Chapter 1

Curvilinear Coordinates and Metrics - By Dorival Pedroso

1.1 Base vectors and metrics

The position of a point in space can be represented by

$$\underset{\cdot}{\boldsymbol{x}} = \underset{\cdot}{\boldsymbol{x}}(x_1, x_2, x_3) \tag{1.1}$$

where (x_1, x_2, x_3) are a set of *physical* (or "laboratory" or "real world") coordinates. The x_i may be curvilinear or not. We also introduce another set of coordinates (r^1, r^2, r^3) called the *reference* (or "computational" or "natural") coordinates. We assume that a mapping from one set of coordinates to the other set is available and that this mapping is invertible. Therefore, the coordinates in one set can be written as a function of the other set using

$$x_i = x_i(r^1, r^2, r^3) \qquad i = 1, 2, 3 \tag{1.2}$$

and, *vice-versa*,

$$r^i = r^i(x_1, x_2, x_3) \qquad i = 1, 2, 3 \tag{1.3}$$

It is worth remarking that the coordinates r^i are not components of any vector.

The lines with constant x_i or r^i are known as the coordinate lines and produce a *coordinate grid* (Fig. 1.1). The grid may be orthogonal or non-orthogonal and the lines may vary in a non-homogeneous way (i.e. they are curvilinear). The partial derivatives of the functions $x_i(r^1, r^2, r^3)$ with respect to each r^i define tangent vectors to the coordinates lines r^i as follows

$$\underset{\cdot}{\boldsymbol{g}}_i = \frac{\partial \underset{\cdot}{\boldsymbol{x}}}{\partial r^i} \tag{1.4}$$

The position $\underset{\cdot}{\boldsymbol{x}}$ can be viewed as a *vector function* of three scalar arguments

$$\underset{\cdot}{\boldsymbol{x}} = \underset{\cdot}{\boldsymbol{x}}(x_1(r^1, r^2, r^3),\ x_2(r^1, r^2, r^3),\ x_3(r^1, r^2, r^3)) = \underset{\cdot}{\boldsymbol{x}}(r^1, r^2, r^3) \tag{1.5}$$

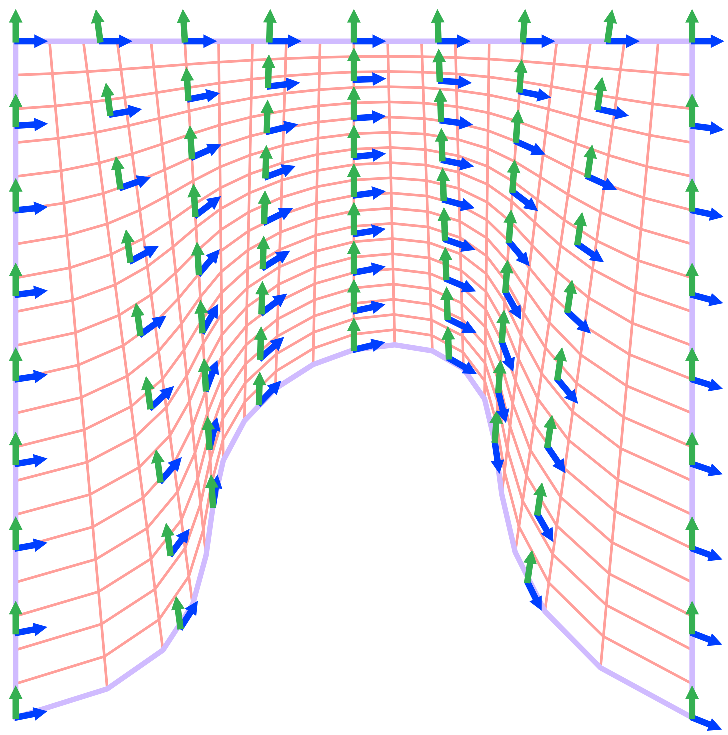

Fig. 1.1: Curvilinear coordinates and covariant base vectors.

where the dependence of x_i on r^i is implicitly understood. On the other hand, each one of the three reference coordinates r^i can be viewed as a *scalar function* of a vector argument according to

$$r^i = r^i(\underset{\sim}{\boldsymbol{x}}) \qquad i = 1, 2, 3 \tag{1.6}$$

The (physical) gradients of each reference coordinate can thus be calculated with the results being indicated as

$$\underset{\sim}{\boldsymbol{g}}^i = \frac{\mathrm{d}r^i}{\mathrm{d}\underset{\sim}{\boldsymbol{x}}} \tag{1.7}$$

Note that, while Eq. (1.4) represents three partial derivatives of vector with respect to scalar, Eq. (1.7) represents three total derivatives of scalar with respect to vector. At this point, we can explain a convention for the indices:

superscripts on the denominator of derivatives become subscripts on the results and vice-versa.

By using the chain rule, we can express the derivative of each coordinate r^i with respect to another coordinate r^j as follows

$$\frac{\partial r^i}{\partial r^j} = \delta^i_j = \frac{\mathrm{d}r^i}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} \cdot \frac{\partial \underset{\cdot}{\boldsymbol{x}}}{\partial r^j} = \underset{\cdot}{\boldsymbol{g}}^i \cdot \underset{\cdot}{\boldsymbol{g}}_j \tag{1.8}$$

(note that the first superscript j becomes a subscript in δ^i_j). Therefore, the vectors $\underset{\cdot}{\boldsymbol{g}}_i$ and $\underset{\cdot}{\boldsymbol{g}}^i$ are orthogonal one with another. Furthermore, both set of vectors may be used as reference basis. The vectors $\underset{\cdot}{\boldsymbol{g}}_i$ are called the *covariant base vectors* and the vectors $\underset{\cdot}{\boldsymbol{g}}^i$ are called the *contravariant base vectors*. For convenience, the position of the indices can be memorised with the following mnemonic: *co-go-below* (instead of contra-go-above). Note that the contravariant base vectors are normal to the surfaces of constant r^i and are directed towards the increasing r^i (i.e. they are gradients).

We define the metric covariant coefficients as

$$g_{ij} = \underset{\cdot}{\boldsymbol{g}}_i \cdot \underset{\cdot}{\boldsymbol{g}}_j = g_{ji} \tag{1.9}$$

and the contravariant metric coefficients as

$$g^{ij} = \underset{\cdot}{\boldsymbol{g}}^i \cdot \underset{\cdot}{\boldsymbol{g}}^j = g^{ji} \tag{1.10}$$

The following properties can be easily demonstrated

$$g^{ik}\, g_{kj} = g^{ki}\, g_{kj} = g^{ik}\, g_{jk} = g^{ki}\, g_{jk} = \delta^i_j \tag{1.11}$$

It is also easily verifiable that

$$\underset{\cdot}{\boldsymbol{g}}_i = g_{ij}\, \underset{\cdot}{\boldsymbol{g}}^j \qquad \text{and} \qquad \underset{\cdot}{\boldsymbol{g}}^i = g^{ij}\, \underset{\cdot}{\boldsymbol{g}}_j \tag{1.12}$$

The matrix formed by the covariant coefficients g_{ij} is denoted here as

$$[g] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{1.13}$$

and the matrix formed by the contravariant coefficients g^{ij} is denoted as

$$[G] = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} \tag{1.14}$$

The determinants of the above matrices are

$$g = \det([g]) \qquad \text{and} \qquad G = \det([G]) \tag{1.15}$$

i.e., we use the symbols g and G without indices or brackets are (scalar) determinant values. Brackets are always used for the matrix representation and the base vectors are in bold font with an underdot. This notation should prevent confusion. Finally, following properties hold

$$[G] = [g]^{-1} \qquad \text{and} \qquad G = \frac{1}{g} \tag{1.16}$$

A differential element vector $\mathrm{d}\underset{\cdot}{\boldsymbol{x}}$ can be written as

$$\mathrm{d}\underset{\cdot}{\boldsymbol{x}} = \frac{\partial \underset{\cdot}{\boldsymbol{x}}}{\partial r^i}\,\mathrm{d}r^i = \mathrm{d}r^i\,\underset{\cdot}{\boldsymbol{g}}_i \tag{1.17}$$

The magnitude of this vector allows the computation of the arc-length differential element $\mathrm{d}s$ as follows

$$(\mathrm{d}s)^2 = \|\mathrm{d}\underset{\cdot}{\boldsymbol{x}}\| = \mathrm{d}\underset{\cdot}{\boldsymbol{x}} \cdot \mathrm{d}\underset{\cdot}{\boldsymbol{x}} = \mathrm{d}r^i\,\mathrm{d}r^j\,\underset{\cdot}{\boldsymbol{g}}_i \cdot \underset{\cdot}{\boldsymbol{g}}_j = g_{ij}\,\mathrm{d}r^i\,\mathrm{d}r^j \tag{1.18}$$

An infinitesimal surface area increment $\mathrm{d}\underset{\cdot}{\boldsymbol{S}}^1$ parallel to the direction 1 can be obtained by using

$$\mathrm{d}\underset{\cdot}{\boldsymbol{S}}^1 = (\mathrm{d}r^2\,\underset{\cdot}{\boldsymbol{g}}_2) \times (\mathrm{d}r^3\,\underset{\cdot}{\boldsymbol{g}}_3) = \mathrm{d}r^2\,\mathrm{d}r^3\,(\underset{\cdot}{\boldsymbol{g}}_2 \times \underset{\cdot}{\boldsymbol{g}}_3) \tag{1.19}$$

The other area increments can be analogously calculated by considering a cyclic permutation of $(1, 2, 3)$. An infinitesimal volume increment $\mathrm{d}V$ can be calculated as

$$\mathrm{d}V = \underset{\cdot}{\boldsymbol{g}}_1 \cdot (\underset{\cdot}{\boldsymbol{g}}_2 \times \underset{\cdot}{\boldsymbol{g}}_3)\,\mathrm{d}r^1\,\mathrm{d}r^2\,\mathrm{d}r^3 \tag{1.20}$$

or

$$\mathrm{d}V = J\,\mathrm{d}r^1\,\mathrm{d}r^2\,\mathrm{d}r^3 \tag{1.21}$$

with

$$J = J(r^1, r^2, r^3) = \underset{\cdot}{\boldsymbol{g}}_1 \cdot (\underset{\cdot}{\boldsymbol{g}}_2 \times \underset{\cdot}{\boldsymbol{g}}_3) \tag{1.22}$$

being the Jacobian coefficient. Since we assumed that the mapping is invertible, the Jacobian is non-zero. Moreover, it is possible to show that

$$J^2 = \det([g]) = g \tag{1.23}$$

It is important to note that, in general, the base vectors and metrics are spatial fields; i.e. they vary with the coordinates themselves—an exception is when the system of reference is homogeneous (orthogonal or not). Therefore, some derivatives related to the metric coefficients may be required when computing gradients or divergences. Without proof and noting that the determinant g is a function $g = g(g_{ij})$ (Eq. 1.15a), we can write

$$\frac{\partial g}{\partial g_{ij}} = g\,g^{ij} \tag{1.24}$$

Likewise, noting that the determinant G is a function $G = G(g^{ij})$ (Eq. 1.15b), we can write

$$\frac{\partial G}{\partial g^{ij}} = G\, g_{ij} \tag{1.25}$$

As a consequence, we can also write (from Eq. 1.23)

$$\frac{\partial J}{\partial g_{ij}} = \frac{1}{2}\, J\, g^{ij} \tag{1.26}$$

1.2 Vector and tensor components

The primary objective of the covariant vectors $\underset{\cdot}{\boldsymbol{g}}_i$ and the contravariant vectors $\underset{\cdot}{\boldsymbol{g}}^i$ is to form a basis of reference such that any vector $\underset{\cdot}{\boldsymbol{u}}$ (at a location $\underset{\cdot}{\boldsymbol{x}}$) can be expressed as

$$\underset{\cdot}{\boldsymbol{u}} = u^i\, \underset{\cdot}{\boldsymbol{g}}_i = u_i\, \underset{\cdot}{\boldsymbol{g}}^i \tag{1.27}$$

The *contravariant components* u^i are the projections onto the contravariant basis vectors and can be obtained by multiplying both sides of the above equation by the contravariant counterpart vector using the single contraction operator (inner vector product)—hereafter, we will call this procedure simply *dotting operation*. In this way, we obtain

$$u^i = \underset{\cdot}{\boldsymbol{u}} \cdot \underset{\cdot}{\boldsymbol{g}}^i \tag{1.28}$$

by recalling the orthogonality condition in Eq. (1.8). Similarly, by dotting both sides by the covariant counterpart vector, we obtain the *covariant components* u_i as

$$u_i = \underset{\cdot}{\boldsymbol{u}} \cdot \underset{\cdot}{\boldsymbol{g}}_i \tag{1.29}$$

We can rewrite the expression for the contravariant components by replacing the vector $\underset{\cdot}{\boldsymbol{u}}$ in Eq. (1.28) with its representation from Eq. (1.27). This procedure leads to

$$u^i = u_j\, \underset{\cdot}{\boldsymbol{g}}^j \cdot \underset{\cdot}{\boldsymbol{g}}^i = g^{ij}\, u_j \tag{1.30}$$

where the definition of the metric coefficients in Eq. (1.10) was considered. Likewise, we can write

$$u_i = u^j\, \underset{\cdot}{\boldsymbol{g}}_j \cdot \underset{\cdot}{\boldsymbol{g}}_i = g_{ij}\, u^j \tag{1.31}$$

Therefore, the metric coefficients act like the Kronecker delta by swapping indices. In particular, the metric coefficients allow us to “raise” or “lower” indices when desired. For example, the index j at the right-hand-side of Eq. (1.31) is “lowered” to i at the left-hand-side.

A second order tensor $\underset{\cdot\cdot}{\boldsymbol{\sigma}}$ can also be expressed as a function of the base vectors. However, now there are four possible representations

$$\boldsymbol{\sigma} = \sigma_{ij}\, \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \sigma^{ij}\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j = \sigma^{i}_{.j}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j = \sigma_{i}^{.j}\, \boldsymbol{g}^i \otimes \boldsymbol{g}_j \tag{1.32}$$

where the dots are used to indicate which index i or j comes first.

The components of the second order tensor $\boldsymbol{\sigma}$ can be related one with another by "dotting" the base vectors from the left and from the right of Eq. (1.32). The four possible combinations are

$$\begin{aligned}
\sigma^{kl} &= \boldsymbol{g}^k \cdot \boldsymbol{\sigma} \cdot \boldsymbol{g}^l = \sigma_{ij}\, (\boldsymbol{g}^k \cdot \boldsymbol{g}^i)(\boldsymbol{g}^j \cdot \boldsymbol{g}^l) = \sigma_{ij}\, g^{ik}\, g^{jl} \\
\sigma_{kl} &= \boldsymbol{g}_k \cdot \boldsymbol{\sigma} \cdot \boldsymbol{g}_l = \sigma^{ij}\, (\boldsymbol{g}_k \cdot \boldsymbol{g}_i)(\boldsymbol{g}_j \cdot \boldsymbol{g}_l) = \sigma^{ij}\, g_{ik}\, g_{jl} \\
\sigma_{k}^{.l} &= \boldsymbol{g}_k \cdot \boldsymbol{\sigma} \cdot \boldsymbol{g}^l = \sigma^{i}_{.j}\, (\boldsymbol{g}_k \cdot \boldsymbol{g}_i)(\boldsymbol{g}^j \cdot \boldsymbol{g}^l) = \sigma^{i}_{.j}\, g_{ik}\, g^{jl} \\
\sigma^{k}_{.l} &= \boldsymbol{g}^k \cdot \boldsymbol{\sigma} \cdot \boldsymbol{g}_l = \sigma_{i}^{.j}\, (\boldsymbol{g}^k \cdot \boldsymbol{g}^i)(\boldsymbol{g}_j \cdot \boldsymbol{g}_l) = \sigma_{i}^{.j}\, g^{ik}\, g_{jl}
\end{aligned} \tag{1.33}$$

The inner (dot) product between vectors in curvilinear coordinates is determined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = (u^i\, \boldsymbol{g}_i) \cdot (v^j\, \boldsymbol{g}_j) = u^i\, v^j\, \boldsymbol{g}_i \cdot \boldsymbol{g}_j = u^i\, v^j\, g_{ij} \tag{1.34}$$

Thus, contrary to the situation in rectangular orthogonal coordinates, the inner product now depends on the metric coefficients (when using contravariant coordinates). Nonetheless, by using the "raise-lower" property of the metric coefficients (Eq. 1.31), we can rewrite the above equation as

$$\boldsymbol{u} \cdot \boldsymbol{v} = u^i\, v_i \qquad \text{or} \qquad \boldsymbol{u} \cdot \boldsymbol{v} = u_i\, v^i \tag{1.35}$$

where the index j was swapped by i and lowered.

The dot product between a second order tensor and a vector is determined by

$$\begin{aligned}
\boldsymbol{\sigma} \cdot \boldsymbol{u} &= \sigma_{ij}\, (\boldsymbol{g}^i \otimes \boldsymbol{g}^j) \cdot \boldsymbol{g}^k\, u_k \\
&= \sigma_{ij}\, u_k\, \boldsymbol{g}^i\, (\boldsymbol{g}^j \cdot \boldsymbol{g}^k) \\
&= \sigma_{ij}\, u_k\, \boldsymbol{g}^i\, g^{jk} \\
&= \sigma_{ij}\, u^j\, \boldsymbol{g}^i
\end{aligned} \tag{1.36}$$

Other expressions can be obtained as well by considering all combinations in Eq. (1.27) and in Eq. (1.32). For example, we can write

$$\boldsymbol{\sigma} \cdot \boldsymbol{u} = \sigma^{ij}\, u_j\, \boldsymbol{g}_i = \sigma^{i}_{.j}\, u^j\, \boldsymbol{g}_i \tag{1.37}$$

The dot product between two second order tensor can be be determined by dotting by substituting the components representation in Eq. (1.32). For instance, we obtain

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \sigma^{ij}\, \varepsilon_{jk}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^k \tag{1.38}$$

with $\boldsymbol{\varepsilon}$ being also a second order tensor. Other combinations using covariant, contravariant, or mixed-variant components are possible.

Let's introduce the identity second order tensor $\underset{\cdot\cdot}{\boldsymbol{I}}$ such that

$$\underset{\cdot}{\boldsymbol{u}} \cdot \underset{\cdot\cdot}{\boldsymbol{I}} \cdot \underset{\cdot}{\boldsymbol{v}} = \underset{\cdot}{\boldsymbol{u}} \cdot \underset{\cdot}{\boldsymbol{v}} \tag{1.39}$$

By replacing $\underset{\cdot}{\boldsymbol{u}}$ with $\underset{\cdot}{\boldsymbol{g}}_i$ and $\underset{\cdot}{\boldsymbol{v}}$ with $\underset{\cdot}{\boldsymbol{g}}_j$, we get

$$\underset{\cdot}{\boldsymbol{g}}_i \cdot \underset{\cdot\cdot}{\boldsymbol{I}} \cdot \underset{\cdot}{\boldsymbol{g}}_j = \underset{\cdot}{\boldsymbol{g}}_i \cdot \underset{\cdot}{\boldsymbol{g}}_j = g_{ij} \tag{1.40}$$

As carried out in Eq. (1.33), the components of a second order tensor are obtained by dotting from left and right, hence the expression above tells us that g_{ij} are components of the identity tensor, i.e.

$$\underset{\cdot\cdot}{\boldsymbol{I}} = g_{ij}\, \underset{\cdot}{\boldsymbol{g}}^i \otimes \underset{\cdot}{\boldsymbol{g}}^j \tag{1.41}$$

By dotting $\underset{\cdot\cdot}{\boldsymbol{I}}$ from left and right with the other bases, we also obtain

$$\underset{\cdot\cdot}{\boldsymbol{I}} = g^{ij}\, \underset{\cdot}{\boldsymbol{g}}_i \otimes \underset{\cdot}{\boldsymbol{g}}_j = \delta_i^j\, \underset{\cdot}{\boldsymbol{g}}^i \otimes \underset{\cdot}{\boldsymbol{g}}_j = \delta_j^i\, \underset{\cdot}{\boldsymbol{g}}_i \otimes \underset{\cdot}{\boldsymbol{g}}^j \tag{1.42}$$

1.3 Christoffel symbols

In general curvilinear coordinates, the base vectors may vary with the position, for instance

$$\underset{\cdot}{\boldsymbol{g}}_i = \underset{\cdot}{\boldsymbol{g}}_i(r^1, r^2, r^3) \qquad \text{and} \qquad \underset{\cdot}{\boldsymbol{g}}^i = \underset{\cdot}{\boldsymbol{g}}^i(r^1, r^2, r^3) \tag{1.43}$$

Therefore, the derivatives of the base vectors with respect to the coordinates will arise when the differential operators such as the gradient or the Laplacian are expanded. Such derivatives are written here as

$$\underset{\cdot}{\boldsymbol{\Gamma}}_{ij} = \frac{\partial \underset{\cdot}{\boldsymbol{g}}_i}{\partial r^j} \qquad \text{and} \qquad \underset{\cdot}{\bar{\boldsymbol{\Gamma}}}{}^i_{\ j} = \frac{\partial \underset{\cdot}{\boldsymbol{g}}^i}{\partial r^j} \tag{1.44}$$

where we introduced the two auxiliary sets of vectors $\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}$ and $\underset{\cdot}{\bar{\boldsymbol{\Gamma}}}{}^i_{\ j}$, each containing nine vectors. Note that, in an Euclidean space, we can write

$$\underset{\cdot}{\boldsymbol{\Gamma}}_{ij} = \underset{\cdot}{\boldsymbol{\Gamma}}_{ji} \qquad \text{and} \qquad \underset{\cdot}{\bar{\boldsymbol{\Gamma}}}{}^i_{\ j} = \underset{\cdot}{\bar{\boldsymbol{\Gamma}}}{}^j_{\ i} \tag{1.45}$$

where the above expressions can be easily verified by taking the second derivative of the position vector in Eq. (1.5).

Hereafter, we will work with the vectors $\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}$ mostly and only by the end of this section attention will be turned to the vectors $\underset{\cdot}{\bar{\boldsymbol{\Gamma}}}{}^i_{\ j}$. Furthermore, we will call the set $\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}$ the "Christoffel vectors."

By dotting the Christoffel vectors $\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}$ by the covariant base vectors, we define the Christoffel symbols of the *first kind* as

$$\Gamma_{ijk} = \underset{\sim}{\boldsymbol{\Gamma}}_{ij} \cdot \underset{\sim}{\boldsymbol{g}}_k = \frac{\partial \underset{\sim}{\boldsymbol{g}}_i}{\partial r^j} \cdot \underset{\sim}{\boldsymbol{g}}_k \tag{1.46}$$

Thus, as for any vector (see Eq. 1.27), the Christoffel vectors can be represented by

$$\underset{\sim}{\boldsymbol{\Gamma}}_{ij} = \frac{\partial \underset{\sim}{\boldsymbol{g}}_i}{\partial r^j} = \Gamma_{ijk}\, \underset{\sim}{\boldsymbol{g}}^k \tag{1.47}$$

Above, note that we have three k covariant components for each i, j vector. Also, note that Γ_{ijk} are not components of a third order tensor.

Similarly, by dotting the Christoffel vectors $\underset{\sim}{\boldsymbol{\Gamma}}_{ij}$ by the contravariant base vectors, we define the Christoffel symbols of the *second kind* as

$$\Gamma_{ij}^{k} = \underset{\sim}{\boldsymbol{\Gamma}}_{ij} \cdot \underset{\sim}{\boldsymbol{g}}^k = \frac{\partial \underset{\sim}{\boldsymbol{g}}_i}{\partial r^j} \cdot \underset{\sim}{\boldsymbol{g}}^k \tag{1.48}$$

Thus, as in Eq. (1.27), the Christoffel vectors can be also represented by

$$\underset{\sim}{\boldsymbol{\Gamma}}_{ij} = \frac{\partial \underset{\sim}{\boldsymbol{g}}_i}{\partial r^j} = \Gamma_{ij}^{k}\, \underset{\sim}{\boldsymbol{g}}_k \tag{1.49}$$

Above, note that we have three k contravariant components for each i, j vector. Also, note that Γ_{ij}^{k} are not the components of a third order tensor.

The Christoffel symbols of first and second kind can be related by considering that both are components of the Christoffel vector. To obtain such relationship, we first multiply both sides of Eq. (1.47) by the contravariant base vectors as follows

$$\underset{\sim}{\boldsymbol{\Gamma}}_{ij} \cdot \underset{\sim}{\boldsymbol{g}}^\ell = \Gamma_{ijk}\, \underset{\sim}{\boldsymbol{g}}^k \cdot \underset{\sim}{\boldsymbol{g}}^\ell = \Gamma_{ijk}\, g^{k\ell} \tag{1.50}$$

Second, we multiply both sides of Eq. (1.49) by the same contravariant vectors, allowing us to write

$$\underset{\sim}{\boldsymbol{\Gamma}}_{ij} \cdot \underset{\sim}{\boldsymbol{g}}^\ell = \Gamma_{ij}^{k}\, \underset{\sim}{\boldsymbol{g}}_k \cdot \underset{\sim}{\boldsymbol{g}}^\ell = \Gamma_{ij}^{k}\, \delta_k^\ell = \Gamma_{ij}^{\ell} \tag{1.51}$$

Finally, by comparing the two last equations, we obtain

$$\Gamma_{ij}^{\ell} = \Gamma_{ijk}\, g^{k\ell} \tag{1.52}$$

i.e. the metric coefficients once again work by swapping the index k by ℓ. In a similar manner, by dotting the Christoffel vectors by the covariant base vectors, we can deduce

$$\Gamma_{ijk} = \Gamma_{ij}^{\ell}\, g_{\ell k} \tag{1.53}$$

Note that, due to Eq. (1.45), the following properties can be observed

$$\Gamma_{ijk} = \Gamma_{jik} \qquad \text{and} \qquad \Gamma_{ij}^{k} = \Gamma_{ji}^{k} \tag{1.54}$$

Be aware however that the index k cannot be swapped here.

As for the base vectors, the metric coefficients depend on the position as well, for instance,

$$g^{ij} = g^{ij}(r^1, r^2, r^3) \qquad \text{and} \qquad g_{ij} = g_{ij}(r^1, r^2, r^3) \tag{1.55}$$

Thus, we may be interested in the following derivatives

$$\frac{\partial g_{ij}}{\partial r^k} = \frac{\partial(\boldsymbol{g}_i \cdot \boldsymbol{g}_j)}{\partial r^k} = \frac{\partial \boldsymbol{g}_i}{\partial r^k} \cdot \boldsymbol{g}_j + \boldsymbol{g}_i \cdot \frac{\partial \boldsymbol{g}_j}{\partial r^k} = \boldsymbol{\Gamma}_{ik} \cdot \boldsymbol{g}_j + \boldsymbol{\Gamma}_{jk} \cdot \boldsymbol{g}_i \tag{1.56}$$

The inner products in the right-hand-side can be conveniently expressed by the Christoffel symbols of the first kind. Thus

$$\frac{\partial g_{ij}}{\partial r^k} = \Gamma_{ikj} + \Gamma_{jki} \tag{1.57}$$

Due to the symmetry properties in Eq. (1.54), the above derivatives can be written in a number of ways. By adding and subtracting the resulting expressions in a systematic manner, we can further derive

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial r^i} + \frac{\partial g_{ki}}{\partial r^j} + \frac{\partial g_{ij}}{\partial r^k}\right) \tag{1.58}$$

which is a useful way to obtain the Christoffel symbols by using the metric coefficients only. From the results, Γ^k_{ij} can be obtained using Eq. (1.52).

Finally, it is also possible to demonstrate that

$$\Gamma^k_{ik} = \Gamma^k_{ki} = \frac{1}{J}\frac{\partial J}{\partial r^i} \tag{1.59}$$

Let's consider now the set of vectors $\bar{\boldsymbol{\Gamma}}^i_j$ in Eq. (1.44)b. By dotting this equation by the covariant base vectors, we define the following auxiliary coefficients

$$\bar{\Gamma}^i_{jk} = \bar{\boldsymbol{\Gamma}}^i_j \cdot \boldsymbol{g}_k = \frac{\partial \boldsymbol{g}^i}{\partial r^j} \cdot \boldsymbol{g}_k \tag{1.60}$$

Now, we differentiate the orthogonality condition $\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$ (Eq. 1.8) with respect to r^k to obtain

$$\begin{aligned}\frac{\partial \boldsymbol{g}^i}{\partial r^k} \cdot \boldsymbol{g}_j + \frac{\partial \boldsymbol{g}_j}{\partial r^k} \cdot \boldsymbol{g}^i &= 0 \\ \bar{\boldsymbol{\Gamma}}^i_k \cdot \boldsymbol{g}_j + \boldsymbol{\Gamma}_{jk} \cdot \boldsymbol{g}^i &= 0 \\ \bar{\Gamma}^i_{kj} + \Gamma^i_{jk} &= 0 \end{aligned} \tag{1.61}$$

Therefore,

$$\bar{\Gamma}^i_{jk} = -\Gamma^i_{jk} \tag{1.62}$$

where Eq. (1.45) was considered to swap the indices k and j. This result indicates that we only need the Γ^i_{jk} coefficients in the final expressions.

1.4 Gradient, Divergent and Laplacian

To proceed, we will need the differential increments $\mathrm{d}r^i$. By dotting both sides of Eq. (1.17) by the contravariant basis and by recalling the orthogonality condition in Eq. (1.8), we obtain

$$\mathrm{d}r^i = \boldsymbol{g}^i \cdot \mathrm{d}\boldsymbol{x} \tag{1.63}$$

1.4.1 Gradient

Given the scalar function

$$\phi = \phi(\boldsymbol{x}) = \phi(\boldsymbol{x}(r^1, r^2, r^3)) = \phi(r^1, r^2, r^3) \tag{1.64}$$

we can write its differential either as

$$\mathrm{d}\phi = \frac{\mathrm{d}\phi}{\mathrm{d}\boldsymbol{x}} \cdot \mathrm{d}\boldsymbol{x} \tag{1.65}$$

or as

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial r^i}\,\mathrm{d}r^i = \frac{\partial \phi}{\partial r^i}\,\boldsymbol{g}^i \cdot \mathrm{d}\boldsymbol{x} \tag{1.66}$$

where Eq. (1.63) was considered. Therefore, by comparing the two above expressions, we can write the gradient of ϕ as

$$\boldsymbol{\nabla}\,\phi = \frac{\mathrm{d}\phi}{\mathrm{d}\boldsymbol{x}} = \frac{\partial \phi}{\partial r^i}\,\boldsymbol{g}^i \tag{1.67}$$

Similarly, given the vector field function

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}(r^1, r^2, r^3)) = \boldsymbol{u}(r^1, r^2, r^3) \tag{1.68}$$

we can write the differential increment $\mathrm{d}\boldsymbol{u}$ either as

$$\mathrm{d}\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\boldsymbol{x}} \cdot \mathrm{d}\boldsymbol{x} \tag{1.69}$$

or as

$$\mathrm{d}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial r^j}\,\mathrm{d}r^j = \frac{\partial \boldsymbol{u}}{\partial r^j}\,(\boldsymbol{g}^j \cdot \mathrm{d}\boldsymbol{x}) = \left(\frac{\partial \boldsymbol{u}}{\partial r^j} \otimes \boldsymbol{g}^j\right) \cdot \mathrm{d}\boldsymbol{x} \tag{1.70}$$

where Eq. (1.63) was considered. Therefore, by comparing the two above expressions, we can write

$$\frac{\mathrm{d}\underset{\cdot}{\boldsymbol{u}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} = \frac{\partial \underset{\cdot}{\boldsymbol{u}}}{\partial r^j} \otimes \underset{\cdot}{\boldsymbol{g}}^j \tag{1.71}$$

Now, noting that, according to Eq. (1.27),

$$\underset{\cdot}{\boldsymbol{u}} = u^i\, \underset{\cdot}{\boldsymbol{g}}_i = u^i(r^1, r^2, r^3)\, \underset{\cdot}{\boldsymbol{g}}_i(r^1, r^2, r^3) \tag{1.72}$$

we obtain

$$\frac{\partial \underset{\cdot}{\boldsymbol{u}}}{\partial r^j} = \frac{\partial u^i}{\partial r^j}\, \underset{\cdot}{\boldsymbol{g}}_i + u^i\, \frac{\partial \underset{\cdot}{\boldsymbol{g}}_i}{\partial r^j} = \frac{\partial u^i}{\partial r^j}\, \underset{\cdot}{\boldsymbol{g}}_i + u^i\, \underset{\cdot}{\boldsymbol{\Gamma}}_{ij} \tag{1.73}$$

where the Christoffel vectors from Eq. (1.44)a are used. Therefore, the gradient in Eq. (1.71) becomes

$$\begin{aligned}\frac{\mathrm{d}\underset{\cdot}{\boldsymbol{u}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} &= \frac{\partial u^i}{\partial r^j}\, \underset{\cdot}{\boldsymbol{g}}_i \otimes \underset{\cdot}{\boldsymbol{g}}^j + u^i\, \underset{\cdot}{\boldsymbol{\Gamma}}_{ij} \otimes \underset{\cdot}{\boldsymbol{g}}^j \\ &= \frac{\partial u^i}{\partial r^j}\, \underset{\cdot}{\boldsymbol{g}}_i \otimes \underset{\cdot}{\boldsymbol{g}}^j + u^i\, \Gamma^k_{ij}\, \underset{\cdot}{\boldsymbol{g}}_k \otimes \underset{\cdot}{\boldsymbol{g}}^j\end{aligned} \tag{1.74}$$

where the Christoffel symbol of the second kind (Eq. 1.47) is considered. The above expression can be also written as

$$\frac{\mathrm{d}\underset{\cdot}{\boldsymbol{u}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} = \left(\frac{\partial u^i}{\partial r^j} + u^k\, \Gamma^i_{kj} \right)\, \underset{\cdot}{\boldsymbol{g}}_i \otimes \underset{\cdot}{\boldsymbol{g}}^j \tag{1.75}$$

after interchanging the repeated indices on the second term of the right-hand-side.

For convenience, the following notation is introduced to indicate covariant vector differentiation

$$u^i_{/j} = \frac{\partial u^i}{\partial r^j} + u^k\, \Gamma^i_{kj} \tag{1.76}$$

In this way, the gradient of vector $\underset{\cdot}{\boldsymbol{u}}$ given in Eq. (1.75) can be written as

$$\underset{\cdot\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot}{\boldsymbol{u}} = \frac{\mathrm{d}\underset{\cdot}{\boldsymbol{u}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} = u^i_{/j}\, \underset{\cdot}{\boldsymbol{g}}_i \otimes \underset{\cdot}{\boldsymbol{g}}^j \tag{1.77}$$

1.4.2 Divergent

The divergence of a vector is the trace of its gradient and is indicated by

$$\operatorname{div} \underset{\cdot}{\boldsymbol{u}} = \underset{\cdot}{\boldsymbol{\nabla}} \boldsymbol{\cdot} \underset{\cdot}{\boldsymbol{u}} = \operatorname{tr}\left(\frac{\mathrm{d}\underset{\cdot}{\boldsymbol{u}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} \right) = \frac{\mathrm{d}\underset{\cdot}{\boldsymbol{u}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} : \underset{\cdot\cdot}{\boldsymbol{I}} \tag{1.78}$$

By recalling that

$$\mathrm{tr}(\underset{\cdot}{\boldsymbol{a}} \otimes \underset{\cdot}{\boldsymbol{b}}) = \underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot}{\boldsymbol{b}} \tag{1.79}$$

we can compute the divergence of $\underset{\cdot}{\boldsymbol{u}}$ by substituting Eq. (1.77) into Eq. (1.78). The procedure leads to

$$\mathrm{div}\, \underset{\cdot}{\boldsymbol{u}} = v^i_{/j}\, \underset{\cdot}{\boldsymbol{g}}_i \cdot \underset{\cdot}{\boldsymbol{g}}^j = v^i_{/j}\, \delta^j_i = v^i_{/i} \tag{1.80}$$

Thus,

$$\mathrm{div}\, \underset{\cdot}{\boldsymbol{u}} = \frac{\partial u^i}{\partial r^i} + u^k\, \Gamma^i_{ki} \tag{1.81}$$

Note that, if the coordinates are homogeneous, the second term is zero (because the bases would not vary with position) and the expression simplifies to the well known divergence operation.

Considering the relationship between the Christoffel symbol of second kind given in Eq. (1.59), the expression for the vector divergence can be written as

$$\mathrm{div}\, \underset{\cdot}{\boldsymbol{u}} = \frac{\partial u^i}{\partial r^i} + u^k\, \frac{1}{J}\, \frac{\partial J}{\partial r^k} \tag{1.82}$$

or, equivalently, as

$$\mathrm{div}\, \underset{\cdot}{\boldsymbol{u}} = \frac{1}{J}\, \frac{\partial J\, u^i}{\partial r^i} \tag{1.83}$$

1.4.3 Laplacian

The Laplacian of a scalar quantity ϕ is the divergence of the gradient of ϕ and is usually symbolised by

$$\nabla^2 \phi = \mathrm{div}\, (\underset{\cdot}{\boldsymbol{\nabla}}\, \phi) = \underset{\cdot}{\boldsymbol{\nabla}} \cdot \underset{\cdot}{\boldsymbol{\nabla}}\, \phi = \Delta \phi \tag{1.84}$$

Here, we will use the first and second notations only.

First, let's differentiate the scalar gradient with respect to $\underset{\cdot}{\boldsymbol{x}}$ as follows (see also Eq. 1.71)

$$\begin{aligned} \frac{\mathrm{d}\, \underset{\cdot}{\boldsymbol{\nabla}}\, \phi}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} &= \frac{\partial\, \underset{\cdot}{\boldsymbol{\nabla}}\, \phi}{\partial r^j} \otimes \frac{\mathrm{d}r^j}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} \\ &= \frac{\partial\, \underset{\cdot}{\boldsymbol{\nabla}}\, \phi}{\partial r^j} \otimes \underset{\cdot}{\boldsymbol{g}}^j \end{aligned} \tag{1.85}$$

Thus

$$\mathrm{div}\, (\underset{\cdot}{\boldsymbol{\nabla}}\, \phi) = \mathrm{tr}\left(\frac{\partial\, \underset{\cdot}{\boldsymbol{\nabla}}\, \phi}{\partial r^j} \otimes \underset{\cdot}{\boldsymbol{g}}^j \right) = \frac{\partial\, \underset{\cdot}{\boldsymbol{\nabla}}\, \phi}{\partial r^j} \cdot \underset{\cdot}{\boldsymbol{g}}^j \tag{1.86}$$

The partial derivative of the gradient of ϕ above can be expanded by considering the gradient expression $\underset{\cdot}{\boldsymbol{\nabla}}\, \phi = (\partial \phi / r^i) \underset{\cdot}{\boldsymbol{g}}^i$ (Eq. 1.67). Thus

$$\frac{\partial \nabla \phi}{\partial r^j} = \frac{\partial^2 \phi}{\partial r^i \partial r^j}\, \boldsymbol{g}^i + \frac{\partial \phi}{\partial r^i}\, \frac{\partial \boldsymbol{g}^i}{\partial r^j} \tag{1.87}$$

or

$$\frac{\partial \nabla \phi}{\partial r^j} = \frac{\partial^2 \phi}{\partial r^i \partial r^j}\, \boldsymbol{g}^i + \frac{\partial \phi}{\partial r^k}\, \bar{\boldsymbol{\Gamma}}^k_{\ j} \tag{1.88}$$

where the index i in the right-hand-side has been replaced by k. In this way, the divergence becomes (using $\boldsymbol{g}^j = g^{ij}\, \boldsymbol{g}_i$; Eqs. 1.11 and 1.12)

$$\begin{aligned}
\operatorname{div}(\nabla \phi) &= \left(\frac{\partial^2 \phi}{\partial r^i \partial r^j}\, \boldsymbol{g}^i + \frac{\partial \phi}{\partial r^k}\, \bar{\boldsymbol{\Gamma}}^k_{\ j} \right) \cdot \boldsymbol{g}^j \\
&= \frac{\partial^2 \phi}{\partial r^i \partial r^j}\, \boldsymbol{g}^i \cdot \boldsymbol{g}^j + \frac{\partial \phi}{\partial r^k}\, \bar{\boldsymbol{\Gamma}}^k_{\ j} \cdot \boldsymbol{g}_i\, g^{ij} \\
&= \frac{\partial^2 \phi}{\partial r^i \partial r^j}\, g^{ij} + \frac{\partial \phi}{\partial r^k}\, \bar{\Gamma}^k_{ij}\, g^{ij}
\end{aligned} \tag{1.89}$$

where we considered the auxiliary Christoffel symbol (Eq. 1.60) and its symmetry. Now, considering the relation between Christoffel symbols (Eq. 1.62), we can write

$$\nabla^2 \phi = \operatorname{div}(\nabla \phi) = \left(\frac{\partial^2 \phi}{\partial r^i \partial r^j} - \frac{\partial \phi}{\partial r^k}\, \Gamma^k_{ij} \right) g^{ij} \tag{1.90}$$

An alternative expression for the Laplacian can be deduced by simply substituting the contravariant components of $\nabla \phi$ into Eq. (1.83). These components are computed as

$$u^i \equiv \nabla \phi \cdot \boldsymbol{g}^i = \frac{\partial \phi}{\partial r^j}\, \boldsymbol{g}^j \cdot \boldsymbol{g}^i = g^{ij}\, \frac{\partial \phi}{\partial r^j} \tag{1.91}$$

and the procedure results in

$$\nabla^2 \phi = \operatorname{div}(\nabla \phi) = \frac{1}{J}\, \frac{\partial}{\partial r^i} \left(J\, g^{ij}\, \frac{\partial \phi}{\partial r^j} \right) \tag{1.92}$$

The following operation is also common in science and engineering

$$\operatorname{div}\left[\nu(\boldsymbol{x})\, \nabla \phi\right] = \nabla \cdot (\nu\, \nabla \phi) \tag{1.93}$$

where the coefficient $\nu = \nu(\boldsymbol{x})$ may be a function of the coordinates and hence cannot be removed from the outer derivative. Therefore, by recalling that $\operatorname{div}(a\, \boldsymbol{b}) = \nabla a \cdot \boldsymbol{b} + a \operatorname{div} \boldsymbol{b}$, we obtain

$$\operatorname{div}(\nu\, \nabla \phi) = \nabla \nu \cdot \nabla \phi + \nu \operatorname{div}(\nabla \phi) \tag{1.94}$$

where the dependency of ν on $\boldsymbol{x}$ is implicitly assumed therein. Now, by considering Eq. (1.67) and Eq. (1.90), we can write

$$\operatorname{div}(\nu \, \underset{\cdot}{\boldsymbol{\nabla}} \, \phi) = \frac{\partial \nu}{\partial r^i} \underset{\cdot}{\boldsymbol{g}}^i \cdot \frac{\partial \phi}{\partial r^j} \underset{\cdot}{\boldsymbol{g}}^j + \nu \left(\frac{\partial^2 \phi}{\partial r^i \partial r^j} - \frac{\partial \phi}{\partial r^k} \, \Gamma^k_{ij} \right) g^{ij} \tag{1.95}$$

Therefore,

$$\operatorname{div}(\nu \, \underset{\cdot}{\boldsymbol{\nabla}} \, \phi) = \left(\frac{\partial \nu}{\partial r^i} \frac{\partial \phi}{\partial r^j} + \nu \, \frac{\partial^2 \phi}{\partial r^i \partial r^j} - \nu \, \frac{\partial \phi}{\partial r^k} \, \Gamma^k_{ij} \right) g^{ij} \tag{1.96}$$

Alternatively, we can also write

$$\begin{aligned} \operatorname{div}(\nu \, \underset{\cdot}{\boldsymbol{\nabla}} \, \phi) &= \operatorname{tr} \left(\frac{\partial \nu \, \underset{\cdot}{\boldsymbol{\nabla}} \, \phi}{\partial r^j} \otimes \underset{\cdot}{\boldsymbol{g}}^j \right) \\ &= \frac{\partial \nu \, \underset{\cdot}{\boldsymbol{\nabla}} \, \phi}{\partial r^j} \cdot \underset{\cdot}{\boldsymbol{g}}^j \\ &= \underset{\cdot}{\boldsymbol{g}}^j \cdot \frac{\partial}{\partial r^j} \left(\nu \, \frac{\partial \phi}{\partial r^i} \, \underset{\cdot}{\boldsymbol{g}}^i \right) \end{aligned} \tag{1.97}$$

1.4.4 Divergent of tensor

TODO

1.5 Examples

1.5.1 Laplacian in 2D

The Laplacian in curvilinear coordinates is given by

$$\begin{aligned} \nabla^2 \phi &= \left(\frac{\partial^2 \phi}{\partial r^i \partial r^j} - \frac{\partial \phi}{\partial r^k} \, \Gamma^k_{ij} \right) g^{ij} \\ &= \frac{\partial^2 \phi}{\partial r^i \partial r^j} \, g^{ij} - \frac{\partial \phi}{\partial r^k} \, L^k \end{aligned} \tag{1.98}$$

where

$$L^k = \Gamma^k_{ij} \, g^{ij} \tag{1.99}$$

Thus, with $a = r^1$ and $b = r^2$, the 2D Laplacian becomes

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial a \partial a} \, g^{11} + \frac{\partial^2 \phi}{\partial b \partial b} \, g^{22} + 2 \, \frac{\partial^2 \phi}{\partial a \partial b} \, g^{12} - \frac{\partial \phi}{\partial a} \, L^1 - \frac{\partial \phi}{\partial b} \, L^2 \tag{1.100}$$

where

$$
\begin{aligned}
L^1 &= \Gamma^1_{11}\, g^{11} + \Gamma^1_{22}\, g^{22} + 2\, \Gamma^1_{12}\, g^{12} \\
L^2 &= \Gamma^2_{11}\, g^{11} + \Gamma^2_{22}\, g^{22} + 2\, \Gamma^2_{12}\, g^{12}
\end{aligned} \tag{1.101}
$$

1.5.2 Laplacian in 3D

With $a = r^1$, $b = r^2$ and $c = r^3$, the 3D Laplacian becomes

$$
\begin{aligned}
\nabla^2\phi = \; & \frac{\partial^2\phi}{\partial a\partial a}\, g^{11} + \frac{\partial^2\phi}{\partial b\partial b}\, g^{22} + \frac{\partial^2\phi}{\partial c\partial c}\, g^{33} \\
& + 2\, \frac{\partial^2\phi}{\partial a\partial b}\, g^{12} + 2\, \frac{\partial^2\phi}{\partial b\partial c}\, g^{23} + 2\, \frac{\partial^2\phi}{\partial c\partial a}\, g^{31} \\
& - \frac{\partial\phi}{\partial a}\, L^1 - \frac{\partial\phi}{\partial b}\, L^2 - \frac{\partial\phi}{\partial c}\, L^3
\end{aligned} \tag{1.102}
$$

where

$$
\begin{aligned}
L^1 &= \Gamma^1_{11} g^{11} + \Gamma^1_{22} g^{22} + \Gamma^1_{33} g^{33} + 2\Gamma^1_{12} g^{12} + 2\Gamma^1_{23} g^{23} + 2\Gamma^1_{31} g^{31} \\
L^2 &= \Gamma^2_{11} g^{11} + \Gamma^2_{22} g^{22} + \Gamma^2_{33} g^{33} + 2\Gamma^2_{12} g^{12} + 2\Gamma^2_{23} g^{23} + 2\Gamma^2_{31} g^{31} \\
L^3 &= \Gamma^3_{11} g^{11} + \Gamma^3_{22} g^{22} + \Gamma^3_{33} g^{33} + 2\Gamma^3_{12} g^{12} + 2\Gamma^3_{23} g^{23} + 2\Gamma^3_{31} g^{31}
\end{aligned} \tag{1.103}
$$

1.5.3 Cylindrical coordinates

Coordinates ($r^1 = \rho$, $r^2 = \alpha$, $r^3 = \zeta$)

$$
\begin{cases}
x = \rho\, c_\alpha \\
y = \rho\, s_\alpha \\
z = \zeta
\end{cases} \tag{1.104}
$$

where $c_\alpha = \cos\alpha$ and $s_\alpha = \sin\alpha$.

Covariant base vectors

$$
\{\underset{\cdot}{\boldsymbol{g}}_1\} = \begin{Bmatrix} \frac{\partial x}{\partial \rho} \\ \frac{\partial y}{\partial \rho} \\ \frac{\partial z}{\partial \rho} \end{Bmatrix} = \begin{Bmatrix} c_\alpha \\ s_\alpha \\ 0 \end{Bmatrix} \tag{1.105}
$$

and

$$
\{\underset{\cdot}{\boldsymbol{g}}_2\} = \begin{Bmatrix} \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial \alpha} \\ \frac{\partial z}{\partial \alpha} \end{Bmatrix} = \begin{Bmatrix} -\rho\, s_\alpha \\ \rho\, c_\alpha \\ 0 \end{Bmatrix} \tag{1.106}
$$

and

$$\{\underset{\cdot}{\boldsymbol{g}}_3\} = \begin{Bmatrix} \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \zeta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{1.107}$$

Metric coefficients

$$[g] = \begin{bmatrix} \underset{\cdot}{\boldsymbol{g}}_1 \cdot \underset{\cdot}{\boldsymbol{g}}_1 & \underset{\cdot}{\boldsymbol{g}}_1 \cdot \underset{\cdot}{\boldsymbol{g}}_2 & \underset{\cdot}{\boldsymbol{g}}_1 \cdot \underset{\cdot}{\boldsymbol{g}}_3 \\ \underset{\cdot}{\boldsymbol{g}}_2 \cdot \underset{\cdot}{\boldsymbol{g}}_1 & \underset{\cdot}{\boldsymbol{g}}_2 \cdot \underset{\cdot}{\boldsymbol{g}}_2 & \underset{\cdot}{\boldsymbol{g}}_2 \cdot \underset{\cdot}{\boldsymbol{g}}_3 \\ \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}_1 & \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}_2 & \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.108}$$

and

$$[G] = [g]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.109}$$

Contravariant base vectors

$$\underset{\cdot}{\boldsymbol{g}}^1 = g^{11}\underset{\cdot}{\boldsymbol{g}}_1 + g^{12}\underset{\cdot}{\boldsymbol{g}}_2 + g^{13}\underset{\cdot}{\boldsymbol{g}}_3 = \underset{\cdot}{\boldsymbol{g}}_1 \qquad \text{thus} \qquad \underset{\cdot}{\boldsymbol{g}}^1 = \begin{Bmatrix} c_\alpha \\ s_\alpha \\ 0 \end{Bmatrix} \tag{1.110}$$

and

$$\underset{\cdot}{\boldsymbol{g}}^2 = g^{21}\underset{\cdot}{\boldsymbol{g}}_1 + g^{22}\underset{\cdot}{\boldsymbol{g}}_2 + g^{23}\underset{\cdot}{\boldsymbol{g}}_3 = \frac{1}{\rho^2}\underset{\cdot}{\boldsymbol{g}}_2 \qquad \text{thus} \qquad \underset{\cdot}{\boldsymbol{g}}^2 = \begin{Bmatrix} -\frac{s_\alpha}{\rho} \\ \frac{c_\alpha}{\rho} \\ 0 \end{Bmatrix} \tag{1.111}$$

and

$$\underset{\cdot}{\boldsymbol{g}}^3 = g^{31}\underset{\cdot}{\boldsymbol{g}}_1 + g^{32}\underset{\cdot}{\boldsymbol{g}}_2 + g^{33}\underset{\cdot}{\boldsymbol{g}}_3 = \underset{\cdot}{\boldsymbol{g}}_3 \qquad \text{thus} \qquad \underset{\cdot}{\boldsymbol{g}}^3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{1.112}$$

Christoffel vectors

$$[\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}] = \begin{bmatrix} \frac{\partial \underset{\cdot}{\boldsymbol{g}}_1}{\partial \rho} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_1}{\partial \alpha} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_1}{\partial \zeta} \\ \frac{\partial \underset{\cdot}{\boldsymbol{g}}_2}{\partial \rho} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_2}{\partial \alpha} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_2}{\partial \zeta} \\ \frac{\partial \underset{\cdot}{\boldsymbol{g}}_3}{\partial \rho} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_3}{\partial \alpha} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_3}{\partial \zeta} \end{bmatrix} \tag{1.113}$$

Thus

$$[\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}] = \begin{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} & \begin{Bmatrix} -s_\alpha \\ c_\alpha \\ 0 \end{Bmatrix} & \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \\ \\ \begin{Bmatrix} -s_\alpha \\ c_\alpha \\ 0 \end{Bmatrix} & \begin{Bmatrix} -\rho\, c_\alpha \\ -\rho\, s_\alpha \\ 0 \end{Bmatrix} & \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \\ \\ \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} & \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} & \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \end{bmatrix} = \begin{bmatrix} \{\underset{\cdot}{\mathbf{0}}\} & \{\frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\} & \{\underset{\cdot}{\mathbf{0}}\} \\ \{\frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\} & \{-\rho\underset{\cdot}{\boldsymbol{g}}_1\} & \{\underset{\cdot}{\mathbf{0}}\} \\ \{\underset{\cdot}{\mathbf{0}}\} & \{\underset{\cdot}{\mathbf{0}}\} & \{\underset{\cdot}{\mathbf{0}}\} \end{bmatrix} \tag{1.114}$$

Christoffel symbols of second kind

$$[\Gamma^k_{ij}] = \begin{bmatrix} \underset{\cdot}{\boldsymbol{\Gamma}}_{11}\cdot\underset{\cdot}{\boldsymbol{g}}^1 & \underset{\cdot}{\boldsymbol{\Gamma}}_{12}\cdot\underset{\cdot}{\boldsymbol{g}}^1 & \underset{\cdot}{\boldsymbol{\Gamma}}_{13}\cdot\underset{\cdot}{\boldsymbol{g}}^1 \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{21}\cdot\underset{\cdot}{\boldsymbol{g}}^1 & \underset{\cdot}{\boldsymbol{\Gamma}}_{22}\cdot\underset{\cdot}{\boldsymbol{g}}^1 & \underset{\cdot}{\boldsymbol{\Gamma}}_{23}\cdot\underset{\cdot}{\boldsymbol{g}}^1 \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{31}\cdot\underset{\cdot}{\boldsymbol{g}}^1 & \underset{\cdot}{\boldsymbol{\Gamma}}_{32}\cdot\underset{\cdot}{\boldsymbol{g}}^1 & \underset{\cdot}{\boldsymbol{\Gamma}}_{33}\cdot\underset{\cdot}{\boldsymbol{g}}^1 \\ \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{11}\cdot\underset{\cdot}{\boldsymbol{g}}^2 & \underset{\cdot}{\boldsymbol{\Gamma}}_{12}\cdot\underset{\cdot}{\boldsymbol{g}}^2 & \underset{\cdot}{\boldsymbol{\Gamma}}_{13}\cdot\underset{\cdot}{\boldsymbol{g}}^2 \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{21}\cdot\underset{\cdot}{\boldsymbol{g}}^2 & \underset{\cdot}{\boldsymbol{\Gamma}}_{22}\cdot\underset{\cdot}{\boldsymbol{g}}^2 & \underset{\cdot}{\boldsymbol{\Gamma}}_{23}\cdot\underset{\cdot}{\boldsymbol{g}}^2 \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{31}\cdot\underset{\cdot}{\boldsymbol{g}}^2 & \underset{\cdot}{\boldsymbol{\Gamma}}_{32}\cdot\underset{\cdot}{\boldsymbol{g}}^2 & \underset{\cdot}{\boldsymbol{\Gamma}}_{33}\cdot\underset{\cdot}{\boldsymbol{g}}^2 \\ \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{11}\cdot\underset{\cdot}{\boldsymbol{g}}^3 & \underset{\cdot}{\boldsymbol{\Gamma}}_{12}\cdot\underset{\cdot}{\boldsymbol{g}}^3 & \underset{\cdot}{\boldsymbol{\Gamma}}_{13}\cdot\underset{\cdot}{\boldsymbol{g}}^3 \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{21}\cdot\underset{\cdot}{\boldsymbol{g}}^3 & \underset{\cdot}{\boldsymbol{\Gamma}}_{22}\cdot\underset{\cdot}{\boldsymbol{g}}^3 & \underset{\cdot}{\boldsymbol{\Gamma}}_{23}\cdot\underset{\cdot}{\boldsymbol{g}}^3 \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{31}\cdot\underset{\cdot}{\boldsymbol{g}}^3 & \underset{\cdot}{\boldsymbol{\Gamma}}_{32}\cdot\underset{\cdot}{\boldsymbol{g}}^3 & \underset{\cdot}{\boldsymbol{\Gamma}}_{33}\cdot\underset{\cdot}{\boldsymbol{g}}^3 \end{bmatrix} \tag{1.115}$$

Thus

$$[\Gamma^k_{ij}] = \begin{bmatrix} 0 & \frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\cdot\underset{\cdot}{\boldsymbol{g}}^1 & 0 \\ \frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\cdot\underset{\cdot}{\boldsymbol{g}}^1 & -\rho\underset{\cdot}{\boldsymbol{g}}_1\cdot\underset{\cdot}{\boldsymbol{g}}^1 & 0 \\ 0 & 0 & 0 \\ \\ 0 & \frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\cdot\underset{\cdot}{\boldsymbol{g}}^2 & 0 \\ \frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\cdot\underset{\cdot}{\boldsymbol{g}}^2 & -\rho\underset{\cdot}{\boldsymbol{g}}_1\cdot\underset{\cdot}{\boldsymbol{g}}^2 & 0 \\ 0 & 0 & 0 \\ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & 0 \\ \\ 0 & \frac{1}{\rho} & 0 \\ \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & 0 \\ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.116}$$

Laplacian in cylindrical coordinates (from Eq. 1.90 and noting that the off-diagonal metric coefficients are zero)

$$
\begin{aligned}
\nabla^2\phi = &\left(\frac{\partial^2\phi}{\partial\rho^2} - \frac{\partial\phi}{\partial\rho}\cancel{\Gamma^1_{11}} - \frac{\partial\phi}{\partial\alpha}\cancel{\Gamma^2_{11}} - \frac{\partial\phi}{\partial\zeta}\cancel{\Gamma^3_{11}}\right) \\
&+ \left(\frac{\partial^2\phi}{\partial\alpha^2} - \frac{\partial\phi}{\partial\rho}\Gamma^1_{22} - \frac{\partial\phi}{\partial\alpha}\cancel{\Gamma^2_{22}} - \frac{\partial\phi}{\partial\zeta}\cancel{\Gamma^3_{22}}\right)\frac{1}{\rho^2} \\
&+ \left(\frac{\partial^2\phi}{\partial\zeta^2} - \frac{\partial\phi}{\partial\rho}\cancel{\Gamma^1_{33}} - \frac{\partial\phi}{\partial\alpha}\cancel{\Gamma^2_{33}} - \frac{\partial\phi}{\partial\zeta}\cancel{\Gamma^3_{33}}\right)
\end{aligned} \tag{1.117}
$$

Thus

$$
\nabla^2\phi = \frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial\zeta^2} \tag{1.118}
$$

1.5.4 Spherical coordinates

Coordinates ($r^1 = \rho$, $r^2 = \theta$, $r^3 = \alpha$)

$$
\begin{cases} x = \rho\, s_\theta\, c_\alpha \\ y = \rho\, s_\theta\, s_\alpha \\ z = \rho\, c_\theta \end{cases} \tag{1.119}
$$

where $c_\theta = \cos\theta$, $s_\theta = \sin\theta$, $c_\alpha = \cos\alpha$, and $s_\alpha = \sin\alpha$.

Covariant base vectors

$$
\{\underset{\cdot}{\boldsymbol{g}}_1\} = \begin{Bmatrix} \frac{\partial x}{\partial\rho} \\ \frac{\partial y}{\partial\rho} \\ \frac{\partial z}{\partial\rho} \end{Bmatrix} = \begin{Bmatrix} s_\theta\, c_\alpha \\ s_\theta\, s_\alpha \\ c_\theta \end{Bmatrix} \tag{1.120}
$$

and

$$
\{\underset{\cdot}{\boldsymbol{g}}_2\} = \begin{Bmatrix} \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\theta} \end{Bmatrix} = \begin{Bmatrix} \rho\, c_\theta\, c_\alpha \\ \rho\, c_\theta\, s_\alpha \\ -\rho\, s_\theta \end{Bmatrix} \tag{1.121}
$$

and

$$
\{\underset{\cdot}{\boldsymbol{g}}_3\} = \begin{Bmatrix} \frac{\partial x}{\partial\alpha} \\ \frac{\partial y}{\partial\alpha} \\ \frac{\partial z}{\partial\alpha} \end{Bmatrix} = \begin{Bmatrix} -\rho\, s_\theta\, s_\alpha \\ \rho\, s_\theta\, c_\alpha \\ 0 \end{Bmatrix} \tag{1.122}
$$

Metric coefficients

$$
[g] = \begin{bmatrix} \underset{\cdot}{\boldsymbol{g}}_1\cdot\underset{\cdot}{\boldsymbol{g}}_1 & \underset{\cdot}{\boldsymbol{g}}_1\cdot\underset{\cdot}{\boldsymbol{g}}_2 & \underset{\cdot}{\boldsymbol{g}}_1\cdot\underset{\cdot}{\boldsymbol{g}}_3 \\ \underset{\cdot}{\boldsymbol{g}}_2\cdot\underset{\cdot}{\boldsymbol{g}}_1 & \underset{\cdot}{\boldsymbol{g}}_2\cdot\underset{\cdot}{\boldsymbol{g}}_2 & \underset{\cdot}{\boldsymbol{g}}_2\cdot\underset{\cdot}{\boldsymbol{g}}_3 \\ \underset{\cdot}{\boldsymbol{g}}_3\cdot\underset{\cdot}{\boldsymbol{g}}_1 & \underset{\cdot}{\boldsymbol{g}}_3\cdot\underset{\cdot}{\boldsymbol{g}}_2 & \underset{\cdot}{\boldsymbol{g}}_3\cdot\underset{\cdot}{\boldsymbol{g}}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 s_\theta^2 \end{bmatrix} \tag{1.123}
$$

and

$$[G] = [g]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2 s_\theta^2} \end{bmatrix} \tag{1.124}$$

Contravariant base vectors

$$\underset{\cdot}{\boldsymbol{g}}^1 = g^{11}\underset{\cdot}{\boldsymbol{g}}_1 + g^{12}\underset{\cdot}{\boldsymbol{g}}_2 + g^{13}\underset{\cdot}{\boldsymbol{g}}_3 = \underset{\cdot}{\boldsymbol{g}}_1 \qquad \text{thus} \qquad \underset{\cdot}{\boldsymbol{g}}^1 = \begin{Bmatrix} s_\theta\, c_\alpha \\ s_\theta\, s_\alpha \\ c_\theta \end{Bmatrix} \tag{1.125}$$

and

$$\underset{\cdot}{\boldsymbol{g}}^2 = g^{21}\underset{\cdot}{\boldsymbol{g}}_1 + g^{22}\underset{\cdot}{\boldsymbol{g}}_2 + g^{23}\underset{\cdot}{\boldsymbol{g}}_3 = \frac{1}{\rho^2}\underset{\cdot}{\boldsymbol{g}}_2 \quad \text{thus} \quad \underset{\cdot}{\boldsymbol{g}}^2 = \begin{Bmatrix} \frac{c_\theta\, c_\alpha}{\rho} \\ \frac{c_\theta\, s_\alpha}{\rho} \\ -\frac{s_\theta}{\rho} \end{Bmatrix} \tag{1.126}$$

and

$$\underset{\cdot}{\boldsymbol{g}}^3 = g^{31}\underset{\cdot}{\boldsymbol{g}}_1 + g^{32}\underset{\cdot}{\boldsymbol{g}}_2 + g^{33}\underset{\cdot}{\boldsymbol{g}}_3 = \frac{1}{\rho^2 s_\theta^2}\underset{\cdot}{\boldsymbol{g}}_3 \quad \text{thus} \quad \underset{\cdot}{\boldsymbol{g}}^3 = \begin{Bmatrix} -\frac{s_\alpha}{\rho\, s_\theta} \\ \frac{c_\alpha}{\rho\, s_\theta} \\ 0 \end{Bmatrix} \tag{1.127}$$

Christoffel vectors

$$[\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}] = \begin{bmatrix} \frac{\partial \underset{\cdot}{\boldsymbol{g}}_1}{\partial \rho} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_1}{\partial \theta} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_1}{\partial \alpha} \\ \frac{\partial \underset{\cdot}{\boldsymbol{g}}_2}{\partial \rho} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_2}{\partial \theta} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_2}{\partial \alpha} \\ \frac{\partial \underset{\cdot}{\boldsymbol{g}}_3}{\partial \rho} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_3}{\partial \theta} & \frac{\partial \underset{\cdot}{\boldsymbol{g}}_3}{\partial \alpha} \end{bmatrix} \tag{1.128}$$

Thus

$$[\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}] = \begin{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} & \begin{Bmatrix} c_\theta\, c_\alpha \\ c_\theta\, s_\alpha \\ -s_\theta \end{Bmatrix} & \begin{Bmatrix} -s_\theta\, s_\alpha \\ s_\theta\, c_\alpha \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} c_\theta\, c_\alpha \\ c_\theta\, s_\alpha \\ -s_\theta \end{Bmatrix} & \begin{Bmatrix} -\rho\, s_\theta\, c_\alpha \\ -\rho\, s_\theta\, s_\alpha \\ -\rho\, c_\theta \end{Bmatrix} & \begin{Bmatrix} -\rho\, c_\theta\, s_\alpha \\ \rho\, c_\theta\, c_\alpha \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} -s_\theta\, s_\alpha \\ s_\theta\, c_\alpha \\ 0 \end{Bmatrix} & \begin{Bmatrix} -\rho\, c_\theta\, s_\alpha \\ \rho\, c_\theta\, c_\alpha \\ 0 \end{Bmatrix} & \begin{Bmatrix} -\rho\, s_\theta\, c_\alpha \\ -\rho\, s_\theta\, s_\alpha \\ 0 \end{Bmatrix} \end{bmatrix} \tag{1.129}$$

and

$$[\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}] = \begin{bmatrix} \{\underset{\cdot}{\boldsymbol{0}}\} & \{\frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\} & \{\frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_3\} \\ \{\frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\} & \{-\rho\underset{\cdot}{\boldsymbol{g}}_1\} & \{\frac{1}{t_\theta}\underset{\cdot}{\boldsymbol{g}}_3\} \\ \{\frac{1}{\rho}\underset{\cdot}{\boldsymbol{g}}_3\} & \{\frac{1}{t_\theta}\underset{\cdot}{\boldsymbol{g}}_3\} & \{-\rho\, s_\theta\, \underset{\cdot}{\boldsymbol{\alpha}}\} \end{bmatrix} \tag{1.130}$$

where $t_\theta = \tan\theta$ and

$$\underset{\cdot}{\boldsymbol{\alpha}} = \begin{Bmatrix} c_\alpha \\ s_\alpha \\ 0 \end{Bmatrix} \tag{1.131}$$

therefore

$$\underset{\cdot}{\boldsymbol{\alpha}} \cdot \underset{\cdot}{\boldsymbol{g}}^1 = s_\theta \, c_\alpha^2 + s_\theta \, s_\alpha^2 = s_\theta \tag{1.132}$$

and

$$\underset{\cdot}{\boldsymbol{\alpha}} \cdot \underset{\cdot}{\boldsymbol{g}}^2 = \frac{1}{\rho} \, c_\theta \, c_\alpha^2 + \frac{1}{\rho} \, c_\theta \, s_\alpha^2 = \frac{1}{\rho} \, c_\theta \tag{1.133}$$

and

$$\underset{\cdot}{\boldsymbol{\alpha}} \cdot \underset{\cdot}{\boldsymbol{g}}^3 = -\frac{1}{\rho} \, \frac{s_\alpha \, c_\alpha}{s_\theta} + \frac{1}{\rho} \, \frac{s_\alpha \, c_\alpha}{s_\theta} = 0 \tag{1.134}$$

Christoffel symbols of second kind

$$[\Gamma_{ij}^k] = \begin{bmatrix}
\underset{\cdot}{\boldsymbol{0}} \cdot \underset{\cdot}{\boldsymbol{g}}^1 & \frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_2 \cdot \underset{\cdot}{\boldsymbol{g}}^1 & \frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^1 \\
\frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_2 \cdot \underset{\cdot}{\boldsymbol{g}}^1 & -\rho \underset{\cdot}{\boldsymbol{g}}_1 \cdot \underset{\cdot}{\boldsymbol{g}}^1 & \frac{1}{t_\theta} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^1 \\
\frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^1 & \frac{1}{t_\theta} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^1 & -\rho \, s_\theta \, \underset{\cdot}{\boldsymbol{\alpha}} \cdot \underset{\cdot}{\boldsymbol{g}}^1 \\
\\
\underset{\cdot}{\boldsymbol{0}} \cdot \underset{\cdot}{\boldsymbol{g}}^2 & \frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_2 \cdot \underset{\cdot}{\boldsymbol{g}}^2 & \frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^2 \\
\frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_2 \cdot \underset{\cdot}{\boldsymbol{g}}^2 & -\rho \underset{\cdot}{\boldsymbol{g}}_1 \cdot \underset{\cdot}{\boldsymbol{g}}^2 & \frac{1}{t_\theta} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^2 \\
\frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^2 & \frac{1}{t_\theta} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^2 & -\rho \, s_\theta \, \underset{\cdot}{\boldsymbol{\alpha}} \cdot \underset{\cdot}{\boldsymbol{g}}^2 \\
\\
\underset{\cdot}{\boldsymbol{0}} \cdot \underset{\cdot}{\boldsymbol{g}}^3 & \frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_2 \cdot \underset{\cdot}{\boldsymbol{g}}^3 & \frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^3 \\
\frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_2 \cdot \underset{\cdot}{\boldsymbol{g}}^3 & -\rho \underset{\cdot}{\boldsymbol{g}}_1 \cdot \underset{\cdot}{\boldsymbol{g}}^3 & \frac{1}{t_\theta} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^3 \\
\frac{1}{\rho} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^3 & \frac{1}{t_\theta} \underset{\cdot}{\boldsymbol{g}}_3 \cdot \underset{\cdot}{\boldsymbol{g}}^3 & -\rho \, s_\theta \, \underset{\cdot}{\boldsymbol{\alpha}} \cdot \underset{\cdot}{\boldsymbol{g}}^3
\end{bmatrix} \tag{1.135}$$

Thus

$$[\Gamma_{ij}^k] = \begin{bmatrix}
0 & 0 & 0 \\
0 & -\rho & 0 \\
0 & 0 & -\rho \, s_\theta^2 \\
\\
0 & \frac{1}{\rho} & 0 \\
\frac{1}{\rho} & 0 & 0 \\
0 & 0 & -s_\theta \, c_\theta \\
\\
0 & 0 & \frac{1}{\rho} \\
0 & 0 & \frac{1}{t_\theta} \\
\frac{1}{\rho} & \frac{1}{t_\theta} & 0
\end{bmatrix} \tag{1.136}$$

Laplacian in spherical coordinates (from Eq. 1.90 and noting that the off-diagonal metric coefficients are zero)

$$
\begin{aligned}
\nabla^2\phi = &\left(\frac{\partial^2\phi}{\partial\rho^2} - \frac{\partial\phi}{\partial\rho}\cancel{\Gamma^1_{11}} - \frac{\partial\phi}{\partial\theta}\cancel{\Gamma^2_{11}} - \frac{\partial\phi}{\partial\alpha}\cancel{\Gamma^3_{11}}\right) \\
&+ \left(\frac{\partial^2\phi}{\partial\theta^2} - \frac{\partial\phi}{\partial\rho}\Gamma^1_{22} - \frac{\partial\phi}{\partial\theta}\cancel{\Gamma^2_{22}} - \frac{\partial\phi}{\partial\alpha}\cancel{\Gamma^3_{22}}\right)\frac{1}{\rho^2} \\
&+ \left(\frac{\partial^2\phi}{\partial\alpha^2} - \frac{\partial\phi}{\partial\rho}\Gamma^1_{33} - \frac{\partial\phi}{\partial\theta}\Gamma^2_{33} - \frac{\partial\phi}{\partial\alpha}\cancel{\Gamma^3_{33}}\right)\frac{1}{\rho^2 s_\theta^2}
\end{aligned} \tag{1.137}
$$

Thus

$$
\begin{aligned}
\nabla^2\phi &= \frac{\partial^2\phi}{\partial\rho^2} + \left(\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial\phi}{\partial\rho}\rho\right)\frac{1}{\rho^2} + \left(\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial\phi}{\partial\rho}\rho s_\theta^2 + \frac{\partial\phi}{\partial\theta}s_\theta c_\theta\right)\frac{1}{\rho^2 s_\theta^2} \\
&= \frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2 s_\theta^2}\frac{\partial^2\phi}{\partial\alpha^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{c_\theta}{\rho^2 s_\theta}\frac{\partial\phi}{\partial\theta}
\end{aligned} \tag{1.138}
$$

or

$$
\nabla^2\phi = \frac{\partial^2\phi}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{c_\theta}{\rho^2 s_\theta}\frac{\partial\phi}{\partial\theta} + \frac{1}{\rho^2 s_\theta^2}\frac{\partial^2\phi}{\partial\alpha^2} \tag{1.139}
$$

1.5.5 Quarter-ring cylindrical coordinates

This example considers a pair of coordinates $r \equiv r^1 \in [-1,+1]$ and $s \equiv r^2 \in [-1+1]$ such that x and y are mapped to a quarter of ring with inner radius a and outer radius b (Fig. 1.2).

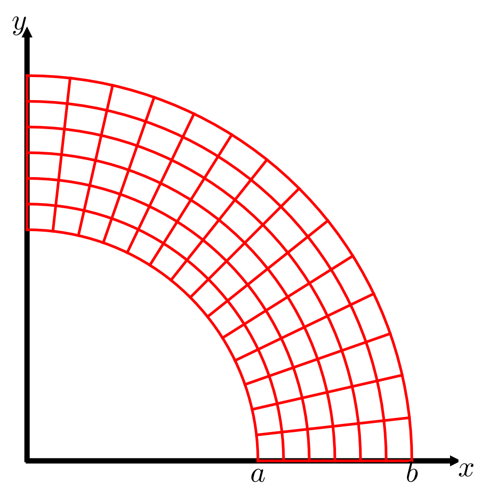

Fig. 1.2: Quarter-ring cylindrical coordinates.

Coordinates:

$$\begin{cases} x = a + \underbrace{(b-a)\,\dfrac{1+r}{2}}_{\rho}\;\underbrace{\cos\left(\dfrac{1+s}{2}\,\dfrac{\pi}{2}\right)}_{c_\alpha} \\ y = a + \underbrace{(b-a)\,\dfrac{1+r}{2}}_{\rho}\;\underbrace{\sin\left(\dfrac{1+s}{2}\,\dfrac{\pi}{2}\right)}_{s_\alpha} \end{cases} \tag{1.140}$$

or

$$\begin{cases} x = a + \rho\, c_\alpha \\ y = a + \rho\, s_\alpha \end{cases} \tag{1.141}$$

By defining $A = (b-a)/2$ and $B = \pi/4$, the covariant base vectors are

$$\{\boldsymbol{g}_1\} = \begin{Bmatrix} \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial r} \end{Bmatrix} = \begin{Bmatrix} A\, c_\alpha \\ A\, s_\alpha \end{Bmatrix} \tag{1.142}$$

and

$$\{\boldsymbol{g}_2\} = \begin{Bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{Bmatrix} = \begin{Bmatrix} -\rho\, B\, s_\alpha \\ \rho\, B\, c_\alpha \end{Bmatrix} \tag{1.143}$$

Metric coefficients

$$[g] = \begin{bmatrix} \boldsymbol{g}_1 \cdot \boldsymbol{g}_1 & \boldsymbol{g}_1 \cdot \boldsymbol{g}_2 \\ \boldsymbol{g}_2 \cdot \boldsymbol{g}_1 & \boldsymbol{g}_2 \cdot \boldsymbol{g}_2 \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ 0 & \rho^2 B^2 \end{bmatrix} \tag{1.144}$$

and

$$[G] = [g]^{-1} = \begin{bmatrix} \frac{1}{A^2} & 0 \\ 0 & \frac{1}{\rho^2 B^2} \end{bmatrix} \tag{1.145}$$

Contravariant base vectors

$$\boldsymbol{g}^1 = g^{11}\boldsymbol{g}_1 + g^{12}\boldsymbol{g}_2 = \frac{1}{A}\boldsymbol{g}_1 \qquad \text{thus} \qquad \boldsymbol{g}^1 = \begin{Bmatrix} \frac{c_\alpha}{A} \\ \frac{s_\alpha}{A} \end{Bmatrix} \tag{1.146}$$

and

$$\boldsymbol{g}^2 = g^{21}\boldsymbol{g}_1 + g^{22}\boldsymbol{g}_2 = \frac{1}{\rho^2 B^2}\boldsymbol{g}_2 \qquad \text{thus} \qquad \boldsymbol{g}^2 = \begin{Bmatrix} -\frac{s_\alpha}{\rho B} \\ \frac{c_\alpha}{\rho B} \end{Bmatrix} \tag{1.147}$$

Christoffel vectors

$$[\boldsymbol{\Gamma}_{ij}] = \begin{bmatrix} \frac{\partial \boldsymbol{g}_1}{\partial r} & \frac{\partial \boldsymbol{g}_1}{\partial s} \\ \frac{\partial \boldsymbol{g}_2}{\partial r} & \frac{\partial \boldsymbol{g}_2}{\partial s} \end{bmatrix} \tag{1.148}$$

Thus

$$[\underset{\cdot}{\boldsymbol{\Gamma}}_{ij}] = \begin{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} & \begin{Bmatrix} -A\,B\,s_\alpha \\ A\,B\,c_\alpha \end{Bmatrix} \\ \\ \begin{Bmatrix} -A\,B\,s_\alpha \\ A\,B\,c_\alpha \end{Bmatrix} & \begin{Bmatrix} -\rho\,B^2\,c_\alpha \\ -\rho\,B^2\,s_\alpha \end{Bmatrix} \end{bmatrix} = \begin{bmatrix} \{\underset{\cdot}{\boldsymbol{0}}\} & \{\frac{A}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\} \\ \{\frac{A}{\rho}\underset{\cdot}{\boldsymbol{g}}_2\} & \{-\frac{\rho\,B^2}{A}\underset{\cdot}{\boldsymbol{g}}_1\} \end{bmatrix} \tag{1.149}$$

Christoffel symbols of second kind

$$[\Gamma^k_{ij}] = \begin{bmatrix} \underset{\cdot}{\boldsymbol{\Gamma}}_{11}\cdot\underset{\cdot}{\boldsymbol{g}}^1 & \underset{\cdot}{\boldsymbol{\Gamma}}_{12}\cdot\underset{\cdot}{\boldsymbol{g}}^1 \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{21}\cdot\underset{\cdot}{\boldsymbol{g}}^1 & \underset{\cdot}{\boldsymbol{\Gamma}}_{22}\cdot\underset{\cdot}{\boldsymbol{g}}^1 \\ \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{11}\cdot\underset{\cdot}{\boldsymbol{g}}^2 & \underset{\cdot}{\boldsymbol{\Gamma}}_{12}\cdot\underset{\cdot}{\boldsymbol{g}}^2 \\ \underset{\cdot}{\boldsymbol{\Gamma}}_{21}\cdot\underset{\cdot}{\boldsymbol{g}}^2 & \underset{\cdot}{\boldsymbol{\Gamma}}_{22}\cdot\underset{\cdot}{\boldsymbol{g}}^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\rho\,B^2}{A} \\ \\ 0 & \frac{A}{\rho} \\ \frac{A}{\rho} & 0 \end{bmatrix} \tag{1.150}$$

Laplacian in cylindrical coordinates (from Eq. 1.90 and noting that the off-diagonal metric coefficients are zero)

$$\begin{aligned} \nabla^2\phi &= \left(\frac{\partial^2\phi}{\partial r^2} - \frac{\partial\phi}{\partial r}\cancel{\Gamma^1_{11}} - \frac{\partial\phi}{\partial s}\cancel{\Gamma^2_{11}}\right)\frac{1}{A^2} \\ &\quad + \left(\frac{\partial^2\phi}{\partial s^2} - \frac{\partial\phi}{\partial r}\,\Gamma^1_{22} - \frac{\partial\phi}{\partial s}\cancel{\Gamma^2_{22}}\right)\frac{1}{\rho^2 B^2} \end{aligned} \tag{1.151}$$

Thus

$$\nabla^2\phi = \frac{1}{A^2}\,\frac{\partial^2\phi}{\partial r^2} + \frac{1}{\rho A}\,\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2 B^2}\,\frac{\partial^2\phi}{\partial\alpha^2} \tag{1.152}$$