Let $\zeta_n = e^{\frac{2\pi i}{n}}$ and define

$$C_n \simeq \left\langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix} \right\rangle \subset \mathrm{SL}_2(\mathbb{C}).$$

We define an action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}[x,y]$ by

$$(A \cdot f)(x,y) := f\big([x \;\; y]A^{-1}\big), \qquad A \in \mathrm{GL}_2(\mathbb{C}),\ f \in \mathbb{C}[x,y].$$

This is a group action since

$$((AB)\cdot f)(x,y) = f\big([x \;\; y](AB)^{-1}\big) = f\big([x \;\; y]B^{-1}A^{-1}\big) = (A \cdot (B \cdot f))(x,y).$$

For $g = \mathrm{diag}(\zeta_n, \zeta_n^{-1}) \in C_n$ we have

$$g \cdot x = \zeta_n^{-1} x, \qquad g \cdot y = \zeta_n \, y.$$

Hence, for a monomial $x^i y^j$,

$$g \cdot (x^i y^j) = (\zeta_n^{-1})^i (\zeta_n)^j x^i y^j = \zeta_n^{j-i} \, x^i y^j.$$

Thus $x^i y^j$ is C_n-invariant iff $j - i \equiv 0 \pmod{n}$.

Introduce

$$u := x^n, \quad v := y^n, \quad w := xy.$$

These elements are invariant, and they satisfy

$$uv - w^n = 0 \quad \text{in } \mathbb{C}[x,y].$$

Conversely, if $x^i y^j$ is invariant, write $i = qn + r$, $j = q'n + r$ with $0 \le r \le n-1$. Then

$$x^i y^j = (x^n)^q (y^n)^{q'} (xy)^r = u^q v^{q'} w^r.$$

So every invariant monomial lies in $\mathbb{C}[u,v,w]$. The natural surjection

$$\phi : \mathbb{C}[u,v,w] \twoheadrightarrow \mathbb{C}[x,y]^{C_n}, \qquad u \mapsto x^n,\ v \mapsto y^n,\ w \mapsto xy$$

has kernel generated by $uv - w^n$. Therefore we obtain

$$\mathbb{C}[x,y]^{C_n} \;\simeq\; \mathbb{C}[u,v,w]/(uv - w^n).$$

Geometrically,

$$\mathrm{Spec}\left(\mathbb{C}[x,y]^{C_n}\right) \;\cong\; \{(u,v,w) \in \mathbb{C}^3 \mid uv = w^n\},$$

which is the Kleinian surface singularity of type A_{n-1}.