

1 Affine Varieties

Def 1.1.

$$A = k[x_1, \dots, x_n]$$

$$Z(T) := \{P \in \mathbb{A}^n \mid \forall f \in T, f(P) = 0\}$$

$Y \subset \mathbb{A}^n$ is algebraic set $\stackrel{\text{def}}{\iff} \exists T \subset A$ s.t. $Y = Z(T)$

Prop 1.2.

The union of two algebraic sets and intersection of any family of algebraic sets are algebraic set.

Proof. If $Y_1 = Z(T_1), Y_2 = Z(T_2)$. then $Y_1 \cup Y_2 = Z(T_1 T_2)$

If $Y_\lambda = Z(T_\lambda)$, then $\bigcap_{\lambda \in \Lambda} Y_\lambda = Z(\bigcup_{\lambda \in \Lambda} T_\lambda)$ ■

Def 1.3.

Zariski topology on $\mathbb{A}^n \stackrel{\text{def}}{\iff} \mathcal{O} = \{Y^c \subset \mathbb{A}^n \mid Y: \text{algebraic set}\}$

$$I(Y) := \{f \in A \mid \forall P \in Y, f(P) = 0\}$$

$Y(\subset X: \text{topological space})$ is irreducible $\stackrel{\text{def}}{\iff} Y$ cannot be expressed as $Y = Y_1 \cup Y_2, (\emptyset \subsetneq Y_1, Y_2 \subsetneq Y: \text{closed})$

Lem 1.4.

Hilbert's basis theorem

$R: \text{Noetherian} \implies R[X]: \text{Noetherian}$

Proof. Let I be an ideal of $R[X]$

$$J := \{a_0 \in R \mid \exists f \in I \text{ s.t. } f(X) = a_0 X^d + \dots + a_d\}$$

in this definition, J can be confirmed as an ideal of R .

\therefore suppose $a_0, b_0 \in J$. By definition, there exists $F(X), G(X) \in I$ s.t.

$$F(X) = a_0 X^r + \dots + a_r$$

$$G(X) = b_0 X^s + \dots + b_s$$

since I is an ideal, $kF(X) \in I$, which means $ka_0 \in J$ and $F(X) + X^{r-s}G(X) \in I$, which means $a_0 + b_0 \in J$

Since R is a Noetherian ring, J is finitely generated. So there exists $a^1, \dots, a^t \in R$ s.t. $J = (a^1, \dots, a^t)$. By definition of J , there exists F_i ($1 \leq i \leq t$) whose leading coefficient is a^i

For $m \geq 0$, we define $J_m \subset J$ as all leading coefficients of polynomial in I of degree at most m . i.e.

$$J_m := \{a_0 \in J \mid r = \deg(f) \leq m, f(X) = a_0 X^r + \dots + a_r, \}$$

J_m can also be verified to be an ideal. Similarly J_m is finitely generated by $a^{m,j}$ ($1 \leq j \leq t_m$) and define similarly $F_{m,j}$ ($m < N, 1 \leq j \leq t_m$).

$$I_0 := (F_i \ (1 \leq i \leq t), F_{m,j} \ (0 \leq m < N, 1 \leq j \leq t_m))$$

Obviously I_0 is finitely generated and $I_0 \subset I$. If we confirm $I \subset I_0$, this proof is over.

Suppose that there exists a polynomial which doesn't belong to I_0 . From these polynomial, we take G as the least degree one and let a be a leading coefficient of G . Since $a \in J$, there exists $k_1, \dots, k_t \in R$ s.t. $a = \sum_{i=1}^t k_i a^i$

We consider two cases $\deg(G) \geq N$ and $m = \deg(G) \leq N$

- In the former case

We define $H_i(X) := k_i X^{N-\deg(F_i)}$.

$$G_0 = G - \sum_{i=1}^t H_i F_i \in I$$

Since $\deg(G_0) \leq N$ and $G \in I$, $G_0 \in I_0$, this means $G = G_0 + \sum_{i=1}^t H_i F_i \in I_0$. Contradiction.

- In the latter case

Since $a \in J_m$, there exists $k_i \in R$ ($1 \leq i \leq t_m$) s.t. $a = \sum_{i=1}^{t_m} k_i a^{(m,i)}$

We define $H_i(X) := k_i X^{N-\deg(F_{m,i})}$.

$$G_0 = G - \sum_{i=1}^t H_i F_{m,i} \in I$$

Similary we can confirm $G \in I_0$. Contradiction. Therefore $I = I_0$, which means I is finitely generated. ■

Lem 1.5.

Weak Hilbert's Nullstellnsatz

k : algebraically closed field, \mathfrak{a} : ideal in $A = k[x_1, \dots, x_n]$

$$Z(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = A$$

Proof. Suppose $\mathfrak{a} \neq A$. Since A is Noetherian, there exists maximal ideal \mathfrak{m} that includes \mathfrak{a} .

A/\mathfrak{m} is isomorphic to some field extension of k , but since k is algebraically closed, $A/\mathfrak{m} = k$. So there exists $a_i \in k$ ($1 \leq i \leq n$) s.t. $X_i - a_i \in \mathfrak{m}$.

$$(X_1 - a_1, \dots, X_n - a_n) \subset \mathfrak{m}$$

However because $(X_1 - a_1, \dots, X_n - a_n)$ is maximal, $\mathfrak{m} = (X_1 - a_1, \dots, X_n - a_n)$

$(a_1, \dots, a_n) \in Z(\mathfrak{m}) \subset Z(\mathfrak{a})$. $Z(\mathfrak{a}) \neq \emptyset$ ■

Thm 1.6.

Hilbert's Nullstellnsatz

k : algebraically closed field, \mathfrak{a} : ideal in $A = k[x_1, \dots, x_n]$

$f \in I(Z(\mathfrak{a}))$ i.e. $\forall P \in Z(\mathfrak{a}), f(P) = 0 \implies \exists r \in \mathbb{N}, f^r \in \mathfrak{a}$

Proof. Since A is a Noetherian ring, there exists $F_i \in A$ ($1 \leq i \leq r$) s.t. $\mathfrak{a} = (F_1, \dots, F_r)$. Suppose that $G \in I(Z(\mathfrak{a})) = I(Z((F_1, \dots, F_r)))$. We define an ideal of $k[X_1, \dots, X_{n+1}]$ as $\mathfrak{b} = (F_1, \dots, F_r, X_{n+1}G - 1)$.

If $\mathfrak{a} \in Z(\mathfrak{a})$, since $G \in I(Z(\mathfrak{a}))$ i.e. $G(\mathfrak{a}) = 0$ then $G X_{n+1} - 1 = -1 \neq 0$. Therefore $Z(\mathfrak{a}) = \emptyset$. From previous theorem, $1 \in \mathfrak{b}$. So, there exists $A_1, \dots, A_r, B \in k[X_1, \dots, X_{n+1}]$ s.t.

$$\sum_{i=1}^r A_i F_i + B(X_{n+1}G - 1) = 1$$

Let $Y = 1/X_{n+1}$, then there exists $C_1, \dots, C_r, D \in k[X_1, \dots, X_n, Y]$, $N \in \mathbb{N}$ s.t.

$$Y^{-N} \left(\sum_{i=1}^r C_i F_i + D(G - Y) \right) = 1$$

$$\sum_{i=1}^r C_i F_i + D(G - Y) = Y^N$$

Especially $Y = G$

$$G^N = \sum_{i=1}^r C_i(X_1, \dots, X_n, G) F_i \in \mathfrak{a}$$
■

Lem 1.7.

$$Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$$

Proof. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, $Z(\mathfrak{a}) \supset \sqrt{\mathfrak{a}}$

Let $P \in Z(\mathfrak{a})$ and $F \in \sqrt{\mathfrak{a}}$, then there exists $n \in \mathbb{N}$ s.t. $F^n(P) = 0$. However since A is integral domain, $F(P) = 0$. ■

Prop 1.8.

- (a) $T_1 \subset T_2 (\subset A) \implies Z(T_1) \supset Z(T_2)$
- (b) $Y_1 \subset Y_2 (\subset \mathbb{A}^n) \implies I(Y_1) \supset I(Y_2)$
- (c) $Y_1, Y_2 \subset \mathbb{A}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$
- (d) $\mathfrak{a} \subset A$, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$
- (e) $Y \subset \mathbb{A}^n$, $Z(I(Y)) = \overline{Y}$

Proof. (a),(b),(c) are obvious.

(d) Let $F \in \sqrt{\mathfrak{a}}$, then $\forall P \in Z(\mathfrak{a}), \exists n \in \mathbb{N}$ s.t. $F^n(P) = 0$ i.e. $F(P) = 0$, which means $F \in I(Z(\mathfrak{a}))$. Inverse proof is Hilbert Nullstellnsatz.

(e) $Y \subset Z(I(Y))$ is obvious, and $Z(I(Y))$ is closed set, so $\overline{Y} \subset Z(I(Y))$

Conversely, let W be any closed set containing Y , then $W = Z(\mathfrak{a})$, some ideal \mathfrak{a} . $Z(\mathfrak{a}) \supset Y$ i.e. $I(Z(\mathfrak{a})) \subset I(Y)$. Obviously $\mathfrak{a} \subset I(Z(\mathfrak{a}))$, hence $\mathfrak{a} \subset I(Y)$, which means $W = Z(\mathfrak{a}) \supset Z(I(Y))$. Thus $Z(I(Y)) = \overline{Y}$ ■

Cor 1.9.

Algebraic sets in $\mathbb{A}^n \xrightarrow{1:1} \text{Radical ideals in } A$

$$\mathfrak{a}: \text{radical ideal} \xrightarrow{\text{def}} \mathfrak{a} = \sqrt{\mathfrak{a}}$$

Algebraic set Y is irreducible $\iff I(Y)$ is prime ideal.

Proof. If Y is irreducible, we show that $I(Y)$ is prime.

If $fg \in I(Y)$, then $Y \subset Z(fg) = Z(f) \cup Z(g)$. $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$, both being closed subsets of Y . Since Y is irreducible, $Y = Y \cap Z(f)$ namely $Y \subset Z(f)$ or $Y \subset Z(g)$, hence $f \in I(Y)$ or $g \in I(Y)$

Conversely, let \mathfrak{p} be a prime ideal and suppose that $Z(\mathfrak{p}) = Y_1 \cup Y_2$, then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$ ■

Def 1.10.

Let $Y \subset \mathbb{A}^n$ be affine algebraic set. We define the affine coordinate $A(Y) := A/I(Y)$

Def 1.11.

X : topological space is Noetherian $\xrightarrow{\text{def}}$ for any sequence $Y_1 \supset Y_2 \supset \dots$ of closed subsets, $\exists r \in \mathbb{N}$ s.t. $Y_r = Y_{r+1} = \dots$

Def 1.12.