Algebra Objects in Near-Group Fusion Categories

Thomas Dunmore

Supervisor: Assoc. Prof. Pinhas Grossman

Co-supervisor: Dr. Anna Romanov

University of New South Wales

t.dunmore@unsw.edu.au

2025/12/09

Overview

Fusion Categories

What Is a Fusion Category?

I like to think of fusion categories as generalizations of the category Rep(G) of finite-dimensional representations of a finite group G over $\mathbb{C}$. To see how this works, let's see what kind of properties Rep(G) has.

- **Linear**: hom-sets are vector spaces.
- **Semisimple**: every representation is a direct sum of irreducibles.
- **Finite**: there are only finitely many irreducibles.
- **Simple unit**: the trivial representation, $\mathbb{1}$, is irreducible.
- **Monoidal**: we have (linear) tensor products with unit $\mathbb{1}$.
- **Rigid**: every representation X has a contragredient dual X^*.

Definition (Fusion Category)

A category with these adjectives is called a fusion category.

Classical Examples of Fusion Categories

Example (Graded Vector Spaces)

The category Vec_G of finite-dimensional vector spaces (over $\mathbb{C}$, say) graded by a finite group G is a fusion category. This category has a simple object for each $g \in G$, monoidal product $g \otimes h \cong gh$ and duals $g^* = g^{-1}$. One obtains distinct fusion categories by "twisting" $\otimes$ by 3-cocycles of G.

Example (Quantum $\mathfrak{sl}_2$)

Given a Lusztig quantum group $U_q^L(\mathfrak{g})$ for some finite-dimensional simple Lie algebra $\mathfrak{g}$ and 2ℓ-th root of unity q, one builds a fusion category by taking a certain quotient of its category of representations. For $\mathfrak{g} = \mathfrak{sl}_2$, we get simple objects $\{W_i\}_{i=0}^{\ell-2}$ satisfying the truncated Clebsch-Gordon rules,

$$W_i \otimes W_j \cong \bigoplus_{k=\max\{i+j-\ell+2,0\}}^{\min\{i,j\}} W_{i+j-2k}.$$

Near-Group Fusion Categories

In general, new fusion categories are very hard to find! Almost all known examples come from those shown above via some kind of construction. One notable class of exceptions is the near-group fusion categories.

Definition (Near-Group Fusion Categories)

A fusion category with only one non-invertible object X (so $X^* \otimes X \not\cong \mathbb{1}$) is called near-group. Then $X \otimes X \cong \bigoplus_{g \in G} g \oplus X^{\oplus m}$, where G is its group of invertibles and $m \in \mathbb{Z}_{\geq 0}$. We say such a near-group has type (G, m).

We are most interested in the case where G is Abelian and $m \in \mathbb{Z}_{>0}|G|$. The other cases have been classified (at least in the unitary setting) by Tambara-Yamagami, Izumi and Evans-Gannon.

Example (First Examples of Near-Groups)

The category $\text{Rep}(S_3)$ is near-group of type $(\mathbb{Z}/2\mathbb{Z}, 1)$, while the Fibonnaci and Yang-Lee fusion categories are near-group of type $(\{\mathbb{1}\}, 1)$.

Module Categories

What Is a Module Category?

All fusion categories define a ring with addition $\oplus$ and multiplication $\otimes$. One can similarly categorify the notion of modules over these rings.

> **Definition (Fusion Module Category)**
>
> A module category over a fusion category $\mathcal{C}$ is a linear, semisimple, finite, category $\mathcal{M}$ with a linear monoidal functor $\mathcal{C} \to \text{End}(\mathcal{M})$. If $\mathcal{M}$ is not a direct sum of non-trivial module categories, I will call it fusion.

This is nice, but why bother? Well, there are several reasons...

- Knowing which modules can and can't exist can place additional combinatorial constraints on the existence of a fusion category.
- The dual category $\mathcal{C}_{\mathcal{M}}^* := \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ of $\mathcal{C}$-module endofunctors of a fusion module category $\mathcal{M}$ is always a fusion category.

Unfortunately, finding new module categories from first principles is often just as hard as finding new fusion categories. Luckily, there's another way.

Algebra Objects and Ostrik's Theorem

Definition (Algebra and Module Objects)

An algebra object is an object A with morphisms $m : A \otimes A \to A$ and $u : \mathbb{1} \to A$ satisfying certain axioms. A module object over A is an object M with an action $a : M \otimes A \to M$ compatible with m and u.

Theorem (Ostrik, 2003)

The category $\mathrm{Mod}_{\mathcal{C}}\text{-}A$ of module objects over a "simple" algebra object $A \in \mathrm{Ob}(\mathcal{C})$ is a fusion module category. Conversely, if M is a simple object in a fusion module category, then $\underline{\mathrm{Hom}}(M, M)$ is a simple algebra object.

Usually, it is significantly easier to come up with potential candidates for algebra objects. In most cases, one can also figure out what the module category should look like.

Examples of Module Categories

Example (Algebras and Modules of Vec_G)

An algebra object in Vec_G is a G-graded associative, unital algebra, and the simple ones are parametrized by pairs (H, ψ) of a subgroup $H \leq G$ and a 2-cocycle ψ of H. The actual object is just the group algebra $\bigoplus_{h \in H} h$ with multiplication "twisted" by ψ, while the corresponding module category looks like $\mathsf{Vec}_{G/H}$ with the usual G-action twisted by ψ.

Example (Dual Fusion Categories of Vec_G)

In any fusion category, the unit $\mathbb{1}$ is an algebra object with dual fusion category $\mathcal{C}$. In Vec_G, the algebra G has dual fusion category $\mathsf{Rep}(G)$.

More interestingly, in $\mathsf{Vec}_{\mathbb{Z}/4\mathbb{Z}}$, the algebra object $\mathbb{Z}/2\mathbb{Z}$ has dual fusion category $\mathsf{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ with non-trivially twisted associativity.

Modules Over Near-Groups

Setting the Stage

Let $\mathcal{C}$ be a near-group of type $(G, m|G|)$ for some finite Abelian group G and $m \in \mathbb{Z}_{>0}$. To help classify its fusion module categories, we can make some observations.

- Any near-group contains (untwisted!) Vec_G as a fusion subcategory.
- Any fusion module category over $\mathcal{C}$ restricts to a (not necessarily indecomposable) Vec_G-module category.
- Any simple algebra object in $\mathcal{C}$ restricts to one in Vec_G, and simple algebra objects in Vec_G lift to ones in $\mathcal{C}$.

The upshot is that our classification of simple algebra objects in Vec_G tells us that every simple algebra object in $\mathcal{C}$ is $H \oplus X^{\oplus a}$ for some $a \in \mathbb{Z}_{\geq 0}$, and every fusion $\mathcal{C}$-module category is a sum of fusion Vec_G-module categories.

Counting Orbits

Since $\mathcal{C}$ has only one non-invertible object, a fusion $\mathcal{C}$-module category can have at most two "G-orbits". Looking at dimensions reveals we always have exactly two, so every fusion $\mathcal{C}$-module category gives us a pair of "Morita equivalent" algebra objects via the internal end. These objects are related as follows.

Proposition

The category of modules over $H \oplus X^{\oplus a}$ has exactly two G-orbits, with the second corresponding to a simple algebra object $H' \oplus X^{\oplus a'}$ satisfying

1. $\frac{a}{|H|} + \frac{a'}{|H'|} = m$;

2. $\sqrt{\frac{|H||H'|}{|G|} + aa'} \in \mathbb{Z}_{>0}$.

On the decategorified level, this is the end of the story: this completely classifies all so-called "NIM-reps" of the underlying fusion ring.

Annoying Examples

Unfortunately, the picture is more complicated upstairs. Here are some examples to illustrate the sort of issues one can encounter.

Example

Let $G = \mathbb{Z}/4\mathbb{Z}$ and $m = 1$. Then $(G, \mathbb{1} \oplus X)$ corresponds to a unique module category, but $(G, G \oplus X^{\oplus 4})$ and $(\mathbb{Z}/2\mathbb{Z} \oplus X, G \oplus X^{\oplus 2})$ do not.

Things are slightly different if we consider the other group of order 4.

Example

Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $m = 1$. Then $(G, \mathbb{1} \oplus X)$ and $(G, G \oplus X^{\oplus 4})$ both correspond to unique module categories, coming from the two non-cohomologous 2-cocycles on G.

What about $\mathbb{Z}/2\mathbb{Z} \oplus X$ when $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$? Unlike before, we can't immediately rule it out precisely because $G \oplus X^{\oplus 4}$ is an algebra object.

A Small Classification

Luckily, these problem cases can be ruled out purely combinatorially when we assume $|G|$ is square-free. In this case, we can say the following.

Proposition

Let $m = 1$ and G be a finite cyclic group whose order is square-free. Then the fusion module categories of $\mathcal{C}$ correspond bijectively to those in Vec_G, with the subgroup H corresponding to the pair $(H, |G/H| \oplus X^{\oplus |G/H|})$.

We've already seen that not all algebra objects follow this particular pattern. However, we have yet to find an example of an algebra object that isn't Morita equivalent to a subgroup of G.

Conjecture

For all G and $m \in \mathbb{Z}_{>0}|G|$, the fusion module categories of $\mathcal{C}$ correspond bijectively to fusion module categories of Vec_G.

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Thank you for listening!