Algorithm 1: AsymDPOP for a_i

When `Initialization`:

1 $util_i \leftarrow \bigotimes_{a_j \in AP(a_i)} f_{ij}$

2 **if** a_i *is a leaf* **then**

3 send UTIL($util_i$) to $P(a_i)$

When `received UTIL(`$util_c$`)` *from* $a_c \in C(a_i)$:

4 $util_i^c \leftarrow util_c$

5 **foreach** $a_j \in (PC(a_i) \cap Desc(a_c)) \cup \{a_c\}$ **do**

6 $util_i^c \leftarrow util_i^c \otimes f_{ij}$

7 $util_i \leftarrow util_i \otimes \min_{EV(a_i,a_c)} util_i^c$

8 **if** a_i *have received all UTIL from* $C(a_i)$ **then**

9 **if** a_i *is the root* **then**

10 $v_i^* \leftarrow \operatorname{argmin}_{x_i} util_i$

11 **PropagateValue**($\{(x_i = v_i^*)\}$)

12 **else**

13 send UTIL($util_i$) to $P(a_i)$

When `received VALUE(`$Assign$`)` *from* $P(a_i)$:

14 **PropagateValue**($Assign$)

Function `PropagateValue(`$Assign$`)`:

15 **foreach** $a_c \in C(a_i)$ **do**

16 $Assign_i^c \leftarrow Assign$

17 **if** $EV(a_i, a_c) \neq \emptyset$ **then**

18 $V^* \leftarrow \operatorname{argmin}_{EV(a_i,a_c)} util_i^c(Assign_{[dims(util_i^c)]})$

19 $Assign_i^c \leftarrow Assign_i^c \cup \{(x_j = V^*_{[x_j]}) | \forall x_j \in EV(a_i, a_c)\}$

20 send VALUE($Assign_i^c$) to a_c

Figure 1: The sketch of AsymDPOP

1 AsymDPOP

1.1 Pseudo Code for AsymDPOP

Fig 1. give the sketch of AsymDPOP, and the execution process can be divided into two phases: utility propagation phase and value propagation phase. The utility propagation phase begins with leaf agents sending their utility tables to their parents via UTIL messages (line 2-3). When an agent a_i receives a UTIL message from a child a_c, it joins its private functions w.r.t its (pseudo) children in branch a_c (line 5-6), and eliminates all the belonging variables whose highest (pseudo) parent is a_i from the utility table (line 7). Here, $EV(a_i, a_c)$is given by

$$EV\left(a_i, a_c\right) = PC\left(a_i\right) \cap Desc\left(a_c\right) \cup \{a_c\} \setminus ID\left(a_i\right)$$

where $ID(a_i)$ is the set of a_i's interfaces descendants which are constrained with $Sep(a_i)$. After receiving all the UTIL messages from its children, a_i propagates the joint utility table to its parent if it is not the root agent. Otherwise, the value propagation phase starts.

This phase begins with the root agent selecting the optimal assignment for itself (line 10). Given the determined assignments either from its parent (line 14) or computed locally (line 11), agent a_i selects the optimal assignments for the eliminated variables in each branch $a_c \in C(a_i)$ by a joint optimization over them (line 15-18), and propagates the assignments together with the determined assignments to a_c (line 19-20). The algorithm terminates when each leaf agent receives a VALUE message.

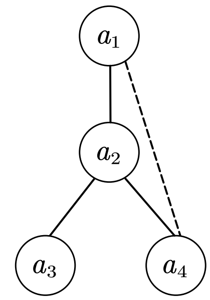

Figure 2: An example of a pseudo-tree in AsymDPOP

1.2 An Example for AsymDPOP

Fig.2 gives a pseudo tree deriving from the ADCOP shown in Fig.1. For better understanding, we take a_2 to explain the concepts in a pseudo tree. Since a_1 is the only ancestor constrained with a_2 via a tree edge, we have $P(a_2) = \{a_1\}$, $PP(a_2) = \emptyset$ and $Sep(a_2) = \{a_1\}$. Similarly, since a_3 and a_4 are descendants constrained with a_2 via tree edge, we have $C(a_2) = \{a_3, a_4\}$, $PC(a_2) = \emptyset$, $Desc(a_2) = \{a_3, a_4\}$. Particularly, since a_4 in $Desc(a_2)$ constraints with a_1 in $Sep(a_2)$, we have $ID(a_2) = \{a_4\}$.

Since a_3 and a_4 are the leaf nodes, they send UTIL messages with their utility tables ($util_3, util_4$) to their parent a_2, where $util_3 = f_{32}$ and $util_4 = f_{41} \otimes f_{42}$.

When received a UTIL message from a_3 (assume a_3's message had arrived earlier), a_2 stores the received message from a_3 ($util_2^3 = util_3 = f_{32}$), and then joins its private function f_{23} to update the $util_2^3$ ($util_2^3 = f_{32} \otimes f_{23}$). According to the definition of $EV(a_i, a_c)$ that all the belonging variables' highest (pseudo) parent is a_i in branch a_c. Here, $EV(a_2, a_3)$ is given by

$$\begin{aligned} EV(a_2, a_3) &= PC(a_2) \cap Desc(a_3) \cup \{a_3\} \setminus ID(a_2) \\ &= \emptyset \cap \emptyset \cup \{a_3\} \setminus \{a_4\} = \{a_3\} \end{aligned}$$

In addition, $ID(a_2)$ is a set of a_2's interface descendant that is constrained with $Sep(a_2)$. Here $Sep(a_2) = \{a_1\}$. Thus, a_2 needs to eliminate the variable x_3 from the $util_2^3$. After that, a_2 joins the eliminated result to the joint utility table $util_2$. Similarly, when a_2 received a UTIL message from a_4, it saves the content as $util_2^4$ ($util_2^4 = util_4 = f_{41} \otimes f_{42}$), and updates it with joining the private function f_{24} ($util_2^4 = f_{41} \otimes f_{42} \otimes f_{24}$). Since $EV(a_2, a_4) = \emptyset$, a_2 joins the $util_2^4$ to the $util_2$ directly without any elimination operation. Since a_2 have received all UTIL messages from its children, it propagates the $util_2$ to its parent a_1. Here we have

$$util_2 = f_{21} \otimes (f_{41} \otimes f_{42} \otimes f_{24}) \otimes (\min_{x_3} f_{23} \otimes f_{32})$$

When receiving the UTIL message from a_2, a_1 saves the content as $util_1^2 = util_2$ and joins the private functions f_{12} and f_{14} to update the $util_1^2$ ($util_1^2 = util_1^2 \otimes (f_{12} \otimes f_{14})$). After eliminating x_2 and x_4 ($EV\left(a_1, a_2\right) = \{a_2, a_4\}$), a_1 joins the eliminated result into $util_1$. Here we have

$$util_1 = \min_{x_2, x_4} f_{12} \otimes f_{14} \otimes (f_{21} \otimes (f_{41} \otimes f_{42} \otimes f_{24}) \otimes (\min_{x_3} f_{23} \otimes f_{32}))$$

Since a_1 is the root agent and received all UTIL messages from its children, it selects the optimal assignment v_1^* for itself and the optimal assignments v_2^* and v_4^* for the eliminated variables x_2 and x_4. That is,

$$v_1^* = \underset{x_1}{argmin}(util_1),$$

$$(v_2^*, v_4^*) = \underset{x_2, x_4}{argmin}(util_1^2(x_1 = v_1^*))$$

Then a_1 propagates a VALUE message including the optimal assignment to its child a_2, where the optimal assignment is $\{(x_1 = v_1^*), (x_2 = v_2^*), (x_3 = v_3^*)\}$.

When receiving the VALUE message from a_1, a_2 assigns the value v_2^* for itself. Since $EV(a_2, a_3) = \{a_3\}$ and $EV(a_2, a_4) = \emptyset$, a_2 selects the optimal assignment for a_3 by performing optimization over $util_2^3$ with the determined assignment of x_2, that is

$$v_3^* = \underset{x_3}{argmin}(util_2^3(x_2 = v_2^*))$$

Then a_2 propagates the optimal assignment $(x_3 = v_3^*)$ and $(x_4 = v_4^*)$ to a_3 and a_4 respectively. Once a_3 and a_4 received the VALUE messages, they assign for themselves. Since all leaf agents received VALUE messages, the algorithm terminates.

2 AsymDPOP with MBPS and MBES

2.1 Pseudo Code for AsymDPOP with MBPS and MBES

Fig.3 gives the sketch of AsymDPOP with MBPS and MBSE. Like Algorithm 1, the execution process consists of two phases: utility propagation phase and value propagation phase. The utility propagation phase also begins with leaf agents though sending a set of utility tables to their parents via UTIL messages (line 2-3). The set of utility tables is obtained from the Function PartitionF. In this function, agent a_i divides its private functions which are constrained with $AP(a_i)$ into several buckets, and joins the private functions in the same bucket into one utility table (line 32-40). If there is any residue, a_i joins them into an arbitrary bucket (line 41-43). Note that k_p is the parameter which controls the minimal number of dimensions of each bucket.

When receiving a UTIL message from a child a_c, it joins its private functions w.r.t its (pseudo) children in branch a_c (line 6-8). Notice that the join operation of its private functions w.r.t. its children does not increase the number of dimensions and should be applied accordingly to the related utility tables. When performing eliminations, it uses Function Min to implement (line 9,24-29). To be more specific, instead of eliminating all the variables in $EV(a_i, a_c)$ directly, it first divides elimination variables into several groups whose variables share at least a common utility table in Function GroupEV (line 24). And then, for each variable group, it divides the variables into several sets (or batches) with the Function PartitionEV in the similar way as Function PartitionF (line 25). Specifically, k_e in PartitionEV specifies the minimal number of variables optimized in a min operator (i.e., the size of a batch). After getting the $EVSet$, it traverses the set to optimize the $util$ that has been passed to

Algorithm 2: AsymDPOP(MBPS+MBES) for a_i

```
   When Initialization:
1      util_i ← ∅
2      if a_i is a leaf then
3          send UTIL(PartitionF()) to P(a_i)
   When received UTIL(util_c) from a_c ∈ C(a_i):
4      util_i^c ← util_c
5      foreach a_j ∈ (PC(a_i) ∩ Desc(a_c)) ∪ {a_c} do
6          if ∃u ∈ util_i^c, s.t.dims(f_ij) ⊂ dims(u) then
7              u ← u ⊗ f_ij
8              break
9      util_i ← util_i ∪ Min(util_i^c, EV(a_i, a_c))
10     if a_i have received all UTIL from C(a_i) then
11         if a_i is the root then
12             v_i^* ← argmin_{x_i}( ⊗_{u∈util_i} u)
13             PropagateValue({(x_i = v_i^*)})
14         else
15             send UTIL(util_i ∪ PartitionF()) to P(a_i)
   When received VALUE(Assign) from P(a_i):
16     PropagateValue(Assign)
   Function PropagateValue(Assign):
17     foreach a_c ∈ C(a_i) do
18         Assign_i^c ← Assign
19         if EV(a_i, a_c) ≠ ∅ then
20             u' ← ⊗_{u∈util_i^c} u(Assign_[dims(u)])
21             V^* ← argmin_{EV(a_i,a_c)} (u')
22             Assign_i^c ← Assign_i^c ∪ {(x_j = V^*_[x_j])|∀x_j ∈
               EV(a_i, a_c)}
23         send VALUE(Assign_i^c) to a_c
   Function Min(util, EV):
24     U ← util, G ← GroupEV(EV)
25     EVSet ← ∪_{∀EV'∈G} PartitionEV(EV', U)
26     foreach EV' ∈ EVSet do
27         F ← {f|∀f ∈ U, dims(f) ∩ EV' ≠ ∅}
28         U ← (U\F) ∪ {min_{EV'} ⊗_{f∈F} f}
29     return U
   Function PartitionF():
30     u ← ∅, U ← ∅
31     order AP(a_i) according to their levels
32     foreach a_j ∈ AP(a_i) do
33         if ∃u' ∈ util_i, s.t.dims(f_ij) ⊂ dims(u') then
34             u' ← u' ⊗ f_ij
35         else
36             if |dims(u)| ≥ k_p then
37                 U ← U ∪ {u}
38                 u ← f_ij
39             else
40                 u ← u ⊗ f_ij
41     if u ≠ ∅ then
42         random select an element u' ∈ U
43         u' ← u' ⊗ u
44     return U
   Function GroupEV(EV, util):
45     Dims ← {dims(u) ∩ EV|∀u ∈ util}
46     while ∃D, D' ∈ Dims, s.t.D ∩ D' ≠ ∅ do
47         D ← D ∪ D', Dims ← Dims\D'
48     return Dims
   Function PartitionEV(EV):
49     EV' ← ∅, EVSet ← ∅
50     foreach ev ∈ EV do
51         if |EV'| ≥ k_e then
52             EVSet ← EVSet ∪ {EV'}
53             EV' ← {ev}
54         else
55             EV' ← EV' ∪ {ev}
56     if EV' ≠ ∅ then
57         random select an element EV'' ∈ EVSet
58         EV'' ← EV'' ∪ EV'
59     return EVSet
```

Figure 3: The sketch of AsymDPOP with MBPS and MBES

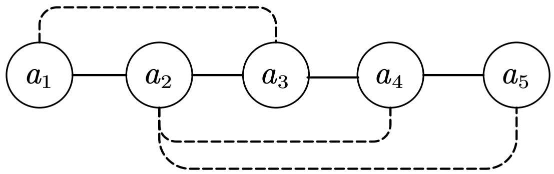

Figure 4: An example of a chain in AsymDPOP with MBPS and MBES

Function Min (line 26). In detail, for each batch, we perform optimization to the functions that are related to the variables in the batch over the batch and replace these functions with the results. The process terminates when the all the variable batch are exhausted (line 27-28). Subsequently, it returns the new set of utility tables which have been eliminated properly (line 29).

When receiving all the UTIL messages from its children, a_i propagates the joint utility tables set to its parent if it is not the root agent. Otherwise, the value propagation phase starts (line 13). The process is roughly as same as Algorithm 1. Agent a_i selects the optimal assignments for itself and the eliminated variables in each branch $a_c \in C(a_i)$. And since it uses the MBPS and MBES, there is may not one joint utility table (in Algorithm 1), but a set of joint utility tables in smaller size. So before selecting the optimal assignments for branch a_c, it joins all the tables in $util_i^c$ with the assignments it received from its parent (line 16) or computed locally (line 13). After computing the assignments for variables of $EV(a_i, a_c)$ in branch a_c (line 21-22), it sends a VALUE message to a_c (line 23). The algorithm terminates when each leaf agent receives a VALUE message.

2.2 An Example of AsymDPOP with MBPS and MBES

For the convenience of further explanation, we denote the index of a joint function f_{ijk} as the dimensions of this function, and the order of index ijk represents the joining operation subsequence of agents.

We take Fig. 4 as an example to demonstrate the Algorithm 2. Suppose that $k_p = 3$, $k_e = 1$. Since a_5 is the leaf node, it sends UTIL message with a set of utility tables returned from Function PartitionF to parent a_4. In PartitionF, it first orders $AP(a_5)$ according to their levels. And it divides its private functions which are related with $AP(a_5)$ into buckets. Because the size of $AP(a_5)$ is two, the biggest dimension of the joint utility table is 3 which is just equal to k_p. So there is only one bucket. And a_5 joins the functions in the bucket to get a 3-ary utility function f_{541} ($f_{541} = f_{54} \otimes f_{51}$), and sends a function set included the joint function f_{541} to its parent a_4.

When a_4 receives the UTIL message from its child a_5, it saves the utility set and updates the element f_{541} to f_{451} by joining its private function f_{45}. And since a_4 is not the highest (pseudo) parent of any decedents ($EV(a_4, a_5) = \emptyset$), there is no elimination operation. Next, since it has received all UTIL messages from its children, it sends a set which contains the joint utility function f_{432} and f_{451} to its parent a_3. Here, f_{432} is given by joining the private functions f_{43} and f_{42}.

When receiving the UTIL message from a_4, similarly, a_3 saves the utility set and updates f_{432} to f_{342} by joining its private function f_{34}. Also a_3 is not the highest (pseudo) parent of a_4 or any other decedents, which means it is no need of any elimination. Then it deals with the residual private functions (f_{31} and f_{32}) with Function PartitionF by joining them into a 3-ary utility function f_{321}. And it sends the utility function set $\{f_{451}, f_{342}, f_{321}\}$ to its parent a_2.

Once a_2 receives the UTIL message from a_3, it also saves the utility set firstly, and joins the relative private function f_{23} into f_{321} getting f_{231}. Since a_2 is the highest (pseudo) parent of a_4 and a_5, that is $EV(a_2, a_3) = \{a_4, a_5\}$, the eliminations is operated by Function Min. Firstly, it groups variables x_4 and x_5 into one group, according to the connection that they share a common utility table f_{451}. Since $k_e = 1$, variables x_4 and x_5 are separated in two batches automatically, and they are eliminated from the utility functions in the set $\{f_{451}, f_{342}, f_{231}\}$ one by one. After eliminating x_5, we get a new utility function set $\{f_{41}, f_{342}, f_{231}\}$. Before eliminate x_4 from this new set, it needs to join the utility function f_{41} with f_{342}. Finally, it gets a set containing f_{231} and f_{213} which is derived by eliminating x_4 from the utility function which is got by joining f_{41} into f_{342}. And then send the new set to its parent a_1.

When a_1 receives the UTIL message, it saves the utility set as routine, then joins the relative private functions f_{13} and f_{12} into f_{231} getting f_{123}. Due to the facts that $EV(a_1, a_2) = \{a_2, a_3\}$ and the variables x_2 and x_3 are both relative to the utility function f_{123}, they are grouped into one set. But since $k_e = 1$, a_1 still eliminates the variables x_2 and x_3 from the utility functions in the set $\{f_{123}, f_{213}\}$ respectively, and gets a new set as its own utility table set. Since it's root agent and has received all UTIL messages from its children, a_1 joins all the functions in the set which is the result of eliminating x_2 and x_3, and choose the optimal value v_1^* for itself. Thus, the value propagation phase starts. a_1 selects the values v_2^* and v_3^* for a_2 and a_3, and propagates the assignment $\{x_1 = v_1^*, x_2 = v_2^*, x_3 = v_3^*\}$ to $a2$. Then $a2$ selects optimal assignments for a_4 and a_5 after receiving VALUE message from a_1, and sends the assignment $\{x_1 = v_1^*, x_2 = v_2^*, x_3 = v_3^*, x_4 = v_4^*, x_5 = v_5^*\}$ to a_3. When a_3 received the VALUE message, it assigns for itself and sends the same assignments it received from a_2 to its child a_4. And a_4 does exactly the same thing as a_3. The algorithm terminates when leaf agent a_5 receives the VALUE message from a_4 and assigns for itself.