Lesson 1

Topic: Stable Matching/Gale Shapley Algorithm

Theoretical Understanding (15 Min)

The Gale-Shapley algorithm, also known as the Stable Marriage Problem algorithm, is a mechanism for solving the problem of matching two sets of elements based on their preferences. This algorithm was proposed by mathematicians David Gale and Lloyd Shapley in 1962 and has applications in various fields, including economics and computer science.

The problem involves two equally-sized sets of elements, often represented as men and women, where each element ranks the members of the other set-in order of preference. The goal is to find a stable matching, where no two elements from different sets prefer each other over their current partners. In other words, there should be no "rogue couples" that would prefer to be with each other rather than their current partners.

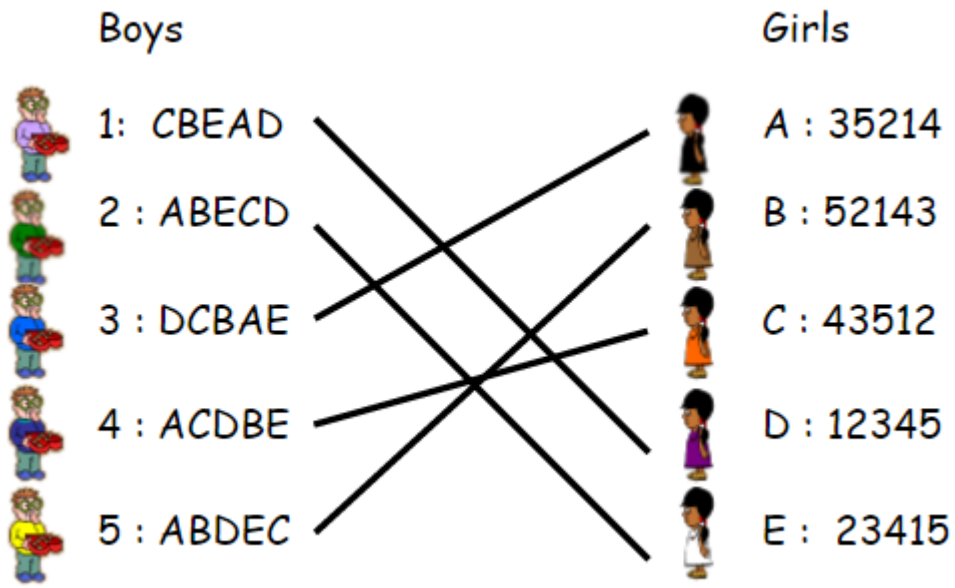
Here's a brief overview of the Gale-Shapley algorithm:

1. Initialization: Each element (man or woman) proposes to their most preferred choice from the other set. Initially, all proposals are pending.
2. Proposal Phase: In each iteration, each woman reviews the proposals she has received and chooses the best one among them. If she is not engaged, she accepts the proposal. If she is already engaged and prefers the new proposal to her current partner, she breaks the engagement and accepts the new proposal. The rejected suitor returns to the pool of available suitors.
3. Rejection Phase: The rejected suitors who were not chosen by any woman will propose to their next preferred choice (the next woman on their list).
4. Iteration and Termination: Steps 2 and 3 are repeated until each woman is engaged to a partner. The process continues until no more changes in engagements occur. At this point, a stable matching has been achieved.

The key insight of the Gale-Shapley algorithm is that it always terminates with a stable matching, meaning there are no rogue couples. One important property of this algorithm is that it's "deferential" to one group (typically the proposing group, often represented by men in the traditional setup), which ensures that once a woman accepts a proposal, she won't later reject it in favour of someone else.

The algorithm guarantees that the resulting matching is both stable and favours the proposing group. However, it's worth noting that the final outcome can be influenced by the order in which proposals are made, and the algorithm doesn't necessarily result in a "fair" outcome from the perspective of the other group.

Overall, the Gale-Shapley algorithm is a fundamental tool for solving matching problems in situations where preferences matter and stability is desired.



We will also try to implement the following example via code section as well.

Complexity Aspect (5 Mins)

The time complexity of the algorithm is O(n^2), where n is the number of individuals in each set. However, the algorithm is relatively efficient for most practical applications.

Overall, the Gale-Shapley Algorithm is a powerful tool in the field of matching theory and has numerous real-world applications, such as matching medical students to residency programs, job seekers to job openings, and kidney donors to recipients.

Understanding algorithm with pseudo code and implementation (15mins)

Algorithm:

1. All individuals have ranked members of opposite set-in order of preference.
2. One of the 2 sets is chosen to make proposal. (Either set will produce stable matchings)
3. One individual from the proposing group who is not already engaged will propose to their most preferable option who has not already rejected them.
4. The person being proposed to will
5. Accept if this is their first offer.
6. Reject if this is worse than their current offer
7. Accept if this is better than their current offer. In this case they will jilt their previous offer.
8. When all members of proposing group are matched the loop terminates. The currently engaged pairs are stably matched.

Pseudo Code:

```
Initialize all men and women to free
while there exists a free man m who still has a woman w to propose to
{
   w = m's highest ranked such woman to whom he has not yet proposed
   if w is free
     (m, w) become engaged
   else some pair (m', w) already exists
     if w prefers m to m'
        (m, w) become engaged
         m' becomes free
     else
        (m', w) remain engaged
}
```

Implementation (Python): Check Code file for the code execution.

References (20 Mins)

1. Stable Marriage Problem – Numberphile: https://youtu.be/Qcv1IqHWAzg?si=7yLtDyNLbOPSrcVP
2. Stable Matching Algo Explanation – Gfg: https://youtu.be/o1olHmxDzTw?si=u5x2FB62gPBsSLEb