1 Affine Varieties

Def 1.1.

$A = k[x_1, \dots, x_n]$

$$Z(T) \coloneqq \{P \in \mathbb{A}^n \mid \forall f \in T,\ f(P) = 0\}$$

$Y \subset \mathbb{A}^n$ is algebraic set $\overset{\text{def}}{\iff} \exists T \subset A$ s.t. $Y = Z(T)$

Prop 1.2.

The union of two algebraic sets and intersection of any family of algebraic sets are algebraic set.

Proof. If $Y_1 = Z(T_1), Y_2 = Z(T_2)$. then $Y_1 \cup Y_2 = Z(T_1 T_2)$
If $Y_\lambda = Z(T_\lambda)$, then $\bigcap_{\lambda \in \Lambda} Y_\lambda = Z(\bigcup_{\lambda \in \Lambda} T_\lambda)$ ■

Def 1.3.

Zariski topology on $\mathbb{A}^n \overset{\text{def}}{\iff} \mathcal{O} = \{Y^c \subset \mathbb{A}^n \mid Y\colon \text{algebraic set}\}$

$$I(Y) \coloneqq \{f \in A \mid \forall P \in Y,\ f(P) = 0\}$$

$Y (\subset X\colon \text{topological space})$ is irreducible $\overset{\text{def}}{\iff}$ Y cannot be expressed as $Y = Y_1 \cup Y_2$, $(\emptyset \subsetneq Y_1, Y_2 \subsetneq Y\colon \text{closed})$

Lem 1.4.

Hilbert's basis theorem
R: Noetherian $\Longrightarrow R[X]$: Noetherian

Proof. Let I be an ideal of $R[X]$

$$J \coloneqq \{a_0 \in R \mid \exists f \in J \text{ s.t. } f(X) = a_0 X^d + \cdots + a_d\}$$

in this definition, J can be confirmed as an ideal of R.
$\because$suppose $a_0, b_0 \in J$. By definition, there exists $F(X), G(X) \in I$ s.t.

$$F(X) = a_0 X^r + \cdots + a_r$$
$$G(X) = b_0 X^s + \cdots + b_s$$

since I is an ideal, $kF(X) \in I$, which means $ka_0 \in J$ and $F(X) + X^{r-s}G(X) \in I$, which means $a_0 + b_0 \in J$
Since R is a Noetherian ring, J is finitely generated. So there exits $a^1, \dots, a^t \in R$ s.t. $J = (a^1, \dots, a^t)$, By definition of J, there exists F_i $(1 \leq i \leq t)$ whose leading coefficient is a^i
For $m \geq 0$, we define $J_m \subset J$ as all leading coefficients of polynomial in I of degree at most m. i.e.

$$J_m \coloneqq \{a_0 \in J \mid r = \deg(f) \leq m,\ f(X) = a_0 X^r + \cdots + a_r, \}$$

J_m can also be verified to be an ideal. Similary J_m is finitedly generated by $a^{m,j}$ $(1 \leq j \leq t_m)$ and define similarly $F_{m,j}$ $(m < N,\ 1 \leq j \leq t_m)$.

$$I_0 \coloneqq (F_i\ (1 \leq i \leq t),\ F_{m,j}\ (0 \leq m < N,\ 1 \leq j \leq t_m))$$

Obviously I_0 is finitely generated and $I_0 \subset I$. If we confirm $I \subset I_0$, this proof is over.
Suppose that there exists a polynomial which doesn't belong to I_0. From these polynomial, we take G as the least degree one and let a be a leading coefficient of G. Since $a \in J$, there exists $k_1, \dots, k_t \in R$ s.t. $a = \sum_{i=1}^t k_i a^i$
We consider two cases $\deg(G) \geq N$ and $m = \deg(G) \leq N$

- In the former case
  We define $H_i(X) \coloneqq k_i X^{N-\deg(F_i)}$.
  $$G_0 = G - \sum_{i=1}^{t} H_i F_i \in I$$
  Since $\deg(G_0) \leq N$ and $G \in I$, $G_0 \in I_0$, this means $G = G_0 + \sum_{i=1}^{t} H_i F_i \in I_0$. Contradiction.

- In the latter case
  Since $a \in J_m$, there exists $k_i \in R$ $(1 \leq i \leq t_m)$ s.t. $a = \sum_{i=1}^{t_m} k_i a^{(m,i)}$
  We define $H_i(X) \coloneqq k_i X^{N-\deg(F_{m,i})}$.
  $$G_0 = G - \sum_{i=1}^{t} H_i F_{m,i} \in I$$
  Similary we can confirm $G \in I_0$.Contradiction.Therefore $I = I_0$, which means I is finitely generated.

■

Lem 1.5.

Weak Hilbert's Nullstellnsatz
k: algebraically closed field, $\mathfrak{a}$: ideal in $A = k[x_1, \ldots, x_n]$
$$Z(\mathfrak{a}) = \emptyset \Longleftrightarrow \mathfrak{a} = A$$

Proof. Suppose $\mathfrak{a} \neq A$. Since A is Noetherian, there exists maximal ideal $\mathfrak{m}$ that includes $\mathfrak{a}$.
$A/\mathfrak{m}$ is isomorphic to some field extension of k, but since k is algebraically closed, $A/\mathfrak{m} = k$. So there exists $a_i \in k$ $(1 \leq i \leq n)$ s.t. $X_i - a_i \in \mathfrak{m}$.
$$(X_1 - a_1, \ldots, X_n - a_n) \subset \mathfrak{m}$$
However because $(X_1 - a_1, \ldots, X_n - a_n)$ is maximal, $\mathfrak{m} = (X_1 - a_1, \ldots, X_n - a_n)$
$(a_1, \ldots, a_n) \in Z(\mathfrak{m}) \subset Z(\mathfrak{a}) . \therefore Z(\mathfrak{a}) \neq \emptyset$ ■

Thm 1.6.

Hilbert's Nullstellnsatz
k: algebraically closed field, $\mathfrak{a}$: ideal in $A = k[x_1, \ldots, x_n]$
$f \in I(Z(\mathfrak{a}))$ i.e. $\forall P \in Z(\mathfrak{a}),\ f(P) = 0 \Longrightarrow \exists r \in \mathbb{N},\ f^r \in \mathfrak{a}$

Proof. Since A is a Noetherian ring, there exists $F_i \in A$ $(1 \leq i \leq r)$ s.t. $\mathfrak{a} = (F_1, \ldots, F_r)$. Suppose that $G \in I(Z(\mathfrak{a})) = I(Z((F_1, \ldots, F_r)))$. We define an ideal of $k[X_1, \ldots, X_{n+1}]$ as $\mathfrak{b} = (F_1, \ldots, F_n, X_{n+1}G - 1)$.
If $\mathfrak{a} \in Z(\mathfrak{a})$, since $G \in I(Z(\mathfrak{a}))$ i.e. $G(\mathfrak{a}) = 0$ then $GX_{n+1} - 1 = -1 \neq 0$. Therefore $Z(\mathfrak{a}) = \emptyset$. From previous theorem, $1 \in \mathfrak{b}$. So, there exists $A_1, \ldots, A_r, B \in k[X_1, \ldots, X_{n+1}]$ s.t.
$$\sum_{i=1}^{r} A_i F_i + B(X_{n+1}G - 1) = 1$$
Let $Y = 1/X_n$, then there exists $C_1, \ldots, C_r, D \in k[X_1, \ldots, X_n, Y]$, $N \in \mathbb{N}$ s.t.
$$Y^{-N}\left(\sum_{i=1}^{r} C_i F_i + D(G - Y)\right) = 1$$
$$\sum_{i=1}^{r} C_i F_i + D(G - Y) = Y^N$$
Especially $Y = G$
$$G^N = \sum_{i=1}^{r} C_i(X_1, \ldots, X_n, G) F_i \in \mathfrak{a}$$

■

Lem 1.7.

$$Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$$

Proof. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, $Z(\mathfrak{a}) \supset \sqrt{\mathfrak{a}}$
Let $P \in Z(\mathfrak{a})$ and $F \in \sqrt{\mathfrak{a}}$, then there exists $n \in \mathbb{N}$ s.t. $F^n(P) = 0$. However since A is integral domain, $F(P) = 0$. ■

Prop 1.8.

(a) $T_1 \subset T_2(\subset A) \Longrightarrow Z(T_1) \supset Z(T_2)$
(b) $Y_1 \subset Y_2(\subset \mathbb{A}^n) \Longrightarrow I(Y_1) \supset I(Y_2)$
(c) $Y_1, Y_2 \subset \mathbb{A}^n,\ I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$
(d) $\mathfrak{a} \subset A,\ I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$
(e) $Y \subset \mathbb{A}^n,\ Z(I(Y)) = \overline{Y}$

Proof. (a),(b),(c) are obvious.
(d) Let $F \in \sqrt{\mathfrak{a}}$, then $\forall P \in Z(\mathfrak{a}), \exists n \in \mathbb{N}$ s.t. $F^n(P) = 0$ i.e. $F(P) = 0$, which means $F \in I(Z(\mathfrak{a}))$. Inverse proof is Hilbert Nullstellnsatz.
(e) $Y \subset Z(I(Y))$ is obvious, and $Z(I(Y))$ is closed set, so $\overline{Y} \subset Z(I(Y))$
Conversely, let W be any closed set containing Y, then $W = Z(\mathfrak{a})$, some ideal $\mathfrak{a}$. $Z(\mathfrak{a}) \supset Y$ i.e. $I(Z(\mathfrak{a})) \subset I(Y)$. Obviously $\mathfrak{a} \subset I(Z(\mathfrak{a}))$, hence $\mathfrak{a} \subset I(Y)$,which means $W = Z(\mathfrak{a}) \supset Z(I(Y))$. Thus $Z(I(Y)) = \overline{Y}$ ■

Cor 1.9.

$$\text{Algebraic sets in } \mathbb{A}^n \xleftrightarrow{1:1} \text{Radical ideals in } A$$

$$\mathfrak{a}\colon \text{radical ideal} \overset{\text{def}}{\iff} \mathfrak{a} = \sqrt{\mathfrak{a}}$$

$$\text{Algebraic set } Y \text{ is irreducible} \iff I(Y) \text{ is prime ideal.}$$

Proof. If Y is irreducible, we show that $I(Y)$ is prime.
If $fg \in I(Y)$, then $Y \subset Z(fg) = Z(f) \cup Z(g)$. $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$, both being closed subsets of Y. Since Y is irreducible, $Y = Y \cap Z(f)$ namely $Y \subset Z(f)$ or $Y \subset Z(g)$, hence $f \in I(Y)$ or $g \in I(Y)$
Conversely, let $\mathfrak{p}$ be a prime ideal and suppose that $Z(\mathfrak{p}) = Y_1 \cup Y_2$, then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$ ■

Def 1.10.

Let $Y \subset \mathbb{A}^n$ be affine algebraic set. We define the affine coordinate $A(Y) \coloneqq A/I(Y)$

Def 1.11.

$X\colon$:topological space is Noetherian $\overset{\text{def}}{\iff}$ for any sequence $Y_1 \supset Y_2 \supset \cdots$ of closed subsets, $\exists r \in \mathbb{N}$ s.t. $Y_r = Y_{r+1} = \cdots$

Prop 1.12.

$X\colon$ Noetherian topological space, for every nonempty closed subsets can be expressed as following

$$Y = Y_1 \cup Y_2 \cdots \cup Y_r$$

$Y_i\colon$ irreducible. If we require $Y_i \not\supset Y_j$, Y_i are uniquely determined.

Proof. First we show the existence of representation of Y. Let $\mathfrak{S}$ be the set of nonempty closed subsets of X which cannot be written as a finite union of irreducible closed subset. If $\mathfrak{S}$ is nonempty, since X is noetherian, there exists minimal element, say Y. Then Y is not irreducible. Thus we can write $Y = Y' \cup Y''$, where Y' and Y'' are proper closed subsets of Y. By minimality of Y, each of Y' and Y'' can be expressed as finite union of

closed irreducible subsets, which is contradiction.
Now suppose $Y = Y_1' \cup \cdots Y_s'$ is another such representation. Then $Y_1' \subset Y_1 \cup \cdots Y_r$.so

$$Y_1' = \bigcup (Y_1' \cap Y_i)$$

But Y_1' is irreducible, so $Y_1' \subset Y_i$ for some i, similarly $Y_1 \subset Y_j'$ for some j. By condition, we find $Y_1' = Y_1$. Proceeding this processes, we obtain the desired result. ■

Cor 1.13.

Every algebraic set in $\mathbb{A}^n$ can be expressed uniquely as a union of varities, no one containing another.

Def 1.14.

X: topological space. We define the dimension of X (denoted $\dim X$) to be the supremum of all integers n such that there exists a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$

Def 1.15.

In a ring A, the height of a pirme ideals $\mathfrak{p}$ is the supremum of all integers n such that there exists a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$
We define the dimension of A (Krull dimension) of A to be the supremum of the heights of all prime ideals.

Prop 1.16.

If Y is an affine algebraic set, then the dimension of Y is equal to the dimension of its affine coordinate ring $A(Y)$

Proof. If Y is an affine algebraic set in $\mathbb{A}^n$, then the closed irreducible subsets of Y corresopond to prime ideals of $A = k[X_1, \ldots, X_n]$ containing $I(Y)$, which corresoponds to prime ideals of $A(Y)$. ■

Thm 1.17.

Let k be a field, and let B be an integral domain which is a finitely generated k-algebra Then:

(a) The dimension of B is equal to the transcendence degree of the quotient field $K(B)$ of B over k
(b) For any prime ideal $\mathfrak{p}$ in B, we have

$$\text{height } \mathfrak{p} + \dim B/\mathfrak{p} = \dim B$$

Proof. ■