

1 Affine Varieties

Def 1.1.

$$A = k[x_1, \dots, x_n]$$

$$Z(T) := \{P \in \mathbb{A}^n \mid \forall f \in T, f(P) = 0\}$$

$Y \subset \mathbb{A}^n$ is algebraic set $\stackrel{\text{def}}{\iff} \exists T \subset A$ s.t. $Y = Z(T)$

Prop 1.2.

The union of two algebraic sets and intersection of any family of algebraic sets are algebraic set.

Proof. If $Y_1 = Z(T_1), Y_2 = Z(T_2)$. then $Y_1 \cup Y_2 = Z(T_1 T_2)$

If $Y_\lambda = Z(T_\lambda)$, then $\bigcap_{\lambda \in \Lambda} Y_\lambda = Z(\bigcup_{\lambda \in \Lambda} T_\lambda)$ ■

Def 1.3.

Zariski topology on $\mathbb{A}^n \stackrel{\text{def}}{\iff} \mathcal{O} = \{Y^c \subset \mathbb{A}^n \mid Y: \text{algebraic set}\}$

$$I(Y) := \{f \in A \mid \forall P \in Y, f(P) = 0\}$$

$Y(\subset X: \text{topological space})$ is irreducible $\stackrel{\text{def}}{\iff} Y$ cannot be expressed as $Y = Y_1 \cup Y_2, (\emptyset \subsetneq Y_1, Y_2 \subsetneq Y: \text{closed})$

Lem 1.4.

Hilbert's basis theorem

$R: \text{Noetherian} \implies R[X]: \text{Noetherian}$

Proof. Let I be an ideal of $R[X]$

$$J := \{a_0 \in R \mid \exists f \in J \text{ s.t. } f(X) = a_0 X^d + \dots + a_d\}$$

in this definition, J can be confirmed as an ideal of R .

\therefore suppose $a_0, b_0 \in J$. By definition, there exists $F(X), G(X) \in I$ s.t.

$$F(X) = a_0 X^r + \dots + a_r$$

$$G(X) = b_0 X^s + \dots + b_s$$

since I is an ideal, $kF(X) \in I$, which means $ka_0 \in J$ and $F(X) + X^{r-s}G(X) \in I$, which means $a_0 + b_0 \in J$

Since R is a Noetherian ring, J is finitely generated. So there exists $a^1, \dots, a^t \in R$ s.t. $J = (a^1, \dots, a^t)$. By definition of J , there exists F_i ($1 \leq i \leq t$) whose leading coefficient is a^i

For $m \geq 0$, we define $J_m \subset J$ as all leading coefficients of polynomial in I of degree at most m . i.e.

$$J_m := \{a_0 \in J \mid r = \deg(f) \leq m, f(X) = a_0 X^r + \dots + a_r, \}$$

J_m can also be verified to be an ideal. Similarly J_m is finitely generated by $a^{m,j}$ ($1 \leq j \leq t_m$) and define similarly $F_{m,j}$ ($m < N, 1 \leq j \leq t_m$).

$$I_0 := (F_i \ (1 \leq i \leq t), F_{m,j} \ (0 \leq m < N, 1 \leq j \leq t_m))$$

Obviously I_0 is finitely generated and $I_0 \subset I$. If we confirm $I \subset I_0$, this proof is over.

Suppose that there exists a polynomial which doesn't belong to I_0 . From these polynomial, we take the least degree and ■

Lem 1.5. —

Weak Hilbert's Nullstellensatz

k : algebraically closed field, \mathfrak{a} : ideal in $A = k[x_1, \dots, x_n]$

$$Z(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = A$$

Proof. Suppose $\mathfrak{a} \neq A$. Since A is Noetherian, there exists maximal ideal \mathfrak{m} that includes \mathfrak{a} .

A/\mathfrak{m} is isomorphic to some field extension of k , but since k is algebraically closed, $A/\mathfrak{m} = k$. So there exists $a_i \in k$ ($1 \leq i \leq n$) s.t. $X_i - a_i \in \mathfrak{m}$.

$$(X_1 - a_1, \dots, X_n - a_n) \subset \mathfrak{m}$$

However because $(X_1 - a_1, \dots, X_n - a_n)$ is maximal, $\mathfrak{m} = (X_1 - a_1, \dots, X_n - a_n)$

$(a_1, \dots, a_n) \in Z(\mathfrak{m}) \subset Z(\mathfrak{a})$. $\therefore Z(\mathfrak{a}) \neq \emptyset$ ■

Thm 1.6. —

Hilbert's Nullstellensatz

k : algebraically closed field, \mathfrak{a} : ideal in $A = k[x_1, \dots, x_n]$

$f \in I(Z(\mathfrak{a}))$ i.e. $\forall P \in Z(\mathfrak{a}), f(P) = 0 \implies \exists r \in \mathbb{N}, f^r \in \mathfrak{a}$

Prop 1.7. —

(a) $T_1 \subset T_2 (\subset A) \implies Z(T_1) \supset Z(T_2)$

(b) $Y_1 \subset Y_2 (\subset \mathbb{A}^n) \implies I(Y_1) \supset I(Y_2)$

(c) $Y_1, Y_2 \subset \mathbb{A}^n, I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$

(d) $\mathfrak{a} \subset A, I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$

(e) $Y \subset \mathbb{A}^n, Z(I(Y)) = \overline{Y}$

Def 1.8. —

X : topological space is Noetherian $\stackrel{\text{def}}{\iff}$ for any sequence $Y_1 \supset Y_2 \supset \dots$ of closed subsets, $\exists r \in \mathbb{N}$ s.t. $Y_r = Y_{r+1} = \dots$