Ribbon Categories and Reshetikhin-Turaev Invariants

Thomas Dunmore

University of New South Wales

t.dunmore@unsw.edu.au

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A Brief History

In 1984, Vaughan Jones stumbled upon a new polynomial invariant for knots. Somewhat unsatisfyingly, all known definitions were intrinsically 2-dimensional, despite links naturally being 3-dimensional objects.

In 1989, Witten solved this “problem” by constructing the Jones polynomial (and other invariants of links and 3-manifolds) from special kinds of $2 + 1$d topological quantum field theories.

The following year, in 1990, Reshetikhin and Turaev gave a mathematical formulation of Witten’s construction in terms of ribbon graphs coloured by representations of quantum groups.

Categories for the Working Mathematician

Remember from Victor's talk that a category is (loosely) a collection of objects along with a collection of arrows (morphisms) between them.

Our running example will be Vec, the category whose objects are finite-dimensional vector spaces (say, over $\mathbb{C}$), and whose morphisms are linear maps (morphisms $\mathbb{C}^m \to \mathbb{C}^n$ are $n \times m$ complex matrices).

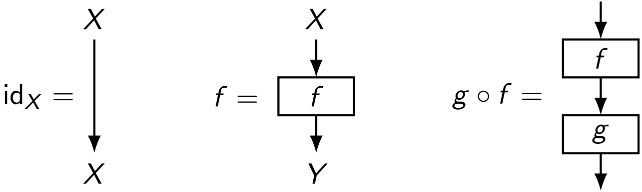
Note that Vec has some nice properties that will be important to us.

- It has finitely many simple objects, and every object is a finite direct sum of them. In Vec, only the one-dimensional vector space $\mathbb{C}$ has no non-trivial subspaces.
- The morphisms between any two objects form a finite-dimensional vector space.

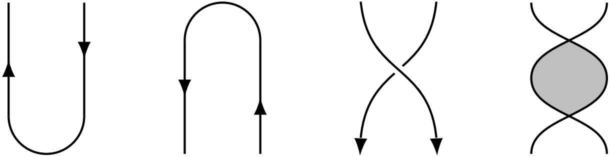
We will ask that all of our categories satisfy these properties.

The Most Useless Graphical Calculus Ever

In any category, we can represent morphisms using diagrams. Great!

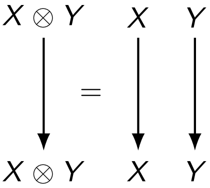


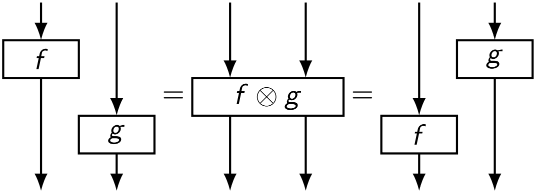
Unfortunately, they tend to be rather boring. If we want to draw knots and links, we're going to need a bit more structure...



Enter: The Second Dimension

To place strings side by side, we define a monoidal structure. That is, we have a product $\otimes$ with unit $\mathbb{1}$, where $X \otimes Y$ denotes the object whose strands are the strands of X and Y placed parallel. Isotopy invariance implies that we must be able to ignore parentheses† and strands of $\mathbb{1}$, and that morphisms should be able to freely move vertically along strands.



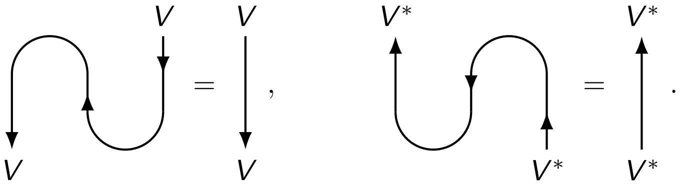


The usual tensor product gives Vec a monoidal structure, where $\mathbb{1} = \mathbb{C}$.

Wiggling Strings

To reproduce even the unknot, we're going to need cups and caps. Our strings are oriented though, so we'll be going up after passing through one!

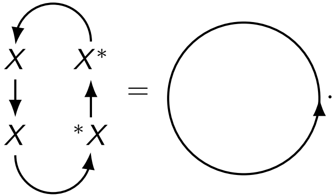
Let X^* denote X with its orientation reversed. Then our cups and caps may be represented by morphisms $\cup : X^* \otimes X \to \mathbb{1}$, $\cap : \mathbb{1} \to X \otimes X^*$. To be invariant under isotopy, the following zig-zag identities must hold:



We call X^* a left dual for X. Swapping the orientations of cups and caps defines right duals, ${}^*X$. A category is called rigid if every object has both left and right duals, and pivotal if these duals coincide in a "natural" way.

Our First (Boring) Knot

In a pivotal category, we can draw oriented circles:



We will ask that $\mathrm{End}(\mathbb{1}) \cong \mathbb{C}$ so we can identify links with numbers.

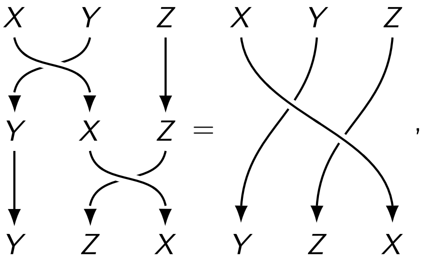
In Vec, the duals of V are given by the dual space, $V^* := \mathrm{Hom}(V, \mathbb{C})$. Given a basis $\{v_i\}_{i=1}^n$ for V with dual basis $\{v_i^*\}_{i=1}^n$ for V^*, we have $\mathrm{ev}_V(v^* \otimes v) = v^*(v)$ and $\mathrm{coev}_V(1) = \sum_{i=1}^n v_i \otimes v_i^*$.

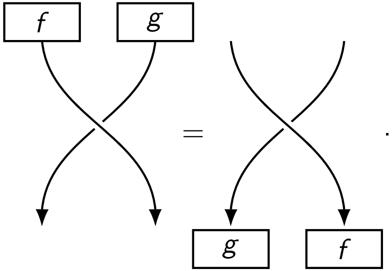
The pivotal structure in Vec is given by the canonical isomorphism between a vector space and its double dual, $v \mapsto (v^* \mapsto v^*(v))$.

Enter: The Third Dimension

Alexander tells us that every link is given by a braid capped off on both ends. To get the other knots, all we need to do now is define crossings.

A braiding is a collection of isomorphisms $X \otimes Y \to Y \otimes X$ that are compatible with the monoidal structure (object "pairings" don't matter) and which allow morphisms to be transported through them. For instance:



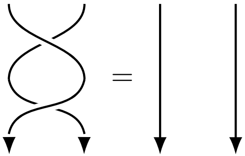


The category Vec is braided using the map $u \otimes v \mapsto v \otimes u$.

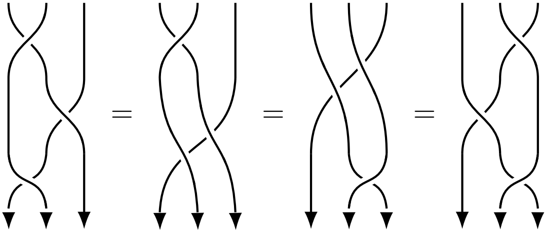
The Reidemeister Moves

Two knots are equivalent if and only if they are related by a sequence of *Reidemeister moves*. Does our graphical calculus give invariants of knots?

Type II:

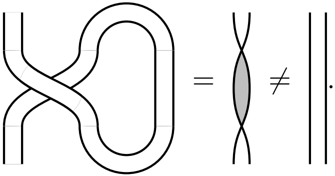


Type III:

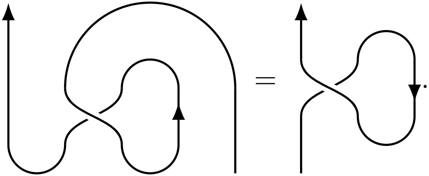


Twists and Ribbons

Unfortunately, the type I move does not hold in general. The reason is clear if we think of our strands as ribbons rather than strings:

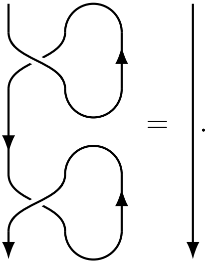


In a pivotal braided category, the diagram above defines a twist $X \to X$. We call it a ribbon structure if we can pull it out of zig-zags:



Invariants of Knots and 3-Manifolds

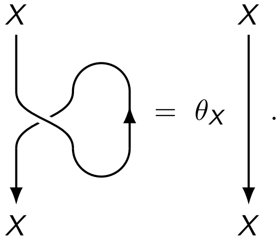
Note that twists satisfy the following modified type I Reidemeister move:



The numbers associated to link diagrams coloured by any object give invariants of framed, oriented links. This also gives invariants of closed, orientable, connected 3-manifolds by a theorem of Lickorish and Wallace: all such manifolds can be obtained by Dehn surgery on a framed link in S^3.

Forgetting the Framing

It turns out that ribbon twists of simple objects are scalings of the identity:



For any link diagram L coloured by X, $\theta_X^{-\mathrm{Wr}(L)} L$ gives an invariant of L as an unframed link. Here, $\mathrm{Wr}(L)$ is the writhe of L (the number of overcrossings minus the number of undercrossings).

While one can always forget the framing, forgetting the orientation is trickier. It is necessary but not sufficient for the object to be self-dual!

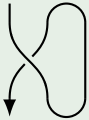
Examples at Long Last

Example (Vector Spaces)

For $\mathbb{C}^n$ in Vec, we get $L \mapsto n^c$ for a link L with c components. Wow!!

Example (Graded Vector Spaces)

Consider the category Vec_G of vector spaces graded by G. This category has 1D simple objects $\mathbb{C}_g$ for each $g \in G$, which satisfy $\mathbb{C}_g \otimes \mathbb{C}_h = \mathbb{C}_{gh}$ and $\mathbb{C}_g^* = \mathbb{C}_{g^{-1}}$. The braidings are given by quadratic forms on G. For instance, if $G = \mathbb{Z}/2\mathbb{Z} = \{\mathbb{1}, g\}$, we can define  $= i \cdot \mathrm{id}_{\mathbb{C}_\mathbb{1}} =$  and

 $= i \cdot \mathrm{id}_{\mathbb{C}_\mathbb{1}}$,  $= i \cdot \mathrm{id}_{\mathbb{C}_g}$.

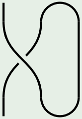
The framed link invariant is $L \mapsto i^{\mathrm{Wr}(L)}$, so the unframed invariant is trivial. In this example, $\mathbb{C}_g$ is self-dual, but the orientation cannot be removed!

The Important Example

Example (Jones Polynomial)

The category $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$ for $q \neq \pm 1$ a root of unity is ribbon. It has a 2D simple object V that is self-dual with $\cup = \begin{bmatrix} 0 & 1 & -q^{-1} & 0 \end{bmatrix}$ and $\cap = \begin{bmatrix} 0 & -q & 1 & 0 \end{bmatrix}^T$. The braiding and ribbon twist for V are given by

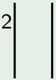
$$= q^{-1/2} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{bmatrix},$$



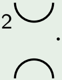
$$= -q^{-3/2} \cdot \mathrm{id}_V.$$

It's an easy exercise to show that this satisfies the following skein relation:

$$= q^{1/2}$$



$$+ \, q^{-1/2}$$



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I've implicitly chosen here the structure that removes the orientation.

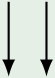
The Jones Polynomial Generalized

Example (HOMFLYPT Polynomial)

More generally, $\mathrm{Rep}(U_q(\mathfrak{sl}_n))$ is ribbon. The standard (n-dimensional) representation V_n has the same evaluation and coevaluation as Vec, but letting $v_{n-1}, v_{n-3}, \ldots, v_{3-n}, v_{1-n}$ be a basis for V_n, its braiding is given by

$$: v_i \otimes v_j \mapsto q^{-1/n} \begin{cases} q v_i \otimes v_j, & \text{if } i = j; \\ v_j \otimes v_i, & \text{if } i > j; \\ v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j, & \text{if } i < j. \end{cases}$$

The ribbon structure is $q^{1/n-n} \cdot \mathrm{id}_{V_2}$. This time, we have skein relation

q^n  $- q^{-1/n}$  $= (q - q)^{-1}$  .

This is the *n*-specialization of the HOMFLYPT polynomial.

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Thank you for listening!