1. The $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, 4)$ Near-Group

Let $\mathcal{C}$ denote the (unitary) near-group fusion category of type (G, n), where $G \coloneqq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $n = |G|$. More precisely, we will denote $G = \{\mathbb{1}, (g, 1), (1, g), (g, g)\}$ and $\mathrm{Irr}(\mathcal{C}) = G \sqcup \{X\}$. Note that $\mathrm{FPdim}(x) = 1$ for $x \in G$ and $\mathrm{FPdim}(X) = 2 + 2\sqrt{2}$. The aim of this document is to explain the current state of our attempt to classify the simple algebra objects in $\mathcal{C}$.

It is easy to show (via the code or just checking manually) that the candidates for simple algebra objects in $\mathcal{C}$ are as follows:

(1). $A_1^1 \coloneqq G \overset{\text{m.e.}}{\sim} G \oplus X^{\oplus 4}$,
(2). $A_1^2 \coloneqq \langle (1, g) \rangle \oplus X \overset{\text{m.e.}}{\sim} G \oplus X^{\oplus 2}$,
(3). $A_1^3 \coloneqq \langle (g, 1) \rangle \oplus X \overset{\text{m.e.}}{\sim} G \oplus X^{\oplus 2}$,
(4). $A_1^4 \coloneqq \langle (g, g) \rangle \oplus X \overset{\text{m.e.}}{\sim} G \oplus X^{\oplus 2}$,
(5). $A_1^5 \coloneqq \langle (1, g) \rangle \overset{\text{m.e.}}{\sim} \langle (1, g) \rangle \oplus X^{\oplus 2}$,
(6). $A_1^6 \coloneqq \langle (1, g) \rangle \overset{\text{m.e.}}{\sim} \langle (g, g) \rangle \oplus X^{\oplus 2}$,
(7). $A_1^7 \coloneqq \langle (1, g) \rangle \overset{\text{m.e.}}{\sim} \langle (g, 1) \rangle \oplus X^{\oplus 2}$,
(8). $A_1^8 \coloneqq \langle (g, g) \rangle \overset{\text{m.e.}}{\sim} \langle (1, g) \rangle \oplus X^{\oplus 2}$,
(9). $A_1^9 \coloneqq \langle (g, g) \rangle \overset{\text{m.e.}}{\sim} \langle (g, g) \rangle \oplus X^{\oplus 2}$,
(10). $A_1^{10} \coloneqq \langle (g, g) \rangle \overset{\text{m.e.}}{\sim} \langle (g, 1) \rangle \oplus X^{\oplus 2}$,
(11). $A_1^{11} \coloneqq \langle (g, 1) \rangle \overset{\text{m.e.}}{\sim} \langle (1, g) \rangle \oplus X^{\oplus 2}$,
(12). $A_1^{12} \coloneqq \langle (g, 1) \rangle \overset{\text{m.e.}}{\sim} \langle (g, g) \rangle \oplus X^{\oplus 2}$,
(13). $A_1^{13} \coloneqq \langle (g, 1) \rangle \overset{\text{m.e.}}{\sim} \langle (g, 1) \rangle \oplus X^{\oplus 2}$,
(14). $A_1^{14} \coloneqq \mathbb{1} \overset{\text{m.e.}}{\sim} G \oplus X^{\oplus 4}$,
(15). $A_1^{15} \coloneqq G \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus X$.

Here, $\langle (g, 1) \rangle \coloneqq \mathbb{1} \oplus (g, 1)$ for example, and we write $\overset{\text{m.e.}}{\sim}$ to denote Morita equivalence of algebra objects. Note that all of these, except for A_1^2, A_1^3 and A_1^4, are Morita equivalent subgroups of G. The main question we are interested in is whether any of these three candidates admit a simple algebra structure, and if so, how many?

I'll divide the rest of this document into three parts.

(I). Which candidates lifting from Vec_G exist?
(II). What are the dual fusion categories at play?
(III). Under what circumstances does $\mathbb{Z}/2\mathbb{Z} \oplus X$ admit a simple algebra structure?

2. Algebras Lifting from Vec_G

We first try to answer the question of which candidates actually exist when the algebra object is a subgroup of G. Thanks to the classification of simple algebra objects in Vec_G, we know that $\langle (g,1)\rangle$, $\langle (1,g)\rangle$ and $\langle (g,g)\rangle$ admit a single algebra structure, while G admits two. By direct computation, one can show that $\mathbb{1} \oplus X$ admits only a single algebra structure, so both A_1^1 and A_1^{15} must exist. While A_1^{15} is self-dual (by which we mean their dual fusion categories are not equivalent to $\mathcal{C}$), A_1^1 has many candidates for its dual fusion category.

Determining which of the algebra structures on the copies of $\mathbb{Z}/2\mathbb{Z}$ exist is a bit trickier. Using the code's `findDualRings()` function, we find that candidates A_1^5, A_1^9 and A_1^{13} cannot be self-dual. The other candidates lifting from copies of $\mathbb{Z}/2\mathbb{Z}$ must be self-dual, however. Checking bimodule compatibility using the code reveals that we have two potential cases.

(1). All three of A_1^5, A_1^9, A_1^{13} exist.

(2). Exactly two of A_1^5, A_1^9, A_1^{13} exist, and the remaining $\mathbb{Z}/2\mathbb{Z}$ subgroup is self-dual.

In both cases, the Morita equivalence class of $\mathcal{C}$ must contain at least one other fusion category.

Is there more we can say here? The Brauer-Picard group could probably help us figure out which case we're in.

3. The Dual Fusion Categories

The code's `findDualRings()` function finds exactly two potential dual fusion categories to A_1^5, A_1^9 and A_1^{13}, which are defined as follows:

$$\begin{gathered}\mathrm{Irr}(\mathcal{D}_1) = G \sqcup G\{X\},\\ (g,1)\otimes X \cong X \otimes (g,1), \quad (1,g)\otimes X \cong X\otimes (g,g), \quad (g,g)\otimes X \cong X \otimes (1,g),\\ X\otimes X \cong \mathbb{1} \oplus X \oplus (g,1)X;\end{gathered}$$

$$\begin{gathered}\mathrm{Irr}(\mathcal{D}_1') = G \sqcup G\{X\},\\ (g,1)\otimes X \cong X \otimes (g,1), \quad (1,g)\otimes X \cong X\otimes (g,g), \quad (g,g)\otimes X \cong X \otimes (1,g),\\ X\otimes X \cong \mathbb{1} \oplus (1,g)X \oplus (g,g)X.\end{gathered}$$

Here, $\mathrm{FPdim}(X) = 1+\sqrt{2}$. Note that the only difference in these rules is $X \otimes X$. We suspect that both of these are $\mathbb{Z}/2\mathbb{Z}$-graded extensions of the $\mathbb{Z}/2\mathbb{Z}$ Haagerup-Izumi fusion category, with $\mathcal{D}_1$ being quasi-trivial.

Both $\mathcal{D}_1$ and $\mathcal{D}_1'$ have 6 simple algebra objects lifting from Vec_G. This completely classifies the simple algebra objects in $\mathcal{D}_1'$. For $\mathcal{D}_1$, however, there is an additional unique algebra structure on $\mathbb{1}\oplus X$ lifting from the $\mathbb{Z}/2\mathbb{Z}$ Haagerup-Izumi subcategory $\langle \mathbb{1}, (g,1), X\rangle$. The algebra object $\mathbb{1}\oplus X$ in $\mathcal{D}_1'$ has a unique candidate for its dual fusion category, which coincides with the unique dual fusion category candidate for $\mathbb{Z}/2\mathbb{Z}\oplus X$ in $\mathcal{C}$. This category is defined as follows:

$$\begin{gathered}\mathrm{Irr}(\mathcal{D}_2) = \mathbb{Z}/2\mathbb{Z} \sqcup \{X\} \sqcup (\mathbb{Z}/2\mathbb{Z})\{Y\} \sqcup \{Z\},\\ g\otimes X \cong X \cong X \otimes g, \quad X\otimes X \cong \mathbb{1}\oplus g,\\ g\otimes Y \cong gY \cong Y\otimes g, \quad Y \otimes Y \cong \mathbb{1} \oplus Y \oplus gY,\\ g\otimes Z \cong Z \cong Z\otimes g, \quad Z\otimes Z \cong \mathbb{1}\oplus g \oplus Y^{\oplus 2} \oplus gY^{\oplus 2},\\ X\otimes Y \cong X\otimes gY \cong Y\otimes X \cong gY\otimes X \cong Z,\\ X\otimes Z \cong Z\otimes X \cong Y \oplus gY,\\ Y\otimes Z \cong Z\otimes Y \cong X \oplus Z^{\oplus 2}.\end{gathered}$$

We have denoted $\mathbb{Z}/2\mathbb{Z} = \{\mathbb{1}, g\}$ here. Because this is a bit messy, we clarify that $\langle \mathbb{1}, g, X\rangle$ is a Tambara-Yamagami fusion category, whereas $\langle \mathbb{1}, g, Y, gY\rangle$ is a $\mathbb{Z}/2\mathbb{Z}$ Haagerup-Izumi fusion category. We also remark that $\mathrm{FPdim}(X) = \sqrt{2}$, $\mathrm{FPdim}(Y) = 1+\sqrt{2}$ and $\mathrm{FPdim}(Z) = 2+\sqrt{2}$.

The simple algebra object candidates in $\mathcal{D}_2$ are a little tricky. It's perhaps worth mentioning what the simple algebra object candidates in these dual fusion categories are. The case of $\mathcal{D}_1'$ is easy: as previously mentioned, every candidate lifts from Vec_G. In particular, since the number of simple algebra objects is an invariant of Morita equivalence, this tells us that $\mathbb{Z}/2\mathbb{Z} \oplus X$ cannot have a simple algebra structure in $\mathcal{C}$ if $\mathcal{C} \overset{\text{m.e.}}{\cong} \mathcal{D}_1'$. Conversely, if $\mathcal{C} \overset{\text{m.e.}}{\cong} \mathcal{D}_1$, then we have 7 simple algebra objects since $\mathbb{1} \oplus X$ lifts uniquely in $\mathcal{D}_1$ (check), so there must be exactly one simple algebra structure on one of the copies of $\mathbb{Z}/2\mathbb{Z} \oplus X$ in $\mathcal{C}$.

In $\mathcal{D}_2$, the simple algebra object candidates are less obvious:

(1). $A_2^1 := \mathbb{1} \oplus Y \oplus Z \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY \oplus Z^{\oplus 2}$,
(2). $A_2^2 := \mathbb{1} \oplus gY \oplus Z \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY \oplus Z^{\oplus 2}$,
(3). $A_2^3 := \mathbb{1} \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus g \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus Y \oplus gY \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y^{\oplus 2} \oplus gY^{\oplus 2}$,
(4). $A_2^4 := \mathbb{1} \oplus gY \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY$,
(5). $A_2^5 := \mathbb{1} \oplus gY \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus Y \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY$,
(6). $A_2^6 := \mathbb{1} \oplus Y \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus g \oplus Y \oplus gY$.

Note that all of these lift uniquely from the Haagerup-Izumi subcategory (check) except for A_2^1 and A_2^2. Checking the bimodules returned by the code, only A_2^1 and A_2^2 can provide Morita equivalences to $\mathcal{C}$ (or they can be self-dual), while A_2^4 and A_2^6 must give Morita equivalences to $\mathcal{D}_1$. The objects A_2^3 and A_2^5 must be self-dual.

4. When Is $\mathbb{Z}/2\mathbb{Z} \oplus X$ an Algebra Object?

Recall that $\mathcal{C}$ is Morita equivalent either to $\mathcal{D}_1$ or $\mathcal{D}_1'$, where the former has 7 simple algebra objects and the latter has 6. Since the number of simple algebra objects is an invariant of Morita equivalence, we can state the following.

(1). There is a (unique) simple algebra structure on $\mathbb{Z}/2\mathbb{Z} \oplus X$ in $\mathcal{C}$ if and only if $\mathcal{C} \overset{\text{m.e.}}{\cong} \mathcal{D}_1$.

Recall that $\mathbb{Z}/2\mathbb{Z} \oplus X \overset{\text{m.e.}}{\sim} G \oplus X^{\oplus 2}$. If we assume that $G \oplus X^{\oplus 2}$ contains G as a subalgebra object, then the factoring argument tells us that the dual fusion category of G contains a simple algebra object with Frobenius-Perron dimension $\mathrm{FPdim}(G \oplus X^{\oplus 2})/\mathrm{FPdim}(G) = 2 + \sqrt{2}$. Note that $\mathcal{C}$ has no object with this dimension, and $G \overset{\text{m.e.}}{\sim} \mathbb{1} \oplus X$ is self-dual. Thus, we obtain another condition on the existence of an algebra structure on $\mathbb{Z}/2\mathbb{Z} \oplus X$.

(2). If there is a simple algebra structure on $\mathbb{Z}/2\mathbb{Z} \oplus X$ and it contains G as a subalgebra object, then this subalgebra object is $A_1^1 \overset{\text{m.e.}}{\sim} G \oplus X^{\oplus 4}$, and is not self-dual.

The code finds several candidates for the dual fusion category of A_1^1, including both $\mathcal{C}$ and $\mathcal{D}_1$. Some of these fusion categories (like $\mathcal{D}_1$, as well as some others) admit an algebra object candidate of Frobenius-Perron dimension $2 + \sqrt{2}$.