

# 1 Affine Varieties

Def 1.1.

$$A = k[x_1, \dots, x_n]$$

$$Z(T) := \{P \in \mathbb{A}^n \mid \forall f \in T, f(P) = 0\}$$

$Y \subset \mathbb{A}^n$  is algebraic set  $\stackrel{\text{def}}{\iff} \exists T \subset A$  s.t.  $Y = Z(T)$

Prop 1.2.

The union of two algebraic sets and intersection of any family of algebraic sets are algebraic set.

*Proof.* If  $Y_1 = Z(T_1), Y_2 = Z(T_2)$ . then  $Y_1 \cup Y_2 = Z(T_1 T_2)$

If  $Y_\lambda = Z(T_\lambda)$ , then  $\bigcap_{\lambda \in \Lambda} Y_\lambda = Z(\bigcup_{\lambda \in \Lambda} T_\lambda)$  ■

Def 1.3.

Zariski topology on  $\mathbb{A}^n \stackrel{\text{def}}{\iff} \mathcal{O} = \{Y^c \subset \mathbb{A}^n \mid Y: \text{algebraic set}\}$

$$I(Y) := \{f \in A \mid \forall P \in Y, f(P) = 0\}$$

$Y(\subset X: \text{topological space})$  is irreducible  $\stackrel{\text{def}}{\iff} Y$  cannot be expressed as  $Y = Y_1 \cup Y_2, (\emptyset \subsetneq Y_1, Y_2 \subsetneq Y: \text{closed})$

Lem 1.4.

Hilbert's basis theorem

$R: \text{Noetherian} \implies R[X]: \text{Noetherian}$

*Proof.* Let  $I$  be an ideal of  $R[X]$

$$J := \{a_0 \in R \mid \exists f \in J \text{ s.t. } f(X) = a_0 X^d + \dots + a_d\}$$

in this definition,  $J$  can be confirmed as an ideal of  $R$ .

$\therefore$  suppose  $a_0, b_0 \in J$ . By definition, there exists  $F(X), G(X) \in I$  s.t.

$$F(X) = a_0 X^r + \dots + a_r$$

$$G(X) = b_0 X^s + \dots + b_s$$

since  $I$  is an ideal,  $kF(X) \in I$ , which means  $ka_0 \in J$  and  $F(X) + X^{r-s}G(X) \in I$ , which means  $a_0 + b_0 \in J$

Since  $R$  is a Noetherian ring,  $J$  is finitely generated. So there exists  $a^1, \dots, a^t \in R$  s.t.  $J = (a^1, \dots, a^t)$ . By definition of  $J$ , there exists  $F_i$  ( $1 \leq i \leq t$ ) whose leading coefficient is  $a^i$

For  $m \geq 0$ , we define  $J_m \subset J$  as all leading coefficients of polynomial in  $I$  of degree at most  $m$ . i.e.

$$J_m := \{a_0 \in J \mid r = \deg(f) \leq m, f(X) = a_0 X^r + \dots + a_r, \}$$

$J_m$  can also be verified to be an ideal. Similarly  $J_m$  is finitely generated by  $a^{m,j}$  ( $1 \leq j \leq t_m$ ) and define similarly  $F_{m,j}$  ( $m < N, 1 \leq j \leq t_m$ ).

$$I_0 := (F_i \ (1 \leq i \leq t), F_{m,j} \ (0 \leq m < N, 1 \leq j \leq t_m))$$

Obviously  $I_0$  is finitely generated and  $I_0 \subset I$ . If we confirm  $I \subset I_0$ , this proof is over.

Suppose that there exists a polynomial which doesn't belong to  $I_0$ . From these polynomial, we take  $G$  as the least degree one and let  $a$  be a leading coefficient of  $G$ . Since  $a \in J$ , there exists  $k_1, \dots, k_t \in R$  s.t.  $a = \sum_{i=1}^t k_i a^i$

We consider two cases  $\deg(G) \geq N$  and  $m = \deg(G) \leq N$

- In the former case

We define  $H_i(X) := k_i X^{N-\deg(F_i)}$ .

$$G_0 = G - \sum_{i=1}^t H_i F_i \in I$$

Since  $\deg(G_0) \leq N$  and  $G \in I$ ,  $G_0 \in I_0$ , this means  $G = G_0 + \sum_{i=1}^t H_i F_i \in I_0$ . Contradiction.

- In the latter case

Since  $a \in J_m$ , there exists  $k_i \in R$  ( $1 \leq i \leq t_m$ ) s.t.  $a = \sum_{i=1}^{t_m} k_i a^{(m,i)}$

We define  $H_i(X) := k_i X^{N-\deg(F_{m,i})}$ .

$$G_0 = G - \sum_{i=1}^t H_i F_{m,i} \in I$$

Similary we can confirm  $G \in I_0$ . Contradiction. Therefore  $I = I_0$ , which means  $I$  is finitely generated. ■

Lem 1.5.

Weak Hilbert's Nullstellnsatz

$k$ : algebraically closed field,  $\mathfrak{a}$ : ideal in  $A = k[x_1, \dots, x_n]$

$$Z(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = A$$

*Proof.* Suppose  $\mathfrak{a} \neq A$ . Since  $A$  is Noetherian, there exists maximal ideal  $\mathfrak{m}$  that includes  $\mathfrak{a}$ .

$A/\mathfrak{m}$  is isomorphic to some field extension of  $k$ , but since  $k$  is algebraically closed,  $A/\mathfrak{m} = k$ . So there exists  $a_i \in k$  ( $1 \leq i \leq n$ ) s.t.  $X_i - a_i \in \mathfrak{m}$ .

$$(X_1 - a_1, \dots, X_n - a_n) \subset \mathfrak{m}$$

However because  $(X_1 - a_1, \dots, X_n - a_n)$  is maximal,  $\mathfrak{m} = (X_1 - a_1, \dots, X_n - a_n)$

$(a_1, \dots, a_n) \in Z(\mathfrak{m}) \subset Z(\mathfrak{a})$ .  $Z(\mathfrak{a}) \neq \emptyset$  ■

Thm 1.6.

Hilbert's Nullstellnsatz

$k$ : algebraically closed field,  $\mathfrak{a}$ : ideal in  $A = k[x_1, \dots, x_n]$

$f \in I(Z(\mathfrak{a}))$  i.e.  $\forall P \in Z(\mathfrak{a}), f(P) = 0 \implies \exists r \in \mathbb{N}, f^r \in \mathfrak{a}$

*Proof.* Since  $A$  is a Noetherian ring, there exists  $F_i \in A$  ( $1 \leq i \leq r$ ) s.t.  $\mathfrak{a} = (F_1, \dots, F_r)$ . Suppose that  $G \in I(Z(\mathfrak{a})) = I(Z((F_1, \dots, F_r)))$ . We define an ideal of  $k[X_1, \dots, X_{n+1}]$  as  $\mathfrak{b} = (F_1, \dots, F_r, X_{n+1}G - 1)$ .

If  $\mathfrak{a} \in Z(\mathfrak{a})$ , since  $G \in I(Z(\mathfrak{a}))$  i.e.  $G(\mathfrak{a}) = 0$  then  $G X_{n+1} - 1 = -1 \neq 0$ . Therefore  $Z(\mathfrak{a}) = \emptyset$ . From previous theorem,  $1 \in \mathfrak{b}$ . So, there exists  $A_1, \dots, A_r, B \in k[X_1, \dots, X_{n+1}]$  s.t.

$$\sum_{i=1}^r A_i F_i + B(X_{n+1}G - 1) = 1$$

Let  $Y = 1/X_{n+1}$ , then there exists  $C_1, \dots, C_r, D \in k[X_1, \dots, X_n, Y]$ ,  $N \in \mathbb{N}$  s.t.

$$Y^{-N} \left( \sum_{i=1}^r C_i F_i + D(G - Y) \right) = 1$$

$$\sum_{i=1}^r C_i F_i + D(G - Y) = Y^N$$

Especially  $Y = G$

$$G^N = \sum_{i=1}^r C_i(X_1, \dots, X_n, G) F_i \in \mathfrak{a}$$
■

Lem 1.7.

$$Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$$

*Proof.* Since  $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$ ,  $Z(\mathfrak{a}) \supset \sqrt{\mathfrak{a}}$

Let  $P \in Z(\mathfrak{a})$  and  $F \in \sqrt{\mathfrak{a}}$ , then there exists  $n \in \mathbb{N}$  s.t.  $F^n(P) = 0$ . However since  $A$  is integral domain,  $F(P) = 0$ . ■

Prop 1.8.

- (a)  $T_1 \subset T_2 (\subset A) \implies Z(T_1) \supset Z(T_2)$
- (b)  $Y_1 \subset Y_2 (\subset \mathbb{A}^n) \implies I(Y_1) \supset I(Y_2)$
- (c)  $Y_1, Y_2 \subset \mathbb{A}^n$ ,  $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$
- (d)  $\mathfrak{a} \subset A$ ,  $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$
- (e)  $Y \subset \mathbb{A}^n$ ,  $Z(I(Y)) = \overline{Y}$

*Proof.* (a),(b),(c) are obvious.

(d) Let  $F \in \sqrt{\mathfrak{a}}$ , then  $\forall P \in Z(\mathfrak{a}), \exists n \in \mathbb{N}$  s.t.  $F^n(P) = 0$  i.e.  $F(P) = 0$ , which means  $F \in I(Z(\mathfrak{a}))$ . Inverse proof is Hilbert Nullstellnsatz.

(e)  $Y \subset Z(I(Y))$  is obvious, and  $Z(I(Y))$  is closed set, so  $\overline{Y} \subset Z(I(Y))$

Conversely, let  $W$  be any closed set containing  $Y$ , then  $W = Z(\mathfrak{a})$ , some ideal  $\mathfrak{a}$ .  $Z(\mathfrak{a}) \supset Y$  i.e.  $I(Z(\mathfrak{a})) \subset I(Y)$ . Obviously  $\mathfrak{a} \subset I(Z(\mathfrak{a}))$ , hence  $\mathfrak{a} \subset I(Y)$ , which means  $W = Z(\mathfrak{a}) \supset Z(I(Y))$ . Thus  $Z(I(Y)) = \overline{Y}$  ■

Cor 1.9.

Algebraic sets in  $\mathbb{A}^n \xrightarrow{1:1} \text{Radical ideals in } A$

$$\mathfrak{a}: \text{radical ideal} \xrightarrow{\text{def}} \mathfrak{a} = \sqrt{\mathfrak{a}}$$

Algebraic set  $Y$  is irreducible  $\iff I(Y)$  is prime ideal.

*Proof.* If  $Y$  is irreducible, we show that  $I(Y)$  is prime.

If  $fg \in I(Y)$ , then  $Y \subset Z(fg) = Z(f) \cup Z(g)$ .  $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$ , both being closed subsets of  $Y$ . Since  $Y$  is irreducible,  $Y = Y \cap Z(f)$  namely  $Y \subset Z(f)$  or  $Y \subset Z(g)$ , hence  $f \in I(Y)$  or  $g \in I(Y)$

Conversely, let  $\mathfrak{p}$  be a prime ideal and suppose that  $Z(\mathfrak{p}) = Y_1 \cup Y_2$ , then  $\mathfrak{p} = I(Y_1) \cap I(Y_2)$  ■

Def 1.10.

Let  $Y \subset \mathbb{A}^n$  be affine algebraic set. We define the affine coordinate  $A(Y) := A/I(Y)$

Def 1.11.

$X$ : topological space is Noetherian  $\xrightarrow{\text{def}}$  for any sequence  $Y_1 \supset Y_2 \supset \dots$  of closed subsets,  $\exists r \in \mathbb{N}$  s.t.  $Y_r = Y_{r+1} = \dots$

Prop 1.12.

$X$ : Noetherian topological space, for every nonempty closed subsets can be expressed as following

$$Y = Y_1 \cup Y_2 \cup \dots \cup Y_r$$

$Y_i$ : irreducible. If we require  $Y_i \not\supset Y_j$ ,  $Y_i$  are uniquely determined.

*Proof.* First we show the existence of representation of  $Y$ . Let  $\mathfrak{S}$  be the set of nonempty closed subsets of  $X$  which cannot be written as a finite union of irreducible closed subset. If  $\mathfrak{S}$  is nonempty, since  $X$  is noetherian, there exists minimal element, say  $Y$ . Then  $Y$  is not irreducible. Thus we can write  $Y = Y' \cup Y''$ , where  $Y'$  and  $Y''$  are proper closed subsets of  $Y$ . By minimality of  $Y$ , each of  $Y'$  and  $Y''$  can be expressed as finite union of

closed irreducible subsets, which is contradiction.

Now suppose  $Y = Y'_1 \cup \cdots \cup Y'_s$  is another such representation. Then  $Y'_1 \subset Y_1 \cup \cdots \cup Y_r$ .so

$$Y'_1 = \bigcup (Y'_1 \cap Y_i)$$

But  $Y'_1$  is irreducible, so  $Y'_1 \subset Y_i$  for some  $i$ , similarly  $Y_1 \subset Y'_j$  for some  $j$ . By condition, we find  $Y'_1 = Y_1$ . Proceeding this processes, we obtain the desired result. ■

Cor 1.13. —

Every algebraic set in  $\mathbb{A}^n$  can be expressed uniquely as a union of varities, no one containing another.

Def 1.14. —

$X$ : topological space. We define the dimension of  $X$  (denoted  $\dim X$ ) to be the supremum of all integers  $n$  such that there exists a chain  $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$

Def 1.15. —

In a ring  $A$ , the height of a prime ideals  $\mathfrak{p}$  is the supremum of all integers  $n$  such that there exists a chain  $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$

We define the dimension of  $A$  (Krull dimension) of  $A$  to be the supremum of the heights of all prime ideals.

Prop 1.16. —

If  $Y$  is an affine algebraic set, then the dimension of  $Y$  is equal to the dimension of its affine coordinate ring  $A(Y)$

*Proof.* If  $Y$  is an affine algebraic set in  $\mathbb{A}^n$ , then the closed irreducible subsets of  $Y$  correspond to prime ideals of  $A = k[X_1, \dots, X_n]$  containing  $I(Y)$ , which corresponds to prime ideals of  $A(Y)$ . ■

Thm 1.17. —

Let  $k$  be a field, and let  $B$  be an integral domain which is a finitely generated  $k$ -algebra Then:

- (a) The dimension of  $B$  is equal to the transcendence degree of the quotient field  $K(B)$  of  $B$  over  $k$
- (b) For any prime ideal  $\mathfrak{p}$  in  $B$ , we have

$$\text{height } \mathfrak{p} + \dim B/\mathfrak{p} = \dim B$$

*Proof.* 後回し ■

Prop 1.18. —

The dimension of  $\mathbb{A}^n$  is  $n$

Prop 1.19. —

$Y$ : quasi-affine variety, then  $\dim Y = \dim \bar{Y}$

*Proof.* If

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

is a sequence of closed irreducible subsets of  $Y$

$$\bar{Z}_0 \subsetneq \bar{Z}_1 \subsetneq \cdots \subsetneq \bar{Z}_n$$

is a sequence of closed subsets of  $\bar{Y}$ . So,  $\dim Y \leq \dim \bar{Y}$  ■