## 1 semi-orthogonal decomposition

Def

(i) We define right(left) orthogonal sub-category as following

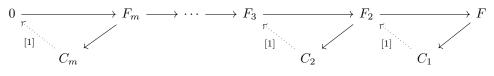
$$\mathcal{C}^{\perp} := \{ F \in \mathcal{D} \mid \forall E \in \mathcal{C}, \operatorname{Hom}_{\mathcal{D}}(E, F) = 0 \}$$
$$^{\perp}\mathcal{C} := \{ E \in \mathcal{D} \mid \forall F \in \mathcal{C}, \operatorname{Hom}_{\mathcal{D}}(E, F) = 0 \}$$

- (ii)  $\mathcal{C}$  is strictly full subcategory of  $\mathcal{D}$  iff  $\forall E \in \mathcal{C}$  and  $F \in \mathcal{D}$  s.t.  $E \simeq F \Longrightarrow F \in \mathcal{C}$
- (iii) Thick closure of  $\mathcal{C}$  in  $\mathcal{D}$  is the minimum strictly full subcategory of  $\mathcal{D}$  which contains  $\mathcal{C}$  and is closed under taking summands.

(iv)

Semi-orthogonal decomposition ————

$$i < j \Longrightarrow \mathcal{C}_i \subset \mathcal{C}_j^{\perp}, \ C_i \in \mathcal{C}_i$$



We denote

$$\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$$

Thm 1.1. -

 $C_i$   $(1 \le i \le m)$  are unique up to isomorphism and

$$p_i \colon \mathcal{D} \to \mathcal{C}_i \ F \mapsto C_i$$
$$p_{i,m} \colon \mathcal{D} \to \langle \mathcal{C}_i, \dots, \mathcal{C}_m \rangle \ F \mapsto F_i$$

is functorial.

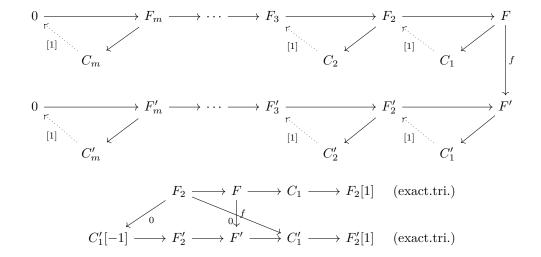
*Proof.* Since  $C_m$  is a strict full subcategory,  $F_m \in C_m$ .  $\forall E \in C_j \ (1 \le j < m), \ \operatorname{Hom}_{\mathcal{D}}(F_m, E) = 0$ 

$$F_m \longrightarrow F_{m-1} \longrightarrow C_{m-1} \longrightarrow F_m[1]$$
 (exact.tri.)

Take  $E \in \mathcal{C}_j$  (j < m-1) and acting cohomological functor  $\operatorname{Hom}_{\mathcal{D}}(-, E)$  this exact triangle,

$$\operatorname{Hom}_{\mathcal{D}}(C_{m-1}, E) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(F_{m-1}, E) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(C_m, E)$$
 (exact)

By definition, since both the left and right sides are 0,  $\operatorname{Hom}_{\mathcal{D}}(F_{m-1}, E) = 0$ Repeating this procedure, we obtain  $\forall E \in \mathcal{C}_i \ (1 \leq i < j) \operatorname{Hom}_{\mathcal{D}}(F_j, E) = 0$ 



From previous lemma, there uniquely exists morphisms which commute the diagram.

This indicates  $p_1(g) \circ p_1(f) = p_1(g \circ f)$  and  $p_{2,m}(g) \circ p_{2,m}(f) = p_{2,m}(g \circ f)$ .

- Thm 1.2. -

If there exists SOD of  $\mathcal{D} = \langle \mathcal{C}_1, \dots, \mathcal{C}_m \rangle$ 

$$K(\mathcal{D}) \to K(\mathcal{C}_1) \oplus \cdots \oplus K(\mathcal{C}_m) \quad [F] \mapsto (p_1(F), \dots, p_m(F))$$

Proof. (well-defined)

Since

$$E \longrightarrow F \longrightarrow G \longrightarrow E[1] \quad (\text{exact}) \Longrightarrow p_1(E) \longrightarrow p_1(F) \longrightarrow p_1(G) \longrightarrow p_1(E[1]) \quad (\text{exact})$$

$$[F] - [E] - [G] = 0 \Longrightarrow [p_i(F)] - [p_i(E)] - [p_i(G)] = 0$$

(injection)

$$[F] \sim [F_2] + [C_1]$$
  
 $\sim [F_3] + [C_2] + [C_1]$   
 $\vdots$   
 $\sim [C_m] + \dots + [C_1]$ 

Therefore  $(p_1(F), \ldots, p_m(F)) = 0$  means [F] = 0 (surjection)

Thm 1.3. —

- (i) Define  $\mathcal{D}' \subset \mathcal{D}$  as  $C_i \in \mathcal{C}_i$
- (ii)