Let $\zeta_n = e^{\frac{2\pi i}{n}}$. The binary dihedral group (often denoted D_n in this context) is the subgroup of $\mathrm{SL}_2(\mathbb{C})$ generated by

$$g = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

satisfying the relations

$$g^n = h^2 = -I, \qquad hgh^{-1} = g^{-1}.$$

It has order $4n$, and is the preimage of the usual dihedral group of order $2n$ under the covering $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$.

We use the same contragredient action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}[x, y]$:

$$(A \cdot f)(x, y) := f([x \quad y]A^{-1}), \qquad A \in \mathrm{GL}_2(\mathbb{C}).$$

Step 1. Action of g. As before,

$$g \cdot x = \zeta_n^{-1} x, \qquad g \cdot y = \zeta_n y,$$

so $x^i y^j$ transforms by ζ_n^{j-i}. The g-invariants are generated by

$$u := x^n, \quad v := y^n, \quad w := xy,$$

with relation

$$uv - w^n = 0,$$

so

$$\mathbb{C}[x, y]^{\langle g \rangle} \simeq \mathbb{C}[u, v, w]/(uv - w^n).$$

Step 2. Action of h. We compute $h^{-1} = -h$, so

$$h \cdot (x, y) = (x, y)h^{-1} = (x, y)(-h) = (-y, x).$$

Thus

$$h \cdot x = -y, \qquad h \cdot y = x.$$

On the invariants u, v, w one finds

$$h \cdot u = h \cdot (x^n) = (-y)^n = (-1)^n v, \qquad h \cdot v = (x)^n = u, \qquad h \cdot w = (-y)(x) = -xy = -w.$$

Step 3. Dihedral invariants. The full invariant ring $\mathbb{C}[x, y]^{\langle g,h \rangle}$ consists of the h-invariant subring of $\mathbb{C}[u, v, w]/(uv - w^n)$. From the action:

$$h : \ (u, v, w) \mapsto ((-1)^n v, \ u, \ -w).$$

Therefore invariants can be taken as follows:
- If n is even: $u + v$, uv, and w^2 are h-invariant, and the relation becomes $(uv) - (w^n) = 0$. - If n is odd: then $u \mapsto -v$, $v \mapsto u$, $w \mapsto -w$, so invariants are generated by $u^2 + v^2$, uv, and w^2, with relations inherited from $uv = w^n$.

In particular, one obtains a presentation of the binary dihedral invariants:

$$\mathbb{C}[x, y]^{D_n} \ \cong \ \begin{cases} \mathbb{C}\big[u + v, \ uv, \ w^2\big] \ / \ \big((uv) - (w^n)\big), & n \text{ even}, \\ \mathbb{C}\big[u^2 + v^2, \ uv, \ w^2\big] \ / \ (\text{relations}), & n \text{ odd}. \end{cases}$$

Geometrically, $\mathrm{Spec}\left(\mathbb{C}[x, y]^{D_n}\right)$ is the Kleinian surface singularity of type D_{n+2}.