1 semi-orthogonal decomposition

Def

(i) We define right(left) orthogonal sub-category as following

$$\mathcal{C}^{\perp} := \{F \in \mathcal{D} \mid \forall E \in \mathcal{C}, \mathrm{Hom}_{\mathcal{D}}(E, F) = 0\}$$
$$^{\perp}\mathcal{C} := \{E \in \mathcal{D} \mid \forall F \in \mathcal{C}, \mathrm{Hom}_{\mathcal{D}}(E, F) = 0\}$$

(ii) $\mathcal{C}$ is strictly full subcategory of $\mathcal{D}$ iff $\forall E \in \mathcal{C}$ and $F \in \mathcal{D}$ s.t. $E \simeq F \Longrightarrow F \in \mathcal{C}$

(iii) Thick closure of $\mathcal{C}$ in $\mathcal{D}$ is the minimum strictly full subcategory of $\mathcal{D}$ which contains $\mathcal{C}$ and is closed under taking summands.

(iv)

Semi-orthogonal decomposition

$i < j \Longrightarrow \mathcal{C}_i \subset \mathcal{C}_j^{\perp}$, $C_i \in \mathcal{C}_i$

$$\begin{tikzcd}
0 \arrow[rr] & & F_m \arrow[r] \arrow[dl] & \cdots \arrow[r] & F_3 \arrow[rr] & & F_2 \arrow[rr] \arrow[dl] & & F \arrow[dl] \\
& C_m \arrow[ul, dotted, "{[1]}"] & & & & C_2 \arrow[ul, dotted, "{[1]}"] & & C_1 \arrow[ul, dotted, "{[1]}"] &
\end{tikzcd}$$

We denote

$$\mathcal{D} = \langle \mathcal{C}_1, \ldots, \mathcal{C}_m \rangle$$

Thm 1.1.

C_i $(1 \leq i \leq m)$ are unique up to isomorphism and

$$p_i \colon \mathcal{D} \to \mathcal{C}_i \;\; F \mapsto C_i$$
$$p_{i,m} \colon \mathcal{D} \to \langle \mathcal{C}_i, \ldots, \mathcal{C}_m \rangle \;\; F \mapsto F_i$$

is functorial.

Proof. Since $\mathcal{C}_m$ is a strict full subcategory, $F_m \in \mathcal{C}_m$. $\forall E \in \mathcal{C}_j$ $(1 \leq j < m)$, $\mathrm{Hom}_{\mathcal{D}}(F_m, E) = 0$

$$F_m \longrightarrow F_{m-1} \longrightarrow C_{m-1} \longrightarrow F_m[1] \quad \text{(exact.tri.)}$$

Take $E \in \mathcal{C}_j$ $(j < m-1)$ and acting cohomological functor $\mathrm{Hom}_{\mathcal{D}}(-, E)$ this exact triangle,

$$\mathrm{Hom}_{\mathcal{D}}(C_{m-1}, E) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_{m-1}, E) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(C_m, E) \quad \text{(exact)}$$

By definition, since both the left and right sides are 0, $\mathrm{Hom}_{\mathcal{D}}(F_{m-1}, E) = 0$
Repeating this procedure, we obtain $\forall E \in \mathcal{C}_i$ $(1 \leq i < j)\, \mathrm{Hom}_{\mathcal{D}}(F_j, E) = 0$

$$\begin{tikzcd}
0 \arrow[rr] & & F_m \arrow[r] \arrow[dl] & \cdots \arrow[r] & F_3 \arrow[rr] & & F_2 \arrow[rr] \arrow[dl] & & F \arrow[dl] \arrow[dd, "f"] \\
& C_m \arrow[ul, dotted, "{[1]}"] & & & & C_2 \arrow[ul, dotted, "{[1]}"] & & C_1 \arrow[ul, dotted, "{[1]}"] & \\
0 \arrow[rr] & & F'_m \arrow[r] \arrow[dl] & \cdots \arrow[r] & F'_3 \arrow[rr] & & F'_2 \arrow[rr] \arrow[dl] & & F' \arrow[dl] \\
& C'_m \arrow[ul, dotted, "{[1]}"] & & & & C'_2 \arrow[ul, dotted, "{[1]}"] & & C'_1 \arrow[ul, dotted, "{[1]}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& F_2 \arrow[r] \arrow[dl, "0"'] \arrow[drr, "0"] & F \arrow[r] \arrow[d, "f"] & C_1 \arrow[r] & F_2[1] & \text{(exact.tri.)} \\
C'_1[-1] \arrow[r] & F'_2 \arrow[r] & F' \arrow[r] & C'_1 \arrow[r] & F'_2[1] & \text{(exact.tri.)}
\end{tikzcd}$$

From previous lemma, there uniquely exists morphisms which commute the diagram.

$$\begin{tikzcd}
 & F_2 \arrow[r] \arrow[d, dotted, "\exists! p_{2,m}(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & F \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & C_1 \arrow[r] \arrow[d, dotted, "\exists! p_1(f)"] & F_2[1] & \text{(exact.tri.)} \\
\text{"}C_1'[-1] \arrow[r] & F_2' \arrow[r] & F' \arrow[r] & C_1' \arrow[r] & F_2'[1] & \text{(exact.tri.)}
\end{tikzcd}$$

This indicates $p_1(g) \circ p_1(f) = p_1(g \circ f)$ and $p_{2,m}(g) \circ p_{2,m}(f) = p_{2,m}(g \circ f)$. ■

Thm 1.2.

If there exists SOD of $\mathcal{D} = \langle \mathcal{C}_1, \dots, C_m \rangle$

$$K(\mathcal{D}) \to K(\mathcal{C}_1) \oplus \cdots \oplus K(\mathcal{C}_m) \qquad [F] \mapsto (p_1(F), \dots, p_m(F))$$

Proof. (well-defined)

Since

$$E \longrightarrow F \longrightarrow G \longrightarrow E[1] \quad \text{(exact)} \Longrightarrow p_1(E) \longrightarrow p_1(F) \longrightarrow p_1(G) \longrightarrow p_1(E[1]) \quad \text{(exact)}$$

$$[F] - [E] - [G] = 0 \Longrightarrow [p_i(F)] - [p_i(E)] - [p_i(G)] = 0$$

(injection)

$$\begin{aligned}
[F] &\sim [F_2] + [C_1] \\
&\sim [F_3] + [C_2] + [C_1] \\
&\vdots \\
&\sim [C_m] + \cdots + [C_1]
\end{aligned}$$

Therefore $(p_1(F), \dots, p_m(F)) = 0$ means $[F] = 0$

(surjection) ■

Lem 1.3.

$$B \sqcup_A C \simeq \operatorname{Cok} \begin{pmatrix} -f \\ g \end{pmatrix}$$

Proof.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "i_B"] \\
C \arrow[r, "i_C"'] & P
\end{tikzcd}$$

$$q \colon B \oplus C \to (B \oplus C)/\operatorname{Im} \begin{pmatrix} -f \\ g \end{pmatrix} =: P$$

$$i_B \coloneqq q \circ \iota_B \qquad i_C \coloneqq q \circ \iota_C$$

$$i_B \circ f - i_C \circ g = q \circ \iota_B \circ f - q \circ \iota_C \circ g = q \circ \begin{pmatrix} -f \\ g \end{pmatrix} = -q \circ \begin{pmatrix} -f \\ g \end{pmatrix} = 0$$

Let $j_B \colon B \to Q$, $j_C \colon C \to Q$ s.t. $j_B \circ f = j_C \circ g$ and $j \coloneqq (j_B, j_C)$, then

$$j \circ (-f, g) = -j_B \circ f + j_C \circ g = 0$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "i_B"] \arrow[ddr, bend left, "j_B"] & \\
C \arrow[r, "i_C"'] \arrow[drr, bend right, "j_C"'] & P \arrow[dr, dashed, "\exists! u"] & \\
 & & Q
\end{tikzcd}$$

By universal property of cokernel, there uniquely exists u commutes the diagram. ■

Thm 1.4.

(i) Define $\mathcal{D}' \subset \mathcal{D}$ as $C_i \in \mathcal{C}_i$

(ii)

$$\chi = (\chi_{ij})$$

$$\chi_{ij} := \sum_{p \in \mathbb{Z}} (-1)^p \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(E_i, E_j[p])$$