1 Introduction

Notation: sometimes we use $A \lesssim B$ to indicate that there is a constant C such that $A \leq C \cdot B$, if this constant C does not depend on properties of both A and B, we will be specific and write directly $A \leq C \cdot B$.

2 Set up

Equation

$$\begin{aligned} \frac{d}{d\tau}u &= \mu + u^2 + f(u, \mu, \varepsilon), \\ \frac{d}{d\tau}\mu &= \varepsilon g(u, \mu, \varepsilon), \end{aligned} \tag{2.1}$$

where $f(u, \mu, \varepsilon) = \mathcal{O}(\varepsilon, u\mu, \mu^2, u^3)$ and $g(u, \mu, \varepsilon) = 1 + \mathcal{O}(u, \mu, \varepsilon)$.

Boundary condition

$$\mu(0) = -\delta, \quad u(T) = \delta, \tag{2.2}$$

where δ, ε are parameters, and T is dependent on ε and δ which will be solved as part of the equation.

2.1 Equation

For demonstration purposes, we focus on the following equation first

$$\begin{aligned} \frac{d}{dt}u &= \mu + u^2 + u^3, \\ \frac{d}{dt}\mu &= \varepsilon, \end{aligned} \tag{2.3}$$

with boundary condition 2.2.

2.2 The Riccati equation

Consider the Riccati equation

$$\frac{d}{ds}u(s) = s + u(s)^2 \tag{2.4}$$

We denote any solution to (2.4) as u_R, it is known to have a unique solution (denoted by $\bar{u}_R$) with the following asymptotics:

$$\bar{u}_R(s) = \begin{cases} (\Omega_0 - s)^{-1} + \mathcal{O}(\Omega_0 - s), \text{ as } s \to \Omega_0, \\ -(-s)^{1/2} - \frac{1}{4}(-s)^{-1} + \mathcal{O}(|s|^{-3/2}), \text{ as } s \to -\infty. \end{cases} \tag{2.5}$$

Here the constant $\Omega_0 \approx 2.3381$ is the smallest positive zero of

$$J_{-1/3}(2z^{3/2}/3) + J_{1/3}(2z^{3/2}/3),$$

where $J_{\pm 1/3}$ are Bessel functions of the first kind. (See [Krupa, Szmolyan])

More generally, we consider family of solutions $u_R(s; u_0)$ of the Riccati equation (2.4) such that $u_R(0; u_0) = u_0$. That is, we take the initial condition u_0 as a parameter to the Riccati equation. For the special Riccati solution $\bar{u}_R$, we denote $\bar{u}_R(0)$ as $\bar{u}_0$. We will use a family of such solutions to build our ansatz, whose properties are summerized in the following proposition:

Proposition 2.1. *There exist blow up time $\Omega_\infty = \Omega_\infty(u_0)$ that depends smoothly on u_0 for $|u_0 - \bar{u}_0| < \eta$, η small, with $\Omega_\infty(\bar{u}_0) = \Omega_0$, and the corresponding solution $u_R(s; u_0)$ is of the form*

$$u_R(s; u_0) = \frac{1}{\Omega_\infty - s} + (\Omega_\infty - s) r(\Omega_\infty - s; u_0), \tag{2.6}$$

where the function r is smooth in both variables and satisfies

$$r(\Omega_\infty - s; u_0) = -\frac{\Omega_\infty}{3} + \mathcal{O}(\Omega_\infty - s), \tag{2.7}$$

as $s \to \Omega_\infty$.

Proof. To get the dependence from u_0 to Ω_∞, we first add the equation $\frac{d}{ds}s = 1$ to equation (2.4) to get a autonomous 2−dimensional system in the (s, u) plane. Consider a small neighbourhood I containing $\bar{u}_0$ on the vertical u-axis, then $u_R(s; u_0)$ is the trajectory that starts at $u_0 \in I$. The map $P_1 : I \to \mathbb{R}$ defined by $P_1(p) = u(2; p)$ is smooth in p, as the blow up time for $\bar{u}_R(s; \bar{u}_0)$ is $\Omega_0 > 2$. Moreover, the image $P_1(I)$ is a finite interval on the vertical line $s = 2$ containing $\bar{u}_R(2; u_0)$ bounded away from 0, since the trajectory $u_R(s; \bar{u}_0)$ crosses the horizontal axis around $s = 1$ and the vector field goes upwards in the first quardrant of the (s, u)-plane.

Denote $\tilde{u}_0 := P_1(u_0)$ for brevity (technically, the interval $P_1(I)$ is a small section of the line $s = 2$, with a little abuse of notation, we identify $\tilde{u}_0$ with the second coordinate of the point $P_1(u_0)$). Again in the Riccati equation (2.4), we make a change of variable by setting $z(s) = 1/u(s)$, the equation z satisfies is:

$$\frac{d}{ds} z(s) = -z^2 s - 1.$$

Let $J = \{1/\tilde{u}_0 \mid \tilde{u}_0 \in P_1(I)\}$ and $z(s; 1/\tilde{u}_0)$ is the trajectory which starts at $1/\tilde{u}_0$. We claim that $z(s; 1/\bar{u}_0)$ reaches 0 at a finite time $\Omega_\infty = \Omega_\infty(1/\bar{u}_0)$. To see this, first notice there is no equilibrium for the two dimensional system $\frac{d}{ds}s = 1, \frac{d}{ds}z = -z^2 s - 1$. Then, on the boundary $s = 2$, the vector field takes the form $(1, -2z^2 - 1)$, which makes any trajectory starting at a point on J moving down towards the right. Moreover, the vector field $(1, -sz^2 - 1)$ always pointing down in the first qurdrant of the (s, z) plane, so trajectories cannot go upwards. Lastly, the vector field crosses the horizontal axis non-tangentially, it identically equals $(1, -1)$ throughout the line $z = 0$, hence, any trajectory which starts at a point on J will cross $z = 0$ in finite time at a unique point $\Omega_\infty = \Omega_\infty(1/\tilde{u}_0)$. The dependence of Ω_∞ on $1/\tilde{u}_0$ is smooth by the smooth dependence on initial conditions.

We now define another map $P_2 : J \to \mathbb{R}$ by $P_2(1/\tilde{u}_0) = \Omega_\infty(1/\tilde{u}_0)$, we get a smooth map $P : I \to \mathbb{R}$ by the composition

$$P = P_2 \circ f \circ P_1,$$

where $f(z) = 1/z$ is the inversion map. Since each of the map in the composition is smooth, $P : u_0 \mapsto \Omega_\infty = \Omega_\infty(u_0)$ is smooth as well.

To get the asymptotic expansion, we set $\xi = \Omega_\infty - s$, then $\tilde{z}(\xi) = z(\Omega_\infty - \xi)$ solves

$$\frac{d}{d\xi}\tilde{z} = \tilde{z}^2(\Omega_\infty - \xi) + 1,$$

and $\tilde{z}(0) = 0$.

Hence we can assume the expansion for $\tilde{z}$ near $\xi = 0$ is of the form

$$\tilde{z} = \xi + z_2\xi^2 + z_3\xi^3 + \mathcal{O}(\xi^4),$$

for some constant z_2, z_3. Differentiating this expansion, use the equation $\tilde{z}$ solves and comparing coefficients, we find that $z_2 = 0, z_3 = \Omega_\infty/3$. Changing back from $\tilde{z}(\xi)$ to $z = z(s)$ with $s = \Omega_\infty - \xi$ and recall $z(s) = 1/u(s)$, we find that $u_R(s; u_0)$ has expansion (2.6) with remainder r satisfies (2.7).

$\blacksquare$

2.3 The t to σ time rescaling

The solution to (2.1) will be based on the solution of the Riccati equation (2.4). We need to rescale time t to obtain appropriate equations as $\varepsilon \to 0$.

We will rescale time t to time σ using the following steps.

- Step 1: Define ψ as

  $$\psi = \varepsilon^{1/3}(t - \varepsilon^{-1}\delta)$$

- Step 2: Fix $M > 0$ large, define σ as

  $$\psi = \psi(\sigma; u_0) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, \text{ for } \sigma \le -M \\ \Omega_\infty(u_0) - e^{-\sigma}, \text{ for } \sigma \ge M, \end{cases}$$

  here Ω_∞ is the blow-up time for u_R found in proposition 2.1.

- Step 3: For $\sigma \in (-M, M)$, we define $\psi(\sigma)$ as the straight line connecting the two points $(M, \Omega_\infty - e^{-M})$ and $(-M, -(\frac{3}{2}M)^{2/3})$. As a result, if we define $\sigma_m = \sigma_m(u_0)$ as the value of σ such that $\psi(\sigma_m; u_0) = 0$, then we have

  $$\frac{|\sigma_m - M|}{M} = \left| \frac{(\frac{3M}{2})^{2/3} - (\Omega_\infty - e^{-M})}{(\frac{3M}{2})^{2/3} - (\Omega_\infty + e^{-M})} - 1 \right| \le CM^{-2/3},$$

  for some constant C independent of u_0.

  Therefore we can write

  $$\sigma_m = M + M_r, \qquad |M_r| \le CM^{1/3} \tag{2.8}$$

We also denote $\varphi(\sigma) := \frac{d}{d\sigma}\psi(\sigma)$.

3 Region A

Region A corresponds to the t-time interval $\{t : t > \varepsilon^{-1}\delta\}$.

3.1 Ansatz in region A

The ansatz in region A takes the form

$$u_A(t) = u_*(t) + w_r(t).$$

The function $u_* = u_*(t; u_0)$ is defined as:

$$u_*(t; u_0) := \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta); u_0), \tag{3.1}$$

where $u_R = u_R(s; u_0)$ is the family of solutions to the Riccati equation which were shown to exist in proposition 2.1, it solves the initial value problem

$$\frac{d}{dt} u_*(t; u_0) = \mu(t) + u_*^2(t; u_0), \qquad u_*(\varepsilon^{-1}\delta; u_0) = \varepsilon^{\frac{1}{3}} u_0 \tag{3.2}$$

with ε and u_0 as parameters. The function w_r is a correction term whose properties are summerized in the following theorem.

Theorem 3.1. *For all δ, α small enough, there exists η, ε_1, C, such that for all $0 < \varepsilon < \varepsilon_1$, and all $|u_0 - \bar{u}_0| < \eta$, there exist a time $T = T(\varepsilon; u_0)$ and a solution to (2.1) of the form*

$$u_A(t; u_0) = u_*(t; u_0) + w_r(t; u_0)$$

exists on the time interval $t \in (\varepsilon^{-1}\delta, T)$, such that w_r and T satisfies

(1) $T = T(\varepsilon; u_0) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0) - \delta^{-1} + T_r$ *with* $|T_r| \leq C\varepsilon^{2/3}\delta^{-3}$,

(2) $w_r(T; u_0) = 0$ *and* $u_*(T, u_0) = \delta$,

(3) $|w_r(\varepsilon^{-1}\delta; u_0)| \leq C\varepsilon^{(2-\alpha)/3}$,

(4) $\sup_t |(T_\infty - t)^{2-\alpha} w_r(t; u_0)| \leq C$,

(5) *The function $w_r(\varepsilon^{-1}\delta; \cdot)$ is smooth, with Lipschitz constant $|Lip\ w_r(\varepsilon^{-1}\delta; \cdot)| \leq C\varepsilon^{2/3}$.*

We will prove this theorem in the following sections.

3.2 The exit time $T(u_0)$

The exit time T is defined by the boundary condition

$$\delta = u_*(T; u_0) = \varepsilon^{1/3} u_R(\varepsilon^{1/3}(T - \varepsilon^{-1}\delta); u_0),$$

since the expansion for u_R is given in (2.6) , if we define $s_T = \varepsilon^{1/3}(T - \varepsilon^{-1}\delta)$, then s_T satisfies

$$\frac{1}{\Omega_\infty - s_T} + (\Omega_\infty - s_T) r(\Omega_\infty - s_T) = \varepsilon^{-1/3}\delta,$$

from which we get the leading order expansion $\Omega_\infty - s_T = \mathcal{O}(\varepsilon^{1/3}\delta^{-1})$. A fixed point argument gives

$$\Omega_\infty - s_T = \varepsilon^{1/3}\delta^{-1} + \mathcal{O}(\varepsilon\delta^{-3}),$$

hence the expansion for $T = T(\varepsilon; u_0)$ is

$$T = T(\varepsilon; u_0) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0) - \delta^{-1} + T_r, \tag{3.3}$$

with $|T_r| \leq C\varepsilon^{2/3}\delta^{-3}$, for some constant C independent of u_0, as $\varepsilon \to 0$.

For conveneience, we define

$$T_\infty = T_\infty(\varepsilon; u_0) = \varepsilon^{-1}\delta + \varepsilon^{-1/3}\Omega_\infty(u_0),$$

so that $T = T_\infty - \delta^{-1} + T_r$.

3.3 Equation for w_r and rescaling

We now plug in the anstaz $u = u_* + w_r$ into equation (2.1), and derive the equation for w_r

$$\begin{aligned} w_r' - 2u_* w_r &= w_r^2 + (u_* + w_r)^3 \\ &= 3u_*^2 w_r + (1 + 3u_*)w_r^2 + w_r^3 + u_*^3 := R_r(w_r) = R_r(w_r; \varepsilon, u_0), \end{aligned} \tag{3.4}$$

moreover, we enforce the boundary condition $u(T; u_0) = \delta$, hence this gives the boundary condition for w_r at $t = T$:

$$w_r(T; u_0) = 0, \tag{3.5}$$

therefore, equation (3.4) is posed on the interval $t \in (\varepsilon^{-1}\delta, T)$, with boundary condition (3.5).

Next, we rescale equation (3.4) into σ-time variable by using the t to σ-time rescaling in section 2.3, and obtain

$$\left(\frac{d}{d\sigma} - a(\sigma; \varepsilon, u_0)\right) W_r = \varepsilon^{-1/3}\varphi \mathcal{R}_r(W_r; \varepsilon, u_0), \tag{3.6}$$

where

- The term $a(\sigma; \varepsilon, u_0)$ is defined as and has asymptotics

  $$a(\sigma; \varepsilon, u_0) := 2\varphi(\sigma) u_R(\psi(\sigma); u_0) = 2 + \mathcal{O}(e^{-2\sigma}) \text{ as } \sigma \to \infty,$$

  we remark that this convergence as $\sigma \to \infty$ is uniform in u_0 due to the definition of our time-rescaling.

- The function $W_r(\sigma)$ is the rescaled version of $w_r(t)$ in the σ-variable, $w_r(t) = w_r(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = W_r(\sigma)$. U_* is similarly the rescaled version of u_*, $U_*(\sigma; u_0) = u_*(t; u_0) = \varepsilon^{1/3} u_R(\psi(\sigma; u_0); u_0)$.

- The function $\mathcal{R}_r$ is a rescaled version of R_r such that $\mathcal{R}_r(W_r; \varepsilon, u_0) = 3U_*^2 W_r + (1 + 3U_*)W_r^2 + U_*^3$, To get the corresponding boundary condition of 3.5, we need to know the corresponding σ-time for the t-time interval $t \in (\varepsilon^{-1}\delta, T)$.

At $t = \varepsilon^{-1}\delta$, the corresponding σ time is at $\sigma = \sigma_m$, from its definition in section 2.3.

At $t = T$, we have $\varepsilon^{1/3}(T - \varepsilon^{-1}\delta) = \Omega_\infty - \varepsilon^{1/3}\delta^{-1} + \varepsilon^{1/3}T_r = \psi(\sigma_T) = \Omega_\infty - e^{-\sigma_T}$ from (3.3), hence, for ε small enough, we get that the corresponding σ-time to $t = T$ is

$$\sigma_T = \sigma_T(u_0) = -\log(\varepsilon^{1/3}(\delta^{-1} - T_r)) = -\log(\varepsilon^{1/3}\delta^{-1}) - \log(1 - \delta T_r), \tag{3.7}$$

Then, the complete boundary value problem we wish to solve is

$$\begin{gathered} \frac{d}{d\sigma}W_r - a(\sigma;\varepsilon,u_0)W_r = \varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r),\ \sigma \in (\sigma_m, \sigma_T) \\ W_r(\sigma_T) = 0. \end{gathered} \tag{3.8}$$

3.4 Linear equation and norms

Our goal now is to solve (3.8) on an appropriate function space, to do so we first slightly enlarge the time interval (σ_m, σ_T) where the boundary value problem is posed.

From the definition of σ_T (3.7), we see that

$$|\sigma_T - (-\log(\varepsilon^{1/3}\delta^{-1}))| \leq |\log(1 - \delta T_r)| \leq C|\delta T_r| \leq C\varepsilon^{2/3}\delta^{-2}, \tag{3.9}$$

for some constant C independent of u_0.

We now define σ_{inf} and σ_{sup} as follows:

$$\sigma_{\text{inf}} = \inf_{|u_0 - \bar{u}_0| < \eta} \sigma_m(u_0), \qquad \sigma_{\text{sup}} = \sup_{|u_0 - \bar{u}_0| < \eta} \sigma_T(u_0)$$

From the definition of σ_m in section 2.3 and (3.9) we have

$$M \approx \sigma_{\text{inf}}, \qquad \sigma_{\text{sup}} \approx \log(\varepsilon^{1/3}\delta^{-1}),$$

with the error indepedent of u_0. Therefore we introduce the function space below:

$$\mathcal{C}_r = \left\{ w(\sigma) : \sup_{\sigma_{\text{sup}} \geq \sigma \geq \sigma_{\text{inf}}} \left| \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} w(\sigma) \right| < \infty \right\}.$$

We establish the invertibility of the linear operator A_r which acts on $w \in \mathcal{C}_r$ as

$$A_r w = \left(\frac{d}{d\sigma}w - a(\sigma;\varepsilon,u_0)w, w(\sigma_T) \right),$$

in the following

Proposition 3.2. *$A_r = A_r(u_0, \varepsilon) : \mathcal{D} \subset \mathcal{C}_r \to \mathcal{C}_r \times \mathbb{R}$ and is invertible, with its inverse smoothly depends and uniformly bounded in u_0, ε.*

Proof. Consider the conjugate opertor of A_r, given by

$$\begin{aligned}\tilde{A}_r v &= \left(e^{(\alpha-2)\sigma}\left(\frac{d}{d\sigma} - a(\sigma;\varepsilon,u_0)\right)e^{(2-\alpha)\sigma}v, v(\sigma_T)\right)\\ &= \left(\frac{d}{d\sigma}v - (\alpha + a(\sigma;\varepsilon,u_0) - 2)v, v(\sigma_T)\right),\end{aligned}$$

for $v(\sigma) = \varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma}w(\sigma) \in \mathcal{C}([\sigma_{\inf}, \sigma_{\sup}])$.

The associated conjugate equation of

$$A_r w = (f, w_T) \text{ with } f \in \mathcal{C}_r, w_T \in \mathbb{R}$$

is

$$\tilde{A}_r v = (\varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma}f, v_T) \text{ with } \varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma}f \in C, v_T = \varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma_T}w_T.$$

Since $\alpha > 0$ and $|a(\sigma;\varepsilon,u_0) - 2| = \mathcal{O}(e^{-2\sigma}) \to 0$ as $\sigma \to \infty$ uniformly in ε, u_0. We apply lemma 6.4 with $L = \sigma_T$ to conclude that there exists a constant C independent of ε, u_0 with

$$\|w\|_{\mathcal{C}_r} = |v|_\infty \le C(|\varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma}f|_\infty + |v_T|) \le C(\|f\|_{\mathcal{C}_r} + |w_T|),$$

notice $|v_T| \le w_T$ by the asymptotics of $\sigma_T = \mathcal{O}(-\log(\varepsilon^{1/3}))$. By the definition of $\sigma_{\inf}$ and $\sigma_{\sup}$, A_r is uniformly invertible in u_0 on $\mathcal{C}_r$, this finishes the proof of the proposition. ■

3.5 Nonlinear estimates

In this section we estimate the nonlinear term

$$\mathcal{R}_r(W_r) = 3U_*^2 W_r + (1 + 3U_*)W_r^2 + W_r^3 + U_*^3,$$

in the $\mathcal{C}_r$ norm to prove

Proposition 3.3. *If $W_r \in \mathcal{C}_r$, then $\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r) \in \mathcal{C}_r$, and*

$$\|\varepsilon^{-1/3}\varphi\mathcal{R}_r\| = \mathcal{O}(\delta^\alpha). \tag{3.10}$$

Proof. Proposition 2.6 shows

$$U_*(\sigma;u_0) = \varepsilon^{\frac{1}{3}}(e^\sigma + e^{-\sigma}r(e^{-\sigma};u_0)) \text{ as } \sigma \to \infty,$$

therefore $|u_*(\sigma)| \lesssim \varepsilon^{\frac{1}{3}}e^\sigma$ for all $\sigma \ge \sigma_{\inf}$.

As $W_r \in \mathcal{C}_r$, we have

$$|W_r(\sigma)| \lesssim \varepsilon^{\frac{2-\alpha}{3}}e^{(2-\alpha)\sigma}.$$

Using these facts, we have

$$\|\varepsilon^{-\frac{1}{3}}\varphi U_*^3\|_{\mathcal{C}_r} \lesssim \varepsilon^{\frac{\alpha}{3}}e^{\alpha\sigma} \lesssim \varepsilon^{\frac{\alpha}{3}}e^{\alpha\sigma_{\sup}} \lesssim \delta^\alpha,$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi W_r^2\|_{\mathcal{C}_r} = \sup|\varepsilon^{-\frac{1}{3}}\varphi W_r| \lesssim \varepsilon^{\frac{1-\alpha}{3}}e^{(1-\alpha)\sigma} \lesssim \varepsilon^{\frac{1-\alpha}{3}}e^{(1-\alpha)\sigma_{\sup}} \lesssim \delta^{1-\alpha},$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi W_r^3\|_{\mathcal{C}_r} = \sup|\varepsilon^{-\frac{1}{3}}\varphi W_r^2| \lesssim \varepsilon^{\frac{3-2\alpha}{3}} e^{(3-2\alpha)\sigma} \lesssim \varepsilon^{\frac{3-2\alpha}{3}} e^{(3-2\alpha)\sigma_{\sup}} \lesssim \delta^{3-2\alpha},$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi U_*^2 W_r\|_{\mathcal{C}_r} = \sup|\varepsilon^{-\frac{1}{3}}\varphi u_r^2| \lesssim \varepsilon^{\frac{1}{3}} e^{\sigma} \lesssim \varepsilon^{\frac{1}{3}} e^{\sigma_{\sup}} \lesssim \delta.$$

■

3.6 Fixed point argument and the proof of Theorem 3.1

In this section we prove theorem 3.1 by setting up an appropriate fixed point argument.

Proof of theorem 3.1. Items (1) and (2) in the assertion of the theorem has been demonstrated in section 3.2 and 3.3. Items (3) and (4) is a direct consequence of the fact that $W_r \in \mathcal{C}_r$, to prove this, we first rewrite equation (3.6) and the boundary condition $W_r(\sigma_T) = w_T$ as

$$F_r(W_r, W_T; \varepsilon, u_0) = 0$$

where $F_r : \mathcal{C}_r \times \mathbb{R} \to \mathcal{C}_r \times \mathbb{R}$ is defined as

$$\begin{aligned} F_r(W_r, w_T; \varepsilon, u_0) &= A_r W_r - \left(\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), w_T\right) \\ &= \left(\frac{d}{d\sigma}W_r - aW_r - \varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), W_r(\sigma_T) - w_T\right). \end{aligned}$$

Let $X = \mathcal{C}_r \times (-\delta_1, \delta_1)$, where $\delta_1 = \mathcal{O}(\delta^{2-\alpha})$ is small, we introduce the solution map $\mathcal{S} : X \to \mathcal{C}_r \times \mathbb{R}$ as follows:

$$\mathcal{S}(W_r, w_T; \varepsilon, u_0) = (W_r - A_r^{-1}F_r(W_r, w_T; \varepsilon, u_0), w_T)$$

From propositions 3.2 and 3.3, we conclude

- $\|\mathcal{S}(0,0;\varepsilon,u_0)\| = \|\left(-A_r^{-1}F_r(0,0;\varepsilon,u_0),0\right)\| \leq \|A_r^{-1}\|\|F_r(0,0;\varepsilon,u_0)\| \lesssim |\|\varepsilon^{-1/3}\varphi\mathcal{R}_r(0)\| \lesssim \delta^{\alpha}$, uniformly in ε and u_0.
- $\mathcal{S}$ is a smooth map in W_r, w_T as well as the parameters ε, u_0.
- The linearization of $\mathcal{S}$ at $(0,0)$, $D_{(W_r,w_T)}\mathcal{S}(0,0)$, is equal to $A_r^{-1}\varepsilon^{-1/3}\varphi(3u_r^2)$, whose norm satisfies

  $$\|A_r^{-1}\varepsilon^{-1/3}\varphi(3u_r^2)\| \lesssim \sup|\varepsilon^{-1/3}\varphi(3u_r^2)| = \mathcal{O}(\delta)$$

  Moreover, for $\|W_r\|$ small enough and $|w_T| \leq \delta_1$, we have $D_{(W_r,w_T)}\mathcal{S}(W_r, w_T; \varepsilon, u_0) = A_r^{-1}(3u_r^2) + \mathcal{O}(\|W_r\|_{\mathcal{C}_r})$, which is uniformly small in ε and u_0.

Therefore, for (W_r, w_T) in a small ball of X, we can apply an iteration scheme and utilize the Banach fix point theorem to the existence of a fixed point, hence a solution to equation (3.6) exists. Moreover, this solution depends smoothly on the parameter ε, u_0. By picking $w_T = 0$, we have shown that a unique fixed point $W_r \in \mathcal{C}_r$ exists and solves equation (3.6).

Finally, to prove item (5) we need to estimate the Lipschitz constant for the map

$$\Psi : u_0 \mapsto w_r(\varepsilon^{-1}\delta; u_0) = W_r(\sigma_m; u_0),$$

which maps from a small interval I containing $\bar{u}_0$ to $\mathbb{R}$. We can write Ψ as the composition of two maps $\Psi = \Psi_1 \circ \Psi_2$ where $\Psi_2 : I \to \mathcal{C}_r$ is the map

$$u_0 \mapsto W_r(\sigma; u_0),$$

and $\Psi_1 : \mathcal{C}_r \to \mathbb{R}$ is the evaluation map

$$W_r(\sigma; u_0) \mapsto W_r(\sigma_m, u_0).$$

To estimate the number $\mathrm{Lip}_{u_0}\Psi$, we need to estimate the number $\mathrm{Lip}_{u_0}\Psi_2$ and Lip Ψ_1.

To estimate $\mathrm{Lip}_{u_0}\Psi_2$, it suffices to estimate the following two quantities

$$C_1 = \mathrm{Lip}_{W_r}\mathcal{S}, \text{ and } C_2 = \mathrm{Lip}_{u_0}\mathcal{S},$$

because W_r is the fixed point of the map $\mathcal{S}$, which implies

$$\mathrm{Lip}_{u_0}\Psi_2 \leq C_2/(1 - C_1).$$

From the definition of $\mathcal{S}$, we see that

$$C_1 \leq \mathrm{Lip}_{W_r}|\varepsilon^{-1/3}\varphi R_r(W_r)| \leq \mathrm{Lip}_{W_r}|\varepsilon^{-1/3}\varphi W_r^2| \leq \sup_{W_r \in \mathcal{C}_r}|\varepsilon^{-1/3}\varphi W_r| = \mathcal{O}(\delta^{1-\alpha}),$$

where the last line follows from proposition 3.3.

To estimate C_2. We notice that

$$C_2 \leq \ \mathrm{Lip}\ _{u_0}|\varepsilon^{-1/3}\varphi U_*^3(\sigma; u_0)|.$$

However if $u_0, \tilde{u}_0$ are two different numbers near $\bar{u}_0$,

$$\|\varepsilon^{-1/3}\varphi[U_*^3(\sigma; u_0) - U_*^3(\sigma; \tilde{u}_0)]\|_{\mathcal{C}_r} \leq \|\varepsilon^{-1/3}\varphi U_*^2\|_{\mathcal{C}_r} \sup|U_*(\sigma; u_0) - U_*(\sigma; \tilde{u}_0)|,$$

proposition 2.6 shows $U_*(\sigma; u_0) = \varepsilon^{1/3}(e^\sigma + e^{-\sigma}r(e^{-\sigma}; u_0))$ for σ large, hence

$$\partial_{u_0}U_*(\sigma; u_0) \leq C\varepsilon^{1/3},$$

for some constant independent of u_0, on the other hand

$$\|\varepsilon^{-1/3}\varphi U_*^2\|_{\mathcal{C}_r} = \mathcal{O}(\varepsilon^{(\alpha-1)/3}),$$

so we conclude that

$$\|\varepsilon^{-1/3}\varphi[U_*^3(\sigma; u_0) - U_*^3(\sigma; \tilde{u}_0)]\|_{\mathcal{C}_r} \leq C\varepsilon^{\alpha/3}|u_0 - \tilde{u}_0|,$$

or $C_2 = \mathcal{O}(\varepsilon^{\alpha/3})$. Hence

$$\mathrm{Lip}_{u_0}\Psi_2 = \mathcal{O}(\varepsilon^{\alpha/3}).$$

On the other hand, the evulation map Ψ_1 is a linear map which satisfies

$$|W(\sigma_m) - \widetilde{W}(\sigma_m)| \leq \|W - \widetilde{W}\|_{\mathcal{C}_r}\varepsilon^{(2-\alpha)/3}e^{(2-\alpha)\sigma_m} \lesssim \varepsilon^{(2-\alpha)/3}\|W - \widetilde{W}\|$$

from the definiton of its norm, therefore

$$\text{Lip } \Psi_1 = \mathcal{O}(\varepsilon^{(2-\alpha)/3})$$

combine the two estimates we conclude that

$$\text{Lip}_{u_0}\Psi \leq \left(\text{Lip}_{u_0}\Psi_2\right)(\text{Lip } \Psi_1) = \mathcal{O}(\varepsilon^{2/3}),$$

which comples the proof of Theore 3.1. ■

4 Region B

Region B corresponds to the t-time interval $t^* < t < \varepsilon^{-1}\delta$.

4.1 Ansatz in region B

The ansatz takes the form

$$u(t) = \bar{u}_*(t) + w_\ell(t).$$

Where $\bar{u}_*$ is the function

$$\bar{u}_*(t) = u_*(t; \bar{u}_0) = \varepsilon^{1/3}u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta); \bar{u}_0) = \varepsilon^{1/3}\bar{u}_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta)). \tag{4.1}$$

$\bar{u}_*$ solves the equation

$$\frac{d}{dt}\bar{u}_*(t) = \mu(t) + \bar{u}_*^2(t), \tag{4.2}$$

that is, $\bar{u}_*$ is merely a rescaled version of the special solution to the Riccati equation.

Similarly to the situation in region A, w_ℓ is a correction term whose properties are summerized in the following

Theorem 4.1. *For all δ, α small enough, there exists ε_2, C, such that for all $0 < \varepsilon < \varepsilon_2$, and a solution to (2.3) of the form*

$$u_B(t) = \bar{u}_*(t) + w_\ell(t),$$

exists on the time interval $t \in (t^*, \varepsilon^{-1}\delta)$, where w_ℓ satisfies

$$w_\ell(t) \leq C\varepsilon^{(2-\alpha)/3}|\varepsilon^{1/3}(t - \varepsilon^{-1}\delta) + 1|; \quad w_\ell(\sigma^*) = w^* = \mathcal{O}(\varepsilon^{1/2-\alpha/3}). \tag{4.3}$$

We prove this theorem in the rest of this section.

4.2 Equation of W_ℓ and rescaling

As before, we plug in the ansatz into (2.3) to derive the equation satisfied by w_ℓ.

$$\begin{aligned} w_\ell' - 2\bar{u}_* w_\ell &= w_\ell^2 + (\bar{u}_* + w_\ell)^3 \\ &= (3\bar{u}_*^2)w_\ell + (1+3\bar{u}_*)w_\ell^2 + w_\ell^3 + \bar{u}_*^3 := R_\ell(w_\ell). \end{aligned} \tag{4.4}$$

We want to solve this equation on $t \in (t^*, \varepsilon^{-1}\delta)$. Following previous steps, we next rescale the equation to the σ-time variable using the time rescaling map $\psi = \psi(\sigma; \bar{u}_0)$ and we obtain

$$\frac{d}{d\sigma} W_\ell - b(\sigma) W_\ell = \varepsilon^{-1/3} \varphi \mathcal{R}_\ell(W_\ell), \tag{4.5}$$

where

- The equation is posed on $\sigma \in (\sigma^*, \sigma_m(\bar{u}_0))$ where $\sigma^* \approx -\varepsilon^{-1/4}$ and $\sigma_m(\bar{u}_0) := \bar{\sigma}_m$ follows the notation used in section 2.3.
- The term $b(\sigma)$ is defined and has asymptotics:

  $$b(\sigma) := 2u_R(\psi(\sigma))\varphi(\sigma) = -2 + \mathcal{O}(|\sigma|^{-1})$$

  as $\sigma \to -\infty$. Again, the convergence is uniform in ε.
- The function $W_\ell(\sigma)$ is the rescaled version of $w_\ell(t)$ in the σ-variable, $w_\ell(t) = w_\ell(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = W_\ell(\sigma)$. $\bar{U}_*$ is similarly the rescaled version of $\bar{u}_*$, $\bar{U}_*(\sigma) = \bar{u}_*(t) = \varepsilon^{1/3}\bar{u}_R(\psi(\sigma))$.
- The function $\mathcal{R}_\ell$ is a rescaled version of R_ℓ such that $\mathcal{R}_\ell(W_r; \varepsilon, u_0) = 3\bar{U}_*^2 W_r + (1+3\bar{U}_*)W_r^2 + \bar{U}_*^3$,

We also need to specify the boundary value at the left end point $\sigma = \sigma^*$, the complete system we want to solve is

$$\begin{aligned} \frac{d}{d\sigma} W_\ell - b(\sigma) W_\ell &= \varepsilon^{-1/3} \varphi \mathcal{R}_\ell(W_\ell),\ \sigma(\sigma^*, \bar{\sigma}_m) \\ W_\ell(\sigma^*) &= w^*. \end{aligned} \tag{4.6}$$

4.3 Linear equation and norms

Similarly, the proof of theorem 4.1 consists of solving (4.5) via a fixed point argument on the following function space:

$$\mathcal{C}_\ell = \left\{ w(\sigma) : \sup_{\sigma^* < \sigma < \bar{\sigma}_m} |\varepsilon^{(\alpha-2)/3} \langle \sigma \rangle^{-2/3} w(\sigma)| < \infty \right\}.$$

To begin with, let us define the operator A_ℓ by

$$A_\ell w = \left(\frac{d}{d\sigma} w - b(\sigma) w, w(\sigma^*) \right),$$

for $w \in \mathcal{D}(A_\ell) \subset \mathcal{C}_\ell$.

Proposition 4.2. *$A_\ell : \mathcal{D}(A_\ell) \subset C_{W_\ell} \to C_{W_\ell} \times \mathbb{R}$, and A_ℓ is bounded invertible with its inverse uniformly bounded in ε.*

Proof. Similar to the proof of proposition 3.2, let $v(\sigma) = \varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}w(\sigma)$, we consider the conjugate linear operator

$$\begin{aligned}\tilde{A}_\ell v &= \left(\langle\sigma\rangle^{-2/3}\left(\frac{d}{d\sigma} - b(\sigma;\varepsilon)\right)\langle\sigma\rangle^{2/3}v, v(\sigma^*)\right)\\ &= \left(\frac{d}{d\sigma}v - \tilde{b}(\sigma;\varepsilon)v, v(\sigma^*)\right),\end{aligned}$$

where $\tilde{b}$ satisfies

$$\tilde{b} = b(\sigma;\varepsilon) - \frac{2}{3}\langle\sigma\rangle^{-1} = -2 + \mathcal{O}(|\sigma|^{-1}) \to -2,$$

uniformly in ε as $\sigma \to -\infty$.

The associated conjugate equation of

$$A_\ell w = (f, w^*) \text{ with } f \in \mathcal{C}_\ell, w^* \in \mathbb{R}$$

is

$$\tilde{A}_\ell v = (\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}f, v^*) \text{ with } \varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}f \in C, v^* = \varepsilon^{(\alpha-2)/3}\langle\sigma^*\rangle^{-2/3}w^*.$$

Again we apply lemma 6.2 to conclude that there exist a constant C independent of ε such that

$$\|w\|_{\mathcal{C}_\ell} = |v|_\infty \le C(|\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}f|_\infty + |v^*|) = C(\|f\|_{\mathcal{C}_\ell} + \varepsilon^{\alpha/3-1/2}|w^*|),$$

which shows the claim, provided that $|w^*| = \mathcal{O}(\varepsilon^{1/2-\alpha/3})$. ∎

4.4 Nonlinear estimates

For $\sigma \in (\sigma^*, \bar{\sigma}_m)$, we will estimate the nonlinear term

$$\mathcal{R}_\ell(W_\ell) = \varepsilon^{-1/3}\varphi(\sigma)\left[(3\bar{U}_*^2)W_\ell + (1 + 3\bar{U}_*)W_\ell^2 + W_\ell^3 + \bar{U}_*^3\right]$$

in the $\mathcal{C}_\ell$ norm. As a result, we have

Proposition 4.3. *If $W_\ell \in \mathcal{C}_\ell$, then $\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell(\sigma)) \in \mathcal{C}_\ell$ and $\|\varepsilon^{-1/3}\varphi\mathcal{R}_\ell\|_{\mathcal{C}_\ell} = \mathcal{O}(\varepsilon^{\alpha/3})$.*

Proof. From the asymptotics (2.5), we have that

$$\bar{U}_*(\sigma) = \varepsilon^{1/3}\bar{u}_R(\psi(\sigma)) \lesssim |\varepsilon\sigma|^{1/3}, \quad \varphi(\sigma) \lesssim |\sigma|^{-1/3}$$

for $\sigma^* \le \sigma \le \bar{\sigma}_m$.

If $W_\ell \in \mathcal{C}_\ell$, then it is true that

$$|W_\ell(\sigma)| \lesssim \varepsilon^{(2-\alpha)/3}\langle\sigma\rangle^{2/3},$$

also recall $|\sigma^*| = \mathcal{O}(\varepsilon^{-1/4})$, from these facts we have

$$\|\varepsilon^{-1/3}\varphi\bar{U}_*^3\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar{\sigma}_m)} \varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}|\varepsilon\sigma|^{2/3} = \mathcal{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-1/3}\varphi W_\ell^2\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar\sigma_m)} \varepsilon^{(1-\alpha)/3}\langle\sigma\rangle^{1/3} \le \varepsilon^{(1-\alpha)/3}\langle\sigma^*\rangle^{1/3} = \mathcal{O}(\varepsilon^{(3-4\alpha)/12}),$$

$$\|\varepsilon^{-1/3}\varphi W_\ell^3\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar\sigma_m)} \varepsilon^{(3-2\alpha)/3}\langle\sigma\rangle \le \varepsilon^{(3-2\alpha)/3}\langle\sigma^*\rangle = \mathcal{O}(\varepsilon^{(9-8\alpha)/12}),$$

$$\|\varepsilon^{-1/3}\varphi \bar U_*^2 W_\ell\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar\sigma_m)} |\varepsilon^{-1/3}\varphi \bar U_*^2| \lesssim |\varepsilon\sigma^*|^{1/3} = \mathcal{O}(\varepsilon^{1/4}),$$

since $\alpha << 1$, we conclude that $\|\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell)\|_{\mathcal{C}_\ell} = \mathcal{O}(\varepsilon^{\alpha/3})$ if $W_\ell \in \mathcal{C}_\ell$. ■

4.5 Fixed point argument and the proof of Theorem 4.1

We are ready to prove theorem 4.1.

Proof of theorem 4.1. The proof consists of rewriting equation (3.8) as a fixed point equation. Using the linear operator A_ℓ, we define $F_\ell : \mathcal{C}_\ell \times \mathbb{R} \to \mathcal{C}_\ell \times \mathbb{R}$ as

$$F_\ell(W_\ell, w^*; \varepsilon) = A_\ell W_\ell - (\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell), w^*)$$

Let $X = \mathcal{C}_\ell \times (-\varepsilon^{1/2-\alpha/3}, \varepsilon^{1/2-\alpha/3})$, we introduce the map $\mathcal{S} : X \to \mathcal{C}_\ell \times \mathbb{R}$ as follows:

$$\mathcal{S}(W_\ell, w^*; \varepsilon) = (W_\ell - A_\ell^{-1}F_\ell(W_\ell, w^*; \varepsilon), w^*)$$

From propositions 4.2 and 4.3, we conclude

- $\|\mathcal{S}(0,0;\varepsilon)\| = \|\left(-A_\ell^{-1}F_\ell(0,0;\varepsilon), 0\right)\| \le \|A_\ell^{-1}\|\|F_\ell(0,0;\varepsilon)\| \lesssim \|\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(0)\| \lesssim \varepsilon^{\alpha/3}$, uniformly in ε.
- $\mathcal{S}$ is a smooth map in W_ℓ, w^* as well as the parameters ε.
- The linearization of $\mathcal{S}$ at $(0,0)$, $D_{(W_\ell,w^*)}\mathcal{S}(0,0)$, is equal to $A_\ell^{-1}\varepsilon^{-1/3}\varphi(3\bar U_*^2)$, whose operator norm satisfies
$$\|A_\ell^{-1}\varepsilon^{-1/3}\varphi(3\bar U_*^2)\| \lesssim \sup|\varepsilon^{-1/3}\varphi(3\bar U_*^2)| = \mathcal{O}(\varepsilon^{1/4}),$$
moreover, for $\|W_\ell\|$ small enough and $|w^*| = \mathcal{O}(\varepsilon^{1/2-\alpha/3})$, we have $D_{(W_\ell,w^*)}\mathcal{S}(W_\ell, w^*; \varepsilon) = A_r^{-1}(3\bar U_*^2) + \mathcal{O}(\|W_\ell\|_{\mathcal{C}_\ell})$, which is uniformly small in ε.

Therefore, for (W_ℓ, w^*) in a small ball of X, we apply Banach's fixed point argument to the map $\mathcal{S}$ to obtain a solution of equation (4.6) with $W_\ell \in \mathcal{C}_\ell$ and $w^* = \mathcal{O}(\varepsilon^{1/2-\alpha/3})$. Scaling back from σ to t-time, we obtain claim (4.3). ■

5 Region C

This region corresponds to the t-time interval $0 < t < t^*$. Recall t^* is the (left) gluing time which corresponds to when $\sigma = \varepsilon^{-1/4} =: \sigma^*$.

5.1 Ansatz in region C

Recall the critical manifold for system (5.1) is the set of points $(u, \mu) \in \mathbb{R}^2$ which satisfies

$$\mu + u^2 + u^3 = 0. \tag{5.1}$$

By the implicit function theorem, whenever $2u + 3u^2 \neq 0$, we can write $u = u(\mu)$, for μ small, let $u_s(t) := u(\mu(t))$

$$0 = \mu(t) + u_s^2(t) + u_s^3(t), \tag{5.2}$$

for $t \in (0, t^*)$.

Then, the ansatz in Region C takes the form

$$u_C(t) = u_s(t) + w_s(t).$$

The properties of the correction term $w_s(t)$ are summerized in the following

Theorem 5.1. *For all δ, α small enough, there exists ε_3, C, such that for all $0 < \varepsilon < \varepsilon_3$, and a solution to (2.3) of the form*

$$u_C(t) = u_s(t) + w_s(t),$$

exists on the time interval $t \in (0, t^*)$, such that w_s satisfies

$$w_s(t) \leq C\varepsilon^{1-\alpha/3}(\varepsilon t - \delta)^{-1}; \quad w_s(0) = w_0 = \mathcal{O}(\delta^{-1}\varepsilon^{1-\alpha/3}). \tag{5.3}$$

This theorem will be proved in the rest of this section.

5.2 Equation of W_s and rescaling

Once again, we plug u_C into (2.3) and use (5.2) to obtain the equation satisfied by w_s.

$$\frac{d}{dt}w_s - 2u_s w_s = (3u_s^2)w_s + (1 + 3u_s)w_s^2 + w_s^3 - \frac{d}{dt}u_s := R_s(w_s), \tag{5.4}$$

which is posed on $t \in (0, t^*)$.

Rescaling to σ time, we obtain

$$\frac{d}{d\sigma}W_s - c(\sigma)W_s = \varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s), \tag{5.5}$$

Where

- The equation is posed on $\sigma \in (\sigma_0, \sigma^*)$. where $\sigma^* \approx -\varepsilon^{-1/4}$ is the left most point in region B and $\sigma_0 = -\frac{2}{3}\delta^{3/2}\varepsilon^{-1}$.
- Like W_r and W_ℓ in region A and region B, the function $W_s(\sigma)$ is the rescaled version of $w_s(t)$ in the σ-variable, $w_s(t) = w_s(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = W_s(\sigma)$.

  It can be calculated that $u_s(t)$ has the following asymptotics

$$u_s(t) = -\sqrt{\delta - \varepsilon t} + \mathcal{O}(|\delta - \varepsilon t|). \tag{5.6}$$

Rescaling to σ variable leads to the asymptotics of the rescaled version of $u_s(t)$:

$$U_s(\sigma) := u_s(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta) = -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathcal{O}(|\varepsilon\sigma|^{2/3}). \tag{5.7}$$

- The function $\mathcal{R}_s$ is likewise a rescaled version of R_s such that

$$\mathcal{R}_s(W_s;\varepsilon) = 3U_s^2 W_s + (1+3U_s)W_s^2 + W_s^3 - \varepsilon^{1/3}\varphi^{-1}\frac{d}{d\sigma}U_s(\sigma),$$

- The term $c(\sigma)$ is defined and has the asymptotics:

$$c(\sigma) = 2\varepsilon^{-\frac{1}{3}}U_s(\sigma)\varphi(\sigma) = -2 + \mathcal{O}(\varepsilon^{1/3}|\sigma|^{1/3}),$$

  as $\sigma \to -\infty$.

Once again we need to specify the boundary value at the left end point $\sigma = \sigma_0$, the complete system we want to solve is

$$\begin{aligned} \frac{d}{d\sigma}W_s - c(\sigma)W_s &= \varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s),\ \sigma \in (\sigma_0, \sigma^*) \\ W_s(\sigma_0) &= w_0. \end{aligned} \tag{5.8}$$

5.3 Linear equation and norms

The proof of theorem 5.1 will be complete if we are able to solve (5.8) using a fixed point argument similar to region A and B. The function space on which we will solve the W_s equation via a fixed point argument is:

$$\mathcal{C}_s = \left\{ w(\sigma) : \sup_{\sigma_0<\sigma<\sigma^*} |\varepsilon^{\frac{\alpha}{3}-1}\langle\varepsilon\sigma\rangle^{\frac{2}{3}}w(\sigma)| < \infty \right\}$$

And similarly we define the linear operator A_s which acts on $w \in \mathcal{D}(A_s) \subset \mathcal{C}_s$ by

$$A_s w = \left(\frac{d}{d\sigma}w - cw, w(\sigma_0)\right).$$

Proposition 5.2. *$A_s : \mathcal{D}(A_s) \subset \mathcal{C}_s \to \mathcal{C}_s \times \mathbb{R}$, and A_s is bounded invertible with its inverse uniformly bounded in ε.*

Proof. Unlike the case for linear operator A_r and A_ℓ, lemma 6.2 cannot be directly used for the operator A_s since from the asymptotics of c we see that $c(\sigma)$ does not converge to -2 as $\sigma \to -\infty$, in fact, it diverges to $-\infty$ as $\sigma \to -\infty$. However, for $\sigma \in (\sigma_0, \sigma^*)$, we have

$$|c(\sigma) - (-2)| \lesssim |\varepsilon\sigma|^{1/3} \lesssim \delta^{1/2},$$

hence for δ small, A_s is a small perturbation of the operator

$$L_s : w \mapsto \left(\frac{d}{d\sigma}w + 2w, w(\sigma_0)\right).$$

To see the invertibility of L_s on the weighted space $\mathcal{C}_s$, let $v(\sigma) = \varepsilon^{\alpha/3-1}\langle\varepsilon\sigma\rangle^{2/3}w(\sigma)$ and consider the conjugate linear opeartor

$$\begin{aligned}\tilde{L}_s : v \mapsto &\left(\langle\varepsilon\sigma\rangle^{-2/3}\left(\frac{d}{d\sigma}+2\right)\langle\varepsilon\sigma\rangle^{2/3}v(\sigma), v(\sigma_0)\right)\\ &= \left(\left(\frac{d}{d\sigma}+2+\frac{2}{3}\varepsilon\langle\varepsilon\sigma\rangle^{-1}\right)v, v(\sigma^*)\right),\end{aligned}$$

which acts on $v \in \mathcal{C}[\sigma_0, \sigma^*]$.

Hence, the conjugate linear equation of

$$L_s w = (f, w_0)$$

is

$$\tilde{L}_s v = (\tilde{f}, v_0)$$

with $v_0 = \varepsilon^{\alpha/3-1}\langle\varepsilon\sigma_0\rangle^{2/3}w_0$ and $\tilde{f} = \varepsilon^{\alpha/3-1}\langle\varepsilon\sigma\rangle^{2/3}f$, which is a differential equation of the form

$$\left(\frac{d}{d\sigma}+2+\mathcal{O}(\varepsilon)\right)v = f, \qquad v(\sigma_0) = v_0.$$

Its linear part is yet another small perturbation of the linear operator $\frac{d}{d\sigma}+2$ on the uniform space $\mathcal{C}[\sigma_0, \sigma^*]$, integrating this equation yields

$$|v|_\infty \leq C(|v_0| + |f|_\infty),$$

for some constant C independent of ε. Equivalently, in terms of w we have

$$\|w\|_{\mathcal{C}_s} \leq C(|\delta\varepsilon^{\alpha/3-1}w_0| + \|f\|_{\mathcal{C}_s}),$$

which shows the invertibility of L_s and uniformity of the inverse in ε, provided that $w_0 = \mathcal{O}(\delta^{-1}\varepsilon^{1-\alpha/3})$. The same property hence holds for A_s as well. ■

5.4 Nonlinear estimates

For $\sigma \in (\sigma_0, \sigma^*)$, we estimate the nonlinear term $\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s(\sigma))$ in the $\mathcal{C}_s$ norm. Similar to proposition 3.3 and 4.3, we have

Proposition 5.3. *If $W_s \in \mathcal{C}_s$, then $\varepsilon^{-\frac{1}{3}}\varphi\mathcal{R}_s(W_s(\sigma)) \in \mathcal{C}_s$ and $\|\varepsilon^{-\frac{1}{3}}\varphi\mathcal{R}_s\|_{\mathcal{C}_s} = \mathcal{O}(\delta^{1/2})$.*

Proof. Recall that

$$\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s;\varepsilon) = \varepsilon^{-1/3}\varphi\left[3U_s^2W_s + (1+3U_s)W_s^2 + W_s^3\right] - \frac{d}{d\sigma}U_s(\sigma),$$

From the definition of $\psi(\sigma)$ we have

$$|\varphi(\sigma)| \lesssim |\sigma|^{-1/3}.$$

From (5.7), we have that

$$\begin{aligned}U_s(\sigma) &= -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathcal{O}(|\varepsilon\sigma|^{2/3})\\ \frac{d}{d\sigma}U_s(\sigma) &= -\frac{1}{2}\varepsilon(\varepsilon\sigma)^{-2/3} + \mathcal{O}(\varepsilon|\varepsilon\sigma|^{-1/3}),\end{aligned}$$

and for $W_s \in \mathcal{C}_s$, it holds that

$$|W_s(\sigma)| \lesssim \varepsilon^{1-\alpha/3}\langle\varepsilon\sigma\rangle^{-2/3}.$$

Hence we have the following estimates:

$$\left\|\frac{d}{d\sigma}U_s(\sigma)\right\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{\alpha/3-1}\langle\sigma\rangle^{2/3}\varepsilon|\varepsilon\sigma|^{-2/3} = \mathcal{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-1/3}\varphi W_s^2(\sigma)\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}\varepsilon^{1-\alpha/3}\langle\varepsilon\sigma\rangle^{-2/3} = \mathcal{O}(\varepsilon^{\frac{1}{4}-\alpha/3}),$$

$$\|\varepsilon^{-1/3}\varphi W_s^3(\sigma)\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}[\varepsilon^{1-\alpha/3}\langle\varepsilon\sigma\rangle^{-2/3}]^2 = \mathcal{O}(\varepsilon^{\frac{3}{4}-2\alpha/3}),$$

$$\|\varepsilon^{-1/3}\varphi U_s^2 W_s(\sigma)\|_{C_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}|\varepsilon\sigma|^{2/3} = \mathcal{O}(\delta^{1/2}),$$

■

5.5 Fixed point argument and the proof of Theorem 5.1

Proof of theorem 5.1. The proof is almost identical to the proof of 4.1. Using the linear operator A_s, we define the nonlinear operator $F_s : \mathcal{C}_s \times \mathbb{R} \to \mathcal{C}_s \times \mathbb{R}$ as

$$F_s(W_s, w_0; \varepsilon) := A_s W_s - (\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s), w_0)$$

Let $X = \mathcal{C}_s \times (-\delta^{-1}\varepsilon^{1-\alpha/3}, \delta^{-1}\varepsilon^{1-\alpha/3})$, we introduce the map $\mathcal{S} : X \to \mathcal{C}_s \times \mathbb{R}$ as follows:

$$\mathcal{S}(W_s, w_0; \varepsilon) := (W_s - A_s^{-1}F_s(W_s, w_0; \varepsilon), w_0).$$

We conclude from proposition 5.2 and 5.3 that:

- $\|\mathcal{S}(0, 0; \varepsilon)\| = \|\left(-A_s^{-1}F_s(0, 0; \varepsilon), 0\right)\| \leq \|A_s^{-1}\|\|F_s(0, 0; \varepsilon)\| \lesssim \|\varepsilon^{-1/3}\varphi\mathcal{R}_s(0)\| \lesssim \varepsilon^{\alpha/3}$, uniformly in ε.
- $\mathcal{S}$ is a smooth map in W_s, w_0 as well as the parameter ε.
- The linearization of $\mathcal{S}$ at $(0, 0)$, $D_{(W_s,w_0)}\mathcal{S}(0, 0)$, is equal to $A_s^{-1}\varepsilon^{-1/3}\varphi(3U_s^2)$, whose operator norm satisfies
$$\|A_s^{-1}\varepsilon^{-1/3}\varphi(3U_s^2)\| \lesssim \sup|\varepsilon^{-1/3}\varphi(3U_s^2)| = \mathcal{O}(\delta^{1/2}),$$
moreover, for $\|W_s\|$ small enough and $|w_0| = \mathcal{O}(\delta^{-1}\varepsilon^{1-\alpha/3})$, we have $D_{(W_s,w_0)}\mathcal{S}(W_s, w_0; \varepsilon) = A_s^{-1}(3U_s^2)+$ $\mathcal{O}(\|W_s\|_{\mathcal{C}_s})$, which is uniformly small in ε.

Therefore, for (W_s, w_0) in a small ball of X, we apply Banach's fixed point argument to the map $\mathcal{S}$ to obtain a solution of equation (5.8) with $W_s \in \mathcal{C}_s$ and $w_0 = \mathcal{O}(\varepsilon^{1-\alpha/3})$. Scaling back from σ to t-time, we obtain claim (5.3). ■

6 Gluing

So far we have showed solutions of the form u_A, u_B, and u_C exists on region A, B and C, respectively.

To show a solution to (2.1) and (2.2) exists on the whole interval $t \in (0, T)$. We first show that the solution u_A, u_B and u_C match at the boundary points of region A, B, and C.

Starting with the left most point in region C, Theorem 5.1 shows (2.1) has a solution of the form

$$u_C(t) = u_s(t) + w_s(t; w_0)$$

exists, provided that we pick $w_0 = \mathcal{O}(\delta^{-1}\varepsilon^{1-\alpha/3})$.

At the t^*, the right end of region C, we see $u_C(t^*) = u_s(t^*) + w_s(t^*; w_0)$ has the following expansion:

$$\begin{aligned} u_s(t^*) &= -\sqrt{\delta - \varepsilon t^*} + \mathcal{O}(|\delta - \varepsilon t^*|) = -\varepsilon^{1/4} + \mathcal{O}(\varepsilon^{1/2}) \\ w_s(t^*) &\lesssim \varepsilon^{1-\alpha/3}(\varepsilon t^* - \delta)^{-1} = \mathcal{O}(\varepsilon^{1/2-\alpha/3}) \\ \implies u_C(t^*) &= -\varepsilon^{1/4} + \mathcal{O}(\varepsilon^{1/2-\alpha/3}) \end{aligned}$$

Notice at t^*, $\bar{u}_*(t^*)$ has the following expansion in ε:

$$\bar{u}_*(t^*) = \varepsilon^{1/3}\bar{u}_R(\varepsilon^{1/3}(t^* - \varepsilon^{-1}\delta)) = -\varepsilon^{1/4} + \mathcal{O}(\varepsilon^{1/2}).$$

Hence, if we set $w^* = u_s(t^*) - \bar{u}_*(t^*) + w_s(t^*)$, then $w^* = \mathcal{O}(\varepsilon^{1/2-\alpha/3})$, we can apply Theorem 4.1 to get a solution of (2.1) of the form

$$u_B(t) = \bar{u}_*(t) + w_\ell(t; w^*),$$

in region B. With the matching condition

$$u_C(t^*) = u_B(t^*). \tag{6.1}$$

These implies, from Theorem 4.1, that at the point $t = \varepsilon^{-1}\delta$, the right end of region B, the value of $u_B(\varepsilon^{-1}\delta) = \bar{u}_*(\varepsilon^{-1}\delta) + w_\ell(\varepsilon^{-1}\delta; w^*)$ has the following expansion in ε:

$$\begin{aligned} \bar{u}_*(\varepsilon^{-1}\delta) &= \varepsilon^{1/3}\bar{u}_R(0) = \varepsilon^{1/3}\bar{u}_0 \\ w_\ell(\varepsilon^{-1}\delta) &= \mathcal{O}(\varepsilon^{(2-\alpha)/3}) \\ \implies u_B(\varepsilon^{-1}\delta) &= \varepsilon^{1/3}\bar{u}_0 + \mathcal{O}(\varepsilon^{(2-\alpha)/3}). \end{aligned}$$

On the other hand, Theorem 3.1 proves a solution to (2.1) of the form $u_A(t; u_0) = u_*(t; u_0) + w_r(t; u_0)$ exists for $t \in [\varepsilon^{-1}\delta, T]$ with $u_A(T; u_0) = \delta$, moreover, at $t = \varepsilon^{-1}\delta$ the value $u_A(\varepsilon^{-1}\delta; u_0)$ has the following expansion in ε:

$$\begin{aligned} u_*(\varepsilon^{-1}\delta; u_0) &= \varepsilon^{1/3}u_R(0; u_0) = \varepsilon^{1/3}u_0 \\ w_r(\varepsilon^{-1}\delta; u_0) &= \mathcal{O}(\varepsilon^{(2-\alpha)/3}) \\ \implies u_A(\varepsilon^{-1}\delta; u_0) &= \varepsilon^{1/3}u_0 + \mathcal{O}(\varepsilon^{(2-\alpha)/3}) \end{aligned}$$

Therefore, the matching condition at $\varepsilon^{-1}\delta$ is

$$u_A(\varepsilon^{-1}\delta; u_0) = u_B(\varepsilon^{-1}\delta), \tag{6.2}$$

Using the expansions we obtained, this amounts to solve the equation

$$0 = \varepsilon^{1/3}(u_0 - \bar{u}_0) + w_r(\varepsilon^{-1}\delta; u_0) - w_\ell(\varepsilon^{-1}\delta), \tag{6.3}$$

in the variable u_0. Let $\phi(\varepsilon; u_0) := w_r(\varepsilon^{-1}\delta; u_0) - w_\ell(\varepsilon^{-1}\delta)$, we conclude from Theorem 3.1 and Theorem 4.1 that

$$\phi(\varepsilon; u_0) = \mathcal{O}(\varepsilon^{2-\alpha)/3})$$

uniformly in u_0 and

$$\mathrm{Lip}_{u_0}\phi(\varepsilon; u_0) = \mathcal{O}(\varepsilon^{2/3}).$$

Hence we divide the right hand side of (6.3) by $\varepsilon^{1/3}$, and apply the implicit function theorem around the point $(u_0 = \bar{u}_0, \varepsilon = 0)$ to conclude that for u_0 such that $|u_0 - \bar{u}_0| = \mathcal{O}(\varepsilon^{(1-\alpha)/3})$ the matching condition (6.2) is satisfied. In conclusion, we have shown

Theorem 6.1 (Gluing of the solutions)**.** *For each $\delta > 0$ and $\eta > 0$ small, there exist ε_0 and u_0 such that for all $0 < \varepsilon < \varepsilon_0$ and $|u_0 - \bar{u}_0| < \eta$, a solution of equation 2.3 exists with $\mu(t) = \varepsilon t - \delta$ and*

$$u(t; u_0) = \begin{cases} u_A(t; u_0), & \textit{for } t \in (\varepsilon^{-1}\delta, T), \\ u_B(t), & \textit{for } t \in (t^*, \varepsilon^{-1}\delta), \\ u_C(t), & \textit{for } t \in (0, t^*) \end{cases},$$

where $u_A(t; u_0)$, $u_B(t)$ and $u_C(t)$ are the solutions that were shown to exist in Theorem 3.1, 4.1 and 5.1. Moreover, u satisfies

$$u(T; u_0) = \delta, \qquad u(0; u_0) = -\sqrt{\delta} + \delta + \mathcal{O}(\varepsilon^{1-\alpha/3}\delta^{-1}).$$

Appendix

In this Appendix we show the main perturbation lemma used to prove the invertibility of the linearized operators at the ansatzs.

Lemma 6.2. *Consider the following boundary value problems*

$$\dot{u}(\sigma) = a(\sigma)u + f(\sigma), \qquad u(L) = u_L, \tag{6.4a}$$

$$\dot{u}(\sigma) = b(\sigma)u + g(\sigma), \qquad u(-M) = u_M, \tag{6.4b}$$

where equation (6.4a) is posed on $\sigma \in [0, L]$ with $L > 0$ and (6.4b) is posed on $\sigma \in [-M, 0]$ with $M > 0$. Assume $a(\sigma), b(\sigma)$ are continuous functions such that

$$a(\sigma) \to a_+ > 0, \qquad \sigma \to \infty, \tag{6.5a}$$

$$b(\sigma) \to b_- < 0, \qquad \sigma \to -\infty, \tag{6.5b}$$

then (6.4) has a unique solutions u_a, u_b which satisfies

$$|u_a|_\infty \leq C_a(u_L + |f|_\infty), \tag{6.6a}$$

$$|u_b|_\infty \leq C_b(u_m + |g|_\infty), \tag{6.6b}$$

for some constants C_a, C_b independent of L and M.

Proof. We only prove the result for (6.4a) since the other case is similar.

Step I

To begin with, consider the asymptotic equation

$$\dot{u} = a_+ u + f(\sigma), \qquad\qquad u(L) = 0. \tag{6.7}$$

posed on $\sigma \in [0, L]$. (6.6a) holds for (6.7) since in this case we have

$$\begin{aligned} u_a(\sigma) &= e^{a_+(\sigma - L)} u_L + \int_L^t e^{a_+(\sigma - s)} f(s) ds \\ &\leq 2|u_L| + \left| \int_L^t e^{a_+(\sigma - s)} ds \right| |f|_\infty \\ &\leq 2|u_L| + \frac{1}{a_+} \left| e^{t-L} - 1 \right| |f|\infty \\ &\leq 2(|u_L| + |f|_\infty). \end{aligned}$$

Step II

Next, give $\eta > 0$ small enough and independent of L, there exist $\sigma_* \leq L$ such that $|a(\sigma) - a_+| < \eta$ for all $\sigma > \sigma*$. It is important to note here that one can choose σ_* independent of L as long as L is large enough. A Neuman series argument shows that in this case the operator

$$u \mapsto \left(\frac{d}{dt} u - a(t)u, u(L) \right)$$

is a $\eta-$perturbation of the asymptotic operator

$$u \mapsto \left(\frac{d}{dt}u - a_+ u, u(L)\right),$$

which acts on the space of coninuous functions $\mathcal{C}([\sigma_*, L])$ with uniform norm (with domain), hence (6.5a) holds with the sup norm taken on $[\sigma_*, L]$.

Step III

Finally, for $\sigma \in [0, \sigma_*]$, the solution is given by the following formula

$$u(\sigma) = \exp\left(\int_{\sigma_*}^{\sigma} a(\tau)d\tau\right) u(\sigma_*) + \int_{\sigma_*}^{\sigma} \exp\left(-\int_{\sigma}^{s} a(\tau)d\tau\right) f(s)ds$$

since $\sigma_* < \infty$ and does not depend on L, there exist a constant C_1 independet of L so that

$$\max\left\{\left|\exp\left(\int_{\sigma_*}^{\sigma} a(\tau)d\tau\right)\right|, \left|\int_{\sigma_*}^{\sigma} \exp\left(-\int_{\sigma}^{s} a(\tau)d\tau\right)\right|\right\} \leq C_1,$$

moreover, the value $u(\sigma_*)$ satisfies

$$u(\sigma_*) \leq \sup_{\sigma \in (\sigma_*, L)} |u(\sigma)| \leq C_2(u_L + |f|_\infty)$$

for some constant C_2 independent of L from the conclusion in step 2. Therefore on $[0, \sigma_*]$ the solution satisfies

$$\sup_{\sigma \in [0, \sigma_*]} |u(\sigma)| \leq C_1 C_2 (u_L + |f|_\infty) + C_1 |f|_\infty \leq C(u_L + |f|_\infty)$$

where obviously C does not depend on L. Therefore we conclude that

$$\sup_{\sigma \in [0, L]} = |u|_\infty \leq C(u_L + |f|_\infty)$$

which is (6.6a). ■