

$$(1) \quad \eta_{\mu\nu} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$(2) \quad F_{\mu\nu} := \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

This tensor is called Faraday tensor and satisfies the following equation.

$$(3) \quad \partial^\gamma F^{\beta\nu} + \partial^\beta F^{\nu\gamma} + \partial^\nu F^{\gamma\beta} = 0$$

This means

$$(4) \quad \begin{cases} \nabla \times E = -\frac{\partial B}{\partial t} \\ \nabla \cdot B = 0 \end{cases}$$

$$(5) \quad \mathcal{J}^\nu := \frac{1}{\mu_0} \partial_\mu F^{\mu\nu}$$

If you denote $\mathcal{J} = {}^t(\rho, J) = {}^t(\rho, j_x, j_y, j_z)$

$$(6) \quad \begin{cases} \nabla \cdot E = \mu_0 \rho \\ \nabla \times B = \mu_0 J + \frac{\partial E}{\partial t} \end{cases}$$

$$(7) \quad \mathcal{T}^{\mu\nu} := \frac{1}{\mu_0} \left(F^{\mu\alpha} \eta_{\alpha\beta} F^{\beta\nu} + \frac{1}{4} \eta^{\mu\nu} F^{\beta\gamma} F_{\beta\gamma} \right)$$

This tensor is called energy-momentum tensor.

Thm

$$(8) \quad \partial_\mu \mathcal{T}^{\mu\nu} = \mathcal{J}_\mu F^{\mu\nu}$$

$$\partial_\mu \mathcal{T}^{\mu\nu} = \frac{1}{\mu_0} \left(\partial_\mu F^{\mu\alpha} \eta_{\alpha\beta} F^{\beta\nu} + \frac{1}{4} \partial_\mu \eta^{\mu\nu} F^{\beta\gamma} F_{\beta\gamma} \right)$$