Chapter 1
Tensor Calculus and Definitions - By Dorival Pedroso

1.1 Introduction

In this work, the simplified statement $f = f(\bigcirc)$ means that f is a function of $\bigcirc$. The context will make it clear when f is a function or a constant.

We will suspend the sum at times and then this will be indicated.

1.2 Notation

In this book, Tensor/Gibbs notation is employed. An alternative is the index notation. To track the kind of an entity by a simple visual inspection of the corresponding symbol, *underdots* are added to the symbol with the number of dots being equal to the tensor order. This notation allows the use of a wider range of symbols because, for example, a, $\underset{\cdot}{\boldsymbol{a}}$, $\underset{\cdot\cdot}{\boldsymbol{a}}$, $\underset{\cdot\cdot\cdot}{\boldsymbol{a}}$, and $\underset{\cdot\cdot\cdot\cdot}{\boldsymbol{a}}$ all using the character 'a', are easily recognisable as being different tensors, namely a 0-order (scalar), a vector, a tensor (second order), a third order tensor and a fourth order tensor, respectively. Inner products are expressed with operator dots such as in $s = \underset{\cdot}{\boldsymbol{a}} \boldsymbol{\cdot} \underset{\cdot}{\boldsymbol{a}}$ and $\sigma = \underset{\cdot\cdot}{\boldsymbol{a}} : \underset{\cdot\cdot}{\boldsymbol{a}}$. The dyadic product is expressed as in $\underset{\cdot\cdot}{\boldsymbol{a}} = \underset{\cdot}{\boldsymbol{a}} \otimes \underset{\cdot}{\boldsymbol{a}}$. With an ortho-normal Cartesian system, the following is valid: $s = a_i\, a_i$, $\sigma = a_{ij}\, a_{ij}$ and $a_{ij} = a_i\, a_j$; considering Einstein's notation (summation over repeated indices). The divergence of a first order tensor (vector) is denoted by $\operatorname{div} \underset{\cdot}{\boldsymbol{a}}$. The divergence of a second order tensor, resulting in a first order tensor, is denoted by $\underset{\cdot}{\mathbf{div}}\, \underset{\cdot\cdot}{\boldsymbol{a}}$.

We will use an *upperdot* for the first time derivative as in

$$\frac{\mathrm{d}\bigcirc}{\mathrm{d}t} := \dot{\bigcirc} \tag{1.1}$$

and two *upperdots* for the second time derivative as in

$$\frac{\mathrm{d}^2\bigcirc}{\mathrm{d}t^2} := \ddot{\bigcirc} \tag{1.2}$$

where $\bigcirc$ can be a scalar, vector, or tensor of any order (i.e. "a tensor").

Sometimes, we also consider the simpler notation for the tensor-vector product; i.e.

$$\underset{\cdot\cdot}{\boldsymbol{F}} \cdot \underset{\cdot}{\boldsymbol{a}} = \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{a}} \tag{1.3}$$

1.3 Orthogonal normalised basis and identities

An orthogonal and normalised basis (orthonormal) is assumed with the basis vectors indicated by $\underset{\cdot}{\boldsymbol{e}}_i$.

Tensors:

$$\underset{\cdot\cdot}{\boldsymbol{a}} = a_{ij}\, \underset{\cdot}{\boldsymbol{e}}_i \otimes \underset{\cdot}{\boldsymbol{e}}_j \tag{1.4}$$

Identity:

$$(\underset{\cdot}{\boldsymbol{a}} \otimes \underset{\cdot}{\boldsymbol{b}}) : (\underset{\cdot}{\boldsymbol{c}} \otimes \underset{\cdot}{\boldsymbol{d}}) = (\underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot}{\boldsymbol{c}})\ (\underset{\cdot}{\boldsymbol{b}} \cdot \underset{\cdot}{\boldsymbol{d}}) \tag{1.5}$$

$$(\underset{\cdot}{\boldsymbol{a}} \otimes \underset{\cdot}{\boldsymbol{b}}) : \underset{\cdot\cdot}{\boldsymbol{c}} = \underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{c}} \cdot \underset{\cdot}{\boldsymbol{b}} \tag{1.6}$$

1.4 Definitions and properties

Transpose

$$\underset{\cdot\cdot}{\boldsymbol{a}} \cdot \underset{\cdot}{\boldsymbol{u}} = \underset{\cdot}{\boldsymbol{u}} \cdot \underset{\cdot\cdot}{\boldsymbol{a}}^T \tag{1.7}$$

The trace is a linear mapping; hence the additivity and homogeneity properties apply:

$$\mathrm{tr}\,(\underset{\cdot\cdot}{\boldsymbol{a}} + \underset{\cdot\cdot}{\boldsymbol{b}}) = \mathrm{tr}\,\underset{\cdot\cdot}{\boldsymbol{a}} + \mathrm{tr}\,\underset{\cdot\cdot}{\boldsymbol{b}} \tag{1.8}$$

and

$$\mathrm{tr}\,(\alpha\, \underset{\cdot\cdot}{\boldsymbol{a}}) = \alpha\ \mathrm{tr}\,\underset{\cdot\cdot}{\boldsymbol{a}} \tag{1.9}$$

The trace of a dyadic product of two vectors results in an inner product as follows

$$\mathrm{tr}\,(\underset{\cdot}{\boldsymbol{a}} \otimes \underset{\cdot}{\boldsymbol{b}}) = \underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot}{\boldsymbol{b}} \tag{1.10}$$

The trace of the single contraction between two second order tensors can be calculated as

$$\mathrm{tr}\,(\underset{\cdot\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}}) = \mathrm{tr}\,(\underset{\cdot\cdot}{\boldsymbol{b}} \cdot \underset{\cdot\cdot}{\boldsymbol{a}}) \tag{1.11}$$

The double contraction and the trace operator are related by the following expressions

$$\underset{\cdot\cdot}{\boldsymbol{a}} : \underset{\cdot\cdot}{\boldsymbol{b}} = \mathrm{tr}\,(\underset{\cdot\cdot}{\boldsymbol{a}}^T \cdot \underset{\cdot\cdot}{\boldsymbol{b}}) = \mathrm{tr}\,(\underset{\cdot\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}}^T) = \mathrm{tr}\,(\underset{\cdot\cdot}{\boldsymbol{b}}^T \cdot \underset{\cdot\cdot}{\boldsymbol{a}}) = \mathrm{tr}\,(\underset{\cdot\cdot}{\boldsymbol{b}} \cdot \underset{\cdot\cdot}{\boldsymbol{a}}^T) \tag{1.12}$$

1.5 Levi-Civita alternator and determinants

Some useful properties involving the Levi-Civita alternator are

$$e_{ijk}\, e_{rsk} = \delta_{ir}\, \delta_{js} - \delta_{is}\, \delta_{jr} \tag{1.13}$$

and

$$e_{ijk} = -e_{jik} = e_{jki} \tag{1.14}$$

The determinant of a tensor can be expressed by using the alternator symbol as follows

$$\det \underset{\cdot\cdot}{\boldsymbol{a}} = e_{ijk}\, a_{0i}\, a_{1j}\, a_{2k} = e_{ijk}\, a_{i0}\, a_{j1}\, a_{k2} \tag{1.15}$$

or, by using

$$\det \underset{\cdot\cdot}{\boldsymbol{a}} = \frac{1}{6} e_{ijk}\, e_{rst}\, a_{ir}\, a_{js}\, a_{kt} \tag{1.16}$$

It can also be shown that

$$e_{rst}\, \det \underset{\cdot\cdot}{\boldsymbol{a}} = e_{ijk}\, a_{ir}\, a_{js}\, a_{kt} \tag{1.17}$$

1.6 Triple product of vectors

The triple product is defined as

$$[\![\underset{\cdot}{\boldsymbol{a}}\ \ \underset{\cdot}{\boldsymbol{b}}\ \ \underset{\cdot}{\boldsymbol{c}}]\!] = \underset{\cdot}{\boldsymbol{a}} \cdot (\underset{\cdot}{\boldsymbol{b}} \times \underset{\cdot}{\boldsymbol{c}}) \tag{1.18}$$

By considering a system of orthogonal coordinates, the triple product can be expanded into

$$[\![\underset{\cdot}{\boldsymbol{a}}\ \ \underset{\cdot}{\boldsymbol{b}}\ \ \underset{\cdot}{\boldsymbol{c}}]\!] = e_{ijk}\, a_i\, b_j\, c_k \tag{1.19}$$

We now consider the simpler notation for the tensor-vector product; i.e. $\underset{\cdot\cdot}{\boldsymbol{F}} \cdot \underset{\cdot}{\boldsymbol{a}} = \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{a}}$. Then, by multiplying each vector in the triple product by a tensor $\underset{\cdot\cdot}{\boldsymbol{F}}$, we can obtain

$$[\![\underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{a}}\ \ \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{b}}\ \ \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{c}}]\!] = e_{ijk}\ \ F_{ir}\, a_r\ \ F_{js}\, b_s\ \ F_{kt}\, c_t \tag{1.20}$$

or, by considering Eq. (1.17),

$$[\![\underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{a}}\ \ \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{b}}\ \ \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{c}}]\!] = \underbrace{e_{ijk}\, F_{ir}\, F_{js}\, F_{kt}}_{e_{rst}\ \det \underset{\cdot\cdot}{\boldsymbol{F}}}\ \ a_r\, b_s\, c_t \tag{1.21}$$

i.e.

$$[\![\underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{a}}\ \ \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{b}}\ \ \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{c}}]\!] = \det \underset{\cdot\cdot}{\boldsymbol{F}}\ e_{rst}\ \ a_r\, b_s\, c_t \tag{1.22}$$

therefore

$$[\![\underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{a}} \;\; \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{b}} \;\; \underset{\cdot\cdot}{\boldsymbol{F}}\underset{\cdot}{\boldsymbol{c}}]\!] = \det \underset{\cdot\cdot}{\boldsymbol{F}} \; [\![\underset{\cdot}{\boldsymbol{a}} \;\; \underset{\cdot}{\boldsymbol{b}} \;\; \underset{\cdot}{\boldsymbol{c}}]\!] \qquad \blacksquare \tag{1.23}$$

1.7 Differentiation of tensors

The derivative of a scalar field $\phi(\underset{\cdot}{\boldsymbol{x}})$ with respect to (w.r.t) a vector $\underset{\cdot}{\boldsymbol{x}}$, resulting in a first order tensor (vector) $\underset{\cdot}{\boldsymbol{w}}$, is defined by means of

$$\underset{\cdot}{\boldsymbol{w}} \equiv \frac{\mathrm{d}\phi}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} = \frac{\partial \phi}{\partial x_i} \, \underset{\cdot}{\boldsymbol{e}}_i \tag{1.24}$$

Note that the number of dots in all terms is equal to one because $\frac{\mathrm{d}\phi}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}}$ has one dot in the denominator and hence must 'produce' an *entity with one dot* (a vector).

The derivative of a vector field $\underset{\cdot}{\boldsymbol{v}}(\underset{\cdot}{\boldsymbol{x}})$ w.r.t to $\underset{\cdot}{\boldsymbol{x}}$, resulting in a second order tensor (simply tensor) $\underset{\cdot\cdot}{\boldsymbol{a}}$, is defined by

$$\underset{\cdot\cdot}{\boldsymbol{a}} \equiv \frac{\mathrm{d}\underset{\cdot}{\boldsymbol{v}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} = \frac{\partial v_i}{\partial x_j} \, \underset{\cdot}{\boldsymbol{e}}_i \otimes \underset{\cdot}{\boldsymbol{e}}_j \tag{1.25}$$

Note now that the number of dots in all terms is equal to two because $\frac{\mathrm{d}\underset{\cdot}{\boldsymbol{v}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}}$ has one dot in the numerator and one dot in the denominator and hence must 'produce' an *entity with two dots* (a tensor).

The derivative of a tensor field $\underset{\cdot\cdot}{\boldsymbol{\sigma}}(\underset{\cdot}{\boldsymbol{x}})$ w.r.t to $\underset{\cdot}{\boldsymbol{x}}$, resulting in a third order tensor $\underset{\cdot\cdot\cdot}{\boldsymbol{\xi}}$ is defined by

$$\underset{\cdot\cdot\cdot}{\boldsymbol{\xi}} \equiv \frac{\mathrm{d}\underset{\cdot\cdot}{\boldsymbol{\sigma}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}} = \frac{\partial \sigma_{ij}}{\partial x_k} \, \underset{\cdot}{\boldsymbol{e}}_i \otimes \underset{\cdot}{\boldsymbol{e}}_j \otimes \underset{\cdot}{\boldsymbol{e}}_k \tag{1.26}$$

Note finally that the number of dots in all terms is equal to three because $\frac{\mathrm{d}\underset{\cdot\cdot}{\boldsymbol{\sigma}}}{\mathrm{d}\underset{\cdot}{\boldsymbol{x}}}$ has two dots in the numerator and one dot in the denominator and hence must 'produce' an *entity with three dots* (a third order tensor).

The above equations illustrate quite well the advantage of the 'underdots' notation. For example, hereafter, $\underset{\cdot}{\boldsymbol{w}}$, $\underset{\cdot\cdot}{\boldsymbol{a}}$ and $\underset{\cdot\cdot\cdot}{\boldsymbol{\xi}}$ are easily recognisable as vector, tensor and third order tensor, respectively, without the need for different fonts or extra 'tricks' such as italic or bold. This advantage is clearly in addition to the implicit verification procedure for the consistency of all terms (number of dots).

Other commonly used symbols for the derivative w.r.t $\underset{\cdot}{\boldsymbol{x}}$ (i.e. the gradient) are listed below, for an operation resulting in a vector ($\underset{\cdot}{\boldsymbol{\nabla}} \phi$) as

$$\underset{\cdot}{\boldsymbol{\nabla}} \phi = \frac{\partial \phi}{\partial \underset{\cdot}{\boldsymbol{x}}}, \tag{1.27}$$

in a second order tensor ($\underset{\cdot\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot}{\boldsymbol{v}}$) as

$$\underset{\cdot\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot}{\boldsymbol{v}} = \frac{\partial \underset{\cdot}{\boldsymbol{v}}}{\partial \underset{\cdot}{\boldsymbol{x}}} \tag{1.28}$$

and in a third order tensor ($\underset{\cdot\cdot\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot\cdot}{\boldsymbol{\sigma}}$) as

$$\underset{\cdot\cdot\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot\cdot}{\boldsymbol{\sigma}} = \frac{\partial \underset{\cdot\cdot}{\boldsymbol{\sigma}}}{\partial \underset{\cdot}{\boldsymbol{x}}} \tag{1.29}$$

Note that the number of dots in the nabla symbol indicates the order of the result; i.e. $\underset{\cdot\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot}{\boldsymbol{v}}$ is of second order and the three dots must *not* be summed.

The divergence is defined as the *trace of the gradient*. For vectors, it is defined as follows

$$\operatorname{div} \underset{\cdot}{\boldsymbol{v}} = \operatorname{tr}\,(\underset{\cdot\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot}{\boldsymbol{v}}) = \frac{\partial \underset{\cdot}{\boldsymbol{v}}}{\partial \underset{\cdot}{\boldsymbol{x}}} : \underset{\cdot\cdot}{\boldsymbol{I}} = \frac{\partial v_i}{\partial x_i} = \underbrace{\underset{\cdot}{\boldsymbol{\nabla}} \boldsymbol{\cdot} \underset{\cdot}{\boldsymbol{v}}}_{\text{not used here}} \tag{1.30}$$

Therein, the 'div' operator has *zero dots* indicating that it 'consumes' the one dot in $\underset{\cdot}{\boldsymbol{v}}$; i.e. the order is reduced by one. Note also that number of dots in $\frac{\partial \underset{\cdot}{\boldsymbol{v}}}{\partial \underset{\cdot}{\boldsymbol{x}}}$ is two as is the case of $\underset{\cdot\cdot}{\boldsymbol{I}}$. Since the colon symbol ':' consumes four dots, the result will be a scalar. In this case only, the result can be made equal to the index notation term. Finally, note that the last term in the above expression is not used in this text because it assumes that the nabla symbol is a vector itself and this may be confusing.

The divergence of a second order tensor is calculated as follows

$$\underset{\cdot}{\mathbf{div}}\, \underset{\cdot\cdot}{\boldsymbol{\sigma}} = \frac{\partial \underset{\cdot\cdot}{\boldsymbol{\sigma}}}{\partial \underset{\cdot}{\boldsymbol{x}}} : \underset{\cdot\cdot}{\boldsymbol{I}} = \frac{\partial \sigma_{ij}}{\partial x_j}\, \underset{\cdot}{\boldsymbol{e}}_i = \underbrace{\sigma_{ij,j}\, \underset{\cdot}{\boldsymbol{e}}_i}_{\text{not used here}} \tag{1.31}$$

In this case, the 'div' operator has one dot indicating the order of the result. Note that $\frac{\partial \underset{\cdot\cdot}{\boldsymbol{\sigma}}}{\partial \underset{\cdot}{\boldsymbol{x}}}$ has three dots, $\underset{\cdot\cdot}{\boldsymbol{I}}$ has two dots, and the inner product ':' consumes four dots; therefore the result must have one dot; i.e. it must be a vector.

1.8 Identities involving the divergence operator

Let $a = a(\underset{\cdot}{\boldsymbol{x}})$ and $\underset{\cdot}{\boldsymbol{b}} = \underset{\cdot}{\boldsymbol{b}}(\underset{\cdot}{\boldsymbol{x}})$ be a scalar and a vector field, respectively. By applying the chain rule, then:

$$\underset{\cdot\cdot}{\boldsymbol{\nabla}}\,(a\, \underset{\cdot}{\boldsymbol{b}}) = \frac{\partial a\, \underset{\cdot}{\boldsymbol{b}}}{\partial \underset{\cdot}{\boldsymbol{x}}} = \underset{\cdot}{\boldsymbol{b}} \otimes \underset{\cdot}{\boldsymbol{\nabla}}\, a + a\, \underset{\cdot\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot}{\boldsymbol{b}} \tag{1.32}$$

Thus, the divergence of the product $a\,\underset{\cdot}{\boldsymbol{b}}$ is given by (using Eqs. (1.8)-(1.10) and Eq. (1.30))

$$\mathrm{div}\,(a\,\underset{\cdot}{\boldsymbol{b}}) = \mathrm{tr}\,[\underset{\cdot}{\boldsymbol{\nabla}}\,(a\,\underset{\cdot}{\boldsymbol{b}})] = \underset{\cdot}{\boldsymbol{\nabla}}\,a \cdot \underset{\cdot}{\boldsymbol{b}} + a\ \mathrm{div}\,\underset{\cdot}{\boldsymbol{b}} \tag{1.33}$$

Now, let $\underset{\cdot}{\boldsymbol{a}} = \underset{\cdot}{\boldsymbol{a}}(\underset{\cdot}{\boldsymbol{x}})$ be a vector field and $\underset{\cdot\cdot}{\boldsymbol{b}} = \underset{\cdot\cdot}{\boldsymbol{b}}(\underset{\cdot}{\boldsymbol{x}})$ be a tensor field, respectively. By the chain rule:

$$\underset{\cdot}{\boldsymbol{\nabla}}\,(\underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}}) = \underset{\cdot}{\boldsymbol{\nabla}}\,(\underset{\cdot\cdot}{\boldsymbol{b}}^T \cdot \underset{\cdot}{\boldsymbol{a}}) = \underset{\cdot\cdot}{\boldsymbol{b}}^T \cdot \underset{\cdot\cdot}{\boldsymbol{\nabla}}\,\underset{\cdot}{\boldsymbol{a}} + \underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot\cdot}{\boldsymbol{\nabla}}\,\underset{\cdot\cdot}{\boldsymbol{b}} \tag{1.34}$$

Therefore, the divergence of the product $\underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}}$ can be calculated as follows

$$\mathrm{div}\,(\underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}}) = \mathrm{tr}\,[\underset{\cdot\cdot}{\boldsymbol{\nabla}}\,(\underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}})] = \mathrm{tr}\,(\underset{\cdot\cdot}{\boldsymbol{b}}^T \cdot \underset{\cdot\cdot}{\boldsymbol{\nabla}}\,\underset{\cdot}{\boldsymbol{a}}) + \underset{\cdot}{\boldsymbol{a}} \cdot (\underset{\cdot\cdot\cdot}{\boldsymbol{\nabla}}\,\underset{\cdot\cdot}{\boldsymbol{b}} : \underset{\cdot\cdot}{\boldsymbol{I}}) \tag{1.35}$$

By considering the properties of the double contraction in Eq. (1.12) and the definition of the divergence of a second order tensor in Eq. (1.31), we obtain

$$\mathrm{div}\,(\underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}}) = \underset{\cdot\cdot}{\boldsymbol{\nabla}}\,\underset{\cdot}{\boldsymbol{a}} : \underset{\cdot\cdot}{\boldsymbol{b}} + \underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot}{\mathbf{div}}\,\underset{\cdot\cdot}{\boldsymbol{b}} \tag{1.36}$$

By isolating the divergence term in Eq. (1.33) and Eq. (1.36), the following two useful expressions are obtained

$$a\ \mathrm{div}\,\underset{\cdot}{\boldsymbol{b}} = \mathrm{div}\,(a\,\underset{\cdot}{\boldsymbol{b}}) - \underset{\cdot}{\boldsymbol{\nabla}}\,a \cdot \underset{\cdot}{\boldsymbol{b}} \tag{1.37}$$

and

$$\underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot}{\mathbf{div}}\,\underset{\cdot\cdot}{\boldsymbol{b}} = \mathrm{div}\,(\underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}}) - \underset{\cdot\cdot}{\boldsymbol{\nabla}}\,\underset{\cdot}{\boldsymbol{a}} : \underset{\cdot\cdot}{\boldsymbol{b}} \tag{1.38}$$

The above expression can easily be verified by considering the indices w.r.t an orthonormal basis as follows

$$\frac{\partial(a_i\,b_{ij})}{\partial x_j} = \frac{\partial a_i}{\partial x_j}\,b_{ij} + a_i\,\frac{\partial b_{ij}}{\partial x_j} \tag{1.39}$$

Hence

$$a_i\,\frac{\partial b_{ij}}{\partial x_j} = \frac{\partial(a_i\,b_{ij})}{\partial x_j} - \frac{\partial a_i}{\partial x_j}\,b_{ij} \tag{1.40}$$

which corresponds to Eq. (1.38).

1.8.1 Index notation

The above identities can be verified for the case where an orthonormal Cartesian system is adopted. In this case, the following two expressions are easily verified

$$\mathrm{div}\,(a\,\underset{\cdot}{\boldsymbol{b}}) = \frac{\partial a b_i}{\partial x_i} = \frac{\partial a}{\partial x_i}\,b_i + a\,\frac{\partial b_i}{\partial x_i} = \underset{\cdot}{\boldsymbol{\nabla}}\,a \cdot \underset{\cdot}{\boldsymbol{b}} + a\ \mathrm{div}\,\underset{\cdot}{\boldsymbol{b}} \tag{1.41}$$

and

$$\operatorname{div}(\underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}}) = \frac{\partial a_i b_{ij}}{\partial x_j} = \frac{\partial a_i}{\partial x_j}\, b_{ij} + a_i\, \frac{\partial b_{ij}}{\partial x_j} = \underset{\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot}{\boldsymbol{a}} : \underset{\cdot\cdot}{\boldsymbol{b}} + \underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot}{\mathbf{div}}\, \underset{\cdot\cdot}{\boldsymbol{b}} \tag{1.42}$$

1.9 Green-Gauss theorem

Definition

$$\int_\Omega \operatorname{div} \underset{\cdot}{\boldsymbol{v}}\ \mathrm{d}\Omega = \int_\Gamma \underset{\cdot}{\boldsymbol{v}} \cdot \underset{\cdot}{\hat{\boldsymbol{n}}}\ \mathrm{d}\Gamma \tag{1.43}$$

Therefore, from Eq. (1.37)

$$\int_\Omega a\ \operatorname{div} \underset{\cdot}{\boldsymbol{b}}\ \mathrm{d}\Omega = \int_\Gamma a\, \underset{\cdot}{\boldsymbol{b}} \cdot \underset{\cdot}{\hat{\boldsymbol{n}}}\ \mathrm{d}\Gamma - \int_\Omega \underset{\cdot}{\boldsymbol{\nabla}}\, a \cdot \underset{\cdot}{\boldsymbol{b}}\ \mathrm{d}\Omega \tag{1.44}$$

and, from Eq. (1.38)

$$\int_\Omega \underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot}{\mathbf{div}}\, \underset{\cdot\cdot}{\boldsymbol{b}}\ \mathrm{d}\Omega = \int_\Gamma \underset{\cdot}{\boldsymbol{a}} \cdot \underset{\cdot\cdot}{\boldsymbol{b}} \cdot \underset{\cdot}{\hat{\boldsymbol{n}}}\ \mathrm{d}\Gamma - \int_\Omega \underset{\cdot}{\boldsymbol{\nabla}}\, \underset{\cdot}{\boldsymbol{a}} : \underset{\cdot\cdot}{\boldsymbol{b}}\ \mathrm{d}\Omega \tag{1.45}$$

1.10 Mandel basis and Voigt representation

The components (a_{ij}) of a second order tensor $\underset{\cdot\cdot}{\boldsymbol{a}}$ can be organised in a matrix $\boldsymbol{a}$ according to (note that we drop the *underdots* for the matrix representation symbol)

$$\boldsymbol{a} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \tag{1.46}$$

Furthermore, any tensor can be represented as a vector in a higher dimensional space. For instance, a 3D second order tensor is a 9D vector. The components of the 9D vector can be expresssed as

$$\boldsymbol{a} = \begin{bmatrix} a_{00} & a_{11} & a_{22} & a_{01} & a_{12} & a_{02} & a_{10} & a_{21} & a_{20} \end{bmatrix}^T \tag{1.47}$$

where the sequence of components is irrelevant but we choose the components along the diagonal first, followed by the off diagonal elements with the upper diagonal ones being selected first as illustrated in Fig. 1.1.

The 9D vector space has the same geometric properties as the 3×3 space; i.e. it is *isomorphic*. For instance, the inner products in both spaces give the same that value and can be calculated using

$$a^T\, a = \underset{\cdot\cdot}{\boldsymbol{a}} : \underset{\cdot\cdot}{\boldsymbol{a}} \tag{1.48}$$

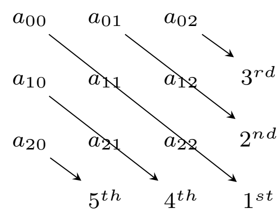

Fig. 1.1: Sequence to select the components of a 3D second order tensor to make up an equivalent 9D vector.

Note that, in Eq. (1.48), since the result is a scalar number, we can write the result of the matrix notation as equal to the result of the tensor notation. It is clear that the use of the 9D vector may facilitate the computer programming some operations.

We now recall that any second order tensor (e.g. $\underset{\cdot\cdot}{\boldsymbol{a}}$ in Eq. 1.4) can be expressed with respect to any other basis dyads. Thus, let's consider the following set of basis vectors known as the Mandel basis vectors [1]

$$\underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{ij} = \begin{cases} \underset{\cdot}{\boldsymbol{e}}_i \otimes \underset{\cdot}{\boldsymbol{e}}_j & \text{if} \quad i = j \\ \frac{1}{\sqrt{2}} \left(\underset{\cdot}{\boldsymbol{e}}_i \otimes \underset{\cdot}{\boldsymbol{e}}_j + \underset{\cdot}{\boldsymbol{e}}_j \otimes \underset{\cdot}{\boldsymbol{e}}_i \right) & \text{if} \quad i < j \\ \frac{1}{\sqrt{2}} \left(\underset{\cdot}{\boldsymbol{e}}_j \otimes \underset{\cdot}{\boldsymbol{e}}_i - \underset{\cdot}{\boldsymbol{e}}_i \otimes \underset{\cdot}{\boldsymbol{e}}_j \right) & \text{if} \quad i > j \end{cases} \tag{1.49}$$

Note that

$$\underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{ij} : \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{kl} = \begin{cases} 1 & \text{if } i = j = k = l \\ 0 & \text{otherwise} \end{cases} \tag{1.50}$$

Hence, the basis is orthogonal and normalised (i.e. orthonormal). See proof in Appendix (1.12).

Clearly, the components of the tensor $\underset{\cdot\cdot}{\boldsymbol{a}}$ with respect to the Mandel basis will be different from the components of the same tensor with respect to the original basis. However, the tensor is invariant itself. With the components of $\underset{\cdot\cdot}{\boldsymbol{a}}$ expressed with respect to the new basis being indicated by $\bar{a}_{ij}$, we can write

$$\underset{\cdot\cdot}{\boldsymbol{a}} = a_{ij} \, \underset{\cdot}{\boldsymbol{e}}_i \otimes \underset{\cdot}{\boldsymbol{e}}_j = \bar{a}_{ij} \, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{ij} \tag{1.51}$$

Thus, by expanding and comparing each dyad, we obtain

$$\bar{a}_{ij} = \begin{cases} a_{ij} & \text{if} \quad i = j \\ \frac{1}{\sqrt{2}} \left(a_{ij} + a_{ji} \right) & \text{if} \quad i < j \\ \frac{1}{\sqrt{2}} \left(a_{ji} - a_{ij} \right) & \text{if} \quad i > j \end{cases} \tag{1.52}$$

Appendix (1.12) demonstrates the above results.

Let's write the 9D vector corresponding to tensor $\underset{\cdot\cdot}{\boldsymbol{a}}$ but now with the components referred to the Mandel basis. In this way, considering the sequence indicated in Fig. 1.1, we obtain

$$\boldsymbol{a} = \begin{Bmatrix} a_{00} \\ a_{11} \\ a_{22} \\ \frac{a_{01}+a_{10}}{\sqrt{2}} \\ \frac{a_{12}+a_{21}}{\sqrt{2}} \\ \frac{a_{02}+a_{20}}{\sqrt{2}} \\ \frac{a_{01}-a_{10}}{\sqrt{2}} \\ \frac{a_{12}-a_{21}}{\sqrt{2}} \\ \frac{a_{02}-a_{20}}{\sqrt{2}} \end{Bmatrix} \tag{1.53}$$

So far, nothing has been said about whether the tensor is symmetric or not. If the tensor is indeed symmetric, we can simplify the above expression by noting that that $a_{ij} = a_{ji}$. We then obtain

$$\boldsymbol{a} = \begin{bmatrix} a_{00} \; a_{11} \; a_{22} \; \sqrt{2}\, a_{01} \; \sqrt{2}\, a_{12} \; \sqrt{2}\, a_{02} \; 0 \; 0 \; 0 \end{bmatrix}^T \tag{1.54}$$

Thus, for symmetric tensors, the last three components w.r.t the Mandel basis will always be zero. Therefore, there is no need to store them in the array. In fact, symmetric second order tensors lie in a 6D manifold. Note further that the inner product in this 6D space is still equal to the inner product of the original symmetric 3×3 tensor; i.e.

$$a^T a = \underset{\ddot{}}{\boldsymbol{a}} : \underset{\ddot{}}{\boldsymbol{a}} \tag{1.55}$$

It is worth noting that other basis tensors can also be chosen. In the literature, a common option is the *standard Voigt* basis that uses something like $\frac{1}{2}\left(\underset{\dot{}}{\boldsymbol{e}}_i \otimes \underset{\dot{}}{\boldsymbol{e}}_j + \underset{\dot{}}{\boldsymbol{e}}_j \otimes \underset{\dot{}}{\boldsymbol{e}}_i\right)$ as basis. However, this basis is **not** orthonormal and hence the isomorphic expressions such as the inner product in Eq. (1.48) are **not** valid in the Voigt representation. Therefore, the Mandel basis is much more convenient.

1.11 Fourth order tensors

$$
\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..0.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..1.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..2.}
\end{bmatrix}_{.0..} &
\begin{bmatrix}
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..0.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..1.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..2.}
\end{bmatrix}_{.1..} &
\begin{bmatrix}
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..0.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..1.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..2.}
\end{bmatrix}_{.2..}
\end{bmatrix}_{0...} \\
\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..0.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..1.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..2.}
\end{bmatrix}_{.0..} &
\begin{bmatrix}
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..0.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..1.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..2.}
\end{bmatrix}_{.1..} &
\begin{bmatrix}
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..0.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..1.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..2.}
\end{bmatrix}_{.2..}
\end{bmatrix}_{1...} \\
\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..0.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..1.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..2.}
\end{bmatrix}_{.0..} &
\begin{bmatrix}
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..0.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..1.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..2.}
\end{bmatrix}_{.1..} &
\begin{bmatrix}
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..0.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..1.} \\
\begin{bmatrix} ...0 & ...1 & ...2 \end{bmatrix}_{..2.}
\end{bmatrix}_{.2..}
\end{bmatrix}_{2...}
\end{bmatrix}
\tag{1.56}
$$

$$
\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix} A_{00\,00} & A_{00\,01} & A_{00\,02} \end{bmatrix} \\
\begin{bmatrix} A_{00\,10} & A_{00\,11} & A_{00\,12} \end{bmatrix} \\
\begin{bmatrix} A_{00\,20} & A_{00\,21} & A_{00\,22} \end{bmatrix}
\end{bmatrix} &
\begin{bmatrix}
\begin{bmatrix} A_{01\,00} & A_{01\,01} & A_{01\,02} \end{bmatrix} \\
\begin{bmatrix} A_{01\,10} & A_{01\,11} & A_{01\,12} \end{bmatrix} \\
\begin{bmatrix} A_{01\,20} & A_{01\,21} & A_{01\,22} \end{bmatrix}
\end{bmatrix} &
\begin{bmatrix}
\begin{bmatrix} A_{02\,00} & A_{02\,01} & A_{02\,02} \end{bmatrix} \\
\begin{bmatrix} A_{02\,10} & A_{02\,11} & A_{02\,12} \end{bmatrix} \\
\begin{bmatrix} A_{02\,20} & A_{02\,21} & A_{02\,22} \end{bmatrix}
\end{bmatrix}
\end{bmatrix} \\
\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix} A_{10\,00} & A_{10\,01} & A_{10\,02} \end{bmatrix} \\
\begin{bmatrix} A_{10\,10} & A_{10\,11} & A_{10\,12} \end{bmatrix} \\
\begin{bmatrix} A_{10\,20} & A_{10\,21} & A_{10\,22} \end{bmatrix}
\end{bmatrix} &
\begin{bmatrix}
\begin{bmatrix} A_{11\,00} & A_{11\,01} & A_{11\,02} \end{bmatrix} \\
\begin{bmatrix} A_{11\,10} & A_{11\,11} & A_{11\,12} \end{bmatrix} \\
\begin{bmatrix} A_{11\,20} & A_{11\,21} & A_{11\,22} \end{bmatrix}
\end{bmatrix} &
\begin{bmatrix}
\begin{bmatrix} A_{12\,00} & A_{12\,01} & A_{12\,02} \end{bmatrix} \\
\begin{bmatrix} A_{12\,10} & A_{12\,11} & A_{12\,12} \end{bmatrix} \\
\begin{bmatrix} A_{12\,20} & A_{12\,21} & A_{12\,22} \end{bmatrix}
\end{bmatrix}
\end{bmatrix} \\
\begin{bmatrix}
\begin{bmatrix}
\begin{bmatrix} A_{20\,00} & A_{20\,01} & A_{20\,02} \end{bmatrix} \\
\begin{bmatrix} A_{20\,10} & A_{20\,11} & A_{20\,12} \end{bmatrix} \\
\begin{bmatrix} A_{20\,20} & A_{20\,21} & A_{20\,22} \end{bmatrix}
\end{bmatrix} &
\begin{bmatrix}
\begin{bmatrix} A_{21\,00} & A_{21\,01} & A_{21\,02} \end{bmatrix} \\
\begin{bmatrix} A_{21\,10} & A_{21\,11} & A_{21\,12} \end{bmatrix} \\
\begin{bmatrix} A_{21\,20} & A_{21\,21} & A_{21\,22} \end{bmatrix}
\end{bmatrix} &
\begin{bmatrix}
\begin{bmatrix} A_{22\,00} & A_{22\,01} & A_{22\,02} \end{bmatrix} \\
\begin{bmatrix} A_{22\,10} & A_{22\,11} & A_{22\,12} \end{bmatrix} \\
\begin{bmatrix} A_{22\,20} & A_{22\,21} & A_{22\,22} \end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\end{bmatrix}
\tag{1.57}
$$

$$
\begin{array}{cccc}
\boxed{\begin{matrix} A_{00\,00} & A_{00\,01} & A_{00\,02} \\ A_{00\,10} & A_{00\,11} & A_{00\,12} \\ A_{00\,20} & A_{00\,21} & A_{00\,22} \end{matrix}} &
\boxed{\begin{matrix} A_{01\,00} & A_{01\,01} & A_{01\,02} \\ A_{01\,10} & A_{01\,11} & A_{01\,12} \\ A_{01\,20} & A_{01\,21} & A_{01\,22} \end{matrix}} &
\boxed{\begin{matrix} A_{02\,00} & A_{02\,01} & A_{02\,02} \\ A_{02\,10} & A_{02\,11} & A_{02\,12} \\ A_{02\,20} & A_{02\,21} & A_{02\,22} \end{matrix}} & A \\
\boxed{\begin{matrix} A_{10\,00} & A_{10\,01} & A_{10\,02} \\ A_{10\,10} & A_{10\,11} & A_{10\,12} \\ A_{10\,20} & A_{10\,21} & A_{10\,22} \end{matrix}} &
\boxed{\begin{matrix} A_{11\,00} & A_{11\,01} & A_{11\,02} \\ A_{11\,10} & A_{11\,11} & A_{11\,12} \\ A_{11\,20} & A_{11\,21} & A_{11\,22} \end{matrix}} &
\boxed{\begin{matrix} A_{12\,00} & A_{12\,01} & A_{12\,02} \\ A_{12\,10} & A_{12\,11} & A_{12\,12} \\ A_{12\,20} & A_{12\,21} & A_{12\,22} \end{matrix}} & B \\
\boxed{\begin{matrix} A_{20\,00} & A_{20\,01} & A_{20\,02} \\ A_{20\,10} & A_{20\,11} & A_{20\,12} \\ A_{20\,20} & A_{20\,21} & A_{20\,22} \end{matrix}} &
\boxed{\begin{matrix} A_{21\,00} & A_{21\,01} & A_{21\,02} \\ A_{21\,10} & A_{21\,11} & A_{21\,12} \\ A_{21\,20} & A_{21\,21} & A_{21\,22} \end{matrix}} &
\boxed{\begin{matrix} A_{22\,00} & A_{22\,01} & A_{22\,02} \\ A_{22\,10} & A_{22\,11} & A_{22\,12} \\ A_{22\,20} & A_{22\,21} & A_{22\,22} \end{matrix}} & C \\
 & 5^{th} & 4^{th} & 1^{st}
\end{array}
$$

$$
\begin{array}{ccc@{\quad}ccc@{\quad}ccc}
A_{00\,00} & A_{00\,01} & A_{00\,02} & A_{01\,00} & A_{01\,01} & A_{01\,02} & A_{02\,00} & A_{02\,01} & A_{02\,02} \\
A_{00\,10} & A_{00\,11} & A_{00\,12} & A_{01\,10} & A_{01\,11} & A_{01\,12} & A_{02\,10} & A_{02\,11} & A_{02\,12} \\
A_{00\,20} & A_{00\,21} & A_{00\,22} & A_{01\,20} & A_{01\,21} & A_{01\,22} & A_{02\,20} & A_{02\,21} & A_{02\,22} \\[1ex]
A_{10\,00} & A_{10\,01} & A_{10\,02} & A_{11\,00} & A_{11\,01} & A_{11\,02} & A_{12\,00} & A_{12\,01} & A_{12\,02} \\
A_{10\,10} & A_{10\,11} & A_{10\,12} & A_{11\,10} & A_{11\,11} & A_{11\,12} & A_{12\,10} & A_{12\,11} & A_{12\,12} \\
A_{10\,20} & A_{10\,21} & A_{10\,22} & A_{11\,20} & A_{11\,21} & A_{11\,22} & A_{12\,20} & A_{12\,21} & A_{12\,22} \\[1ex]
A_{20\,00} & A_{20\,01} & A_{20\,02} & A_{21\,00} & A_{21\,01} & A_{21\,02} & A_{22\,00} & A_{22\,01} & A_{22\,02} \\
A_{20\,10} & A_{20\,11} & A_{20\,12} & A_{21\,10} & A_{21\,11} & A_{21\,12} & A_{22\,10} & A_{22\,11} & A_{22\,12} \\
A_{20\,20} & A_{20\,21} & A_{20\,22} & A_{21\,20} & A_{21\,21} & A_{21\,22} & A_{22\,20} & A_{22\,21} & A_{22\,22}
\end{array}
$$

Fig. 1.2: Sequence to select the components of a 3D fourth order tensor to make up an equivalent 9x9 matrix.

$$
\begin{bmatrix}
A_{00\,00} & A_{00\,11} & A_{00\,22} & A_{00\,01} & A_{00\,12} & A_{00\,02} & A_{00\,10} & A_{00\,21} & A_{00\,20} \\
A_{11\,00} & A_{11\,11} & A_{11\,22} & A_{11\,01} & A_{11\,12} & A_{11\,02} & A_{11\,10} & A_{11\,21} & A_{11\,20} \\
A_{22\,00} & A_{22\,11} & A_{22\,22} & A_{22\,01} & A_{22\,12} & A_{22\,02} & A_{22\,10} & A_{22\,21} & A_{22\,20} \\
A_{01\,00} & A_{01\,11} & A_{01\,22} & A_{01\,01} & A_{01\,12} & A_{01\,02} & A_{01\,10} & A_{01\,21} & A_{01\,20} \\
A_{12\,00} & A_{12\,11} & A_{12\,22} & A_{12\,01} & A_{12\,12} & A_{12\,02} & A_{12\,10} & A_{12\,21} & A_{12\,20} \\
A_{02\,00} & A_{02\,11} & A_{02\,22} & A_{02\,01} & A_{02\,12} & A_{02\,02} & A_{02\,10} & A_{02\,21} & A_{02\,20} \\
A_{10\,00} & A_{10\,11} & A_{10\,22} & A_{10\,01} & A_{10\,12} & A_{10\,02} & A_{10\,10} & A_{10\,21} & A_{10\,20} \\
A_{21\,00} & A_{21\,11} & A_{21\,22} & A_{21\,01} & A_{21\,12} & A_{21\,02} & A_{21\,10} & A_{21\,21} & A_{21\,20} \\
A_{20\,00} & A_{20\,11} & A_{20\,22} & A_{20\,01} & A_{20\,12} & A_{20\,02} & A_{20\,10} & A_{20\,21} & A_{20\,20}
\end{bmatrix}
\tag{1.58}
$$

1.12 Appendix: Additional derivations

For convenience, here we write the dyad $\underset{\cdot}{\boldsymbol{e}}_i \otimes \underset{\cdot}{\boldsymbol{e}}_j$ simply as $\underset{\cdot}{\boldsymbol{e}}_i \underset{\cdot}{\boldsymbol{e}}_j$.

To verify that that the Mandel basis is normalised, the inner product of the basis tensor with itself is expanded. The result should be 1. Thus, for the case when $i = j$ (**no implied sums on** i **or** j)

$$
\begin{aligned}
\underbrace{\underset{\cdot}{\bar{\boldsymbol{E}}}_{ij} : \underset{\cdot}{\bar{\boldsymbol{E}}}_{ij}}_{\text{no sums}} &= (\underset{\cdot}{\boldsymbol{e}}_i \underset{\cdot}{\boldsymbol{e}}_j) : (\underset{\cdot}{\boldsymbol{e}}_i \underset{\cdot}{\boldsymbol{e}}_j) \\
&= (\underset{\cdot}{\boldsymbol{e}}_i \cdot \underset{\cdot}{\boldsymbol{e}}_i)(\underset{\cdot}{\boldsymbol{e}}_j \cdot \underset{\cdot}{\boldsymbol{e}}_j) \\
&= 1
\end{aligned}
\tag{1.59}
$$

For the case when $i < j$ we obtain

$$\begin{aligned}
\underbrace{\underset{\cdot}{\bar{\boldsymbol{E}}}_{ij}:\underset{\cdot}{\bar{\boldsymbol{E}}}_{ij}}_{\text{no sums}} &= \frac{1}{\sqrt{2}}\left(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j+\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i\right):\frac{1}{\sqrt{2}}\left(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j+\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i\right)\\
&= \frac{1}{2}\left[(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j):(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j)+2\,(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j):(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i)+(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i):(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i)\right]\\
&= \frac{1}{2}\left[(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_i)(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_j)+2\,(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_j)(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_i)+(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_j)(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_i)\right]\\
&= \frac{1}{2}\left[1+0+1\right]\\
&= 1
\end{aligned}\tag{1.60}$$

And for the case when $i>j$ we obtain

$$\begin{aligned}
\underbrace{\underset{\cdot}{\bar{\boldsymbol{E}}}_{ij}:\underset{\cdot}{\bar{\boldsymbol{E}}}_{ij}}_{\text{no sums}} &= \frac{1}{\sqrt{2}}\left(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i-\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j\right):\frac{1}{\sqrt{2}}\left(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i-\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j\right)\\
&= \frac{1}{2}\left[(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i):(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i)-2\,(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i):(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j)+(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j):(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j)\right]\\
&= \frac{1}{2}\left[(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_j)(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_i)-2\,(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_i)(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_j)+(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_i)(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_j)\right]\\
&= \frac{1}{2}\left[1-0+1\right]\\
&= 1
\end{aligned}\tag{1.61}$$

Note the importance of the normalising factor $\frac{1}{\sqrt{2}}$ to achieve the value of 1 in the end.

To verify that that the Mandel basis is orthogonal, the inner product of the basis tensor with any other one should be 0. For the case of basis $i<j$ *versus* basis $i>j$, we obtain (**no implied sums on** i **or** j)

$$\begin{aligned}
\underbrace{\underset{\cdot}{\bar{\boldsymbol{E}}}_{ij}:\underset{\cdot}{\bar{\boldsymbol{E}}}_{ij}}_{\text{no sums}} &= \frac{1}{\sqrt{2}}\left(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j+\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i\right):\frac{1}{\sqrt{2}}\left(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i-\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j\right)\\
&= \frac{1}{2}\left[(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j):(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i)-(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j):(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j)+(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i):(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i)-(\underset{\cdot}{\boldsymbol{e}}_j\underset{\cdot}{\boldsymbol{e}}_i):(\underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j)\right]\\
&= \frac{1}{2}\left[(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_j)(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_i)-(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_i)(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_j)+(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_j)(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_i)-(\underset{\cdot}{\boldsymbol{e}}_j\cdot\underset{\cdot}{\boldsymbol{e}}_i)(\underset{\cdot}{\boldsymbol{e}}_i\cdot\underset{\cdot}{\boldsymbol{e}}_j)\right]\\
&= \frac{1}{2}\left[0-1+1-0\right]\\
&= 0 \qquad \blacksquare
\end{aligned}\tag{1.62}$$

The second order tensor given w.r.t the new (Mandel) basis is

$$
\begin{aligned}
\underset{\cdot\cdot}{\boldsymbol{a}} &= \bar{a}_{ij}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{ij} \\
&= \bar{a}_{00}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{00} + \bar{a}_{01}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{01} + \bar{a}_{02}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{02} \\
&+ \bar{a}_{10}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{10} + \bar{a}_{11}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{11} + \bar{a}_{12}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{12} \\
&+ \bar{a}_{20}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{20} + \bar{a}_{21}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{21} + \bar{a}_{22}\, \underset{\cdot\cdot}{\bar{\boldsymbol{E}}}_{22}
\end{aligned} \tag{1.63}
$$

Thus

$$
\begin{aligned}
\underset{\cdot\cdot}{\boldsymbol{a}} = \quad & \bar{a}_{00}\, \underset{\cdot}{\boldsymbol{e}}_0\underset{\cdot}{\boldsymbol{e}}_0 + \frac{\bar{a}_{01}}{\sqrt{2}}(\underset{\cdot}{\boldsymbol{e}}_0\underset{\cdot}{\boldsymbol{e}}_1 + \underset{\cdot}{\boldsymbol{e}}_1\underset{\cdot}{\boldsymbol{e}}_0) + \frac{\bar{a}_{02}}{\sqrt{2}}(\underset{\cdot}{\boldsymbol{e}}_0\underset{\cdot}{\boldsymbol{e}}_2 + \underset{\cdot}{\boldsymbol{e}}_2\underset{\cdot}{\boldsymbol{e}}_0) \\
+ & \frac{\bar{a}_{10}}{\sqrt{2}}(\underset{\cdot}{\boldsymbol{e}}_0\underset{\cdot}{\boldsymbol{e}}_1 - \underset{\cdot}{\boldsymbol{e}}_1\underset{\cdot}{\boldsymbol{e}}_0) + \bar{a}_{11}\, \underset{\cdot}{\boldsymbol{e}}_1\underset{\cdot}{\boldsymbol{e}}_1 + \frac{\bar{a}_{12}}{\sqrt{2}}(\underset{\cdot}{\boldsymbol{e}}_1\underset{\cdot}{\boldsymbol{e}}_2 + \underset{\cdot}{\boldsymbol{e}}_2\underset{\cdot}{\boldsymbol{e}}_1) \\
+ & \frac{\bar{a}_{20}}{\sqrt{2}}(\underset{\cdot}{\boldsymbol{e}}_0\underset{\cdot}{\boldsymbol{e}}_2 - \underset{\cdot}{\boldsymbol{e}}_2\underset{\cdot}{\boldsymbol{e}}_0) + \frac{\bar{a}_{21}}{\sqrt{2}}(\underset{\cdot}{\boldsymbol{e}}_1\underset{\cdot}{\boldsymbol{e}}_2 - \underset{\cdot}{\boldsymbol{e}}_2\underset{\cdot}{\boldsymbol{e}}_1) + \bar{a}_{22}\, \underset{\cdot}{\boldsymbol{e}}_2\underset{\cdot}{\boldsymbol{e}}_2
\end{aligned} \tag{1.64}
$$

Thus

$$
\begin{aligned}
\underset{\cdot\cdot}{\boldsymbol{a}} = \quad & \bar{a}_{00}\, \underset{\cdot}{\boldsymbol{e}}_0\underset{\cdot}{\boldsymbol{e}}_0 + \frac{\bar{a}_{01} + \bar{a}_{10}}{\sqrt{2}}\underset{\cdot}{\boldsymbol{e}}_0\underset{\cdot}{\boldsymbol{e}}_1 + \frac{\bar{a}_{02} + \bar{a}_{20}}{\sqrt{2}}\underset{\cdot}{\boldsymbol{e}}_0\underset{\cdot}{\boldsymbol{e}}_2 \\
+ & \frac{\bar{a}_{01} - \bar{a}_{10}}{\sqrt{2}}\underset{\cdot}{\boldsymbol{e}}_1\underset{\cdot}{\boldsymbol{e}}_0 + \bar{a}_{11}\, \underset{\cdot}{\boldsymbol{e}}_1\underset{\cdot}{\boldsymbol{e}}_1 + \frac{\bar{a}_{12} + \bar{a}_{21}}{\sqrt{2}}\underset{\cdot}{\boldsymbol{e}}_1\underset{\cdot}{\boldsymbol{e}}_2 \\
+ & \frac{\bar{a}_{02} - \bar{a}_{20}}{\sqrt{2}}\underset{\cdot}{\boldsymbol{e}}_2\underset{\cdot}{\boldsymbol{e}}_0 + \frac{\bar{a}_{12} - \bar{a}_{21}}{\sqrt{2}}\underset{\cdot}{\boldsymbol{e}}_2\underset{\cdot}{\boldsymbol{e}}_1 + \bar{a}_{22}\, \underset{\cdot}{\boldsymbol{e}}_2\underset{\cdot}{\boldsymbol{e}}_2
\end{aligned} \tag{1.65}
$$

Since $\underset{\cdot\cdot}{\boldsymbol{a}} = a_{ij}\, \underset{\cdot}{\boldsymbol{e}}_i\underset{\cdot}{\boldsymbol{e}}_j$ as well, by comparing each dyad term, we find that

$$
\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \bar{a}_{00} & \frac{\bar{a}_{01}+\bar{a}_{10}}{\sqrt{2}} & \frac{\bar{a}_{02}+\bar{a}_{20}}{\sqrt{2}} \\ \frac{\bar{a}_{01}-\bar{a}_{10}}{\sqrt{2}} & \bar{a}_{11} & \frac{\bar{a}_{12}+\bar{a}_{21}}{\sqrt{2}} \\ \frac{\bar{a}_{02}-\bar{a}_{20}}{\sqrt{2}} & \frac{\bar{a}_{12}-\bar{a}_{21}}{\sqrt{2}} & \bar{a}_{22} \end{bmatrix} \tag{1.66}
$$

For instance,

$$
\sqrt{2}\, a_{01} = \bar{a}_{01} + \bar{a}_{10} \tag{1.67}
$$

and

$$
\sqrt{2}\, a_{10} = \bar{a}_{01} - \bar{a}_{10} \tag{1.68}
$$

By adding the last two equations, we obtain

$$
\sqrt{2}\, (a_{01} + a_{10}) = 2\, \bar{a}_{01} \tag{1.69}
$$

which results in

$$
\bar{a}_{01} = \frac{a_{01} + a_{10}}{\sqrt{2}} \tag{1.70}
$$

Analogously, by subtracting Eqs. (1.67) and (1.68), we obtain

$$
\sqrt{2}\, (a_{01} - a_{10}) = 2\, \bar{a}_{10} \tag{1.71}
$$

which results in

$$\bar{a}_{10} = \frac{a_{01} - a_{10}}{\sqrt{2}} \tag{1.72}$$

Therefore, by induction, we can verify Eq. (1.52), i.e., we can write

$$\begin{bmatrix} \bar{a}_{00} & \bar{a}_{01} & \bar{a}_{02} \\ \bar{a}_{10} & \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{20} & \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} = \begin{bmatrix} a_{00} & \frac{a_{01}+a_{10}}{\sqrt{2}} & \frac{a_{02}+a_{20}}{\sqrt{2}} \\ \frac{a_{01}-a_{10}}{\sqrt{2}} & a_{11} & \frac{a_{12}+a_{21}}{\sqrt{2}} \\ \frac{a_{02}-a_{20}}{\sqrt{2}} & \frac{a_{12}-a_{21}}{\sqrt{2}} & a_{22} \end{bmatrix} \tag{1.73}$$

References

1. Helnwein, P.: Some remarks on the compressed matrix representation of symmetric second-order and fourth-order tensors. Computer Methods in Applied Mechanics and Engineering **190**(2223), 2753 – 2770 (2001). DOI 10.1016/S0045-7825(00)00263-2