $(-\infty, t_* + 1)$.

- Plug in the anstaz

$$\begin{aligned}&\chi_-' u_- + \chi_- u_-' + \chi_+' u_+ + \chi_+ u_+' + W_-'(t) + W_+'(t) = \\ &= \mu + (\chi_- u_- + W_- + \chi_+ u_+ + W_+)^2 + (\chi_- u_- + W_- + \chi_+ u_+ + W_+)^3.\end{aligned}$$

where $' = \frac{d}{dt}$.
Useful identities

$$\chi_- + \chi_+ = 1, \quad \frac{d}{dt}(\chi_- + \chi_+)(t) = 0,$$

and

$$\frac{d}{dt}u_- = \mu + u_-^2, \quad \frac{d}{dt}u_+ = u_+^2.$$

Equation after cancellation:

$$\begin{aligned}-\chi_+'(u_- - u_+) + W_-' + W_+' = {} & \chi_+\mu + \chi_-\chi_+(u_+ - u_-)u_- + 2\chi_- u_- W_- + W_-^2 \\ & + \chi_-\chi_+(u_- - u_+)u_+ + 2\chi_+ u_+ W_+ + W_+^2 \\ & + 2\chi_+ u_+ W_- + 2\chi_- u_- W_+ + 2W_- W_+ \\ & + (\chi_- u_- + W_- + \chi_+ u_+ + W_+)^3\end{aligned}$$

- Distribute terms in the $-$ side

$$\begin{aligned}W_-' = {} & 2u_- W_- + 2\chi_-(u_- - u_+)W_+ + (\chi_-\chi_+)u_-(u_- - u_+) + \frac{1}{2}\chi_-'(u_+ - u_-) + 2\chi_- W_- W_+ \\ & + W_-^2 + (\chi_- u_- + W_-)^3 \\ & + 3\chi_-^2\chi_+ u_-^2 u_+ + 6\chi_-\chi_+ u_+ u_- W_- + 6\chi_- u_- W_- W_+ + 3\chi_-(W_- + W_+)W_- W_+ \\ & + 3(\chi_+ u_+)W_-^2 + 3(\chi_- u_-)^2 W_+\end{aligned}$$

Recall $|u_+ - u_-|(t) = \mathcal{O}(\varepsilon^{2/3})$ by the Gronwall inequality argument for $t \in supp\chi_-\chi_+$, since $W_- = \varepsilon^{1/3}w_-$ and $\varepsilon^{-1/3}\varphi\frac{d}{dt} = \frac{d}{d\sigma}$,

- Distribute terms in the $+$ side

$$\begin{aligned}W_+' = {} & \chi_+\mu + 2u_+ W_+ + 2\chi_+(u_+ - u_-)W_- + (\chi_-\chi_+)u_+(u_- - u_+) + \frac{1}{2}\chi_+'(u_- - u_+) + 2\chi_+ W_- W_+ \\ & + W_+^2 + (\chi_+ u_+ + W_+)^3 \\ & + 3\chi_+^2\chi_- u_+^2 u_- + 6\chi_+\chi_- u_+ u_- W_+ + 6\chi_+ u_+ W_+ W_- + 3\chi_+(W_- + W_+)W_- W_+ \\ & + 3(\chi_- u_-)W_+^2 + 3(\chi_+ u_+)^2 W_-\end{aligned}$$

- Linear equation.

  Now the equation in W_- and W_+ can be written in the following form

$$W_-' - 2u_- W_- = \mathcal{R}_-,$$
$$W_+' - 2u_+ W_+ = \mathcal{R}_+$$

  with $\mathcal{R}_\pm$ defined as in the distribution of terms.

  First fix

$$\eta \in (1,2), \nu \in (0,1).$$

  To be able to solve the linear equation, we first introduce the following weighted spaces, for the $-$ side we have

$$\mathcal{C}_v = \{u(t) \in \mathcal{C}(0,T) \mid \sup |v(t)u(t)| < \infty\}$$

  where the weight $v(t)$ is defined as follows:

$$v(t) = \begin{cases} \varepsilon^{-\frac{1}{3}(1-\nu)}(T_\infty - t)^\nu, & \text{for } t > \varepsilon^{-1}\delta \\ \varepsilon^{1/3} + (\delta - \varepsilon t), & \text{for } t < \varepsilon^{-1}\delta \end{cases}$$

  We can similarly define $\mathcal{C}_V$, with the other weight $V(t)$

$$V(t) = \begin{cases} \varepsilon^{-\frac{1}{3}(1-\nu)}(T_\infty - t)^{\nu+1}, & \text{for } t > \varepsilon^{-1}\delta \\ \varepsilon^{\frac{2}{3}} + (\delta - \varepsilon t)^{\frac{3}{2}}, & \text{for } t < \varepsilon^{-1}\delta \end{cases}$$

  and for the $+$ side we have:

$$\mathcal{C}_\eta(0,T) = \{u(t) \in \mathcal{C}(0,T) \mid \sup_{t\in(0,T)} |(T_\infty - t)^\eta u(t)| < \infty\}$$

  then we can show the Fredholm properties of the linear operators as follows:

  Theorem 4.1. *For $t \in (0,T)$, the linear operator on the $-$ side*

$$\frac{d}{dt} - 2u_-(t) : \mathcal{C}_v(0,T) \to \mathcal{C}_V(0,T)$$

  and the linear operator on the $+$ side

$$\frac{d}{dt} - 2u_+(t) : \mathcal{C}_\eta(0,T) \to \mathcal{C}_{\eta+1}(0,T)$$

  are Fredholm, and their indices are X, Y, respectively..

  Proof. For the W_- equation, recall we had the following scaling $s = \psi(\sigma)$, $\varphi = \partial_\sigma \psi$. We get the equation

$$\frac{d}{d\sigma}\tilde{W}_- - a(\sigma)\tilde{W}_- = \varepsilon^{-1/3}\varphi\tilde{\mathcal{R}}_-$$

here $\tilde{W}_-(\sigma) = W_-(\varepsilon^{-\frac{1}{3}}\psi(\sigma) + \varepsilon^{-1}\delta) = W_-(t)$, and similarly for $\tilde{\mathcal{R}}_-$. Now recall $a(\sigma) \to \pm 2$ as $\sigma \to \pm\infty$. In these variables, the weight satisfies

$$v(t) \sim \begin{cases} \varepsilon^{-\frac{1}{3}} e^{-\nu\sigma}, \text{ for } \sigma >> 0 \\ \varepsilon^{-\frac{2}{3}} [(-\sigma)^{\frac{2}{3}} + 1]^{-1}, \text{ for } \sigma << 0. \end{cases}$$

and

$$V(t) \sim \begin{cases} \varepsilon^{-\frac{2}{3}} e^{-(\nu+1)\sigma}, \text{ for } \sigma >> 0 \\ [\varepsilon|\sigma| + \varepsilon^{2/3}]^{-1} \text{ for } \sigma << 0. \end{cases}$$

Then for $\nu \neq 2$, the linear operators $\frac{d}{d\sigma} - a(\sigma)$ is Fredholm on the weighted spaces. Since $0 < \nu < 1$ and w has algebraic decay for $\sigma < 0$, we conclude that the Fredholm index is...

For the W_+ equation, we used the rescaling $u_+(t) = e^{\xi}$, and in the ξ equation, the $+$ side equation becomes

$$\frac{d}{d\xi}\tilde{W}_+ - 2\tilde{W}_+ = e^{-\xi}\mathcal{R}_+$$

the weight for W_+ is just $u_+(t)^{\eta} = e^{\eta\xi}$ and because of $1 < \eta < 2$, we see the linear opertator $\frac{d}{d\xi} - 2$ is Fredholm on this weighted function space. $\square$

- Fixed point argument

  To close the argument, we need to check the residual part $\mathcal{R}_\pm$ are compatible with the function space for which we know the Fredholm properties of the linear part.

  /////////////Estimate for W_- equation/////////////

$$
\begin{aligned}
W_-' = & 2u_-W_- + 2\chi_-(u_- - u_+)W_+ + (\chi_-\chi_+)u_-(u_- - u_+) + \frac{1}{2}\chi_-'(u_+ - u_-) + 2\chi_-W_-W_+ \\
& + W_-^2 + (\chi_-u_- + W_-)^3 \\
& + 3\chi_-^2\chi_+u_-^2u_+ + 6\chi_-\chi_+u_+u_-W_- + 6\chi_-u_-W_-W_+ + 3\chi_-(W_- + W_+)W_-W_+ \\
& + 3(\chi_+u_+)W_-^2 + 3(\chi_-u_-)^2W_+
\end{aligned}
$$

(a) Linear term.
As seen, it is

$$
\frac{d}{dt} - 2u_-,
$$

which is Fredholm on the function space as in the theorem.

(b) $(u_- - u_+)$ terms.

Theorem 4.2. *If $|t - t_*| \le C$ for some positive number C, we have*

$$
|u_-(t) - u_+(t)| \le C'\varepsilon^{2/3}
$$

where C' is another constant depending on C.

Proof. Let $v(t) := u_-(t) - u_+(t)$, by the differential equation of u_- and u_+ solves, we have

$$
\begin{aligned}
v'(t) &= \mu + u_-^2 - u_+^2 = \varepsilon t - \delta + v(u_- + u_+)(t) \\
&= \varepsilon(t - t_*) + \varepsilon t_* - \delta + v(u_- + u_+)
\end{aligned}
$$

For $|t - t_*| \le C$, we have $(u_- + u_+)(t) \le C\varepsilon^{1/6}$ for some constant C.
Hence by Grownwall we have that

$$
v(t) \le C'\left[\frac{1}{2}(t^2 - t_*^2) - \delta(t - t_*)\right] \le C'\varepsilon^{2/3}
$$

□

Theorem 4.3. *The $u_- - u_+$ term*

$$
(\chi_-\chi_+)u_-(u_- - u_+) + \frac{1}{2}\chi_-'(u_+ - u_-) + 2\chi_-(u_- - u_+)W_+ := \mathcal{R}_{-,d}
$$

satisfies

$$
\|\mathcal{R}_{-,d}\|_{\mathcal{C}_W} = o_\varepsilon(1)
$$

Proof. For $t \in \operatorname{supp}(\chi_-\chi_+) = \operatorname{supp}\chi_-'$, we have $|t - t_*| \le 1$. Note $u_-(t_*) = \mathcal{O}(\varepsilon^{1/6})$ and $|u_-(t) - u_-(t_*)| = \mathcal{O}(\varepsilon^{1/3})$ from mean value theorem, from previous theorem we conclude that

$$
|(\chi_-\chi_+)u_-(u_- - u_+)|_\infty = \mathcal{O}(\varepsilon^{\frac{5}{6}}).
$$

and

$$|\chi_-'(u_+ - u_-)|_\infty = \mathcal{O}(\varepsilon^{\frac{2}{3}}).$$

The third term $2\chi_-(u_- - u_+)W_+$, however, only gets multiplied by χ_-. We cannot just estimate this term in a compact interval around the gluing point t_*. Instead we will utilize the following facts:

- for $0 < t < \varepsilon^{-1}\delta$, we have

$$|u_-(t)| \leq C(\sqrt{\delta - \varepsilon t} + \varepsilon^{1/3})$$

- for $\varepsilon^{-1}\delta < t < t_*$, we have

$$|u_-(t)| \leq C|u_+(t)|$$

The constant C in the above statements are independent of ε, δ.
To see how these statements proves the theorem, we estimate, for $\varepsilon^{-1}\delta < t < t_*$ we have $I(t) = \varepsilon^{-\frac{1-\nu}{3}}(T_\infty - t)^{1+\nu}$,

$$\begin{aligned}|I(t)\chi_-(u_+ - u_-)W_+|_\infty \leq |I(t)\chi_- u_+ W_+|_\infty \\ + |I(t)\chi_- u_- W_+|_\infty.\end{aligned}$$

Since $|u_+| \leq C(T_\infty - t)^{-1}$, and the estimate $|W_+| \leq C(T_\infty - t)^{-\eta}$ holds, we have, for the first of the two parts

$$|I(t)\chi_- u_+ W_+|_\infty \leq |\varepsilon^{-\frac{1-\nu}{3}}(T_\infty - t)^{\nu-\eta}|_\infty \leq \varepsilon^{-\frac{1-\nu}{3}}(T_\infty - t_*)^{\nu-\eta} = \delta^{2(\eta-1)}.$$

where we used the fact that $\varepsilon + \eta = 2$ and $1 < \eta < 2$. For the term $I(t)\chi_- u_- W_+$, we use the claim to replace u_- by u_+ and hence reduce the proof to the same estimate.
Similarly, for $0 < t < \varepsilon^{-1}\delta$, where the weight $I(t)$ equals $[\varepsilon^{\frac{3}{2}}(\varepsilon^{-1}\delta - t)^{\frac{3}{2}} + \varepsilon^{\frac{2}{3}}]^{-1}$.
We again have to estimate $|I(t)\chi_- u_+ W_+|_\infty$ and $|I(t)\chi_- u_- W_+|_\infty$, first we have

$$\begin{aligned}|I(t)\chi_- u_+ W_+|_\infty &\leq I(t)(T_\infty - t)^{-(1+\eta)} \leq I(\varepsilon^{-1}\delta)(T_\infty - \varepsilon^{-1}\delta)^{-(1+\eta)} \\ &\leq C\varepsilon^{-\frac{2}{3}}\varepsilon^{\frac{1+\eta}{3}} = \mathcal{O}(\varepsilon^{\frac{\eta-1}{3}})\end{aligned}$$

For the u_- term, we use the claim to get

$$|I(t)\chi_- u_- W_+|_\infty \leq \frac{\sqrt{\delta - \varepsilon t} + \varepsilon^{1/3}}{(\delta - \varepsilon t)^{\frac{3}{2}} + \varepsilon^{2/3}}(T_\infty - \varepsilon^{-1}\delta)^{-\eta} \leq \varepsilon^{-1/3}\varepsilon^{\frac{\eta}{3}} = \mathcal{O}(\varepsilon^{\frac{\eta-1}{3}})$$

□

(c) Quadratic and cubic terms

$$\mathcal{R}_{-,c} := W_-^2 + (\chi_- u_- + W_-)^3$$

Theorem 4.4. *Qurdratic terms satisfies...*

Proof. For $t > \varepsilon^{-1}\delta$, near the gluing point, we have

$$\|W_-^2\|_{\mathcal{C}_V} \leq \|W_-\|_{C_v}|(T_\infty - t)W_-|_\infty = \mathcal{O}(\varepsilon^{\frac{1-\nu}{6}})\|W_-\|_{\mathcal{C}_v}$$

For $t < \varepsilon^{-1}\delta$, let $\tau = \varepsilon^{-1}\delta - t$, then

$$\|W_-^2\|_{\mathcal{C}_V} \leq \left|[\varepsilon\tau + \varepsilon^{2/3}]^{-1}[\varepsilon\tau + \varepsilon^{\frac{1}{3}}]^2\right|_\infty = \mathcal{O}(\delta)$$

what about norms in the middle? that is near $\varepsilon^{-1}\delta$, seems I can only get $\mathcal{O}(1)$ bound here. For the cubic term, we use $(\chi_- u_- + W_-)^3 \leq C[(\chi_- u_-)^3 + W_-^3]$ for some constant C to estimate each term separately. First we estimate (for $t > \varepsilon^{-1}\delta$)

$$\|\chi_- u_-^3\|_{\mathcal{C}_V} \leq |\varepsilon^{\frac{1-\nu}{3}}(T_\infty - t)^{1+\nu}\varepsilon u_R^3|_\infty \leq \varepsilon^{-\frac{1-v}{3}}\varepsilon^{-\frac{1+v}{6}}\varepsilon^{\frac{1}{2}} = \varepsilon^{\frac{\nu}{6}}$$

On the interval $t < \varepsilon^{-1}\delta$, we have

$$\|\chi_- u_-^3\|_{\mathcal{C}_V} = \mathcal{O}(1)$$

using the asymptotics $u_R(s) \to -(-s)^{\frac{1}{2}}$ as $s \to \infty$. At $t = 0$ the leading order terms of $V(t)$ gives $\delta^{\frac{3}{2}}$. While $u_-(t)$ gives

$$u_-^3 = \varepsilon u_R^3(s) = \varepsilon s^{\frac{3}{2}} = \varepsilon[\varepsilon^{\frac{1}{3}}(\delta\varepsilon^{-1} - t)]^{\frac{3}{2}} = \delta^{\frac{3}{2}}$$

at $t = 0$.
Finally for the cubic term, we have

$$\|W_-^3\|_{\mathcal{C}_V} \leq \|W_-\|_{\mathcal{C}_v}|(T_\infty - t)W_-^2|_\infty = \mathcal{O}(\varepsilon^{\frac{3-2\nu}{6}})\|W_-\|_{\mathcal{C}_v}$$

for t near t_*.
And ($\tau = \varepsilon^{-1}\delta - t$, again),

$$\|W_-^3\|_{\mathcal{C}_V} \leq \left|[\varepsilon\tau + \varepsilon^{2/3}]^{-1}[\varepsilon\tau + \varepsilon^{\frac{1}{3}}]^3\right|_\infty = \mathcal{O}(\varepsilon^2)$$

□

(d) Mixed terms
Includes

$$\begin{aligned}\mathcal{R}_{-,m} := {} & 3(\chi_-^2\chi_+)u_-^2u_+ + 6(\chi_-\chi_+)u_+u_-W_- \\ & + 6\chi_-(3^{-1} + u_- + 2^{-1}(W_- + W_+))W_-W_+ \\ & + 3(\chi_+u_+)W_-^2 + 3(\chi_+u_+)^2W_-\end{aligned}$$

Theorem 4.5. *Cross terms satisfy the estimate*

$$\|\mathcal{R}_{-,m}\|_{\mathcal{C}_V} = \mathcal{O}(\varepsilon^{\frac{\nu}{6}})$$

Proof. For the first two terms $3(\chi_-^2\chi_+)u_-^2u_+$ and $6(\chi_-\chi_+)u_+u_-W_-$, the support of them are just (t_*-1, t_*+1), on this interval, by theorem $(u_- - u_+)$, we know that $|u_- - u_+(t)| \leq C\varepsilon^{2/3}$ for some constant C. Moreover, recall that $u_-(t_*) = u_+(t_*) = \mathcal{O}(\varepsilon^{1/6})$, so we get

$$\begin{aligned}\|3(\chi_-^2\chi_+)u_-^2u_+\|_{\mathcal{C}_V} &\leq |3(\chi_-^2\chi_+)[u_-^3 + u_-^2(u_+ - u_-)]\varepsilon^{\frac{\nu-1}{3}}(T_\infty - t)^{1+\nu}|_\infty \\ &= \mathcal{O}(\varepsilon^{\frac{1}{2}}\varepsilon^{\frac{\nu-1}{3}}\varepsilon^{-\frac{1+\nu}{6}}) = \mathcal{O}(\varepsilon^{\frac{\nu}{6}})\end{aligned}$$

and

$$\begin{aligned}\|6(\chi_-\chi_+)u_-u_+W_-\|_{\mathcal{C}_V} &\leq |6(\chi_-\chi_+)[u_-^2 + u_-(u_+ - u_-)](T_\infty - t)|_\infty\|W_-\|_{\mathcal{C}_v} \\ &= \mathcal{O}(\varepsilon^{\frac{1}{6}})\|W_-\|_{\mathcal{C}_v}\end{aligned}$$

Next, we have the mixed W_-W_+ term, it is enough to just estimate

$$\|\chi_-W_-W_+\|_{\mathcal{C}_V}$$

for $t > \varepsilon^{-1}\delta$, we have

$$\|\chi_-W_-W_+\|_{\mathcal{C}_V} \leq |(T_\infty - t)W_+|_\infty\|W_-\|_{\mathcal{C}_v} = |(T_\infty - t)^{-\eta+1}|\|W_-\|_{\mathcal{C}_v} = \mathcal{O}(\varepsilon^{\frac{\eta-1}{6}})\|W_-\|_{\mathcal{C}_v}$$

while for $t < \varepsilon^{-1}\delta$,

$$\begin{aligned}\|\chi_-W_-W_+\|_{\mathcal{C}_V} &\leq |\varepsilon^{-1/2}(\varepsilon^{-1}\delta - t)^{-1/2}W_+|_\infty\|W_-\|_{\mathcal{C}_v} \\ &\leq (T_\infty - t)^{-\eta}(\delta - \varepsilon t)^{-1/2}\|W_-\|_{\mathcal{C}_v} \\ &= \mathcal{O}(\varepsilon^{\frac{\eta}{3}})\|W_-\|_{\mathcal{C}_v}\end{aligned}$$

We then focus on the last two terms, first

$$\|3(\chi_+u_+)W_-^2\|_{\mathcal{C}_V} \leq (T_\infty - t)^{-1}\|W_-^2\|_{\mathcal{C}_V} \leq (T_\infty - t_*)^{-1}\|W_-^2\|_{\mathcal{C}_v} = \mathcal{O}(\varepsilon^{1/6})\|W_-^2\|_{\mathcal{C}_v}$$

since we already dealt with $\|W_-^2\|_{\mathcal{C}_v}$ in theorem (quadratic and cubic terms), we are good here.

Next, we estimate

$$\|3(\chi_+u_+)^2W_-\|_{\mathcal{C}_V} \leq |(T_\infty - t)u_+^2|_\infty\|W_-\|_{\mathcal{C}_v} = \mathcal{O}(\varepsilon^{1/6})\|W_-\|_{\mathcal{C}_v}$$

for $t > \varepsilon^{-1}\delta$. Due to the χ_+ cut off, for $t < \varepsilon^{-1}\delta$, this term becomes 0. □

////////////estimate for W_+ equation////////////

$$
\begin{aligned}
W_+' &= \chi_+\mu + 2[u_+ + \chi_-(u_- - u_+)]W_+ + (\chi_-\chi_+)u_+(u_- - u_+) + \frac{1}{2}\chi_+'(u_- - u_+) + 2\chi_+W_-W_+ \\
&\quad + W_+^2 + (\chi_+u_+ + W_+)^3 \\
&\quad + 3\chi_+^2\chi_-u_+^2u_- + 6\chi_+\chi_-u_+u_-W_+ + 6\chi_+u_+W_+W_- + 3\chi_+(W_- + W_+)W_-W_+ \\
&\quad + 3(\chi_-u_-)W_+^2 + 3(\chi_-u_-)^2W_+
\end{aligned}
$$

(a) Linear term, recall $u_+(t) = e^{\xi}$

$$\frac{d}{dt} - 2u_+$$

(b) $\chi_+\mu$ term

Theorem 4.6. *The nonhomogeneous term $\chi_+\mu$ satisfies*

Proof. a calculation shows

$$\|\chi_+\mu\|_{\eta+1} = |(T_\infty - t)^{1+\eta}\chi_+\mu(t)|_\infty = \mathcal{O}(\varepsilon^{\frac{2}{3}}\varepsilon^{\frac{-(1+\eta)}{6}}) = \mathcal{O}(\varepsilon^{\frac{3-\eta}{6}})$$

where we solved directly that $\mu(t) = \varepsilon t - \delta = \varepsilon(T_\infty - u_+(t)^{-1}) - \delta = \varepsilon^{\frac{2}{3}}\Omega_0 - \varepsilon u_+^{-1}$, and that $T_\infty - t \in (\delta^{-1}, \delta^{-1}\varepsilon^{\frac{-1}{6}})$ for $t \in (t_*, T)$. □

(c) $(u_- - u_+)$ term (difference term)
Includes

$$\mathcal{R}_{+,d} := 2\chi_-(u_- - u_+)W_+ + (\chi_-\chi_+)u_+(u_- - u_+) + \frac{1}{2}\chi_+'(u_- - u_+)$$

Theorem 4.7. *The difference term $\mathcal{R}_{+,d}$ satisfies*

$$\|\mathcal{R}_{+,d}\|_{\eta+1} = \mathcal{O}()$$

Proof. The two term $(\chi_-\chi_+)u_+(u_- - u_+)$ and $\frac{1}{2}\chi_+'(u_- - u_+)$ have compact support inside $(t_* - 1, t_* + 1)$. Again, recall $u_+(t_*) = \mathcal{O}(\varepsilon^{1/6})$ and $|u_- - u_+| = \mathcal{O}(\varepsilon^{2/3})$, for all t with $|t - t_*| \le 1$, we have that

$$|(\chi_-\chi_+)u_+(u_- - u_+) + \frac{1}{2}\chi_+'(u_- - u_+)|_\infty = \mathcal{O}(\varepsilon^{5/6}).$$

For the linear term $\chi_-(u_- - u_+)W_+$, we calculate

$$\|\chi_-(u_+ - u_-)W_+\|_{\eta+1} \le |(T_\infty - t)(u_+ - u_-)|_\infty \|W_+\|_\eta$$

again use $u_+ - u_- = o(\frac{1}{T_\infty - t})$. □

(d) quadratic and cubic terms
Includes

$$\mathcal{R}_{+,c} = W_+^2 + (\chi_+u_+ + W_+)^3$$

Theorem 4.8. *we have*

$$\|\mathcal{R}_{+,c}\|_{\eta+1} = \mathcal{O}(\varepsilon^{\eta_*}),$$

with $\eta_* = \min(\eta - 1, 2 - \eta)$.

Proof. First, we have

$$\|W_+^2\|_{\eta+1} = |(T_\infty - t)^{1+\eta} W_+^2|_\infty \le (T_\infty - t)^{1+\eta}(T_\infty - t)^{-2\eta}\|W_+\|_\eta^2 \le (T_\infty - t)^{1-\eta}\|W_+\|_\eta^2$$

now since $t \in (0, T)$ we have $|T_\infty - t| = \mathcal{O}(\varepsilon^{-1})$, this gives

$$\|W_+^2\|_{\eta+1} = \mathcal{O}(\varepsilon^{\eta-1})$$

since $\eta > 1$, this is desired estimate.
Next we have

$$\|(\chi_+ u_+)^3\|_{\eta+1} = |(T_\infty - t)^{1+\eta} u_+^3|_\infty \le C(T_\infty - t)^{1+\eta}(T_\infty - t)^{-3} = \mathcal{O}(\varepsilon^{2-\eta})$$

this uses $\eta < 2$, again, is desired.
Finally, for the cubic term

$$\|W_+^3\|_{\eta+1} \le \|W_+\|_\eta^3 (T_\infty - t)^{1+\eta-3\eta} = \mathcal{O}(\varepsilon^{2\eta-1})$$

□

(e) mixed terms
Includes

$$\begin{aligned}\mathcal{R}_{+,m} := {} & 3(\chi_+^2\chi_-)u_+^2 u_- + 6(\chi_+\chi_-)u_+ u_- W_+ \\ & + 6\chi_+(3^{-1} + u_+ + 2^{-1}(W_- + W_+))W_- W_+ \\ & + 3(\chi_- u_-)W_+^2 + 3(\chi_- u_-)^2 W_+\end{aligned}$$

Theorem 4.9. *We have*

$$\|\mathcal{R}_{+,m}\|_{\eta+1} = ...$$

Proof. The first two terms $3(\chi_+^2\chi_-)u_+^2 u_-$ and $6(\chi_+\chi_-)u_+ u_- W_+$ are localized around $(t_* - 1, t_* + 1)$, hence we can treat them the same way as in the $\mathcal{R}_{-,m}$ case, using theorem 4.2
The middle term $6\chi_+(3^{-1} + u_+ + 2^{-1}(W_- + W_+))W_- W_+$ is estimated as follows:
We only need to consider the term $\chi_+ W_- W_+$, and compute

$$\|\chi_+ W_- W_+\|_{\eta+1} \le |(T_\infty - t)W_-|_\infty \|W_+\|_\eta \le [\varepsilon^{\frac{1}{3}}(T_\infty - t)]^{1-\nu}\|W_+\|_\eta$$

because of the cut off χ_+, we know that $t \in (t_* - 1, T)$, so $|T_\infty - t| = \mathcal{O}(\varepsilon^{-\frac{1}{6}})$, this shows

$$\|W_- W_+\|_{\eta+1} = \mathcal{O}(\varepsilon^{\frac{1-\nu}{6}})$$

Finally, the last two terms $3(\chi_- u_-)W_+^2$ and $3(\chi_- u_-)^2 W_+$ is treated as follows:
For t in the support of χ_-, we have $|u_-| \le C\delta^{\frac{1}{2}}$, so

$$\|(\chi_- u_-)W_+^2\|_{\eta+1} \le |u_-| \|W_+^2\|_{\eta+1},$$

where $\|W_+^2\|_{\eta+1}$ is estimated before.
For $3(\chi_- u_-)^2 W_+$, we have

$$\|3(\chi_- u_-)^2 W_+\|_{\eta+1} = |(T_\infty - t)(\chi_- u_-)^2|_\infty \|W_+\|_\eta$$

same question, we need to use $u_- - u_+ = o(u_+)$ through out... □