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Article *in* Journal of Guidance, Control, and Dynamics · January 2014

DOI: 10.2514/1.62549

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The QUEST algorithm[1,2] has been the most widely and most frequently implemented algorithm for batch three-axis attitude estimation. It is also an important component of many attitude Kalman filters, where it serves as a preprocessor of star-tracker data.[3] The QUEST algorithm was first implemented for the Magsat Mission in 1979.[4] The FORTRAN code for the QUEST algorithm written by the second author of this Note has gone through very few changes. The only change has been a rearrangement of terms by F. Landis Markley of the NASA Goddard Space Flight Center (unpublished and only affecting the computer code) to improve numerical significance. The FORTRAN code for the QUEST algorithm has been made available at the home site of the second author of this Note[a].

QUEST has supported hundreds of missions, both in Earth orbit and at the far reaches of the solar system, without an anomaly. Recently, however, a robustness issue of QUEST was raised.[5] Specifically, it was shown that the Newton-Raphson iteration employed by the QUEST algorithm to solve the characteristic equation for the maximum eigenvalue of the Davenport matrix K did not converge in an extreme case.

It should be pointed out that the issue was about the implementation detail of the characteristic polynomial, not the mathematical procedure of the algorithm. In Refs. [1] and [2], where the QUEST algorithm was first presented, the characteristic polynomial was given by

$$\psi_{\text{QUEST}}(\lambda) = \lambda^4 - (a+b)\lambda^2 - c\lambda + (ab + cs - d)$$

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[a] http://home.comcast.net/~mdshuster3/HTF_NASA_QUEST_FORTRAN_Code_1987_FLM.pdf

where a, b, c, d, s are coefficient parameters and λ is the unknown. The FORTRAN code for the QUEST algorithm uses this form of the characteristic polynomial in the Newton-Raphson iteration. So does the implementation of the QUEST algorithm of Ref. [5]. We emphasize that the two QUEST implementations are not the same. The former has several data validation tests (which are part of the FORTRAN code) but the latter does not have any. It was the latter that was shown to be numerically not robust in the extreme case of Ref. [5].

This Note shows that even in the extreme case of Ref. [5] and even without the data checks in the FORTRAN code, the implementation of the QUEST algorithm will be as robust as those of the other fast attitude estimation algorithms reviewed in Ref. [5] if the following substitution is made to the characteristic polynomial:

$$\lambda^4 - (a+b)\lambda^2 - c\lambda + (ab + cs - d) \;\rightarrow\; (\lambda^2 - a)(\lambda^2 - b) - c\lambda + (cs - d)$$

I. The Wahba Problem and the Characteristic Polynomial

The modern fast batch attitude estimation algorithms all depend on the solution of the Wahba problem,[6] namely, to find the attitude, expressed here by the attitude matrix A,[7] which minimizes the loss function

$$J(A) \equiv \frac{1}{2}\sum_{k=1}^{n} a_k ||\hat{\mathbf{W}}_k - A\hat{\mathbf{V}}_k||^2 \tag{1}$$

where $\hat{\mathbf{W}}_k$, $k = 1, \ldots, n$, are a set of n measured directions. $\hat{\mathbf{V}}_k$, $k = 1, \ldots, n$, are a set of n corresponding reference directions, a_k, $k = 1, \ldots, n$, are a set of positive weights, usually chosen as $a_k = 1/\sigma_k^2$, with σ_k^2 the variance parameters of the measurement vectors, and $\|\cdot\|$ denotes the vector norm. We will assume, without loss of generality, that these weights have unit sum.

The loss function can be rewritten as

$$J(A) = \lambda_o - g_A(A) \tag{2}$$

where

$$\lambda_\mathrm{o} \equiv \sum_{k=1}^{n} a_k = 1 \tag{3}$$

and the gain function $g_A(A)$ is given by

$$g_A(A) = \mathrm{tr}\left[B^\mathrm{T} A\right] \tag{4}$$

with the attitude profile matrix B given by

$$B \equiv \sum_{k=1}^{n} a_k \, \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^{\mathrm{T}} \tag{5}$$

From Eq. (2), minimizing the loss function is equivalent to maximizing the gain function.

Paul Davenport showed that the gain function can be written alternately in terms of the quaternion $\bar{q}$[1,2]

$$g_{\bar{q}}(\bar{q}) \equiv g_A(A(\bar{q})) = \bar{q}^{\mathrm{T}} K \bar{q} \tag{6}$$

where the Davenport matrix K is given by

$$K = \begin{bmatrix} S - sI & \mathbf{Z} \\ \mathbf{Z}^{\mathrm{T}} & s \end{bmatrix} \tag{7}$$

with

$$S \equiv B + B^{\mathrm{T}}\,, \quad s \equiv \operatorname{tr} B, \quad \text{and} \quad \mathbf{Z} \equiv [B_{23} - B_{32},\, B_{31} - B_{13},\, B_{12} - B_{21}]^{\mathrm{T}} \tag{8}$$

As a result, the maximization of $g_A(A)$ and $g_{\bar{q}}(\bar{q})$, can be accomplished by finding the solution of the characteristic-value problem

$$K \bar{q}^* = \lambda_{\max} \bar{q}^* \tag{9}$$

where $\lambda_{\max}$ is the largest characteristic value of the 4×4 real symmetric matrix K and is also the maximum value of $g_{\bar{q}}(\bar{q})$ and $g_A(A)$. The maximization of $g_A(A)$ and $g_{\bar{q}}(\bar{q})$ has led to numerous solutions of the Wahba problem, of which the most prominent have been (in chronological order) Davenport's original q-method, QUEST, Markley's SVD algorithm, FOAM, ESOQ, and ESOQ2. These are all reviewed briefly in Ref. [5]. QUEST, FOAM, ESOQ, and ESOQ2 are referred to as fast attitude estimation algorithms because they are faster than the numerically more robust Davenport's original q-method and Markley's SVD algorithm.

From Eq. (6), we see that if a value can be found for the maximum characteristic value of the Davenport matrix K, then the construction of the attitude, either as the attitude matrix (FOAM) or as the quaternion (QUEST, ESOQ, ESOQ2) can be accomplished easily. Thus, the central part of the fast attitude estimation algorithms has been the computation of $\lambda_{\max}$.

An important early result was that for unit-sum weights and the QUEST measurement model,[8,9]

$$\lambda_{\max} = 1 - \frac{1}{2}\, \sigma_{\mathrm{tot}}^2 \, \chi^2(2n-3) \tag{10}$$

where σ_{tot}^2 is a cumulative variance characteristic of the measurement vectors, and $\chi^2(2n-3)$

is a χ^2 random variable with $2n-3$ degrees of freedom. For n direction measurements with a common accuracy σ in the QUEST measurement model, σ_{tot}^2 has the value σ^2/n. Thus, for example, for a star tracker with a single-star direction accuracy of 1 arcsec and observing three stars, σ_{tot}^2 will have the value $\sigma_{\mathrm{tot}}^2 \approx 7.72 \times 10^{-12}$. It follows that λ_{max} differs from unity in this case by terms of order 10^{-11}. As first noted in 1978,[1] this means that our zero-th-order approximation for λ_{max}, namely

$$\lambda_{\mathrm{max}}(0) = \lambda_{\mathrm{o}} = 1 \tag{11}$$

should generally be adequate for computing the attitude. It also suggests that one may use this zero-th-order approximation as an excellent starting value for further refinement.

The instrument for this refinement for all of the fast attitude estimation algorithms has been the application of the Newton-Raphson method to the characteristic polynomial for λ, namely,

$$\psi(\lambda) = \det\left[\lambda\, I_{4\times 4} - K\right] \tag{12}$$

For the QUEST algorithm, this has had the form[1,2]

$$\psi_{\mathrm{QUEST}}(\lambda) = \lambda^4 - (a+b)\lambda^2 - c\lambda + (ab + cs - d) \tag{13}$$

with

$$a = s^2 - \mathrm{tr}\big(\mathrm{adj}\, S\big) \tag{14a}$$

$$b = s^2 + \mathbf{Z}^{\mathrm{T}}\mathbf{Z} \tag{14b}$$

$$c = \det S + \mathbf{Z}^{\mathrm{T}} S \mathbf{Z} \tag{14c}$$

$$d = \mathbf{Z}^{\mathrm{T}} S^2 \mathbf{Z} \tag{14d}$$

The function "adj" denotes the matrix adjoint, and "det" the determinant.

II. The Extreme Case

Reference [5] includes an extreme case (scenario 2) to test the accuracy and robustness of the attitude estimation algorithms. It examines the case where there are three measured directions, one along the spacecraft body x-axis and two in the body xy-plane separated from the negative body x-axis by approximately 4.5 degrees. The exact values of the three directions are

$$\hat{\mathbf{W}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{W}}_2 = \begin{bmatrix} -0.99712 \\ 0.07584 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \hat{\mathbf{W}}_3 = \begin{bmatrix} -0.99712 \\ -0.07584 \\ 0 \end{bmatrix} \tag{15}$$

The values of the components have been chosen so that each column vector has exactly unit norm. The three measurements of scenario 2 of Ref. [5] are modeled according to the QUEST measurement model with variance parameters given by

$$\sigma_1 = 1 \text{ arcsec}, \qquad \sigma_2 = \sigma_3 = 1 \ \text{deg} \tag{16}$$

The cumulative variance in Eq. (10) is $\sigma_{\text{tot}}^2 \approx \sigma_1^2 \approx 2.35 \times 10^{-11}$ and so the statistical error of $\lambda_{\max}$ is in the order of 10^{-11}. From Ref. [2], the inverse covariance covariance matrix is given by

$$P_{\theta\theta}^{-1} = \text{diag}\left[\frac{1}{\sigma_x^2}, \frac{1}{\sigma_y^2}, \frac{1}{\sigma_z^2}\right] = \text{diag}\left[\frac{2\sin^2\alpha}{\sigma_2^2}, \left(\frac{1}{\sigma_1^2} + \frac{2\cos^2\alpha}{\sigma_2^2}\right), \left(\frac{1}{\sigma_1^2} + \frac{2}{\sigma_2^2}\right)\right] \tag{17}$$

where α is the angle between $-\hat{\mathbf{W}}_1$ and $\hat{\mathbf{W}}_2$ or $\hat{\mathbf{W}}_3$, approximately 4.5 degrees. Substituting the values from Eqs. (15) and (16) leads to attitude estimate error levels about each axis of

$$\sigma_x \approx 9.32 \ \text{deg} \qquad \text{and} \qquad \sigma_{yz} \equiv \sqrt{\sigma_y^2 + \sigma_z^2} \approx 1.41 \text{ arcsec} \tag{18}$$

in agreement with equation (90) of Ref. [5]. The attitude accuracy about the x-axis is extremely bad compared to that about the other two axes, $\sigma_x/\sigma_y = \sigma_x/\sigma_z = 34,000$.

III. The Robustness of the QUEST Characteristic Polynomial

The QUEST characteristic polynomial can be written in either the *expanded* form

$$\psi_{\text{QUEST-exp}}(\lambda) = \lambda^4 - (a+b)\lambda^2 - c\lambda + (ab + cs - d) \tag{19a}$$

or the equivalent *partially-factored* form

$$\psi_{\text{QUEST-fac}}(\lambda) = (\lambda^2 - a)(\lambda^2 - b) - c\lambda + (cs - d) \tag{19b}$$

They are also equivalent to the FOAM characteristic polynomial, given by[10]

$$\psi_{\text{FOAM-fac}}(\lambda) = \left(\lambda^2 - \|B\|_F^2\right)^2 - 8\lambda \det B - 4\,\|\text{adj}\, B\|_F^2 \tag{20}$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. In the numerical tests of Ref. [5], the FOAM characteristic polynomial was used by FOAM, ESOQ, and ESOQ2. Note that it is partially-factored form, too.

The expanded and partially-factored forms of the QUEST characteristic polynomial are identical and yield the same result for infinitely precise arithmetic. For finite 64-bit arithmetic (IEEE double precision[11]) and for the parameter values of the extreme case, they do not.

To see how the different forms of the QUEST characteristic polynomial lead to large errors in the Newton-Raphson sequence for λ_{max} for the extreme case, examine the first correction to λ_{max} in that sequence in a typical run with

$$a = 0.9999999974609042 \tag{21a}$$
$$b = 0.9999999971236696 \tag{21b}$$
$$c = 0 \tag{21c}$$
$$d = 5.946137136732494 \times 10^{-18} \tag{21d}$$

For the extreme case, the coefficient c in Eq. (13) vanishes identically because the measurements are coplanar.

For both forms of the QUEST characteristic polynomial, the first correction in the Newton-Raphson sequence is

$$\Delta\lambda_{\text{max}}(1) = -\psi(1)/\psi'(1) \tag{22a}$$
$$\lambda_{\text{max}}(1) = \lambda_{\text{o}} + \Delta\lambda_{\text{max}}(1) \tag{22b}$$

where $\psi'(\lambda)$ denotes the first derivative of $\psi(\lambda)$ with respect to λ and $\lambda_{\text{o}} = 1$.

For the partially-factored form

$$\begin{aligned}\psi_{\text{QUEST-fac}}(1) &= (1-a)(1-b) - c + (cs - d)\\ &= (1 - 0.9999999974609042)(1 - 0.9999999971236696)\\ &\quad - 0 + (0 - 5.946137136732494 \times 10^{-18})\\ &= (2.5390958 \times 10^{-9})\,(2.8763304 \times 10^{-9})\\ &\quad - 5.946137136732494 \times 10^{-18}\\ &= 1.3571414 \times 10^{-18}\end{aligned} \tag{23}$$

$$\begin{aligned}\psi'_{\text{QUEST-fac}}(1) &= 2(1-a) + 2(1-b) - c\\ &= 2(2.5390958 \times 10^{-9}) + 2(2.8763304 \times 10^{-9}) - 0\\ &= 1.0830853 \times 10^{-8}\end{aligned} \tag{24}$$

In IEEE double-precision arithmetic,[11] both $\psi_{\text{QUEST-fac}}(1)$ and $\psi'_{\text{QUEST-fac}}(1)$ as calculated above have up to seven significant figures, and consequently, $\Delta\lambda_{\max}$ calculated from Eq. (22a) will have seven significant figures. Note, however, that Eqs. (22) do not include the cumulative round-off error from the many arithmetic operations in the computations. Therefore, the number of *accurate* significant figures may be less than seven, probably close to five. However, the magnitude of $\Delta\lambda_{\max}(1)$ is approximately 10^{-10}, so that the contribution of $\Delta\lambda_{\max}(1)$ can increase the numerical error in $\lambda_{\max}$ by 10^{-15} or 10^{-16}, which is close to the limit of IEEE double-precision representation. The numerical error in $\lambda_{\max}$ (10^{-15} or 10^{-16}) in this case is far smaller than the statistical error (10^{-11}) in $\lambda_{\max}$.

Examine now the same calculation for the expanded form of the QUEST characteristic equation. In that case, $\psi'_{\text{QUEST-exp}}(1)$ will have the same value as $\psi'_{\text{QUEST-fac}}(1)$ in Eq. (24) to eight significant figures, but

$$\begin{aligned}
\psi_{\text{QUEST-exp}}(1) &= 1-(a+b)-c+(ab+cs-d) \\
&= 1-1.9999999945845738-0+0.9999999945845738 \\
&\quad +(0-5.946137136732494\times 10^{-18}) \\
&= 1-1.9999999945845738+0.9999999945845738-5.946137136732494\times 10^{-18} \\
&= \bigcirc \times 10^{-16}
\end{aligned} \tag{25}$$

where $\bigcirc$ denotes a number of order 1 with no significant figures. Thus, the numerical error in $\Delta\lambda_{\max}(1) = -\psi(1)/\psi'(1)$ is in the order of 10^{-8}, much larger than the statistical error (10^{-11}). This explains why the Newton-Raphson iteration with the expanded form of the characteristic polynomial does not converge and yields inaccurate $\lambda_{\max}$ in the extreme case. This also shows that the robustness issue of the QUEST characteristic polynomial is not due to errors in the coefficients used in the polynomial, but due to the order in which the terms are combined.

IV. Further Analysis

The numerical analysis in the previous section uses the specific numerical values of a, b, c, d, and s from a typical run to show that the expanded form performs poorly because of the significant precision loss. One can verify that loss of precision will always occur for Eq. (25) (but not Eq. (23)) if $a \approx b \approx \lambda_{\max}^2$, $ab \gg |cs-d|$, and $ab \gg |c\lambda_{\max}|$. In this section, we will show that the inequalities hold if the accuracy of a single vector observation dominates.

The profile matrix B for these cases is of the form

$$B = U \begin{bmatrix} 1/\tau^2 & 0 & 0 \\ 0 & \epsilon_1/\tau^2 & 0 \\ 0 & 0 & \epsilon_2/\tau^2 \end{bmatrix} V^T \tag{26}$$

where U and V are proper orthogonal matrices (with determinant +1) and $|\epsilon_2| \leq \epsilon_1 \ll 1$. Note that when $\epsilon_1 = \epsilon_2 = 0$, the profile matrix is rank one and corresponds to a single vector observation and the three-axis attitude is unobservable. Reference [12] shows that the optimal attitude is given by

$$A_{\text{opt}} = UV^T \tag{27}$$

and

$$\lambda_{\max} = (1 + \epsilon_1 + \epsilon_2)/\tau^2 \tag{28}$$

Using the definitions of a, b, c, d, s and after some straightforward algebras, we have

$$a = 1/\tau^4 + \mathcal{O}(\epsilon_1, |\epsilon_2|)/\tau^4 \tag{29a}$$
$$b = 1/\tau^4 + \mathcal{O}(\epsilon_1, |\epsilon_2|)/\tau^4 \tag{29b}$$
$$c = 8\epsilon_1\epsilon_2/\tau^6 \tag{29c}$$
$$d = \mathcal{O}(\epsilon_1^2, \epsilon_2^2, \epsilon_1|\epsilon_2|)/\tau^8 \tag{29d}$$
$$s = \mathcal{O}(1)/\tau^2 \tag{29e}$$

Here the notation $\mathcal{O}$ denotes a positive or negative quantity of the same order of magnitude as the maximum argument. It follows that

$$ab = 1/\tau^8 + \mathcal{O}(\epsilon_1, |\epsilon_2|)/\tau^8 \tag{30a}$$
$$c\lambda_{\max} = 8\epsilon_1\epsilon_2(1 + \epsilon_1 + \epsilon_2)/\tau^8 \tag{30b}$$
$$cs - d = \mathcal{O}(\epsilon_1^2, \epsilon_2^2, \epsilon_1|\epsilon_2|)/\tau^8 \tag{30c}$$

From Eqs. (28) and (29), we immediately have $a \approx b \approx \lambda_{\max}^2$. From Eqs. (30), because $|\epsilon_2| \leq \epsilon_1 \ll 1$, $ab \gg |cs - d|$ and $ab \gg |c\lambda_{\max}|$.

V. Numerical Result

To see that the use of the partially-factored form of the QUEST characteristic polynomial allows the convergence of the Newton-Raphson sequence for $\lambda_{\max}$ and accurate attitude estimation for the extreme case, we have recomputed Table 2 of Ref. [5] but with all characteristic polynomials in partially-factored form. QUEST uses the QUEST characteristic polynomial given by Eq. (19b). FOAM, ESOQ, and ESOQ2 use the FOAM characteristic polynomial given by Eq. (20). The root-mean-square values of the loss function and the estimate-to-true attitude errors, denoted in our Table 1 by ΔJ, σ_x, and σ_{yz}, respectively, are computed over 1,000 test runs the same way as Ref. [5], which do not use unit-sum weights but $a_k = 1/\sigma_k^2$, $k = 1, 2, 3$, where σ_k is the accuracy parameter of the QUEST measurement model.[2] We have not included results for ESOQ1.1 or ESOQ2.1 which are not important for the discussion and are unchanged from the values in Ref. [5]. We have, however, given the results for the four iterative algorithms for up to five iterations, the better to observe the convergence properties. As can be seen from our Table 1, all algorithms perform equally admirably in the extreme case when the characteristic polynomial is partially factored. We have

Table 1. Estimation Results for Scenario 2 of Reference 5 for Partially-Factored Characteristic Polynomials

Algorithm	Iterations	ΔJ	σ_x (deg)	σ_{yz} (arcsec)
q-Davenport	—	—	9.30	1.43
M-SVD	—	1.40 e-5	9.30	1.43
FOAM	0	—	9.42	1.41
	1	0.102	9.42	1.41
	2	0.952 e-3	9.30	1.43
	3	1.40 e-5	9.30	1.43
	4	1.37 e-5	9.30	1.43
	5	1.38 e-5	9.30	1.43
QUEST	0	—	9.25	1.43
	1	0.102	9.29	1.43
	2	0.953 e-3	9.30	1.43
	3	1.33 e-5	9.30	1.43
	4	1.33 e-5	9.30	1.43
	5	1.33 e-5	9.30	1.43
ESOQ	0	—	9.25	1.43
	1	0.102	9.29	1.43
	2	0.952 e-3	9.30	1.43
	3	1.40 e-5	9.30	1.43
	4	1.37 e-5	9.30	1.43
	5	1.38 e-5	9.30	1.43
ESOQ2	0	—	9.42	1.43
	1	0.102	9.30	1.43
	2	0.952 e-3	9.30	1.43
	3	1.40 e-5	9.30	1.43
	4	1.37 e-5	9.30	1.43
	5	1.38 e-5	9.30	1.43

* We use the designations "q-Davenport" and "M-SVD" to avoid confusing Davenport's original q-algorithm with the QUEST algorithm and others which are also implementations of Davenport's q-algorithm and to avoid confusing Markley's SVD algorithm from the SVD algorithm of Numerical Linear Algebra.[13]

also repeated the calculations of our Table 1 for the extreme case but with the characteristic polynomials of all fast attitude estimation algorithms in expanded form. As expected, they perform uniformly horribly for the extreme case, with σ_x in the order of 50 degrees. This clearly shows 1) that the partially-factored form of the QUEST characteristic polynomial is numerically more robust than the expanded form and 2) that for the extreme case, the numerical round-off errors dominate the performance of Newton-Raphson method and the fast attitude estimation algorithms.

When the Newton-Raphson sequence converges to the maximum eigenvalue $\lambda_{\max}$ of K, all the fast attitude estimation algorithms will yield the same attitude quaternion solution (the eigenvector of K corresponding to $\lambda_{\max}$) and the same estimate-to-true attitude error. When approximate values of $\lambda_{\max}$ are used, for example, $\lambda_{\max} = 1$, the attitude quaternions (not an eigenvector of K) and attitude errors obtained using different algorithms are different in general. For small approximation errors in $\lambda_{\max}$, Table 1 shows that the difference is mainly in the rotation about the least observable direction. The expanded forms of the characteristic polynomials lead to much larger errors in $\lambda_{\max}$ than the statistical errors. The large errors in $\lambda_{\max}$ lead to large attitude errors, which can be as large as close to 180 degrees.

For this case, the Newton-Raphson sequence does not always yield more accurate attitude estimates. For the specific 1,000 test runs, the attitude errors of QUEST and ESOQ obtained with one or more Newton-Raphson iterations are not smaller than the attitude errors obtained with $\lambda_{\max} = 1$. That is mainly due to the closeness of $\lambda_{\max}$ to 1 and the random variations of the results.

Although not necessary for the extreme case, which has $c = 0$ identically, we point out that it is numerically more robust to replace the expression for c in Eq. (14c) by $c = 8 \det B$.

Finally, note that all the numerical results in the extreme case are obtained in IEEE double-precision arithmetic using approximately 16 decimal digits (52-bit mantissa) and that all the conclusions drawn from the numerical results are valid for double-precision floating-point numbers only. Were single-precision floating-point numbers used in the tests, which only have approximately 7 decimal digits (23-bit mantissa), no attitude estimation algorithms, with either partially-factored or expanded characteristic polynomials, would yield any accurate results in the extreme case because the round-off errors would reduce the rank of the profile matrix to one and make the three-axis attitude unobservable. Were quadruple-precision floating-point numbers with approximately 34 decimal digits (112-bit mantissa) used, the algorithms with expanded characteristic polynomials would perform as well as those with partially-factorized polynomials even in the extreme case.

VI. Conclusion

With a rearrangement of terms in the QUEST characteristic polynomial, the poor convergence of the Newton-Raphson sequence for $\lambda_{\max}$ shown in scenario 2 of Reference [5] was fixed. It is recommended that the partially-factorized form of the QUEST characteristic polynomial be used in the implementation of the QUEST algorithm.

VII. Acknowledgment

The authors are very grateful to F. Landis Markley, who graciously supplied them with the MATLAB® m-files used in Reference [5], offered numerous advice, and answered many technical questions.

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