

# Ribbon Categories and Reshetikhin-Turaev Invariants

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# A Brief History

In 1984, Vaughan Jones stumbled upon a new polynomial invariant for knots. Somewhat unsatisfyingly, all known definitions were intrinsically 2-dimensional, despite links naturally being 3-dimensional objects.

In 1989, Witten solved this “problem” by constructing the Jones polynomial (and other invariants of links and 3-manifolds) from special kinds of 2 + 1d topological quantum field theories.

The following year, in 1990, Reshetikhin and Turaev gave a mathematical formulation of Witten’s construction in terms of ribbon graphs coloured by representations of quantum groups.

# Categories for the Working Mathematician

Remember from Victor's talk that a category is (loosely) a collection of objects along with a collection of arrows (morphisms) between them.

Our running example will be **Vec**, the category whose objects are **finite-dimensional** vector spaces (say, over  $\mathbb{C}$ ), and whose morphisms are linear maps (morphisms  $\mathbb{C}^m \rightarrow \mathbb{C}^n$  are  $n \times m$  complex matrices).

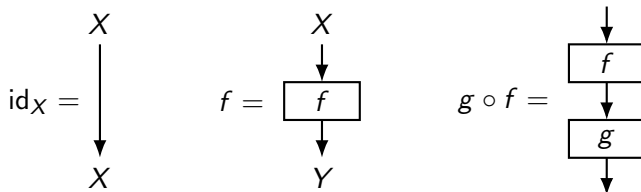
Note that **Vec** has some nice properties that will be important to us.

- It has finitely many simple objects, and every object is a finite direct sum of them. In **Vec**, only the one-dimensional vector space  $\mathbb{C}$  has no non-trivial subspaces.
- The morphisms between any two objects form a finite-dimensional vector space.

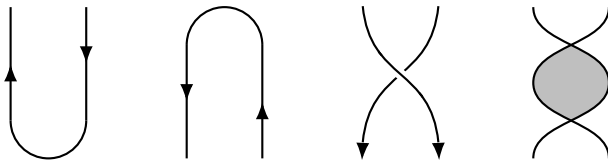
We will ask that all of our categories satisfy these properties.

# The Most Useless Graphical Calculus Ever

In any category, we can represent morphisms using diagrams. Great!

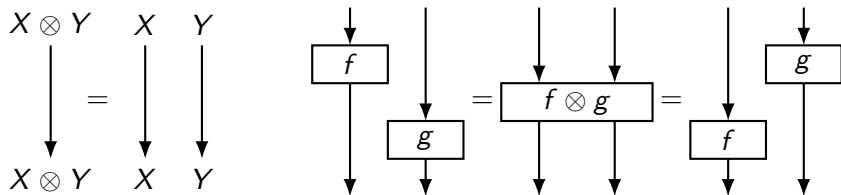


Unfortunately, they tend to be rather boring. If we want to draw knots and links, we're going to need a bit more structure...



# Enter: The Second Dimension

To place strings side by side, we define a **monoidal structure**. That is, we have a product  $\otimes$  with unit  $\mathbb{1}$ , where  $X \otimes Y$  denotes the object whose strands are the strands of  $X$  and  $Y$  placed parallel. Isotopy invariance implies that we must be able to ignore parentheses<sup>†</sup> and strands of  $\mathbb{1}$ , and that morphisms should be able to freely move vertically along strands.



The usual tensor product gives  $\text{Vec}$  a monoidal structure, where  $\mathbb{1} = \mathbb{C}$ .

# Wiggling Strings

To reproduce even the unknot, we're going to need cups and caps. Our strings are oriented though, so we'll be going up after passing through one!

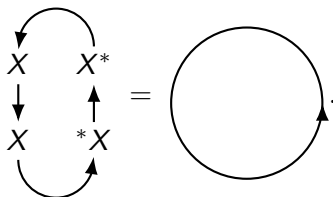
Let  $X^*$  denote  $X$  with its orientation reversed. Then our cups and caps may be represented by morphisms  $\smile : X^* \otimes X \rightarrow \mathbb{1}$ ,  $\frown : \mathbb{1} \rightarrow X \otimes X^*$ . To be invariant under isotopy, the following **zig-zag identities** must hold:

$$\begin{array}{c} \text{Diagram 1: } \downarrow_V \text{ (cup)} \rightarrow \downarrow_V \text{ (cup)} = \downarrow_V \\ \text{Diagram 2: } \uparrow_{V^*} \text{ (cup)} \rightarrow \uparrow_{V^*} \text{ (cup)} = \uparrow_{V^*} \end{array}$$

We call  $X^*$  a **left dual** for  $X$ . Swapping the orientations of cups and caps defines **right duals**,  ${}^*X$ . A category is called **rigid** if every object has both left and right duals, and **pivotal** if these duals coincide in a “natural” way.

# Our First (Boring) Knot

In a pivotal category, we can draw oriented circles:



We will ask that  $\text{End}(\mathbb{1}) \cong \mathbb{C}$  so we can identify links with numbers.

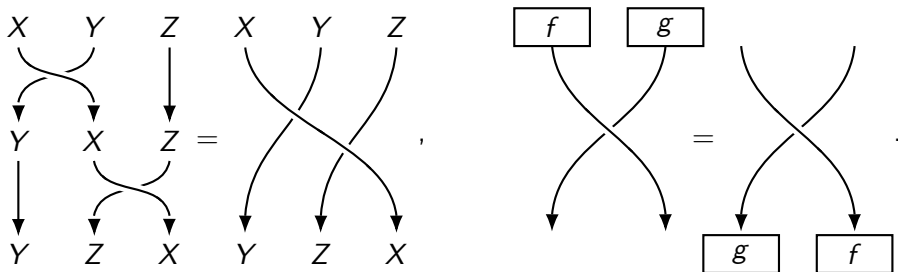
In  $\text{Vec}$ , the duals of  $V$  are given by the **dual space**,  $V^* := \text{Hom}(V, \mathbb{C})$ . Given a basis  $\{v_i\}_{i=1}^n$  for  $V$  with dual basis  $\{v_i^*\}_{i=1}^n$  for  $V^*$ , we have  $\text{ev}_V(v^* \otimes v) = v^*(v)$  and  $\text{coev}_V(1) = \sum_{i=1}^n v_i \otimes v_i^*$ .

The pivotal structure in  $\text{Vec}$  is given by the canonical isomorphism between a vector space and its double dual,  $v \mapsto (v^* \mapsto v^*(v))$ .

# Enter: The Third Dimension

Alexander tells us that every link is given by a braid capped off on both ends. To get the other knots, all we need to do now is define crossings.

A **braiding** is a collection of isomorphisms  $X \otimes Y \rightarrow Y \otimes X$  that are compatible with the monoidal structure (object “pairings” don’t matter) and which allow morphisms to be transported through them. For instance:

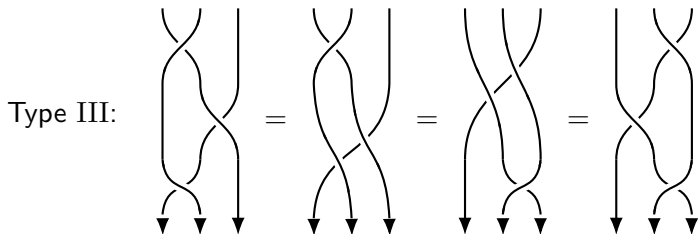
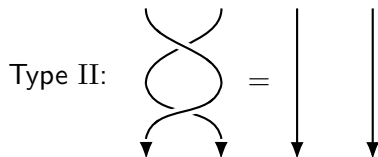


The category  $\text{Vec}$  is braided using the map  $u \otimes v \mapsto v \otimes u$ .



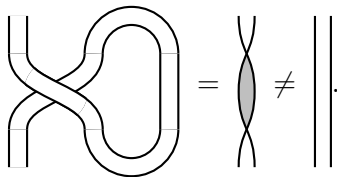
# The Reidemeister Moves

Two knots are equivalent if and only if they are related by a sequence of *Reidemeister moves*. Does our graphical calculus give invariants of knots?

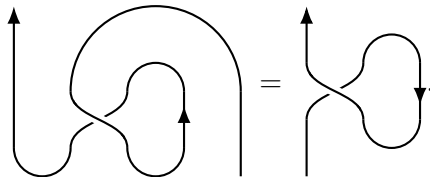


# Twists and Ribbons

Unfortunately, the type I move does not hold in general. The reason is clear if we think of our strands as **ribbons** rather than strings:

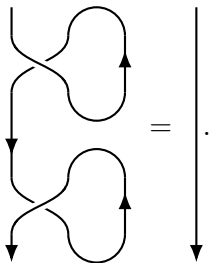


In a pivotal braided category, the diagram above defines a **twist**  $X \rightarrow X$ . We call it a **ribbon structure** if we can pull it out of zig-zags:



# Invariants of Knots and 3-Manifolds

Note that twists satisfy the following **modified** type I Reidemeister move:



The numbers associated to link diagrams coloured by any object give invariants of **framed, oriented links**. This also gives invariants of closed, orientable, connected 3-manifolds by a theorem of Lickorish and Wallace: all such manifolds can be obtained by Dehn surgery on a framed link in  $S^3$ .

# Forgetting the Framing

It turns out that ribbon twists of simple objects are scalings of the identity:

$$\text{Twist of } X = \theta_X \cdot \text{Identity on } X$$

For any link diagram  $L$  coloured by  $X$ ,  $\theta_X^{-\text{Wr}(L)} L$  gives an invariant of  $L$  as an [unframed link](#). Here,  $\text{Wr}(L)$  is the [writhe](#) of  $L$  (the number of overcrossings minus the number of undercrossings).

While one can always forget the framing, forgetting the orientation is trickier. It is not actually sufficient for the object to be self-dual!

# Examples at Long Last

## Example (Vector Spaces)

For  $\mathbb{C}^n$  in  $\text{Vec}$ , we get  $L \mapsto n^c$  for a link  $L$  with  $c$  components. Wow!!

## Example (Graded Vector Spaces)

Consider the category  $\text{Vec}_G$  of vector spaces **graded** by  $G$ . This category has a 1D simple object  $\mathbb{C}_g$  for each  $g \in G$  which satisfy  $\mathbb{C}_g \otimes \mathbb{C}_h = \mathbb{C}_{gh}$  and  $\mathbb{C}_g^* = \mathbb{C}_{g^{-1}}$ . The braidings are given by **quadratic forms** on  $G$ . For instance, if  $G = \mathbb{Z}/2\mathbb{Z} = \{1, g\}$ , we can define  $\smile = i \cdot \text{id}_{\mathbb{C}_1} = \curvearrowright$  and

$$\begin{array}{c} \diagup \quad \diagdown \\ \downarrow \quad \downarrow \end{array} = i \cdot \text{id}_{\mathbb{C}_1}, \quad \begin{array}{c} \diagup \quad \diagdown \\ \downarrow \quad \downarrow \end{array} = i \cdot \text{id}_{\mathbb{C}_g}.$$

The framed link invariant is  $L \mapsto i^{\text{Wr}(L)}$ , so the unframed invariant is trivial. In this example,  $\mathbb{C}_g$  is self-dual, but the orientation cannot be removed!

# The Important Example

## Example (Jones Polynomial)

The category  $\text{Rep}(U_q(\mathfrak{sl}_2))$  for  $q \neq \pm 1$  a root of unity is ribbon. It has a 2D simple object  $V$  that is self-dual with  $\smile = \begin{bmatrix} 0 & 1 & -q^{-1} & 0 \end{bmatrix}$  and  $\frown = \begin{bmatrix} 0 & -q & 1 & 0 \end{bmatrix}^T$ . The braiding and ribbon twist for  $V$  are given by

$$\text{Crossing} = q^{-1/2} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \quad \text{Twist} = -q^{-3/2} \cdot \text{id}_V.$$

It's an easy exercise to show that this satisfies the following skein relation:

$$\text{Crossing} = q^{1/2} \left| \begin{array}{c} \text{Two parallel strands} \end{array} \right| + q^{-1/2} \left| \begin{array}{c} \text{Cap and cup} \end{array} \right|.$$

I've implicitly chosen here the structure that removes the orientation.

# The Jones Polynomial Generalized

We can play this game for other simple representations and Lie algebras  $\mathfrak{g}$ .

## Example (Coloured Jones Polynomial)

If  $\mathfrak{g} = \mathfrak{sl}_2$ , the simple  $n$ -dimensional representations for  $n > 2$  give coloured Jones polynomials.

## Example (HOMFLYPT Polynomial)

If  $\mathfrak{g} = \mathfrak{sl}_n$ , the fundamental representations give the HOMFLYPT polynomial.

## Example (Kauffman Polynomial)

If  $\mathfrak{g} = \mathfrak{so}_n$ , the fundamental representations give the Kauffman polynomial.

# References

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Thank you for listening!