Chapter 1
Transfinite Mapping - By Dorival Pedroso

1.1 Transfinite mapping in 2D

The bilinear transfinite mapping in 2D is

$$\begin{aligned}\underset{\cdot}{\boldsymbol{x}}(r,s) = \\ +(1-r)\,\underset{\cdot}{\boldsymbol{b}}_0(s)/2 + (1+r)\,\underset{\cdot}{\boldsymbol{b}}_1(s)/2 \\ +(1-s)\,\underset{\cdot}{\boldsymbol{b}}_2(r)/2 + (1+s)\,\underset{\cdot}{\boldsymbol{b}}_3(r)/2 \\ -(1-r)\,(1-s)\,\underset{\cdot}{\boldsymbol{p}}_0/4 - (1+r)\,(1-s)\,\underset{\cdot}{\boldsymbol{p}}_1/4 \\ -(1+r)\,(1+s)\,\underset{\cdot}{\boldsymbol{p}}_2/4 - (1-r)\,(1+s)\,\underset{\cdot}{\boldsymbol{p}}_3/4\end{aligned} \tag{1.1}$$

1.1.1 First order derivatives

Derivative of $\underset{\cdot}{\boldsymbol{x}}$ with respect to r

$$\begin{aligned}\frac{\partial \underset{\cdot}{\boldsymbol{x}}(r,s)}{\partial r} = \\ -\underset{\cdot}{\boldsymbol{b}}_0(s)/2 + \underset{\cdot}{\boldsymbol{b}}_1(s)/2 \\ +(1-s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_2(r)}{\partial r}/2 + (1+s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_3(r)}{\partial r}/2 \\ +(1-s)\,\underset{\cdot}{\boldsymbol{p}}_0/4 - (1-s)\,\underset{\cdot}{\boldsymbol{p}}_1/4 \\ -(1+s)\,\underset{\cdot}{\boldsymbol{p}}_2/4 + (1+s)\,\underset{\cdot}{\boldsymbol{p}}_3/4\end{aligned} \tag{1.2}$$

Derivative of $\boldsymbol{x}$ with respect to s

$$\begin{aligned} \frac{\partial \boldsymbol{x}(r,s)}{\partial s} &= \\ +(1-r)\,\frac{\partial \boldsymbol{b}_0(s)}{\partial s}/2 + (1+r)\,\frac{\partial \boldsymbol{b}_1(s)}{\partial s}/2 & \\ -\boldsymbol{b}_2(r)/2 + \boldsymbol{b}_3(r)/2 & \\ +(1-r)\,\boldsymbol{p}_0/4 + (1+r)\,\boldsymbol{p}_1/4 & \\ -(1+r)\,\boldsymbol{p}_2/4 - (1-r)\,\boldsymbol{p}_3/4 & \end{aligned} \tag{1.3}$$

1.1.2 Second order derivatives

Derivative of $\partial\boldsymbol{x}/\partial r$ with respect to r

$$\begin{aligned} \frac{\partial^2 \boldsymbol{x}(r,s,t)}{\partial r^2} &= \\ (1-s)\,\frac{\partial^2 \boldsymbol{b}_2(r)}{\partial r^2}/2 + (1+s)\,\frac{\partial^2 \boldsymbol{b}_3(r)}{\partial r^2}/2 & \end{aligned} \tag{1.4}$$

Derivative of $\partial\boldsymbol{x}/\partial s$ with respect to s

$$\begin{aligned} \frac{\partial^2 \boldsymbol{x}(r,s,t)}{\partial s^2} &= \\ (1-r)\,\frac{\partial^2 \boldsymbol{b}_0(s)}{\partial s^2}/2 + (1+r)\,\frac{\partial^2 \boldsymbol{b}_1(s)}{\partial s^2}/2 & \end{aligned} \tag{1.5}$$

Derivative of $\partial\boldsymbol{x}/\partial r$ with respect to s

$$\begin{aligned} \frac{\partial^2 \boldsymbol{x}(r,s,t)}{\partial r \partial s} &= \\ -\frac{\partial \boldsymbol{b}_0(s)}{\partial s}/2 + \frac{\partial \boldsymbol{b}_1(s)}{\partial s}/2 & \\ -\frac{\partial \boldsymbol{b}_2(r)}{\partial r}/2 + \frac{\partial \boldsymbol{b}_3(r)}{\partial r}/2 & \\ -\boldsymbol{p}_0/4 + \boldsymbol{p}_1/4 & \\ -\boldsymbol{p}_2/4 + \boldsymbol{p}_3/4 & \end{aligned} \tag{1.6}$$

1.2 Transfinite mapping in 3D

Let's define

$$m = -1 \qquad \text{and} \qquad p = +1 \tag{1.7}$$

The bilinear transfinite mapping in 3D is

$$
\begin{aligned}
\boldsymbol{x}(r,s,t) = \\
&+(1-r)\,\boldsymbol{b}_0(s,t)/2 + (1+r)\,\boldsymbol{b}_1(s,t)/2 \\
&+(1-s)\,\boldsymbol{b}_2(r,t)/2 + (1+s)\,\boldsymbol{b}_3(r,t)/2 \\
&+(1-t)\,\boldsymbol{b}_4(r,s)/2 + (1+t)\,\boldsymbol{b}_5(r,s)/2 \\
\\
&-(1-r)\,(1-s)\,\boldsymbol{b}_0(m,t)/4 - (1-r)\,(1+s)\,\boldsymbol{b}_0(p,t)/4 \\
&-(1+r)\,(1-s)\,\boldsymbol{b}_1(m,t)/4 - (1+r)\,(1+s)\,\boldsymbol{b}_1(p,t)/4 \\
\\
&-(1-r)\,(1-t)\,\boldsymbol{b}_0(s,m)/4 - (1-r)\,(1+t)\,\boldsymbol{b}_0(s,p)/4 \\
&-(1+r)\,(1-t)\,\boldsymbol{b}_1(s,m)/4 - (1+r)\,(1+t)\,\boldsymbol{b}_1(s,p)/4 \\
\\
&-(1-s)\,(1-t)\,\boldsymbol{b}_2(r,m)/4 - (1-s)\,(1+t)\,\boldsymbol{b}_2(r,p)/4 \\
&-(1+s)\,(1-t)\,\boldsymbol{b}_3(r,m)/4 - (1+s)\,(1+t)\,\boldsymbol{b}_3(r,p)/4 \\
\\
&+(1-r)\,(1-s)\,(1-t)\,\boldsymbol{p}_0/8 + (1+r)\,(1-s)\,(1-t)\,\boldsymbol{p}_1/8 \\
&+(1+r)\,(1+s)\,(1-t)\,\boldsymbol{p}_2/8 + (1-r)\,(1+s)\,(1-t)\,\boldsymbol{p}_3/8 \\
&+(1-r)\,(1-s)\,(1+t)\,\boldsymbol{p}_4/8 + (1+r)\,(1-s)\,(1+t)\,\boldsymbol{p}_5/8 \\
&+(1+r)\,(1+s)\,(1+t)\,\boldsymbol{p}_6/8 + (1-r)\,(1+s)\,(1+t)\,\boldsymbol{p}_7/8
\end{aligned}
\tag{1.8}
$$

1.2.1 First order derivatives

Derivative of $\boldsymbol{x}$ with respect to r

$$
\begin{aligned}
\frac{\partial \boldsymbol{x}(r,s,t)}{\partial r} = \\
\\
-\boldsymbol{b}_0(s,t)/2 + \boldsymbol{b}_1(s,t)/2 \\
+(1-s)\,\frac{\partial \boldsymbol{b}_2(r,t)}{\partial r}/2 + (1+s)\,\frac{\partial \boldsymbol{b}_3(r,t)}{\partial r}/2 \\
+(1-t)\,\frac{\partial \boldsymbol{b}_4(r,s)}{\partial r}/2 + (1+t)\,\frac{\partial \boldsymbol{b}_5(r,s)}{\partial r}/2 \\
\\
+(1-s)\,\boldsymbol{b}_0(m,t)/4 + (1+s)\,\boldsymbol{b}_0(p,t)/4 \\
-(1-s)\,\boldsymbol{b}_1(m,t)/4 - (1+s)\,\boldsymbol{b}_1(p,t)/4 \\
\\
+(1-t)\,\boldsymbol{b}_0(s,m)/4 + (1+t)\,\boldsymbol{b}_0(s,p)/4 \\
-(1-t)\,\boldsymbol{b}_1(s,m)/4 - (1+t)\,\boldsymbol{b}_1(s,p)/4 \\
\\
-(1-s)\,(1-t)\,\frac{\partial \boldsymbol{b}_2(r,m)}{\partial r}/4 - (1-s)\,(1+t)\,\frac{\partial \boldsymbol{b}_2(r,p)}{\partial r}/4 \\
-(1+s)\,(1-t)\,\frac{\partial \boldsymbol{b}_3(r,m)}{\partial r}/4 - (1+s)\,(1+t)\,\frac{\partial \boldsymbol{b}_3(r,p)}{\partial r}/4 \\
\\
-(1-s)\,(1-t)\,\boldsymbol{p}_0/8 + (1-s)\,(1-t)\,\boldsymbol{p}_1/8 \\
+(1+s)\,(1-t)\,\boldsymbol{p}_2/8 - (1+s)\,(1-t)\,\boldsymbol{p}_3/8 \\
-(1-s)\,(1+t)\,\boldsymbol{p}_4/8 + (1-s)\,(1+t)\,\boldsymbol{p}_5/8 \\
+(1+s)\,(1+t)\,\boldsymbol{p}_6/8 - (1+s)\,(1+t)\,\boldsymbol{p}_7/8
\end{aligned}
\tag{1.9}
$$

Derivative of $\boldsymbol{x}$ with respect to s

$$
\begin{aligned}
\frac{\partial \boldsymbol{x}(r,s,t)}{\partial s} = \\
+(1-r)\,\frac{\partial \boldsymbol{b}_0(s,t)}{\partial s}/2 + (1+r)\,\frac{\partial \boldsymbol{b}_1(s,t)}{\partial s}/2 \\
-\boldsymbol{b}_2(r,t)/2 + \boldsymbol{b}_3(r,t)/2 \\
+(1-t)\,\frac{\partial \boldsymbol{b}_4(r,s)}{\partial s}/2 + (1+t)\,\frac{\partial \boldsymbol{b}_5(r,s)}{\partial s}/2 \\
+(1-r)\,\boldsymbol{b}_0(m,t)/4 - (1-r)\,\boldsymbol{b}_0(p,t)/4 \\
+(1+r)\,\boldsymbol{b}_1(m,t)/4 - (1+r)\,\boldsymbol{b}_1(p,t)/4 \\
-(1-r)\,(1-t)\,\frac{\partial \boldsymbol{b}_0(s,m)}{\partial s}/4 - (1-r)\,(1+t)\,\frac{\partial \boldsymbol{b}_0(s,p)}{\partial s}/4 \\
-(1+r)\,(1-t)\,\frac{\partial \boldsymbol{b}_1(s,m)}{\partial s}/4 - (1+r)\,(1+t)\,\frac{\partial \boldsymbol{b}_1(s,p)}{\partial s}/4 \\
+(1-t)\,\boldsymbol{b}_2(r,m)/4 + (1+t)\,\boldsymbol{b}_2(r,p)/4 \\
-(1-t)\,\boldsymbol{b}_3(r,m)/4 - (1+t)\,\boldsymbol{b}_3(r,p)/4 \\
-(1-r)\,(1-t)\,\boldsymbol{p}_0/8 - (1+r)\,(1-t)\,\boldsymbol{p}_1/8 \\
+(1+r)\,(1-t)\,\boldsymbol{p}_2/8 + (1-r)\,(1-t)\,\boldsymbol{p}_3/8 \\
-(1-r)\,(1+t)\,\boldsymbol{p}_4/8 - (1+r)\,(1+t)\,\boldsymbol{p}_5/8 \\
+(1+r)\,(1+t)\,\boldsymbol{p}_6/8 + (1-r)\,(1+t)\,\boldsymbol{p}_7/8
\end{aligned}
\tag{1.10}
$$

Derivative of $\underset{\cdot}{\boldsymbol{x}}$ with respect to t

$$
\begin{aligned}
\frac{\partial \underset{\cdot}{\boldsymbol{x}}(r,s,t)}{\partial t} = \\
+(1-r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(s,t)}{\partial t}/2 + (1+r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(s,t)}{\partial t}/2 \\
+(1-s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_2(r,t)}{\partial t}/2 + (1+s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_3(r,t)}{\partial t}/2 \\
-\underset{\cdot}{\boldsymbol{b}}_4(r,s)/2 + \underset{\cdot}{\boldsymbol{b}}_5(r,s)/2 \\
-(1-r)\,(1-s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(m,t)}{\partial t}/4 - (1-r)\,(1+s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(p,t)}{\partial t}/4 \\
-(1+r)\,(1-s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(m,t)}{\partial t}/4 - (1+r)\,(1+s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(p,t)}{\partial t}/4 \\
+(1-r)\,\underset{\cdot}{\boldsymbol{b}}_0(s,m)/4 - (1-r)\,\underset{\cdot}{\boldsymbol{b}}_0(s,p)/4 \\
+(1+r)\,\underset{\cdot}{\boldsymbol{b}}_1(s,m)/4 - (1+r)\,\underset{\cdot}{\boldsymbol{b}}_1(s,p)/4 \\
+(1-s)\,\underset{\cdot}{\boldsymbol{b}}_2(r,m)/4 - (1-s)\,\underset{\cdot}{\boldsymbol{b}}_2(r,p)/4 \\
+(1+s)\,\underset{\cdot}{\boldsymbol{b}}_3(r,m)/4 - (1+s)\,\underset{\cdot}{\boldsymbol{b}}_3(r,p)/4 \\
-(1-r)\,(1-s)\,\underset{\cdot}{\boldsymbol{p}}_0/8 - (1+r)\,(1-s)\,\underset{\cdot}{\boldsymbol{p}}_1/8 \\
-(1+r)\,(1+s)\,\underset{\cdot}{\boldsymbol{p}}_2/8 - (1-r)\,(1+s)\,\underset{\cdot}{\boldsymbol{p}}_3/8 \\
+(1-r)\,(1-s)\,\underset{\cdot}{\boldsymbol{p}}_4/8 + (1+r)\,(1-s)\,\underset{\cdot}{\boldsymbol{p}}_5/8 \\
+(1+r)\,(1+s)\,\underset{\cdot}{\boldsymbol{p}}_6/8 + (1-r)\,(1+s)\,\underset{\cdot}{\boldsymbol{p}}_7/8
\end{aligned}
\tag{1.11}
$$

1.2.2 Second order derivatives

Derivative of $\partial \underset{\cdot}{\boldsymbol{x}}/\partial r$ with respect to r

$$\begin{aligned}
\frac{\partial^2 \underset{\cdot}{\boldsymbol{x}}(r,s,t)}{\partial r^2} &= \\
+(1-s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_2(r,t)}{\partial r^2}/2 + (1+s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_3(r,t)}{\partial r^2}/2 \\
+(1-t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_4(r,s)}{\partial r^2}/2 + (1+t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_5(r,s)}{\partial r^2}/2 \\
-(1-s)\,(1-t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_2(r,m)}{\partial r^2}/4 - (1-s)\,(1+t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_2(r,p)}{\partial r^2}/4 \\
-(1+s)\,(1-t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_3(r,m)}{\partial r^2}/4 - (1+s)\,(1+t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_3(r,p)}{\partial r^2}/4
\end{aligned} \tag{1.12}$$

Derivative of $\partial \underset{\cdot}{\boldsymbol{x}}/\partial s$ with respect to s

$$\begin{aligned}
\frac{\partial^2 \underset{\cdot}{\boldsymbol{x}}(r,s,t)}{\partial s^2} &= \\
+(1-r)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_0(s,t)}{\partial s^2}/2 + (1+r)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_1(s,t)}{\partial s^2}/2 \\
+(1-t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_4(r,s)}{\partial s^2}/2 + (1+t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_5(r,s)}{\partial s^2}/2 \\
-(1-r)\,(1-t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_0(s,m)}{\partial s^2}/4 - (1-r)\,(1+t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_0(s,p)}{\partial s^2}/4 \\
-(1+r)\,(1-t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_1(s,m)}{\partial s^2}/4 - (1+r)\,(1+t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_1(s,p)}{\partial s^2}/4
\end{aligned} \tag{1.13}$$

Derivative of $\partial \underset{\cdot}{\boldsymbol{x}}/\partial t$ with respect to t

$$\begin{aligned}
\frac{\partial^2 \underset{\cdot}{\boldsymbol{x}}(r,s,t)}{\partial t^2} &= \\
+(1-r)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_0(s,t)}{\partial t^2}/2 + (1+r)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_1(s,t)}{\partial t^2}/2 \\
+(1-s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_2(r,t)}{\partial t^2}/2 + (1+s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_3(r,t)}{\partial t^2}/2 \\
-(1-r)\,(1-s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_0(m,t)}{\partial t^2}/4 - (1-r)\,(1+s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_0(p,t)}{\partial t^2}/4 \\
-(1+r)\,(1-s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_1(m,t)}{\partial t^2}/4 - (1+r)\,(1+s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_1(p,t)}{\partial t^2}/4
\end{aligned} \tag{1.14}$$

Derivative of $\partial \underset{\cdot}{\boldsymbol{x}} / \partial r$ with respect to s

$$
\begin{aligned}
\frac{\partial^2 \underset{\cdot}{\boldsymbol{x}}(r,s,t)}{\partial r \partial s} = \\
-\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(s,t)}{\partial s}/2 + \frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(s,t)}{\partial s}/2 \\
-\frac{\partial \underset{\cdot}{\boldsymbol{b}}_2(r,t)}{\partial r}/2 + \frac{\partial \underset{\cdot}{\boldsymbol{b}}_3(r,t)}{\partial r}/2 \\
+(1-t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_4(r,s)}{\partial r \partial s}/2 + (1+t)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_5(r,s)}{\partial r \partial s}/2 \\
\\
-\underset{\cdot}{\boldsymbol{b}}_0(m,t)/4 + \underset{\cdot}{\boldsymbol{b}}_0(p,t)/4 \\
+\underset{\cdot}{\boldsymbol{b}}_1(m,t)/4 - \underset{\cdot}{\boldsymbol{b}}_1(p,t)/4 \\
\\
+(1-t)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(s,m)}{\partial s}/4 + (1+t)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(s,p)}{\partial s}/4 \\
-(1-t)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(s,m)}{\partial s}/4 - (1+t)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(s,p)}{\partial s}/4 \\
\\
+(1-t)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_2(r,m)}{\partial r}/4 + (1+t)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_2(r,p)}{\partial r}/4 \\
-(1-t)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_3(r,m)}{\partial r}/4 - (1+t)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_3(r,p)}{\partial r}/4 \\
\\
+(1-t)\,\underset{\cdot}{\boldsymbol{p}}_0/8 - (1-t)\,\underset{\cdot}{\boldsymbol{p}}_1/8 \\
+(1-t)\,\underset{\cdot}{\boldsymbol{p}}_2/8 - (1-t)\,\underset{\cdot}{\boldsymbol{p}}_3/8 \\
+(1+t)\,\underset{\cdot}{\boldsymbol{p}}_4/8 - (1+t)\,\underset{\cdot}{\boldsymbol{p}}_5/8 \\
+(1+t)\,\underset{\cdot}{\boldsymbol{p}}_6/8 - (1+t)\,\underset{\cdot}{\boldsymbol{p}}_7/8
\end{aligned}
\tag{1.15}
$$

Derivative of $\partial \underset{\cdot}{\boldsymbol{x}}/\partial r$ with respect to t

$$
\begin{aligned}
\frac{\partial^2 \underset{\cdot}{\boldsymbol{x}}(r,s,t)}{\partial r \partial t} &= \\[1em]
&-\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(s,t)}{\partial t}/2 + \frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(s,t)}{\partial t}/2 \\
&+(1-s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_2(r,t)}{\partial r \partial t}/2 + (1+s)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_3(r,t)}{\partial r \partial t}/2 \\
&-\frac{\partial \underset{\cdot}{\boldsymbol{b}}_4(r,s)}{\partial r}/2 + \frac{\partial \underset{\cdot}{\boldsymbol{b}}_5(r,s)}{\partial r}/2 \\[1em]
&+(1-s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(m,t)}{\partial t}/4 + (1+s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(p,t)}{\partial t}/4 \\
&-(1-s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(m,t)}{\partial t}/4 - (1+s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(p,t)}{\partial t}/4 \\[1em]
&-\underset{\cdot}{\boldsymbol{b}}_0(s,m)/4 + \underset{\cdot}{\boldsymbol{b}}_0(s,p)/4 \\
&+\underset{\cdot}{\boldsymbol{b}}_1(s,m)/4 - \underset{\cdot}{\boldsymbol{b}}_1(s,p)/4 \\[1em]
&+(1-s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_2(r,m)}{\partial r}/4 - (1-s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_2(r,p)}{\partial r}/4 \\
&+(1+s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_3(r,m)}{\partial r}/4 - (1+s)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_3(r,p)}{\partial r}/4 \\[1em]
&+(1-s)\,\underset{\cdot}{\boldsymbol{p}}_0/8 - (1-s)\,\underset{\cdot}{\boldsymbol{p}}_1/8 \\
&-(1+s)\,\underset{\cdot}{\boldsymbol{p}}_2/8 + (1+s)\,\underset{\cdot}{\boldsymbol{p}}_3/8 \\
&-(1-s)\,\underset{\cdot}{\boldsymbol{p}}_4/8 + (1-s)\,\underset{\cdot}{\boldsymbol{p}}_5/8 \\
&+(1+s)\,\underset{\cdot}{\boldsymbol{p}}_6/8 - (1+s)\,\underset{\cdot}{\boldsymbol{p}}_7/8 \qquad (1.16)
\end{aligned}
$$

Derivative of $\partial \underset{\cdot}{\boldsymbol{x}}/\partial s$ with respect to t

$$
\begin{aligned}
\frac{\partial^2 \underset{\cdot}{\boldsymbol{x}}(r,s,t)}{\partial s \partial t} = & \\
\\
+(1-r)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_0(s,t)}{\partial s \partial t}/2 + (1+r)\,\frac{\partial^2 \underset{\cdot}{\boldsymbol{b}}_1(s,t)}{\partial s \partial t}/2 & \\
-\frac{\partial \underset{\cdot}{\boldsymbol{b}}_2(r,t)}{\partial t}/2 + \frac{\partial \underset{\cdot}{\boldsymbol{b}}_3(r,t)}{\partial t}/2 & \\
-\frac{\partial \underset{\cdot}{\boldsymbol{b}}_4(r,s)}{\partial s}/2 + \frac{\partial \underset{\cdot}{\boldsymbol{b}}_5(r,s)}{\partial s}/2 & \\
\\
+(1-r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(m,t)}{\partial t}/4 - (1-r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(p,t)}{\partial t}/4 & \\
+(1+r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(m,t)}{\partial t}/4 - (1+r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(p,t)}{\partial t}/4 & \\
\\
+(1-r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(s,m)}{\partial s}/4 - (1-r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_0(s,p)}{\partial s}/4 & \\
+(1+r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(s,m)}{\partial s}/4 - (1+r)\,\frac{\partial \underset{\cdot}{\boldsymbol{b}}_1(s,p)}{\partial s}/4 & \\
\\
-\underset{\cdot}{\boldsymbol{b}}_2(r,m)/4 + \underset{\cdot}{\boldsymbol{b}}_2(r,p)/4 & \\
+\underset{\cdot}{\boldsymbol{b}}_3(r,m)/4 - \underset{\cdot}{\boldsymbol{b}}_3(r,p)/4 & \\
\\
+(1-r)\,\underset{\cdot}{\boldsymbol{p}}_0/8 + (1+r)\,\underset{\cdot}{\boldsymbol{p}}_1/8 & \\
-(1+r)\,\underset{\cdot}{\boldsymbol{p}}_2/8 - (1-r)\,\underset{\cdot}{\boldsymbol{p}}_3/8 & \\
-(1-r)\,\underset{\cdot}{\boldsymbol{p}}_4/8 - (1+r)\,\underset{\cdot}{\boldsymbol{p}}_5/8 & \\
+(1+r)\,\underset{\cdot}{\boldsymbol{p}}_6/8 + (1-r)\,\underset{\cdot}{\boldsymbol{p}}_7/8 & \qquad (1.17)
\end{aligned}
$$