

# 1. THE $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, 4)$ NEAR-GROUP

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Let  $\mathcal{C}$  denote the (unitary) near-group fusion category of type  $(G, n)$ , where  $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and  $n = |G|$ . More precisely, we will denote  $G = \{1, (g, 1), (1, g), (g, g)\}$  and  $\text{Irr}(\mathcal{C}) = G \sqcup \{X\}$ . Note that  $\text{FPdim}(x) = 1$  for  $x \in G$  and  $\text{FPdim}(X) = 2 + 2\sqrt{2}$ . The aim of this document is to explain the current state of our attempt to classify the simple algebra objects in  $\mathcal{C}$ .

It is easy to show (via the code or just checking manually) that the candidates for simple algebra objects in  $\mathcal{C}$  are as follows:

- (1).  $A_1^1 := G \stackrel{\text{m.e.}}{\sim} G \oplus X^{\oplus 4}$ ,
- (2).  $A_1^2 := \langle (1, g) \rangle \oplus X \stackrel{\text{m.e.}}{\sim} G \oplus X^{\oplus 2}$ ,
- (3).  $A_1^3 := \langle (g, 1) \rangle \oplus X \stackrel{\text{m.e.}}{\sim} G \oplus X^{\oplus 2}$ ,
- (4).  $A_1^4 := \langle (g, g) \rangle \oplus X \stackrel{\text{m.e.}}{\sim} G \oplus X^{\oplus 2}$ ,
- (5).  $A_1^5 := \langle (1, g) \rangle \stackrel{\text{m.e.}}{\sim} \langle (1, g) \rangle \oplus X^{\oplus 2}$ ,
- (6).  $A_1^6 := \langle (1, g) \rangle \stackrel{\text{m.e.}}{\sim} \langle (g, g) \rangle \oplus X^{\oplus 2}$ ,
- (7).  $A_1^7 := \langle (1, g) \rangle \stackrel{\text{m.e.}}{\sim} \langle (g, 1) \rangle \oplus X^{\oplus 2}$ ,
- (8).  $A_1^8 := \langle (g, g) \rangle \stackrel{\text{m.e.}}{\sim} \langle (1, g) \rangle \oplus X^{\oplus 2}$ ,
- (9).  $A_1^9 := \langle (g, g) \rangle \stackrel{\text{m.e.}}{\sim} \langle (g, g) \rangle \oplus X^{\oplus 2}$ ,
- (10).  $A_1^{10} := \langle (g, g) \rangle \stackrel{\text{m.e.}}{\sim} \langle (g, 1) \rangle \oplus X^{\oplus 2}$ ,
- (11).  $A_1^{11} := \langle (g, 1) \rangle \stackrel{\text{m.e.}}{\sim} \langle (1, g) \rangle \oplus X^{\oplus 2}$ ,
- (12).  $A_1^{12} := \langle (g, 1) \rangle \stackrel{\text{m.e.}}{\sim} \langle (g, g) \rangle \oplus X^{\oplus 2}$ ,
- (13).  $A_1^{13} := \langle (g, 1) \rangle \stackrel{\text{m.e.}}{\sim} \langle (g, 1) \rangle \oplus X^{\oplus 2}$ ,
- (14).  $A_1^{14} := 1 \stackrel{\text{m.e.}}{\sim} G \oplus X^{\oplus 4}$ ,
- (15).  $A_1^{15} := G \stackrel{\text{m.e.}}{\sim} 1 \oplus X$ .

Here,  $\langle (g, 1) \rangle := 1 \oplus (g, 1)$  for example, and we write  $\stackrel{\text{m.e.}}{\sim}$  to denote Morita equivalence of algebra objects. Note that all of these, except for  $A_1^2$ ,  $A_1^3$  and  $A_1^4$ , are Morita equivalent subgroups of  $G$ . The main question we are interested in is whether any of these three candidates admit a simple algebra structure, and if so, how many?

I'll divide the rest of this document into three parts.

- (I). Which candidates lifting from  $\text{Vec}_G$  exist?
- (II). What are the dual fusion categories at play?
- (III). Under what circumstances does  $\mathbb{Z}/2\mathbb{Z} \oplus X$  admit a simple algebra structure?

## 2. ALGEBRAS LIFTING FROM $\mathbf{Vec}_G$

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We first try to answer the question of which candidates actually exist when the algebra object is a subgroup of  $G$ . Thanks to the classification of simple algebra objects in  $\mathbf{Vec}_G$ , we know that  $\langle(g, 1)\rangle$ ,  $\langle(1, g)\rangle$  and  $\langle(g, g)\rangle$  admit a single algebra structure, while  $G$  admits two. By direct computation, one can show that  $\mathbb{1} \oplus X$  admits only a single algebra structure, so both  $A_1^1$  and  $A_1^{15}$  must exist. While  $A_1^{15}$  is self-dual (by which we mean their dual fusion categories are not equivalent to  $\mathcal{C}$ ),  $A_1^1$  has many candidates for its dual fusion category.

Determining which of the algebra structures on the copies of  $\mathbb{Z}/2\mathbb{Z}$  exist is a bit trickier. Using the code's `findDualRings()` function, we find that candidates  $A_1^5$ ,  $A_1^9$  and  $A_1^{13}$  cannot be self-dual. The other candidates lifting from copies of  $\mathbb{Z}/2\mathbb{Z}$  must be self-dual, however. Checking bimodule compatibility using the code reveals that we have two potential cases.

- (1). All three of  $A_1^5$ ,  $A_1^9$ ,  $A_1^{13}$  exist.
- (2). Exactly two of  $A_1^5$ ,  $A_1^9$ ,  $A_1^{13}$  exist, and the remaining  $\mathbb{Z}/2\mathbb{Z}$  subgroup is self-dual.

In both cases, the Morita equivalence class of  $\mathcal{C}$  must contain at least one other fusion category.

Is there more we can say here? The Brauer-Picard group could probably help us figure out which case we're in.

### 3. THE DUAL FUSION CATEGORIES

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The code's `findDualRings()` function finds exactly two potential dual fusion categories to  $A_1^5$ ,  $A_1^9$  and  $A_1^{13}$ , which are defined as follows:

$$\begin{aligned} \text{Irr}(\mathcal{D}_1) &= G \sqcup G\{X\}, \\ (g, 1) \otimes X &\cong X \otimes (g, 1), \quad (1, g) \otimes X \cong X \otimes (g, g), \quad (g, g) \otimes X \cong X \otimes (1, g), \\ X \otimes X &\cong \mathbb{1} \oplus X \oplus (g, 1)X; \\ \text{Irr}(\mathcal{D}'_1) &= G \sqcup G\{X\}, \\ (g, 1) \otimes X &\cong X \otimes (g, 1), \quad (1, g) \otimes X \cong X \otimes (g, g), \quad (g, g) \otimes X \cong X \otimes (1, g), \\ X \otimes X &\cong \mathbb{1} \oplus (1, g)X \oplus (g, g)X. \end{aligned}$$

Here,  $\text{FPdim}(X) = 1 + \sqrt{2}$ . Note that the only difference in these rules is  $X \otimes X$ . We suspect that both of these are  $\mathbb{Z}/2\mathbb{Z}$ -graded extensions of the  $\mathbb{Z}/2\mathbb{Z}$  Haagerup-Izumi fusion category, with  $\mathcal{D}_1$  being quasi-trivial.

Both  $\mathcal{D}_1$  and  $\mathcal{D}'_1$  have 6 simple algebra objects lifting from  $\text{Vec}_G$ . This completely classifies the simple algebra objects in  $\mathcal{D}'_1$ . For  $\mathcal{D}_1$ , however, there is an additional unique algebra structure on  $\mathbb{1} \oplus X$  lifting from the  $\mathbb{Z}/2\mathbb{Z}$  Haagerup-Izumi subcategory  $\langle \mathbb{1}, (g, 1), X \rangle$ . The algebra object  $\mathbb{1} \oplus X$  in  $\mathcal{D}'_1$  has a unique candidate for its dual fusion category, which coincides with the unique dual fusion category candidate for  $\mathbb{Z}/2\mathbb{Z} \oplus X$  in  $\mathcal{C}$ . This category is defined as follows:

$$\begin{aligned} \text{Irr}(\mathcal{D}_2) &= \mathbb{Z}/2\mathbb{Z} \sqcup \{X\} \sqcup (\mathbb{Z}/2\mathbb{Z})\{Y\} \sqcup \{Z\}, \\ g \otimes X &\cong X \cong X \otimes g, \quad X \otimes X \cong \mathbb{1} \oplus g, \\ g \otimes Y &\cong gY \cong Y \otimes g, \quad Y \otimes Y \cong \mathbb{1} \oplus Y \oplus gY, \\ g \otimes Z &\cong Z \cong Z \otimes g, \quad Z \otimes Z \cong \mathbb{1} \oplus g \oplus Y^{\oplus 2} \oplus gY^{\oplus 2}, \\ X \otimes Y &\cong X \otimes gY \cong Y \otimes X \cong gY \otimes X \cong Z, \\ X \otimes Z &\cong Z \otimes X \cong Y \oplus gY, \\ Y \otimes Z &\cong Z \otimes Y \cong X \oplus Z^{\oplus 2}. \end{aligned}$$

We have denoted  $\mathbb{Z}/2\mathbb{Z} = \{\mathbb{1}, g\}$  here. Because this is a bit messy, we clarify that  $\langle \mathbb{1}, g, X \rangle$  is a Tambara-Yamagami fusion category, whereas  $\langle \mathbb{1}, g, Y, gY \rangle$  is a  $\mathbb{Z}/2\mathbb{Z}$  Haagerup-Izumi fusion category. We also remark that  $\text{FPdim}(X) = \sqrt{2}$ ,  $\text{FPdim}(Y) = 1 + \sqrt{2}$  and  $\text{FPdim}(Z) = 2 + \sqrt{2}$ .

The simple algebra object candidates in  $\mathcal{D}_2$  are a little tricky. It's perhaps worth mentioning what the simple algebra object candidates in these dual fusion categories are. The case of  $\mathcal{D}'_1$  is easy: as previously mentioned, every candidate lifts from  $\mathbf{Vec}_G$ . In particular, since the number of simple algebra objects is an invariant of Morita equivalence, this tells us that  $\mathbb{Z}/2\mathbb{Z} \oplus X$  cannot have a simple algebra structure in  $\mathcal{C}$  if  $\mathcal{C} \stackrel{\text{m.e.}}{\cong} \mathcal{D}'_1$ . Conversely, if  $\mathcal{C} \stackrel{\text{m.e.}}{\cong} \mathcal{D}_1$ , then we have 7 simple algebra objects since  $\mathbb{1} \oplus X$  lifts uniquely in  $\mathcal{D}_1$  (check), so there must be exactly one simple algebra structure on one of the copies of  $\mathbb{Z}/2\mathbb{Z} \oplus X$  in  $\mathcal{C}$ .

In  $\mathcal{D}_2$ , the simple algebra object candidates are less obvious:

- (1).  $A_2^1 := \mathbb{1} \oplus Y \oplus Z \stackrel{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY \oplus Z^{\oplus 2}$ ,
- (2).  $A_2^2 := \mathbb{1} \oplus gY \oplus Z \stackrel{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY \oplus Z^{\oplus 2}$ ,
- (3).  $A_2^3 := \mathbb{1} \stackrel{\text{m.e.}}{\sim} \mathbb{1} \oplus g \stackrel{\text{m.e.}}{\sim} \mathbb{1} \oplus Y \oplus gY \stackrel{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y^{\oplus 2} \oplus gY^{\oplus 2}$ ,
- (4).  $A_2^4 := \mathbb{1} \oplus gY \stackrel{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY$ ,
- (5).  $A_2^5 := \mathbb{1} \oplus gY \stackrel{\text{m.e.}}{\sim} \mathbb{1} \oplus Y \stackrel{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY$ ,
- (6).  $A_2^6 := \mathbb{1} \oplus Y \stackrel{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY$ .

While most of these lift from the Haagerup-Izumi subcategory,  $A_2^1$  and  $A_2^2$  are exceptions. Mention which of these give Morita equivalences to  $\mathcal{C}$  and which give Morita equivalences to  $\mathcal{D}_1$ .

## 4. WHEN IS $\mathbb{Z}/2\mathbb{Z} \oplus X$ AN ALGEBRA OBJECT?

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Recall that  $\mathcal{C}$  is Morita equivalent either to  $\mathcal{D}_1$  or  $\mathcal{D}'_1$ , where the former has 7 simple algebra objects and the latter has 6. Since the number of simple algebra objects is an invariant of Morita equivalence, we can state the following.

- (1). There is a (unique) simple algebra structure on  $\mathbb{Z}/2\mathbb{Z} \oplus X$  in  $\mathcal{C}$  if and only if  $\mathcal{C} \stackrel{\text{m.e.}}{\cong} \mathcal{D}_1$ .

Recall that  $\mathbb{Z}/2\mathbb{Z} \oplus X \stackrel{\text{m.e.}}{\sim} G \oplus X^{\oplus 2}$ . If we assume that  $G \oplus X^{\oplus 2}$  contains  $G$  as a subalgebra object, then the factoring argument tells us that the dual fusion category of  $G$  contains a simple algebra object with Frobenius-Perron dimension  $\text{FPdim}(G \oplus X^{\oplus 2})/\text{FPdim}(G) = 2 + \sqrt{2}$ . Note that  $\mathcal{C}$  has no object with this dimension, and  $G \stackrel{\text{m.e.}}{\sim} 1 \oplus X$  is self-dual. Thus, we obtain another condition on the existence of an algebra structure on  $\mathbb{Z}/2\mathbb{Z} \oplus X$ .

- (2). If there is a simple algebra structure on  $\mathbb{Z}/2\mathbb{Z} \oplus X$  and it contains  $G$  as a subalgebra object, then this subalgebra object is  $A_1^1 \stackrel{\text{m.e.}}{\sim} G \oplus X^{\oplus 4}$ , and is not self-dual.

The code finds several candidates for the dual fusion category of  $A_1^1$ , including both  $\mathcal{C}$  and  $\mathcal{D}_1$ . Some of these fusion categories (like  $\mathcal{D}_1$ , as well as some others) admit an algebra object candidate of Frobenius-Perron dimension  $2 + \sqrt{2}$ .