Some Results on Modules over Near-Groups

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2025/08/11

Overview

Fusion Categories

What Is a Fusion Category?

I like to think of fusion categories as generalizations of the category $\mathrm{Rep}(G)$ of finite-dimensional representations of a finite group G over $\mathbb{C}$. To see how this works, let's see what kind of properties $\mathrm{Rep}(G)$ has.

- Linear: hom-sets are vector spaces.
- Semisimple: every representation is a direct sum of irreducibles.
- Finite: there are only finitely many irreducibles.
- Simple unit: the trivial representation, $\mathbb{1}$, is irreducible.
- Monoidal: we have (linear) tensor products with unit $\mathbb{1}$.
- Rigid: every representation X has a contragredient dual X^*.

Definition (Fusion Category)

A category with these adjectives is called a fusion category.

Classical Examples of Fusion Categories

Example (Graded Vector Spaces)

The category Vec_G of finite-dimensional vector spaces (over $\mathbb{C}$, say) graded by a finite group G is a fusion category. This category has a simple object for each $g \in G$, monoidal product $g \otimes h \cong gh$ and duals $g^* = g^{-1}$. One obtains distinct fusion categories by "twisting" $\otimes$ by 3-cocycles of G.

Example (Quantum $\mathfrak{sl}_2$)

Given a Lusztig quantum group $U_q^L(\mathfrak{g})$ for some finite-dimensional simple Lie algebra $\mathfrak{g}$ and 2ℓ-th root of unity q, one builds a fusion category by taking a certain quotient of its category of representations. For $\mathfrak{g} = \mathfrak{sl}_2$, we get simple objects $\{W_i\}_{i=0}^{\ell-2}$ satisfying the truncated Clebsch-Gordon rules,

$$W_i \otimes W_j \cong \bigoplus_{k=\max\{i+j-\ell+2,0\}}^{\min\{i,j\}} W_{i+j-2k}.$$

Near-Group Fusion Categories

In general, new fusion categories are very hard to find! Almost all known examples come from these via some kind of construction. One notable class of exceptions is the near-group fusion categories.

Definition (Near-Group Fusion Categories)

A fusion category with only one non-invertible object X (so $X^* \otimes X \not\cong \mathbb{1}$) is called near-group. Then $X \otimes X \cong \bigoplus_{g \in G} g \oplus X^{\oplus m}$, where G is its group of invertibles and $m \in \mathbb{Z}_{\geq 0}$. We say such a near-group has type (G, m).

We are most interested in the case where G is Abelian and $m \in \mathbb{Z}_{>0}|G|$. The other cases have been classified (at least in the unitary setting) by Tambara-Yamagami, Izumi and Evans-Gannon.

Example (First Examples of Near-Groups)

The category $\mathrm{Rep}(S_3)$ is near-group of type $(\mathbb{Z}/2\mathbb{Z}, 1)$, while the Fibonnaci and Yang-Lee fusion categories are near-group of type $(\{\mathbb{1}\}, 1)$.

Module Categories

What Is a Module Category?

All fusion categories define a ring with addition $\oplus$ and multiplication $\otimes$. One can similarly categorify the notion of modules over these rings.

Definition (Fusion Module Category)

A module category over a fusion category $\mathcal{C}$ is a linear, semisimple, finite, category $\mathcal{M}$ with a linear monoidal functor $\mathcal{C} \to \mathrm{End}(\mathcal{M})$. If $\mathcal{M}$ is not a direct sum of non-trivial module categories, I will call it fusion.

This is nice, but why bother? Well, there are several reasons...

- Knowing which modules can and can't exist can place additional combinatorial constraints on the existence of a fusion category.
- The dual category $\mathcal{C}^*_{\mathcal{M}} := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ of $\mathcal{C}$-module endofunctors of a fusion module category $\mathcal{M}$ is always a fusion category.

Unfortunately, finding new module categories from first principles is often just as hard as finding new fusion categories. Luckily, there's another way.

Algebra Objects and Ostrik's Theorem

Definition (Algebra and Module Objects)

An algebra object is an object A with morphisms $m : A \otimes A \to A$ and $u : \mathbb{1} \to A$ satisfying certain axioms. A module object over A is an object M with an action $a : M \otimes A \to M$ compatible with m and u.

Theorem (Ostrik, 2003)

Given a "simple" algebra object $A \in \mathrm{Ob}(\mathcal{C})$, the category $\mathrm{Mod}_{\mathcal{C}}$-$A$ of A-module objects is a fusion module category, and every fusion module category is of this form.

Usually, it is significantly easier to come up with potential candidates for algebra objects. In most cases, one can also figure out what the module category should look like.

Examples of Module Categories

Example (Algebras and Modules of Vec_G)

An algebra object in Vec_G is a G-graded associative, unital algebra, and the simple ones are parametrized by pairs (H, ψ) of a subgroup $H \leq G$ and a 2-cocycle ψ of H. The actual object is just the group algebra $\bigoplus_{h \in H} h$ with multiplication "twisted" by ψ, while the corresponding module category looks like $\mathrm{Vec}_{G/H}$ with the usual G-action twisted by ψ.

Example (Dual Fusion Categories of Vec_G)

In any fusion category, the unit $\mathbb{1}$ is an algebra object with dual fusion category $\mathcal{C}$. In Vec_G, the algebra G has dual fusion category $\mathrm{Rep}(G)$.

More interestingly, in $\mathrm{Vec}_{\mathbb{Z}/4\mathbb{Z}}$, the algebra object $\mathbb{Z}/2\mathbb{Z}$ has dual fusion category $\mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ with non-trivially twisted associativity.

Modules Over Near-Groups

Setting the Stage

Let $\mathcal{C}$ be a near-group of type $(G, m|G|)$ for some finite Abelian group G and $m \in \mathbb{Z}_{>0}$. To help classify its fusion module categories, we can make some observations.

- Any near-group contains (untwisted!) Vec_G as a fusion subcategory.
- Any fusion module category over $\mathcal{C}$ restricts to a (not necessarily indecomposable) Vec_G-module category.
- Any simple algebra object in $\mathcal{C}$ restricts to one in Vec_G, and simple algebra objects in Vec_G lift to ones in $\mathcal{C}$.

The upshot is that our classification of simple algebra objects in Vec_G tells us that every simple algebra object in $\mathcal{C}$ is $H \oplus X^{\oplus a}$ for some $a \in \mathbb{Z}_{\geq 0}$.

Question: For a given integer a, does $H \oplus X^{\oplus a}$ admit an algebra structure? If so, how many?

Counting Orbits

On the algebra object side, there are a few bounds we can place on a.

Lemma

If $H \oplus X^{\oplus a}$ is a simple algebra object, then $a \leq m|H|$, and $|H| \mid |G/H|a^2$.

On the module category side, while a fusion $\mathcal{C}$-module category may not restrict to a fusion Vec_G-module category, it will restrict to a direct sum of them, and the summands tell us what the action of G is. How many of these "G-orbits" can we have?

Proposition

The category of modules over $H \oplus X^{\oplus a}$ has exactly two G-orbits, with the second corresponding to a simple algebra object $H' \oplus X^{\oplus a'}$ satisfying

1. $\frac{a}{|H|} + \frac{a'}{|H'|} = m$;
2. $\sqrt{\frac{|H'|}{|H|}(|H| + m|G|a - |G/H|a^2)} \in \mathbb{Z}_{>0}$.

Annoying Examples

The conditions in the previous proposition are necessary, but not sufficient! Here is an example to illustrate this sad reality.

Example

Let $G = \mathbb{Z}/4\mathbb{Z}$ and $m = 1$. Then the pair $(\mathbb{Z}/2\mathbb{Z} \oplus X,\ \mathbb{Z}/4\mathbb{Z} \oplus X^{\oplus 2})$ satisfies the criteria of the previous proposition, but one can show that these objects do not admit algebra structures.

Even worse, an object can appear in multiple distinct pairs!

Example

Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $m = 1$. Then the object $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has two distinct simple algebra structures (coming its two non-cohomologous 2-cocycles), with corresponding pairs $(G, \mathbb{1} \oplus X)$ and $(G, G \oplus X^{\oplus 4})$.

A Small Classification

I don't want everyone to leave frightened, so let's conclude with a case where these problems vanish.

Proposition

Let $m = 1$ and G be a finite cyclic group whose order is square-free. Then the fusion module categories of $\mathcal{C}$ correspond bijectively to those in Vec_G.

We have yet to find a near-group category with a simple algebra object that doesn't fit this pattern. Thus, we have the following conjecture.

Conjecture

For all G and m, the fusion module categories of $\mathcal{C}$ correspond bijectively to fusion module categories of Vec_G.

While this conjecture still stands, we have some candidates in mind...

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Thank you for listening!