1 semi-orthogonal decomposition

Def

(i) We define right(left) orthogonal sub-category as following

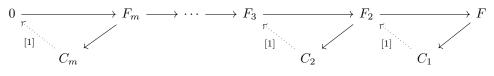
$$\mathcal{C}^{\perp} := \{ F \in \mathcal{D} \mid \forall E \in \mathcal{C}, \operatorname{Hom}_{\mathcal{D}}(E, F) = 0 \}$$
$$^{\perp}\mathcal{C} := \{ E \in \mathcal{D} \mid \forall F \in \mathcal{C}, \operatorname{Hom}_{\mathcal{D}}(E, F) = 0 \}$$

- (ii) \mathcal{C} is strictly full subcategory of \mathcal{D} iff $\forall E \in \mathcal{C}$ and $F \in \mathcal{D}$ s.t. $E \simeq F \Longrightarrow F \in \mathcal{C}$
- (iii) Thick closure of \mathcal{C} in \mathcal{D} is the minimum strictly full subcategory of \mathcal{D} which contains \mathcal{C} and is closed under taking summands.

(iv)

Semi-orthogonal decomposition ————

$$i < j \Longrightarrow \mathcal{C}_i \subset \mathcal{C}_j^{\perp}, \ C_i \in \mathcal{C}_i$$



We denote

$$\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$$

Thm 1.1. -

 C_i $(1 \le i \le m)$ are unique up to isomorphism and

$$p_i \colon \mathcal{D} \to \mathcal{C}_i \ F \mapsto C_i$$
$$p_{i,m} \colon \mathcal{D} \to \langle \mathcal{C}_i, \dots, \mathcal{C}_m \rangle \ F \mapsto F_i$$

is functorial.

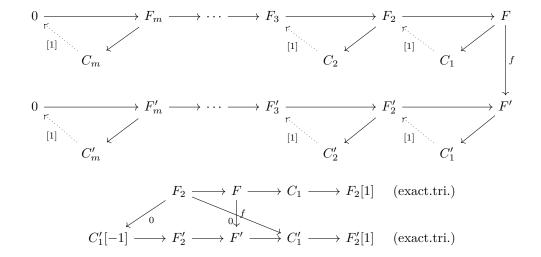
Proof. Since C_m is a strict full subcategory, $F_m \in C_m$. $\forall E \in C_j \ (1 \le j < m), \ \operatorname{Hom}_{\mathcal{D}}(F_m, E) = 0$

$$F_m \longrightarrow F_{m-1} \longrightarrow C_{m-1} \longrightarrow F_m[1]$$
 (exact.tri.)

Take $E \in \mathcal{C}_j$ (j < m-1) and acting cohomological functor $\operatorname{Hom}_{\mathcal{D}}(-, E)$ this exact triangle,

$$\operatorname{Hom}_{\mathcal{D}}(C_{m-1}, E) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(F_{m-1}, E) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(C_m, E) \quad (\operatorname{exact})$$

By definition, since both the left and right sides are 0, $\operatorname{Hom}_{\mathcal{D}}(F_{m-1}, E) = 0$ Repeating this procedure, we obtain $\forall E \in \mathcal{C}_i \ (1 \leq i < j) \operatorname{Hom}_{\mathcal{D}}(F_j, E) = 0$



From previous lemma, there uniquely exists morphisms which commute the diagram.

This indicates $p_1(g) \circ p_1(f) = p_1(g \circ f)$ and $p_{2,m}(g) \circ p_{2,m}(f) = p_{2,m}(g \circ f)$.

Thm 1.2.

If there exists SOD of $\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$

$$K(\mathcal{D}) \to K(\mathcal{C}_1) \oplus \cdots \oplus K(\mathcal{C}_m) \quad [F] \mapsto (p_1(F), \dots, p_m(F))$$

Proof. (well-defined)

Since

$$E \longrightarrow F \longrightarrow G \longrightarrow E[1] \quad (\text{exact}) \Longrightarrow p_1(E) \longrightarrow p_1(F) \longrightarrow p_1(G) \longrightarrow p_1(E[1]) \quad (\text{exact})$$

$$[F] - [E] - [G] = 0 \Longrightarrow [p_i(F)] - [p_i(E)] - [p_i(G)] = 0$$

(injection)

$$[F] \sim [F_2] + [C_1]$$

 $\sim [F_3] + [C_2] + [C_1]$
 \vdots
 $\sim [C_m] + \dots + [C_1]$

Therefore $(p_1(F), \ldots, p_m(F)) = 0$ means [F] = 0 (surjection)

- Lem 1.3. -

$$B \sqcup_A C \simeq \operatorname{Cok} \begin{pmatrix} -f \\ g \end{pmatrix}$$

Proof.

$$A \xrightarrow{f} B$$

$$g \downarrow \qquad \qquad \downarrow i_{B}$$

$$C \xrightarrow{i_{C}} P$$

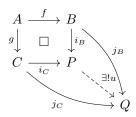
$$q \colon B \oplus C \to (B \oplus C) / \operatorname{Im} \begin{pmatrix} -f \\ g \end{pmatrix} \eqqcolon P$$

$$i_{B} \coloneqq q \circ \iota_{B} \quad i_{C} \coloneqq q \circ \iota_{C}$$

$$i_B \circ f - i_C \circ g = q \circ \iota_B \circ f - q \circ \iota_C \circ g = q \circ \begin{pmatrix} -f \\ g \end{pmatrix} = -q \circ \begin{pmatrix} -f \\ g \end{pmatrix} = 0$$

Let $j_B: B \to Q$, $j_C: C \to Q$ s.t. $j_B \circ f = j_C \circ g$ and $j := (j_B, j_C)$, then

$$j \circ (-f, g) = -j_B \circ f + j_C \circ g = 0$$



Thm 1.4.

(i) Define $\mathcal{D}' \subset \mathcal{D}$ as $C_i \in \mathcal{C}_i$

(ii)

$$\chi = (\chi_{ij})$$

$$\chi_{ij} \coloneqq \sum_{p \in \mathbb{Z}} (-1)^p \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(E_i, E_j[p])$$