1 Affine Varieties

Def 1.1.

$A = k[x_1, \dots, x_n]$

$$Z(T) \coloneqq \{P \in \mathbb{A}^n \mid \forall f \in T,\ f(P) = 0\}$$

$Y \subset \mathbb{A}^n$ is algebraic set $\overset{\text{def}}{\iff} \exists T \subset A$ s.t. $Y = Z(T)$

Prop 1.2.

The union of two algebraic sets and intersection of any family of algebraic sets are algebraic set.

Proof. If $Y_1 = Z(T_1), Y_2 = Z(T_2)$. then $Y_1 \cup Y_2 = Z(T_1 T_2)$
If $Y_\lambda = Z(T_\lambda)$, then $\bigcap_{\lambda \in \Lambda} Y_\lambda = Z(\bigcup_{\lambda \in \Lambda} T_\lambda)$ ■

Def 1.3.

Zariski topology on $\mathbb{A}^n \overset{\text{def}}{\iff} \mathcal{O} = \{Y^c \subset \mathbb{A}^n \mid Y\colon \text{algebraic set}\}$

$$I(Y) \coloneqq \{f \in A \mid \forall P \in Y,\ f(P) = 0\}$$

$Y (\subset X\colon$ topological space$)$ is irreducible$\overset{\text{def}}{\iff}$ Y cannot be expressed as $Y = Y_1 \cup Y_2,\ (\emptyset \subsetneq Y_1, Y_2 \subsetneq Y\colon \text{closed})$

Lem 1.4.

Hilbert's basis theorem
$R\colon$ Noetherian $\Longrightarrow R[X]\colon$ Noetherian

Proof. Let I be an ideal of $R[X]$

$$J \coloneqq \{a_0 \in R \mid \exists f \in J \text{ s.t. } f(X) = a_0 X^d + \cdots + a_d\}$$

in this definition, J can be confirmed as an ideal of R.
$\because$suppose $a_0, b_0 \in J$. By definition, there exists $F(X), G(X) \in I$ s.t.

$$F(X) = a_0 X^r + \cdots + a_r$$
$$G(X) = b_0 X^s + \cdots + b_s$$

since I is an ideal, $kF(X) \in I$, which means $ka_0 \in J$ and $F(X) + X^{r-s}G(X) \in I$, which means $a_0 + b_0 \in J$
Since R is a Noetherian ring, J is finitely generated. So there exits $a^1, \dots, a^t \in R$ s.t. $J = (a^1, \dots, a^t)$, By definition of J, there exists $F_i\ (1 \le i \le t)$ whose leading coefficient is a^i
For $m \ge 0$, we define $J_m \subset J$ as all leading coefficients of polynomial in I of degree at most m. i.e.

$$J_m \coloneqq \{a_0 \in J \mid r = \deg(f) \le m,\ f(X) = a_0 X^r + \cdots + a_r,\}$$

J_m can also be verified to be an ideal. Similary J_m is finitedly generated by $a^{m,j}$ $(1 \le j \le t_m)$ and define similarly $F_{m,j}$ $(m < N,\ 1 \le j \le t_m)$.

$$I_0 \coloneqq (F_i\ (1 \le i \le t),\ F_{m,j}\ (0 \le m < N,\ 1 \le j \le t_m))$$

Obviously I_0 is finitely generated and $I_0 \subset I$. If we confirm $I \subset I_0$, this proof is over.
Suppose that there exists a polynomial which doesn't belong to I_0. From these polynomial, we take G as the least degree one and let a be a leading coefficient of G. Since $a \in J$, there exists $k_1, \dots, k_t \in R$ s.t. $a = \sum_{i=1}^t k_i a^i$
We consider two cases $\deg(G) \ge N$ and $m = \deg(G) \le N$

- In the former case
  We define $H_i(X) \coloneqq k_i X^{N-\deg(F_i)}$.

$$G_0 = G - \sum_{i=1}^{t} H_i F_i \in I$$

  Since $\deg(G_0) \leq N$ and $G \in I$, $G_0 \in I_0$, this means $G = G_0 + \sum_{i=1}^{t} H_i F_i \in I_0$. Contradiction.

- In the latter case
  Since $a \in J_m$, there exists $k_i \in R$ $(1 \leq i \leq t_m)$ s.t. $a = \sum_{i=1}^{t_m} k_i a^{(m,i)}$
  We define $H_i(X) \coloneqq k_i X^{N-\deg(F_{m,i})}$.

$$G_0 = G - \sum_{i=1}^{t} H_i F_{m,i} \in I$$

  Similary we can confirm $G \in I_0$.Contradiction.Therefore $I = I_0$, which means I is finitely generated.

■

Lem 1.5.

Weak Hilbert's Nullstellnsatz
k: algebraically closed field, $\mathfrak{a}$: ideal in $A = k[x_1, \dots, x_n]$

$$Z(\mathfrak{a}) = \emptyset \Longleftrightarrow \mathfrak{a} = A$$

Proof. Suppose $\mathfrak{a} \neq A$. Since A is Noetherian, there exists maximal ideal $\mathfrak{m}$ that includes $\mathfrak{a}$.
$A/\mathfrak{m}$ is isomorphic to some field extension of k, but since k is algebraically closed, $A/\mathfrak{m} = k$. So there exists $a_i \in k$ $(1 \leq i \leq n)$ s.t. $X_i - a_i \in \mathfrak{m}$.

$$(X_1 - a_1, \dots, X_n - a_n) \subset \mathfrak{m}$$

However because $(X_1 - a_1, \dots, X_n - a_n)$ is maximal, $\mathfrak{m} = (X_1 - a_1, \dots, X_n - a_n)$
$(a_1, \dots, a_n) \in Z(\mathfrak{m}) \subset Z(\mathfrak{a}) \therefore Z(\mathfrak{a}) \neq \emptyset$ ■

Thm 1.6.

Hilbert's Nullstellnsatz
k: algebraically closed field, $\mathfrak{a}$: ideal in $A = k[x_1, \dots, x_n]$
$f \in I(Z(\mathfrak{a}))$ i.e. $\forall P \in Z(\mathfrak{a}),\ f(P) = 0 \Longrightarrow \exists r \in \mathbb{N},\ f^r \in \mathfrak{a}$

Proof. Since A is a Noetherian ring, there exists $F_i \in A$ $(1 \leq i \leq r)$ s.t. $\mathfrak{a} = (F_1, \dots, F_r)$. Suppose that $G \in I(Z(\mathfrak{a})) = I(Z((F_1, \dots, F_r)))$. We define an ideal of $k[X_1, \dots, X_{n+1}]$ as $\mathfrak{b} = (F_1, \dots, F_n, X_{n+1}G - 1)$.
If $\mathfrak{a} \in Z(\mathfrak{a})$, since $G \in I(Z(\mathfrak{a}))$ i.e. $G(\mathfrak{a}) = 0$ then $GX_{n+1} - 1 = -1 \neq 0$. Therefore $Z(\mathfrak{a}) = \emptyset$. From previous theorem, $1 \in \mathfrak{b}$. So, there exists $A_1, \dots, A_r, B \in k[X_1, \dots, X_{n+1}]$ s.t.

$$\sum_{i=1}^{r} A_i F_i + B(X_{n+1}G - 1) = 1$$

Let $Y = 1/X_n$, then there exists $C_1, \dots, C_r, D \in k[X_1, \dots, X_n, Y]$, $N \in \mathbb{N}$ s.t.

$$Y^{-N} \left(\sum_{i=1}^{r} C_i F_i + D(G - Y) \right) = 1$$

$$\sum_{i=1}^{r} C_i F_i + D(G - Y) = Y^N$$

Especially $Y = G$

$$G^N = \sum_{i=1}^{r} C_i(X_1, \dots, X_n, G) F_i \in \mathfrak{a}$$

■

Lem 1.7.

$$Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$$

Proof. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, $Z(\mathfrak{a}) \supset \sqrt{\mathfrak{a}}$
Let $P \in Z(\mathfrak{a})$ and $F \in \sqrt{\mathfrak{a}}$, then there exists $n \in \mathbb{N}$ s.t. $F^n(P) = 0$. However since A is integral domain, $F(P) = 0$. ■

Prop 1.8.

(a) $T_1 \subset T_2 (\subset A) \Longrightarrow Z(T_1) \supset Z(T_2)$
(b) $Y_1 \subset Y_2 (\subset \mathbb{A}^n) \Longrightarrow I(Y_1) \supset I(Y_2)$
(c) $Y_1, Y_2 \subset \mathbb{A}^n,\ I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$
(d) $\mathfrak{a} \subset A,\ I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$
(e) $Y \subset \mathbb{A}^n,\ Z(I(Y)) = \overline{Y}$

Proof. (a),(b),(c) are obvious.
(d) Let $F \in \sqrt{\mathfrak{a}}$, then $\forall P \in Z(\mathfrak{a}), \exists n \in \mathbb{N}$ s.t. $F^n(P) = 0$ i.e. $F(P) = 0$, which means $F \in I(Z(\mathfrak{a}))$. Inverse proof is Hilbert Nullstellnsatz.
(e) ■

Def 1.9.

$X :$:topological space is Noetherian $\overset{\text{def}}{\Longleftrightarrow}$ for any sequence $Y_1 \supset Y_2 \supset \cdots$ of closed subsets, $\exists r \in \mathbb{N}$ s.t. $Y_r = Y_{r+1} = \cdots$